2

Stochastic discounting

In this chapter we define a mathematically consistent model for calculating time values of cash flows. The key objects are so-called deflators which play the role of stochastic discount factors. Our definition (via deflators) leads to market values which are consistent with the usual financial theory that involves risk neutral valuation. Typically, in financial mathematics the pricing formulas are based on equivalent martingale measures (see, for example, Föllmer-Schied [FS04]), economists use the notion of state price density processes (see Malamud et al. [MTW08]) and actuaries use the terminology of deflators under the real world probability measure (see Duffie [Du96] and Bühlmann et al. [BDES98]). In this chapter we describe these terminologies.

Moreover, we would like to emphasize that in financial mathematics one usually works under risk neutral measures (equivalent martingale measures) for pricing financial assets. In actuarial mathematics, however, one should also understand the processes under the real world probability measure (physical measure) which makes it necessary that we understand the connection between these two probability measures as well as the transform of measure techniques.

2.1 Basic discrete time model

In this chapter we develop the theoretical foundations of market-consistent valuation. We work in a discrete time setting which has the advantage that the mathematical machinery becomes simpler for the calculation of the price processes (for continuous time models we refer to the standard literature on financial mathematics, see for example Jeanblanc et al. [JYC09]).

Choose $n \in \mathbb{N}$ fixed. This is the final time horizon. Then, w.l.o.g. we consider cash flows on the yearly grid $t = 0, 1, \ldots, n$.

We choose a probability space $(\Omega, \mathcal{F}, P)$ and an increasing sequence of σ-fields $\mathbb{F} = (\mathcal{F}_t)_{t=0,\ldots,n}$ with

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$$\{\emptyset, \Omega\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n \tag{2.1}$$

and for simplicity, we assume $\mathcal{F}_n = \mathcal{F}$. We call $(\Omega, \mathcal{F}, P, \mathbb{F})$ a filtered probability space with filtration $\mathbb{F}$. The σ-field $\mathcal{F}_t$ plays the role of the information available/known at time t. This includes demographic information, insurance technical information on insurance contracts, financial and economic information and any other information (weather conditions, legal changes, politics, etc.) that is available at time t.

Moreover, we assume that we have a sequence of $\mathbb{F}$-adapted random variables

$$\mathbf{X} = (X_0, X_1, \ldots, X_n) \tag{2.2}$$

on the filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F})$. That is, X_t is an $\mathcal{F}_t$-measurable random variable for all $t = 0, \ldots, n$.

Interpretation and aim. $\mathbf{X}$ is a (random) cash flow, with single payments X_t at time t. If we have information $\mathcal{F}_t$, then X_k is known for all $k \leq t$, otherwise it may be random. Henceforth, on the one hand, we need to predict future payments X_s, $s > t$, based on the information $\mathcal{F}_t$ available at time t. On the other hand, our goal is to determine the (time) value of such cash flows $\mathbf{X}$ at any time $t = 0, \ldots, n$, see also Figure 2.1.

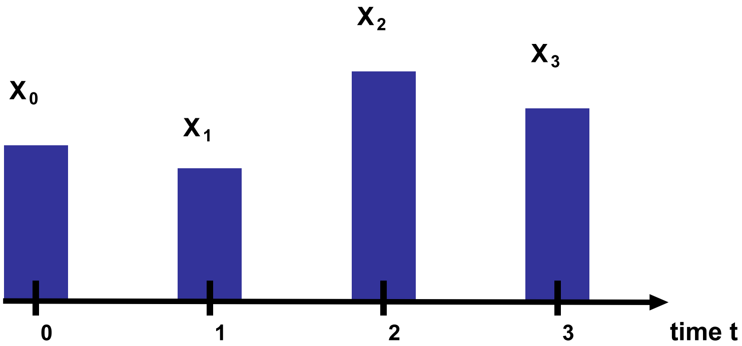

Fig. 2.1. Cash flow $\mathbf{X} = (X_0, X_1, \ldots, X_n)$

We make some technical assumptions.

Assumption 2.1 *Assume that every component of* $\mathbf{X}$ *is square integrable.*

For a general square integrable cash flow $\mathbf{X} = (X_0, X_1, \ldots, X_n)$ we write

$$\mathbf{X} = (X_0, X_1, \ldots, X_n) \in L^2_{n+1}(P), \tag{2.3}$$

where $L^2_{n+1}(P)$ is a Hilbert space with

$$E\left[\sum_{t=0}^{n} X_t^2\right] < \infty \qquad \text{for all } \mathbf{X} \in L^2_{n+1}(P), \tag{2.4}$$

$$< \mathbf{X}, \mathbf{Y} > \; = E\left[\sum_{t=0}^{n} X_t Y_t\right] \qquad \text{for all } \mathbf{X}, \mathbf{Y} \in L^2_{n+1}(P), \tag{2.5}$$

$$\|\mathbf{X}\| = \; < \mathbf{X}, \mathbf{X} >^{1/2} \; < \infty \qquad \text{for all } \mathbf{X} \in L^2_{n+1}(P). \tag{2.6}$$

If the cash flow $\mathbf{X} = (X_0, X_1, \ldots, X_n)$ is $\mathbb{F}$-adapted and square integrable, then we write

$$\mathbf{X} = (X_0, X_1, \ldots, X_n) \in L^2_{n+1}(P, \mathbb{F}). \tag{2.7}$$

Technical remark. The equality $\|\mathbf{X} - \mathbf{Y}\| = 0$ implies that $\mathbf{X} = \mathbf{Y}$, P-a.s. As usually done in Hilbert spaces, we identify random variables which are equal P-a.s.

Example 2.1 (Life insurance).

We consider a general life insurance policy financed by a regular premium income stream $(\Pi_0, \ldots, \Pi_n)$, where Π_t denotes the premium payment made at time t. Furthermore cash outflows comprise the expenses and the benefit payments occurring in the time interval $(t-1, t]$. If we map all cash flows occurring in the time interval $(t-1, t]$ to the right end point t of the time interval, we obtain a discrete time cash flow for $t \in \{0, \ldots, n\}$:

$$X_t = -\Pi_t + \text{benefits and expenses paid within } (t-1, t]. \tag{2.8}$$

Henceforth, $\mathbf{X}$ denotes the cash flow generated by this single policy.

□

Example 2.2 (Non-life insurance).

In non-life insurance the insurance company usually receives a (risk) premium at the beginning of a well-defined insurance period. Within this insurance period certain (well-defined, random) financial losses are covered. We denote the premium payment by $\Pi = -X_0$. The occurrence of an insured event (covered claim) during the insurance period typically entails a sequence of future cash outflows, namely claims payments until the claim is settled. That is, usually the insurance company cannot immediately settle a claim. It takes quite some time until the ultimate claim amount is known. The delay in the settlement is due to the fact that, for example, it takes time until the total medical expenses are known, until the claim is settled at court, until the damaged building is fixed, until the recovery process is understood, etc. (see also Wüthrich-Merz [WM08]).

Since one does not wait with the payments until the ultimate claim amount is known (e.g. medical expenses and salaries are paid when they occur) a claim consists of several single payments X_t which reflect the on-going recovery process. Hence, the total or ultimate claim amount (nominal) is given by

$$C_n = \sum_{t=1}^{n} X_t, \tag{2.9}$$

where X_t ($t \leq n$) denote the single claims payments and X_n denotes the final payment when the claim is closed/settled. Henceforth, at time t we have information $\mathcal{F}_t$ and the payments X_k, $k \leq t$, are already made, whereas the future payments X_s, $s > t$, need to be predicted based on the information $\mathcal{F}_t$ available at time t.

The underwriting loss (nominal loss) can then be written as

$$UL = \sum_{t=0}^{n} X_t = -\Pi + C_n. \tag{2.10}$$

Remark. UL does not necessarily need to be negative to run successfully this non-life insurance business. The nominal underwriting loss UL does not consider the financial income during the settlement of the claim. That is, the delay in the payments allows for discounting of the payments, which in the profit and loss statement is considered similar to investment incomes on financial assets at the insurance company (see next sections).

□

2.2 Market-consistent valuation in the basic discrete time model

We now value the (stochastic) cash flow $\mathbf{X}$. We proceed as in Bühlmann [Bü92, Bü95] using a positive, continuous, linear (valuation) functional.

Definition 2.2 (Positivity)

- $\mathbf{X} \geq 0 \iff X_t \geq 0$, *$P$-a.s., for all* $t = 0, \ldots, n$.
- $\mathbf{X} > 0 \iff \mathbf{X} \geq 0$ *and there exists* $k \in \{0, \ldots, n\}$ *such that* $X_k > 0$ *with positive probability.*
- $\mathbf{X} \gg 0 \iff X_t > 0$, *$P$-a.s., for all* $t = 0, \ldots, n$.

Assumption 2.3 *Assume that* $Q : L^2_{n+1}(P) \to \mathbb{R}$ *is a positive, continuous, linear functional on* $L^2_{n+1}(P)$.

This means that the functional Q satisfies the following properties:

(1) Positivity: $\mathbf{X} > 0$ implies $Q[\mathbf{X}] > 0$.

(2) Continuity: For any sequence $\mathbf{X}^{(k)} \in L^2_{n+1}(P)$ with $\mathbf{X}^{(k)} \to \mathbf{X}$ in $L^2_{n+1}(P)$ as $k \to \infty$, we have $Q\left[\mathbf{X}^{(k)}\right] \to Q[\mathbf{X}]$ in $\mathbb{R}$ as $k \to \infty$.

(3) Linearity: For all $\mathbf{X}, \mathbf{Y} \in L^2_{n+1}(P)$ and $a, b \in \mathbb{R}$ we have

$$Q[a\mathbf{X} + b\mathbf{Y}] = aQ[\mathbf{X}] + bQ[\mathbf{Y}]. \tag{2.11}$$

Terminology.
The mapping $\mathbf{X} \mapsto Q[\mathbf{X}]$ assigns a monetary value $Q[\mathbf{X}] \in \mathbb{R}$ at time 0 to the cash flow $\mathbf{X}$. That is, the valuation function Q attaches a value to any $\mathbf{X} \in L^2_{n+1}(P)$, which can be seen as the price of $\mathbf{X}$ at time 0. As we will see below, this valuation/pricing will be done in a market-consistent way which leads to a risk neutral valuation scheme and $Q[\mathbf{X}]$ is the (market-consistent) price for $\mathbf{X}$ at time 0.

Remark. Assumptions (1) and (3) ensure that one can develop an arbitrage-free pricing system (see Lemma 2.8 and Remark 2.14).

Lemma 2.4 *Assumptions (1) and (3) imply (2).*

Proof. Define $\mathbf{Y}^{(k)} = \mathbf{X}^{(k)} - \mathbf{X}$. Due to the linearity of Q it suffices to prove that $\mathbf{Y}^{(k)} \to 0$ in $L^2_{n+1}(P)$ implies that $Q\left[\mathbf{Y}^{(k)}\right] \to 0$.

In the first step we assume that $\mathbf{Y}^{(k)} \geq 0$. Then we claim

$$\mathbf{Y}^{(k)} \to 0 \text{ in } L^2_{n+1}(P) \text{ implies } Q\left[\mathbf{Y}^{(k)}\right] \to 0 \text{ as } k \to \infty. \tag{2.12}$$

Assume (2.12) does not hold true, hence (using the positivity of the linear functional) there exists $\varepsilon > 0$ and an infinite subsequence k' of k such that for all k'

$$Q\left[\mathbf{Y}^{(k')}\right] \geq \varepsilon. \tag{2.13}$$

Choose an infinite subsequence k'' of k' with

$$\sum_{k''} \left\| \mathbf{Y}^{(k'')} \right\| < \infty. \tag{2.14}$$

We define

$$\mathbf{Y} = \sum_{k''} \mathbf{Y}^{(k'')}. \tag{2.15}$$

Due to the completeness of $L^2_{n+1}(P)$ we know that $\mathbf{Y} \in L^2_{n+1}(P)$. But

$$Q[\mathbf{Y}] \geq Q\left[\sum_{k''=1}^{K} \mathbf{Y}^{(k'')}\right] \geq K\,\varepsilon \qquad \text{for every } K. \tag{2.16}$$

This implies that $Q[\mathbf{Y}] = \infty$ is not finite, which is a contradiction.

Second step: Decompose $\mathbf{Y}^{(k)} = \mathbf{Y}_+^{(k)} - \mathbf{Y}_-^{(k)}$ into a positive and a so-called negative part. Since $\|\mathbf{Y}_+^{(k)}\| \leq \|\mathbf{Y}^{(k)}\| \to 0$ and $\|\mathbf{Y}_-^{(k)}\| \leq \|\mathbf{Y}^{(k)}\| \to 0$ we see that both $\mathbf{Y}_+^{(k)}$ and $\mathbf{Y}_-^{(k)}$ tend to 0. Because $\mathbf{Y}_+^{(k)} \geq 0$ and $\mathbf{Y}_-^{(k)} \geq 0$ we have - as proved in the first step -

$$Q\left[\mathbf{Y}_+^{(k)}\right] \to 0 \quad \text{and} \quad Q\left[\mathbf{Y}_-^{(k)}\right] \to 0. \tag{2.17}$$

Using once more the linearity of Q completes the proof.

□

Theorem 2.5 (Riesz' representation theorem) *Under Assumption 2.3 there exists* $\boldsymbol{\varphi} \in L^2_{n+1}(P)$ *such that for all* $\mathbf{X} \in L^2_{n+1}(P)$ *we have*

$$Q[\mathbf{X}] \;=\; <\mathbf{X}, \boldsymbol{\varphi}> \;=\; E\left[\sum_{t=0}^{n} X_t \; \varphi_t\right]. \tag{2.18}$$

Definition 2.6 *The vector* $\boldsymbol{\varphi}$ *(and its single components* φ_t*) is called (state price)* **deflator**.

The terminology (state price) deflator was introduced by Duffie [Du96] and Bühlmann et al. [BDES98]. In economic theory deflators are called "state price densities" and in financial mathematics "financial pricing kernels" or "stochastic interest rates".

Remarks. The deflator has the following properties:

- The positivity of Q ensures that $\boldsymbol{\varphi} \gg 0$.
- Assume $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$ is $\mathbb{F}$-adapted. Then $\boldsymbol{\varphi}$ may also be chosen $\mathbb{F}$-adapted: replace φ_t by $\widetilde{\varphi}_t = E[\varphi_t | \mathcal{F}_t]$. Then we have for all $\mathbb{F}$-adapted, square integrable cash flows $\mathbf{X}$

$$\begin{aligned} Q[\mathbf{X}] &= E\left[\sum_{t=0}^{n} X_t \; \varphi_t\right] = \sum_{t=0}^{n} E[X_t \; \varphi_t] = \sum_{t=0}^{n} E[E[X_t \; \varphi_t | \mathcal{F}_t]] \\ &= \sum_{t=0}^{n} E[X_t \; E[\varphi_t | \mathcal{F}_t]] = E\left[\sum_{t=0}^{n} X_t \; E[\varphi_t | \mathcal{F}_t]\right] \\ &= E\left[\sum_{t=0}^{n} X_t \; \widetilde{\varphi}_t\right] \;=\; <\mathbf{X}, \widetilde{\boldsymbol{\varphi}}>, \end{aligned} \tag{2.19}$$

where in the third step on the first line we have used the tower property for conditional expectations (see Williams [Wi91]), and in the fourth step we have used that X_t is $\mathcal{F}_t$-measurable. Henceforth, because we will only work on $L^2_{n+1}(P, \mathbb{F})$ we may and will assume that $\boldsymbol{\varphi}$ is $\mathbb{F}$-adapted, throughout.

- There is exactly one $\mathbb{F}$-adapted deflator $\boldsymbol{\varphi}$ in $L^2_{n+1}(P,\mathbb{F})$ for a given Q (up to measure 0): assume that there are two $\mathbb{F}$-adapted random vectors $\boldsymbol{\varphi}$ and $\boldsymbol{\varphi}^*$ satisfying for all $\mathbf{X} \in L^2_{n+1}(P,\mathbb{F})$

$$Q[\mathbf{X}] =< \mathbf{X}, \boldsymbol{\varphi} >=< \mathbf{X}, \boldsymbol{\varphi}^* > . \tag{2.20}$$

  But then we choose $\mathbf{X} = \boldsymbol{\varphi} - \boldsymbol{\varphi}^* \in L^2_{n+1}(P,\mathbb{F})$. This and (2.20) imply

$$0 =< \mathbf{X}, \boldsymbol{\varphi} - \boldsymbol{\varphi}^* >= \|\boldsymbol{\varphi} - \boldsymbol{\varphi}^*\|^2, \tag{2.21}$$

  which immediately gives $\boldsymbol{\varphi} = \boldsymbol{\varphi}^*$, P-a.s.
- Furthermore, we assume that Q is such that $\varphi_0 \equiv 1$. This means that for a (deterministic) payment x_0 at time 0, we have $Q[(x_0, 0, \ldots, 0)] = x_0$. This means that for x_0 the functional Q delivers simply its nominal value.
- We have assumed that $\mathbf{X} \in L^2_{n+1}(P,\mathbb{F})$ in order to find the state price deflator $\boldsymbol{\varphi}$. This can be generalized to cash flows $\mathbf{X} \in L^p_{n+1}(P,\mathbb{F})$ $(1 \le p \le \infty)$ and then the deflator $\boldsymbol{\varphi}$ would be in $L^q_{n+1}(P,\mathbb{F})$ with $1/p + 1/q = 1$. Or even more generally we can take $\boldsymbol{\varphi} \in L^1_{n+1}(P,\mathbb{F})$ fixed and then define the set of cash flows that can be priced by

$$\mathcal{L}_{\boldsymbol{\varphi}} = \left\{ \mathbf{X} \in L^1_{n+1}(P,\mathbb{F}) : \;\; < \mathbf{X}, \boldsymbol{\varphi} > \;\; < \infty \right\}. \tag{2.22}$$

  For these cash flows we then define the pricing functional Q on $\mathcal{L}_{\boldsymbol{\varphi}}$ by $Q[\mathbf{X}] =< \mathbf{X}, \boldsymbol{\varphi} >$.

2.2.1 Task of modelling

Find the appropriate pricing functional Q or equivalently find the appropriate $\mathbb{F}$-adapted state price deflator $\boldsymbol{\varphi}$!

In the more general setup, one would define/choose $\boldsymbol{\varphi} \in L^1_{n+1}(P,\mathbb{F})$ and then value the cash flows $\mathbf{X} \in \mathcal{L}_{\boldsymbol{\varphi}}$, see (2.22). The choice of $\boldsymbol{\varphi}$ will include market risk aversion as well as individual risk aversion, this will be described in the following chapters, and we will also describe the connection between the state price deflators and the risk neutral martingale measures.

The $\mathbb{F}$-adaptedness will be crucial in the sequel. It essentially means that the deflator φ_t (stochastic discount factor) is known at time t, and hence, allows for a direct connection of the $\mathcal{F}_t$-measurable cash flow X_t with the behaviour φ_t of the financial market at time t. Especially, this means that φ_t will allow for the modelling of embedded options and guarantees in X_t that depend on economic and financial scenarios.

Examples of state price deflators can be found in Bühlmann [Bü95], for example the Ehrenfest Urn with limit Ornstein-Uhlenbeck model, in Filipovic-Zabczyk [FZ02] or one can easily discretize, for example, the Vasicek model, see Brigo-Mercurio [BM06] and Exercise 2.3 below.

Exercise 2.3 (Discrete time Vasicek [Va77] model).

Choose a filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F})$ and assume that $(\varepsilon_t)_{t=0,\dots,n}$ is $\mathbb{F}$-adapted, that ε_t is independent of $\mathcal{F}_{t-1}$ for all $t = 1, \dots, n$ and standard Gaussian distributed. Then, we define the stochastic process $(r_t)_{t=0,\dots,n}$ by $r_0 > 0$ (fixed) and for $t \geq 1$

$$r_t = b + \beta r_{t-1} + \rho \varepsilon_t, \tag{2.23}$$

for given $b, \beta, \rho > 0$. This $(r_t)_{t=0,\dots,n}$ describes the spot rate dynamics of the Vasicek model under the (real world) probability measure P, see Brigo-Mercurio [BM06] Section 3.2.1.
Next, we choose $\lambda \in \mathbb{R}$ and define the deflator in the Vasicek model by

$$\varphi_t = \exp\left\{ -\sum_{k=1}^{t} \left[r_{k-1} + \frac{\lambda^2}{2}\, r_{k-1}^2 \right] - \sum_{k=1}^{t} \lambda\, r_{k-1}\, \varepsilon_k \right\}. \tag{2.24}$$

Prove that $\boldsymbol{\varphi} \in L^1_{n+1}(P, \mathbb{F})$ is a deflator. Moreover, prove that the cash flow $\mathbf{X} = (0, \dots, 0, 1, 0, \dots, 0) \in \mathcal{L}_{\varphi}$, see (2.22).

□

2.2.2 Understanding deflators

A **deflator** φ_t transports cash amount at time t to value at time 0, see Figure 2.2. This transportation is a stochastic transportation (stochastic discounting). This implies, a cash flow $\mathbf{X}_t = (0, \dots, 0, X_t, 0, \dots, 0)$ does not necessarily need to be independent (or uncorrelated) of φ_t, which then gives

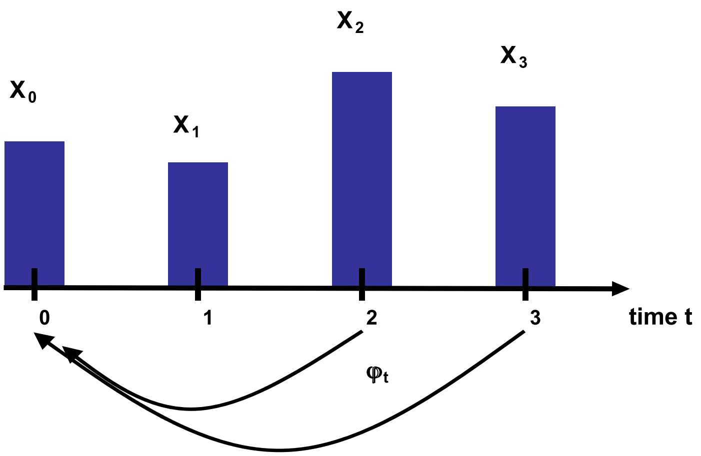

Fig. 2.2. Deflator φ and cash flow $\mathbf{X}$

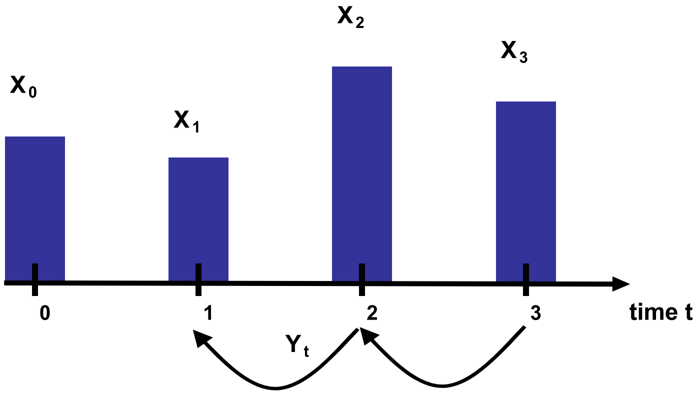

Fig. 2.3. Span-deflators Y_t and cash flow $\mathbf{X}$

$$Q[\mathbf{X}_t] = E[X_t\ \varphi_t] \neq E[X_t]\, E[\varphi_t]. \tag{2.25}$$

$Q[\mathbf{X}_t]$ describes the value/price of $\mathbf{X}_t$ at time 0, where X_t is stochastically discounted with the deflator φ_t.

We decompose the deflator $\boldsymbol{\varphi}$ into its **span-deflators**. Since $\boldsymbol{\varphi} \gg 0$ we can build the following ratios for all $t > 0$, P-a.s.:

$$Y_t = \frac{\varphi_t}{\varphi_{t-1}}. \tag{2.26}$$

Moreover, we define $Y_0 = 1$. Thus, $\mathbf{Y} = (Y_t)_{t=0,\dots,n}$ is $\mathbb{F}$-adapted and satisfies

$$\varphi_t = Y_0\ Y_1\ \cdots\ Y_t = \prod_{k=0}^{t} Y_k. \tag{2.27}$$

$\mathbf{Y} = (Y_t)_{t=0,\dots,n}$ is called span-deflator. Span-deflators Y_t, $t \geq 1$, transport cash amount at time t to value at time $t-1$, see Figure 2.3.

Question. How is the deflator $\boldsymbol{\varphi}$ related to zero coupon bonds and classical financial discounting?

Denote by $\mathbf{Z}^{(t)} = (0, \dots, 0, 1, 0, \dots, 0)$ the cash flow of the **zero coupon bond** paying the amount 1 at time t. The value at time 0 of this zero coupon bond is given by

$$D_{0,t} = Q\left[\mathbf{Z}^{(t)}\right] = E[\varphi_t]. \tag{2.28}$$

In the financial literature $D_{0,t}$ is often denoted by $P(0,t)$, which is the value at time 0 of a default-free contract paying 1 at time t.

Hence, also $D_{0,t}$ transports cash amount at time t to value in 0. But $D_{0,t}$ is $\mathcal{F}_0$-measurable, whereas φ_t is a $\mathcal{F}_t$-measurable random variable. This means that the deterministic discount factor $D_{0,t}$ is known at the beginning of the time period $(0,t]$, whereas φ_t is only known at the end of the time period $(0,t]$. As long as we deal with deterministic cash flows $\mathbf{X}$, we can either work with zero coupon bond prices $D_{0,t}$ or with deflators φ_t to determine the value of $\mathbf{X}$ at time 0. But as soon as the cash flows $\mathbf{X}$ are stochastic we need to work with deflators (see (2.25)) since X_t and φ_t may be influenced by the same factors (are dependent). An easy example is that X_t is an option that depends on the actual realization of φ_t. Various life insurance policies contain such embedded options and financial guarantees, that is, the insurance payout depends on the development of economic and financial market factors (which are also risk drivers of φ_t).

Classical **actuarial discounting** is taking a constant interest rate i. That is, in classical actuarial models φ_t has the following form

$$\varphi_t = (1+i)^{-t}. \tag{2.29}$$

This deflator gives a consistent theory but it is far from the economic observations in practice. This indicates that we have to be very careful with this deterministic model in a total balance sheet approach, since it implies that we obtain values far away from those consistent with the financial market values on the asset classes.

Exercise 2.4 (Price of the zero coupon bond in the Vasicek model).

We revisit the discrete time Vasicek model presented in Exercise 2.3. Calculate for this model the zero coupon bond price $D_{0,t}$. We claim that this price is given by

$$D_{0,t} = \exp\left\{a(t) - r_0\ b(t)\right\}, \tag{2.30}$$

for appropriate functions $a(t)$ and $b(t)$.

Hint: the claim is proved by induction using properties of log-normal distributions.

Give an interpretation to r_0 in terms of $D_{0,1}$.

□

2.2.3 Toy example for deflators

In this subsection we give a toy example which is based on finite probability spaces: in a first step we need a market model for calibration purposes. In a second step we construct deflators (the example is taken from Jarvis et al. [JSV01]).

We consider a one-period model, and we assume that there are two possible states at time 1, namely $\Omega = \{\omega_1, \omega_2\}$. For this example on finite probability spaces finding the deflators is essentially an exercise in linear algebra. Here, we would also like to mention that finite models often have the advantage that one can easier find the crucial mathematical and economic structures (see also Malamud et al. [MTW08]).

Step 1. In a first step we construct the **state space securities** SS_1 and SS_2. A state space security for state ω_i pays one unit if state ω_i occurs at time 1. These state space securities are used to construct an arbitrage-free pricing model. That is,

	SS_1	SS_2
market price Q at time 0	?	?
payout if in state ω_1 at time 1	1	0
payout if in state ω_2 at time 1	0	1

Since we have two states ω_1 and ω_2 we need two linearly independent assets A and B to calibrate the model. Assume that assets A and B have the following price and payout structure:

	asset A	asset B
market price Q at time 0	1.65	1
payout if in state ω_1 at time 1	3	2
payout if in state ω_2 at time 1	1	0.5

With this information we can now construct the two state space securities SS_1 and SS_2, respectively. That is, we can calculate the prices of SS_1 and SS_2 at time 0. To this end (for SS_1) we construct a portfolio that consists of x units of asset A and y units of asset B. The goal is to determine x and y such that the resulting portfolio pays 1 if state ω_1 occurs at time 1 and 0 otherwise. That is, this portfolio exactly replicates the state price security SS_1. Mathematically speaking we need to solve the linear equation $SS_1 = xA + yB$ for SS_1, and a similar linear equation for SS_2. The solution to these two linear equations provides the following table (with the corresponding prices at time 0):

	units of asset A	units of asset B	market price Q
ω_1 state security SS_1	-1	2	0.35
ω_2 state security SS_2	4	-6	0.60

Note that this is similar to the derivation of the Arbitrage Pricing Theory model (see Ingersoll [Ing87], Chapter 7). Basically, we need that asset A and asset B are linearly independent and that the pricing functional Q is linear. Hence, if we have another risky asset $\mathbf{X}$ which pays 2 in state ω_1 and 1 in state ω_2, its price is given by

$$Q\left[\mathbf{X}\right] = 2 \cdot 0.35 + 1 \cdot 0.6 = 1.3. \tag{2.31}$$

We now consider the zero coupon bond $\mathbf{Z}^{(1)}$. The zero coupon bond pays in both states ω_1 and ω_2 the amount 1:

$$D_{0,1} = Q\left[\mathbf{Z}^{(1)}\right] = 1 \cdot 0.35 + 1 \cdot 0.6 = 0.95, \tag{2.32}$$

which leads to a risk-free return of $(0.95)^{-1} - 1 = 5.26\%$.

Step 2. Now we construct the deflators. Denote by $Q(\omega_i)$ the market price of the ω_i state space security SS_i at time 0, i.e. $Q(\omega_1) = 0.35$ and $Q(\omega_2) = 0.60$. Moreover, let $X_1(\omega_i)$ denote the payout at time 1 of the risky asset $\mathbf{X} = (0, X_1)$, if we are in state ω_i at time 1. Hence the market price of $\mathbf{X}$ at time 0 is given by (see (2.31))

$$Q[\mathbf{X}] = \sum_{i=1}^{2} Q(\omega_i)\ X_1(\omega_i). \tag{2.33}$$

Note: so far we have not used any probabilities!

Now we assume that we are in state ω_1 at time 1 with probability $p(\omega_1) \in (0,1)$ and in state ω_2 with probability $p(\omega_2) = 1 - p(\omega_1)$. Hence (2.33) can be rewritten as follows

$$\begin{aligned} Q[\mathbf{X}] &= \sum_{i=1}^{2} Q(\omega_i)\ X_1(\omega_i) \\ &= \sum_{i=1}^{2} p(\omega_i)\ \frac{Q(\omega_i)}{p(\omega_i)}\ X_1(\omega_i) \\ &= E\left[\frac{Q}{p}\ X_1\right]. \end{aligned} \tag{2.34}$$

Henceforth, define the random variable

$$\varphi_1 = \frac{Q}{p}, \tag{2.35}$$

which immediately implies the pricing formula

$$Q[\mathbf{X}] = E[\varphi_1\ X_1]. \tag{2.36}$$

For an explicit choice of probabilities $p(\omega_i)$, the deflator φ_1 takes the following values:

	value of deflator φ_1	probability $p(\omega_i)$
state ω_1 at time 1	0.7	0.5
state ω_2 at time 1	1.2	0.5

Hence, alternatively to (2.32) we obtain for the value of the zero coupon bond

$$Q\left[\mathbf{Z}^{(1)}\right] = E[\varphi_1] = \sum_{i=1}^{2} \varphi_1(\omega_i)\ p(\omega_i) = 0.7 \cdot 0.5 + 1.2 \cdot 0.5 = 0.95. \tag{2.37}$$

Note that in our example the deflator φ_1 is not necessarily smaller than 1. With probability 1/2 we will observe that the deflator has a value of 1.2. This may be counter-intuitive from an economic point of view but makes perfect sense in our model world. Henceforth, the model and parameters need to be specified carefully in order to get economically meaningful models.

2.3 Valuation at time $t > 0$

Postulate: Correct prices should eliminate the possibility to play games with cash flows (see also Remark 2.14).

Assume an $\mathbb{F}$-adapted deflator $\boldsymbol{\varphi} \in L^2_{n+1}(P, \mathbb{F})$ is given. We then define the price process for a random vector $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$ as follows: for $t = 0, \ldots, n$

$$Q_t[\mathbf{X}] = Q[\mathbf{X}|\mathcal{F}_t] = \frac{1}{\varphi_t} \; E\left[\sum_{k=0}^{n} \varphi_k \; X_k \middle| \mathcal{F}_t \right]. \tag{2.38}$$

Note, $\boldsymbol{\varphi} \gg 0$ implies that $Q_t[\mathbf{X}]$ is well-defined. The right-hand side of (2.38) can be decoupled because the payments X_k (and the deflators φ_k) are $\mathcal{F}_t$-measurable for $k \le t$.

Terminology.
The mapping $\mathbf{X} \mapsto Q_t[\mathbf{X}]$ assigns a monetary value $Q_t[\mathbf{X}]$ at time t to the cash flow $\mathbf{X}$, i.e. attaches an $\mathcal{F}_t$-measurable price to the cash flow $\mathbf{X}$. Of course this price is stochastic seen from time 0, it depends on $\mathcal{F}_t$. As we see below, this valuation process $(Q_t)_{t=0,\ldots,n}$ is done in a market-consistent way which leads to a risk neutral valuation scheme (see also Lemma 2.8 and Remark 2.14).

First, we note that by our assumption we have $Q[\mathbf{X}] = Q_0[\mathbf{X}]$.

The justification of our price process definition $(Q_t)_{t=0,\ldots,n}$ uses an equilibrium principle or an arbitrage argument. Assume that we pay for cash flow $\mathbf{X}$ at time t the price $Q_t[\mathbf{X}]$. Hence, we generate a payment cash flow

$$Q_t[\mathbf{X}] \;\; \mathbf{Z}^{(t)} = (0, \ldots, 0, Q_t[\mathbf{X}], 0, \ldots, 0), \tag{2.39}$$

if we pay the price for $\mathbf{X}$ at time t. From today's point of view this payment stream has value

$$Q_0\left[Q_t[\mathbf{X}] \;\; \mathbf{Z}^{(t)}\right], \tag{2.40}$$

since we have only information $\mathcal{F}_0$ at time 0 about the price $Q_t[\mathbf{X}]$ of $\mathbf{X}$ at time t. Equilibrium requires, that

$$Q_0[\mathbf{X}] = Q_0\left[Q_t[\mathbf{X}] \;\; \mathbf{Z}^{(t)}\right], \tag{2.41}$$

since (based on today's information $\mathcal{F}_0$) the two payment streams should have the same value. That is, we today agree either to buy and pay $\mathbf{X}$ today or to buy and pay $\mathbf{X}$ at time t (at its current price $Q_t[\mathbf{X}]$ at that time). Since we use the same information $\mathcal{F}_0$ for these two contracts and we obtain the same cash flow $\mathbf{X}$ the two contracts should have the same price.

Suppose now that we play the following game: We decide to buy and pay cash flow $\mathbf{X}$ only if an event $F_t \in \mathcal{F}_t$ occurs. Since from today's point of view we do not know whether the event F_t occurs or not, we should have the following price equilibrium, see also (2.41),

$$Q_0\left[\mathbf{X}\ 1_{F_t}\right] = Q_0\left[Q_t\left[\mathbf{X}\right]\ \mathbf{Z}^{(t)}\ 1_{F_t}\right], \tag{2.42}$$

note, however, that $\mathbf{X}\ 1_{F_t}$ is not $\mathbb{F}$-adapted (to avoid this we could also do an argument similar to (2.51) below). Using deflators, we rewrite (2.42)

$$E\left[\sum_{k=0}^{n} \varphi_k\ X_k\ 1_{F_t}\right] = E\left[\varphi_t\ Q_t\left[\mathbf{X}\right]\ 1_{F_t}\right]. \tag{2.43}$$

Since $(\varphi_t\ Q_t[\mathbf{X}])$ is $\mathcal{F}_t$-measurable and equation (2.43) must hold true for all $F_t \in \mathcal{F}_t$, this is exactly the definition of the conditional expectation given the σ-field $\mathcal{F}_t$. Henceforth, (2.43) implies (2.38), P-a.s., and justifies that (2.38) is an economically meaningful definition. A more financial mathematically based argumentation would say that deflated price processes need to be $(P, \mathbb{F})$-martingales in order to have an arbitrage-free pricing model, see Lemma 2.8 and Remark 2.14 below.

We close this section with some remarks on "pure" financial risks. We have defined the traditional discount factors

$$D_{0,m} = Q_0\left[\mathbf{Z}^{(m)}\right] = E\left[\varphi_m\right] \tag{2.44}$$

at time 0 for a zero coupon bond with maturity m. For $t < m$, let $D_{t,m}$ stand for the discount factor from time m back to time t, *fixed* at time 0. The terminology forward refers to this fixing at an earlier time point. We must have

$$D_{0,t}\ D_{t,m} = D_{0,m}. \tag{2.45}$$

The left-hand side of (2.45) is the price at time 0 for receiving $D_{t,m}$ at time t, and $D_{t,m}$ is the price for receiving 1 at time m (fixed at time 0 and to be paid at time t). The right-hand side of (2.45) is the price at time 0 for receiving 1 at time m.

Hence we define forward discount factors for $t < m$:

$$D_{t,m} = \frac{D_{0,m}}{D_{0,t}}. \tag{2.46}$$

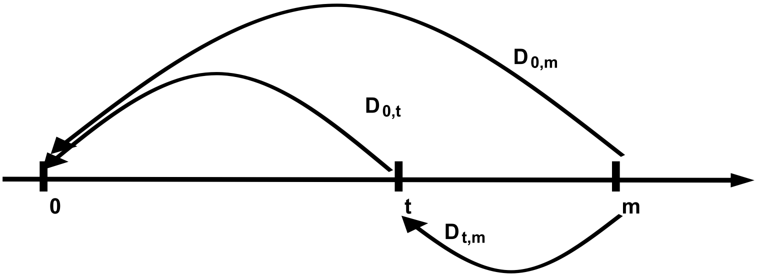

Fig. 2.4. Forward discount factor $D_{t,m}$ for $t < m$

This is the forward price of a zero coupon bond with maturity m fixed at time 0 to be paid at time t ($\mathcal{F}_0$-measurable).

On the other hand, the value/price at time t of a zero coupon bond with maturity m is given by ($\mathcal{F}_t$-measurable)

$$Q_t\left[\mathbf{Z}^{(m)}\right] = \frac{1}{\varphi_t}\, E\left[\varphi_m \middle| \mathcal{F}_t\right] = E\left[\frac{\varphi_m}{\varphi_t}\middle| \mathcal{F}_t\right]. \tag{2.47}$$

This is exactly (2.38) for a single deterministic payment of 1 in m.

Remark. In financial mathematics literature one often uses the notation

$$P(t,m) = Q_t\left[\mathbf{Z}^{(m)}\right] = E\left[\varphi_m/\varphi_t \middle| \mathcal{F}_t\right] = E^*\left[\exp\left\{-\sum_{s=t}^{m-1} r_s\right\}\middle| \mathcal{F}_t\right],$$

where $(r_t)_{t=0,\dots,n}$ stands for the spot rate process, see also Exercise 2.3, and E^* is the expectation under the risk neutral measure $P^* \sim P$ (see also Exercise 2.6). Note that $D_{0,m} = P(0,m) = Q_0\left[\mathbf{Z}^{(m)}\right]$.

Exercise 2.5.

We revisit the discrete time Vasicek model presented in Exercise 2.3. Calculate for this model the zero coupon bond price $P(t,m)$ at time $t < m$. We claim that this price is given by

$$P(t,m) = Q_t\left[\mathbf{Z}^{(m)}\right] = \exp\left\{a(m-t) - r_t\, b(m-t)\right\}, \tag{2.48}$$

for appropriate functions $a(\cdot)$ and $b(\cdot)$ and $\mathcal{F}_t$-measurable spot rate r_t, see also (2.30).

Give an interpretation to r_t in terms of $P(t, t+1)$.

Remark. The zero coupon bond price representation (2.48) is called an affine term structure, because its logarithm is an affine function of the observed spot rate r_t for all $t = 0, \dots, m-1$.

□

2.4 The meaning of basic reserves

In the previous section we have considered the valuation of cash flows $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$ at any time $t = 0, \ldots, n$. In the insurance industry however, we are mainly interested in the valuation of the future cash flows $(0, \ldots, 0, X_{t+1}, \ldots, X_n)$ if we are at time t. For these cash flows we need to build reserves in our balance sheet, because they refer to the outstanding (loss) liabilities. This means that we need to predict X_k, $k > t$, and assign market-consistent values to them, based on the information $\mathcal{F}_t$.

Note that from an economic point of view the terminology *reserves* is not quite correct (because reserves refer rather to shareholder value) and one should call the reserves instead *provisions* because they belong to the insured (policyholder).

Postulate: Correct basic reserves should eliminate the possibility to play games with insurance liabilities.

Throughout: assume an $\mathbb{F}$-adapted deflator $\boldsymbol{\varphi} \in L^2_{n+1}(P, \mathbb{F})$ is given.

Assume that an insurance contract is represented by the (stochastic) cash flow $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$. We define for $k \leq n$ the outstanding liabilities at time $k-1$ by

$$\mathbf{X}_{(k)} = (0, \ldots, 0, X_k, \ldots, X_n) \in L^2_{n+1}(P, \mathbb{F}), \tag{2.49}$$

this is the remaining cash flow after time $k-1$. $\mathbf{X}_{(k)}$ represents the amounts for which we have to build reserves at time $k-1$, such that we are able to meet all future payments arising of this contract. Henceforth, the **reserves at time** $t \leq k-1$ for the outstanding liabilities $\mathbf{X}_{(k)}$ are defined as

$$R_t^{(k)} = R\left[\mathbf{X}_{(k)} \middle| \mathcal{F}_t\right] = Q_t[\mathbf{X}_{(k)}] = \frac{1}{\varphi_t}\, E\left[\left.\sum_{s=k}^{n} \varphi_s\, X_s \right| \mathcal{F}_t\right]. \tag{2.50}$$

On the one hand, $R_t^{(k)}$ corresponds to the conditionally expected monetary value of the cash flow $\mathbf{X}_{(k)}$ viewed from time t. On the other hand, $R_t^{(k)}$ is used to predict the monetary value of the random variable $\mathbf{X}_{(k)}$. Therefore, $R_t^{(k)}$ is often called discounted "best-estimate" reserves, see also (2.113) below.

We justify that (2.50) is a reasonable definition for the reserves. We argue for $R_t^{(k)}$ in a similar fashion as in the last section. We want to avoid that we can play games with insurance contracts. In particular, we consider the following game: assume we have two insurance companies A and B that have the following business strategies.

- Company A keeps the contract until the ultimate payment is made.

- Company B decides (at time 0) to sell the run-off of the outstanding liabilities at time $t-1$ at price $R\left[\mathbf{X}_{(t)}\middle|\mathcal{F}_{t-1}\right]$ if an event $F_{t-1}\in\mathcal{F}_{t-1}$ occurs.

This implies that the two strategies generate the following cash flows:

	0	…	$t-1$	t	…	n
$\mathbf{X}^{(A)}=$	$(X_0,$	$\ldots,$	$X_{t-1},$	$X_t,$	$\ldots,$	$X_n)$
$\mathbf{X}^{(B)}=$	$(X_0,$	$\ldots,$	$X_{t-1}+R\left[\mathbf{X}_{(t)}\middle|\mathcal{F}_{t-1}\right]1_{F_{t-1}},$	$X_t 1_{F^c_{t-1}},$	$\ldots,$	$X_n 1_{F^c_{t-1}})$

Hence, the price difference at time 0 of these two strategies is given by

$$Q_0\left[\mathbf{X}^{(A)}-\mathbf{X}^{(B)}\right]=E\left[-\varphi_{t-1}R\left[\mathbf{X}_{(t)}\middle|\mathcal{F}_{t-1}\right]1_{F_{t-1}}\right]+E\left[\sum_{s=t}^{n}\varphi_s X_s 1_{F_{t-1}}\right]. \tag{2.51}$$

As in (2.42), we have that the two strategies based on the information $\mathcal{F}_0$ should have the same initial value (because they are based on the same information), i.e. $Q_0\left[\mathbf{X}^{(A)}-\mathbf{X}^{(B)}\right]=0$. This implies that for all events $F_{t-1}\in\mathcal{F}_{t-1}$ we need to have the equality

$$E\left[\varphi_{t-1}R\left[\mathbf{X}_{(t)}\middle|\mathcal{F}_{t-1}\right]1_{F_{t-1}}\right]=E\left[\sum_{s=t}^{n}\varphi_s X_s 1_{F_{t-1}}\right]. \tag{2.52}$$

Hence, using the definition of conditional expectations, this justifies the following definition of the reserves:

$$R^{(t)}_{t-1}=R\left[\mathbf{X}_{(t)}\middle|\mathcal{F}_{t-1}\right]=\frac{1}{\varphi_{t-1}}\;E\left[\left.\sum_{s=t}^{n}\varphi_s\;X_s\right|\mathcal{F}_{t-1}\right]=Q_{t-1}\left[\mathbf{X}_{(t)}\right], \tag{2.53}$$

which justifies (2.50) for $k=t$. The case $k>t$ is then easily obtained by the fact that we should have the martingale property for deflated price processes given by Lemma 2.8 (see below) which says

$$\begin{aligned}\varphi_{t-1}R^{(k)}_{t-1}&=\varphi_{t-1}Q_{t-1}\left[\mathbf{X}_{(k)}\right] \qquad (2.54)\\&=E\left[\varphi_{k-1}Q_{k-1}\left[\mathbf{X}_{(k)}\right]\middle|\mathcal{F}_{t-1}\right]=E\left[\varphi_{k-1}R^{(k)}_{k-1}\middle|\mathcal{F}_{t-1}\right].\end{aligned}$$

Observe that we have the following self-financing property:

Corollary 2.7 (Self-financing property) *The following recursion holds*

$$E\left[\varphi_t\;\left(R^{(t+1)}_t+X_t\right)\middle|\mathcal{F}_{t-1}\right]=\varphi_{t-1}\;R^{(t)}_{t-1}. \tag{2.55}$$

Remark.

- The classical actuarial theory with $\varphi_t = (1+i)^{-t}$ for some constant interest rate i (see (2.29)) forms a consistent theory but the deflators are not market-consistent, because they are often far from observed economic behaviours.
- Corollary 2.7 basically says that if we want to avoid arbitrage opportunities of reserves then we need to define them as conditional expectations of the random cash flows.

Proof of Corollary 2.7. We have the following identity (using the $\mathcal{F}_t$-measurability of X_t and the tower property of conditional expectations, see Williams [Wi91], Chapter 9)

$$E\left[\varphi_t \left(R_t^{(t+1)} + X_t\right)\middle|\, \mathcal{F}_{t-1}\right] = E\left[\sum_{k=t}^{n} \varphi_k X_k \middle|\, \mathcal{F}_{t-1}\right] = \varphi_{t-1} R_{t-1}^{(t)}. \quad (2.56)$$

This completes the proof of the corollary.

□

2.5 Equivalent martingale measures

Assume a fixed $\mathbb{F}$-adapted deflator $\boldsymbol{\varphi} \in L^2_{n+1}(P, \mathbb{F})$ is given.

The price process defined in (2.38) gives in a natural way a martingale (that is, it satisfies the efficient market hypothesis in its strong form, see Remark 2.14 below):

Lemma 2.8 *The deflated price process (2.38)*

$$(\varphi_t Q_t[\mathbf{X}])_{t=0,\dots,n} \qquad \textit{forms an } \mathbb{F}\textit{-martingale under } P. \quad (2.57)$$

Proof. Since $\mathcal{F}_t \subset \mathcal{F}_{t+1}$ we have with the tower property of conditional expectations, see Williams [Wi91],

$$\begin{aligned} E\left[\varphi_{t+1} Q_{t+1}[\mathbf{X}]\middle|\, \mathcal{F}_t\right] &= E\left[E\left[\sum_{k=0}^{n} \varphi_k X_k \middle|\, \mathcal{F}_{t+1}\right]\middle|\, \mathcal{F}_t\right] \quad (2.58)\\ &= E\left[\sum_{k=0}^{n} \varphi_k X_k \middle|\, \mathcal{F}_t\right] = \varphi_t Q_t[\mathbf{X}]. \end{aligned}$$

This finishes the proof of the lemma.

□

Remarks on deflating and discounting.

- From the martingale property we immediately have

$$Q_t[\mathbf{X}] = \frac{1}{\varphi_t} E\left[\varphi_{t+1} Q_{t+1}[\mathbf{X}] \middle| \mathcal{F}_t\right] = E\left[\left.\frac{\varphi_{t+1}}{\varphi_t} Q_{t+1}[\mathbf{X}]\right| \mathcal{F}_t\right]. \tag{2.59}$$

This implies for the span-deflated price

$$Q_t[\mathbf{X}] = E\left[Y_{t+1} Q_{t+1}[\mathbf{X}] \middle| \mathcal{F}_t\right], \tag{2.60}$$

with span-deflator Y_{t+1} defined in (2.26). The (stochastic) span-deflator Y_{t+1} is $\mathcal{F}_{t+1}$-measurable, i.e. it is known only at the end of the time period $(t, t+1]$, and not at the beginning of that time period.

- We define the span-discount known at the beginning of the time period $(t, t+1]$, i.e. which is observable on the market at time t:

$$D(\mathcal{F}_t) = E\left[Y_{t+1} | \mathcal{F}_t\right] = E\left[\left.\frac{\varphi_{t+1}}{\varphi_t}\right| \mathcal{F}_t\right]. \tag{2.61}$$

  It is often convenient to rewrite (2.60) using the span-discount $D(\mathcal{F}_t)$ instead of the span-deflator Y_{t+1}. The reason is that the span-discounts are eventually observable whereas span-deflators are always "hidden variables". The basic idea is to change the probability measure P to P^* such that we can change from span-deflators Y_{t+1} to observable span-discounts $D(\mathcal{F}_t)$ at time t.
- If the time interval $(t, t+1]$ is one year then $D(\mathcal{F}_t)$ is exactly the price of the zero coupon bond with maturity 1 year at time t, i.e., on this yearly grid this corresponds to the one-year risk-free investment at time t. Henceforth, on a yearly grid $D(\mathcal{F}_t)^{-1}$ describes the development of the value of the bank account. That is, if we invest 1 into the bank account at time 0, then the value of this investment at time $t \geq 1$ is given by (yearly role over)

$$B_t = \prod_{s=0}^{t-1} D(\mathcal{F}_s)^{-1} = \prod_{s=0}^{t-1} E\left[Y_{s+1} | \mathcal{F}_s\right]^{-1} = \exp\left\{\sum_{s=0}^{t-1} r_s\right\}, \tag{2.62}$$

  where we have defined

$$r_t = -\log E\left[Y_{t+1} | \mathcal{F}_t\right]. \tag{2.63}$$

  We remark that $(r_t)_{t=0,\ldots,n-1}$ is the spot rate process in discrete time and we have already met it in Exercise 2.3.
- The change of probability measure mentioned above will then correspond to a change of discount factors from the deflator $\boldsymbol{\varphi}$ to the bank account numeraire $(B_t^{-1})_{t=0,\ldots,n}$.

We define the process $\boldsymbol{\xi} = (\xi_s)_{s=0,\ldots,n}$ by $\xi_0 = 1$ and for $s = 1, \ldots, n$

$$\xi_s = \prod_{t=0}^{s-1} \frac{Y_{t+1}}{D(\mathcal{F}_t)} = \varphi_s \; B_s. \tag{2.64}$$

Corollary 2.9 *We have* $\boldsymbol{\xi} \gg 0$ *is a normalized density process w.r.t.* P.

Proof. Positivity is immediately clear. Moreover, $\boldsymbol{\xi}$ is a P-martingale (which immediately follows from Lemma 2.8 because $(B_t)_{t=0,\ldots,n}$ is the price process of the bank account) with normalization $E[\xi_n] = 1$. This proves the claim.

□

For $A \in \mathcal{F}_n$ we define

$$P^*[A] = \int_A \xi_n \, dP = E[\xi_n 1_A]. \tag{2.65}$$

Lemma 2.10 *We have the following statements:*

(1) P^* *is a probability measure on* $(\Omega, \mathcal{F}_n)$ *equivalent to* P.
(2) We have

$$\left.\frac{dP^*}{dP}\right|_{\mathcal{F}_s} = \xi_s \qquad P\text{-a.s.} \tag{2.66}$$

(3) Moreover, for $s \le t$ *and* $A \in \mathcal{F}_t$

$$P^*[A|\mathcal{F}_s] = \frac{1}{\xi_s} E[\xi_t \, 1_A | \mathcal{F}_s] \qquad P\text{-a.s.} \tag{2.67}$$

Proof. The proof of statement (1) follows from Corollary 2.9. The normalization implies that $P^*[\Omega] = E[\xi_n] = 1$, which says that P^* is a probability measure on $(\Omega, \mathcal{F}_n)$. Moreover, $\xi_n > 0$ P-a.s. implies that $P^* \sim P$, i.e. they are equivalent measures.

Next we prove statement (2). Note that for any $\mathcal{F}_s$-measurable set C we have

$$P^*[C] = E[\xi_n 1_C] = E[E[\xi_n | \mathcal{F}_s] 1_C] = E[\xi_s 1_C], \tag{2.68}$$

using the martingale property of $\boldsymbol{\xi}$ in the last step. Therefore, ξ_s is the density on $\mathcal{F}_s$.

Finally we prove (3). Note that we have for any $\mathcal{F}_s$-measurable set C

$$\begin{aligned} E^*[1_C \, 1_A] &= E[1_C \, \xi_n \, 1_A] \\ &= E[1_C \, E[\xi_n \, 1_A | \mathcal{F}_s]] \\ &= E\left[\xi_s \left(1_C \, \frac{1}{\xi_s} \, E[\xi_n \, 1_A | \mathcal{F}_s]\right)\right] \\ &= E^*\left[1_C \, \frac{1}{\xi_s} \, E[\xi_n \, 1_A | \mathcal{F}_s]\right] \\ &= E^*\left[1_C \, \frac{1}{\xi_s} \, E[1_A \, E[\xi_n | \mathcal{F}_t] | \mathcal{F}_s]\right] \\ &= E^*\left[1_C \, \frac{1}{\xi_s} \, E[\xi_t \, 1_A | \mathcal{F}_s]\right] \\ &= E^*[1_C \, P^*[A|\mathcal{F}_s]], \end{aligned} \tag{2.69}$$

by the definition of conditional expectations w.r.t. P^*. This completes the proof of the lemma.

□

Item (3) of Lemma 2.10 immediately implies the next corollary:

Corollary 2.11 *For $s < t$ we have*

$$E^* \left[Q_t [\mathbf{X}] \middle| \mathcal{F}_s \right] = \frac{1}{\xi_s} \, E \left[\xi_t \, Q_t [\mathbf{X}] \middle| \mathcal{F}_s \right]. \tag{2.70}$$

If we apply (2.60) and Corollary 2.11 to $s = t - 1$ we obtain

$$\begin{aligned} E^* \left[Q_t [\mathbf{X}] \middle| \mathcal{F}_{t-1} \right] &= \frac{1}{\xi_{t-1}} \, E \left[\xi_t \, Q_t [\mathbf{X}] \middle| \mathcal{F}_{t-1} \right] \\ &= \frac{1}{\xi_{t-1}} \, E \left[\xi_{t-1} \, \frac{Y_t}{D(\mathcal{F}_{t-1})} \, Q_t [\mathbf{X}] \middle| \mathcal{F}_{t-1} \right] \\ &= \frac{1}{D(\mathcal{F}_{t-1})} \, E \left[Y_t \, Q_t [\mathbf{X}] \middle| \mathcal{F}_{t-1} \right] \\ &= \frac{1}{D(\mathcal{F}_{t-1})} Q_{t-1} [\mathbf{X}], \end{aligned} \tag{2.71}$$

or

$$D(\mathcal{F}_{t-1}) \; E^* \left[Q_t [\mathbf{X}] \middle| \mathcal{F}_{t-1} \right] = Q_{t-1} [\mathbf{X}]. \tag{2.72}$$

Hence for the bank account numeraire

$$B_t^{-1} = \prod_{s=0}^{t-1} D \left(\mathcal{F}_s \right) \tag{2.73}$$

we find

$$E^* \left[B_t^{-1} \, Q_t [\mathbf{X}] \middle| \mathcal{F}_{t-1} \right] = B_t^{-1} \, E^* \left[Q_t [\mathbf{X}] \middle| \mathcal{F}_{t-1} \right] = B_{t-1}^{-1} Q_{t-1} [\mathbf{X}]. \tag{2.74}$$

Note that the discount factor B_t^{-1} is now measurable w.r.t. $\mathcal{F}_{t-1}$. Hence, in contrast to φ_t (see (2.57)) we have now an $\mathcal{F}_{t-1}$-measurable discount factor (bank account numeraire) which describes the growth of the bank account. This gives the following corollary (compare to Lemma 2.8):

Corollary 2.12 *Under the probability measure P^* the process*

$$\left(B_t^{-1} Q_t [\mathbf{X}] \right)_{t=0,\ldots,n} \tag{2.75}$$

is an $\mathbb{F}$-martingale w.r.t. P^*.

Remark 2.13 (Real world and risk neutral measure)

- Henceforth, the price process is now a martingale for discounting with the bank account numeraire B_t^{-1} under the equivalent measure $P^* \sim P$. Therefore, this measure is often called equivalent martingale measure or risk neutral measure.

- As a consequence we can either work under the **real world probability measure** P (*physical measure* or *objective measure*) where the price processes need to be deflated with $\boldsymbol{\varphi}$. Alternatively, we can also work under the **equivalent martingale measure** P^* (*risk neutral measure*). In that case the price processes need to be discounted with the bank account numeraire B_t^{-1}.
- If we work with financial instruments only, then it is often easier to work under P^*. If we additionally have insurance products then usually one works under P. Therefore, actuaries need to well-understand the connection between these two measures.
- For the equivalent martingale measure P^* we choose the bank account numeraire B_t^{-1} for discounting. In general, if $(A_t)_{t=0,\dots,n}$ is any strictly positive, normalized price process, then we could choose A_t^{-1} as a numeraire and find the appropriate equivalent measure $P^A \sim P$ such that the price processes $(A_t^{-1} Q_t [\mathbf{X}])_{t=0,\dots,n}$ are $\mathbb{F}$-martingales w.r.t. P^A. For more on this subject we refer to Brigo-Mercurio [BM06], Sections 2.2-2.3.

In the one-period model we obtain

$$Q_0 [\mathbf{X}] = D(\mathcal{F}_0)\ E^* [Q_1 [\mathbf{X}]] = E [Y_1\ Q_1 [\mathbf{X}]] . \tag{2.76}$$

Exercise 2.6.

Prove that the price of the zero coupon bond with maturity m at time $t < m$ is given by

$$P(t, m) = Q_t \left[\mathbf{Z}^{(m)}\right] = E^* \left[\exp\left\{ - \sum_{s=t}^{m-1} r_s \right\} \middle| \mathcal{F}_t \right], \tag{2.77}$$

where $(r_t)_{t=0,\dots,n}$ was defined in (2.63).

□

Remark 2.14 (Fundamental Theorem of Asset Pricing)

- The **efficient market hypothesis** in its strong form assumes that the deflated price processes

  $$\widetilde{Q}_t = \varphi_t\ Q_t [\mathbf{X}], \qquad t = 0, \dots, n, \tag{2.78}$$

  form $\mathbb{F}$-martingales under P. This implies for the expected net gains ($t > s$)

  $$E \left[\widetilde{Q}_t - \widetilde{Q}_s \middle| \mathcal{F}_s \right] = 0, \tag{2.79}$$

  which means that there exists no arbitrage strategy defined the "right way" (which roots in the idea of risk neutral valuation).

- The **efficient market hypothesis** in its weak form assumes that "there is no free lunch", i.e. there does not exist any (appropriately defined) self-financing trading strategy with positive expected gains and without any downside risk. In a finite discrete time model, this is equivalent to the existence of an equivalent martingale measure for the deflated price processes (which rules out arbitrage) (see e.g. Theorem 2.6 in Lamberton-Lapeyre [LL91]), the proof for a finite probability space is essentially an exercise in linear algebra. In a more general setting the characterization is more delicate (see Delbaen-Schachermayer [DS94] and Föllmer-Schied [FS04]).
  That is, the existence of an equivalent martingale measure rules out appropriately defined arbitrage (which is the easier direction). The opposite that no-arbitrage defined the right way implies the existence of an equivalent martingale measure is rather delicate and was proved by Delbaen-Schachermayer [DS94] in its most general form.
- In complete markets, the equivalent martingale measure is unique, which implies that we have a perfect replication of contingent claims and the calculation of the prices is straight forward (see e.g. Theorem 3.4 in Lamberton-Lapeyre [LL91]).
- In incomplete markets, where we have more than one equivalent martingale measure, we need an economic model to decide which measure to use (e.g. utility theory, super-hedge or efficient hedging (utility based models accepting some risks), see also Föllmer-Schied [FS04] or Malamud et al. [MTW08]).

Toy example (revisited).
In this subsection we revisit the toy example from Subsection 2.2.3. We transform our probability measure according to Lemma 2.10 (here we work in a one-period model with $Q_0 = Q$):

$$p^*(\omega_i) = \xi_1(\omega_i)\ p(\omega_i) = \frac{\varphi_1(\omega_i)}{E\left[\varphi_1\right]}\ p(\omega_i) = \frac{Q(\omega_i)}{Q\left[\mathbf{Z}^{(1)}\right]}. \tag{2.80}$$

Hence, from (2.33) and (2.36)

$$Q\left[\mathbf{X}\right] = E\left[\varphi_1\ X_1\right] = \sum_{i=1}^{2} Q(\omega_i)\ X_1(\omega_i), \tag{2.81}$$

$$Q\left[\mathbf{X}\right] = B_1^{-1}\ E^*\left[X_1\right] = \sum_{i=1}^{2} Q(\omega_i)\ X_1(\omega_i), \tag{2.82}$$

with (see (2.73))

$$B_1^{-1} = E\left[\varphi_1\right] = Q\left[\mathbf{Z}^{(1)}\right], \tag{2.83}$$

which is deterministic at time 0. Hence under P^* we have

$$Q[\mathbf{X}] = B_1^{-1}\ E^*[X_1] = Q\left[\mathbf{Z}^{(1)}\right] E^*[X_1]. \tag{2.84}$$

This leads to the following table with $p^*(\omega_1) = 0.368$:

	$\mathbf{Z}^{(1)}$	asset A	asset B
market price Q_0	0.95	1.65	1.00
payout state ω_1	1	3	2
payout state ω_2	1	1	0.5
P^* expected payout	1	1.737	1.053
P^* expected return	5.26%	5.26%	5.26%

which is the martingale property of the discounted cash flow $Q\left[\mathbf{Z}^{(1)}\right] X_1$ w.r.t. P^*.

□

Exercise 2.7.

We revisit the discrete time Vasicek model given in Exercise 2.3. The spot rate dynamics $(r_t)_{t=0,\dots,n}$ was given by $r_0 > 0$ (fixed) and for $t \geq 1$

$$r_t = b + \beta r_{t-1} + \rho\varepsilon_t, \tag{2.85}$$

for given $b, \beta, \rho > 0$, and $(\varepsilon_t)_{t=0,\dots,n}$ is $\mathbb{F}$-adapted with ε_t independent of $\mathcal{F}_{t-1}$ for all $t = 1, \dots, n$ and standard Gaussian distributed under the *real world probability measure* P.

The deflator $\boldsymbol{\varphi}$ was then defined by

$$\varphi_t = \exp\left\{-\sum_{k=1}^{t}\left[r_{k-1} + \frac{\lambda^2}{2} r_{k-1}^2\right] - \sum_{k=1}^{t} \lambda r_{k-1}\varepsilon_k\right\}, \tag{2.86}$$

for $\lambda \in \mathbb{R}$.

- Calculate the span-discount $D(\mathcal{F}_t)$ from the span-deflator

$$Y_{t+1} = \frac{\varphi_{t+1}}{\varphi_t} = \exp\left\{-\left[r_t + \frac{\lambda^2}{2} r_t^2\right] - \lambda r_t \varepsilon_{t+1}\right\} \tag{2.87}$$

  and show that the model is well-defined.
- Prove that the density process $(\xi_t)_{t=0,\dots,n}$ is given by

$$\xi_t = \exp\left\{-\sum_{k=1}^{t} \frac{\lambda^2}{2} r_{k-1}^2 - \sum_{k=1}^{t} \lambda r_{k-1}\varepsilon_k\right\}, \tag{2.88}$$

  where an empty sum is set equal to zero.

- Prove that
$$\varepsilon_t^* = \varepsilon_t + \lambda r_{t-1} \tag{2.89}$$
has, conditionally given $\mathcal{F}_{t-1}$, a standard Gaussian distribution under the *equivalent martingale measure* $P^* \sim P$, given by the density in Lemma 2.10.
Hint: use the moment generating function and Lemma 2.10.
- Prove that (2.89) implies for the spot rate process $(r_t)_{t=0,\ldots,n}$: $r_0 > 0$ (fixed) and for $t \geq 1$
$$r_t = b + (\beta - \lambda\rho) r_{t-1} + \rho \varepsilon_t^*, \tag{2.90}$$
where $(\varepsilon_t^*)_{t=0,\ldots,n}$ is $\mathbb{F}$-adapted with ε_t^* independent of $\mathcal{F}_{t-1}$ for all $t = 1, \ldots, n$ and standard Gaussian distributed under the equivalent martingale measure P^*.
- Calculate the zero coupon bond prices $t < m$ (see also Exercise 2.5)
$$P(t,m) = E^* \left[\exp\left\{ -\sum_{s=t}^{m-1} r_s \right\} \middle| \mathcal{F}_t \right] = \exp\left\{ a(m-t) - r_t\, b(m-t) \right\}. \tag{2.91}$$

□

Remark on Exercise 2.7. In (2.88) we calculate the density process $(\xi_t)_{t=0,\ldots,n}$ for the discrete time Vasicek model. It depends on the parameter $\lambda \in \mathbb{R}$. We see that if $\lambda = 0$, then the density process is identical equal to 1, and henceforth $P^* = P$. Therefore, λ models the difference between the real world probability measure P and the equivalent martingale measure P^* which is in economic theory explained through the market risk aversion. Therefore, λ is often called **market price of risk** parameter and explains the aggregate market risk aversion (in our Vasicek model). In general, a higher risk aversion explains lower prices because the more risk averse some is, the less he is willing to accept risky positions.

Conclusions:

- We have found three different ways to value cash flows $\mathbf{X}$:
  1. via a positive linear functional Q,
  2. via deflators $\boldsymbol{\varphi}$ under P,
  3. via the bank account numeraire B_t^{-1} under risk neutral measures P^*.
- The advantage of using risk neutral measures is that the discount factor is a priori known, which means that we have state independent discount factors. The main disadvantage of using the risk neutral measure is that the concept is not straight forward (especially parameter estimation and modelling of insurance liabilities), and that the risk neutral measure changes under currency changes.

- By contrast, deflators are calculated using the real world probability measure (expressing market risk aversion). Moreover, as shown below, they clearly describe the dependence structures (also between deflator and cash flow). From a practical point of view, deflators allow for the modelling of embedded (financial) options and guarantees in insurance policies, and are therefore preferred especially by actuaries that value life insurance products.

2.6 Insurance technical and financial variables

2.6.1 Choice of numeraire

Choose a cash flow $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$. For practical purposes in insurance applications it makes sense to factorize the payments X_k into an appropriate financial basis $\mathcal{U}_k$, $k = 0, \ldots, n$, and the number of units Λ_k of this basis. Assume that we can split the payments X_k as follows

$$X_k = \Lambda_k \; U_k^{(k)}, \quad k = 0, \ldots, n, \tag{2.92}$$

where the variable $U_t^{(k)}$ denotes the value/price of one unit of the financial instrument $\mathcal{U}_k$ at time $t = 0, \ldots, n$, and

$$\Lambda_k = \frac{X_k}{U_k^{(k)}}, \quad k = 0, \ldots, n, \tag{2.93}$$

gives the number of units that we need to hold (insurance technical variable). This means that we measure insurance liabilities in units $\mathcal{U}_k$ which have price/value $U_k^{(k)}$ at time k and insurance technical variable Λ_k.

We denote the price processes of the financial instruments $\mathcal{U}_k$ by

$$U_0^{(k)}, U_1^{(k)}, \ldots, U_k^{(k)}, U_{k+1}^{(k)}, \ldots, U_n^{(k)}. \tag{2.94}$$

Assume that the price process $(U_t^{(k)})_{t=0,\ldots,k}$ is strictly positive, P-a.s., then $(U_t^{(k)})_{t=0,\ldots,k}$ is called **numeraire** in which we study the liability X_k (see also Remark 2.13), that is, every payment X_k is studied with its appropriate numeraire.

Examples of units/numeraires.

- Currencies like CHF, USD, EURO
- Indexed CHF (inflation index, salary index, claims inflation index, medical expenses index, etc.)
- stock index, real estates, etc.
- strictly positive asset portfolio

Examples of insurance technical events.

- death benefit, annuity payments, disability benefit
- car accident compensation, fire claim
- medical expenses, workmen's compensation

We would like to factorize the filtered probability space $(\Omega, \mathcal{F}_n, P, \mathbb{F})$ into a product space such that we get an independent decoupling:

$$\mathcal{T} = (\mathcal{T}_t)_{t=0,\dots,n} \quad \sigma\text{-filtration for the insurance technical events,} \tag{2.95}$$

$$\mathcal{G} = (\mathcal{G}_t)_{t=0,\dots,n} \quad \sigma\text{-filtration for the financial events,} \tag{2.96}$$

with for all $t = 0, \dots, n$

$$\mathcal{F}_t = \sigma(\mathcal{T}_t, \mathcal{G}_t) = \text{smallest } \sigma\text{-field containing all sets of } \mathcal{T}_t \text{ and } \mathcal{G}_t. \tag{2.97}$$

We assume that under P the two σ-filtrations $\mathcal{T}$ and $\mathcal{G}$ are independent, i.e. $\mathbb{F}$ can be decoupled into a product of independent σ-fields, one covering insurance technical risks $\mathcal{T}$ and one covering financial risks $\mathcal{G}$. That is, we obtain a product probability space with product measure

$$dP = dP_{\mathcal{T}} \times dP_{\mathcal{G}}, \tag{2.98}$$

with $P_{\mathcal{T}}$ describing insurance technical risks $\mathbf{\Lambda} = (\Lambda_0, \dots, \Lambda_n)$ which will be $\mathcal{T}$-adapted and with $P_{\mathcal{G}}$ describing financial risks $(U_t^{(k)})_{t=0,\dots,n}$ which will be $\mathcal{G}$-adapted. This decoupling is crucial in the sequel of this manuscript and explained in the next assumption.

Assumption 2.15 *We assume that* $\mathbf{\Lambda} = (\Lambda_0, \dots, \Lambda_n) \in L^2_{n+1}(P_{\mathcal{T}}, \mathcal{T})$ *and that* $(U_t^{(k)})_{t=0,\dots,n} \in L^2_{n+1}(P_{\mathcal{G}}, \mathcal{G})$ *for all* $k = 0, \dots, n$*. Moreover, we assume for the given deflator* $\boldsymbol{\varphi} \in L^2_{n+1}(P, \mathbb{F})$ *that it factorizes* $\varphi_k = \varphi_k^{(\mathcal{T})} \varphi_k^{(\mathcal{G})}$ *such that* $\boldsymbol{\varphi}^{(\mathcal{T})}$ *is* $\mathcal{T}$*-adapted and* $\boldsymbol{\varphi}^{(\mathcal{G})}$ *is* $\mathcal{G}$*-adapted.*

The valuation of the cash flow $\mathbf{X} = (\Lambda_0 \; U_0^{(0)}, \dots, \Lambda_n \; U_n^{(n)}) \in L^2_{n+1}(P, \mathbb{F})$ is then under Assumption 2.15 given by

$$\begin{aligned} \varphi_t \; Q_t[\mathbf{X}] &= E\left[\sum_{k=0}^n \varphi_k \; X_k \middle| \mathcal{F}_t \right] \\ &= E\left[\sum_{k=0}^n \varphi_k^{(\mathcal{T})} \; \Lambda_k \; \varphi_k^{(\mathcal{G})} \; U_k^{(k)} \middle| \mathcal{T}_t, \mathcal{G}_t \right] \\ &= \sum_{k=0}^n E_{\mathcal{T}}\left[\varphi_k^{(\mathcal{T})} \Lambda_k \middle| \mathcal{T}_t \right] \; E_{\mathcal{G}}\left[\varphi_k^{(\mathcal{G})} U_k^{(k)} \middle| \mathcal{G}_t \right], \end{aligned} \tag{2.99}$$

where $E_{\mathcal{T}}$ is the expectation w.r.t. $P_{\mathcal{T}}$ and $E_{\mathcal{G}}$ is the expectation w.r.t. $P_{\mathcal{G}}$. In the sequel we drop the subscripts $\mathcal{T}$ and $\mathcal{G}$ if it does not cause any confusion. Note that the conditional expectations can be dropped for $k \leq t$.

Remarks.

- The expression $E_{\mathcal{T}}\left[\varphi_k^{(\mathcal{T})}\Lambda_k\middle|\mathcal{T}_t\right]$ describes the price of the insurance cover in units of currency. $\varphi_k^{(\mathcal{T})}$ defines the *loading* (**probability distortion**) of the insurance technical price. This is further outlined in Subsection 2.6.2.
- The expression $E_{\mathcal{G}}\left[\varphi_k^{(\mathcal{G})}U_k^{(k)}\middle|\mathcal{G}_t\right]$ relates to the price for one unit $\mathcal{U}_k$ at time t, see also Subsection 2.6.2 on probability distortion.
  $\varphi_k^{(\mathcal{G})}$ should be obtained from financial market data. For example, we can use the Vasicek model, proposed in Exercise 2.3, and fit the model to financial market parameters, see Wüthrich-Bühlmann [WB08].
- We have separated the pricing problem into two independent pricing problems, one for pricing insurance cover in units and one for pricing units. This split looks very natural, but in practice one needs to be careful with its applications. Especially in non-life insurance, it is very difficult to find such an orthogonal split, since the severities of the claims often depend on the financial market and the split is non-trivial. For example, if we consider workmen's compensation (which pays the salary when someone is injured or sick), it is very difficult to describe the dependence structure between 1) salary height, 2) length of sickness (which may have mental cause), 3) state of the job market, 4) state of the financial market 5) political environment.
- The financial economy including insurance products could also be defined in other ways that would allow for similar splits. For an example we refer to Malamud et al. [MTW08]. There one starts with a complete financial market described by the financial σ-field. Then one introduces insurance products that enlarge the underlying σ-field. This enlargement in general makes the market incomplete (but still arbitrage-free) and adds idiosyncratic risks to the economic model. Finally, one defines the "hedgeable" σ-field that exactly describes the part of the insurance claims that can be described via financial market movements. The remaining parts are then the insurance technical risks.

2.6.2 Probability distortion

In this section we discuss the factorization of the deflator $\varphi_k = \varphi_k^{(\mathcal{T})}\varphi_k^{(\mathcal{G})}$ from Assumption 2.15. The choice of the probability distortion $\boldsymbol{\varphi}^{(\mathcal{T})}$ needs some care in order to obtain a reasonable model.

(1) Firstly, we observe that $\boldsymbol{\varphi}^{(\mathcal{T})} \gg 0$, which follows from $\boldsymbol{\varphi} \gg 0$. Moreover, $\boldsymbol{\varphi}^{(\mathcal{T})} \in L^2_{n+1}(P_{\mathcal{T}}, \mathcal{T})$, which follows from $\boldsymbol{\varphi} \in L^2_{n+1}(P, \mathbb{F})$ and the independence and $\mathcal{T}$-adaptedness in Assumption 2.15.

(2) Secondly, to avoid ambiguity, we set for all $t = 0, \ldots, n$

$$E\left[\varphi_t^{(\mathcal{T})}\right] = 1. \tag{2.100}$$

Otherwise, the decoupling into a product $\varphi_t = \varphi_t^{(\mathcal{T})}\varphi_t^{(\mathcal{G})}$ is not unique, which can easily be seen by multiplying and dividing both terms by the same positive constant.

(3) Thirdly, we assume that the sequence $(\varphi_t^{(\mathcal{T})})_{t=0,\ldots,n}$ is a $\mathcal{T}$-martingale under P, i.e.

$$E\left[\varphi_{t+1}^{(\mathcal{T})}\middle|\mathcal{T}_t\right] = \varphi_t^{(\mathcal{T})}. \tag{2.101}$$

Of course, the normalization (2.100) is then an easy consequence from the requirement

$$E\left[\varphi_n^{(\mathcal{T})}\right] = 1. \tag{2.102}$$

Under Assumption 2.15 and assuming (1)-(3) for the probability distortion $(\varphi_t^{(\mathcal{T})})_{t=0,\ldots,n}$ we see that

$$(\varphi_t^{(\mathcal{T})})_{t=0,\ldots,n} \text{ is a density process w.r.t. } \mathcal{T} \text{ and } P_{\mathcal{T}}, \tag{2.103}$$

see also (2.104). This allows for the definition of an equivalent probability measure $P_{\mathcal{T}}^* \sim P_{\mathcal{T}}$ via the density

$$\left.\frac{dP_{\mathcal{T}}^*}{dP_{\mathcal{T}}}\right|_{\mathcal{T}_n} = \varphi_n^{(\mathcal{T})}. \tag{2.104}$$

Moreover, we define the *price process for the insurance technical variable Λ_k* as follows: for $t \le k$

$$\Lambda_{t,k} = \frac{1}{\varphi_t^{(\mathcal{T})}}\, E\left[\varphi_k^{(\mathcal{T})}\Lambda_k\middle|\mathcal{T}_t\right]. \tag{2.105}$$

Lemma 2.16 *Assume Assumption 2.15 and (2.103) hold true. The probability distorted process*

$$\left(\varphi_t^{(\mathcal{T})}\Lambda_{t,k}\right)_{t=0,\ldots,k} \quad \textit{forms a } \mathcal{T}\textit{-martingale under } P_{\mathcal{T}}. \tag{2.106}$$

The process

$$(\Lambda_{t,k})_{t=0,\ldots,k} \quad \textit{forms a } \mathcal{T}\textit{-martingale under } P_{\mathcal{T}}^*. \tag{2.107}$$

Proof Lemma 2.16. The first claim follows similarly to Lemma 2.8 and uses the tower property of conditional expectations, see Williams [Wi91]. The second claim follows similarly to Corollary 2.12 and equality (2.71). Note that here the numeraire is equal to 1 (due to our choice of the density process) which proves the claim.

□

An immediate consequence of Lemma 2.16 is the following corollary:

Corollary 2.17 *Under the assumptions of Lemma 2.16 we have*

$$\Lambda_{t,k} = \frac{1}{\varphi_t^{(\mathcal{T})}} \; E\left[\varphi_k^{(\mathcal{T})} \Lambda_k \middle| \mathcal{T}_t \right] = E^*_{\mathcal{T}}\left[\Lambda_k \middle| \mathcal{T}_t \right]. \tag{2.108}$$

This has further consequences:

Theorem 2.18 *Under the assumptions of Lemma 2.16 and (2.57) we obtain that the price process $(U_t^{(k)})_{t=0,\dots,k}$ of the financial instrument $\mathcal{U}_k$ satisfies for $t<k$*

$$U_t^{(k)} = \frac{1}{\varphi_t^{(\mathcal{G})}} \; E\left[\varphi_{t+1}^{(\mathcal{G})} U_{t+1}^{(k)} \middle| \mathcal{G}_t \right]. \tag{2.109}$$

Proof of Theorem 2.18. We define the cash flow $\mathbf{X} = U_k^{(k)} \mathbf{Z}^{(k)} = (0,\dots,0,U_k^{(k)},0,\dots,0) \in L^2_{n+1}(P,\mathbb{F})$. Note that in fact the cash flow $\mathbf{X} \in L^2_{n+1}(P,\mathcal{G})$. The martingale property (2.57), Assumption 2.15 and Corollary 2.17 imply for $t<k$

$$\begin{aligned}
\varphi_t Q_t\left[\mathbf{X}\right] = E\left[\varphi_k Q_k\left[\mathbf{X}\right] \middle| \mathcal{F}_t \right] \; &= \; E\left[\varphi_k U_k^{(k)} \middle| \mathcal{F}_t \right] \\
&= E\left[\varphi_k^{(\mathcal{T})} \varphi_k^{(\mathcal{G})} U_k^{(k)} \middle| \mathcal{F}_t \right] \\
&= E\left[\varphi_k^{(\mathcal{T})} \middle| \mathcal{T}_t \right] E\left[\varphi_k^{(\mathcal{G})} U_k^{(k)} \middle| \mathcal{G}_t \right] \\
&= \varphi_t^{(\mathcal{T})} \; E\left[\varphi_k^{(\mathcal{G})} U_k^{(k)} \middle| \mathcal{G}_t \right].
\end{aligned} \tag{2.110}$$

This implies that

$$U_t^{(k)} = Q_t\left[\mathbf{X}\right] = \frac{1}{\varphi_t^{(\mathcal{G})}} E\left[\varphi_k^{(\mathcal{G})} U_k^{(k)} \middle| \mathcal{G}_t \right]. \tag{2.111}$$

Henceforth $(\varphi_t^{(\mathcal{G})} U_t^{(k)})_{t=0,\dots,k}$ is a $\mathcal{G}$-martingale under P, which proves the claim.

□

Corollary 2.17 and Theorem 2.18 imply that we can study the insurance technical variables $\mathbf{\Lambda}$ and the price processes of the financial instruments $\mathcal{U}_k$ independently. The valuation of the outstanding loss liabilities

$$\mathbf{X}_{(k)} = (0,\dots,0,\Lambda_k U_k^{(k)},\dots,\Lambda_n U_n^{(n)}) \in L^2_{n+1}(P,\mathbb{F}) \tag{2.112}$$

at time $t \le k$ can then easily be done (see also (2.99)). The basic reserves are given by (see also (2.50))

$$\begin{aligned}
R_t^{(k)} = Q_t\left[\mathbf{X}_{(k)}\right] &= \frac{1}{\varphi_t} \; \sum_{s=k}^{n} E\left[\varphi_s^{(\mathcal{T})} \Lambda_s \middle| \mathcal{T}_t \right] \; E\left[\varphi_s^{(\mathcal{G})} U_s^{(s)} \middle| \mathcal{G}_t \right] \\
&= \sum_{s=k}^{n} \Lambda_{t,s} \; U_t^{(s)}.
\end{aligned} \tag{2.113}$$

Conclusions.
Under the product space Assumption 2.15, the assumption (2.103) that the insurance technical deflator is a density process w.r.t. $\mathcal{T}$ and P, and under the no-arbitrage assumption (2.57) we obtain that we can separate the valuation problem into two independent valuation problems:

(1) the insurance technical processes $(\Lambda_{t,k})_{t=0,\ldots,k}$, $k = 0, \ldots, n$, describe the probability distorted developments of the predictions of Λ_k if we increase the information $\mathcal{T}_t \to \mathcal{T}_{t+1}$;

(2) the financial processes $(U_t^{(k)})_{t=0,\ldots,k}$, $k = 0, \ldots, n$, describe the price processes of the financial instruments $\mathcal{U}_k$ at the financial market $(\Omega, \mathcal{G}_n, P_{\mathcal{G}}, \mathcal{G})$.

Example 2.8 (Best-Estimate Predictions).

Choose $\boldsymbol{\varphi}^{(\mathcal{T})} \equiv 1$. Hence, $\boldsymbol{\varphi}^{(\mathcal{T})}$ gives a suitable probability distortion (normalized martingale). This implies for the insurance technical process at time $t \leq k$

$$\Lambda_{t,k} = E\left[\Lambda_k \middle| \mathcal{T}_t\right], \tag{2.114}$$

i.e. $\Lambda_{t,k}$ is simply the "best-estimate" prediction of Λ_k based on the information $\mathcal{T}_t$ (conditional expectation which has minimal conditional variance).

□

Exercise 2.9 (Esscher Premium).

We choose a positive random variable Y on the underlying filtered probability space $(\Omega, \mathcal{T}_n, P_{\mathcal{T}}, \mathcal{T})$ such that for some $\alpha > 0$ the following moment generating function exists

$$M_Y(2\alpha) = E\left[\exp\{2\alpha Y\}\right] < \infty. \tag{2.115}$$

Then we define the probability distortion

$$\varphi_t^{(\mathcal{T})} = \frac{E\left[\exp\{\alpha Y\} \middle| \mathcal{T}_t\right]}{E\left[\exp\{\alpha Y\}\right]} = \frac{E\left[\exp\{\alpha Y\} \middle| \mathcal{T}_t\right]}{M_Y(\alpha)}. \tag{2.116}$$

(1) Prove that $\boldsymbol{\varphi}^{(\mathcal{T})} \gg 0$. Moreover, prove that $\boldsymbol{\varphi}^{(\mathcal{T})} \in L^2_{n+1}(P_{\mathcal{T}}, \mathcal{T})$.

(2) Show that $(\varphi_t^{(\mathcal{T})})_{t=0,\ldots,n}$ is a density process w.r.t. $\mathcal{T}$ and $P_{\mathcal{T}}$.

Assume that $\mathbf{X}_k = (0, \ldots, 0, X_k, 0, \ldots, 0)$ with $X_k = \Lambda_k U_k^{(k)}$. Choose $Y = \Lambda_k$ and assume that $t < k$. Prove under Assumption 2.15 and (2.57) that

$$Q_t\left[\mathbf{X}_k\right] = \frac{1}{E\left[\exp\{\alpha\Lambda_k\} \middle| \mathcal{T}_t\right]}\; E\left[\Lambda_k\, e^{\alpha\Lambda_k} \middle| \mathcal{T}_t\right]\; U_t^{(k)}. \tag{2.117}$$

If we define the conditional moment generating function by

$$M_{\Lambda_k|\mathcal{T}_t}(\alpha) = E\left[\exp\left\{\alpha \Lambda_k\right\}\middle|\, \mathcal{T}_t\right], \tag{2.118}$$

then the term

$$\Lambda_{t,k} = \frac{d}{dr} \log M_{\Lambda_k|\mathcal{T}_t}(r)\Big|_{r=\alpha} = M_{\Lambda_k|\mathcal{T}_t}(\alpha)^{-1} E\left[\Lambda_k \; e^{\alpha \Lambda_k}\middle|\, \mathcal{T}_t\right] \tag{2.119}$$

describes the Esscher premium of Λ_k at time $t < k$, see Gerber-Pafumi [GP98].

Claim: prove that the Esscher premium (2.119) is strictly increasing in α.

Remark: α plays the role of the risk aversion.

□

Exercise 2.10 (Expected Shortfall).

Choose a continuous integrable random variable Y on the filtered probability space $(\Omega, \mathcal{T}_n, P_\mathcal{T}, \mathcal{T})$. Denote the distribution of Y by $F_Y(x) = P\left[Y \le x\right]$ and the generalized inverse by F_Y^{-1}, where $F_Y^{-1}(u) = \inf\{x | F_Y(x) \ge u\}$. Henceforth, the Value-at-Risk of Y at level $1 - \alpha \in (0, 1)$ is then given by

$$\text{VaR}_{1-\alpha}(Y) = F_Y^{-1}(1-\alpha). \tag{2.120}$$

We obtain

$$\begin{aligned} P\left[Y > \text{VaR}_{1-\alpha}(Y)\right] &= 1 - P\left[Y \le \text{VaR}_{1-\alpha}(Y)\right] \\ &= 1 - F_Y\left(\text{VaR}_{1-\alpha}(Y)\right) \\ &= 1 - F_Y\left(F_Y^{-1}(1-\alpha)\right) = \alpha. \end{aligned} \tag{2.121}$$

Choose $c \in (0,1)$ and define (note that Y is $\mathcal{T}_n$-measurable)

$$\varphi_n^{(\mathcal{T})} = (1-c) + \frac{c}{\alpha} 1_{\{Y > \text{VaR}_{1-\alpha}(Y)\}}, \tag{2.122}$$

and for $t < n$

$$\varphi_t^{(\mathcal{T})} = E\left[\varphi_n^{(\mathcal{T})}\middle|\, \mathcal{T}_t\right]. \tag{2.123}$$

(1) Prove that $\boldsymbol{\varphi}^{(\mathcal{T})} \gg 0$. Moreover, prove that $\boldsymbol{\varphi}^{(\mathcal{T})} \in L^2_{n+1}(P_\mathcal{T}, \mathcal{T})$.

(2) Show that $(\varphi_t^{(\mathcal{T})})_{t=0,\ldots,n}$ is a density process w.r.t. $\mathcal{T}$ and $P_\mathcal{T}$.

Assume that $\mathbf{X}_k = (0, \ldots, 0, X_k, 0, \ldots, 0)$ with $X_k = \Lambda_k U_k^{(k)}$. Choose $Y = \Lambda_k$ and assume that $t < k$. Under Assumption 2.15 and (2.57) show that

$$Q_t\left[\mathbf{X}_k\right] = \left\{\beta_t E\left[\Lambda_k|\, \mathcal{T}_t\right] + (1-\beta_t) E\left[\Lambda_k|\, \Lambda_k > \text{VaR}_{1-\alpha}(\Lambda_k), \mathcal{T}_t\right]\right\} U_t^{(k)}, \tag{2.124}$$

with so-called credibility weights

$$\beta_t = \frac{1-c}{(1-c) + c\ \frac{P\left[\Lambda_k > \mathrm{VaR}_{1-\alpha}(\Lambda_k) | \mathcal{T}_t\right]}{\alpha}}. \tag{2.125}$$

We define the probability

$$\alpha_t = P\left[\Lambda_k > \mathrm{VaR}_{1-\alpha}(\Lambda_k) \middle| \mathcal{T}_t\right]. \tag{2.126}$$

This implies

$$\alpha_t = P\left[\Lambda_k > \mathrm{VaR}_{1-\alpha}(\Lambda_k) \middle| \mathcal{T}_t\right] = P\left[\Lambda_k > \mathrm{VaR}_{1-\alpha_t}(\Lambda_k | \mathcal{T}_t) \middle| \mathcal{T}_t\right], \tag{2.127}$$

which says

$$\mathrm{VaR}_{1-\alpha}(\Lambda_k) = \mathrm{VaR}_{1-\alpha_t}(\Lambda_k | \mathcal{T}_t), \tag{2.128}$$

where $\mathrm{VaR}_{1-\alpha_t}(\Lambda_k | \mathcal{T}_t)$ denotes the Value-at-Risk of $\Lambda_k | \mathcal{T}_t$ at level $1 - \alpha_t$. Henceforth, the credibility weight is given by

$$\beta_t = \frac{1-c}{(1-c) + c\ \frac{\alpha_t}{\alpha}} \tag{2.129}$$

and for the price of the insurance technical variable we obtain

$$\Lambda_{t,k} = \beta_t E\left[\Lambda_k \middle| \mathcal{T}_t\right] + (1-\beta_t) E\left[\Lambda_k \middle| \Lambda_k > \mathrm{VaR}_{1-\alpha_t}(\Lambda_k | \mathcal{T}_t), \mathcal{T}_t\right]. \tag{2.130}$$

The last term is called expected shortfall of $\Lambda_k | \mathcal{T}_t$ at level $1 - \alpha_t$, see McNeil et al. [MFE05]. Value-at-Risk and expected shortfall are probably the two most popular risk measures in the insurance industry.

Choose the special case $t = 0$. Then we have $\alpha_0 = \alpha$ (note that $\mathcal{T}_0 = \{\emptyset, \Omega\}$), which implies $\beta_t = 1 - c$ and

$$\begin{aligned} \Lambda_{0,k} &= (1-c) E\left[\Lambda_k\right] + c E\left[\Lambda_k \middle| \Lambda_k > \mathrm{VaR}_{1-\alpha}(\Lambda_k)\right] \\ &= E\left[\Lambda_k\right] + c \left\{ E\left[\Lambda_k \middle| \Lambda_k > \mathrm{VaR}_{1-\alpha}(\Lambda_k)\right] - E\left[\Lambda_k\right] \right\}. \end{aligned} \tag{2.131}$$

Henceforth, the basic reserve for Λ_k at time 0 is given by its expected value $E\left[\Lambda_k\right]$ plus a loading where $c \in (0,1)$ plays the role of the cost-of-capital rate and

$$E\left[\Lambda_k \middle| \Lambda_k > \mathrm{VaR}_{1-\alpha}(\Lambda_k)\right] - E\left[\Lambda_k\right] \tag{2.132}$$

is the capital-at-risk (unexpected loss) measured by the expected shortfall on the level $1 - \alpha$. This is in line with actual solvency developments, see for example SST [SST06], Pelsser [Pe10], Salzmann-Wüthrich [SW10] and Section 5.3 below.

□

2.7 Conclusions on Chapter 2

We have developed **theoretical foundations of market-consistent valuation** based on (possibly distorted) **expected values** (see (2.99) and (2.113)). The distorted probabilities will lead to the **price for risk**. The framework as developed is not the "full story" since it only gives the price for risk (the so-called (probability distorted) pure risk premium) for an insurance company.

However, it does not provide enough information on the risk bearing. This means, that we have not described how the risk bearing should be organized in order to protect against insolvencies.

An insurance company can take the following measures to protect itself against financial impacts of adverse scenarios:

1. buying options and reinsurance, if available,
2. hedging options internally,
3. setting up sufficient risk bearing capital (solvency margin).

In practice, one has to be extremely careful in each application whether the price for risk resulting from the mathematical model is already sufficient to finance adverse scenarios.

Remark on the existing literature. There is a wide range of literature on the definition of market-consistent values. Usually all these definitions are not very mathematical and slightly differ from each other, e.g. market-consistent values should be realistic values, should serve for the exchange of two portfolios, etc. One has to be very careful with these definitions, e.g. do they include cost-of-capital charges, etc.

Our model gives a mathematical framework for a market-consistent valuation. Charges for the risk bearing can be integrated via distorted probabilities, however (as mentioned above) this does not solve the question of the organization of the risk bearing.