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Market-Consistent Actuarial Valuation

Second, revised and enlarged edition

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Preface to the first edition

The balance sheet of an insurance company is often difficult to interpret. This derives from the fact that assets and liabilities are measured by different yardsticks. Assets are mostly valued at market prices; liabilities - as far as they relate to contractual obligations to the insured - are measured by established actuarial methods. Since, in general, there is no trading market for insurance policies, the question arises how these actuarial methods need to be changed to give values - as if these markets existed. The answer to this question is "Market-Consistent Actuarial Valuation". These lecture notes explain the logical mathematical framework that leads to market-consistent values for insurance liabilities.

In Chapter 1 we motivate the use of market-consistent values. Solvency requirements by regulators are one major reason for it.

Chapter 2 introduces stochastic discounting, which in a market-consistent actuarial valuation framework replaces discounting with the classical technical interest rate. In this chapter we introduce the notion of "Financial Variables", (which follow the laws of financial markets) and the notion of "Technical Variables", (which are purely depending on insurance events).

In Chapter 3 the concept of the "Valuation Portfolio" (VaPo) is introduced and explained in the life insurance context. The basic idea is not to calculate in monetary values but in units, which are appropriately chosen financial instruments. For life insurance products this choice is quite natural. The risk due to technical variables is included in the protected (against technical risk) VaPo, denoted by VaPo^{prot}.

Financial risk is treated in Chapter 4. It derives from the fact that the actual investment portfolio of the insurance company differs from the VaPo^{prot}. Ways to control the financial risk are - among others - derivative securities such as Margrabe Options and/or (additional) Risk Bearing Capital.

In Chapter 5 the notion of the Valuation Portfolio (VaPo) and the protected (against technical risk) Valuation Portfolio (VaPo^{prot}) is extended to the non-life insurance sector. The basic difference to life insurance derives from the fact that in property-casualty insurance the technical risk is much more important. The discussion of appropriate risk measures (in particular the quadratic prediction error) is therefore a central issue.

The final Chapter 6 contains selected topics. We mention only the treatment of the "Legal Quote" in life insurance.

These lecture notes stem from a course on Market-Consistent Actuarial Valuation, so far given twice at ETH Zürich, namely in 2004/05 by HB and HJF and in 2006 by MW and HJF. MW has greatly improved on the first version of these notes. But obviously also this version is not to be considered as final. For this reason we are grateful that the newly created EAA Lecture Notes series gives us the opportunity to share these notes with many friends and colleagues, whom we invite to participate in the process of discussions and further improvement of the present text as well as of further clarification of our way of understanding and modelling.

The authors wish to thank Professor Paul Embrechts for his interest and constant encouragement while they were working on this project. His support has been a great stimulus for us.

It is also a great honour for us that our text appears as the first volume of the newly founded EAA Lecture Notes series. We are grateful to Peter Diethelm, who as Managing Director has been the driving force in getting this series started.

Zürich, May 2007

Mario Wüthrich
Hans Bühlmann
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Preface to the second edition

The financial crisis of 2007-2010 has shown that the topic of market-consistent valuation and solvency has nothing lost from its topicality. On the contrary it has shown that we need a much deeper understanding of the models used, their limitations, etc. in order to model real world problems. In this spirit the first edition of these lecture notes has initiated a very active discussion among academics and practitioners about actuarial modelling and the use of models.

Since the first edition of these notes this course was again held at ETH Zurich in 2008 and 2010. Moreover, we have also presented part of these notes in various European countries, such as Germany, UK, France, The Netherlands, Sweden. These presentations have stimulated several interesting discussions which we have implemented into the new version. The main new features are:

In Chapter 2 we elaborate on the separation of financial deflators and probability distortions. For the financial deflator we then give an explicit simple example in terms of the discrete time Vasicek model. Probability distortions on the other hand can be understood in various ways. We give different examples that lead to the Esscher premium, to the cost-of-capital loading for expected shortfall and to first order life tables (in Chapter 3).

In Chapter 3 we introduce the approximate valuation portfolio which is useful in the case where we are not able to construct an exact valuation portfolio. This is done using selected scenarios evaluated with the help of an appropriate distance function. This is in line with the state-of-art concepts used in life insurance practice.

Finally, in Chapter 6 we add two sections that discuss losses and gains from insurance technical risks. This is closely related to the actual discussion of the claims development result in non-life insurance, but of course also applies to life insurance problems.

Zürich,
May 2010

Mario Wüthrich
Hans Bühlmann
Hansjörg Furrer

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Introduction

1.1 Three pillar approach

The recent years have shown that (financial) companies need to have a good management, a good business strategy, a good financial strength and a sound risk management in order to survive financial distress periods. It is essential that the risks are known, assessed, controlled and, whenever possible, quantified by the management and the respective specialist units.

Especially in the past few years, we have observed several failures of financial companies (for example Barings Bank, HIH Insurance Australia, Lehman Brothers, Washington Mutual, etc.). From 1996 until 2002 many companies were facing severe solvency and liquidity problems. As a consequence, supervision and politics have started several initiatives to analyze these problems and to improve qualitative and quantitative risk management within the companies (Basel II, Solvency 2 and local initiatives like the Swiss Solvency Test [SST06], for an overview we refer to Sandström [Sa06]). The next financial crisis of 2007-2010 however has shown that there is still a long way to go (leading economists view this last financial crisis as the worst one since the Great Depression of the 1930s). This crisis has (again) shown that risks need to be understood and managed properly on the one hand, and on the other hand that mathematical models, their assumptions and their limitations need to be well-understood in order to solve real world problems.

Concerning insurance companies: the goal behind all the solvency initiatives is to protect the policyholder (and the injured third party, respectively) from the consequences of an insolvency of an insurance company. Hence, in most cases it is not primarily the object of the regulator to avoid insolvencies of insurance companies, but given an insolvency of an insurance company occurs, the regulator has to ensure that all liabilities are covered with assets and can be fulfilled in an appropriate way (this is not the shareholder's point of view).

One special project was carried out by the "London working group". The London working group has analyzed 21 cases of solvency problems (actual

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failures and 'near misses') in 17 European countries. Their findings can be found in the famous Sharma Report [Sha02]. The main lessons learned are:

- In most cases bad management was the source of the problem.
- Lack of a comprehensive risk landscape
- Misspecified business strategies which were not adapted to local situations

From this perspective, what can we really do?

Sharma says: "Capital is only the second strategy of defense in a company, the first is a good risk management".

Supervision has started several initiatives to strengthen the financial basis and to improve risk management thinking within the industry and the companies (e.g. Basel II, Solvency 2, Swiss Solvency Test [SST06], Laeven-Valencia [LV08], Besar et al. [BBCMP09]). Most of the new approaches and requirements are formulated in three pillars:

1. Pillar 1: Minimum financial requirements (quantitative requirements)
2. Pillar 2: Supervisory review process, adequate risk management (qualitative requirements)
3. Pillar 3: Market discipline and public transparency

Consequences: regulators, academia as well as actuaries, mathematicians and risk managers of financial institutions are in search for new solvency regulations. These guidelines should be **risk-adjusted**. Moreover they should be based on a **market-consistent valuation** of the balance sheet (full balance sheet approach).

From this perspective we derive the valuation portfolio which reflects a market-consistent actuarial valuation of our balance sheet. Moreover, we describe the uncertainties within this portfolio which corresponds to a risk-adjusted analysis of our assets and liabilities.

1.2 Solvency

The International Association of Insurance Supervisors IAIS [IAIS05] defines solvency as follows

"the ability of an insurer to meet its obligations (liabilities) under all contracts at any time. Due to the very nature of insurance business, it is impossible to guarantee solvency with certainty. In order to come to a practicable definition, it is necessary to make clear under which circumstances the appropriateness of the assets to cover claims is to be considered, ...".

Hence the aim of solvency is to protect the policyholder (or the injured third party, respectively). As it is formulated in Swiss law: it is not the main goal of the regulator to avoid insolvencies of insurance companies, but in case of an insolvency the policyholder's demands must still be met. Avoiding

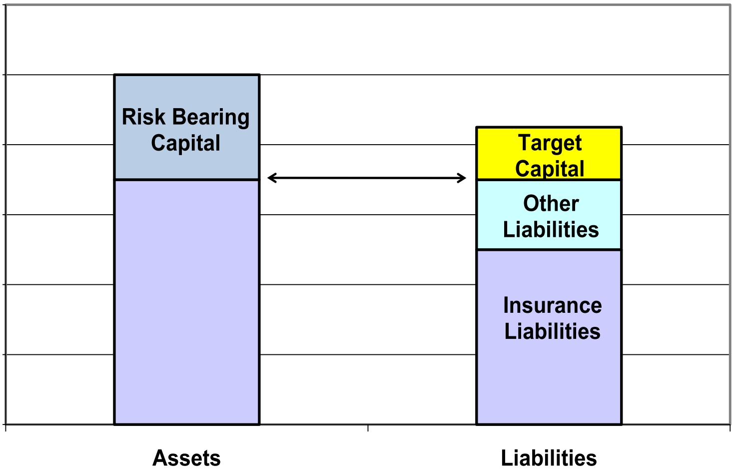

Fig. 1.1. Balance sheet of an insurance company

insolvencies must be the main task of the management and the board of an insurance company. Moreover, modern risk-based solvency requirements will ensure a certain stability of the financial market.

In this lecture we give a mathematical approach and interpretation to the solvency definition of the IAIS [IAIS05].

Let us start with two definitions (see also Fig. 1.1):

1. **Available Solvency Surplus** (see [IAIS05]), or **Risk Bearing Capital** RBC (see [SST06]) is the difference between the market-consistent value of the assets minus the market-consistent value of the liabilities. This corresponds to the Available Risk Margin, the Available Risk Capacity or the Financial Strength of a company.
2. **Required Solvency Margin** (see [IAIS05]) or **Target Capital** TC (see [SST06]) is the Required Risk Capital (from a regulatory point of view) in order to be able to run the business such that also certain adverse scenarios are covered (see solvency definition of the IAIS [IAIS05]). This is the Necessary Risk Capacity, Required Risk Capacity or the Minimal Financial Requirement for writing certain business.

Then, the general solvency requirement is:

$$\mathrm{TC} \overset{!!!}{\leq} \mathrm{RBC}. \tag{1.1}$$

Hence, given the amount of risk TC a company is exposed to, the regulators require that this risk is bounded by the available surplus RBC. That is, RBC defines the risk capacity of a company, which has to be compared to the required solvency margin TC.

Otherwise if (1.1) is not satisfied, the authorities force the company to take certain actions to improve the financial strength or to reduce the risks within the portfolio, such as: write less risky business, buy protection instruments (like derivative securities), sell part of the business or even close the company, and make sure that another company guarantees the smooth runoff of the liabilities.

1.3 From the past to the future

A short overview on the historical developments of solvency requirements can be found in Sandström [Sa07].

In the past, the evaluation of the Risk Bearing Capital RBC was not based on market-consistent valuation techniques of assets and liabilities (for example, insurance liabilities were measured by means of statutory accounting principles). Moreover, the Target Capital TC was volume- and not risk-based. For example, the Solvency 1 regulations in non-life insurance were simply of the form

$$\text{Target Capital TC } = 16\% \text{ of premium,} \tag{1.2}$$

and in traditional life insurance they are essentially given by

$$\begin{aligned}\text{Target Capital TC } &= 4\% \text{ of the mathematical reserves (financial risk)}\\ &\quad +3‰ \text{ of capital at risk (technical risk).}\end{aligned} \tag{1.3}$$

These solvency regulations are very simple and robust, easy to understand and to use. They are rule-based but not risk-based. As such, they are not tailored to the specifics of the written business and neglect the differences between the asset and the liability profiles. Moreover, risk mitigation techniques such as reinsurance are only allowed to a limited extent as eligible elements for the solvency margin.

Our goal in this lecture is to give a mathematical theory to a market-consistent valuation approach. Moreover, our model builds a bridge of understanding between actuaries and asset managers. In the past, actuaries were responsible for the liabilities in the balance sheet and asset managers were concerned with the active side of the balance sheet. But these two parties do not always speak the same language which makes it difficult to design a successful asset and liability management (ALM) strategy. In this lecture we introduce a language which allows actuaries and asset managers to communicate in a successful way which leads to a canonical risk-adjusted full balance sheet approach to the solvency problem.

1.4 Full balance sheet approach

A typical balance sheet of an insurance company contains the following positions:

Assets	Liabilities
cash and cash equivalents	deposits
debt securities	policyholder deposits
bonds	reinsurance deposits
loans	borrowings
mortgages	money market
real estate	hybrid debt
equity	convertible debt
equity securities	insurance liabilities
private equity	mathematical reserves
investments in associates	claims reserves
hedge funds	premium reserves
derivatives	derivatives
futures, swaptions, equity options	
insurance and other receivables	insurance and other payables
reinsurance assets	reinsurance liabilities
property and equipment	employee benefit plan
intangible assets	provisions
goodwill	
deferred acquisition costs	
income tax assets	income tax liabilities
other assets	other liabilities

It is necessary that assets and liabilities are measured in a consistent way. Market values have no absolute significance, depending on the purpose other values may be better (for example statutorial values). But market values guarantee the switching property (at market price).

Applications of these lectures are found in:

- pricing and reserving of insurance products,
- value based management tools, dynamical financial analysis tools,
- risk management tools,
- for solvency purposes which are based on a market-consistent valuation,
- finding prices for trading insurance policies and for loss portfolio transfers.

In the past actuaries we have mostly been using deterministic models for discounting liabilities. As soon as interest rates are assumed to be stochastic, life is much more complicated. This is illustrated by the following example. Let $r > 0$ be a stochastic interest rate, then (by Jensen's inequality applied to the convex function $u(x) = (1 + x)^{-1}$)

$$1 = E\left[\frac{1+r}{1+r}\right] \neq E\left[1+r\right]\ E\left[\frac{1}{1+r}\right] > 1, \tag{1.4}$$

that is, in a stochastic environment we cannot simply exchange the expectation of the stochastic return $1 + r$ with the expectation of the stochastic discount $(1 + r)^{-1}$. This problem arises as soon as we work with random variables. In the next chapter we define a consistent model for stochastic discounting (deflating) cash flows.

1.5 Recent financial failures and difficulties

We close this chapter with some recent failures in the insurance industry. This list is far from being complete. For instance, it does not contain companies which were taken over by other companies just before they would have collapsed.

- 1988-1991: Lloyd's London loss of more than USD 3 billion due to asbestos and other health IBNR claims.
- 1991: Executive Life Insurance Company due to junk bonds.
- 1993: Confederation Life Insurance, Canada, loss of USD 1.3 billion due to fatal errors in asset investments.
- 1997 Nissan Mutual Life, Japan, too high guarantees on rates cost 300 billion Yen.
- 2000: Dai-ichi Mutual Fire and Marine Insurance Company, Japan, is liquidated, strategic mismanagement of their insurance merchandise.
- 2001: HIH Insurance Australia is liquidated due to loss of USD 4 billion.
- 2001: Independent Insurance UK is liquidated due to rapid growth, insufficient reserves and not adequate premiums.
- 2001: Taisei Fire and Marine, Japan, loss of 100 billion Yen due to large reinsurance claims, e.g. world trade center September 11, 2001.
- 2002 Gerling Global Re, Germany, seemed to be undercapitalized and underreserved for many years. Further problems arose by the acquisition of Constitution Re.
- 2003: Equitable Life Assurance Society UK is liquidated due to concentration and interest rate risks.
- 2003: KBV Krankenkasse, Switzerland, is liquidated due to financial losses caused by fraud.

This was the list presented in our first edition of the lecture notes in 2007. Meanwhile we have many additional bad examples that have failed in the financial crisis 2007-2010. For example, we may mention:

- Yamato Life Insurance, loss of 22 billion Yen.
- AIG, loss of USD 145 billion by August 2009, see Donnelly-Embrechts [DE10].

- In 2008: 25 US banks failed with, for example, Washington Mutual with a loss of USD 307 billion.
- In 2009: 140 US banks failed.
- Investment banks have disappeared: Merrill Lynch, Lehman Brothers, Bear Stearns, etc.
- Fraud cases like the Madoff Investment Securities LLC.

A detailed anatomy of the credit crisis 2007–2010 can be found in the Geneva Reports [GR10].

This shows that the topic of the current lecture notes has not lost any of its topicality and there is a strong need for sound quantitative methods (and the understanding of their limitations).

2

Stochastic discounting

In this chapter we define a mathematically consistent model for calculating time values of cash flows. The key objects are so-called deflators which play the role of stochastic discount factors. Our definition (via deflators) leads to market values which are consistent with the usual financial theory that involves risk neutral valuation. Typically, in financial mathematics the pricing formulas are based on equivalent martingale measures (see, for example, Föllmer-Schied [FS04]), economists use the notion of state price density processes (see Malamud et al. [MTW08]) and actuaries use the terminology of deflators under the real world probability measure (see Duffie [Du96] and Bühlmann et al. [BDES98]). In this chapter we describe these terminologies.

Moreover, we would like to emphasize that in financial mathematics one usually works under risk neutral measures (equivalent martingale measures) for pricing financial assets. In actuarial mathematics, however, one should also understand the processes under the real world probability measure (physical measure) which makes it necessary that we understand the connection between these two probability measures as well as the transform of measure techniques.

2.1 Basic discrete time model

In this chapter we develop the theoretical foundations of market-consistent valuation. We work in a discrete time setting which has the advantage that the mathematical machinery becomes simpler for the calculation of the price processes (for continuous time models we refer to the standard literature on financial mathematics, see for example Jeanblanc et al. [JYC09]).

Choose $n \in \mathbb{N}$ fixed. This is the final time horizon. Then, w.l.o.g. we consider cash flows on the yearly grid $t = 0, 1, \ldots, n$.

We choose a probability space $(\Omega, \mathcal{F}, P)$ and an increasing sequence of σ-fields $\mathbb{F} = (\mathcal{F}_t)_{t=0,\ldots,n}$ with

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$$\{\emptyset, \Omega\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n \tag{2.1}$$

and for simplicity, we assume $\mathcal{F}_n = \mathcal{F}$. We call $(\Omega, \mathcal{F}, P, \mathbb{F})$ a filtered probability space with filtration $\mathbb{F}$. The σ-field $\mathcal{F}_t$ plays the role of the information available/known at time t. This includes demographic information, insurance technical information on insurance contracts, financial and economic information and any other information (weather conditions, legal changes, politics, etc.) that is available at time t.

Moreover, we assume that we have a sequence of $\mathbb{F}$-adapted random variables

$$\mathbf{X} = (X_0, X_1, \ldots, X_n) \tag{2.2}$$

on the filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F})$. That is, X_t is an $\mathcal{F}_t$-measurable random variable for all $t = 0, \ldots, n$.

Interpretation and aim. $\mathbf{X}$ is a (random) cash flow, with single payments X_t at time t. If we have information $\mathcal{F}_t$, then X_k is known for all $k \leq t$, otherwise it may be random. Henceforth, on the one hand, we need to predict future payments X_s, $s > t$, based on the information $\mathcal{F}_t$ available at time t. On the other hand, our goal is to determine the (time) value of such cash flows $\mathbf{X}$ at any time $t = 0, \ldots, n$, see also Figure 2.1.

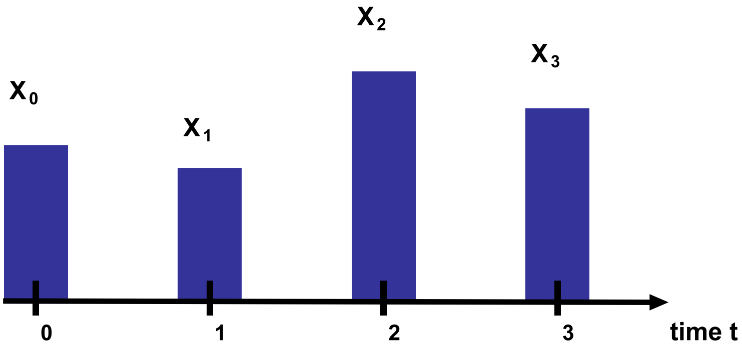

Fig. 2.1. Cash flow $\mathbf{X} = (X_0, X_1, \ldots, X_n)$

We make some technical assumptions.

Assumption 2.1 *Assume that every component of* $\mathbf{X}$ *is square integrable.*

For a general square integrable cash flow $\mathbf{X} = (X_0, X_1, \ldots, X_n)$ we write

$$\mathbf{X} = (X_0, X_1, \ldots, X_n) \in L^2_{n+1}(P), \tag{2.3}$$

where $L^2_{n+1}(P)$ is a Hilbert space with

$$E\left[\sum_{t=0}^{n} X_t^2\right] < \infty \qquad \text{for all } \mathbf{X} \in L^2_{n+1}(P), \tag{2.4}$$

$$< \mathbf{X}, \mathbf{Y} > \;=\; E\left[\sum_{t=0}^{n} X_t Y_t\right] \qquad \text{for all } \mathbf{X}, \mathbf{Y} \in L^2_{n+1}(P), \tag{2.5}$$

$$\|\mathbf{X}\| = \;< \mathbf{X}, \mathbf{X} >^{1/2} \;< \infty \qquad \text{for all } \mathbf{X} \in L^2_{n+1}(P). \tag{2.6}$$

If the cash flow $\mathbf{X} = (X_0, X_1, \ldots, X_n)$ is $\mathbb{F}$-adapted and square integrable, then we write

$$\mathbf{X} = (X_0, X_1, \ldots, X_n) \in L^2_{n+1}(P, \mathbb{F}). \tag{2.7}$$

Technical remark. The equality $\|\mathbf{X} - \mathbf{Y}\| = 0$ implies that $\mathbf{X} = \mathbf{Y}$, P-a.s. As usually done in Hilbert spaces, we identify random variables which are equal P-a.s.

Example 2.1 (Life insurance).

We consider a general life insurance policy financed by a regular premium income stream $(\Pi_0, \ldots, \Pi_n)$, where Π_t denotes the premium payment made at time t. Furthermore cash outflows comprise the expenses and the benefit payments occurring in the time interval $(t-1, t]$. If we map all cash flows occurring in the time interval $(t-1, t]$ to the right end point t of the time interval, we obtain a discrete time cash flow for $t \in \{0, \ldots, n\}$:

$$X_t = -\Pi_t + \text{benefits and expenses paid within } (t-1, t]. \tag{2.8}$$

Henceforth, $\mathbf{X}$ denotes the cash flow generated by this single policy.

□

Example 2.2 (Non-life insurance).

In non-life insurance the insurance company usually receives a (risk) premium at the beginning of a well-defined insurance period. Within this insurance period certain (well-defined, random) financial losses are covered. We denote the premium payment by $\Pi = -X_0$. The occurrence of an insured event (covered claim) during the insurance period typically entails a sequence of future cash outflows, namely claims payments until the claim is settled. That is, usually the insurance company cannot immediately settle a claim. It takes quite some time until the ultimate claim amount is known. The delay in the settlement is due to the fact that, for example, it takes time until the total medical expenses are known, until the claim is settled at court, until the damaged building is fixed, until the recovery process is understood, etc. (see also Wüthrich-Merz [WM08]).

Since one does not wait with the payments until the ultimate claim amount is known (e.g. medical expenses and salaries are paid when they occur) a claim consists of several single payments X_t which reflect the on-going recovery process. Hence, the total or ultimate claim amount (nominal) is given by

$$C_n = \sum_{t=1}^{n} X_t, \tag{2.9}$$

where X_t $(t \leq n)$ denote the single claims payments and X_n denotes the final payment when the claim is closed/settled. Henceforth, at time t we have information $\mathcal{F}_t$ and the payments X_k, $k \leq t$, are already made, whereas the future payments X_s, $s > t$, need to be predicted based on the information $\mathcal{F}_t$ available at time t.

The underwriting loss (nominal loss) can then be written as

$$UL = \sum_{t=0}^{n} X_t = -\Pi + C_n. \tag{2.10}$$

Remark. UL does not necessarily need to be negative to run successfully this non-life insurance business. The nominal underwriting loss UL does not consider the financial income during the settlement of the claim. That is, the delay in the payments allows for discounting of the payments, which in the profit and loss statement is considered similar to investment incomes on financial assets at the insurance company (see next sections).

□

2.2 Market-consistent valuation in the basic discrete time model

We now value the (stochastic) cash flow $\mathbf{X}$. We proceed as in Bühlmann [Bü92, Bü95] using a positive, continuous, linear (valuation) functional.

Definition 2.2 (Positivity)

- $\mathbf{X} \geq 0 \iff X_t \geq 0$, *$P$-a.s., for all* $t = 0, \ldots, n$.
- $\mathbf{X} > 0 \iff \mathbf{X} \geq 0$ *and there exists* $k \in \{0, \ldots, n\}$ *such that* $X_k > 0$ *with positive probability.*
- $\mathbf{X} \gg 0 \iff X_t > 0$, *$P$-a.s., for all* $t = 0, \ldots, n$.

Assumption 2.3 *Assume that* $Q : L^2_{n+1}(P) \to \mathbb{R}$ *is a positive, continuous, linear functional on* $L^2_{n+1}(P)$.

This means that the functional Q satisfies the following properties:

(1) Positivity: $\mathbf{X} > 0$ implies $Q[\mathbf{X}] > 0$.

(2) Continuity: For any sequence $\mathbf{X}^{(k)} \in L^2_{n+1}(P)$ with $\mathbf{X}^{(k)} \to \mathbf{X}$ in $L^2_{n+1}(P)$ as $k \to \infty$, we have $Q\left[\mathbf{X}^{(k)}\right] \to Q\left[\mathbf{X}\right]$ in $\mathbb{R}$ as $k \to \infty$.

(3) Linearity: For all $\mathbf{X}, \mathbf{Y} \in L^2_{n+1}(P)$ and $a, b \in \mathbb{R}$ we have

$$Q\left[a\mathbf{X} + b\mathbf{Y}\right] = aQ\left[\mathbf{X}\right] + bQ\left[\mathbf{Y}\right]. \tag{2.11}$$

Terminology.
The mapping $\mathbf{X} \mapsto Q[\mathbf{X}]$ assigns a monetary value $Q[\mathbf{X}] \in \mathbb{R}$ at time 0 to the cash flow $\mathbf{X}$. That is, the valuation function Q attaches a value to any $\mathbf{X} \in L^2_{n+1}(P)$, which can be seen as the price of $\mathbf{X}$ at time 0. As we will see below, this valuation/pricing will be done in a market-consistent way which leads to a risk neutral valuation scheme and $Q[\mathbf{X}]$ is the (market-consistent) price for $\mathbf{X}$ at time 0.

Remark. Assumptions (1) and (3) ensure that one can develop an arbitrage-free pricing system (see Lemma 2.8 and Remark 2.14).

Lemma 2.4 *Assumptions (1) and (3) imply (2).*

Proof. Define $\mathbf{Y}^{(k)} = \mathbf{X}^{(k)} - \mathbf{X}$. Due to the linearity of Q it suffices to prove that $\mathbf{Y}^{(k)} \to 0$ in $L^2_{n+1}(P)$ implies that $Q\left[\mathbf{Y}^{(k)}\right] \to 0$.

In the first step we assume that $\mathbf{Y}^{(k)} \geq 0$. Then we claim

$$\mathbf{Y}^{(k)} \to 0 \text{ in } L^2_{n+1}(P) \text{ implies } Q\left[\mathbf{Y}^{(k)}\right] \to 0 \text{ as } k \to \infty. \tag{2.12}$$

Assume (2.12) does not hold true, hence (using the positivity of the linear functional) there exists $\varepsilon > 0$ and an infinite subsequence k' of k such that for all k'

$$Q\left[\mathbf{Y}^{(k')}\right] \geq \varepsilon. \tag{2.13}$$

Choose an infinite subsequence k'' of k' with

$$\sum_{k''} \left\| \mathbf{Y}^{(k'')} \right\| < \infty. \tag{2.14}$$

We define

$$\mathbf{Y} = \sum_{k''} \mathbf{Y}^{(k'')}. \tag{2.15}$$

Due to the completeness of $L^2_{n+1}(P)$ we know that $\mathbf{Y} \in L^2_{n+1}(P)$. But

$$Q\left[\mathbf{Y}\right] \geq Q\left[\sum_{k''=1}^{K} \mathbf{Y}^{(k'')}\right] \geq K\,\varepsilon \qquad \text{for every } K. \tag{2.16}$$

This implies that $Q\left[\mathbf{Y}\right] = \infty$ is not finite, which is a contradiction.

Second step: Decompose $\mathbf{Y}^{(k)} = \mathbf{Y}_+^{(k)} - \mathbf{Y}_-^{(k)}$ into a positive and a so-called negative part. Since $\|\mathbf{Y}_+^{(k)}\| \leq \|\mathbf{Y}^{(k)}\| \to 0$ and $\|\mathbf{Y}_-^{(k)}\| \leq \|\mathbf{Y}^{(k)}\| \to 0$ we see that both $\mathbf{Y}_+^{(k)}$ and $\mathbf{Y}_-^{(k)}$ tend to 0. Because $\mathbf{Y}_+^{(k)} \geq 0$ and $\mathbf{Y}_-^{(k)} \geq 0$ we have - as proved in the first step -

$$Q\left[\mathbf{Y}_+^{(k)}\right] \to 0 \quad \text{and} \quad Q\left[\mathbf{Y}_-^{(k)}\right] \to 0. \tag{2.17}$$

Using once more the linearity of Q completes the proof.

□

Theorem 2.5 (Riesz' representation theorem) *Under Assumption 2.3 there exists* $\boldsymbol{\varphi} \in L^2_{n+1}(P)$ *such that for all* $\mathbf{X} \in L^2_{n+1}(P)$ *we have*

$$Q[\mathbf{X}] \;=\; <\mathbf{X}, \boldsymbol{\varphi}> \;=\; E\left[\sum_{t=0}^{n} X_t \; \varphi_t\right]. \tag{2.18}$$

Definition 2.6 *The vector* $\boldsymbol{\varphi}$ *(and its single components* φ_t*) is called (state price)* **deflator***.*

The terminology (state price) deflator was introduced by Duffie [Du96] and Bühlmann et al. [BDES98]. In economic theory deflators are called "state price densities" and in financial mathematics "financial pricing kernels" or "stochastic interest rates".

Remarks. The deflator has the following properties:

- The positivity of Q ensures that $\boldsymbol{\varphi} \gg 0$.
- Assume $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$ is $\mathbb{F}$-adapted. Then $\boldsymbol{\varphi}$ may also be chosen $\mathbb{F}$-adapted: replace φ_t by $\widetilde{\varphi}_t = E[\varphi_t | \mathcal{F}_t]$. Then we have for all $\mathbb{F}$-adapted, square integrable cash flows $\mathbf{X}$

$$\begin{aligned} Q[\mathbf{X}] &= E\left[\sum_{t=0}^{n} X_t \; \varphi_t\right] = \sum_{t=0}^{n} E[X_t \; \varphi_t] = \sum_{t=0}^{n} E[E[X_t \; \varphi_t | \mathcal{F}_t]] \\ &= \sum_{t=0}^{n} E[X_t \; E[\varphi_t | \mathcal{F}_t]] = E\left[\sum_{t=0}^{n} X_t \; E[\varphi_t | \mathcal{F}_t]\right] \\ &= E\left[\sum_{t=0}^{n} X_t \; \widetilde{\varphi}_t\right] = \; <\mathbf{X}, \widetilde{\boldsymbol{\varphi}}>, \end{aligned} \tag{2.19}$$

where in the third step on the first line we have used the tower property for conditional expectations (see Williams [Wi91]), and in the fourth step we have used that X_t is $\mathcal{F}_t$-measurable. Henceforth, because we will only work on $L^2_{n+1}(P, \mathbb{F})$ we may and will assume that $\boldsymbol{\varphi}$ is $\mathbb{F}$-adapted, throughout.

- There is exactly one $\mathbb{F}$-adapted deflator $\boldsymbol{\varphi}$ in $L^2_{n+1}(P, \mathbb{F})$ for a given Q (up to measure 0): assume that there are two $\mathbb{F}$-adapted random vectors $\boldsymbol{\varphi}$ and $\boldsymbol{\varphi}^*$ satisfying for all $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$

$$Q[\mathbf{X}] = < \mathbf{X}, \boldsymbol{\varphi} > = < \mathbf{X}, \boldsymbol{\varphi}^* > . \tag{2.20}$$

But then we choose $\mathbf{X} = \boldsymbol{\varphi} - \boldsymbol{\varphi}^* \in L^2_{n+1}(P, \mathbb{F})$. This and (2.20) imply

$$0 = < \mathbf{X}, \boldsymbol{\varphi} - \boldsymbol{\varphi}^* > = \|\boldsymbol{\varphi} - \boldsymbol{\varphi}^*\|^2, \tag{2.21}$$

which immediately gives $\boldsymbol{\varphi} = \boldsymbol{\varphi}^*$, P-a.s.
- Furthermore, we assume that Q is such that $\varphi_0 \equiv 1$. This means that for a (deterministic) payment x_0 at time 0, we have $Q[(x_0, 0, \ldots, 0)] = x_0$. This means that for x_0 the functional Q delivers simply its nominal value.
- We have assumed that $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$ in order to find the state price deflator $\boldsymbol{\varphi}$. This can be generalized to cash flows $\mathbf{X} \in L^p_{n+1}(P, \mathbb{F})$ $(1 \le p \le \infty)$ and then the deflator $\boldsymbol{\varphi}$ would be in $L^q_{n+1}(P, \mathbb{F})$ with $1/p + 1/q = 1$. Or even more generally we can take $\boldsymbol{\varphi} \in L^1_{n+1}(P, \mathbb{F})$ fixed and then define the set of cash flows that can be priced by

$$\mathcal{L}_{\boldsymbol{\varphi}} = \left\{ \mathbf{X} \in L^1_{n+1}(P, \mathbb{F}) : \ < \mathbf{X}, \boldsymbol{\varphi} > \ < \infty \right\}. \tag{2.22}$$

For these cash flows we then define the pricing functional Q on $\mathcal{L}_{\boldsymbol{\varphi}}$ by $Q[\mathbf{X}] = < \mathbf{X}, \boldsymbol{\varphi} >$.

2.2.1 Task of modelling

Find the appropriate pricing functional Q or equivalently find the appropriate $\mathbb{F}$-adapted state price deflator $\boldsymbol{\varphi}$!

In the more general setup, one would define/choose $\boldsymbol{\varphi} \in L^1_{n+1}(P, \mathbb{F})$ and then value the cash flows $\mathbf{X} \in \mathcal{L}_{\boldsymbol{\varphi}}$, see (2.22). The choice of $\boldsymbol{\varphi}$ will include market risk aversion as well as individual risk aversion, this will be described in the following chapters, and we will also describe the connection between the state price deflators and the risk neutral martingale measures.

The $\mathbb{F}$-adaptedness will be crucial in the sequel. It essentially means that the deflator φ_t (stochastic discount factor) is known at time t, and hence, allows for a direct connection of the $\mathcal{F}_t$-measurable cash flow X_t with the behaviour φ_t of the financial market at time t. Especially, this means that φ_t will allow for the modelling of embedded options and guarantees in X_t that depend on economic and financial scenarios.

Examples of state price deflators can be found in Bühlmann [Bü95], for example the Ehrenfest Urn with limit Ornstein-Uhlenbeck model, in Filipovic-Zabczyk [FZ02] or one can easily discretize, for example, the Vasicek model, see Brigo-Mercurio [BM06] and Exercise 2.3 below.

Exercise 2.3 (Discrete time Vasicek [Va77] model).

Choose a filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F})$ and assume that $(\varepsilon_t)_{t=0,\dots,n}$ is $\mathbb{F}$-adapted, that ε_t is independent of $\mathcal{F}_{t-1}$ for all $t = 1, \dots, n$ and standard Gaussian distributed. Then, we define the stochastic process $(r_t)_{t=0,\dots,n}$ by $r_0 > 0$ (fixed) and for $t \geq 1$

$$r_t = b + \beta r_{t-1} + \rho \varepsilon_t, \tag{2.23}$$

for given $b, \beta, \rho > 0$. This $(r_t)_{t=0,\dots,n}$ describes the spot rate dynamics of the Vasicek model under the (real world) probability measure P, see Brigo-Mercurio [BM06] Section 3.2.1.
Next, we choose $\lambda \in \mathbb{R}$ and define the deflator in the Vasicek model by

$$\varphi_t = \exp\left\{ -\sum_{k=1}^{t} \left[r_{k-1} + \frac{\lambda^2}{2}\, r_{k-1}^2 \right] - \sum_{k=1}^{t} \lambda\, r_{k-1}\, \varepsilon_k \right\}. \tag{2.24}$$

Prove that $\boldsymbol{\varphi} \in L^1_{n+1}(P, \mathbb{F})$ is a deflator. Moreover, prove that the cash flow $\mathbf{X} = (0, \dots, 0, 1, 0, \dots, 0) \in \mathcal{L}_\varphi$, see (2.22).

□

2.2.2 Understanding deflators

A **deflator** φ_t transports cash amount at time t to value at time 0, see Figure 2.2. This transportation is a stochastic transportation (stochastic discounting). This implies, a cash flow $\mathbf{X}_t = (0, \dots, 0, X_t, 0, \dots, 0)$ does not necessarily need to be independent (or uncorrelated) of φ_t, which then gives

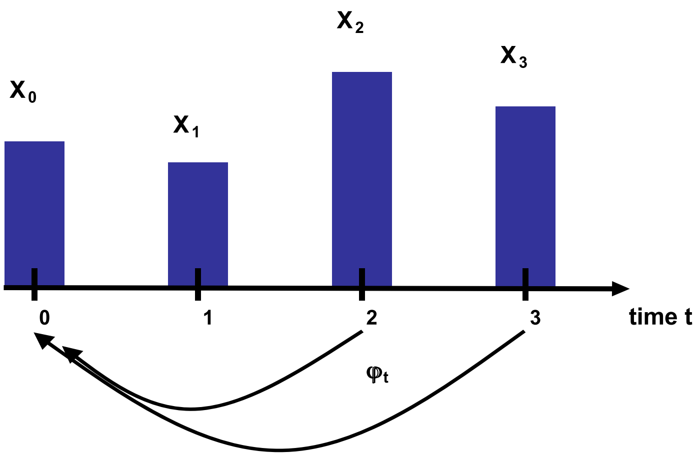

Fig. 2.2. Deflator φ and cash flow $\mathbf{X}$

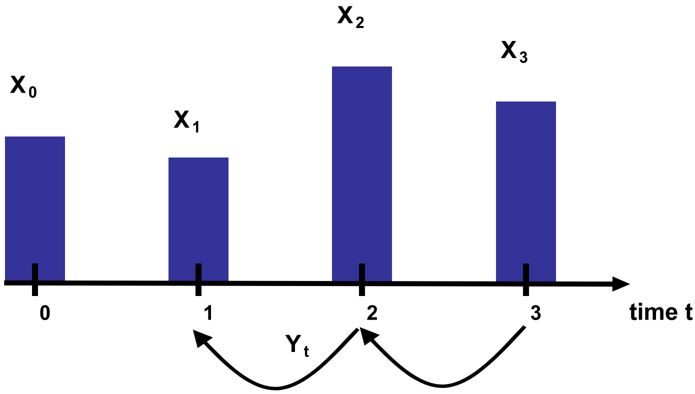

Fig. 2.3. Span-deflators Y_t and cash flow $\mathbf{X}$

$$Q\left[\mathbf{X}_t\right] = E\left[X_t\ \varphi_t\right] \neq E\left[X_t\right] E\left[\varphi_t\right]. \tag{2.25}$$

$Q\left[\mathbf{X}_t\right]$ describes the value/price of $\mathbf{X}_t$ at time 0, where X_t is stochastically discounted with the deflator φ_t.

We decompose the deflator $\boldsymbol{\varphi}$ into its **span-deflators**. Since $\boldsymbol{\varphi} \gg 0$ we can build the following ratios for all $t > 0$, P-a.s.:

$$Y_t = \frac{\varphi_t}{\varphi_{t-1}}. \tag{2.26}$$

Moreover, we define $Y_0 = 1$. Thus, $\mathbf{Y} = (Y_t)_{t=0,\ldots,n}$ is $\mathbb{F}$-adapted and satisfies

$$\varphi_t = Y_0\ Y_1\ \cdots\ Y_t = \prod_{k=0}^{t} Y_k. \tag{2.27}$$

$\mathbf{Y} = (Y_t)_{t=0,\ldots,n}$ is called span-deflator. Span-deflators Y_t, $t \geq 1$, transport cash amount at time t to value at time $t-1$, see Figure 2.3.

Question. How is the deflator $\boldsymbol{\varphi}$ related to zero coupon bonds and classical financial discounting?

Denote by $\mathbf{Z}^{(t)} = (0, \ldots, 0, 1, 0, \ldots, 0)$ the cash flow of the **zero coupon bond** paying the amount 1 at time t. The value at time 0 of this zero coupon bond is given by

$$D_{0,t} = Q\left[\mathbf{Z}^{(t)}\right] = E\left[\varphi_t\right]. \tag{2.28}$$

In the financial literature $D_{0,t}$ is often denoted by $P(0,t)$, which is the value at time 0 of a default-free contract paying 1 at time t.

Hence, also $D_{0,t}$ transports cash amount at time t to value in 0. But $D_{0,t}$ is $\mathcal{F}_0$-measurable, whereas φ_t is a $\mathcal{F}_t$-measurable random variable. This means that the deterministic discount factor $D_{0,t}$ is known at the beginning of the time period $(0,t]$, whereas φ_t is only known at the end of the time period $(0,t]$. As long as we deal with deterministic cash flows $\mathbf{X}$, we can either work with zero coupon bond prices $D_{0,t}$ or with deflators φ_t to determine the value of $\mathbf{X}$ at time 0. But as soon as the cash flows $\mathbf{X}$ are stochastic we need to work with deflators (see (2.25)) since X_t and φ_t may be influenced by the same factors (are dependent). An easy example is that X_t is an option that depends on the actual realization of φ_t. Various life insurance policies contain such embedded options and financial guarantees, that is, the insurance payout depends on the development of economic and financial market factors (which are also risk drivers of φ_t).

Classical **actuarial discounting** is taking a constant interest rate i. That is, in classical actuarial models φ_t has the following form

$$\varphi_t = (1+i)^{-t}. \tag{2.29}$$

This deflator gives a consistent theory but it is far from the economic observations in practice. This indicates that we have to be very careful with this deterministic model in a total balance sheet approach, since it implies that we obtain values far away from those consistent with the financial market values on the asset classes.

Exercise 2.4 (Price of the zero coupon bond in the Vasicek model).

We revisit the discrete time Vasicek model presented in Exercise 2.3. Calculate for this model the zero coupon bond price $D_{0,t}$. We claim that this price is given by

$$D_{0,t} = \exp\left\{a(t) - r_0\ b(t)\right\}, \tag{2.30}$$

for appropriate functions $a(t)$ and $b(t)$.

Hint: the claim is proved by induction using properties of log-normal distributions.

Give an interpretation to r_0 in terms of $D_{0,1}$.

□

2.2.3 Toy example for deflators

In this subsection we give a toy example which is based on finite probability spaces: in a first step we need a market model for calibration purposes. In a second step we construct deflators (the example is taken from Jarvis et al. [JSV01]).

We consider a one-period model, and we assume that there are two possible states at time 1, namely $\Omega = \{\omega_1, \omega_2\}$. For this example on finite probability spaces finding the deflators is essentially an exercise in linear algebra. Here, we would also like to mention that finite models often have the advantage that one can easier find the crucial mathematical and economic structures (see also Malamud et al. [MTW08]).

Step 1. In a first step we construct the **state space securities** SS_1 and SS_2. A state space security for state ω_i pays one unit if state ω_i occurs at time 1. These state space securities are used to construct an arbitrage-free pricing model. That is,

	SS_1	SS_2
market price Q at time 0	?	?
payout if in state ω_1 at time 1	1	0
payout if in state ω_2 at time 1	0	1

Since we have two states ω_1 and ω_2 we need two linearly independent assets A and B to calibrate the model. Assume that assets A and B have the following price and payout structure:

	asset A	asset B
market price Q at time 0	1.65	1
payout if in state ω_1 at time 1	3	2
payout if in state ω_2 at time 1	1	0.5

With this information we can now construct the two state space securities SS_1 and SS_2, respectively. That is, we can calculate the prices of SS_1 and SS_2 at time 0. To this end (for SS_1) we construct a portfolio that consists of x units of asset A and y units of asset B. The goal is to determine x and y such that the resulting portfolio pays 1 if state ω_1 occurs at time 1 and 0 otherwise. That is, this portfolio exactly replicates the state price security SS_1. Mathematically speaking we need to solve the linear equation $SS_1 = xA + yB$ for SS_1, and a similar linear equation for SS_2. The solution to these two linear equations provides the following table (with the corresponding prices at time 0):

	units of asset A	units of asset B	market price Q
ω_1 state security SS_1	-1	2	0.35
ω_2 state security SS_2	4	-6	0.60

Note that this is similar to the derivation of the Arbitrage Pricing Theory model (see Ingersoll [Ing87], Chapter 7). Basically, we need that asset A and asset B are linearly independent and that the pricing functional Q is linear. Hence, if we have another risky asset $\mathbf{X}$ which pays 2 in state ω_1 and 1 in state ω_2, its price is given by

$$Q[\mathbf{X}] = 2 \cdot 0.35 + 1 \cdot 0.6 = 1.3. \tag{2.31}$$

We now consider the zero coupon bond $\mathbf{Z}^{(1)}$. The zero coupon bond pays in both states ω_1 and ω_2 the amount 1:

$$D_{0,1} = Q\left[\mathbf{Z}^{(1)}\right] = 1 \cdot 0.35 + 1 \cdot 0.6 = 0.95, \tag{2.32}$$

which leads to a risk-free return of $(0.95)^{-1} - 1 = 5.26\%$.

Step 2. Now we construct the deflators. Denote by $Q(\omega_i)$ the market price of the ω_i state space security SS_i at time 0, i.e. $Q(\omega_1) = 0.35$ and $Q(\omega_2) = 0.60$. Moreover, let $X_1(\omega_i)$ denote the payout at time 1 of the risky asset $\mathbf{X} = (0, X_1)$, if we are in state ω_i at time 1. Hence the market price of $\mathbf{X}$ at time 0 is given by (see (2.31))

$$Q[\mathbf{X}] = \sum_{i=1}^{2} Q(\omega_i)\ X_1(\omega_i). \tag{2.33}$$

Note: so far we have not used any probabilities!

Now we assume that we are in state ω_1 at time 1 with probability $p(\omega_1) \in (0,1)$ and in state ω_2 with probability $p(\omega_2) = 1 - p(\omega_1)$. Hence (2.33) can be rewritten as follows

$$\begin{aligned} Q[\mathbf{X}] &= \sum_{i=1}^{2} Q(\omega_i)\ X_1(\omega_i) \\ &= \sum_{i=1}^{2} p(\omega_i)\ \frac{Q(\omega_i)}{p(\omega_i)}\ X_1(\omega_i) \\ &= E\left[\frac{Q}{p}\ X_1\right]. \end{aligned} \tag{2.34}$$

Henceforth, define the random variable

$$\varphi_1 = \frac{Q}{p}, \tag{2.35}$$

which immediately implies the pricing formula

$$Q[\mathbf{X}] = E[\varphi_1\ X_1]. \tag{2.36}$$

For an explicit choice of probabilities $p(\omega_i)$, the deflator φ_1 takes the following values:

	value of deflator φ_1	probability $p(\omega_i)$
state ω_1 at time 1	0.7	0.5
state ω_2 at time 1	1.2	0.5

Hence, alternatively to (2.32) we obtain for the value of the zero coupon bond

$$Q\left[\mathbf{Z}^{(1)}\right] = E[\varphi_1] = \sum_{i=1}^{2} \varphi_1(\omega_i)\ p(\omega_i) = 0.7 \cdot 0.5 + 1.2 \cdot 0.5 = 0.95. \tag{2.37}$$

Note that in our example the deflator φ_1 is not necessarily smaller than 1. With probability 1/2 we will observe that the deflator has a value of 1.2. This may be counter-intuitive from an economic point of view but makes perfect sense in our model world. Henceforth, the model and parameters need to be specified carefully in order to get economically meaningful models.

2.3 Valuation at time $t > 0$

Postulate: Correct prices should eliminate the possibility to play games with cash flows (see also Remark 2.14).

Assume an $\mathbb{F}$-adapted deflator $\boldsymbol{\varphi} \in L^2_{n+1}(P, \mathbb{F})$ is given. We then define the price process for a random vector $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$ as follows: for $t = 0, \ldots, n$

$$Q_t\left[\mathbf{X}\right] = Q\left[\mathbf{X} | \mathcal{F}_t\right] = \frac{1}{\varphi_t}\ E\left[\left.\sum_{k=0}^{n} \varphi_k\ X_k \right| \mathcal{F}_t\right]. \tag{2.38}$$

Note, $\boldsymbol{\varphi} \gg 0$ implies that $Q_t\left[\mathbf{X}\right]$ is well-defined. The right-hand side of (2.38) can be decoupled because the payments X_k (and the deflators φ_k) are $\mathcal{F}_t$-measurable for $k \leq t$.

Terminology.
The mapping $\mathbf{X} \mapsto Q_t[\mathbf{X}]$ assigns a monetary value $Q_t[\mathbf{X}]$ at time t to the cash flow $\mathbf{X}$, i.e. attaches an $\mathcal{F}_t$-measurable price to the cash flow $\mathbf{X}$. Of course this price is stochastic seen from time 0, it depends on $\mathcal{F}_t$. As we see below, this valuation process $(Q_t)_{t=0,\ldots,n}$ is done in a market-consistent way which leads to a risk neutral valuation scheme (see also Lemma 2.8 and Remark 2.14).

First, we note that by our assumption we have $Q[\mathbf{X}] = Q_0[\mathbf{X}]$.

The justification of our price process definition $(Q_t)_{t=0,\ldots,n}$ uses an equilibrium principle or an arbitrage argument. Assume that we pay for cash flow $\mathbf{X}$ at time t the price $Q_t\left[\mathbf{X}\right]$. Hence, we generate a payment cash flow

$$Q_t\left[\mathbf{X}\right]\ \mathbf{Z}^{(t)} = \left(0, \ldots, 0, Q_t\left[\mathbf{X}\right], 0, \ldots, 0\right), \tag{2.39}$$

if we pay the price for $\mathbf{X}$ at time t. From today's point of view this payment stream has value

$$Q_0\left[Q_t\left[\mathbf{X}\right]\ \mathbf{Z}^{(t)}\right], \tag{2.40}$$

since we have only information $\mathcal{F}_0$ at time 0 about the price $Q_t\left[\mathbf{X}\right]$ of $\mathbf{X}$ at time t. Equilibrium requires, that

$$Q_0\left[\mathbf{X}\right] = Q_0\left[Q_t\left[\mathbf{X}\right]\ \mathbf{Z}^{(t)}\right], \tag{2.41}$$

since (based on today's information $\mathcal{F}_0$) the two payment streams should have the same value. That is, we today agree either to buy and pay $\mathbf{X}$ today or to buy and pay $\mathbf{X}$ at time t (at its current price $Q_t[\mathbf{X}]$ at that time). Since we use the same information $\mathcal{F}_0$ for these two contracts and we obtain the same cash flow $\mathbf{X}$ the two contracts should have the same price.

Suppose now that we play the following game: We decide to buy and pay cash flow $\mathbf{X}$ only if an event $F_t \in \mathcal{F}_t$ occurs. Since from today's point of view we do not know whether the event F_t occurs or not, we should have the following price equilibrium, see also (2.41),

$$Q_0[\mathbf{X}\ 1_{F_t}] = Q_0\left[Q_t[\mathbf{X}]\ \mathbf{Z}^{(t)}\ 1_{F_t}\right], \tag{2.42}$$

note, however, that $\mathbf{X}\ 1_{F_t}$ is not $\mathbb{F}$-adapted (to avoid this we could also do an argument similar to (2.51) below). Using deflators, we rewrite (2.42)

$$E\left[\sum_{k=0}^{n} \varphi_k\ X_k\ 1_{F_t}\right] = E\left[\varphi_t\ Q_t[\mathbf{X}]\ 1_{F_t}\right]. \tag{2.43}$$

Since $(\varphi_t\ Q_t[\mathbf{X}])$ is $\mathcal{F}_t$-measurable and equation (2.43) must hold true for all $F_t \in \mathcal{F}_t$, this is exactly the definition of the conditional expectation given the σ-field $\mathcal{F}_t$. Henceforth, (2.43) implies (2.38), P-a.s., and justifies that (2.38) is an economically meaningful definition. A more financial mathematically based argumentation would say that deflated price processes need to be $(P, \mathbb{F})$-martingales in order to have an arbitrage-free pricing model, see Lemma 2.8 and Remark 2.14 below.

We close this section with some remarks on "pure" financial risks. We have defined the traditional discount factors

$$D_{0,m} = Q_0\left[\mathbf{Z}^{(m)}\right] = E[\varphi_m] \tag{2.44}$$

at time 0 for a zero coupon bond with maturity m. For $t < m$, let $D_{t,m}$ stand for the discount factor from time m back to time t, *fixed* at time 0. The terminology forward refers to this fixing at an earlier time point. We must have

$$D_{0,t}\ D_{t,m} = D_{0,m}. \tag{2.45}$$

The left-hand side of (2.45) is the price at time 0 for receiving $D_{t,m}$ at time t, and $D_{t,m}$ is the price for receiving 1 at time m (fixed at time 0 and to be paid at time t). The right-hand side of (2.45) is the price at time 0 for receiving 1 at time m.

Hence we define forward discount factors for $t < m$:

$$D_{t,m} = \frac{D_{0,m}}{D_{0,t}}. \tag{2.46}$$

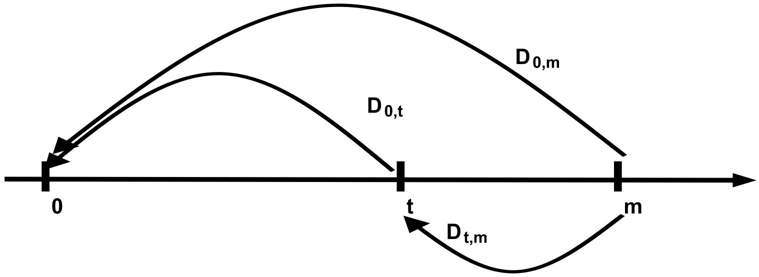

Fig. 2.4. Forward discount factor $D_{t,m}$ for $t < m$

This is the forward price of a zero coupon bond with maturity m fixed at time 0 to be paid at time t ($\mathcal{F}_0$-measurable).

On the other hand, the value/price at time t of a zero coupon bond with maturity m is given by ($\mathcal{F}_t$-measurable)

$$Q_t\left[\mathbf{Z}^{(m)}\right] = \frac{1}{\varphi_t}\, E\left[\varphi_m \middle| \mathcal{F}_t\right] = E\left[\frac{\varphi_m}{\varphi_t}\middle| \mathcal{F}_t\right]. \tag{2.47}$$

This is exactly (2.38) for a single deterministic payment of 1 in m.

Remark. In financial mathematics literature one often uses the notation

$$P(t,m) = Q_t\left[\mathbf{Z}^{(m)}\right] = E\left[\varphi_m/\varphi_t \middle| \mathcal{F}_t\right] = E^*\left[\exp\left\{-\sum_{s=t}^{m-1} r_s\right\}\middle| \mathcal{F}_t\right],$$

where $(r_t)_{t=0,\ldots,n}$ stands for the spot rate process, see also Exercise 2.3, and E^* is the expectation under the risk neutral measure $P^* \sim P$ (see also Exercise 2.6). Note that $D_{0,m} = P(0,m) = Q_0\left[\mathbf{Z}^{(m)}\right]$.

Exercise 2.5.

We revisit the discrete time Vasicek model presented in Exercise 2.3. Calculate for this model the zero coupon bond price $P(t,m)$ at time $t < m$. We claim that this price is given by

$$P(t,m) = Q_t\left[\mathbf{Z}^{(m)}\right] = \exp\left\{a(m-t) - r_t\, b(m-t)\right\}, \tag{2.48}$$

for appropriate functions $a(\cdot)$ and $b(\cdot)$ and $\mathcal{F}_t$-measurable spot rate r_t, see also (2.30).

Give an interpretation to r_t in terms of $P(t,t+1)$.

Remark. The zero coupon bond price representation (2.48) is called an affine term structure, because its logarithm is an affine function of the observed spot rate r_t for all $t = 0,\ldots,m-1$.

□

2.4 The meaning of basic reserves

In the previous section we have considered the valuation of cash flows $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$ at any time $t = 0, \ldots, n$. In the insurance industry however, we are mainly interested in the valuation of the future cash flows $(0, \ldots, 0, X_{t+1}, \ldots, X_n)$ if we are at time t. For these cash flows we need to build reserves in our balance sheet, because they refer to the outstanding (loss) liabilities. This means that we need to predict X_k, $k > t$, and assign market-consistent values to them, based on the information $\mathcal{F}_t$.

Note that from an economic point of view the terminology *reserves* is not quite correct (because reserves refer rather to shareholder value) and one should call the reserves instead *provisions* because they belong to the insured (policyholder).

Postulate: Correct basic reserves should eliminate the possibility to play games with insurance liabilities.

Throughout: assume an $\mathbb{F}$-adapted deflator $\boldsymbol{\varphi} \in L^2_{n+1}(P, \mathbb{F})$ is given.

Assume that an insurance contract is represented by the (stochastic) cash flow $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$. We define for $k \leq n$ the outstanding liabilities at time $k-1$ by

$$\mathbf{X}_{(k)} = (0, \ldots, 0, X_k, \ldots, X_n) \in L^2_{n+1}(P, \mathbb{F}), \tag{2.49}$$

this is the remaining cash flow after time $k-1$. $\mathbf{X}_{(k)}$ represents the amounts for which we have to build reserves at time $k-1$, such that we are able to meet all future payments arising of this contract. Henceforth, the **reserves at time** $t \leq k-1$ for the outstanding liabilities $\mathbf{X}_{(k)}$ are defined as

$$R_t^{(k)} = R\left[\mathbf{X}_{(k)} \middle| \mathcal{F}_t\right] = Q_t[\mathbf{X}_{(k)}] = \frac{1}{\varphi_t}\, E\left[\left.\sum_{s=k}^{n} \varphi_s\, X_s \right| \mathcal{F}_t\right]. \tag{2.50}$$

On the one hand, $R_t^{(k)}$ corresponds to the conditionally expected monetary value of the cash flow $\mathbf{X}_{(k)}$ viewed from time t. On the other hand, $R_t^{(k)}$ is used to predict the monetary value of the random variable $\mathbf{X}_{(k)}$. Therefore, $R_t^{(k)}$ is often called discounted "best-estimate" reserves, see also (2.113) below.

We justify that (2.50) is a reasonable definition for the reserves. We argue for $R_t^{(k)}$ in a similar fashion as in the last section. We want to avoid that we can play games with insurance contracts. In particular, we consider the following game: assume we have two insurance companies A and B that have the following business strategies.

- Company A keeps the contract until the ultimate payment is made.

- Company B decides (at time 0) to sell the run-off of the outstanding liabilities at time $t-1$ at price $R\left[\mathbf{X}_{(t)}\middle|\mathcal{F}_{t-1}\right]$ if an event $F_{t-1} \in \mathcal{F}_{t-1}$ occurs.

This implies that the two strategies generate the following cash flows:

	0	…	$t-1$	t	…	n
$\mathbf{X}^{(A)} =$	$(X_0,$	$\ldots,$	$X_{t-1},$	$X_t,$	$\ldots,$	$X_n)$
$\mathbf{X}^{(B)} =$	$(X_0,$	$\ldots,$	$X_{t-1} + R\left[\mathbf{X}_{(t)}\middle|\mathcal{F}_{t-1}\right] 1_{F_{t-1}},$	$X_t 1_{F^c_{t-1}},$	$\ldots,$	$X_n 1_{F^c_{t-1}})$

Hence, the price difference at time 0 of these two strategies is given by

$$Q_0\left[\mathbf{X}^{(A)} - \mathbf{X}^{(B)}\right] = E\left[-\varphi_{t-1} R\left[\mathbf{X}_{(t)}\middle|\mathcal{F}_{t-1}\right] 1_{F_{t-1}}\right] + E\left[\sum_{s=t}^{n} \varphi_s X_s 1_{F_{t-1}}\right]. \tag{2.51}$$

As in (2.42), we have that the two strategies based on the information $\mathcal{F}_0$ should have the same initial value (because they are based on the same information), i.e. $Q_0\left[\mathbf{X}^{(A)} - \mathbf{X}^{(B)}\right] = 0$. This implies that for all events $F_{t-1} \in \mathcal{F}_{t-1}$ we need to have the equality

$$E\left[\varphi_{t-1} R\left[\mathbf{X}_{(t)}\middle|\mathcal{F}_{t-1}\right] 1_{F_{t-1}}\right] = E\left[\sum_{s=t}^{n} \varphi_s X_s 1_{F_{t-1}}\right]. \tag{2.52}$$

Hence, using the definition of conditional expectations, this justifies the following definition of the reserves:

$$R_{t-1}^{(t)} = R\left[\mathbf{X}_{(t)}\middle|\mathcal{F}_{t-1}\right] = \frac{1}{\varphi_{t-1}}\; E\left[\left.\sum_{s=t}^{n} \varphi_s\; X_s \right| \mathcal{F}_{t-1}\right] = Q_{t-1}\left[\mathbf{X}_{(t)}\right], \tag{2.53}$$

which justifies (2.50) for $k = t$. The case $k > t$ is then easily obtained by the fact that we should have the martingale property for deflated price processes given by Lemma 2.8 (see below) which says

$$\begin{aligned} \varphi_{t-1} R_{t-1}^{(k)} &= \varphi_{t-1} Q_{t-1}\left[\mathbf{X}_{(k)}\right] \\ &= E\left[\varphi_{k-1} Q_{k-1}\left[\mathbf{X}_{(k)}\right]\middle|\mathcal{F}_{t-1}\right] = E\left[\varphi_{k-1} R_{k-1}^{(k)}\middle|\mathcal{F}_{t-1}\right]. \end{aligned} \tag{2.54}$$

Observe that we have the following self-financing property:

Corollary 2.7 (Self-financing property) *The following recursion holds*

$$E\left[\varphi_t\; \left(R_t^{(t+1)} + X_t\right)\middle|\mathcal{F}_{t-1}\right] = \varphi_{t-1}\; R_{t-1}^{(t)}. \tag{2.55}$$

Remark.

- The classical actuarial theory with $\varphi_t = (1+i)^{-t}$ for some constant interest rate i (see (2.29)) forms a consistent theory but the deflators are not market-consistent, because they are often far from observed economic behaviours.
- Corollary 2.7 basically says that if we want to avoid arbitrage opportunities of reserves then we need to define them as conditional expectations of the random cash flows.

Proof of Corollary 2.7. We have the following identity (using the $\mathcal{F}_t$-measurability of X_t and the tower property of conditional expectations, see Williams [Wi91], Chapter 9)

$$E\left[\varphi_t \left(R_t^{(t+1)} + X_t\right)\Big| \mathcal{F}_{t-1}\right] = E\left[\sum_{k=t}^{n} \varphi_k X_k \Bigg| \mathcal{F}_{t-1}\right] = \varphi_{t-1} R_{t-1}^{(t)}. \qquad (2.56)$$

This completes the proof of the corollary.

□

2.5 Equivalent martingale measures

Assume a fixed $\mathbb{F}$-adapted deflator $\boldsymbol{\varphi} \in L^2_{n+1}(P, \mathbb{F})$ is given.

The price process defined in (2.38) gives in a natural way a martingale (that is, it satisfies the efficient market hypothesis in its strong form, see Remark 2.14 below):

Lemma 2.8 *The deflated price process (2.38)*

$$(\varphi_t Q_t[\mathbf{X}])_{t=0,\ldots,n} \qquad \textit{forms an } \mathbb{F}\textit{-martingale under } P. \qquad (2.57)$$

Proof. Since $\mathcal{F}_t \subset \mathcal{F}_{t+1}$ we have with the tower property of conditional expectations, see Williams [Wi91],

$$\begin{aligned} E\left[\varphi_{t+1} Q_{t+1}[\mathbf{X}] \middle| \mathcal{F}_t\right] &= E\left[E\left[\sum_{k=0}^{n} \varphi_k X_k \Bigg| \mathcal{F}_{t+1}\right] \Bigg| \mathcal{F}_t\right] \qquad (2.58) \\ &= E\left[\sum_{k=0}^{n} \varphi_k X_k \Bigg| \mathcal{F}_t\right] = \varphi_t Q_t[\mathbf{X}]. \end{aligned}$$

This finishes the proof of the lemma.

□

Remarks on deflating and discounting.

- From the martingale property we immediately have

$$Q_t[\mathbf{X}] = \frac{1}{\varphi_t} E\left[\varphi_{t+1} Q_{t+1}[\mathbf{X}] \middle| \mathcal{F}_t\right] = E\left[\frac{\varphi_{t+1}}{\varphi_t} Q_{t+1}[\mathbf{X}] \middle| \mathcal{F}_t\right]. \tag{2.59}$$

This implies for the span-deflated price

$$Q_t[\mathbf{X}] = E\left[Y_{t+1} Q_{t+1}[\mathbf{X}] \middle| \mathcal{F}_t\right], \tag{2.60}$$

with span-deflator Y_{t+1} defined in (2.26). The (stochastic) span-deflator Y_{t+1} is $\mathcal{F}_{t+1}$-measurable, i.e. it is known only at the end of the time period $(t, t+1]$, and not at the beginning of that time period.

- We define the span-discount known at the beginning of the time period $(t, t+1]$, i.e. which is observable on the market at time t:

$$D(\mathcal{F}_t) = E\left[Y_{t+1} | \mathcal{F}_t\right] = E\left[\frac{\varphi_{t+1}}{\varphi_t} \middle| \mathcal{F}_t\right]. \tag{2.61}$$

It is often convenient to rewrite (2.60) using the span-discount $D(\mathcal{F}_t)$ instead of the span-deflator Y_{t+1}. The reason is that the span-discounts are eventually observable whereas span-deflators are always "hidden variables". The basic idea is to change the probability measure P to P^* such that we can change from span-deflators Y_{t+1} to observable span-discounts $D(\mathcal{F}_t)$ at time t.

- If the time interval $(t, t+1]$ is one year then $D(\mathcal{F}_t)$ is exactly the price of the zero coupon bond with maturity 1 year at time t, i.e., on this yearly grid this corresponds to the one-year risk-free investment at time t. Henceforth, on a yearly grid $D(\mathcal{F}_t)^{-1}$ describes the development of the value of the bank account. That is, if we invest 1 into the bank account at time 0, then the value of this investment at time $t \geq 1$ is given by (yearly role over)

$$B_t = \prod_{s=0}^{t-1} D(\mathcal{F}_s)^{-1} = \prod_{s=0}^{t-1} E\left[Y_{s+1} | \mathcal{F}_s\right]^{-1} = \exp\left\{\sum_{s=0}^{t-1} r_s\right\}, \tag{2.62}$$

where we have defined

$$r_t = -\log E\left[Y_{t+1} | \mathcal{F}_t\right]. \tag{2.63}$$

We remark that $(r_t)_{t=0,\ldots,n-1}$ is the spot rate process in discrete time and we have already met it in Exercise 2.3.

- The change of probability measure mentioned above will then correspond to a change of discount factors from the deflator $\boldsymbol{\varphi}$ to the bank account numeraire $(B_t^{-1})_{t=0,\ldots,n}$.

We define the process $\boldsymbol{\xi} = (\xi_s)_{s=0,\ldots,n}$ by $\xi_0 = 1$ and for $s = 1, \ldots, n$

$$\xi_s = \prod_{t=0}^{s-1} \frac{Y_{t+1}}{D(\mathcal{F}_t)} = \varphi_s \, B_s. \tag{2.64}$$

Corollary 2.9 *We have* $\boldsymbol{\xi} \gg 0$ *is a normalized density process w.r.t.* P.

Proof. Positivity is immediately clear. Moreover, $\boldsymbol{\xi}$ is a P-martingale (which immediately follows from Lemma 2.8 because $(B_t)_{t=0,\ldots,n}$ is the price process of the bank account) with normalization $E[\xi_n] = 1$. This proves the claim.

□

For $A \in \mathcal{F}_n$ we define

$$P^*[A] = \int_A \xi_n \, dP = E[\xi_n 1_A]. \tag{2.65}$$

Lemma 2.10 *We have the following statements:*

(1) P^* *is a probability measure on* $(\Omega, \mathcal{F}_n)$ *equivalent to* P.
(2) We have

$$\left.\frac{dP^*}{dP}\right|_{\mathcal{F}_s} = \xi_s \qquad P\text{-a.s.} \tag{2.66}$$

(3) Moreover, for $s \le t$ *and* $A \in \mathcal{F}_t$

$$P^*[A|\mathcal{F}_s] = \frac{1}{\xi_s} E[\xi_t \, 1_A|\mathcal{F}_s] \qquad P\text{-a.s.} \tag{2.67}$$

Proof. The proof of statement (1) follows from Corollary 2.9. The normalization implies that $P^*[\Omega] = E[\xi_n] = 1$, which says that P^* is a probability measure on $(\Omega, \mathcal{F}_n)$. Moreover, $\xi_n > 0$ P-a.s. implies that $P^* \sim P$, i.e. they are equivalent measures.

Next we prove statement (2). Note that for any $\mathcal{F}_s$-measurable set C we have

$$P^*[C] = E[\xi_n 1_C] = E[E[\xi_n|\mathcal{F}_s] 1_C] = E[\xi_s 1_C], \tag{2.68}$$

using the martingale property of $\boldsymbol{\xi}$ in the last step. Therefore, ξ_s is the density on $\mathcal{F}_s$.

Finally we prove (3). Note that we have for any $\mathcal{F}_s$-measurable set C

$$\begin{aligned}
E^*[1_C \, 1_A] &= E[1_C \, \xi_n \, 1_A] \\
&= E[1_C \, E[\xi_n \, 1_A|\mathcal{F}_s]] \\
&= E\left[\xi_s \left(1_C \, \frac{1}{\xi_s} \, E[\xi_n \, 1_A|\mathcal{F}_s]\right)\right] \\
&= E^*\left[1_C \, \frac{1}{\xi_s} \, E[\xi_n \, 1_A|\mathcal{F}_s]\right] \\
&= E^*\left[1_C \, \frac{1}{\xi_s} \, E[1_A \, E[\xi_n|\mathcal{F}_t]|\mathcal{F}_s]\right] \\
&= E^*\left[1_C \, \frac{1}{\xi_s} \, E[\xi_t \, 1_A|\mathcal{F}_s]\right] \\
&= E^*[1_C \, P^*[A|\mathcal{F}_s]],
\end{aligned} \tag{2.69}$$

by the definition of conditional expectations w.r.t. P^*. This completes the proof of the lemma.

□

Item (3) of Lemma 2.10 immediately implies the next corollary:

Corollary 2.11 *For $s < t$ we have*

$$E^*\left[Q_t[\mathbf{X}]\right|\mathcal{F}_s] = \frac{1}{\xi_s}\ E\left[\xi_t\ Q_t[\mathbf{X}]\right|\mathcal{F}_s]. \tag{2.70}$$

If we apply (2.60) and Corollary 2.11 to $s = t-1$ we obtain

$$\begin{aligned} E^*\left[Q_t[\mathbf{X}]\right|\mathcal{F}_{t-1}] &= \frac{1}{\xi_{t-1}}\ E\left[\xi_t\ Q_t[\mathbf{X}]\right|\mathcal{F}_{t-1}] \\ &= \frac{1}{\xi_{t-1}}\ E\left[\xi_{t-1}\ \frac{Y_t}{D(\mathcal{F}_{t-1})}\ Q_t[\mathbf{X}]\right|\mathcal{F}_{t-1}\right] \\ &= \frac{1}{D(\mathcal{F}_{t-1})}\ E\left[Y_t\ Q_t[\mathbf{X}]\right|\mathcal{F}_{t-1}] \\ &= \frac{1}{D(\mathcal{F}_{t-1})} Q_{t-1}[\mathbf{X}], \end{aligned} \tag{2.71}$$

or

$$D(\mathcal{F}_{t-1})\ E^*\left[Q_t[\mathbf{X}]\right|\mathcal{F}_{t-1}] = Q_{t-1}[\mathbf{X}]. \tag{2.72}$$

Hence for the bank account numeraire

$$B_t^{-1} = \prod_{s=0}^{t-1} D(\mathcal{F}_s) \tag{2.73}$$

we find

$$E^*\left[B_t^{-1}\ Q_t[\mathbf{X}]\right|\mathcal{F}_{t-1}] = B_t^{-1}\ E^*\left[Q_t[\mathbf{X}]\right|\mathcal{F}_{t-1}] = B_{t-1}^{-1} Q_{t-1}[\mathbf{X}]. \tag{2.74}$$

Note that the discount factor B_t^{-1} is now measurable w.r.t. $\mathcal{F}_{t-1}$. Hence, in contrast to φ_t (see (2.57)) we have now an $\mathcal{F}_{t-1}$-measurable discount factor (bank account numeraire) which describes the growth of the bank account. This gives the following corollary (compare to Lemma 2.8):

Corollary 2.12 *Under the probability measure P^* the process*

$$\left(B_t^{-1} Q_t[\mathbf{X}]\right)_{t=0,\ldots,n} \tag{2.75}$$

is an $\mathbb{F}$-martingale w.r.t. P^*.

Remark 2.13 (Real world and risk neutral measure)

- Henceforth, the price process is now a martingale for discounting with the bank account numeraire B_t^{-1} under the equivalent measure $P^* \sim P$. Therefore, this measure is often called equivalent martingale measure or risk neutral measure.

- As a consequence we can either work under the **real world probability measure** P (*physical measure* or *objective measure*) where the price processes need to be deflated with $\boldsymbol{\varphi}$. Alternatively, we can also work under the **equivalent martingale measure** P^* (*risk neutral measure*). In that case the price processes need to be discounted with the bank account numeraire B_t^{-1}.
- If we work with financial instruments only, then it is often easier to work under P^*. If we additionally have insurance products then usually one works under P. Therefore, actuaries need to well-understand the connection between these two measures.
- For the equivalent martingale measure P^* we choose the bank account numeraire B_t^{-1} for discounting. In general, if $(A_t)_{t=0,\ldots,n}$ is any strictly positive, normalized price process, then we could choose A_t^{-1} as a numeraire and find the appropriate equivalent measure $P^A \sim P$ such that the price processes $(A_t^{-1} Q_t [\mathbf{X}])_{t=0,\ldots,n}$ are $\mathbb{F}$-martingales w.r.t. P^A. For more on this subject we refer to Brigo-Mercurio [BM06], Sections 2.2-2.3.

In the one-period model we obtain

$$Q_0 [\mathbf{X}] = D(\mathcal{F}_0)\ E^* [Q_1 [\mathbf{X}]] = E [Y_1\ Q_1 [\mathbf{X}]] . \tag{2.76}$$

Exercise 2.6.

Prove that the price of the zero coupon bond with maturity m at time $t < m$ is given by

$$P(t, m) = Q_t \left[\mathbf{Z}^{(m)}\right] = E^* \left[\exp\left\{ - \sum_{s=t}^{m-1} r_s \right\} \middle| \mathcal{F}_t \right], \tag{2.77}$$

where $(r_t)_{t=0,\ldots,n}$ was defined in (2.63).

□

Remark 2.14 (Fundamental Theorem of Asset Pricing)

- The **efficient market hypothesis** in its strong form assumes that the deflated price processes

$$\widetilde{Q}_t = \varphi_t\ Q_t [\mathbf{X}] , \qquad t = 0, \ldots , n, \tag{2.78}$$

  form $\mathbb{F}$-martingales under P. This implies for the expected net gains $(t > s)$

$$E \left[\widetilde{Q}_t - \widetilde{Q}_s \middle| \mathcal{F}_s \right] = 0, \tag{2.79}$$

  which means that there exists no arbitrage strategy defined the “right way” (which roots in the idea of risk neutral valuation).

- The **efficient market hypothesis** in its weak form assumes that "there is no free lunch", i.e. there does not exist any (appropriately defined) self-financing trading strategy with positive expected gains and without any downside risk. In a finite discrete time model, this is equivalent to the existence of an equivalent martingale measure for the deflated price processes (which rules out arbitrage) (see e.g. Theorem 2.6 in Lamberton-Lapeyre [LL91]), the proof for a finite probability space is essentially an exercise in linear algebra. In a more general setting the characterization is more delicate (see Delbaen-Schachermayer [DS94] and Föllmer-Schied [FS04]).
  That is, the existence of an equivalent martingale measure rules out appropriately defined arbitrage (which is the easier direction). The opposite that no-arbitrage defined the right way implies the existence of an equivalent martingale measure is rather delicate and was proved by Delbaen-Schachermayer [DS94] in its most general form.
- In complete markets, the equivalent martingale measure is unique, which implies that we have a perfect replication of contingent claims and the calculation of the prices is straight forward (see e.g. Theorem 3.4 in Lamberton-Lapeyre [LL91]).
- In incomplete markets, where we have more than one equivalent martingale measure, we need an economic model to decide which measure to use (e.g. utility theory, super-hedge or efficient hedging (utility based models accepting some risks), see also Föllmer-Schied [FS04] or Malamud et al. [MTW08]).

Toy example (revisited).
In this subsection we revisit the toy example from Subsection 2.2.3. We transform our probability measure according to Lemma 2.10 (here we work in a one-period model with $Q_0 = Q$):

$$p^*(\omega_i) = \xi_1(\omega_i)\ p(\omega_i) = \frac{\varphi_1(\omega_i)}{E\left[\varphi_1\right]}\ p(\omega_i) = \frac{Q(\omega_i)}{Q\left[\mathbf{Z}^{(1)}\right]}. \tag{2.80}$$

Hence, from (2.33) and (2.36)

$$Q\left[\mathbf{X}\right] = E\left[\varphi_1\ X_1\right] = \sum_{i=1}^{2} Q(\omega_i)\ X_1(\omega_i), \tag{2.81}$$

$$Q\left[\mathbf{X}\right] = B_1^{-1}\ E^*\left[X_1\right] = \sum_{i=1}^{2} Q(\omega_i)\ X_1(\omega_i), \tag{2.82}$$

with (see (2.73))

$$B_1^{-1} = E\left[\varphi_1\right] = Q\left[\mathbf{Z}^{(1)}\right], \tag{2.83}$$

which is deterministic at time 0. Hence under P^* we have

$$Q[\mathbf{X}] = B_1^{-1} \; E^*[X_1] = Q\left[\mathbf{Z}^{(1)}\right] E^*[X_1]. \tag{2.84}$$

This leads to the following table with $p^*(\omega_1) = 0.368$:

	$\mathbf{Z}^{(1)}$	asset A	asset B
market price Q_0	0.95	1.65	1.00
payout state ω_1	1	3	2
payout state ω_2	1	1	0.5
P^* expected payout	1	1.737	1.053
P^* expected return	5.26%	5.26%	5.26%

which is the martingale property of the discounted cash flow $Q\left[\mathbf{Z}^{(1)}\right] X_1$ w.r.t. P^*.

□

Exercise 2.7.

We revisit the discrete time Vasicek model given in Exercise 2.3. The spot rate dynamics $(r_t)_{t=0,\ldots,n}$ was given by $r_0 > 0$ (fixed) and for $t \geq 1$

$$r_t = b + \beta r_{t-1} + \rho \varepsilon_t, \tag{2.85}$$

for given $b, \beta, \rho > 0$, and $(\varepsilon_t)_{t=0,\ldots,n}$ is $\mathbb{F}$-adapted with ε_t independent of $\mathcal{F}_{t-1}$ for all $t = 1, \ldots, n$ and standard Gaussian distributed under the *real world probability measure* P.

The deflator $\boldsymbol{\varphi}$ was then defined by

$$\varphi_t = \exp\left\{ -\sum_{k=1}^{t} \left[r_{k-1} + \frac{\lambda^2}{2} r_{k-1}^2 \right] - \sum_{k=1}^{t} \lambda r_{k-1} \varepsilon_k \right\}, \tag{2.86}$$

for $\lambda \in \mathbb{R}$.

- Calculate the span-discount $D(\mathcal{F}_t)$ from the span-deflator

$$Y_{t+1} = \frac{\varphi_{t+1}}{\varphi_t} = \exp\left\{ -\left[r_t + \frac{\lambda^2}{2} r_t^2 \right] - \lambda r_t \varepsilon_{t+1} \right\} \tag{2.87}$$

  and show that the model is well-defined.
- Prove that the density process $(\xi_t)_{t=0,\ldots,n}$ is given by

$$\xi_t = \exp\left\{ -\sum_{k=1}^{t} \frac{\lambda^2}{2} r_{k-1}^2 - \sum_{k=1}^{t} \lambda r_{k-1} \varepsilon_k \right\}, \tag{2.88}$$

  where an empty sum is set equal to zero.

- Prove that
$$\varepsilon_t^* = \varepsilon_t + \lambda r_{t-1} \tag{2.89}$$
has, conditionally given $\mathcal{F}_{t-1}$, a standard Gaussian distribution under the *equivalent martingale measure* $P^* \sim P$, given by the density in Lemma 2.10.
Hint: use the moment generating function and Lemma 2.10.
- Prove that (2.89) implies for the spot rate process $(r_t)_{t=0,\ldots,n}$: $r_0 > 0$ (fixed) and for $t \geq 1$
$$r_t = b + (\beta - \lambda\rho) r_{t-1} + \rho \varepsilon_t^*, \tag{2.90}$$
where $(\varepsilon_t^*)_{t=0,\ldots,n}$ is $\mathbb{F}$-adapted with ε_t^* independent of $\mathcal{F}_{t-1}$ for all $t = 1, \ldots, n$ and standard Gaussian distributed under the equivalent martingale measure P^*.
- Calculate the zero coupon bond prices $t < m$ (see also Exercise 2.5)
$$P(t,m) = E^* \left[\exp\left\{ - \sum_{s=t}^{m-1} r_s \right\} \middle| \mathcal{F}_t \right] = \exp\left\{ a(m-t) - r_t \, b(m-t) \right\}. \tag{2.91}$$

□

Remark on Exercise 2.7. In (2.88) we calculate the density process $(\xi_t)_{t=0,\ldots,n}$ for the discrete time Vasicek model. It depends on the parameter $\lambda \in \mathbb{R}$. We see that if $\lambda = 0$, then the density process is identical equal to 1, and henceforth $P^* = P$. Therefore, λ models the difference between the real world probability measure P and the equivalent martingale measure P^* which is in economic theory explained through the market risk aversion. Therefore, λ is often called **market price of risk** parameter and explains the aggregate market risk aversion (in our Vasicek model). In general, a higher risk aversion explains lower prices because the more risk averse some is, the less he is willing to accept risky positions.

Conclusions:

- We have found three different ways to value cash flows $\mathbf{X}$:
  1. via a positive linear functional Q,
  2. via deflators $\boldsymbol{\varphi}$ under P,
  3. via the bank account numeraire B_t^{-1} under risk neutral measures P^*.
- The advantage of using risk neutral measures is that the discount factor is a priori known, which means that we have state independent discount factors. The main disadvantage of using the risk neutral measure is that the concept is not straight forward (especially parameter estimation and modelling of insurance liabilities), and that the risk neutral measure changes under currency changes.

- By contrast, deflators are calculated using the real world probability measure (expressing market risk aversion). Moreover, as shown below, they clearly describe the dependence structures (also between deflator and cash flow). From a practical point of view, deflators allow for the modelling of embedded (financial) options and guarantees in insurance policies, and are therefore preferred especially by actuaries that value life insurance products.

2.6 Insurance technical and financial variables

2.6.1 Choice of numeraire

Choose a cash flow $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$. For practical purposes in insurance applications it makes sense to factorize the payments X_k into an appropriate financial basis $\mathcal{U}_k$, $k = 0, \ldots, n$, and the number of units Λ_k of this basis. Assume that we can split the payments X_k as follows

$$X_k = \Lambda_k \, U_k^{(k)}, \quad k = 0, \ldots, n, \tag{2.92}$$

where the variable $U_t^{(k)}$ denotes the value/price of one unit of the financial instrument $\mathcal{U}_k$ at time $t = 0, \ldots, n$, and

$$\Lambda_k = \frac{X_k}{U_k^{(k)}}, \quad k = 0, \ldots, n, \tag{2.93}$$

gives the number of units that we need to hold (insurance technical variable). This means that we measure insurance liabilities in units $\mathcal{U}_k$ which have price/value $U_k^{(k)}$ at time k and insurance technical variable Λ_k.

We denote the price processes of the financial instruments $\mathcal{U}_k$ by

$$U_0^{(k)}, U_1^{(k)}, \ldots, U_k^{(k)}, U_{k+1}^{(k)}, \ldots, U_n^{(k)}. \tag{2.94}$$

Assume that the price process $(U_t^{(k)})_{t=0,\ldots,k}$ is strictly positive, P-a.s., then $(U_t^{(k)})_{t=0,\ldots,k}$ is called **numeraire** in which we study the liability X_k (see also Remark 2.13), that is, every payment X_k is studied with its appropriate numeraire.

Examples of units/numeraires.

- Currencies like CHF, USD, EURO
- Indexed CHF (inflation index, salary index, claims inflation index, medical expenses index, etc.)
- stock index, real estates, etc.
- strictly positive asset portfolio

Examples of insurance technical events.

- death benefit, annuity payments, disability benefit
- car accident compensation, fire claim
- medical expenses, workmen's compensation

We would like to factorize the filtered probability space $(\Omega, \mathcal{F}_n, P, \mathbb{F})$ into a product space such that we get an independent decoupling:

$$\mathcal{T} = (\mathcal{T}_t)_{t=0,\ldots,n} \quad \sigma\text{-filtration for the insurance technical events,} \tag{2.95}$$

$$\mathcal{G} = (\mathcal{G}_t)_{t=0,\ldots,n} \quad \sigma\text{-filtration for the financial events,} \tag{2.96}$$

with for all $t = 0, \ldots, n$

$$\mathcal{F}_t = \sigma\left(\mathcal{T}_t, \mathcal{G}_t\right) = \text{smallest } \sigma\text{-field containing all sets of } \mathcal{T}_t \text{ and } \mathcal{G}_t. \tag{2.97}$$

We assume that under P the two σ-filtrations $\mathcal{T}$ and $\mathcal{G}$ are independent, i.e. $\mathbb{F}$ can be decoupled into a product of independent σ-fields, one covering insurance technical risks $\mathcal{T}$ and one covering financial risks $\mathcal{G}$. That is, we obtain a product probability space with product measure

$$dP = dP_{\mathcal{T}} \times dP_{\mathcal{G}}, \tag{2.98}$$

with $P_{\mathcal{T}}$ describing insurance technical risks $\mathbf{\Lambda} = (\Lambda_0, \ldots, \Lambda_n)$ which will be $\mathcal{T}$-adapted and with $P_{\mathcal{G}}$ describing financial risks $(U_t^{(k)})_{t=0,\ldots,n}$ which will be $\mathcal{G}$-adapted. This decoupling is crucial in the sequel of this manuscript and explained in the next assumption.

Assumption 2.15 *We assume that* $\mathbf{\Lambda} = (\Lambda_0, \ldots, \Lambda_n) \in L^2_{n+1}(P_{\mathcal{T}}, \mathcal{T})$ *and that* $(U_t^{(k)})_{t=0,\ldots,n} \in L^2_{n+1}(P_{\mathcal{G}}, \mathcal{G})$ *for all* $k = 0, \ldots, n$. *Moreover, we assume for the given deflator* $\boldsymbol{\varphi} \in L^2_{n+1}(P, \mathbb{F})$ *that it factorizes* $\varphi_k = \varphi_k^{(\mathcal{T})} \varphi_k^{(\mathcal{G})}$ *such that* $\boldsymbol{\varphi}^{(\mathcal{T})}$ *is* $\mathcal{T}$*-adapted and* $\boldsymbol{\varphi}^{(\mathcal{G})}$ *is* $\mathcal{G}$*-adapted.*

The valuation of the cash flow $\mathbf{X} = (\Lambda_0\ U_0^{(0)}, \ldots, \Lambda_n\ U_n^{(n)}) \in L^2_{n+1}(P, \mathbb{F})$ is then under Assumption 2.15 given by

$$\begin{aligned} \varphi_t\ Q_t\left[\mathbf{X}\right] &= E\left[\left.\sum_{k=0}^n \varphi_k\ X_k \right| \mathcal{F}_t\right] \\ &= E\left[\left.\sum_{k=0}^n \varphi_k^{(\mathcal{T})}\ \Lambda_k\ \varphi_k^{(\mathcal{G})}\ U_k^{(k)} \right| \mathcal{T}_t, \mathcal{G}_t\right] \\ &= \sum_{k=0}^n E_{\mathcal{T}}\left[\left.\varphi_k^{(\mathcal{T})} \Lambda_k \right| \mathcal{T}_t\right]\ E_{\mathcal{G}}\left[\left.\varphi_k^{(\mathcal{G})} U_k^{(k)} \right| \mathcal{G}_t\right], \end{aligned} \tag{2.99}$$

where $E_{\mathcal{T}}$ is the expectation w.r.t. $P_{\mathcal{T}}$ and $E_{\mathcal{G}}$ is the expectation w.r.t. $P_{\mathcal{G}}$. In the sequel we drop the subscripts $\mathcal{T}$ and $\mathcal{G}$ if it does not cause any confusion. Note that the conditional expectations can be dropped for $k \leq t$.

Remarks.

- The expression $E_{\mathcal{T}}\left[\varphi_k^{(\mathcal{T})}\Lambda_k\middle|\mathcal{T}_t\right]$ describes the price of the insurance cover in units of currency. $\varphi_k^{(\mathcal{T})}$ defines the *loading* (**probability distortion**) of the insurance technical price. This is further outlined in Subsection 2.6.2.
- The expression $E_{\mathcal{G}}\left[\varphi_k^{(\mathcal{G})}U_k^{(k)}\middle|\mathcal{G}_t\right]$ relates to the price for one unit $\mathcal{U}_k$ at time t, see also Subsection 2.6.2 on probability distortion. $\varphi_k^{(\mathcal{G})}$ should be obtained from financial market data. For example, we can use the Vasicek model, proposed in Exercise 2.3, and fit the model to financial market parameters, see Wüthrich-Bühlmann [WB08].
- We have separated the pricing problem into two independent pricing problems, one for pricing insurance cover in units and one for pricing units. This split looks very natural, but in practice one needs to be careful with its applications. Especially in non-life insurance, it is very difficult to find such an orthogonal split, since the severities of the claims often depend on the financial market and the split is non-trivial. For example, if we consider workmen's compensation (which pays the salary when someone is injured or sick), it is very difficult to describe the dependence structure between 1) salary height, 2) length of sickness (which may have mental cause), 3) state of the job market, 4) state of the financial market 5) political environment.
- The financial economy including insurance products could also be defined in other ways that would allow for similar splits. For an example we refer to Malamud et al. [MTW08]. There one starts with a complete financial market described by the financial σ-field. Then one introduces insurance products that enlarge the underlying σ-field. This enlargement in general makes the market incomplete (but still arbitrage-free) and adds idiosyncratic risks to the economic model. Finally, one defines the "hedgeable" σ-field that exactly describes the part of the insurance claims that can be described via financial market movements. The remaining parts are then the insurance technical risks.

2.6.2 Probability distortion

In this section we discuss the factorization of the deflator $\varphi_k = \varphi_k^{(\mathcal{T})}\varphi_k^{(\mathcal{G})}$ from Assumption 2.15. The choice of the probability distortion $\boldsymbol{\varphi}^{(\mathcal{T})}$ needs some care in order to obtain a reasonable model.

(1) Firstly, we observe that $\boldsymbol{\varphi}^{(\mathcal{T})} \gg 0$, which follows from $\boldsymbol{\varphi} \gg 0$. Moreover, $\boldsymbol{\varphi}^{(\mathcal{T})} \in L^2_{n+1}(P_{\mathcal{T}}, \mathcal{T})$, which follows from $\boldsymbol{\varphi} \in L^2_{n+1}(P, \mathbb{F})$ and the independence and $\mathcal{T}$-adaptedness in Assumption 2.15.

(2) Secondly, to avoid ambiguity, we set for all $t = 0, \ldots, n$

$$E\left[\varphi_t^{(\mathcal{T})}\right] = 1. \tag{2.100}$$

Otherwise, the decoupling into a product $\varphi_t = \varphi_t^{(\mathcal{T})} \varphi_t^{(\mathcal{G})}$ is not unique, which can easily be seen by multiplying and dividing both terms by the same positive constant.

(3) Thirdly, we assume that the sequence $(\varphi_t^{(\mathcal{T})})_{t=0,\dots,n}$ is a $\mathcal{T}$-martingale under P, i.e.

$$E\left[\varphi_{t+1}^{(\mathcal{T})} \middle| \mathcal{T}_t\right] = \varphi_t^{(\mathcal{T})}. \tag{2.101}$$

Of course, the normalization (2.100) is then an easy consequence from the requirement

$$E\left[\varphi_n^{(\mathcal{T})}\right] = 1. \tag{2.102}$$

Under Assumption 2.15 and assuming (1)-(3) for the probability distortion $(\varphi_t^{(\mathcal{T})})_{t=0,\dots,n}$ we see that

$$(\varphi_t^{(\mathcal{T})})_{t=0,\dots,n} \text{ is a density process w.r.t. } \mathcal{T} \text{ and } P_{\mathcal{T}}, \tag{2.103}$$

see also (2.104). This allows for the definition of an equivalent probability measure $P^*_{\mathcal{T}} \sim P_{\mathcal{T}}$ via the density

$$\left.\frac{dP^*_{\mathcal{T}}}{dP_{\mathcal{T}}}\right|_{\mathcal{T}_n} = \varphi_n^{(\mathcal{T})}. \tag{2.104}$$

Moreover, we define the *price process for the insurance technical variable* Λ_k as follows: for $t \le k$

$$\Lambda_{t,k} = \frac{1}{\varphi_t^{(\mathcal{T})}} \; E\left[\varphi_k^{(\mathcal{T})} \Lambda_k \middle| \mathcal{T}_t\right]. \tag{2.105}$$

Lemma 2.16 *Assume Assumption 2.15 and (2.103) hold true. The probability distorted process*

$$\left(\varphi_t^{(\mathcal{T})} \Lambda_{t,k}\right)_{t=0,\dots,k} \quad \textit{forms a } \mathcal{T}\textit{-martingale under } P_{\mathcal{T}}. \tag{2.106}$$

The process

$$(\Lambda_{t,k})_{t=0,\dots,k} \quad \textit{forms a } \mathcal{T}\textit{-martingale under } P^*_{\mathcal{T}}. \tag{2.107}$$

Proof Lemma 2.16. The first claim follows similarly to Lemma 2.8 and uses the tower property of conditional expectations, see Williams [Wi91]. The second claim follows similarly to Corollary 2.12 and equality (2.71). Note that here the numeraire is equal to 1 (due to our choice of the density process) which proves the claim.

□

An immediate consequence of Lemma 2.16 is the following corollary:

Corollary 2.17 *Under the assumptions of Lemma 2.16 we have*

$$\Lambda_{t,k} = \frac{1}{\varphi_t^{(\mathcal{T})}}\ E\left[\varphi_k^{(\mathcal{T})}\Lambda_k \middle| \mathcal{T}_t\right] = E^*_{\mathcal{T}}\left[\Lambda_k \middle| \mathcal{T}_t\right]. \tag{2.108}$$

This has further consequences:

Theorem 2.18 *Under the assumptions of Lemma 2.16 and (2.57) we obtain that the price process $(U_t^{(k)})_{t=0,\dots,k}$ of the financial instrument $\mathcal{U}_k$ satisfies for $t<k$*

$$U_t^{(k)} = \frac{1}{\varphi_t^{(\mathcal{G})}}\ E\left[\varphi_{t+1}^{(\mathcal{G})} U_{t+1}^{(k)} \middle| \mathcal{G}_t\right]. \tag{2.109}$$

Proof of Theorem 2.18. We define the cash flow $\mathbf{X} = U_k^{(k)}\mathbf{Z}^{(k)} = (0,\dots,0,U_k^{(k)},0,\dots,0) \in L^2_{n+1}(P,\mathbb{F})$. Note that in fact the cash flow $\mathbf{X} \in L^2_{n+1}(P,\mathcal{G})$. The martingale property (2.57), Assumption 2.15 and Corollary 2.17 imply for $t<k$

$$\begin{aligned} \varphi_t Q_t\left[\mathbf{X}\right] &= E\left[\varphi_k Q_k\left[\mathbf{X}\right]\middle| \mathcal{F}_t\right] \;=\; E\left[\varphi_k U_k^{(k)} \middle| \mathcal{F}_t\right] \\ &= E\left[\varphi_k^{(\mathcal{T})}\varphi_k^{(\mathcal{G})} U_k^{(k)} \middle| \mathcal{F}_t\right] \\ &= E\left[\varphi_k^{(\mathcal{T})} \middle| \mathcal{T}_t\right] E\left[\varphi_k^{(\mathcal{G})} U_k^{(k)} \middle| \mathcal{G}_t\right] \\ &= \varphi_t^{(\mathcal{T})}\ E\left[\varphi_k^{(\mathcal{G})} U_k^{(k)} \middle| \mathcal{G}_t\right]. \end{aligned} \tag{2.110}$$

This implies that

$$U_t^{(k)} = Q_t\left[\mathbf{X}\right] = \frac{1}{\varphi_t^{(\mathcal{G})}} E\left[\varphi_k^{(\mathcal{G})} U_k^{(k)} \middle| \mathcal{G}_t\right]. \tag{2.111}$$

Henceforth $(\varphi_t^{(\mathcal{G})} U_t^{(k)})_{t=0,\dots,k}$ is a $\mathcal{G}$-martingale under P, which proves the claim.

□

Corollary 2.17 and Theorem 2.18 imply that we can study the insurance technical variables $\mathbf{\Lambda}$ and the price processes of the financial instruments $\mathcal{U}_k$ independently. The valuation of the outstanding loss liabilities

$$\mathbf{X}_{(k)} = (0,\dots,0,\Lambda_k U_k^{(k)},\dots,\Lambda_n U_n^{(n)}) \in L^2_{n+1}(P,\mathbb{F}) \tag{2.112}$$

at time $t \le k$ can then easily be done (see also (2.99)). The basic reserves are given by (see also (2.50))

$$\begin{aligned} R_t^{(k)} = Q_t\left[\mathbf{X}_{(k)}\right] &= \frac{1}{\varphi_t}\ \sum_{s=k}^{n} E\left[\varphi_s^{(\mathcal{T})}\Lambda_s \middle| \mathcal{T}_t\right]\ E\left[\varphi_s^{(\mathcal{G})} U_s^{(s)} \middle| \mathcal{G}_t\right] \\ &= \sum_{s=k}^{n} \Lambda_{t,s}\ U_t^{(s)}. \end{aligned} \tag{2.113}$$

Conclusions.
Under the product space Assumption 2.15, the assumption (2.103) that the insurance technical deflator is a density process w.r.t. $\mathcal{T}$ and P, and under the no-arbitrage assumption (2.57) we obtain that we can separate the valuation problem into two independent valuation problems:

(1) the insurance technical processes $(\Lambda_{t,k})_{t=0,\ldots,k}$, $k=0,\ldots,n$, describe the probability distorted developments of the predictions of Λ_k if we increase the information $\mathcal{T}_t \to \mathcal{T}_{t+1}$;

(2) the financial processes $(U_t^{(k)})_{t=0,\ldots,k}$, $k=0,\ldots,n$, describe the price processes of the financial instruments $\mathcal{U}_k$ at the financial market $(\Omega, \mathcal{G}_n, P_{\mathcal{G}}, \mathcal{G})$.

Example 2.8 (Best-Estimate Predictions).

Choose $\boldsymbol{\varphi}^{(\mathcal{T})} \equiv 1$. Hence, $\boldsymbol{\varphi}^{(\mathcal{T})}$ gives a suitable probability distortion (normalized martingale). This implies for the insurance technical process at time $t \le k$

$$\Lambda_{t,k} = E\left[\Lambda_k \middle| \mathcal{T}_t\right], \tag{2.114}$$

i.e. $\Lambda_{t,k}$ is simply the "best-estimate" prediction of Λ_k based on the information $\mathcal{T}_t$ (conditional expectation which has minimal conditional variance).

□

Exercise 2.9 (Esscher Premium).

We choose a positive random variable Y on the underlying filtered probability space $(\Omega, \mathcal{T}_n, P_{\mathcal{T}}, \mathcal{T})$ such that for some $\alpha > 0$ the following moment generating function exists

$$M_Y(2\alpha) = E\left[\exp\{2\alpha Y\}\right] < \infty. \tag{2.115}$$

Then we define the probability distortion

$$\varphi_t^{(\mathcal{T})} = \frac{E\left[\exp\{\alpha Y\} \middle| \mathcal{T}_t\right]}{E\left[\exp\{\alpha Y\}\right]} = \frac{E\left[\exp\{\alpha Y\} \middle| \mathcal{T}_t\right]}{M_Y(\alpha)}. \tag{2.116}$$

(1) Prove that $\boldsymbol{\varphi}^{(\mathcal{T})} \gg 0$. Moreover, prove that $\boldsymbol{\varphi}^{(\mathcal{T})} \in L^2_{n+1}(P_{\mathcal{T}}, \mathcal{T})$.

(2) Show that $(\varphi_t^{(\mathcal{T})})_{t=0,\ldots,n}$ is a density process w.r.t. $\mathcal{T}$ and $P_{\mathcal{T}}$.

Assume that $\mathbf{X}_k = (0,\ldots,0,X_k,0,\ldots,0)$ with $X_k = \Lambda_k U_k^{(k)}$. Choose $Y = \Lambda_k$ and assume that $t < k$. Prove under Assumption 2.15 and (2.57) that

$$Q_t\left[\mathbf{X}_k\right] = \frac{1}{E\left[\exp\{\alpha\Lambda_k\} \middle| \mathcal{T}_t\right]}\; E\left[\Lambda_k\, e^{\alpha\Lambda_k} \middle| \mathcal{T}_t\right]\; U_t^{(k)}. \tag{2.117}$$

If we define the conditional moment generating function by

$$M_{\Lambda_k|\mathcal{T}_t}(\alpha) = E\left[\exp\left\{\alpha\Lambda_k\right\}\middle|\,\mathcal{T}_t\right], \tag{2.118}$$

then the term

$$\Lambda_{t,k} = \frac{d}{dr}\log M_{\Lambda_k|\mathcal{T}_t}(r)\Big|_{r=\alpha} = M_{\Lambda_k|\mathcal{T}_t}(\alpha)^{-1}E\left[\Lambda_k\, e^{\alpha\Lambda_k}\middle|\,\mathcal{T}_t\right] \tag{2.119}$$

describes the Esscher premium of Λ_k at time $t < k$, see Gerber-Pafumi [GP98].

Claim: prove that the Esscher premium (2.119) is strictly increasing in α.

Remark: α plays the role of the risk aversion.

□

Exercise 2.10 (Expected Shortfall).

Choose a continuous integrable random variable Y on the filtered probability space $(\Omega, \mathcal{T}_n, P_{\mathcal{T}}, \mathbb{T})$. Denote the distribution of Y by $F_Y(x) = P[Y \le x]$ and the generalized inverse by F_Y^{-1}, where $F_Y^{-1}(u) = \inf\{x | F_Y(x) \ge u\}$. Henceforth, the Value-at-Risk of Y at level $1-\alpha \in (0,1)$ is then given by

$$\mathrm{VaR}_{1-\alpha}(Y) = F_Y^{-1}(1-\alpha). \tag{2.120}$$

We obtain

$$\begin{aligned} P\left[Y > \mathrm{VaR}_{1-\alpha}(Y)\right] &= 1 - P\left[Y \le \mathrm{VaR}_{1-\alpha}(Y)\right] \\ &= 1 - F_Y\left(\mathrm{VaR}_{1-\alpha}(Y)\right) \\ &= 1 - F_Y\left(F_Y^{-1}(1-\alpha)\right) = \alpha. \end{aligned} \tag{2.121}$$

Choose $c \in (0,1)$ and define (note that Y is $\mathcal{T}_n$-measurable)

$$\varphi_n^{(\mathcal{T})} = (1-c) + \frac{c}{\alpha} 1_{\{Y > \mathrm{VaR}_{1-\alpha}(Y)\}}, \tag{2.122}$$

and for $t < n$

$$\varphi_t^{(\mathcal{T})} = E\left[\varphi_n^{(\mathcal{T})}\middle|\,\mathcal{T}_t\right]. \tag{2.123}$$

(1) Prove that $\boldsymbol{\varphi}^{(\mathcal{T})} \gg 0$. Moreover, prove that $\boldsymbol{\varphi}^{(\mathcal{T})} \in L^2_{n+1}(P_{\mathcal{T}}, \mathbb{T})$.

(2) Show that $(\varphi_t^{(\mathcal{T})})_{t=0,\dots,n}$ is a density process w.r.t. $\mathbb{T}$ and $P_{\mathcal{T}}$.

Assume that $\mathbf{X}_k = (0,\dots,0,X_k,0,\dots,0)$ with $X_k = \Lambda_k U_k^{(k)}$. Choose $Y = \Lambda_k$ and assume that $t < k$. Under Assumption 2.15 and (2.57) show that

$$Q_t\left[\mathbf{X}_k\right] = \left\{\beta_t E\left[\Lambda_k\middle|\,\mathcal{T}_t\right] + (1-\beta_t)E\left[\Lambda_k\middle|\,\Lambda_k > \mathrm{VaR}_{1-\alpha}(\Lambda_k), \mathcal{T}_t\right]\right\} U_t^{(k)}, \tag{2.124}$$

with so-called credibility weights

$$\beta_t = \frac{1-c}{(1-c) + c\ \frac{P[\Lambda_k > \mathrm{VaR}_{1-\alpha}(\Lambda_k)|\mathcal{T}_t]}{\alpha}}. \tag{2.125}$$

We define the probability

$$\alpha_t = P\left[\Lambda_k > \mathrm{VaR}_{1-\alpha}(\Lambda_k)\right|\mathcal{T}_t\left.\right]. \tag{2.126}$$

This implies

$$\alpha_t = P\left[\Lambda_k > \mathrm{VaR}_{1-\alpha}(\Lambda_k)\right|\mathcal{T}_t\left.\right] = P\left[\Lambda_k > \mathrm{VaR}_{1-\alpha_t}(\Lambda_k|\mathcal{T}_t)\right|\mathcal{T}_t\left.\right], \tag{2.127}$$

which says

$$\mathrm{VaR}_{1-\alpha}(\Lambda_k) = \mathrm{VaR}_{1-\alpha_t}(\Lambda_k|\mathcal{T}_t), \tag{2.128}$$

where $\mathrm{VaR}_{1-\alpha_t}(\Lambda_k|\mathcal{T}_t)$ denotes the Value-at-Risk of $\Lambda_k|\mathcal{T}_t$ at level $1-\alpha_t$. Henceforth, the credibility weight is given by

$$\beta_t = \frac{1-c}{(1-c) + c\ \frac{\alpha_t}{\alpha}} \tag{2.129}$$

and for the price of the insurance technical variable we obtain

$$\Lambda_{t,k} = \beta_t E\left[\Lambda_k\right|\mathcal{T}_t\left.\right] + (1-\beta_t) E\left[\Lambda_k\right|\Lambda_k > \mathrm{VaR}_{1-\alpha_t}(\Lambda_k|\mathcal{T}_t), \mathcal{T}_t\left.\right]. \tag{2.130}$$

The last term is called expected shortfall of $\Lambda_k|\mathcal{T}_t$ at level $1-\alpha_t$, see McNeil et al. [MFE05]. Value-at-Risk and expected shortfall are probably the two most popular risk measures in the insurance industry.

Choose the special case $t = 0$. Then we have $\alpha_0 = \alpha$ (note that $\mathcal{T}_0 = \{\emptyset, \Omega\}$), which implies $\beta_t = 1-c$ and

$$\begin{aligned} \Lambda_{0,k} &= (1-c)E\left[\Lambda_k\right] + cE\left[\Lambda_k\right|\Lambda_k > \mathrm{VaR}_{1-\alpha}(\Lambda_k)\left.\right] \\ &= E\left[\Lambda_k\right] + c\left\{E\left[\Lambda_k\right|\Lambda_k > \mathrm{VaR}_{1-\alpha}(\Lambda_k)\left.\right] - E\left[\Lambda_k\right]\right\}. \end{aligned} \tag{2.131}$$

Henceforth, the basic reserve for Λ_k at time 0 is given by its expected value $E\left[\Lambda_k\right]$ plus a loading where $c \in (0,1)$ plays the role of the cost-of-capital rate and

$$E\left[\Lambda_k\right|\Lambda_k > \mathrm{VaR}_{1-\alpha}(\Lambda_k)\left.\right] - E\left[\Lambda_k\right] \tag{2.132}$$

is the capital-at-risk (unexpected loss) measured by the expected shortfall on the level $1-\alpha$. This is in line with actual solvency developments, see for example SST [SST06], Pelsser [Pe10], Salzmann-Wüthrich [SW10] and Section 5.3 below.

□

2.7 Conclusions on Chapter 2

We have developed **theoretical foundations of market-consistent valuation** based on (possibly distorted) **expected values** (see (2.99) and (2.113)). The distorted probabilities will lead to the **price for risk**. The framework as developed is not the "full story" since it only gives the price for risk (the so-called (probability distorted) pure risk premium) for an insurance company.

However, it does not provide enough information on the risk bearing. This means, that we have not described how the risk bearing should be organized in order to protect against insolvencies.

An insurance company can take the following measures to protect itself against financial impacts of adverse scenarios:

1. buying options and reinsurance, if available,
2. hedging options internally,
3. setting up sufficient risk bearing capital (solvency margin).

In practice, one has to be extremely careful in each application whether the price for risk resulting from the mathematical model is already sufficient to finance adverse scenarios.

Remark on the existing literature. There is a wide range of literature on the definition of market-consistent values. Usually all these definitions are not very mathematical and slightly differ from each other, e.g. market-consistent values should be realistic values, should serve for the exchange of two portfolios, etc. One has to be very careful with these definitions, e.g. do they include cost-of-capital charges, etc.

Our model gives a mathematical framework for a market-consistent valuation. Charges for the risk bearing can be integrated via distorted probabilities, however (as mentioned above) this does not solve the question of the organization of the risk bearing.

3

Valuation portfolio in life insurance

In this chapter we define the valuation portfolio for a life insurance liability portfolio. The construction is done with the help of an explicit example. We proceed in two steps: First, we assume that the cash flows have deterministic insurance technical risk, i.e. we have a deterministic mortality table, and only the value of the financial instruments describe a stochastic process. Then, we map the cash flows onto these financial instruments. In the second step, we introduce stochastic mortality rates yielding stochastic insurance technical risk. In that case we follow the construction in step 1, but we add loadings for the insurance technical risks coming from the stochastic mortality table. This construction gives us a replicating portfolio (protected against insurance technical risks) in terms of financial instruments.

3.1 Deterministic life insurance model

To define the valuation portfolio VaPo we start with a deterministic life insurance model where no insurance technical risk is involved (see also Baumgartner et al. [BBK04]). We assume that we have a deterministic mortality table (second order life table) giving the mortalities *without loadings*. Let l_x denote the number of insured lives aged x and d_x the number of insured lives aged x who die before reaching age $x+1$.

$$\begin{array}{lcl} l_x & & \\ \downarrow & \longrightarrow & d_x = l_x - l_{x+1} \\ l_{x+1} & & \\ \downarrow & \longrightarrow & d_{x+1} = l_{x+1} - l_{x+2} \\ l_{x+2} & & \\ \downarrow & \longrightarrow & d_{x+2} = l_{x+2} - l_{x+3} \\ \vdots & & \vdots \end{array}$$

M.V. Wüthrich et al., *Market-Consistent Actuarial Valuation*,
EAA Series,
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Example 3.1 (Endowment insurance policy).

We assume that the initial sum insured (death benefit) is CHF 1, the age at policy inception is $x = 50$ and the contract term is $n = 5$. Moreover, we assume that:

- The annual premium $\Pi_t = \Pi$, $t = 50, \ldots, 54$, is due in non-indexed CHF at the beginning of each year.
- The benefits are indexed by a well-defined index I_t, $t = 50, 51, \ldots, 55$, with initial value $I_{50} = 1$.
  - Death benefit is the indexed maximum of I_t and $(1+i)^{t-50}$ for some fixed minimal guaranteed interest rate i.
  - Survival benefit is I_{55}, i.e. no minimal guarantee in the case of survival.

The benefits are always paid at the end of each period $(t-1, t]$.

This means that the survival benefit is given by a financial instrument **I** whose price is a stochastic process $(I_t)_{t \geq 50}$ with initial value $I_{50} = 1$. This index can be any financial instrument like a stock, a fund, etc. Hence, to hedge the survival benefit we need to buy one unit of index **I** at the price $I_{50} = 1$ and it generates the (random) survival benefit I_{55} at time $t = 55$.

Thus, the endowment contract gives the following cash flow diagram for $\mathbf{X} = (X_{50}, \ldots, X_{55}) \in L^2_{n+1}(P, \mathcal{G})$: for initially l_{50} persons alive we have (if we only consider 1 person we divide by l_{50})

time	cash flow	premium	death benefit	survival benefit
50	X_{50}	$-l_{50}\ \Pi$		
51	X_{51}	$-l_{51}\ \Pi$	$d_{50}\ \left(I_{51} \vee (1+i)^1\right)$	
52	X_{52}	$-l_{52}\ \Pi$	$d_{51}\ \left(I_{52} \vee (1+i)^2\right)$	
53	X_{53}	$-l_{53}\ \Pi$	$d_{52}\ \left(I_{53} \vee (1+i)^3\right)$	
54	X_{54}	$-l_{54}\ \Pi$	$d_{53}\ \left(I_{54} \vee (1+i)^4\right)$	
55	X_{55}		$d_{54}\ \left(I_{55} \vee (1+i)^5\right)$	$l_{55}\ I_{55}$

Cash inflows (premium) have a negative sign, cash outflows have a positive sign, and $x \vee y = \max\{x, y\}$.

□

Task: Value this endowment policy at the beginning of the contract and at every successive year!

3.2 Valuation portfolio for the deterministic life insurance model

For the life insurance portfolio considered in Example 3.1 (with deterministic mortality rates) we now want to construct the valuation portfolio. Roughly speaking the valuation portfolio (VaPo) is a portfolio of financial instruments that replicates the future cash flows arising from the insurance contracts. The procedure is the following: to replicate the insurance cash flow $\mathbf{X} \in L^2_{n+1}(P, \mathcal{G})$ we specify in a first step the set of financial instruments that will be used for the replication purposes. Second, for each financial instrument the appropriate number of units must be determined, this gives the VaPo for $\mathbf{X}$. Thirdly, we define the market-consistent value of the cash flow $\mathbf{X}$ to be equal to the value of the VaPo. This convention is consistent with the well-known "law of one price"-principle which says that in an arbitrage-free economy two instruments with the same cash flows must have the same price.

Step 1. Define units, choose a financial basis.

- The premium Π is due at time $t = 50, \dots, 54$ in non-indexed CHF. Hence, as units we choose the zero coupon bonds $Z^{(50)}, \dots, Z^{(54)}$ (the units are denoted by $Z^{(t)}$, whereas the cash flow of the zero coupon bond $Z^{(t)}$ is denoted by $\mathbf{Z}^{(t)}$, see (2.28) and (3.4)).
- Survival benefit: Unit is the indexed fund $\mathbf{I}$ with price process $(I_t)_{t=50,\dots,55}$.
- Death benefit $I_t \vee (1+i)^{t-50}$ can be measured in an indexed fund $\mathbf{I}$ plus a put option on $\mathbf{I}$ with strike time t and strike $(1+i)^{t-50}$. We denote this put option by $\mathrm{Put}^{(t)} = \mathrm{Put}^{(t)}(\mathbf{I}, (1+i)^{t-50})$.

Hence we have the following units (financial instruments)

$$(\mathcal{U}_1, \dots, \mathcal{U}_{11}) \\ = \left(Z^{(50)}, \dots, Z^{(54)}, \mathbf{I}, \mathrm{Put}^{(51)}\left(\mathbf{I}, (1+i)^1\right), \dots, \mathrm{Put}^{(55)}\left(\mathbf{I}, (1+i)^5\right)\right), \tag{3.1}$$

i.e. we have that the total number of different units equals 11. These units play the role of the basis (financial instruments) in which we measure the insurance liabilities.

Step 2. Determine the number/amount of each unit needed.

At the beginning of the policy we need:

<u>*Valuation Scheme A*</u> (for l_{50} persons)

time	premium	death benefit	survival benefit
50	$-l_{50}\ \Pi\ Z^{(50)}$		
51	$-l_{51}\ \Pi\ Z^{(51)}$	$d_{50}\left(\mathbf{I} + \mathrm{Put}^{(51)}\left(\mathbf{I}, (1+i)^1\right)\right)$	
52	$-l_{52}\ \Pi\ Z^{(52)}$	$d_{51}\left(\mathbf{I} + \mathrm{Put}^{(52)}\left(\mathbf{I}, (1+i)^2\right)\right)$	
53	$-l_{53}\ \Pi\ Z^{(53)}$	$d_{52}\left(\mathbf{I} + \mathrm{Put}^{(53)}\left(\mathbf{I}, (1+i)^3\right)\right)$	
54	$-l_{54}\ \Pi\ Z^{(54)}$	$d_{53}\left(\mathbf{I} + \mathrm{Put}^{(54)}\left(\mathbf{I}, (1+i)^4\right)\right)$	
55		$d_{54}\left(\mathbf{I} + \mathrm{Put}^{(55)}\left(\mathbf{I}, (1+i)^5\right)\right)$	$l_{55}\ \mathbf{I}$

This immediately leads to the summary of units:

Valuation Scheme B (for l_{50} persons)

unit $\mathcal{U}_i$	number of units
$Z^{(50)}$	$-l_{50}\ \Pi$
$Z^{(51)}$	$-l_{51}\ \Pi$
$Z^{(52)}$	$-l_{52}\ \Pi$
$Z^{(53)}$	$-l_{53}\ \Pi$
$Z^{(54)}$	$-l_{54}\ \Pi$
$\mathbf{I}$	$d_{50} + d_{51} + d_{52} + d_{53} + d_{54} + l_{55} = l_{50}$
$\mathrm{Put}^{(51)}\left(\mathbf{I}, (1+i)^1\right)$	d_{50}
$\mathrm{Put}^{(52)}\left(\mathbf{I}, (1+i)^2\right)$	d_{51}
$\mathrm{Put}^{(53)}\left(\mathbf{I}, (1+i)^3\right)$	d_{52}
$\mathrm{Put}^{(54)}\left(\mathbf{I}, (1+i)^4\right)$	d_{53}
$\mathrm{Put}^{(55)}\left(\mathbf{I}, (1+i)^5\right)$	d_{54}

Observe that the number of units of $\mathbf{I}$ needed is exactly l_{50} because every person insured receives one index $\mathbf{I}$, no matter whether he dies during the term of the contract or not.

Our valuation portfolio VaPo($\mathbf{X}$) is a point in an 11-dimensional vector space (see also (3.2) in Section 3.3 below) where we have specified a basis of financial instruments $\mathcal{U}_i$ (dimension of vector space) and the number of instruments we need to hold to replicate the insurance liabilities.

Step 3. To obtain the (monetary) value for our cash flow we need to apply an accounting principle to this VaPo($\mathbf{X}$), see Section 3.3 below.

□

Conclusion. In a first and second step, we decompose the liability cash flow $\mathbf{X} = (X_{50}, \ldots, X_{55})$ into a 11-dimensional vector VaPo($\mathbf{X}$), whose basis consists of financial instruments $\mathcal{U}_1, \ldots, \mathcal{U}_{11}$. Only in a third step, we calculate the monetary value of the cash flow $\mathbf{X}$ by applying an accounting principle to the units $\mathcal{U}_i$, and thus to VaPo($\mathbf{X}$).

Hence we have found the following general valuation procedure:

3.3 General valuation procedure for deterministic insurance technical risks

1. For every policy with cash flow $\mathbf{X} \in L^2_{n+1}(P, \mathcal{G})$ with deterministic insurance technical risk we construct the VaPo($\mathbf{X}$) as follows: Define units $\mathcal{U}_i$ (basis of a multidimensional vector space) and determine the (deterministic) number $\lambda_i(\mathbf{X}) \in \mathbb{R}$ of each unit $\mathcal{U}_i$:

$$\mathbf{X} \;\mapsto\; \mathrm{VaPo}(\mathbf{X}) = \sum_i \lambda_i(\mathbf{X})\, \mathcal{U}_i. \tag{3.2}$$

From a theoretical point of view the VaPo mapping needs to be a multi-dimensional positive continuous linear function that maps the insurance liabilities $\mathbf{X}$ onto a valuation portfolio $\mathrm{VaPo}(\mathbf{X})$ which replicates the insurance liabilities in terms of financial instruments.

2. Apply then an accounting principle $\mathcal{A}_t$ to the valuation portfolio to obtain a monetary value at time $t \geq 0$

$$\mathrm{VaPo}(\mathbf{X}) \mapsto \mathcal{A}_t\left(\mathrm{VaPo}(\mathbf{X})\right) = Q_t\left[\mathbf{X}\right] \in \mathbb{R}. \tag{3.3}$$

This mapping must be a positive, continuous, linear functional.
Moreover, the sequence of accounting principles $(\mathcal{A}_t)_{t=0,\dots,n}$ must satisfy certain consistency properties in order to have an arbitrage-free pricing system. In fact, we require a martingale property (2.57) for deflated price processes. This is further discussed below.

For the zero coupon bond with maturity m we have at time 0 ($\mathcal{U}_1 = Z^{(m)}$)

$$Q_0[\mathbf{Z}^{(m)}] = \mathcal{A}_0\left(\mathrm{VaPo}(\mathbf{Z}^{(m)})\right) = \mathcal{A}_0\left(\lambda_1(\mathbf{Z}^{(m)})\; Z^{(m)}\right) = \mathcal{A}_0\left(Z^{(m)}\right). \tag{3.4}$$

The construction of the VaPo adds enormously to the understanding and communication between actuaries and asset managers and investors, respectively. In a first step the actuary decomposes the insurance portfolio into financial instruments, in a second step the asset manager evaluates the financial instruments. Indeed, it is the key step to a successful *asset and liability management* (ALM) technique, and it clearly highlights the sources of uncertainties involved in the process. It also allocates the responsibilities for the uncertainties to the different parties involved.

Remark 1. For a cash flow $\mathbf{X} \in L^2_{n+1}(P, \mathcal{G})$ with no insurance technical risk involved we obtain for the value at time 0

$$Q_0\left[\mathbf{X}\right] = \mathcal{A}_0\left(\mathrm{VaPo}(\mathbf{X})\right) = \sum_i \lambda_i(\mathbf{X})\; \mathcal{A}_0\left(\mathcal{U}_i\right) \quad \in \mathbb{R}, \tag{3.5}$$

which should be a positive, continuous, linear functional on $L^2_{n+1}(P, \mathcal{G})$. One has to be a little bit careful with the positivity: In order to obtain a positive linear functional, we must have that $U_t^{(i)} = \mathcal{A}_t\left(\mathcal{U}_i\right) > 0$ for all i as long as a policy is in force, which must be kept in mind whenever the units are selected.

Remark 2. By linearity the individual policies can be added up to a portfolio, i.e. individual cash flows $\mathbf{X}^{(k)} \in L^2_{n+1}(P, \mathcal{G})$ easily merge to $\sum_k \mathbf{X}^{(k)}$ which has value

$$Q_t\left[\sum_k \mathbf{X}^{(k)}\right] = \sum_k \mathcal{A}_t\left(\mathrm{VaPo}(\mathbf{X}^{(k)})\right). \tag{3.6}$$

This means that we can value portfolios of a single contract as well as of the whole insurance company. Note that this aggregation needs to be done very carefully as soon as also insurance technical risks are involved.

Examples of accounting principles $\mathcal{A}_t$. An accounting principle $\mathcal{A}_t$ attaches a value to the financial instruments. There are different ways to choose an appropriate accounting principle. In fact, choosing an appropriate accounting principle very much depends on the problem under consideration. We give two examples.

- Classical actuarial discounting. In many situation, for example in (traditional) communication with regulators, the value of the financial instruments are determined by a mathematical model (such as amortized costs, etc.). If we choose the model where we discount with a fixed constant interest rate we denote the accounting principle by $\mathcal{D}_t$.
- In modern actuarial valuation, the financial instruments are often valued at an economic value, market value or value according to the IASB accounting rule. In general, this means that the value of the asset is essentially the price at which it can be exchanged at the financial market. If we use such an economic accounting principle we use the symbol $\mathcal{E}_t$.

Both principles $\mathcal{D}_t$ and $\mathcal{E}_t$ need to fulfill some time consistency properties in order to have an arbitrage-free pricing system. That is, assume we choose the economic accounting principles $\mathcal{E}_t$, $t = 0, \ldots, n$. Then, for cash flows $\mathbf{X} \in L^2_{n+1}(P, \mathcal{G})$ (with deterministic insurance technical risk) we have the following value at time 0

$$Q_0[\mathbf{X}] = \mathcal{E}_0(\mathrm{VaPo}(\mathbf{X})) = \sum_i \lambda_i(\mathbf{X})\ \mathcal{E}_0(\mathcal{U}_i). \tag{3.7}$$

Using Riesz' representation theorem (Theorem 2.5) we find the state price deflator $\boldsymbol{\varphi} \in L^2_{n+1}(P, \mathbb{F})$ with

$$< \mathbf{X}, \boldsymbol{\varphi} > \ = \ Q_0[\mathbf{X}] \ = \ \mathcal{E}_0(\mathrm{VaPo}(\mathbf{X})). \tag{3.8}$$

Using price definition (2.38), Lemma 2.8 then implies that we need to have

$$\left(\varphi_t \mathcal{E}_t(\mathrm{VaPo}(\mathbf{X}))\right)_{t=0,\ldots,n} \qquad \text{forms an } \mathbb{F}\text{-martingale under } P. \tag{3.9}$$

3.4 Self-financing property of the VaPo (deterministic insurance technical risk)

In (2.49) we have defined $\mathbf{X}_{(k)} \in L^2_{n+1}(P, \mathcal{G})$ as the remaining cash flow after time $k-1$. Moreover, define the cash flow

$$\mathbf{X}_k = X_k\ \mathbf{Z}^{(k)} = (0, \ldots, 0, X_k, 0 \ldots, 0) \in L^2_{n+1}(P, \mathcal{G}). \tag{3.10}$$

Hence, note

$$\mathbf{X}_{(k)} = \mathbf{X}_{(k+1)} + \mathbf{X}_k, \tag{3.11}$$

and using the linearity of the valuation portfolio (3.2) we have the following lemma.

Lemma 3.1 (Self-financing property as portfolio) *For* $\mathbf{X} \in L^2_{n+1}(P, \mathcal{G})$

$$\text{VaPo}\left(\mathbf{X}_{(k)}\right) = \text{VaPo}\left(\mathbf{X}_{(k+1)}\right) + \text{VaPo}\left(\mathbf{X}_k\right). \tag{3.12}$$

Of course, in this lemma we assume that the vector space is spanned by the financial instruments determined by $\mathbf{X}_{(k)}$.

Remark. At time k, the last term in (3.12) is simply cash value, which we abbreviate

$$\text{VaPo}\left(\mathbf{X}_k\right) = X_k \qquad \text{at time } k, \tag{3.13}$$

i.e. we omit in this case to write the unit because it is just 1 at time k.

Studying now the values given by the accounting principle $\mathcal{A}_t$, we have by the linearity of $\mathcal{A}_t$ the following lemma:

Lemma 3.2 (Self-financing property in value) *For* $\mathbf{X} \in L^2_{n+1}(P, \mathcal{G})$ *and* $t \leq k$

$$\mathcal{A}_t\left(\text{VaPo}\left(\mathbf{X}_{(k)}\right)\right) = \mathcal{A}_t\left(\text{VaPo}\left(\mathbf{X}_{(k+1)}\right)\right) + \mathcal{A}_t\left(\text{VaPo}\left(\mathbf{X}_k\right)\right). \tag{3.14}$$

In particular, if the valuation portfolio of $\mathbf{X}_k$ is evaluated at time k then

$$X_k = \frac{1}{\varphi_k}\; E\left[\varphi_k\; X_k \middle| \mathcal{F}_k\right] = Q_k\left[\mathbf{X}_k\right] = \mathcal{A}_k\left(\text{VaPo}(\mathbf{X}_k)\right), \tag{3.15}$$

hence

$$\mathcal{A}_k\left(\text{VaPo}\left(\mathbf{X}_{(k)}\right)\right) = \mathcal{A}_k\left(\text{VaPo}\left(\mathbf{X}_{(k+1)}\right)\right) + X_k, \tag{3.16}$$

which tells again that the VaPo for $\mathbf{X}_k$ at time k is simply X_k. This observation is fundamental and should hold independently of the value assigned to the VaPo by the accounting principle $\mathcal{A}_t$.

For a more detailed analysis of the self-financing property in monetary value over time we refer to Subsection 6.2.

3.5 VaPo protected against insurance technical risks

So far we have considered an ideal situation which is an important point of reference to measure deviations.

ideal	realistic	deviation
deterministic mortality	stochastic mortality	technical risk
VaPo	real investment portfolio $\mathcal{S}$	financial risk

The ideal situation is often called base scenario and one then studies deviations from this base scenario.

In this section we want to consider insurance technical risks. They come from the fact that the insurance liabilities are not deterministic, i.e. $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$. This means in our example that we have stochastic mortality rates.

For the deviations from the deterministic model (which are expectations, best-estimates for the liabilities) we add a protection. Such a protection can be obtained e.g. via reinsurance products, risk loadings or risk bearing capital. The VaPo with this additional protection will be called **VaPo protected against insurance technical risks**.

3.5.1 Construction of the VaPo protected against insurance technical risks

Let us return to our Example 3.1. The stochastic mortality table reads as:

$$\begin{array}{lll} l_x & & \\ \downarrow & \longrightarrow & D_x = l_x - L_{x+1} \\ L_{x+1} & & \\ \downarrow & \longrightarrow & D_{x+1} = L_{x+1} - L_{x+2} \\ L_{x+2} & & \\ \downarrow & \longrightarrow & D_{x+2} = L_{x+2} - L_{x+3} \\ \vdots & & \vdots \end{array}$$

where now L_{x+k} and D_{x+k-1} are random variables for $k \geq 1$. From

$$D_{50} = l_{50} - L_{51}, \tag{3.17}$$

$$d_{50} = l_{50} - l_{51}, \tag{3.18}$$

we obtain

$$D_{50} - d_{50} = l_{51} - L_{51}, \tag{3.19}$$

which describes the deviations of D_{50} and L_{51} from their expected values d_{50} and l_{51}, respectively. In fact, in a first step we use the expected value d_{50} as a predictor for the random variable D_{50}, and in a second step we need to study the prediction uncertainty or the deviation of the random variable D_{50} around its predictor d_{50}.

The Valuation Scheme A then reads as follows for the stochastic mortality table:

time	premium	death benefit	survival benefit
50	$-l_{50}\ \Pi\ Z^{(50)}$		
51	$-L_{51}\ \Pi\ Z^{(51)}$	$D_{50}\ \left(\mathbf{I} + \mathrm{Put}^{(51)}\left(\mathbf{I}, (1+i)^1\right)\right)$	
52	$-L_{52}\ \Pi\ Z^{(52)}$	$D_{51}\ \left(\mathbf{I} + \mathrm{Put}^{(52)}\left(\mathbf{I}, (1+i)^2\right)\right)$	
53	$-L_{53}\ \Pi\ Z^{(53)}$	$D_{52}\ \left(\mathbf{I} + \mathrm{Put}^{(53)}\left(\mathbf{I}, (1+i)^3\right)\right)$	
54	$-L_{54}\ \Pi\ Z^{(54)}$	$D_{53}\ \left(\mathbf{I} + \mathrm{Put}^{(54)}\left(\mathbf{I}, (1+i)^4\right)\right)$	
55		$D_{54}\ \left(\mathbf{I} + \mathrm{Put}^{(55)}\left(\mathbf{I}, (1+i)^5\right)\right)$	$L_{55}\ \mathbf{I}$

Let us define the expected survival probabilities and the expected death probabilities (second order life table) for $t \geq x$:

$$p_t = \frac{l_{t+1}}{l_t} \qquad \text{and} \qquad q_t = 1 - p_t = \frac{d_t}{l_t}. \tag{3.20}$$

Denote by $\mathrm{VaPo}(\mathbf{X}_{(t+1)})$ the valuation portfolio for the cash flows after time t with deterministic insurance technical risks (deterministic mortality table as defined in Section 3.2). I.e. $\mathrm{VaPo}(\mathbf{X}_{(t+1)})$ denotes the valuation portfolio with the expected cash flows (L_t is replaced by its mean l_t).

If we allow for a stochastic survival in period $(50, 51]$ we have the following deviations from the expected VaPo (deterministic insurance technical risks): For $t = 51$ we obtain the following deviations form the expected payments

$$(D_{50} - d_{50})\ \left(\mathbf{I} + \mathrm{Put}^{(51)}\left(\mathbf{I}, (1+i)^1\right)\right), \tag{3.21}$$

$$(l_{51} - L_{51})\ \Pi\ Z^{(51)}, \tag{3.22}$$

$$(L_{51} - l_{51})\ \frac{\mathrm{VaPo}(\mathbf{X}_{(52)})}{l_{51}}, \tag{3.23}$$

if $\mathrm{VaPo}(\mathbf{X}_{(52)})$ denotes the deterministic cash flows of our endowment policy after time $t = 51$ (according to Section 3.2). This means that we have deviations in the payments at time $t = 51$ due to the stochastic mortality, and then at $t = 51$, we start with a new basis of L_{51} insured lives (instead of l_{51}), which gives a new expected VaPo after time $t = 51$ of (use the linearity of the VaPo)

$$L_{51}\ \frac{\mathrm{VaPo}(\mathbf{X}_{(52)})}{l_{51}}. \tag{3.24}$$

Using (3.19) and equations (3.21)-(3.23) we see that we need additional reserves of

$$(D_{50} - d_{50})\ \left(\mathbf{I} + \mathrm{Put}^{(51)}\left(\mathbf{I}, (1+i)^1\right) + \Pi\ Z^{(51)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(52)})}{l_{51}}\right) \tag{3.25}$$

for the deviations from the expected mortality table within $(50, 51]$. Note that this deviation is stochastic seen from time $t = 50$. Hence the portfolio at risk is

$$\mathbf{I} + \mathrm{Put}^{(51)}\left(\mathbf{I}, (1+i)^1\right) + \varPi \; Z^{(51)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(52)})}{l_{51}}. \tag{3.26}$$

We can now iterate this procedure:
For $t = 52$ we have the following deviation from the expected VaPo. The expected VaPo starts now after $t = 51$ with the new basis of L_{51} insured lives (we have to build the additional VaPo reserves for the new basis in (3.25)). Note that conditionally, given L_{51}, we expect $q_{51} L_{51}$ persons to die within the time interval $(51, 52]$ and we observe D_{51} at time $t = 52$. This gives the following deviations

$$(D_{51} - q_{51} \; L_{51}) \; \left(\mathbf{I} + \mathrm{Put}^{(52)}\left(\mathbf{I}, (1+i)^2\right)\right), \tag{3.27}$$

$$(p_{51} \; L_{51} - L_{52}) \; \varPi \; Z^{(52)}, \tag{3.28}$$

$$(L_{52} - p_{51} \; L_{51}) \; \frac{\mathrm{VaPo}(\mathbf{X}_{(53)})}{p_{51} \; L_{51}} \; \frac{L_{51}}{l_{51}}, \tag{3.29}$$

where the last term can be simplified to

$$\frac{\mathrm{VaPo}(\mathbf{X}_{(53)})}{p_{51} \; L_{51}} \; \frac{L_{51}}{l_{51}} = \frac{\mathrm{VaPo}(\mathbf{X}_{(53)})}{l_{52}}. \tag{3.30}$$

Hence we need for the deviation in $(51, 52]$ additional reserves of

$$(D_{51} - q_{51} \; L_{51}) \; \left(\mathbf{I} + \mathrm{Put}^{(52)}\left(\mathbf{I}, (1+i)^2\right) + \varPi \; Z^{(52)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(53)})}{l_{52}}\right). \tag{3.31}$$

And analogously for $t = 53, 54, 55$ we obtain the deviations

$$\begin{aligned} &(D_{52} - q_{52} \; L_{52}) \; \left(\mathbf{I} + \mathrm{Put}^{(53)}\left(\mathbf{I}, (1+i)^3\right) + \varPi \; Z^{(53)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(54)})}{l_{53}}\right), \\ &(D_{53} - q_{53} \; L_{53}) \; \left(\mathbf{I} + \mathrm{Put}^{(54)}\left(\mathbf{I}, (1+i)^4\right) + \varPi \; Z^{(54)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(55)})}{l_{54}}\right), \\ &(D_{54} - q_{54} \; L_{54}) \; \left(\mathbf{I} + \mathrm{Put}^{(55)}\left(\mathbf{I}, (1+i)^5\right) - \mathbf{I}\right). \end{aligned} \tag{3.32}$$

Remark. One can see that when adding up the terms inside in (3.25) and (3.31)-(3.32) the unit **I** cancels since $\mathrm{VaPo}(\mathbf{X}_{(t+1)})$ contains exactly l_t units of **I** for $t = 50$. This is immediately clear because the number of units **I** we need to buy at the beginning of the policy does not depend on the mortality table (see Valuation Scheme B on page 46), i.e. no matter whether a person dies or stays alive it receives **I**.

Hence we find the following portfolios at risk:

$$\begin{aligned}
t = 51:\ & \mathbf{I} + \mathrm{Put}^{(51)}\left(\mathbf{I}, (1+i)^1\right) + \Pi\ Z^{(51)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(52)})}{l_{51}}, \\
t = 52:\ & \mathbf{I} + \mathrm{Put}^{(52)}\left(\mathbf{I}, (1+i)^2\right) + \Pi\ Z^{(52)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(53)})}{l_{52}}, \\
t = 53:\ & \mathbf{I} + \mathrm{Put}^{(53)}\left(\mathbf{I}, (1+i)^3\right) + \Pi\ Z^{(53)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(54)})}{l_{53}}, \\
t = 54:\ & \mathbf{I} + \mathrm{Put}^{(54)}\left(\mathbf{I}, (1+i)^4\right) + \Pi\ Z^{(54)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(55)})}{l_{54}}, \\
t = 55:\ & \mathbf{I} + \mathrm{Put}^{(55)}\left(\mathbf{I}, (1+i)^5\right) - \mathbf{I}.
\end{aligned} \tag{3.33}$$

The interpretation of (3.33) is the following. Consider for example the period $(52, 53]$, if more people die than expected ($D_{52} > q_{52}\ L_{52}$) we have to pay an additional death benefit of

$$(D_{52} - q_{52}\ L_{52})\ \left(\mathbf{I} + \mathrm{Put}^{(53)}\left(\mathbf{I}, (1+i)^3\right)\right). \tag{3.34}$$

On the other hand for all these people the contracts are terminated which means that our liabilities are reduced by

$$(D_{52} - q_{52}\ L_{52})\ \left(-\Pi\ Z^{(53)} + \frac{\mathrm{VaPo}(\mathbf{X}_{(54)})}{l_{53}}\right). \tag{3.35}$$

These insurance technical risks are now protected against adverse developments by adding a security loading. This gives us the following **reinsurance premium loadings as a portfolio**:

$$\begin{aligned}
\mathrm{RPP}_{50} &= l_{50}\ (q^*_{50} - q_{50}) \\
&\qquad \left(\mathbf{I} + \mathrm{Put}^{(51)}\left(\mathbf{I}, (1+i)^1\right) + \Pi\ Z^{(51)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(52)})}{l_{51}}\right), \\
\mathrm{RPP}_{51} &= l_{51}\ (q^*_{51} - q_{51}) \\
&\qquad \left(\mathbf{I} + \mathrm{Put}^{(52)}\left(\mathbf{I}, (1+i)^2\right) + \Pi\ Z^{(52)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(53)})}{l_{52}}\right), \\
\mathrm{RPP}_{52} &= l_{52}\ (q^*_{52} - q_{52}) \\
&\qquad \left(\mathbf{I} + \mathrm{Put}^{(53)}\left(\mathbf{I}, (1+i)^3\right) + \Pi\ Z^{(53)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(54)})}{l_{53}}\right), \\
\mathrm{RPP}_{53} &= l_{53}\ (q^*_{53} - q_{53}) \\
&\qquad \left(\mathbf{I} + \mathrm{Put}^{(54)}\left(\mathbf{I}, (1+i)^4\right) + \Pi\ Z^{(54)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(55)})}{l_{54}}\right), \\
\mathrm{RPP}_{54} &= l_{54}\ (q^*_{54} - q_{54})\ \left(\mathbf{I} + \mathrm{Put}^{(55)}\left(\mathbf{I}, (1+i)^5\right) - \mathbf{I}\right),
\end{aligned} \tag{3.36}$$

where $q^*_t - q_t$ denote the loadings charged by the reinsurer against insurance technical risks, and l_t is the number of units we need to buy. Here, q^*_t can be interpreted as the yearly renewable term (YRT) rates charged by the reinsurer.

Valuation Portfolio protected against insurance technical risks is now defined as

$$\mathrm{VaPo}^{prot}(\mathbf{X}) = \mathrm{VaPo}(\mathbf{X}) + \sum_{t=50}^{54} \mathrm{RPP}_t. \tag{3.37}$$

Remarks.

- For a monetary reinsurance premium we need to apply an accounting principle $\mathcal{A}_t$ to the reinsurance premium portfolio (yearly renewable term):

$$\begin{aligned} \Pi_t^R &= \mathcal{A}_t(\mathrm{RPP}_t) \\ &= l_t \; (q_t^* - q_t) \\ &\quad \times \mathcal{A}_t \left(\mathbf{I} + \mathrm{Put}^{(t+1)}\left(\mathbf{I}, (1+i)^{t-50+1}\right) + \Pi Z^{(t+1)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(t+2)})}{l_{t+1}} \right). \end{aligned} \tag{3.38}$$

- The last term in (3.38) highlights that the choice of the loadings $q_t^* - q_t$ needs some care. The monetary value of the portfolio at risk (3.33) may have both signs. Therefore the sign of the loading may depend on the monetary value of the portfolio at risk. For example, for death benefits we decrease the survival probabilities p_t, whereas for annuities we increase the survival probabilities.
- There are different possibilities to determine the premium: We could choose an actuarial accounting principle $\mathcal{D}_t$ or an economic accounting principle $\mathcal{E}_t$ (which gives an economic yearly renewable term, see also page 48). This idea opens interesting *new reinsurance products*: Offer a reinsurance cover against insurance technical risks in terms of a valuation portfolio.
- A static hedging strategy is to invest the reinsurance premium into the valuation portfolios of the reinsurer.

3.5.2 Probability distortion of life tables

The choice of the death probabilities q_t^* may look artificial at the first sight. They often come from a first order life table. A first order life table refers to survival or death probabilities that are chosen prudent (i.e. with some safety margin), whereas the second order life table refers to best-estimate survival and death probabilities. However, the choice of a first order life table fits perfectly into our modelling framework. Indeed, the first order life tables can be explained by probability distortions: In (2.105) we have considered the term $\Lambda_{t,k} = \frac{1}{\varphi_t^{(\mathcal{T})}} E\left[\varphi_k^{(\mathcal{T})} \Lambda_k \middle| \mathcal{T}_t\right]$, $k > t$, referring to the price of the insurance cover in units.

To explain this term, we revisit our Example 3.1 with a stochastic mortality table: for illustrative purposes we choose $t = 52$. The σ-field $\mathcal{T}_{52}$ tells us that there are L_{52} persons alive at time $t = 52$, i.e. L_{52} is $\mathcal{T}_{52}$-measurable. Moreover, we choose $k = 53$ and we assume that Λ_{53} models the death benefit. Thus we study (set $\varphi_{52}^{(\mathcal{T})} = 1$)

$$E\left[\varphi_{53}^{(\mathcal{T})} \Lambda_{53} \middle| \mathcal{T}_{52}\right] = E\left[\varphi_{53}^{(\mathcal{T})} D_{52} \middle| \mathcal{T}_{52}\right], \tag{3.39}$$

which describes for how many financial units we build insurance technical reserves.

In a first step we choose $\varphi_{53}^{(\mathcal{T})} \equiv 1$, then we obtain

$$E\left[\varphi_{53}^{(\mathcal{T})} \Lambda_{53} \middle| \mathcal{T}_{52}\right] = E\left[D_{52} \middle| \mathcal{T}_{52}\right] = q_{52}\ L_{52}, \tag{3.40}$$

i.e. q_{52} describes the single death probability within $(52, 53]$ and (3.40) leads to the VaPo that covers expected liabilities.

We now model the probability distortion (insurance technical deflator) $\varphi_{53}^{(\mathcal{T})}$ so that we obtain the first order life table q_{52}^*. Note that

$$E\left[\varphi_{53}^{(\mathcal{T})} \Lambda_{53} \middle| \mathcal{T}_{52}\right] = E\left[\varphi_{53}^{(\mathcal{T})} D_{52} \middle| \mathcal{T}_{52}\right] = \sum_{i=1}^{L_{52}} E\left[\varphi_{53}^{(\mathcal{T})} I_i \middle| \mathcal{T}_{52}\right], \tag{3.41}$$

where I_i is the indicator whether person i dies within $(52, 53]$.

We assume that single life times (of persons all of the same age) are i.i.d. Then we assume that the probability distortion $\varphi_{53}^{(\mathcal{T})}$ is of the form

$$\varphi_{53}^{(\mathcal{T})} = \prod_{i=1}^{L_{52}} \varphi_{53}^{(\mathcal{T})}(I_i), \tag{3.42}$$

such that each factor of this product has expectation 1. Henceforth, we write

$$\sum_{i=1}^{L_{52}} E\left[\varphi_{53}^{(\mathcal{T})} I_i \middle| \mathcal{T}_{52}\right] = \sum_{i=1}^{L_{52}} E\left[\varphi_{53}^{(\mathcal{T})}(I_i)\ I_i \middle| \mathcal{T}_{52}\right]. \tag{3.43}$$

The factors of the probability distortions are now chosen as follows: Take $q_{52}^* \in (0, 1)$ and define

$$\varphi_{53}^{(\mathcal{T})}(1) = \frac{q_{52}^*}{q_{52}}, \tag{3.44}$$

$$\varphi_{53}^{(\mathcal{T})}(0) = \frac{1 - q_{52}^*}{1 - q_{52}}. \tag{3.45}$$

We then obtain the required normalization

$$E\left[\varphi_{53}^{(\mathcal{T})}(I_i) \middle| \mathcal{T}_{52}\right] = q_{52}\ \frac{q_{52}^*}{q_{52}} + p_{52}\ \frac{1 - q_{52}^*}{1 - q_{52}} = 1, \tag{3.46}$$

and the first order life table

$$E\left[\varphi_{53}^{(\mathcal{T})}(I_i)I_i\middle|\,\mathcal{T}_{52}\right] = q_{52}\;\frac{q_{52}^*}{q_{52}} = q_{52}^*, \tag{3.47}$$

i.e., note that we have set $\varphi_{52}^{(\mathcal{T})} = 1$, and

$$E\left[\varphi_{53}^{(\mathcal{T})}\Lambda_{53}\middle|\,\mathcal{T}_{52}\right] = q_{52}^*\;L_{52}. \tag{3.48}$$

In other words the transition from the second order life table p_t to the first order life table p_t^* exactly refers to a probability distortion $\varphi_{t+1}^{(\mathcal{T})}$.

Exercise 3.2 (Life-Time Annuity).

Consider a life-time annuity for a man aged x at time 0. We assume that the life-time annuity contract is paid by a single premium installment π_0 at the beginning of the insurance period (initial lump sum) and that the insured receives an annual payment of M until he dies.

- Determine the valuation portfolio VaPo based on the second order life table p_t, $t \geq x$.
- Calculate the portfolios at risk and the VaPo protected against insurance technical risks.
- Determine the sign of the loadings $p_t^* - p_t$.
- Express the second order life table p_t^* with the help of probability distortions $\varphi_{t+1}^{(\mathcal{T})}$.

□

3.6 Back to the basic model

In Chapter 2 we have chosen a deflator

$$\boldsymbol{\varphi} = (\varphi_0, \ldots, \varphi_n) \in L^2_{n+1}(P, \mathbb{F}) \tag{3.49}$$

to value cash flows $\mathbf{X} = (X_0, \ldots, X_n) \in L^2_{n+1}(P, \mathbb{F})$. The basic assumption was that $\boldsymbol{\varphi}$ and $\mathbf{X}$ are $\mathbb{F}$-adapted. Moreover, we have assumed that on our filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F})$ we can decompose $\mathbb{F}$ into independent filtrations $\mathcal{T}$ and $\mathcal{G}$ such that

$$X_k = \Lambda_k\; U_k^{(k)}, \tag{3.50}$$

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}^{(\mathcal{T})}\;\boldsymbol{\varphi}^{(\mathcal{G})}, \tag{3.51}$$

where $\boldsymbol{\Lambda}, \boldsymbol{\varphi}^{(\mathcal{T})} \in L^2_{n+1}(P, \mathcal{T})$ and $\boldsymbol{\varphi}^{(\mathcal{G})}, (U_t^{(k)})_{t=0,\ldots,n} \in L^2_{n+1}(P_{\mathcal{G}}, \mathcal{G})$ for all $k = 0, \ldots, n$, see Assumption 2.15. This means that we can split the problem into two independent problems, one measuring insurance technical risks $\mathcal{T}$ and one describing (financial) price processes on $\mathcal{G}$.

To avoid ambiguity we have assumed that the expectation of the probability distortion is 1, (see also (2.100))

$$E\left[\varphi_t^{(\mathcal{T})}\right] = 1 \tag{3.52}$$

for all $t = 0, \dots, n$, and moreover, we have assumed that $(\varphi_t^{(\mathcal{T})})_{t=0,\dots,n}$ is a $\mathcal{T}$-martingale under P, see (2.101).

The VaPo construction in this chapter has now led to a multidimensional approach, i.e. the cash flow $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$ is decomposed into a vector consisting of different financial instruments/units (see also (3.2))

$$\mathbf{X} \;\mapsto\; \sum_{i=1}^{p} \Lambda_i(\mathbf{X})\, \mathcal{U}_i, \tag{3.53}$$

if $\mathcal{U}_1, \dots, \mathcal{U}_p$ represent the p financial instruments by which $\mathbf{X}$ can be described, and Λ_i the (random) number of units $\mathcal{U}_i$ needed. The value/price process of $\mathcal{U}_i$ is denoted by $(U_t^{(i)})_{t=0,\dots,n}$ and is independent of $\mathcal{T}$. If we now use vector notation, (3.53) can be rewritten as (we have linear mappings)

$$\mathbf{X} = \begin{pmatrix} X_0 \\ X_1 \\ \vdots \\ X_n \end{pmatrix} \mapsto \sum_{i=1}^{p} \begin{pmatrix} \Lambda_i(\mathbf{X}_0) \\ \Lambda_i(\mathbf{X}_1) \\ \vdots \\ \Lambda_i(\mathbf{X}_n) \end{pmatrix}^T \begin{pmatrix} \mathcal{U}_i \\ \mathcal{U}_i \\ \vdots \\ \mathcal{U}_i \end{pmatrix}, \tag{3.54}$$

where $\mathbf{X}_t = X_t\, \mathbf{Z}^{(t)} = (0, \dots, 0, X_t, 0, \dots, 0)$.

For the **VaPo construction seen from time 0** we have then replaced the random $\Lambda_i(\mathbf{X}_k)$ by deterministic numbers (expected values):

$$\Lambda_i(\mathbf{X}_k) \;\mapsto\; l_{i,k} = l_{i,k}^{(0)} = E\left[\Lambda_i(\mathbf{X}_k) \middle|\, \mathcal{T}_0\right]. \tag{3.55}$$

If $\Lambda_i(\mathbf{X}_k)$ is deterministic as in Section 3.1, then we have $\Lambda_i(\mathbf{X}_k) = \lambda_i(\mathbf{X}_k) = l_{i,k}$ (see (3.2)).

For the **VaPo protected against insurance technical risks** (seen from time 0) we replace $\Lambda_i(\mathbf{X}_k)$ by the following deterministic numbers (distorted expected values):

$$\Lambda_i(\mathbf{X}_k) \;\mapsto\; l_{i,k}^{*} = l_{i,k}^{*,0} = E\left[\varphi_k^{(\mathcal{T})}\; \Lambda_i(\mathbf{X}_k) \middle|\, \mathcal{T}_0\right], \tag{3.56}$$

which adds a loading to $l_{i,k}$ for insurance technical risks. If $\Lambda_i(\mathbf{X}_k)$ is deterministic as in Section 3.1, i.e. $\mathbf{X} \in L^2_{n+1}(P, \mathcal{G})$, then we have $\Lambda_i(\mathbf{X}_k) = \lambda_i(\mathbf{X}_k) = l_{i,k} = l_{i,k}^{*}$ due to (2.100), i.e. we do not need a loading for insurance technical risks. The loading in $l^*_{i,k}$ could also have been chosen directly, not via the definition of a probability distortion. This gives now

$$\mathrm{VaPo}\,(\mathbf{X}) = \sum_{i=1}^{p} \begin{pmatrix} l_{i,0} \\ \vdots \\ l_{i,n} \end{pmatrix}^T \begin{pmatrix} \mathcal{U}_i \\ \vdots \\ \mathcal{U}_i \end{pmatrix}. \tag{3.57}$$

This can also be written as

$$\mathrm{VaPo}\,(\mathbf{X}) = \sum_{i=1}^{p} l_i\, \mathcal{U}_i, \tag{3.58}$$

with

$$l_i = \sum_{t=0}^{n} l_{i,t}. \tag{3.59}$$

The VaPo protected against insurance technical risks is given by

$$\mathrm{VaPo}^{prot}\,(\mathbf{X}) = \sum_{i=1}^{p} \begin{pmatrix} l^*_{i,0} \\ \vdots \\ l^*_{i,n} \end{pmatrix}^T \begin{pmatrix} \mathcal{U}_i \\ \vdots \\ \mathcal{U}_i \end{pmatrix}, \tag{3.60}$$

or equivalently

$$\mathrm{VaPo}^{prot}\,(\mathbf{X}) = \sum_{i=1}^{p} l^*_i\, \mathcal{U}_i, \tag{3.61}$$

with

$$l^*_i = \sum_{t=0}^{n} l^*_{i,t}. \tag{3.62}$$

Remark. Observe that (3.57) and (3.58) provide two representations for $\mathrm{VaPo}\,(\mathbf{X})$. Firstly, we have the **cash flow representation**, which corresponds to Valuation Scheme A in Section 3.2. That is, (3.57) implies

$$\mathrm{VaPo}\,(\mathbf{X}) = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}^T \begin{pmatrix} \sum_{i=1}^{p} l_{i,0}\, \mathcal{U}_i \\ \vdots \\ \sum_{i=1}^{p} l_{i,n}\, \mathcal{U}_i \end{pmatrix}. \tag{3.63}$$

Secondly, we have the **instrument representation** (3.58) which corresponds to Valuation Scheme B in Section 3.2.

Analogously, we have the two representations (3.60) and (3.61) for the VaPo protected against insurance technical risks $\mathrm{VaPo}^{prot}\,(\mathbf{X})$.

For many purposes the instrument representation (3.58) of the VaPo suffices. Sometimes, however, it may be necessary to work with the cash flow representation (3.63), see for example Section 3.7 below.

Applying an accounting principle $\mathcal{A}_0$ to the VaPo (or equivalently to the financial instruments $\mathcal{U}_i$) gives then a monetary value for the basic reserves at time 0.

Remark. It is important to see that the valuation portfolio construction in (3.55) is seen from time 0. If the cash flows have no insurance technical risks (as in Section 3.3) there are no deviations in $\Lambda_i(\mathbf{X})$ over time, which means that $l_{i,k}$ is constant in time. But if we have insurance technical risks involved, then

$$l_{i,k}^{(m)} = E\left[\Lambda_i(\mathbf{X}_k)\middle|\,\mathcal{T}_m\right], \tag{3.64}$$

$$l_{i,k}^{*,m} = \frac{1}{\varphi_m^{(\mathcal{T})}}\; E\left[\varphi_k^{(\mathcal{T})}\;\Lambda_i(\mathbf{X}_k)\middle|\,\mathcal{T}_m\right] \tag{3.65}$$

are functions of time (see also Chapter 6). This then leads to time dependent valuation portfolios

$$\mathrm{VaPo}_{(m)}\left(\mathbf{X}\right) \quad \text{and} \quad \mathrm{VaPo}_{(m)}^{prot}\left(\mathbf{X}\right). \tag{3.66}$$

We then also need to study the changes in these valuation portfolios over time, i.e.

$$\mathrm{VaPo}_{(m)}\left(\mathbf{X}\right) - \mathrm{VaPo}_{(m-1)}\left(\mathbf{X}\right) \tag{3.67}$$

and

$$\mathrm{VaPo}_{(m)}^{prot}\left(\mathbf{X}\right) - \mathrm{VaPo}_{(m-1)}^{prot}\left(\mathbf{X}\right), \tag{3.68}$$

which considers the update of information $\mathcal{T}_{m-1} \mapsto \mathcal{T}_m$ and is similar to the claims development result in non-life insurance, see for example Merz-Wüthrich [MW08] and Salzmann-Wüthrich [SW10].

3.7 Approximate valuation portfolio

In Section 3.2 we have constructed the VaPo for a rather simple example. We have considered a small homogeneous portfolio and its liabilities were easily described by financial instruments. In practice the situation is often more complicated. Life insurance companies have high-dimensional portfolios which usually involve embedded options and guarantees as well as management decisions. I.e. the valuation portfolio becomes path dependent and the determination of the liability cash flows and the appropriate financial instruments is not straightforward. In such situations one often tries to approximate the VaPo by a financial portfolio. Here, we will define the approximate VaPo (denoted by VaPo^{approx}) which plays the role of a replicating portfolio.

Let us choose a filtered probability space $(\Omega, \mathcal{F}_n, P, \mathbb{F})$ and assume that we have an insurance liability cash flow $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$.

In order to construct an approximate VaPo we choose a set of basic tradable financial instruments $\mathcal{U}_1, \ldots, \mathcal{U}_q$ from which we believe that they can replicate the liabilities in an appropriate way and for which we can *easily* describe their price processes

$$U_t^{(i)} = \mathcal{A}_t(\mathcal{U}_i), \qquad \text{for } t = 0, \ldots, n, \tag{3.69}$$

i.e. we want to choose q financial instruments for which we have a good understanding.

We now want to approximate the cash flow representation (3.63) of

$$\mathrm{VaPo}\,(\mathbf{X}) = \sum_{k=0}^{n} \mathrm{VaPo}\,(\mathbf{X}_k)\,. \tag{3.70}$$

That is, for all single cash flows X_k, $k = 0, \ldots, n$, our goal is to choose $\mathbf{y}_k \in \mathbb{R}^q$ such that

$$\mathrm{VaPo}\,(\mathbf{Y}_k) = \sum_{i=1}^{q} y_{i,k}\, \mathcal{U}_i \tag{3.71}$$

approximates $\mathrm{VaPo}\,(\mathbf{X}_k)$. Or in vector notation, we choose $\mathbf{y} \in \mathbb{R}^{q\times(n+1)}$ such that

$$\mathrm{VaPo}\,(\mathbf{Y}) = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}^T \begin{pmatrix} \sum_{i=1}^q y_{i,0}\, \mathcal{U}_i \\ \vdots \\ \sum_{i=1}^q y_{i,n}\, \mathcal{U}_i \end{pmatrix} \tag{3.72}$$

approximates $\mathrm{VaPo}\,(\mathbf{X})$, see (3.63). That is, our aim is to choose $\mathbf{y} \in \mathbb{R}^{q\times(n+1)}$ such that $\mathbf{X}$ and $\mathbf{Y}$ are “close”. Of course, close will depend on some distance function.

If there is no insurance technical risk and if $\mathcal{U}_1, \ldots, \mathcal{U}_q$ is a complete financial basis for the liabilities we can achieve

$$\mathbf{X} = \mathbf{Y} \qquad P\text{-a.s.} \tag{3.73}$$

In general, we are not able to achieve (3.73) nor is it possible to evaluate the random vectors $\mathbf{X}$ and $\mathbf{Y}$ for all sample points $\omega \in \Omega$. Therefore, one then chooses a finite set of so-called scenarios $\Omega_K = \{\omega_1, \ldots, \omega_K\} \subset \Omega$ and one evaluates the random vectors $\mathbf{X}$ and $\mathbf{Y}$ in these scenarios. We introduce a distance function

$$\mathrm{dist}\,(\mathbf{X}(\cdot), \mathbf{Y}(\cdot), \Omega_K) \in \mathbb{R}, \tag{3.74}$$

then the approximate valuation portfolio is given by

$$\mathbf{y}^* = \arg \min_{\mathbf{y}\in\mathbb{R}^{q\times(n+1)}} \mathrm{dist}\,(\mathbf{X}(\cdot), \mathbf{Y}(\cdot), \Omega_K)\,, \tag{3.75}$$

and for $k = 0, \ldots, n$ we obtain

$$\mathrm{VaPo}^{approx}\left(\mathbf{X}_k\right)=\sum_{i=1}^{q} y_{i,k}^{*}\,\mathcal{U}_i, \tag{3.76}$$

or, respectively,

$$\mathrm{VaPo}^{approx}\left(\mathbf{X}\right)=\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}^{T}\begin{pmatrix}\sum_{i=1}^{q} y_{i,0}^{*}\,\mathcal{U}_i\\ \vdots\\ \sum_{i=1}^{q} y_{i,n}^{*}\,\mathcal{U}_i\end{pmatrix} \tag{3.77}$$

Remark. It is important to realize that the approximate valuation portfolio $\mathbf{y}^*$ depends on the choice of (a) the financial instruments $\mathcal{U}_1,\ldots,\mathcal{U}_q$, (b) the choice of the scenarios Ω_K, and (c) the choice of the distance function. Based on the purpose of the approximate valuation portfolio (e.g. profit testing, solvency, extremal behaviour) these choices will vary and there is *no* obvious best choice.

Example 3.3 (Cash flow matching).

We assume that we want to match the entire cash flow $\mathbf{X}$ as good as possible and we use the L^2-distance measure. Assume that there are positive deterministic weight functions $\chi_t:\Omega_K\to\mathbb{R}_+$ given for $t=0,\ldots,n$. Our distance function is defined by

$$\mathrm{dist}\left(\mathbf{X}(\cdot),\mathbf{Y}(\cdot),\Omega_K\right)=\sum_{k=1}^{K}\sum_{t=0}^{n}\chi_t(\omega_k)\left(X_t(\omega_k)-Y_t(\omega_k)\right)^2. \tag{3.78}$$

For $\chi_t(\cdot)$ we can make different choices. Often one wants to account for time values, therefore one chooses the financial deflator $\boldsymbol{\varphi}^{(\mathcal{G})}$ (see Assumption 2.15) and a normalized positive deterministic weight function $p:\Omega_K\to\mathbb{R}_+$ with $\sum_{k=1}^{K}p(\omega_k)=1$ and defines for $t=0,\ldots,n$

$$\chi_t(\omega_k)=p(\omega_k)\,\left(\varphi_t^{(\mathcal{G})}(\omega_k)\right)^2. \tag{3.79}$$

The distance function is then rewritten as

$$\begin{aligned}\mathrm{dist}\left(\mathbf{X}(\cdot),\mathbf{Y}(\cdot),\Omega_K\right)&=\sum_{k=1}^{K}p(\omega_k)\sum_{t=0}^{n}\left(\varphi_t^{(\mathcal{G})}(\omega_k)\right)^2\left(X_t(\omega_k)-Y_t(\omega_k)\right)^2\\ &=E_K\left[\sum_{t=0}^{n}\left(\varphi_t^{(\mathcal{G})}X_t-\varphi_t^{(\mathcal{G})}Y_t\right)^2\right],\end{aligned} \tag{3.80}$$

where E_K denotes the expected value under the discrete probability measure P_K which assigns probability weight $p(\omega_k)$ to the scenarios in Ω_K.

The distance function defined in (3.80) tries to match pointwise in time the values of the cash flows $\mathbf{X}$ and $\mathbf{Y}$ as good as possible. Other approaches often work under equivalent probability measures (risk neutral measures or forward measures) so that the discount factors become measurable at the beginning of the corresponding periods.

□

Exercise 3.4.

Calculate the approximate valuation portfolio explicitly under distance function (3.80).

Hint: Note that we have a quadratic form in (3.80). Set the gradient equal to zero and calculate the Hessian matrix (see Ingersoll [Ing87], formula (37) on page 8).

□

Example 3.5 (Time value matching).

We assume that we want to match the time value of $\mathbf{X}$ as good as possible and we use the L^2-distance measure. For a positive deterministic weight function χ_t similar to (3.79) we define the distance function

$$\begin{aligned} \mathrm{dist}\left(\mathbf{X}(\cdot), \mathbf{Y}(\cdot), \Omega_K\right) &= \sum_{k=1}^{K} p(\omega_k) \left\{ \sum_{t=0}^{n} \varphi_t^{(\mathcal{G})}(\omega_k) \left(X_t(\omega_k) - Y_t(\omega_k) \right) \right\}^2 \\ &= E_K \left[\left(\sum_{t=0}^{n} \varphi_t^{(\mathcal{G})} X_t - \varphi_t^{(\mathcal{G})} Y_t \right)^2 \right], \end{aligned} \tag{3.81}$$

where E_K denotes the expected value under the discrete probability measure P_K which assigns probability weight $p(\omega_k)$ to the scenarios in Ω_K.

The distance function defined in (3.81) tries to match time value of entire cash flows $\mathbf{X}$ and $\mathbf{Y}$ as good as possible. Note that the difference is, that we match the entire time value of $\mathbf{X}$ in (3.81) whereas in (3.80) we match cash flow X_k individually in k.

□

Exercise 3.6.

Calculate the approximate valuation portfolio explicitly under distance function (3.81).

Hint: Note that we have a quadratic form in (3.81). Set the gradient equal to zero and calculate the Hessian matrix (see Ingersoll [Ing87], formula (37) on page 8).

□

3.8 Conclusions on Chapter 3

We have decomposed the cash flow $\mathbf{X}$ in a two-step procedure:

1. Choose a multidimensional vector space whose basis consists of financial instruments $\mathcal{U}_1, \ldots, \mathcal{U}_p$.
2. Express the cash flow $\mathbf{X}$ as a vector in this vector space. The number of each unit is determined by the expected number of units (where the expectation is calculated with possibly distorted probabilities).

Calculating the monetary value of the valuation portfolio is then the third step where we use an accounting principle to give values to the vectors in the multidimensional vector space.

We should mention that we have constructed our VaPo for a very basic example. In practice the VaPo construction is much more difficult because, for example, (a) modelling embedded options and guarantees can become very difficult, see Section 3.7; (b) often one has not the necessary information on single policies in the portfolio (e.g. collective policies). Moreover, in practice one faces a lot of problems about data storing and data management since the volume of the data can become very large.

Finally, we mention that we can also construct the VaPo if the financial instruments do not exist on the financial market, e.g. a 41-years zero coupon bond. The VaPo construction still works. However, calculating the monetary value of the VaPo is not straightforward if the instruments do not exist on the financial market.

3.9 Examples

In this section we give a numerical example to the deterministic Example 3.1 (endowment insurance policy). Note that the mathematical details for the evaluation of the accounting principles are given in Chapter 4, below.

For the deterministic mortality table we choose Table 3.1.

time	survival	death
50	$l_{50} = 1'000$	
51	$l_{51} = 996$	$d_{50} = 4$
52	$l_{52} = 991$	$d_{51} = 5$
53	$l_{53} = 986$	$d_{52} = 5$
54	$l_{54} = 981$	$d_{53} = 5$
55	$l_{55} = 975$	$d_{54} = 6$

Table 3.1. Deterministic mortality table, portfolio of 1'000 insured lives

Example 3.7 (Equity-linked life insurance).

We choose an equity-linked life insurance product. Assume that $(I_s)_s$ denotes the price process of the equity index $\mathbf{I}$ (see (4.9)) in the economic world $\mathcal{E}$. That is, we choose an accounting principle $\mathcal{E}$ that corresponds to financial market prices, moreover $\mathcal{E}_s$ denotes these market prices at time s, henceforth

$$I_s = \mathcal{E}_s(\mathbf{I}) = \mathcal{E}(\mathbf{I}|\,\mathcal{G}_s), \tag{3.82}$$

and that $Z_s^{(t)} = P(s,t)$, $s = 0, \ldots, t$, denotes the price process of the zero coupon bond paying 1 at time t. I.e.

$$Z_s^{(t)} = Q_s\left[\mathbf{Z}^{(t)}\right] = Q\left[\mathbf{Z}^{(t)}\middle|\,\mathcal{G}_s\right] = \mathcal{E}_s\left(Z^{(t)}\right) = \mathcal{E}\left(Z^{(t)}\middle|\,\mathcal{G}_s\right), \tag{3.83}$$

where $\mathbf{Z}^{(t)}$ is the cash flow of the zero coupon bond $Z^{(t)}$ (see (4.10)). Assume that the zero coupon bond yield curves $R(s,t)$ (continuously-compounded spot rates) at time $s \le t$ are given by

$$Z_s^{(t)} = \exp\{-(t-s)\; R(s,t)\} \quad\Longleftrightarrow\quad R(s,t) = -\frac{1}{t-s}\;\log Z_s^{(t)}. \tag{3.84}$$

Considering historical data we observe (source of zero coupon bond yield curves given by the Schweizerische Nationalbank [SNB]): see Table 3.2.

		$R(s,t)$				
s	$\ln(I_s/I_{s-1})$	$t-s=1$	$t-s=2$	$t-s=3$	$t-s=4$	$t-s=5$
1996	12.99%	1.94%	2.42%	2.79%	3.12%	3.42%
1997	13.35%	1.82%	1.92%	2.20%	2.48%	2.74%
1998	22.11%	1.71%	1.81%	1.95%	2.10%	2.27%
1999	5.41%	2.21%	2.06%	2.21%	2.31%	2.42%
2000	2.02%	3.37%	3.52%	3.53%	3.56%	3.60%
2001	8.60%	2.00%	2.85%	2.90%	2.96%	3.02%
2002	-12.41%	0.69%	1.84%	2.14%	2.38%	2.57%
2003	-14.83%	0.58%	0.79%	1.14%	1.46%	1.72%
2004	15.87%	0.99%	1.11%	1.42%	1.70%	1.94%
2005	1.83%	1.41%	1.14%	1.32%	1.48%	1.62%
average	5.49%	1.67%	1.95%	2.16%	2.35%	2.53%

Table 3.2. Equity index and yield curve of the zero coupon bond

We assume that our endowment insurance policy starts in year 2000, i.e. we identify the starting point at age $x = 50$ with the year $t_0 = 2000$.

Assume that the guaranteed interest rate is $i = 2\%$.

To adopt the option pricing formula to the case of non-constant interest rates we transform our price process I_s by a change of numeraire (see also Subsection 4.3.2) and consider for $t_0 \le s \le t$

$$\widetilde{I}_s = \frac{I_s}{Z_s^{(t)}} \qquad \text{for fixed } t, \tag{3.85}$$

that is, we consider the t-forward risk neutral measure for the zero-coupon bond numeraire $Z_s^{(t)}$, see for example Section 2.5 in Brigo-Mercurio [BM06].

Now we need to choose a stochastic model for the price process $\widetilde{I}_s$: In order to apply classical financial mathematics we switch to a continuous time model. We assume that, under the t-forward risk neutral measure, $\widetilde{I}_s$ is a martingale satisfying the following stochastic differential equation

$$d\widetilde{I}_s = \sigma \; \widetilde{I}_s \; dW_s, \tag{3.86}$$

where W_s is a standard Brownian motion under the t-forward risk neutral measure. Hence using Ito calculus, $\widetilde{I}_s$ can be rewritten as follows (see e.g. Subsection 3.4.3 in Lamberton-Lapeyre [LL91])

$$\widetilde{I}_s = \widetilde{I_{t_0}} \; \exp\left\{ -\frac{\sigma^2}{2} \; (s - t_0) + \sigma \; W_{s-t_0} \right\}. \tag{3.87}$$

Using the general option pricing formula for European put options (see e.g. Section 9.4 in Elliott-Kopp [EK99]) we obtain the price process

$$\mathcal{E}_s \left(\mathrm{Put}^{(t)} \left(\mathbf{I}, (1+i)^{t-t_0} \right) \right) = K_s^{(t)} \; \Phi \left(-d_2(s,t) \right) - I_s \; \Phi \left(-d_1(s,t) \right), \tag{3.88}$$

with Φ standard Gaussian distribution and

$$K_s^{(t)} = (1+i)^{t-t_0} \; Z_s^{(t)}, \tag{3.89}$$

$$d_1(s,t) = \frac{\log \left(I_s / K_s^{(t)} \right) + \sigma^2 (t-s)/2}{\sigma \sqrt{t-s}}, \tag{3.90}$$

$$d_2(s,t) = d_1(s,t) - \sigma \sqrt{t-s}. \tag{3.91}$$

Remark. For $Z_s^{(t)} = \exp\{-r \; (t-s)\}$ with $r > 0$ constant, (3.88) is the well-known Black-Scholes formula.

We choose I_s and $Z_s^{(t)}$ according to Table 3.2 with $I_{t_0} = 1$ and $\sigma = 15\%$ and obtain the following prices for the put options (observe that in year $t_0 = 2000$ we have a rather high yield $R(t_0, t)$, which gives a low price for our put option): see Table 3.3.

	$t-s=1$	$t-s=2$	$t-s=3$	$t-s=4$	$t-s=5$
$s=2000$	0.053	0.069	0.080	0.088	0.093
$s=2001$	0.034	0.051	0.066	0.076	
$s=2002$	0.117	0.131	0.144		
$s=2003$	0.249	0.267			
$s=2004$	0.140				

Table 3.3. Prices put options $\mathcal{E}_s(\mathrm{Put}^{(t)}(\mathbf{I}, (1+i)^{t-t_0}))$

Now we calculate the monetary value of the valuation portfolio of $\mathbf{X}$: Assume that the survival and death benefit (before index-linking) equal 100'000. Hence we require (premium equivalence principle)

$$\mathcal{E}_{t_0}\left(\mathrm{VaPo}\left(\mathbf{X}\right)\right) = Q_{t_0}\left[\mathbf{X}\right] \overset{(!)}{=} 0, \tag{3.92}$$

which gives the market-consistent pure risk premium $\Pi = 21'667$ (per policy).

Now we consider the valuation portfolios at different times $t_0 \leq s \leq t-1$. Denote by $\mathbf{X}_{(s+1)} = (0, \ldots, X_{s+1}, \ldots, X_t)$ the cash flow (outstanding liabilities) after time s.

$$\begin{aligned} \mathcal{E}_s^{(+)} &= \mathcal{E}_s\left(\mathrm{VaPo}\left(\mathbf{X}_{(s+1)}\right) - l_{50+s-t_0}\ \Pi\ Z^{(s)}\right) \\ &= \mathcal{E}_s\left(\mathrm{VaPo}\left(\mathbf{X}_{(s+1)}\right)\right) - l_{50+s-t_0}\ \Pi = Q_s\left[\mathbf{X}_{(s+1)}\right] - l_{50+s-t_0}\ \Pi, \end{aligned} \tag{3.93}$$

is the monetary value before the premium $l_{50+s-t_0}\ \Pi$ has been paid at time s, and

$$\mathcal{E}_s^{(-)} = \mathcal{E}_s\left(\mathrm{VaPo}\left(\mathbf{X}_{(s+1)}\right)\right) = Q_s\left[\mathbf{X}_{(s+1)}\right], \tag{3.94}$$

is the monetary value after the premium $l_{50+s-t_0}\ \Pi$ has been paid at time s. Of course $\mathcal{E}_{t_0}^{(+)} = \mathcal{E}_{t_0}\left(\mathrm{VaPo}\left(\mathbf{X}\right)\right) = 0$ (premium equivalence principle). This gives the following results for the monetary values of the valuation portfolios: see Table 3.4.

	$\mathcal{E}_s^{(+)}$	$\mathcal{E}_s^{(-)}$
$s = 2000$	0	21'666'637
$s = 2001$	26'370'714	47'950'684
$s = 2002$	32'423'186	53'894'823
$s = 2003$	39'619'061	60'982'365
$s = 2004$	74'244'766	95'499'737

Table 3.4. Development of the monetary values of the valuation portfolios

For the valuation portfolio protected against insurance technical risks, we proceed as follows: we define p_t and q_t as in (3.20). Moreover we choose $q_t^* = 1.5 \cdot q_t$ (first order life table). Hence we consider the premium for the yearly renewable term Π_s^R defined in (3.38) for our accounting principle $\mathcal{E}_{t_0}$. This gives the following monetary reinsurance loadings at time t_0: see Table 3.5.

	Π_s^R
$s = 2000$	167'885
$s = 2001$	162'340
$s = 2002$	115'180
$s = 2003$	68'723
$s = 2004$	27'818

Table 3.5. monetary yearly renewable terms premium

□

Example 3.8 (Wage index).

In non-life insurance the products are rather linked to other indices like the inflation index, wage index, the consumer price index or a medical expenses index. As index we choose the wage index (source Schweizerische Nationalbank [SNB]): see Table 3.6.

s	$\frac{I_s}{I_{s-1}}-1$	$R(s,t)$ $t-s=1$	$t-s=2$	$t-s=3$	$t-s=4$	$t-s=5$
1996	1.30%	1.94%	2.42%	2.79%	3.12%	3.42%
1997	1.26%	1.82%	1.92%	2.20%	2.48%	2.74%
1998	0.47%	1.71%	1.81%	1.95%	2.10%	2.27%
1999	0.69%	2.21%	2.06%	2.21%	2.31%	2.42%
2000	0.29%	3.37%	3.52%	3.53%	3.56%	3.60%
2001	1.26%	2.00%	2.85%	2.90%	2.96%	3.02%
2002	2.48%	0.69%	1.84%	2.14%	2.38%	2.57%
2003	1.79%	0.58%	0.79%	1.14%	1.46%	1.72%
2004	1.40%	0.99%	1.11%	1.42%	1.70%	1.94%
2005	0.93%	1.41%	1.14%	1.32%	1.48%	1.62%
average	1.19%	1.67%	1.95%	2.16%	2.35%	2.53%

Table 3.6. Wage inflation index and yield curve of the zero coupon bond

This time we choose as minimal guaranteed interest rate of $i = 1.5\%$. For the volatility we choose $\sigma = 1\%$. This implies that the market-consistent pure risk premium Π equals $\Pi = 21'624$ (per policy) and the prices for the put options can be found in Table 3.7.

	$t-s=1$	$t-s=2$	$t-s=3$	$t-s=4$	$t-s=5$
$s=2000$	$1.16 \cdot 10^{-4}$	$8.26 \cdot 10^{-6}$	$8.42 \cdot 10^{-7}$	$7.36 \cdot 10^{-8}$	$4.74 \cdot 10^{-9}$
$s=2001$	$2.82 \cdot 10^{-3}$	$2.28 \cdot 10^{-4}$	$6.14 \cdot 10^{-5}$	$1.39 \cdot 10^{-5}$	
$s=2002$	$4.60 \cdot 10^{-3}$	$1.21 \cdot 10^{-3}$	$4.75 \cdot 10^{-4}$		
$s=2003$	$3.72 \cdot 10^{-3}$	$8.27 \cdot 10^{-3}$			
$s=2004$	$2.43 \cdot 10^{-3}$				

Table 3.7. Put option prices $\mathcal{E}_s(\text{Put}^{(t)}(\mathbf{I}, (1+i)^{t-t_0}))$

Observe that the premium Π and the put prices are smaller in the wage index example than in the equity-linked example. This comes from the fact that the choice of σ is much smaller in the second example.

The monetary values of the valuation portfolios are provided in Table 3.8.

	$\mathcal{E}_s^{(+)}$	$\mathcal{E}_s^{(-)}$
$s=2000$	0	21'624'505
$s=2001$	18'723'288	40'261'295
$s=2002$	39'780'582	61'210'467
$s=2003$	61'740'997	83'062'759
$s=2004$	83'857'251	105'070'890

Table 3.8. Development of the monetary values of the valuation portfolios

And the reinsurance loadings are given in Table 3.9.

	Π_s^R
$s = 2000$	157'404
$s = 2001$	145'186
$s = 2002$	95'278
$s = 2003$	46'890
$s = 2004$	0.0014

Table 3.9. Monetary yearly renewable terms premium

The reinsurance premium looks rather small compared to the pure risk premium $l_t \; \Pi \; Z_{t_0}^{(s)}$. This comes from the fact that σ is rather small, that the minimal guarantee $i = 1.5\%$ is rather low compared to the yield $R(t_0, \cdot)$ in year $t_0 = 2000$, and from the fact the randomness of D_t is rather small compared to the total volume l_t.

□

4

Financial risks

In the previous chapter we have defined the valuation portfolio VaPo for life insurance policies. This valuation portfolio VaPo can be viewed as a replicating portfolio for the insurance liabilities in terms of financial instruments. In this chapter we analyze financial risks which come from the fact that the VaPo and the real existing asset portfolio on the asset side of the balance sheet may differ.

4.1 Asset and liability management

We assume that the VaPo, the VaPo^{prot} and the VaPo^{approx} consist of financial instruments $\mathcal{U}_i$ which can be bought at the financial market (this is reasonable for life insurance). We should now compare these valuation portfolios to the **existing asset portfolio** S which our insurance company holds on the asset side of its balance sheet, see Figure 4.1.

In the sequel we drop the upper indices "*prot*" and "*approx*".

Definition 4.1 *If we buy* VaPo *as assets the resulting portfolio is called replicating portfolio.*

It is convenient to use VaPo for both: 1) the portfolio of liabilities, 2) the replicating portfolio of assets, since they are physically the same portfolio.

Definition 4.2 *Financial risks derive from the fact that the existing asset portfolio* S *and the replicating portfolio* VaPo *differ.*

Financial risk management and asset and liability management (ALM) is concerned with maximizing financial returns under the constraint that one has to cover the given liabilities VaPo. Goal is to obtain solvency at any time, where solvency is defined relative to an accounting principle.

M.V. Wüthrich et al., *Market-Consistent Actuarial Valuation*, EAA Series,
DOI 10.1007/978-3-642-14852-1_4,

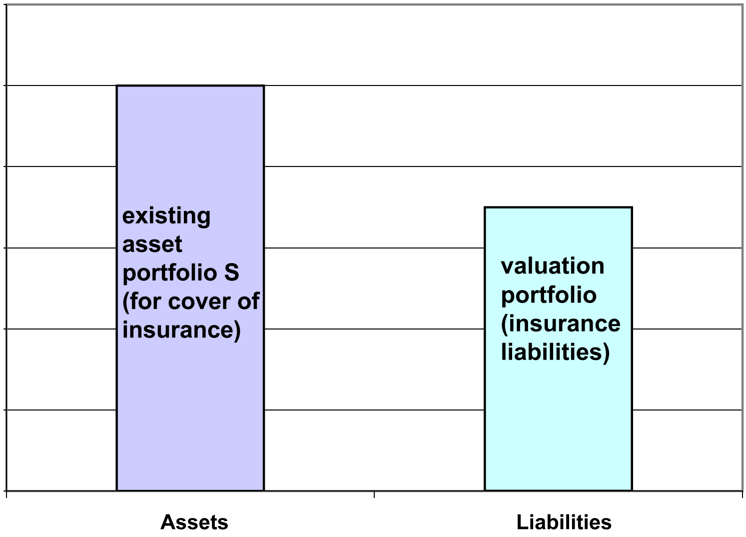


Fig. 4.1. Existing asset portfolio S for the cover of insurance liabilities and the valuation portfolio protected against insurance technical risks VaPo^{prot}

In the sequel we choose the economic accounting principle $\mathcal{E}_t$ (see Section 3.3). Here, this corresponds to the prices that are paid at the financial market for the different financial instruments. Since these prices are time dependent, we attach a subscript t to the accounting principle denoting the time point at which the prices of the financial instruments are evaluated. In order to have a meaningful pricing system we again need consistency properties for the accounting principles $\mathcal{E}_t$ over time $t = 0, \ldots, n$. Similarly to Section 3.3 this means that we want the accounting principles to be continuous, positive, linear functionals such that the martingale property (3.9) is fulfilled for the corresponding deflator $\boldsymbol{\varphi} \in L^2_{n+1}(P, \mathbb{F})$.

Definition 4.3 *Choose $t_0 \in \{0, \ldots, n-1\}$. An insurance company is solvent at time t_0, iff*

$$\mathcal{E}_{t_0}[S] \geq \mathcal{E}_{t_0}[\mathrm{VaPo}], \tag{4.1}$$

this is the accounting condition (actual market-consistent balance sheet), and

$$\mathcal{E}_t[S] \geq \mathcal{E}_t[\mathrm{VaPo}] \qquad \textit{for all } t = t_0 + 1, \ldots, n, \tag{4.2}$$

this is the insurance contract condition.

□

Remarks.

- Definition 4.3 is *our* definition for solvency. Pay attention to the fact that there is not a unique definition for solvency, indeed the solvency rules slightly differ from country to country. In particular, they depend on the risk classes considered, the risk measure used, the security level used, the stochastic model used, etc. Moreover, importantly, we could replace the economic accounting principle $\mathcal{E}_t$ by any other appropriate accounting principle $\mathcal{A}_t$ and we would obtain different solvency results.

- The accounting condition (4.1) is necessary but not sufficient for solvency. It basically says that the market-consistent value of the outstanding insurance liabilities described by the VaPo is covered by the asset value.
- Either there is no insurance technical risk involved (i.e. the valuation portfolio is deterministic with respect to insurance technical risks) or if there is insurance technical risk involved, we consider the valuation portfolio protected against insurance technical risks. Hence in both situations the valuation portfolios are assumed to be deterministic (w.r.t. insurance technical risks), i.e. the cash flow generated by the VaPo is in $L^2_{n+1}(P, \mathcal{G})$. Therefore, requirement (4.2) only considers financial risks. This view will be refined in Chapter 6.
- Note that viewed from time t_0, the values $\mathcal{E}_t[S]$ and $\mathcal{E}_t[\text{VaPo}]$ are random variables for $t = t_0+1, \ldots, n$. Therefore, we require the insurance contract condition (4.2) to hold with $P[\cdot|\mathcal{G}_{t_0}]$-probability 1 (below we write shortly P-a.s.). At the first sight this seems rather restrictive and in practice one often relaxes (4.2) to hold with high probability. However, we will see below how we can achieve our definition of solvency.
- In many solvency considerations the time interval under consideration is 1. This means that one assumes that the accounting condition needs to be fulfilled at time t_0 and that the insurance contract condition is fulfilled at time $t_0 + 1$ (this is the so-called one-year solvency view). After $t_0 + 1$ we iterate this one-year procedure with a new accounting condition at $t_0 + 1$ and so on (until the run-off of all liabilities is done). In the sequel, for the protection against financial risks, we will also take this point of view. Then the problem of solvency decouples into one-period problems (that need to be calculated recursively and involve multiperiod risk measures, see also Salzmann-Wüthrich [SW10]).

Task of financial risk management. If (4.1) is satisfied, how do we need to choose our asset portfolio S such that our company is solvent?

S is a dynamic portfolio, which can be restructured at any time $t = t_0, \ldots, n$.

a) Prudent solution. Choose S at time t_0 as follows

$$S = \text{VaPo} + F, \tag{4.3}$$

where VaPo is the replicating portfolio of the liabilities and F is the free reserve or excess capital which must satisfy $\mathcal{E}_t[F] \geq 0$ for all $t \geq t_0$, P-a.s. Hence solvency is guaranteed which, from a mathematical point of view, shows that solvency is possible.

b) Realistic situation.

- S does not (entirely) contain VaPo.
- ALM mismatch (between S and VaPo) is often wanted, because taking additional financial risks on the asset side of the balance sheet opens the possibility for receiving higher investment returns.
- This mismatch asks for additional protections against financial risks to achieve solvency. In fact regulators ask for a substantially increased target capital for the protection against financial risks. It turns out in the Swiss Solvency Test [SST06] that the financial risk is the dominant term for life insurance companies, whereas in a typical non-life insurance company the target capital for financial risks has about the same size as the target capital for insurance technical risks.

4.2 Procedure to control financial risks

As described on page 71 we decouple the solvency problem into one-period problems. For simplicity we only study the first accounting year $\{t_0, t_0 + 1\}$. We decompose our portfolio at the beginning t_0 of the accounting year into three parts

$$S = \widetilde{S} + M + F, \tag{4.4}$$

where $\widetilde{S}$ is any asset portfolio which satisfies the accounting condition (4.1), and M is a margin which is determined below, i.e.

$$\begin{array}{ll} \mathcal{E}_{t_0}\left[\text{VaPo}\right] = \mathcal{E}_{t_0}\left[\widetilde{S}\right] & \text{accounting condition,} \\ M & \text{margin,} \\ F & \text{free reserves, excess capital.} \end{array} \tag{4.5}$$

At the end $t_0 + 1$ of the accounting year (before adding additional insurance contracts to our balance sheet), we should be able to

(1) switch from $\widetilde{S} + M$ to VaPo if necessary,

(2) $\mathcal{E}_t[F]$ is not allowed to become negative for $t = t_0, t_0 + 1$, P-a.s.

I.e. the margin M is calculated such that we are able to switch from $\widetilde{S} + M$ to VaPo at the end of the accounting period $\{t_0, t_0 + 1\}$, if necessary.

Formalizing (1). A Margrabe option gives the right to exchange one asset for another. It is named after William Margrabe [Ma78].

Hence, in terms of financial instruments, M is chosen to be a Margrabe option that allows for switching from the asset portfolio $\widetilde{S}$ to the asset portfolio VaPo whenever

$$\mathcal{E}_{t_0+1}\left[\text{VaPo}\right] > \mathcal{E}_{t_0+1}\left[\widetilde{S}\right]. \tag{4.6}$$

This means that the decomposition (4.4) is chosen such that

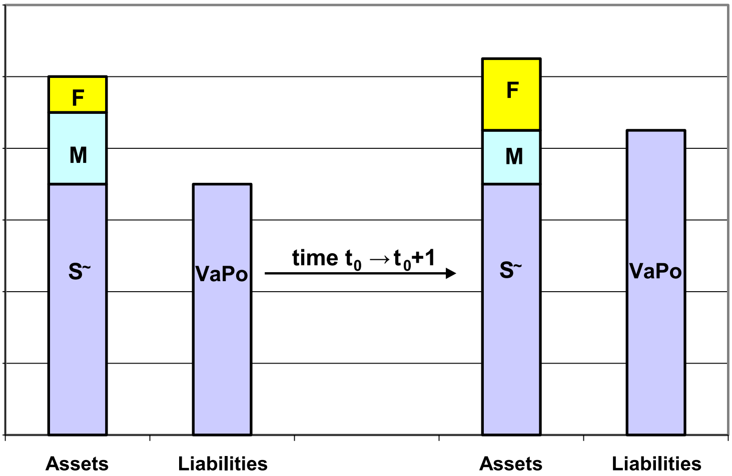


Fig. 4.2. Time evolution of the asset and liability portfolios

(1) $\mathcal{E}_{t_0}[\text{VaPo}] = \mathcal{E}_{t_0}\left[\widetilde{S}\right]$ (accounting condition);

(2) M allows for switching from $\widetilde{S} + M$ to VaPo whenever (4.6) holds at time $t_0 + 1$, note that $\mathcal{E}_t[M] \geq 0$ for $t = t_0, t_0 + 1$, P-a.s.;

(3) $\mathcal{E}_t[F] \geq 0$ for $t = t_0, t_0 + 1$, P-a.s.

In order to calculate the price of the Margrabe option, we consider in the sequel the two price processes generated by $\widetilde{S}$ and VaPo:

$$Y_t = \mathcal{E}_t\left[\widetilde{S}\right], \tag{4.7}$$

$$V_t = \mathcal{E}_t[\text{VaPo}]. \tag{4.8}$$

For explicit calculations it will be useful to consider a continuous time model $t \in [t_0, t_0 + 1]$ because this allows for applying classical financial mathematics like the geometric Brownian motion framework.

4.3 Financial modelling

4.3.1 Stochastic discounting of financial variables

In this subsection we recall the situation from Section 2.6. Our aim is to model the financial market $\mathcal{G}$, where throughout we assume Assumption 2.15 and that the probability distortion $\boldsymbol{\varphi}^{(\mathcal{T})}$ is a density process according to (2.103).

We choose a filtered probability space $(\Omega, \mathcal{G}_n, P, \mathcal{G})$ with financial filtration $\mathcal{G} = (\mathcal{G}_t)_{t=0,\dots,n}$. On this probability space we choose a fixed financial deflator

$\boldsymbol{\varphi}^{(\mathcal{G})} \in L^2_{n+1}(P, \mathcal{G})$, and we assume that the price processes of the financial instruments $\mathcal{U}_i$ satisfy $(U_t^{(i)})_{t=0,\ldots,n} \in L^2_{n+1}(P, \mathcal{G})$.

In order to have an economically meaningful pricing framework we obtain from Theorem 2.18 that the deflated price process $(\varphi_t^{(\mathcal{G})} U_t^{(i)})_{t=0,\ldots,n}$ has at time $s \le t$ value

$$E\left[\varphi_t^{(\mathcal{G})} \, U_t^{(i)} \middle| \mathcal{G}_s \right] = \varphi_s^{(\mathcal{G})} \, U_s^{(i)}. \tag{4.9}$$

This means that deflated price processes are $(\mathcal{G}, P)$-martingales which corresponds to the fundamental theorem of asset pricing, see Remarks 2.14.

If we denote by $\mathbf{Z}^{(t)} = (0, \ldots, 0, 1, 0, \ldots, 0) \in L^2_{n+1}(P, \mathcal{G})$ the cash flow of the zero coupon bond paying 1 at time t. Then the value $Z_s^{(t)}$ of the zero coupon bond at time $s \le t$ is given by (see (2.47), $Z_t^{(t)} = 1$)

$$\begin{aligned} Z_s^{(t)} = Q_s\left[\mathbf{Z}^{(t)}\right] &= \frac{1}{\varphi_s^{(\mathcal{G})}} \, E\left[\varphi_t^{(\mathcal{G})} \middle| \mathcal{G}_s \right] \\ &= \frac{1}{\varphi_s^{(\mathcal{G})}} \, E\left[\varphi_t^{(\mathcal{G})} \, Z_t^{(t)} \middle| \mathcal{G}_s \right] = E\left[\frac{\varphi_t^{(\mathcal{G})}}{\varphi_s^{(\mathcal{G})}} \middle| \mathcal{G}_s \right]. \end{aligned} \tag{4.10}$$

Remarks.

- We have seen in Lemma 2.8 and Theorem 2.18 that $(\varphi_s^{(\mathcal{G})} Z_s^{(t)})_{s=0,\ldots,t}$ forms an $\mathcal{G}$-martingale under P.
- At time 0 we have (see (2.28))

$$Z_0^{(t)} = Q_0\left[\mathbf{Z}^{(t)}\right] = E\left[\varphi_t^{(\mathcal{G})}\right] = D_{0,t}. \tag{4.11}$$

If we consider the equivalent martingale measure $P^* \sim P$ for the bank account numeraire B_t^{-1}, see Lemma 2.10 and Corollary 2.12, then we get discount factors that are known (measurable) at the beginning of the period under consideration. In particular, in view of (2.76), we have for the one-period model $t = 0, 1$

$$U_0^{(i)} = D_{0,1} \, E^*\left[U_1^{(i)}\right] = E\left[\varphi_1^{(\mathcal{G})} \, U_1^{(i)}\right], \tag{4.12}$$

with $D_{0,1} = B_1^{-1}$.

Exercise 4.1 (Pricing of financial assets).

We revisit the discrete time Vasicek model from Exercise 2.3. In all the assumptions and statements we replace the filtration $\mathbb{F}$ by the financial filtration $\mathcal{G}$.

Moreover, we assume that we have a two-dimensional process $(\varepsilon_t, \delta_t)_{t=0,\ldots,n}$ that is $\mathcal{G}$-adapted and $(\varepsilon_t, \delta_t)$ is independent of $\mathcal{G}_{t-1}$ with standard Gaussian distribution and correlation ϱ for $t = 1, \ldots, n$.

Assume that the financial asset $\mathcal{U}$ has price process given by $U_0 > 0$ (fixed) and for $t = 1, \ldots, n$

$$U_t = U_{t-1} \ \exp\left\{\mu_t - \sigma\delta_t\right\}, \tag{4.13}$$

for given $\mu_t \in \mathbb{R}$ and $\sigma > 0$.

Determine the necessary properties of $\mu_t \in \mathbb{R}$ so that the price process

$$(\varphi_t^{(\mathcal{G})} U_t^{(i)})_{t=0,\ldots,n} \qquad \text{is a } (\mathcal{G}, P)\text{-martingale}, \tag{4.14}$$

for the financial deflator $(\varphi_t^{(\mathcal{G})})_t$ given in (2.24).

Hint: use the properties of log-normal distributions.

□

4.3.2 Modelling Margrabe options

Recall definitions (4.7)-(4.8). It is often convenient to make another change of numeraire. Assume that $(V_t)_{t=0,\ldots,n} = (\mathcal{E}_t\left[\text{VaPo}\right])_{t=0,\ldots,n} \gg 0$, then the price process $(V_t)_{t=0,\ldots,n}$ may serve as a numeraire as follows: we define

$$\widetilde{Y}_t = \frac{Y_t}{V_t} = \frac{\mathcal{E}_t\left[\widetilde{S}\right]}{\mathcal{E}_t\left[\text{VaPo}\right]}, \tag{4.15}$$

these are the assets measured relative to the liabilities. The advantage of using $\widetilde{Y}_t$ is that the deflator disappears, since both expressions have the same time value. Growth of $\widetilde{Y}_t$ means that we have an extensive growth of the assets Y_t relative to the liabilities V_t.

In the sequel we identify the solvency problem in $\{t_0, t_0+1\}$ for which we would like to price the Margrabe option exercised at t_0+1. This is basically a one-period problem similar two (4.12). If we price the Margrabe option at time t_0, we need to model/calculate (see (2.71)) for the cash flow $\mathbf{X} = (0, \ldots, 0, (V_{t_0+1} - Y_{t_0+1})_+, 0, \ldots, 0)$:

$$\begin{aligned} Q_{t_0}[\mathbf{X}] &= \frac{1}{\varphi_{t_0}^{(\mathcal{G})}} \ E\left[\varphi_{t_0+1}^{(\mathcal{G})} \left(V_{t_0+1} - Y_{t_0+1}\right)_+ \middle| \mathcal{G}_{t_0}\right] \\ &= D(\mathcal{G}_{t_0}) \ E^*\left[\left(V_{t_0+1} - Y_{t_0+1}\right)_+ \middle| \mathcal{G}_{t_0}\right] \\ &= Z_{t_0}^{(t_0+1)} \ E^*\left[\left(V_{t_0+1} - Y_{t_0+1}\right)_+ \middle| \mathcal{G}_{t_0}\right] \\ &= V_{t_0} \ E^{**}\left[\left(1 - \widetilde{Y}_{t_0+1}\right)_+ \middle| \mathcal{G}_{t_0}\right], \end{aligned} \tag{4.16}$$

where the equivalent probability measure $P^{**} \sim P^*$ is defined by the density

$$dP^{**}(\cdot|\mathcal{G}_{t_0}) = Z_{t_0}^{(t_0+1)} \ \frac{V_{t_0+1}}{V_{t_0}} \ dP^*(\cdot|\mathcal{G}_{t_0}). \tag{4.17}$$

Observe that $Z_{t_0}^{(t_0+1)} \frac{V_{t_0+1}}{V_{t_0}}$ is (by assumption) strictly positive with probability 1 and, moreover, it is a density w.r.t. P^* because

$$E^{**}\left[1|\mathcal{G}_{t_0}\right] = E^*\left[Z_{t_0}^{(t_0+1)} \frac{V_{t_0+1}}{V_{t_0}} \middle| \mathcal{G}_{t_0} \right] = 1, \tag{4.18}$$

due to the martingale property of discounted price processes w.r.t. the equivalent martingale measure P^*. Hence from the right-hand side of (4.16) we need to model

$$V_{t_0}\ E^{**}\left[\left(1-\widetilde{Y}_{t_0+1}\right)_+ \middle| \mathcal{G}_{t_0} \right], \tag{4.19}$$

where prices are relative to the initial value $V_{t_0} = \mathcal{E}_{t_0}(\text{VaPo})$ of the valuation portfolio.

Example 4.2.

We assume that $\widetilde{Y}_{t_0+1} = \exp\{W\}$ has a log-normal distribution with parameters μ and σ^2, conditionally given $\mathcal{G}_{t_0}$, w.r.t. P^{**}. Since deflated price processes are martingales we obtain

$$\begin{aligned} Y_{t_0} &= Z_{t_0}^{(t_0+1)}\ E^*\left[Y_{t_0+1}\middle|\,\mathcal{G}_{t_0}\right] \\ &= V_{t_0}\ E^{**}\left[\widetilde{Y}_{t_0+1}\middle|\,\mathcal{G}_{t_0}\right] \\ &= V_{t_0}\ \exp\left\{\mu+\sigma^2/2\right\}. \end{aligned} \tag{4.20}$$

The accounting condition $Y_{t_0} = V_{t_0}$ then implies the drift condition satisfies $\mu = -\sigma^2/2$.

Henceforth, (4.19) is simply the price of a European put option for log-normal prices. We calculate

$$\begin{aligned} E^{**}\left[\left(1-\widetilde{Y}_{t_0+1}\right)_+\middle|\,\mathcal{G}_{t_0}\right] &= E^{**}\left[\left(1-\widetilde{Y}_{t_0+1}\right)1_{\{\widetilde{Y}_{t_0+1}\le 1\}}\middle|\,\mathcal{G}_{t_0}\right] \\ &= P^{**}\left[\widetilde{Y}_{t_0+1}\le 1\middle|\,\mathcal{G}_{t_0}\right] - E^{**}\left[\widetilde{Y}_{t_0+1}1_{\{\widetilde{Y}_{t_0+1}\le 1\}}\middle|\,\mathcal{G}_{t_0}\right] \\ &= P^{**}\left[W\le 0|\,\mathcal{G}_{t_0}\right] - \int_0^1 \frac{1}{\sqrt{2\pi}\sigma}\frac{1}{y}\ y\ \exp\left\{-\frac{1}{2}\frac{\left(\log y+\sigma^2/2\right)^2}{\sigma^2}\right\}dy \\ &= P^{**}\left[W\le 0|\,\mathcal{G}_{t_0}\right] - \int_0^1 \frac{1}{\sqrt{2\pi}\sigma}\frac{1}{y}\ \exp\left\{-\frac{1}{2}\frac{\left(\log y-\sigma^2/2\right)^2}{\sigma^2}\right\}dy \\ &= P^{**}\left[W\le 0|\,\mathcal{G}_{t_0}\right] - P^{**}\left[\widetilde{W}\le 0\middle|\,\mathcal{G}_{t_0}\right], \end{aligned} \tag{4.21}$$

with $W|\mathcal{G}_{t_0} \overset{P^{**}}{\sim} \mathcal{N}(-\sigma^2/2\ ,\ \sigma^2)$ and $\widetilde{W}|\mathcal{G}_{t_0} \overset{P^{**}}{\sim} \mathcal{N}(\sigma^2/2\ ,\ \sigma^2)$. This immediately implies

$$E^{**}\left[\left(1-\widetilde{Y}_{t_0+1}\right)_+\middle|\,\mathcal{G}_{t_0}\right] = \Phi(\sigma/2) - \Phi(-\sigma/2), \tag{4.22}$$

where $\Phi(\cdot)$ denotes the standard Gaussian distribution. Note that (4.22) is the Black-Scholes price for a European put option, see Lamberton-Lapeyre [LL91], Section 3.2.

We find the following relative loadings (depending on the volatility of the assets relative to the liabilities):

σ	price relative to V_{t_0}
0.05	1.99%
0.10	3.99%
0.20	7.97%
0.30	11.92%

□

4.3.3 Conclusions

We have decoupled the solvency problem into recursive one-period problems. To protect against financial risks one has to invest each year the price of the Margrabe option. This price measures the ALM mismatch between the real asset portfolio $\widetilde{S}$ and the liability portfolio VaPo.

The agents who are entitled to receive the earnings beyond the VaPo should also finance this option:

- With-profit policies share the price: Between the policyholder and the shareholder according to their participation.
- Non-participating policy: Shareholder has to pay the full price.

As the price of the Margrabe option is relative to the VaPo, we can easily make a similar calculation for the VaPo protected against insurance technical risks. And if the VaPo protected against insurance technical risks cannot be financed we need to a) have more capital, b) do better ALM, and/or c) reduce insurance technical risks.

4.4 Pricing Margrabe options

A first version of pricing Margrabe options has been given in (4.19). For pricing and hedging Margrabe options in general we go over to a continuous time model $t \in [0, n]$ for a fixed final time horizon $n \in \mathbb{N}$. This has the advantage that we can use classical financial mathematics. However, we restrict the continuous time setting to the present subsection.

We choose a filtered probability space $(\Omega, \mathcal{G}_n, P, \mathbb{G})$ with financial filtration $\mathbb{G} = (\mathcal{G}_t)_{t \in [0,n]}$ satisfying the usual conditions (completeness and right-continuity, see for example Filipović [Fi09], page 59) and $\mathcal{G}_0 = \{\emptyset, \Omega\}$.

The VaPo protected against financial risks will provide us each year with the price of a Margrabe option, which can be used to

1. buy the option (often not realistic),
2. hedge the option,
3. cover the cost of target capital TC.

This is discussed below.

Below, we describe pricing of financial instruments using Esscher transforms which are well-known in actuarial mathematics. This approach might be rather unusual in classical financial mathematics, but it is straightforward to actuaries who are familiar with the Esscher premium.

4.4.1 Pricing using Esscher transforms

To derive the price of a Margrabe option we closely follow the outline in Gerber-Shiu [GS94b].

Choose $\delta > 0$ fixed. We assume that our financial market consists of L financial assets $\mathcal{U}_1, \ldots, \mathcal{U}_L$ with strictly positive $\mathcal{G}$-adapted price processes. Assume that $U_t^{(1)}, \ldots, U_t^{(L)} > 0$ denote the (cum dividend) prices of these L assets $\mathcal{U}_1, \ldots, \mathcal{U}_L$ at time $t \in [0, n]$. We define the logarithmized price processes by

$$W_t^{(i)} = \log\left(\frac{U_t^{(i)}}{U_0^{(i)}}\right) \in \mathbb{R}, \tag{4.23}$$

for $i = 1, \ldots, L$ and $t \geq 0$. Moreover, for fixed $t \in [0, n]$

$$\mathbf{W}_t = \left(W_t^{(1)}, \ldots, W_t^{(L)}\right)^T \tag{4.24}$$

is a stochastic vector in $\mathbb{R}^L$ with distribution function

$$F(\mathbf{x}, t) = P\left[W_t^{(i)} \leq x_i,\ i = 1, \ldots, L\right] \tag{4.25}$$

for all $t \in [0, n]$ and $\mathbf{x} \in \mathbb{R}^L$.

We define the moment generating function of $\mathbf{W}_t$ as follows, for $\mathbf{z} \in \mathbb{R}^L$ and $t \in [0, n]$

$$M(\mathbf{z}, t) = E\left[\exp\left\{\mathbf{z}^T\ \mathbf{W}_t\right\}\right], \tag{4.26}$$

whenever it exists.

Assumptions. Assume that the stochastic process $\{\mathbf{W}_t\}_{t \in [0,n]}$ has stationary, independent increments and that, hence,

$$M(\mathbf{z}, t) = [M(\mathbf{z}, 1)]^t. \tag{4.27}$$

Moreover, we assume that $\mathbf{W}_t$ has a density for $t \in [0, n]$:

$$f(\mathbf{x}, t) = \frac{\partial^L}{\partial x_1 \cdots \partial x_L} F(\mathbf{x}, t). \tag{4.28}$$

□

The modified (probability distorted) density under the Esscher transform is for $\mathbf{h} \in \mathbb{R}^L$ defined as follows:

$$f(\mathbf{x}, t; \mathbf{h}) = \frac{\exp\left\{\mathbf{h}^T \ \mathbf{x}\right\} \ f(\mathbf{x}, t)}{M(\mathbf{h}, t)}, \tag{4.29}$$

the corresponding moment generating function is given by

$$M(\mathbf{z}, t; \mathbf{h}) = \frac{M(\mathbf{z} + \mathbf{h}, t)}{M(\mathbf{h}, t)}. \tag{4.30}$$

Define the transformed distribution function: $F_{\mathbf{h}}(\cdot, \cdot) = F(\cdot, \cdot; \mathbf{h})$, where $F(\cdot, \cdot; \mathbf{h})$ denotes the distribution function to the density $f(\cdot, \cdot; \mathbf{h})$. Then the Esscher transform of the process $\{\mathbf{W}_t\}_{t \in [0,n]}$ has again stationary, independent increments with

$$M(\mathbf{z}, t; \mathbf{h}) = [M(\mathbf{z}, 1; \mathbf{h})]^t . \tag{4.31}$$

Our goal is to choose $\mathbf{h}^* \in \mathbb{R}^L$ such that the (discounted) price processes

$$\left\{ e^{-\delta t} U_t^{(i)} \right\}_{t \in [0,n]} \tag{4.32}$$

are martingales w.r.t. $F^* = F_{\mathbf{h}^*}(\cdot, \cdot) = F(\cdot, \cdot; \mathbf{h}^*)$ and $\mathcal{G}$:
Choose $t > 0$ and $s \in [0, t]$ then

$$\begin{aligned} E^* \left[e^{-\delta t} U_t^{(i)} \middle| \mathcal{G}_s \right] &= e^{-\delta t} \ U_0^{(i)} \ E^* \left[\exp\left\{ W_t^{(i)} \right\} \middle| \mathcal{G}_s \right] \\ &= e^{-\delta t} \ U_0^{(i)} \ E^* \left[\exp\left\{ W_t^{(i)} - W_s^{(i)} + W_s^{(i)} \right\} \middle| \mathcal{G}_s \right] \\ &= e^{-\delta t} \ U_0^{(i)} \ \exp\left\{ W_s^{(i)} \right\} \ E^* \left[\exp\left\{ W_t^{(i)} - W_s^{(i)} \right\} \middle| \mathcal{G}_s \right] \\ &= e^{-\delta s} \ U_s^{(i)} \ e^{-\delta(t-s)} \ E^* \left[\exp\left\{ W_t^{(i)} - W_s^{(i)} \right\} \middle| \mathcal{G}_s \right]. \end{aligned} \tag{4.33}$$

Since we have stationary and independent increments, $\mathbf{h}^* \in \mathbb{R}^L$ must satisfy for all $s \leq t$

$$E^* \left[\exp\left\{ W_t^{(i)} - W_s^{(i)} \right\} \middle| \mathcal{G}_s \right] = E^* \left[\exp\left\{ W_{t-s}^{(i)} \right\} \right] = e^{\delta(t-s)}. \tag{4.34}$$

This implies that

$$\begin{aligned} e^{\delta(t-s)} &= E^* \left[\exp\left\{ W_{t-s}^{(i)} \right\} \right] \\ &= E_{\mathbf{h}^*} \left[\exp\left\{ W_{t-s}^{(i)} \right\} \right] \\ &= M(\mathbf{1}_i, t-s; \mathbf{h}^*) = [M(\mathbf{1}_i, 1; \mathbf{h}^*)]^{t-s} , \end{aligned} \tag{4.35}$$

where $\mathbf{1}_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T \in \mathbb{R}^L$. But this immediately implies the requirement

$$e^{\delta} = M(\mathbf{1}_i, 1; \mathbf{h}^*). \tag{4.36}$$

It can be shown (see Gerber-Shiu [GS94a]) that there is a unique solution $\mathbf{h}^* \in \mathbb{R}^L$, which satisfies (4.36) for all $i = 1, \ldots, L$. Hence the Esscher transformed measures give an equivalent martingale measure for the discounted price processes

$$\left\{ e^{-\delta t} U_t^{(i)} \right\}_{t \in [0,n]}. \tag{4.37}$$

Remarks.

- The parameter $\mathbf{h}^*$ is called the risk-neutral Esscher transform parameter and the corresponding equivalent martingale measure with marginals $F^* = F_{\mathbf{h}^*}(\cdot, \cdot) = F(\cdot, \cdot; \mathbf{h}^*)$ the *risk-neutral Esscher measure.*
- Since $\mathbf{h}^*$ is unique we have that the risk-neutral Esscher measure is unique. However, there may be other equivalent martingale measures, i.e. the market is not necessarily complete.
- The risk-neutral Esscher measure allows here for discounting with a constant interest rate numeraire $e^{-\delta t}$.

The following theorem is helpful for many problems in option pricing.

Theorem 4.4 *Let $g : \mathbb{R}^L \to \mathbb{R}$ be a measurable function. Then for all $t \in [0, n]$ we have the following identity*

$$E_{\mathbf{h}^*}\left[e^{-\delta t} U_t^{(i)} \; g\left(U_t^{(1)}, \ldots, U_t^{(L)} \right) \right] = U_0^{(i)} \; E_{\mathbf{h}^* + \mathbf{1}_i}\left[g\left(U_t^{(1)}, \ldots, U_t^{(L)} \right) \right]. \tag{4.38}$$

Proof. We consider (see (4.30))

$$\begin{aligned} \exp\{x_i\} \; f(\mathbf{x}, t; \mathbf{h}^*) &= \exp\left\{ \mathbf{x}^T \mathbf{1}_i \right\} \; f(\mathbf{x}, t; \mathbf{h}^*) \\ &= \frac{\exp\left\{ \mathbf{x}^T \left(\mathbf{h}^* + \mathbf{1}_i \right) \right\} \; f(\mathbf{x}, t)}{M(\mathbf{h}^*, t)} \\ &= f(\mathbf{x}, t; \mathbf{h}^* + \mathbf{1}_i) \; \frac{M(\mathbf{h}^* + \mathbf{1}_i, t)}{M(\mathbf{h}^*, t)} \\ &= f(\mathbf{x}, t; \mathbf{h}^* + \mathbf{1}_i) \; M(\mathbf{1}_i, t; \mathbf{h}^*). \end{aligned} \tag{4.39}$$

By the choice of $\mathbf{h}^*$ this last expression is equal to

$$e^{x_i} \; f(\mathbf{x}, t; \mathbf{h}^*) = e^{\delta t} \; f(\mathbf{x}, t; \mathbf{h}^* + \mathbf{1}_i). \tag{4.40}$$

But this immediately implies that

$$\begin{aligned} & E_{\mathbf{h}^*}\left[e^{-\delta t} U_t^{(i)} \; g\left(U_t^{(1)}, \ldots, U_t^{(L)} \right) \right] \\ &\quad = U_0^{(i)} \; E_{\mathbf{h}^*}\left[e^{-\delta t} \; e^{W_t^{(i)}} \; g\left(U_t^{(1)}, \ldots, U_t^{(L)} \right) \right] \\ &\quad = U_0^{(i)} \; E_{\mathbf{h}^* + \mathbf{1}_i}\left[g\left(U_t^{(1)}, \ldots, U_t^{(L)} \right) \right], \end{aligned} \tag{4.41}$$

which completes the proof.

□

This gives us the following corollary for the Esscher price of the Margrabe option.

Corollary 4.5 (Margrabe Option) *Assume* $L = 2$. *The Esscher price (using the risk-neutral Esscher measure) at time* 0 *of an option to exchange* $U_t^{(2)}$ *for* $U_t^{(1)}$ *at time* $t \in [0, n]$ *is*

$$U_0^{(1)} \; P_{\mathbf{h}^*+\mathbf{1}_1}\left[U_t^{(1)} > U_t^{(2)}\right] - U_0^{(2)} \; P_{\mathbf{h}^*+\mathbf{1}_2}\left[U_t^{(1)} > U_t^{(2)}\right]. \tag{4.42}$$

Proof. The value of the option at time 0 is

$$\begin{aligned}
&E_{\mathbf{h}^*}\left[e^{-\delta t}\;\left(U_t^{(1)} - U_t^{(2)}\right)_+\right] \qquad (4.43)\\
&= E_{\mathbf{h}^*}\left[e^{-\delta t}\;\left(U_t^{(1)} - U_t^{(2)}\right)\; 1_{\{U_t^{(1)} > U_t^{(2)}\}}\right]\\
&= E_{\mathbf{h}^*}\left[e^{-\delta t}\; U_t^{(1)}\; 1_{\{U_t^{(1)} > U_t^{(2)}\}}\right] - E_{\mathbf{h}^*}\left[e^{-\delta t}\; U_t^{(2)}\; 1_{\{U_t^{(1)} > U_t^{(2)}\}}\right]\\
&= U_0^{(1)}\; E_{\mathbf{h}^*+\mathbf{1}_1}\left[1_{\{U_t^{(1)} > U_t^{(2)}\}}\right] - U_0^{(2)}\; E_{\mathbf{h}^*+\mathbf{1}_2}\left[1_{\{U_t^{(1)} > U_t^{(2)}\}}\right].
\end{aligned}$$

This completes the proof of the corollary.

□

Remark. Theorem 4.4 can also be used to price a European call option with strike K at time τ: Assume that $(U_t^{(1)})_{t\in[0,n]}$ is the price process of the underlying asset and define for $t \in [0, n]$ the process $U_t^{(2)} = K\; e^{\delta(t-\tau)}$. $U_t^{(2)}$ describes the price process of an initial investment of $Ke^{-\delta\tau}$ into the numeraire asset. Then the Esscher price of the European call can be calculated from Theorem 4.4 using the function $g(x_1, x_2) = (x_1 - x_2)_+$.

4.4.2 Application of the Esscher transform to the multi-dimensional Wiener process

So far, the underlying process $\{\mathbf{W}_t\}_{t\in[0,n]}$ is a general process having stationary and independent increments with (4.27) fulfilled and such that the densities exist. Is there any process satisfying these assumptions?

We now choose a specific underlying process for the price processes $\mathbf{U}_t = (U_t^{(1)}, \ldots, U_t^{(L)})$, $t \in [0, n]$. Assume that $\mathbf{W}_t = \log(\mathbf{U}_t/\mathbf{U}_0)$ is described by a $\mathcal{G}$-adapted multidimensional Wiener process with non-singular covariance matrix Σ and mean parameter $\boldsymbol{\mu} \in \mathbb{R}^L$ (see (4.23)-(4.24)). Henceforth, $\mathbf{W}_t$ has for $t \in (0, n]$ density given by

$$f(\mathbf{x}, t) = \frac{1}{(2\pi)^{L/2}\; |t\Sigma|^{1/2}}\; \exp\left\{-(\mathbf{x} - t\boldsymbol{\mu})^T (2t\Sigma)^{-1} (\mathbf{x} - t\boldsymbol{\mu})\right\}. \tag{4.44}$$

The moment generating function is then given by ($\mathbf{z} \in \mathbb{R}^L$)

$$M(\mathbf{z},t) = E\left[\exp\left\{\mathbf{z}\ \mathbf{W}_t\right\}\right] = \exp\left\{t\ \left[\mathbf{z}^T\boldsymbol{\mu} + \mathbf{z}^T\Sigma\mathbf{z}/2\right]\right\}, \tag{4.45}$$

and for $\mathbf{h} \in \mathbb{R}^L$ we obtain

$$\begin{aligned} M(\mathbf{z},t;\mathbf{h}) &= \frac{M(\mathbf{z}+\mathbf{h},t)}{M(\mathbf{h},t)} \\ &= \exp\left\{t\ \left[(\mathbf{z}+\mathbf{h})^T\boldsymbol{\mu} + (\mathbf{z}+\mathbf{h})^T\Sigma(\mathbf{z}+\mathbf{h})/2\right]\right\} \\ &\quad \cdot \exp\left\{-t\ \left[\mathbf{h}^T\boldsymbol{\mu} + \mathbf{h}^T\Sigma\mathbf{h}/2\right]\right\} \\ &= \exp\left\{t\ \left[\mathbf{z}^T\ (\boldsymbol{\mu} + \Sigma\mathbf{h}) + \mathbf{z}^T\Sigma\mathbf{z}/2\right]\right\}. \end{aligned} \tag{4.46}$$

Henceforth, the Esscher transform of an L-dimensional Wiener process is again an L-dimensional Wiener process with modified mean vector $\boldsymbol{\mu} \mapsto \boldsymbol{\mu} + \Sigma\mathbf{h}$ and **unchanged** covariance matrix Σ.

Equation (4.36) implies that for all $i = 1, \dots, L$

$$\delta = \mathbf{1}_i^T\ (\boldsymbol{\mu} + \Sigma\mathbf{h}^*) + \mathbf{1}_i^T\Sigma\mathbf{1}_i/2, \tag{4.47}$$

which gives us (if we bring to last term to the other side)

$$\boldsymbol{\mu} + \Sigma\mathbf{h}^* = (\delta - \sigma_{1,1}/2, \dots, \delta - \sigma_{L,L}/2)\,. \tag{4.48}$$

This implies by adding $\Sigma\mathbf{1}_i$

$$\boldsymbol{\mu} + \Sigma(\mathbf{h}^* + \mathbf{1}_i) = (\delta + \sigma_{1,i} - \sigma_{1,1}/2, \dots, \delta + \sigma_{L,i} - \sigma_{L,L}/2)\,. \tag{4.49}$$

Note that the right-hand side of (4.48)-(4.49) is independent of $\boldsymbol{\mu}$.

If we apply Corollary 4.5 to the 2-dimensional Wiener process $\mathbf{W}_t = (W_t^{(1)}, W_t^{(2)})$, $t \in [0,n]$, we obtain

$$\begin{aligned} & E_{\mathbf{h}^*}\left[e^{-\delta t}\ \left(U_t^{(1)} - U_t^{(2)}\right)_+\right] \\ &= U_0^{(1)}\ P_{\mathbf{h}^*+\mathbf{1}_1}\left[U_t^{(1)} > U_t^{(2)}\right] - U_0^{(2)}\ P_{\mathbf{h}^*+\mathbf{1}_2}\left[U_t^{(1)} > U_t^{(2)}\right] \\ &= U_0^{(1)}\ P_{\mathbf{h}^*+\mathbf{1}_1}\left[W(t) < \zeta\right] - U_0^{(2)}\ P_{\mathbf{h}^*+\mathbf{1}_2}\left[W(t) < \zeta\right], \end{aligned} \tag{4.50}$$

with $\zeta = \log(U_0^{(1)}/U_0^{(2)})$ and $W(t) = W_t^{(2)} - W_t^{(1)}$. $W(1)$ has the following distributions, using (4.49),

$$\mathcal{N}\left(-\sigma_{1,1}/2 + \sigma_{1,2} - \sigma_{2,2}/2\ ,\ \sigma_{1,1} - 2\sigma_{1,2} + \sigma_{2,2}\right) \quad \text{under } P_{\mathbf{h}^*+\mathbf{1}_1}, \tag{4.51}$$

$$\mathcal{N}\left(\sigma_{1,1}/2 - \sigma_{1,2} + \sigma_{2,2}/2\ ,\ \sigma_{1,1} - 2\sigma_{1,2} + \sigma_{2,2}\right) \quad \text{under } P_{\mathbf{h}^*+\mathbf{1}_2}. \tag{4.52}$$

With $\mathcal{N}(\mu, \sigma^2)$ we denote the one-dimensional Gaussian distribution with mean μ and variance σ^2, $\Phi(\cdot)$ denotes the standard Gaussian distribution ($\mu = 0$ and $\sigma^2 = 1$) and φ its density.

Let us define

$$v^2 = \sigma_{1,1} - 2\sigma_{1,2} + \sigma_{2,2} = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2, \tag{4.53}$$

with $\sigma_1^2 = \sigma_{1,1}$, $\sigma_2^2 = \sigma_{2,2}$ and $\rho = \sigma_{1,2}/(\sigma_1\sigma_2)$. Then we see that $W(t)$ has the following distributions

$$\mathcal{N}\left(-v^2t/2, tv^2\right) \quad \text{under } P_{\mathbf{h}^*+\mathbf{1}_1}, \tag{4.54}$$

$$\mathcal{N}\left(v^2t/2, tv^2\right) \quad \text{under } P_{\mathbf{h}^*+\mathbf{1}_2}. \tag{4.55}$$

So we immediately have the next corollary, which gives the price of the Margrabe option at time 0 under the risk-neutral Esscher measure for a 2-dimensional Wiener process:

Corollary 4.6 (Margrabe Option for Wiener process) *The price of the Margrabe option at time 0 for an exercise at time t is given by*

$$E_{\mathbf{h}^*}\left[e^{-\delta t}\ \left(U_t^{(1)} - U_t^{(2)}\right)_+\right] \tag{4.56}$$

$$= U_0^{(1)}\ \Phi\left(\frac{\zeta + v^2t/2}{vt^{1/2}}\right) - U_0^{(2)}\ \Phi\left(\frac{\zeta - v^2t/2}{vt^{1/2}}\right).$$

If we come back to (4.4) modified to a continuous time setup: We define the price process of the Margrabe option by M_t, $t \in [t_0, t_0+1]$, to exchange $\tilde{S}$ with the valuation portfolio VaPo at time t_0+1 whenever (4.6). We choose

$$U_t^{(1)} = V_t = \mathcal{E}_t\left[\text{VaPo}\right] \qquad \text{and} \qquad U_t^{(2)} = Y_t = \mathcal{E}_t\left[\widetilde{S}\right], \tag{4.57}$$

then

$$\zeta_t = \log\left(\frac{U_t^{(1)}}{U_t^{(2)}}\right) = \log\left(\frac{V_t}{Y_t}\right) = -\log \widetilde{Y}_t. \tag{4.58}$$

Hence the price process of the Margrabe option is given by (see Corollary 4.6)

$$M_t = V_t\ \left[\Phi\left(\frac{\zeta_t + v_t^2/2}{v_t}\right) - e^{-\zeta_t}\ \Phi\left(\frac{\zeta_t - v_t^2/2}{v_t}\right)\right] \tag{4.59}$$

$$= Y_t\ \left[e^{\zeta_t}\ \Phi\left(\frac{\zeta_t + v_t^2/2}{v_t}\right) - \Phi\left(\frac{\zeta_t - v_t^2/2}{v_t}\right)\right],$$

where $v_t^2 = v^2\ (t_0 + 1 - t)$, which implies at $t = t_0$

$$M_{t_0} = V_{t_0}\ \left[\Phi\left(v/2\right) - \Phi\left(-v/2\right)\right], \tag{4.60}$$

see also (4.22) and again corresponds to the Black-Scholes formula for geometric Brownian motion.

4.4.3 Hedging Margrabe options

We have seen that the price process of the Margrabe option M_t for 2-dimensional Wiener processes is given by (4.59). We consider now the price process $Y_t + M_t$ which allows for switching from $\widetilde{S}$ to VaPo in $t \in [t_0, t_0 + 1]$. Since in practice, we are not able to buy such a Margrabe option, we need to use a hedging strategy to protect ourselves against financial losses from the ALM mismatch.

Define the function

$$H(t,x) = x\ \Phi\left(\frac{\log x + v_t^2/2}{v_t}\right) - \Phi\left(\frac{\log x - v_t^2/2}{v_t}\right). \tag{4.61}$$

Then we have

$$M_t = Y_t\ H(t, e^{\zeta_t}) = Y_t\ H\left(t, \widetilde{Y}_t^{-1}\right) \tag{4.62}$$

and with Itô calculus one sees that we have to study

$$\begin{aligned} &\frac{\partial}{\partial x} H(t,x) \qquad (4.63)\\ &= \Phi\left(\frac{\log x + v_t^2/2}{v_t}\right) + \varphi\left(\frac{\log x + v_t^2/2}{v_t}\right)/v_t - \varphi\left(\frac{\log x - v_t^2/2}{v_t}\right)/(x v_t).\\ &= \Phi\left(\frac{\log x + v_t^2/2}{v_t}\right), \end{aligned}$$

which is a well-known expression for the European call option in the Black-Scholes model (see e.g. Remarque 3.6 in Lamberton-Lapeyre [LL91] on p. 79).

Hence for the hedging strategy $\psi = (\widetilde{\lambda}_t, \lambda_t)$ we obtain the following natural candidate (see e.g. Section 3.3 in Lamberton-Lapeyre [LL91]): Invest

$$\widetilde{\lambda}_t = \left.\frac{\partial}{\partial x} H(t,x)\right|_{x = e^{\zeta_t} = \widetilde{Y}_t^{-1}} = \Phi\left(\frac{\zeta_t + v_t^2/2}{v_t}\right) \tag{4.64}$$

into the asset V_t and

$$\lambda_t = 1 - \Phi\left(\frac{\zeta_t - v_t^2/2}{v_t}\right) \tag{4.65}$$

into asset Y_t.

Hence the value of our portfolio is at any time t

$$\begin{aligned} \widetilde{\lambda}_t\ V_t + \lambda_t\ Y_t &= V_t\ \left[\widetilde{\lambda}_t + e^{-\zeta_t}\ \lambda_t\right] \qquad (4.66)\\ &= V_t\ \left[\Phi\left(\frac{\zeta_t + v_t^2/2}{v_t}\right) + e^{-\zeta_t}\ \left(1 - \Phi\left(\frac{\zeta_t - v_t^2/2}{v_t}\right)\right)\right]\\ &= e^{-\zeta_t}\ V_t + M_t = Y_t + M_t, \end{aligned}$$

which means that we can switch to the VaPo at any time $t \in [t_0, t_0 + 1]$.

Remark. The choice (4.64) is done since it reflects the relative change of the value of the Margrabe option as a function of time.

Example 4.3.

We choose $v = 0.05$ as in Example 4.2. This immediately implies that the price of the Margrabe option is $M_{t_0} = 2\% \cdot V_{t_0}$. This leads for a specific realization of $\widetilde{Y}_t$ to the following development of the price process: see Figure 4.3.

If we plot the process for three different realizations of the $\widetilde{Y}_t$ we obtain a picture as shown in Figure 4.4.

Observe that the path of $M_t + Y_t$ never falls below V_t, i.e. we have full financial coverage of all liabilities during the whole investment period.

□

v 0.05

Y~=Y/V	lambda*Y	lambda~*V	M+Y	lambda units of Y	lambda units of V
1.0000	51.0%	51.0%	102.00%	51.0%	51.0%
1.0077	51.4%	51.0%	102.39%	57.3%	44.6%
1.0215	58.5%	44.6%	103.12%	68.8%	32.9%
1.0295	70.8%	32.9%	103.66%	75.6%	25.8%
1.0100	76.3%	25.8%	102.13%	60.4%	41.2%
1.0423	63.0%	41.2%	104.14%	86.5%	14.3%
1.0443	90.4%	14.3%	104.68%	89.3%	11.3%
1.0192	91.0%	11.3%	102.38%	72.8%	28.3%
1.0252	74.6%	28.3%	102.92%	81.0%	19.8%
1.0062	81.5%	19.8%	101.30%	60.2%	40.7%
0.9901	59.7%	40.7%	100.37%	31.7%	69.1%
0.9941	31.5%	69.1%	100.54%	34.4%	66.2%
0.9845	33.8%	66.2%	100.00%	0.0%	100.0%

Fig. 4.3. One realization of $\widetilde{Y}_t$ with its associated value process $M_t + Y_t$

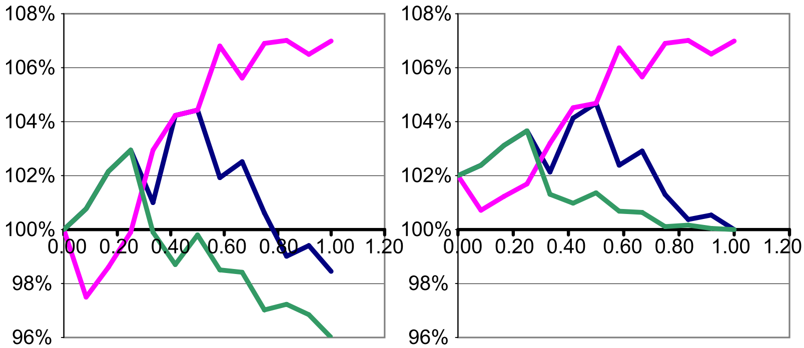


Fig. 4.4. We have plotted three different realizations of $\widetilde{Y}$ and the corresponding value processes $M_t + Y_t$

Conclusions. If we cannot buy the Margrabe option we need to hedge for switching into the VaPo at the end of each year.

The hedging strategy can be made cheaper, if we only hedge for switching at the point when the money is needed → the price of the Margrabe option becomes cheaper.

(a) Money is needed in n years. Switch at the end of every year. Henceforth, the price is

$$n \;\; \text{price Margrabe option}(\sigma). \tag{4.67}$$

(b) Money is needed in n years. Switch at the end of the period. Henceforth, the price is

$$\text{price Margrabe option}(\sqrt{n}\,\sigma). \tag{4.68}$$

Roughly speaking: approach (a) corresponds to a yearly guarantee whereas approach (b) only corresponds to a final wealth guarantee. Therefore it is clear that approach (a) needs to be more expensive.

Observe also, that in case (a) we may take out $(Y_t - V_t)_+$ every year, whereas in case (b) these profits must be left in the risk process.

4.4.4 Target capital

An alternative to controlling financial risk by Margrabe options uses the so-called target capital to absorb the fluctuations caused by financial risk. The target capital is obtained as follows. If our risk measure is Value-at-Risk, we choose q_{t_0} such that for given small $\varepsilon > 0$

$$\begin{aligned} P\left[(1+q_{t_0})\,Y_{t_0+1} \geq V_{t_0+1}\middle|\,\mathcal{G}_{t_0}\right] &= P\left[(1+q_{t_0})\,\widetilde{Y}_{t_0+1} \geq 1\middle|\,\mathcal{G}_{t_0}\right] \\ &\geq 1-\varepsilon, \end{aligned} \tag{4.69}$$

i.e. only with small probability ε we have a shortfall which cannot be financed by the target capital, which says that adverse scenarios that are not completely covered by the target capital need to have a probability of at most ε.

From the theoretical point of view, this is not an ideal solution, but it is the solution, which is (at the moment) applied in many practical solvency applications like Solvency 2 and the Swiss Solvency Test [SST06].

Under the simple standard model from last subsection, we obtain for the log-normal distribution $\log \widetilde{Y}_{t_0+1}|\mathcal{G}_{t_0} \sim \mathcal{N}(\mu, \sigma^2)$

$$P\left[\log \widetilde{Y}_{t_0+1} \geq -\log(1+q_{t_0})\middle|\,\mathcal{G}_{t_0}\right] = 1-\varepsilon. \tag{4.70}$$

Henceforth, q_{t_0} is given by

$$-\log(1+q_{t_0}) = \sigma\,\Phi^{-1}(\varepsilon) + \mu. \tag{4.71}$$

Example 4.4.

We choose the following example: assume that we can choose our standard deviation parameter $\sigma = \upsilon$ and the default probability $\varepsilon > 0$. The default probability is chosen such that it matches to the Standard & Poors ratings. The result is shown in Figure 4.5. Observe that for the target capital we also

Standard & Poors

Rating	default prob.	normal quantile	sigma mu	0.05 0.012	0.1 0.024	0.15 0.036	0.2 0.048
AAA	0.01%	-3.72		19.0%	41.6%	68.5%	100.5%
AA	0.03%	-3.43		17.3%	37.6%	61.4%	89.3%
A	0.07%	-3.19		15.9%	34.4%	55.8%	80.6%
BBB	0.18%	-2.91		14.3%	30.6%	49.3%	70.6%
BB	1.08%	-2.30		10.8%	22.8%	36.2%	50.9%
B	6.41%	-1.52		6.6%	13.7%	21.2%	29.2%
B-	11.61%	-1.19		4.9%	10.0%	15.4%	21.0%
Price Margrabe Option				2.0%	4.0%	6.0%	8.0%

Fig. 4.5. Target capital calculation depending on the choice of ε, μ and σ

need to specify the expected return μ. In discussions with economists and in the developments of the Swiss Solvency Test it has turned out that it is highly non-trivial to estimate μ for the different asset classes. For example, for the Swiss Solvency Test 2005, even experts have had so different opinions about estimations of μ that at the end one has put the expected investment return equal to the risk-free rate. But Example 4.4 shows that for the target capital calculation it only makes sense to consider the expected return and the expected volatility simultaneously. Higher expected returns will increase the uncertainty because one needs to invest into more risky assets. This example also shows that the choice of the asset portfolio is more crucial than the choice of the security level ε.

□

5

Valuation portfolio in non-life insurance

5.1 Introduction

To illustrate the problem we assume that we have a non-life insurance contract, which protects the policyholder against claims within a fixed calender year, see Figure 5.1. Assume that we receive a premium Π at the beginning of this calender year. Hence the policyholder exchanges the premium Π against a contract, which gives him a cover against well-specified random events (claims) occurring within a fixed time period.

Assume that we have a claim within this fixed time period. In that case the insurance company will replace the financial damage caused by that claim (according to the insurance contract).

In general, the insurance company is not able to assess the claim immediately at the occurrence date due to:

1. Usually, there is a reporting delay (time gap between claim occurrence and claim reporting to the insurance company). This time gap can be small (a few days), for example, in motor hull insurance, but it can also be quite large (months or years). Especially, in general liability insurance we can have large reporting delays: typical examples are asbestos claims that were caused several years ago but are only noticed and reported today.
2. Usually it takes quite some time to settle a claim (time difference between reporting date and settlement date). This is due to several different reasons, for example, for bodily injury claims we first have to observe the recovery process before finally deciding on the claim and on the compensation, or other claims can only be settled at court which usually takes quite some time until the final settlement. In most cases a (more complex) claim is settled by several payments X_k $(k \geq 1)$: Whenever a bill for that specific claim comes it is paid by the insurance company.

M.V. Wüthrich et al., *Market-Consistent Actuarial Valuation*, EAA Series,
DOI 10.1007/978-3-642-14852-1_5,

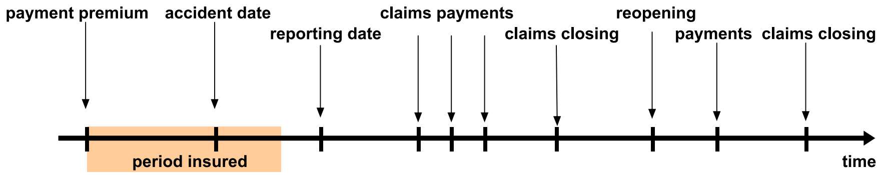


Fig. 5.1. Claims development process

Assume that a contract (or a portfolio of contracts) generates a cash flow

$$\mathbf{X} = (X_0, \ldots, X_N), \tag{5.1}$$

where X_k denotes the payments at time/in period $k \in \mathbb{N}$ ($X_k = 0$ if there is no payment at time k), and N is the (random) number of payments, i.e. the last payment takes place at time/in period N.

Remarks.

- In general, non-life insurance payments are done continuously over time. For modelling however, we choose a yearly grid $k = 0, 1, 2, \ldots,$ and we map all payments within accounting year $(k, k+1]$ to its endpoint $k+1$, that is, X_{k+1} will denote the payments within accounting year $(k, k+1]$.
- The settlement date is random for non-life insurance claims and therefore also the time point N of the last payment is random. In our case, we assume that $n \in \mathbb{N}$ is sufficiently large, such that $N \leq n$, P-a.s.

With these remarks we set

$$X_0 = -\Pi, \qquad \text{premium paid at the beginning of the contract,} \tag{5.2}$$

$$X_k, \quad k \in \{1, \ldots, n\}, \quad \text{nominal claims payments in period } (k-1, k]. \tag{5.3}$$

We denote cumulative nominal claims payments until time $k \in \{1, \ldots, n\}$ by

$$C_k = \sum_{j=1}^{k} X_j, \tag{5.4}$$

henceforth the ultimate loss/claim is given by $C_N = C_n$.

We choose a filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F})$ and assume that the insurance liability cash flow satisfies $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$.

Problems in practice.

(1) Predict the ultimate claim amount C_n for given information $\mathcal{F}_k$ at time k. This is in general a very difficult problem, which is known under the name "claims reserving problem". It is not further discussed here, but there is a vast literature on the claims reserving problem. For a reference we refer to Wüthrich-Merz [WM08].

(2) Split the total ultimate claim C_n into the different (annual) payments $X_1, \ldots, X_n$, i.e. estimate a payout/cash flow pattern for C_n.

A first idea is to directly predict the single cash flows X_i, $i \in \{k+1, \ldots, n\}$, given $\mathcal{F}_k$. But long time experience indicates that this approach does in most cases not lead to reasonable estimates and predictions for the total claim amount C_n. Therefore one usually proceeds by (1) and then (2), as above. Usually, the following information is used for these predictions:

For (1): paid claims experience, incurred losses experience, claims handling directives, financial parameters, other external knowledge and expert opinion.

For (2): paid claims experience to split C_n into the different periods.

In this lecture we restrict ourselves to *paid claims* data also for (1). In practice, of course, we would take into account any other additional information. This means that in this lecture we assume that the insurance technical information $\mathcal{T}$ is basically generated by the payments $\mathbf{X}$ (after "subtracting" the financial information $\mathcal{G}$, e.g. readjusting for inflation).

To construct the VaPo in non-life insurance we proceed as in Chapter 3 with the two steps:

Step 1. Choose an appropriate basis $\mathcal{U}_1, \mathcal{U}_2, \ldots,$ of financial instruments:

- zero coupon bonds $Z^{(t)}$ paying 1 at time $t \in \{0, \ldots, n\}$, or
- inflation protected zero coupon bonds,
- etc.

Step 2. Determine the number of units $\Lambda_i(\mathbf{X}_k)$, $l_{i,k}$ and $l^*_{i,k}$, respectively, of financial instrument $\mathcal{U}_i$ we need to reserve in order to meet all our future obligations (which are covered by past premium), see Section 3.6.

Questions:
How should we determine our future liabilities of old contracts (outstanding loss liabilities)? We are at time t: How should we reserve? How should we construct the VaPo?

Assumption 5.1

We assume that the appropriate financial basis is given by the zero coupon bonds $Z^{(t)}$, $t = 0, \ldots, n$, i.e. we assume that the price processes $(Z^{(t)}_s)_{s=0,\ldots,n} \in L^2_{n+1}(P)$ of the zero coupon bonds are independent of the random cash flow $\mathbf{X} \in L^2_{n+1}(P)$ (which generates in our case the insurance technical information $\mathcal{T}$).

□

For a comment on Assumption 5.1 we refer to Remark 5.2.

Assumption 5.1 implies that we will work on a product probability space.

1. The financial market is modelled with $(\Omega, \mathcal{G}_n, P_{\mathcal{G}}, \mathbb{G})$ and all financial instruments $\mathcal{U}_i$ will have price processes $(U_t^{(i)})_{t=0,\ldots,n} \in L^2_{n+1}(P_{\mathcal{G}}, \mathbb{G})$.
2. The insurance technical events are modelled on $(\Omega, \mathcal{T}_n, P_{\mathcal{T}}, \mathbb{T})$. The filtration $\mathbb{T}$ will be generated by a square integrable cash flow $\mathbf{X} \in L^2_{n+1}(P_{\mathcal{T}})$, i.e. $\mathcal{T}_t = \sigma\{X_0, \ldots, X_t\}$.
3. The product space is then denoted by $(\Omega, \mathcal{F}, P, \mathbb{F})$.

Remark. Under Assumption 5.1 we know that the ZCB $Z^{(k)}$ is the right financial instrument for cash flow X_k that leads to the independent decoupling according to Assumption 2.15. This then immediately implies that (see (2.93))

$$\mathbf{X} = \mathbf{\Lambda} = (\Lambda_0, \ldots, \Lambda_n) \in L^2_{n+1}(P_{\mathcal{T}}, \mathbb{T}). \tag{5.5}$$

(Random) cash flow after time $t < n$.

period	instrument	cash flow		number of units
$t+1$	$Z^{(t+1)}$	X_{t+1}	$\longrightarrow$	$l_{t+1}^{(t)}$
$t+2$	$Z^{(t+2)}$	X_{t+2}	$\longrightarrow$	$l_{t+2}^{(t)}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$t+k$	$Z^{(t+k)}$	X_{t+k}	$\longrightarrow$	$l_{t+k}^{(t)}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
n	$Z^{(n)}$	X_n	$\longrightarrow$	$l_n^{(t)}$

Task: Replace the stochastic cash flows X_{t+k} by $\mathcal{F}_t$-measurable numbers $l_{t+k}^{(t)}$ (where the upper index denotes the information that was used to determine the number of units). In the notation of Chapter 3 we denote the units as follows

$$\mathcal{U}_i = Z^{(i)}, \quad i = t+1, \ldots, n. \tag{5.6}$$

Then the number of units $\mathcal{U}_i$ at time t are given by (VaPo without protection against insurance technical risks, see (3.64))

$$\sum_{k=0}^{n} l_{i,k}^{(t)} = \sum_{k=0}^{n} E\left[\Lambda_i(\mathbf{X}_k)|\mathcal{T}_t\right] = E\left[\Lambda_i(\mathbf{X}_i)|\mathcal{T}_t\right] = E\left[X_i|\mathcal{T}_t\right] = l_{i,i}^{(t)}, \tag{5.7}$$

where we have used (5.5). Hence we use the following abbreviations

$$l_i = l_i^{(t)} = l_{i,i}^{(t)} \qquad \text{and} \qquad l_i^* = l_i^{*,t} = l_{i,i}^{*,t}, \tag{5.8}$$

where l_i^* is the number of units used for the valuation portfolio protected against insurance technical risks, that is, if we use probability distortions $\boldsymbol{\varphi}^{(\mathcal{T})} \in L^2_{n+1}(P, \mathbb{T})$ satisfying (2.103), then

$$l_i^* = \frac{1}{\varphi_t^{(\mathcal{T})}}\, E\left[\varphi_i^{(\mathcal{T})}\, X_i \middle| \mathcal{T}_t\right]. \tag{5.9}$$

Below we give an explicit construction for the choice of l_i and l_i^*.

Remark 5.2

- The zero coupon bonds $Z^{(t+k)}$ are the financial basis, which represent our cash flow, valuation portfolio, respectively. The choice of the basis was rather obvious in life insurance. In non-life insurance this is one of the crucial, non-trivial steps: find a decoupling such that the price processes of the units $\mathcal{U}_i$ and the number of units are independent.
  Indeed, the nominal payments X_i may depend on the state of the economy, on the job market, on the financial market, etc. (but also on the line of business we have chosen). Therefore we would need an inflation protected zero coupon bond which reflects what kind of business we write, and how this business is correlated with the economy and the financial market (immunization against financial risks).
- The mapping $X_{t+k} \mapsto l_{t+k}$ is considered both with and without protection (margin) for insurance technical risks, hence as in the life insurance chapter we have either l_{t+k} and l_{t+k}^*, see also (5.9). l_{t+k}^* exactly describes against which shortfalls the insurance company provides protection.
- The mapping $X_{t+k} \mapsto l_{t+k}$ should also incorporate that actual information considered, hence as in (3.64)-(3.65), we have $l_{t+k} = l_{t+k}^{(t)}$ and $l_{t+k}^* = l_{t+k}^{*,t}$. These depend on the available information $\mathcal{T}_t$ at time t, which in our case is generated by X_k, $k \leq t$. But in practice we would, of course, include any information available at time t.
- The financial risk is treated exactly in the same way as the ALM risk of the life insurance VaPo. Therefore we will no further address this problem here and refer to Chapter 4.

5.2 Construction of the VaPo in non-life insurance

In the sequel we work under Assumptions 5.1 and 2.15 for a given deflator $\varphi \in L^2_{n+1}(P, \mathbb{F})$. Moreover, we assume that the probability distortion $\varphi^{(\mathcal{T})} \in L^2_{n+1}(P, \mathcal{T})$ is a density process according to (2.103).

Then, we define for $k = t, \ldots, n$

$$E_k^{(t)} = E\left[X_k \middle| \mathcal{T}_t \right], \tag{5.10}$$

$$V_k^{(t)} = \operatorname{Var}\left(X_k \middle| \mathcal{T}_t \right). \tag{5.11}$$

$E_k^{(t)}, E_k^{(t+1)}, \ldots$ denotes the sequence of successive best-estimate predictions (minimum variance forecasts) for X_k (conditional expectations). Moreover, the sequence forms a martingale, which means that the sequence has uncorrelated increments. This is important if one works with variance and standard deviation loadings, see Salzmann-Wüthrich [SW10]. The so-called best-estimate prediction of the ultimate nominal claim $C_n = \sum_{k=1}^n X_k$ at time t is then given by

$$E\left[C_n \middle| \mathcal{T}_t\right] = C_t + \sum_{k=t+1}^{n} E_k^{(t)}. \tag{5.12}$$

Note that we have assumed that $N \le n$, P-a.s., which means that all liabilities are settled at time n.

The best-estimate nominal reserves at time t for the remaining liabilities after and including time $k \ge t+1$ (including k) are given by

$$\widetilde{R}_t^{(k)} = \sum_{l=k}^{n} E_l^{(t)} = E\left[C_n - C_{k-1} \middle| \mathcal{T}_t\right]. \tag{5.13}$$

Remark.
$\widetilde{R}_t^{(k)}$ are nominal reserves (not discounted values). If we choose a constant deflator $\varphi_k \equiv 1$ for all $k = 0, \ldots, n$, then we obtain in view of (2.50)

$$\begin{aligned} R_{k-1}^{(k)} &= R\left[\mathbf{X}_{(k)} \middle| \mathcal{F}_{k-1}\right] = Q_{k-1}[\mathbf{X}_{(k)}] \\ &= \sum_{l=k}^{n} E\left[X_l \middle| \mathcal{F}_{k-1}\right] = \sum_{l=k}^{n} E_l^{(k-1)} = \widetilde{R}_{k-1}^{(k)}, \end{aligned} \tag{5.14}$$

and analogously for constant $\varphi_k \equiv 1$

$$\widetilde{R}_t^{(k)} = Q_t[\mathbf{X}_{(k)}] = Q\left[\mathbf{X}_{(k)} \middle| \mathcal{F}_t\right]. \tag{5.15}$$

Note that for $\varphi_k \equiv 1$ time value of money does not matter and hence the price of the zero coupon bond is equal to 1.

We define the valuation portfolio at time t, $\mathrm{VaPo}_{(t)}(\mathbf{X}_{(t+1)})$, for the outstanding loss liabilities $\mathbf{X}_{(t+1)} = (0, \ldots, 0, X_{t+1}, \ldots, X_n)$ as follows:

period	instrument	cash flow		number of units
$t+1$	$Z^{(t+1)}$	X_{t+1}	$\longrightarrow$	$l_{t+1} = l_{t+1}^{(t)} = E_{t+1}^{(t)}$
$t+2$	$Z^{(t+2)}$	X_{t+2}	$\longrightarrow$	$l_{t+2} = l_{t+2}^{(t)} = E_{t+2}^{(t)}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$t+k$	$Z^{(t+k)}$	X_{t+k}	$\longrightarrow$	$l_{t+k} = l_{t+k}^{(t)} = E_{t+k}^{(t)}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
n	$Z^{(n)}$	X_n	$\longrightarrow$	$l_n = l_n^{(t)} = E_n^{(t)}$

That is,

$$\mathrm{VaPo}_{(t)} = \mathrm{VaPo}_{(t)}(\mathbf{X}_{(t+1)}) = \sum_{k=1}^{n-t} l_{t+k}\, Z^{(t+k)}. \tag{5.16}$$

Remarks.

- So far the VaPo contains only the expected liabilities (written as a portfolio). Of course we need to protect this VaPo against insurance technical risks $l_i \mapsto l_i^*$, which will be the main subject of the remaining chapter.
- If we valuate by nominal values (the accounting principle $\mathcal{A}$ is simply adding nominal values, which corresponds to taking the deflator $\varphi_k \equiv 1$), we simply obtain the classical undiscounted best-estimate claims reserves. Hence for constructing the valuation portfolio in non-life insurance one proceeds as follows: (1) Estimate nominal best-estimate claims reserves $\widetilde{R}_t^{(t+1)}$ for $\mathbf{X}_{(t+1)}$ given the information $\mathcal{T}_t$; (2) allocate them to different time periods and appropriate financial instruments, i.e. estimate a cash flow pattern which allocates $l_{t+k}^{(t)} = E_{t+k}^{(t)}$ to, e.g., $Z^{(t+k)}$, preserving $\widetilde{R}_t^{(t+1)} = \sum_{l=1}^{n-t} E_{t+l}^{(t)}$ obtained in step (1).

5.3 VaPo protected against insurance technical risks, pragmatic approach

Our main goal is to choose a risk measure which describes the uncertainties in X_{t+k} relative to l_{t+k}. This risk measure should protect against adverse developments (relative to l_{t+k}) in the claims developments. Before we discuss the probability distortion based approach (5.9) we discuss pragmatic approaches used in practice.

We assume that we can consider the uncertainties in the payments independently. Hence we attach to each unit a security margin determined by a **standard deviation loading**. Note that the standard deviation loading is very common in insurance pricing. Recently, it has also been used for solvency discussions, for example, in Pelsser [Pe10] and Salzmann-Wüthrich [SW10].

We choose $i, \beta > 0$ and define the valuation portfolio protected against insurance technical risks for the outstanding loss liabilities $\mathbf{X}_{(t+1)}$ at time t as follows:

period	instrument	cash flow	number of units
$t+1$	$Z^{(t+1)}$	X_{t+1}	$\longrightarrow\ l_{t+1}^* = E_{t+1}^{(t)} + i\ \beta\ \sqrt{V_{t+1}^{(t)}}$
$t+2$	$Z^{(t+2)}$	X_{t+2}	$\longrightarrow\ l_{t+2}^* = E_{t+2}^{(t)} + i\ \beta\ \sqrt{V_{t+2}^{(t)}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$t+k$	$Z^{(t+k)}$	X_{t+k}	$\longrightarrow\ l_{t+k}^* = E_{t+k}^{(t)} + i\ \beta\ \sqrt{V_{t+k}^{(t)}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$Z^{(n)}$	X_n	$\longrightarrow\ l_n^* = E_n^{(t)} + i\ \beta\ \sqrt{V_n^{(t)}}$

That is,

$$\mathrm{VaPo}_{(t)}^{prot} = \mathrm{VaPo}_{(t)}^{prot}(\mathbf{X}_{(t+1)}) = \sum_{k=1}^{n-t} l_{t+k}^{*}\ Z^{(t+k)}. \tag{5.17}$$

Remarks.

- The certainty equivalent $l_{t+k}^{*} = l_{t+k}^{*,t}$ also depends on the time of its evaluation, i.e. on the information $\mathcal{T}_t$ on which it is based on. For increasing information $\mathcal{T}_t \to \mathcal{T}_{t+1}$ the uncertainty decreases.
- $\beta\ \sqrt{V_{t+k}^{(t)}}$ stands for the target capital (risk measure). Regulatory constraints impose that the company needs to hold a risk measure in order to run the business in accounting year $t+k$ (evaluated at time t). It measures the uncertainties of $X_{t+k}|\mathcal{T}_t$ relative to $E_{t+k}^{(t)}$, and covers adverse developments in the claims payments X_{k+t}. Observe that we have chosen a standard deviation loading, i.e. the risk measure is proportional to the standard deviation of $X_{t+k}|\mathcal{T}_t$ with proportionality factor β.
- We define this target capital $\beta\ \sqrt{V_{t+k}^{(t)}}$ individually for each accounting year. One should carefully make such a choice because it should also respect the dependence structures between the different accounting periods. Below we give other definitions.
- The parameter i denotes the **cost-of-capital rate**. If we want to mobilize the target capital $\beta\ \sqrt{V_{t+k}^{(t)}}$ from the financial market, we need to promise a return on that target capital, which is higher than the risk-free rate. Because if our business runs badly, which means that we have an adverse development in X_{t+k}, we use the target capital $\beta\ \sqrt{V_{t+k}^{(t)}}$ to cover this adverse development. Hence, the investor's capital is exposed to risk for which he wants to obtain a price i (of course i could also depend on the time $t+k$ and in general it depends on economic factors, for simplicity we choose i constant).
- This means that we decompose the liabilities into $E_{t+k}^{(t)}$ (best-estimate of outstanding loss liabilities towards to insured/injured) and $i\ \beta\ \sqrt{V_{t+k}^{(t)}}$ (price to the investor/shareholder for risk bearing beyond the best-estimate liabilities).
- It is important to distinguish between
  - price for capital exposed to risk: $i\ \beta\ \sqrt{V_{t+k}^{(t)}}$
  - availability of the capital exposed to risk: $\beta\ \sqrt{V_{t+k}^{(t)}}$.

  Observe that we only hold the money that is needed to recruit the risk measure (we hold the **price** of risk measure). It is then the task of the regulator to make sure that the insurance company really recruits/holds that target capital for risk bearing. Moreover, holding the price for the target capital does not guarantee its availability when it is due. Henceforth,

i has to be so large that the risk measure can really be recruited at that price.

- $\beta \sqrt{V_{t+k}^{(t)}}$ can be motivated by the Swiss Solvency Test approach: risk is considered one a 1-year time horizon. As risk measure one considers expected shortfall at some level $\alpha > 0$: $\mathrm{VaR}_\alpha(X)$ is the α-quantile of X, then the expected shortfall of X at level α is given by (losses are assumed to be centered, continuous and positive)
$$\mathrm{ES}_\alpha(X) = E\left[X \middle| X > \mathrm{VaR}_\alpha(X)\right], \tag{5.18}$$
see also (2.132). If X is normally distributed, then both, the Value-at-Risk $\mathrm{VaR}_\alpha(X)$ and the expected shortfall $\mathrm{ES}_\alpha(X)$, are multiples β of the standard deviation of X.
- In this construction we consider each accounting year, X_{t+k} respectively, individually. Pay attention to the fact that the single cash flows are not necessarily independent, which may have various impacts on simultaneous risk capital calculations for all future accounting years.
- **We conclude.** Both, the choice of the risk measure and the cost-of-capital rate, are rather ad-hoc. Many solvency systems, for example the Swiss Solvency Test [SST06], use similar ad-hoc solutions. To obtain a unified approach using economic theory, financial mathematics and actuarial sciences much more research (and deeper mathematical methods) are needed. First attempts are done for example in Pelsser [Pe10], Salzmann-Wüthrich [SW10] and based on the idea of indifference pricing in Malamud et al. [MTW08]. In most of these developments the models and methods need to be further refined so that, e.g., the role of the regulator is modelled realistically. The notoriously difficult thing is that they involve multiperiod risk measures that can only be calculated recursively and often involve nested simulations.

5.4 VaPo protected against insurance technical risks, theoretical considerations

The comprehensive approach to calculate the valuation portfolio protected against insurance technical risks uses probability distortions, see (5.9). One could even go one step beyond and use probability distortion together with utility theory. This has been done in Tsanakas-Christofides [TC06]. In this section we would like to motivate the choice of the risk margin by the utility theory approach.

Choose $l_{t+k}^* = l_{t+k}^{*(\alpha)}$, where $l_{t+k}^{*(\alpha)}$ is the certainty equivalent at time t for payments X_{t+k} in period $t+k$ for a certain risk aversion α (for the explicit definition see below). Hence the margin is then defined by
$$l_{t+k}^{*(\alpha)} - E_{t+k}^{(t)}. \tag{5.19}$$

We construct the certainty equivalent $l_{t+k}^{*(\alpha)}$ as follows:

Utility Theory. If we have two random variables W and V we introduce a preference ordering. Say: we prefer V over W, write $V \succeq W$.

Neumann-Morgenstern, 1944 (Theory of games and economic behaviour): All reasonable preference orderings can be understood as expected utilities. Choose a twice differentiable utility function $u : \mathbb{R} \to \mathbb{R}$ with $u(0) = 0$ and $u' > 0$. If we have risk aversion we need to choose u such that $u'' < 0$. Hence we define the preference ordering by

$$V \succeq W \quad \Leftrightarrow \quad E[u(V)] \geq E[u(W)]. \tag{5.20}$$

$u(x)$ indicates the utility that is located in the monetary unit x. One of the most popular utility functions is the exponential utility function: $u(x) = 1 - \exp\{-\alpha x\}$ with risk aversion constant $\alpha > 0$ (see Figure 5.2).

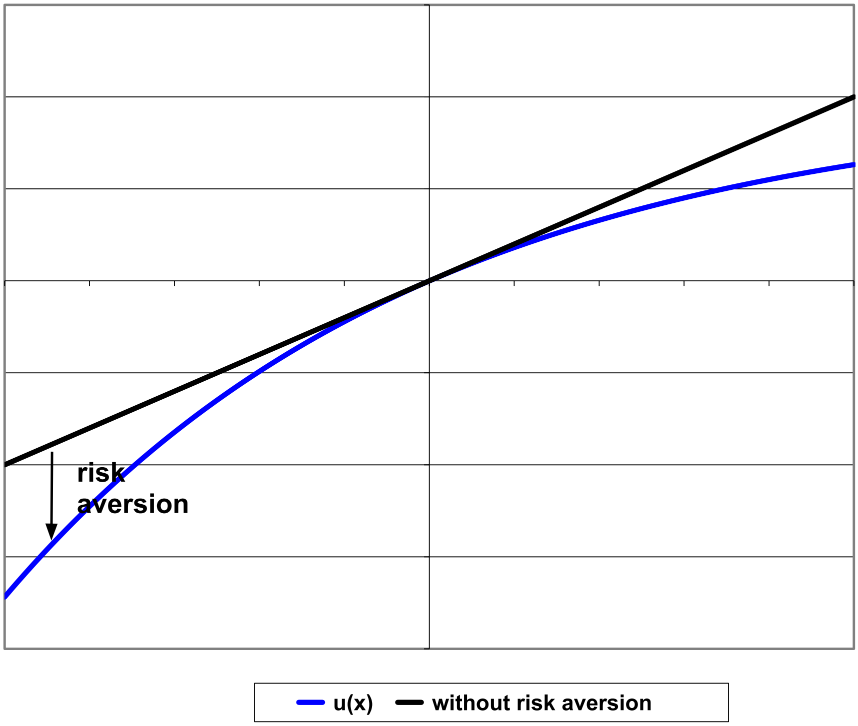


Fig. 5.2. Risk aversion for utility function $u(x) = 1 - \exp\{-\alpha x\}$, $\alpha > 0$

The exponential utility function has constant risk aversion given by

$$-\frac{u''(x)}{u'(x)} = \alpha. \tag{5.21}$$

Henceforth, the risk aversion is for the exponential utility function parametrized by α.

A second popular utility function used is the so-called power utility function that is not further discussed here (see Cochrane [Co01]).

The zero utility principle from our point of view with utility function u means that we are willing to replace the random variable $X_{t+k}|\mathcal{T}_t$ by a fixed amount $l_{t+k}^{*(\alpha)}$ such that the expected utility is the same:

$$E\left[u(-X_{t+k})|\,\mathcal{T}_t\right] = u(-l_{t+k}^{*(\alpha)}), \tag{5.22}$$

i.e. we are willing to pay the $\mathcal{T}_t$-measurable premium $l_{t+k}^{*(\alpha)}$ according to our preference order $u(\cdot)$. Note that $l_{t+k}^{*(\alpha)} \geq E_{t+k}^{(t)}$, since (using concavity, Jensen's inequality and $u' > 0$)

$$u(-l_{t+k}^{*(\alpha)}) = E\left[u(-X_{t+k})|\,\mathcal{T}_t\right] \leq u\left(-E\left[X_{k+t}|\,\mathcal{T}_t\right]\right) = u\left(-E_{t+k}^{(t)}\right). \tag{5.23}$$

If we work with the exponential utility function then the price we are willing to pay is given by

$$\begin{aligned} E\left[u(-X_{t+k})|\,\mathcal{T}_t\right] &= 1 - E\left[\exp\{-\alpha(-X_{t+k})\}|\,\mathcal{T}_t\right] \\ &= 1 - \exp\{-\alpha(-l_{t+k}^{*(\alpha)})\}. \end{aligned} \tag{5.24}$$

This implies that

$$\begin{aligned} l_{t+k}^{*(\alpha)} &= \frac{1}{\alpha}\log E\left[\exp\{-\alpha(-X_{t+k})\}|\,\mathcal{T}_t\right] \\ &= E_{t+k}^{(t)} + \frac{1}{\alpha}\log E\left[\exp\left\{\alpha\left(X_{t+k} - E_{t+k}^{(t)}\right)\right\}\Big|\,\mathcal{T}_t\right] \\ &\approx E_{t+k}^{(t)} + \frac{\alpha}{2}\,V_{t+k}^{(t)} \geq E_{t+k}^{(t)} \qquad \text{for } \alpha > 0, \end{aligned} \tag{5.25}$$

where in the last step we have used a Taylor approximation.

Hence this gives the following VaPo protected against insurance technical risks (**Variance loading**):

period	instrument	cash flow		number of units
$t+1$	$Z^{(t+1)}$	X_{t+1}	$\longrightarrow$	$l^*_{t+1} = E^{(t)}_{t+1} + \frac{\alpha}{2}\ V^{(t)}_{t+1}$
$t+2$	$Z^{(t+2)}$	X_{t+2}	$\longrightarrow$	$l^*_{t+2} = E^{(t)}_{t+2} + \frac{\alpha}{2}\ V^{(t)}_{t+2}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$t+k$	$Z^{(t+k)}$	X_{t+k}	$\longrightarrow$	$l^*_{t+k} = E^{(t)}_{t+k} + \frac{\alpha}{2}\ V^{(t)}_{t+k}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
n	$Z^{(n)}$	X_n	$\longrightarrow$	$l^*_n = E^{(t)}_n + \frac{\alpha}{2}\ V^{(t)}_n$

How do we quantify α? Using risk theory we can determine α from ruin probabilities. For details we refer to the literature on risk theory (see e.g. Mikosch [Mi04]).

Remark. It may be disturbing that the pragmatic solution for the VaPo protected against insurance technical risks uses a standard deviation approach whereas utility considerations suggest a variance loading. This dilemma is also known in the framework of premium calculation principles. The variance loading violates the positive homogeneity property, which is often desirable in practice, see also Pelsser [Pe10].

Question. How do we estimate $E^{(t)}_{t+k}$ and $V^{(t)}_{t+k}$ (at time t)? Pay especially attention to the fact, that these parameters need to be estimated. This immediately implies that the certainty equivalents l^*_{t+k} should also contain a **margin for parameter and model risk**. Parameter risk is often quantified using an appropriate Ansatz or taking a Bayesian point of view. Model risk, however, is almost never really treated.

Remark. The variance loading approach can also be seen as an Esscher premium approach, see also Exercise 2.9, and then we are back in the situation with probability distortions. The Esscher premium is defined as follows: For $\alpha > 0$ set

$$H_\alpha(X_{t+k}|\mathcal{T}_t) = \frac{E\left[\exp\{\alpha\ X_{t+k}\}\ X_{t+k}\right|\mathcal{T}_t]}{E\left[\exp\{\alpha\ X_{t+k}\}\right|\mathcal{T}_t]}, \tag{5.26}$$

subject to the condition that the moment generating function exists for α,

$$E\left[\exp\{\alpha\ X_{t+k}\}\right|\mathcal{T}_t] < \infty, \tag{5.27}$$

see also (2.117)-(2.118). For $\alpha \to 0$ we have

$$H_\alpha(X_{t+k}|\mathcal{T}_t) = E\left[X_{t+k}\right|\mathcal{T}_t] + \alpha\ \mathrm{Var}\left[X_{t+k}\right|\mathcal{T}_t] + o(\alpha). \tag{5.28}$$

For normally distributed random variables the Esscher premium is exact ($o(\alpha) = 0$). In Wang [Wa02] and Landsman [La04] there is also defined the Esscher premium for exponential and elliptical tilting which enables to consider loadings for dependent random variables.

Pragmatic vs. theoretical approach. Observe that we have not said anything about the dependence structure between accounting years. If we choose the variance loading in each accounting year and then simply add the risk measures to obtain the overall loading, we assume that the accounting year payments are independent. On the other hand, if we add risk measures from standard deviation loadings, we are rather on the safe side, because this approach would be implied by assuming maximal positive correlation between accounting year payments.

Final Remark. In our VaPo construction we completely neglected the fact that accounting rules may also influence solvency requirements. Especially, the question "when does the target capital need to be available" is crucial. This question has motivated many new developments, especially in the field of the so-called claims development result, see Merz-Wüthrich [MW08], Bühlmann et al. [BFGMW09] and Wüthrich-Bühlmann [WB08].

5.5 Loss development triangles

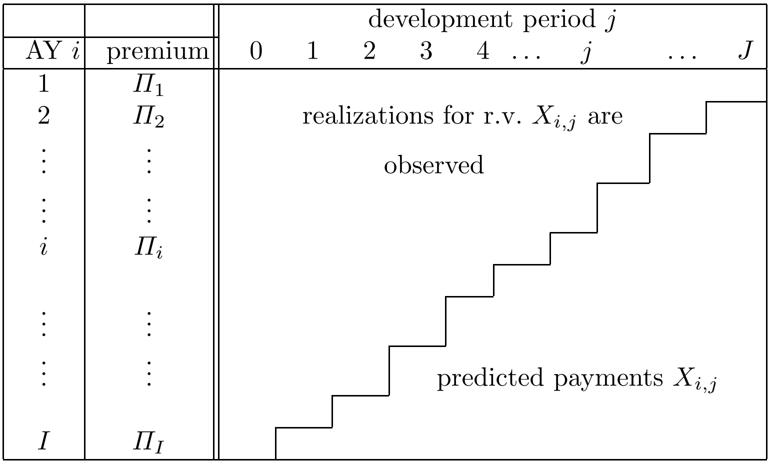
5.5.1 Definitions

Pooling data and claims occurrence principle:

Usually, in non-life insurance, data are pooled so that one obtains homogeneous groups. For example, for pricing one builds homogeneous subportfolios which are then evaluated. For claims reserving one typically builds different subportfolios consisting of different lines of business, claims types, etc. These subportfolios are then further structured by a time component like the accident year.

There are different methodologies to set an accident year, e.g. underwriting year principle, accident date principle, claims-made principle, etc. The insurance contract rules exactly which claims within which time period are covered by the premium. In order to make a meaningful analysis it is important that the choice of the premium principle and the accident date principle are compatible, when pooling data for the study of loss development triangles.

Then claims data are typically structured in a triangular form with the vertical axis labelling accident years $i \in \{1, \ldots, I\}$ and the horizontal axis labelling development years $j \in \{0, \ldots, J\}$, see Wüthrich-Merz [WM08]. We assume that all claims are settled after J development periods.



$X_{i,j}$ denotes the payments for accident year i in development period $j \in \{0, \dots, J\}$ and $X_{i,-1} = -\Pi_i$ denotes the premium received for accident year i (at the beginning of accident year i, see also Figure 5.1). Cumulative claims payments for accident year i within the first j development periods are given by

$$C_{i,j} = \sum_{k=0}^{j} X_{i,k}. \tag{5.29}$$

If we want to have all claims payments within a fixed accounting year we should consider

$$X_k = \sum_{i+j=k} X_{i,j}, \tag{5.30}$$

these are the diagonals of our loss development squares. This implies (if we neglect the premium payments $X_{i,-1} = -\Pi_i$) for accounting year k

$$X_k = \sum_{i=1 \vee (k-J)}^{I \wedge k} X_{i,k-i}. \tag{5.31}$$

Example 5.1 (Non-life development triangles).

For our example we use the Taylor-Ashe [TA83] data, which were also used by Verrall [Ve90], [Ve91] and Mack [Ma93] (see Table 1 in [Ma93]).

□

Incremental payments $X_{i,j}$

	0	1	2	3	4	5	6	7	8	9
1	357'848	766'940	610'542	482'940	527'326	574'398	146'342	139'950	227'229	67'948
2	352'118	884'021	933'894	1'183'289	445'745	320'996	527'804	266'172	425'046	
3	290'507	1'001'799	926'219	1'016'654	750'816	146'923	495'992	280'405		
4	310'608	1'108'250	776'189	1'562'400	272'482	352'053	206'286			
5	443'160	693'190	991'983	769'488	504'851	470'639				
6	396'132	937'085	847'498	805'037	705'960					
7	440'832	847'631	1'131'398	1'063'269						
8	359'480	1'061'648	1'443'370							
9	376'686	986'608								
10	344'014									

Cumulative payments $C_{i,j}$

	0	1	2	3	4	5	6	7	8	9
1	357'848	1'124'788	1'735'330	2'218'270	2'745'596	3'319'994	3'466'336	3'606'286	3'833'515	3'901'463
2	352'118	1'236'139	2'170'033	3'353'322	3'799'067	4'120'063	4'647'867	4'914'039	5'339'085	
3	290'507	1'292'306	2'218'525	3'235'179	3'985'995	4'132'918	4'628'910	4'909'315		
4	310'608	1'418'858	2'195'047	3'757'447	4'029'929	4'381'982	4'588'268			
5	443'160	1'136'350	2'128'333	2'897'821	3'402'672	3'873'311				
6	396'132	1'333'217	2'180'715	2'985'752	3'691'712					
7	440'832	1'288'463	2'419'861	3'483'130						
8	359'480	1'421'128	2'864'498							
9	376'686	1'363'294								
10	344'014									

Accounting year payments X_k

1	2	3	4	5	6	7	8	9	10
357'848	1'119'058	1'785'070	2'729'241	4'188'244	3'902'308	5'150'454	3'911'256	5'221'066	5'993'545

5.5.2 Chain-ladder method

Probably the most popular method to predict future claims payments is the so-called chain-ladder method. For our exposition we revisit this method since we will use it to construct the valuation portfolio. For other methods and more background information we refer to Wüthrich-Merz [WM08].

Assume I is the last accident year (=accounting year) for which we have an observation $X_{i,0}$. Define the observations in the first k columns of the claims development triangle by

$$\mathcal{B}_k = \sigma\left\{X_{i,j} : i + j \le I, j \le k\right\} = \sigma\left\{C_{i,j} : i + j \le I, j \le k\right\}. \tag{5.32}$$

Hence, $\mathcal{B}_J$ is the σ-field in the upper triangle, where we have collected all observations up to time I.

Model Assumptions 5.3 (Chain-ladder model) *We assume that*

- *the filtration $\mathcal{T}_t$ is generated by $\{C_{i,j} : i + j \le t\}$,*
- *payments $X_{i,j}$ in different accident years i are independent,*
- *$(C_{i,j})_{j=0,\dots,J}$ is a Markov process and there exist $f_j > 0$, $j \in \{0,\dots,J-1\}$, and $\sigma_j^2 > 0$, $j \in \{0,\dots,J-1\}$, such that for all $i \in \{1,\dots,I\}$ and $j \in \{1,\dots,J\}$*

$$E\left[C_{i,j} \middle| C_{i,j-1}\right] = f_{j-1}\; C_{i,j-1}, \tag{5.33}$$

$$\operatorname{Var}\left(C_{i,j} \middle| C_{i,j-1}\right) = \sigma_{j-1}^2\; C_{i,j-1}. \tag{5.34}$$

Remarks.

- There is a huge literature on the chain-ladder method. One of the first rigorous probabilistic approaches to the chain-ladder method is due to Mack [Ma93]. Mack has given a distribution-free stochastic model for the chain-ladder method in which he derived an estimate for the mean square error of prediction.
- f_j are called chain-ladder factors, development factors, age-to-age factors or link ratios.
- Define the individual development factors

$$F_{i,j} = \frac{C_{i,j+1}}{C_{i,j}}, \tag{5.35}$$

  then $F_{i,j}$ are conditionally, given $C_{i,j}$, unbiased estimators for f_j with conditional variance

$$\operatorname{Var}\left(F_{i,j} \middle| C_{i,j}\right) = \sigma_j^2 / C_{i,j}. \tag{5.36}$$

The chain-ladder model immediately implies, how we should predict the ultimate claim $C_{i,J}$ and the incremental payments $X_{i,j}$ for $i + j > I$:

Lemma 5.4 *Under Model Assumptions 5.3 we have for all* $i = 1, \ldots, I$, $j = 0, \ldots, J-1$ *and* $k \geq 1$

$$E\left[C_{i,J} \middle| C_{i,j}\right] = C_{i,j} \prod_{l=j}^{J-1} f_l, \tag{5.37}$$

$$E\left[X_{i,j+k} \middle| C_{i,j}\right] = C_{i,j} \prod_{l=j}^{j+k-2} f_l \;\left(f_{j+k-1} - 1\right). \tag{5.38}$$

Proof. This is an exercise using conditional expectations. Using the tower property of conditional expectations (see Williams [Wi91]), the Markov property of cumulative payments $C_{i,j}$ and (5.33) we obtain

$$\begin{aligned} E\left[C_{i,J} \middle| C_{i,j}\right] &= E\left[E\left[C_{i,J} \middle| C_{i,J-1}\right] \middle| C_{i,j}\right] \\ &= f_{J-1}\; E\left[C_{i,J-1} \middle| C_{i,j}\right]. \end{aligned} \tag{5.39}$$

If we iterate this procedure until we reach j we obtain the first result. The second assertion easily follows from $X_{i,j+k} = C_{i,j+k} - C_{i,j+k-1}$ and a similar calculation to (5.39).

□

Note that the information given at time $t = I$ is

$$\mathcal{T}_I = \mathcal{B}_J. \tag{5.40}$$

Therefore, from Lemma 5.4 we see that under the chain-ladder model assumptions we obtain for $\mathcal{T}_t = \mathcal{B}_J$ at time $t = I$

$$\begin{aligned} E_{t+k}^{(t)} = E\left[X_{t+k} \middle| \mathcal{T}_t\right] &= \sum_{i+j=t+k} E\left[X_{i,j} \middle| \mathcal{B}_J\right] \\ &= \sum_{i+j=t} C_{i,j}\; f_j \cdots f_{j+k-2}\; \left(f_{j+k-1} - 1\right), \end{aligned} \tag{5.41}$$

hence there remains to estimate the chain-ladder factors f_j in order to estimate the $\mathrm{VaPo}_{(t)}$, see (5.16).

Remark. We use $E_{t+k}^{(t)}$ as a prediction for X_{t+k} based on the information $\mathcal{T}_t$. Since the true model parameters f_j are not known they need to be estimated.

This leads to the estimator $\widehat{E_{t+k}^{(t)}}$ for $E_{t+k}^{(t)}$ which is at the same time a $\mathcal{T}_t$-measurable predictor for X_{t+k}.

We introduce the following notations (assume $J+1=I=t$)

$$i^*(j) = I - j \quad \text{and} \quad j^*(i) = J + 1 - i, \tag{5.42}$$

hence $X_{i^*(j),j}$ and $X_{i,j^*(i)}$ belong to the last observed accounting year, which is the last diagonal in the observed claims development triangle. Estimators for f_j and σ_j^2 are then given by

$$\widehat{f}_j = \frac{\sum_{i=1}^{i^*(j+1)} C_{i,j+1}}{\sum_{i=1}^{i^*(j+1)} C_{i,j}} = \sum_{i=1}^{i^*(j+1)} \frac{C_{i,j}}{\sum_{m=1}^{i^*(j+1)} C_{m,j}} F_{i,j}, \tag{5.43}$$

$$\widehat{\sigma}_j^2 = \frac{1}{i^*(j+1)-1} \sum_{i=1}^{i^*(j+1)} C_{i,j} \left(\frac{C_{i,j+1}}{C_{i,j}} - \widehat{f}_j \right)^2 .$$

Note that the estimator $\widehat{f}_j$ is a weighted average of the observed individual development factors $F_{i,j}$.

Hence choose on the set $\mathcal{T}_t$ the following chain-ladder estimators for l_{t+k}

$$l_{t+k} = \widehat{E_{t+k}^{(t)}} = \sum_{i+j=t} C_{i,j}\ \widehat{f}_j \cdots \widehat{f}_{j+k-2} \left(\widehat{f}_{j+k-1} - 1 \right). \tag{5.44}$$

Lemma 5.5 *Conditionally, given $\mathcal{B}_j$, $\widehat{f}_j$ are unbiased estimators for f_j.*

This immediately implies that $\widehat{f}_j$ are (unconditionally) unbiased estimators for f_j.

Proof of Lemma 5.5. Using the $\mathcal{B}_j$-measurability of the $C_{i,j}$'s we obtain

$$E\left[\widehat{f}_j \middle| \mathcal{B}_j \right] = E\left[\frac{\sum_{i=1}^{i^*(j+1)} C_{i,j+1}}{\sum_{i=1}^{i^*(j+1)} C_{i,j}} \middle| \mathcal{B}_j \right] \tag{5.45}$$

$$= \frac{E\left[\sum_{i=1}^{i^*(j+1)} C_{i,j+1} \middle| \mathcal{B}_j \right]}{\sum_{i=1}^{i^*(j+1)} C_{i,j}} = f_j.$$

This finishes the proof of Lemma 5.5.

□

Lemma 5.6 *Choose $l \leq k < j$. Then $\widehat{f}_j$ and $\widehat{f}_k$ are conditionally uncorrelated, given $\mathcal{B}_l$.*

This also immediately implies the unconditional uncorrelatedness of the estimators $\widehat{f}_j$ and $\widehat{f}_k$.

Proof of Lemma 5.6. Assume $j > k \geq l$. Then using the tower property of conditional expectations

$$E\left[\widehat{f}_j\ \widehat{f}_k \middle| \mathcal{B}_l\right] = E\left[\widehat{f}_k\ E\left[\widehat{f}_j \middle| \mathcal{B}_j\right] \middle| \mathcal{B}_l\right] = E\left[\widehat{f}_k\ f_j \middle| \mathcal{B}_l\right] = f_k\ f_j. \tag{5.46}$$

In view of Lemma 5.5, this finishes the proof of Lemma 5.6.

□

An immediate consequence is the next corollary:

Corollary 5.7 *Choose $j > j^*(i)$. The chain-ladder estimator given by*

$$\widehat{X}_{i,j} = C_{i,j^*(i)}\ \widehat{f}_{j^*(i)} \cdots \widehat{f}_{j-2}\ \left(\widehat{f}_{j-1} - 1\right) \tag{5.47}$$

is conditionally unbiased for $E\left[X_{i,j}\middle|\mathcal{T}_t\right]$, given $\mathcal{B}_{j^*(i)}$ or $C_{i,j^*(i)}$, respectively.

Proof. We have

$$\begin{aligned} E\left[\widehat{X}_{i,j}\middle|\mathcal{B}_{j^*(i)}\right] &= C_{i,j^*(i)}\ E\left[\widehat{f}_{j^*(i)} \cdots \widehat{f}_{j-2}\ \left(\widehat{f}_{j-1} - 1\right)\middle|\mathcal{B}_{j^*(i)}\right] \\ &= E\left[X_{i,j}\middle|\mathcal{T}_t\right], \end{aligned} \tag{5.48}$$

where in the last step we have used the conditional unbiasedness and uncorrelatedness of the $\widehat{f}_j$, the independence of different accident years and the Markov property of our time series.

□

Hence this motivates to (a) estimate $E^{(t)}_{t+k}$, (b) predict X_{t+k} by

$$\widehat{E^{(t)}_{t+k}} = \sum_{i+j=t+k} \widehat{X}_{i,j}, \tag{5.49}$$

which is exactly (5.44).

Lemma 5.8 *$\widehat{f}_j$ is the $\mathcal{B}_{j+1}$-measurable unbiased estimator, which has minimal variance among all linear combinations of unbiased estimators of $F_{i,j} = C_{i,j+1}/C_{i,j}$.*

Proof. See Lemmas 3.3 and 3.4 in Wüthrich-Merz [WM08].

□

Lemma 5.9 *$\widehat{\sigma}_j^2$ are conditionally unbiased estimators for σ_j^2, given $\mathcal{B}_j$.*

Proof of Lemma 5.9. We have (add and subtract f_j), $i+j \le I$,

$$E\left[\left(\frac{C_{i,j+1}}{C_{i,j}} - \widehat{f}_j\right)^2 \middle| \mathcal{B}_j\right] = E\left[\left(\frac{C_{i,j+1}}{C_{i,j}} - f_j\right)^2 \middle| \mathcal{B}_j\right] \tag{5.50}$$
$$-2\; E\left[\left(\frac{C_{i,j+1}}{C_{i,j}} - f_j\right)\left(\widehat{f}_j - f_j\right) \middle| \mathcal{B}_j\right] + E\left[\left(\widehat{f}_j - f_j\right)^2 \middle| \mathcal{B}_j\right].$$

Hence we calculate the terms on the r.h.s. of the equality above.

$$E\left[\left(\frac{C_{i,j+1}}{C_{i,j}} - f_j\right)^2 \middle| \mathcal{B}_j\right] = \mathrm{Var}\left(\frac{C_{i,j+1}}{C_{i,j}} \middle| \mathcal{B}_j\right) = \frac{1}{C_{i,j}}\; \sigma_j^2. \tag{5.51}$$

The next term is (using the independence of different accident years)

$$\begin{aligned} E\left[\left(\frac{C_{i,j+1}}{C_{i,j}} - f_j\right)\; \left(\widehat{f}_j - f_j\right) \middle| \mathcal{B}_j\right] &= \mathrm{Cov}\left(\frac{C_{i,j+1}}{C_{i,j}}, \widehat{f}_j \middle| \mathcal{B}_j\right) \\ &= \frac{C_{i,j}}{\sum_i C_{i,j}}\; \mathrm{Var}\left(\frac{C_{i,j+1}}{C_{i,j}} \middle| \mathcal{B}_j\right) \\ &= \frac{\sigma_j^2}{\sum_i C_{i,j}}. \end{aligned} \tag{5.52}$$

Whereas for the last term we obtain

$$E\left[\left(\widehat{f}_j - f_j\right)^2 \middle| \mathcal{B}_j\right] = \mathrm{Var}\left(\widehat{f}_j \middle| \mathcal{B}_j\right) = \frac{\sigma_j^2}{\sum_i C_{i,j}}. \tag{5.53}$$

Putting all pieces together gives

$$E\left[\left(\frac{C_{i,j+1}}{C_{i,j}} - \widehat{f}_j\right)^2 \middle| \mathcal{B}_j\right] = \sigma_j^2\; \left(\frac{1}{C_{i,j}} - \frac{1}{\sum_i C_{i,j}}\right). \tag{5.54}$$

Hence we have

$$E\left[\widehat{\sigma}_j^2 \middle| \mathcal{B}_j\right] = \frac{1}{i^*(j+1)-1} \sum_{i=1}^{i^*(j+1)} C_{i,j}\; E\left[\left(\frac{C_{i,j+1}}{C_{i,j}} - \widehat{f}_j\right)^2 \middle| \mathcal{B}_j\right] = \sigma_j^2, \tag{5.55}$$

which proves the claim of Lemma 5.9.

□

In the sequel, we will also need the following equality

$$E\left[\widehat{f}_j^2 \middle| \mathcal{B}_j\right] = \mathrm{Var}\left(\widehat{f}_j \middle| \mathcal{B}_j\right) + f_j^2 = \frac{\sigma_j^2}{\sum_{i=1}^{i^*(j+1)} C_{i,j}} + f_j^2, \tag{5.56}$$

which is obtained from (5.53).

In a first step we would now like to calculate the valuation portfolio for our Example 5.1. From (5.16) we obtain

$$\begin{aligned} \mathrm{VaPo}_{(t)} = \mathrm{VaPo}_{(t)}(\mathbf{X}_{(t+1)}) &= \sum_{k\geq 1} l_{t+k}\ Z^{(t+k)} = \sum_{k\geq 1} \widehat{E_{t+k}^{(t)}}\ Z^{(t+k)} \\ &= \sum_{k\geq 1}\left[\sum_{i+j=t} C_{i,j}\ \widehat{f}_j \cdots \widehat{f}_{j+k-2}\ \left(\widehat{f}_{j+k-1}-1\right)\right] Z^{(t+k)}. \end{aligned} \tag{5.57}$$

Note that this corresponds to the $\mathcal{T}_t$-measurable VaPo represented in terms of the financial instruments $Z^{(t+k)}$.

Example 5.1 (revisited).

Observed individual chain-ladder factors $F_{i,j}$.

	0	1	2	3	4	5	6	7	8
1	3.1432	1.5428	1.2783	1.2377	1.2092	1.0441	1.0404	1.0630	1.0177
2	3.5106	1.7555	1.5453	1.1329	1.0845	1.1281	1.0573	1.0865	
3	4.4485	1.7167	1.4583	1.2321	1.0369	1.1200	1.0606		
4	4.5680	1.5471	1.7118	1.0725	1.0874	1.0471			
5	2.5642	1.8730	1.3615	1.1742	1.1383				
6	3.3656	1.6357	1.3692	1.2364					
7	2.9228	1.8781	1.4394						
8	3.9533	2.0157							
9	3.6192								
10									
$\widehat{f}_j$	3.4906	1.7473	1.4574	1.1739	1.1038	1.0863	1.0539	1.0766	1.0177
$\widehat{\sigma}_j$	400.35	194.26	204.85	123.22	117.18	90.48	21.13	33.87	21.13

Note that we do not have enough data to estimate the last variance parameter σ_8^2. Therefore to estimate σ_8^2 we have chosen the formula given in Mack [Ma93] ($J = 9$):

$$\widehat{\sigma}_{J-1}^2 = \min\left\{\widehat{\sigma}_{J-2}^4/\widehat{\sigma}_{J-3}^2\ ;\ \widehat{\sigma}_{J-2}^2\ ;\ \widehat{\sigma}_{J-3}^2\right\}. \tag{5.58}$$

We now complete the claims triangle by predicting the cumulative payments $C_{i,j}$, $i + j > I$, using the chain-ladder predictor

$$\widehat{C}_{i,j} = C_{i,j^*(i)}\ \widehat{f}_{j^*(i)} \cdots \widehat{f}_{j-1}. \tag{5.59}$$

This gives the following completed (predicted) claims development triangle for the cumulative payments $C_{i,j}$:

	0	1	2	3	4	5	6	7	8	9
1	357'848	1'124'788	1'735'330	2'218'270	2'745'596	3'319'994	3'466'336	3'606'286	3'833'515	3'901'463
2	352'118	1'236'139	2'170'033	3'353'322	3'799'067	4'120'063	4'647'867	4'914'039	5'339'085	5'433'719
3	290'507	1'292'306	2'218'525	3'235'179	3'985'995	4'132'918	4'628'910	4'909'315	5'285'148	5'378'826
4	310'608	1'418'858	2'195'047	3'757'447	4'029'929	4'381'982	4'588'268	4'835'458	5'205'637	5'297'906
5	443'160	1'136'350	2'128'333	2'897'821	3'402'672	3'873'311	4'207'459	4'434'133	4'773'589	4'858'200
6	396'132	1'333'217	2'180'715	2'985'752	3'691'712	4'074'999	4'426'546	4'665'023	5'022'155	5'111'171
7	440'832	1'288'463	2'419'861	3'483'130	4'088'678	4'513'179	4'902'528	5'166'649	5'562'182	5'660'771
8	359'480	1'421'128	2'864'498	4'174'756	4'900'545	5'409'337	5'875'997	6'192'562	6'666'635	6'784'799
9	376'686	1'363'294	2'382'128	3'471'744	4'075'313	4'498'426	4'886'502	5'149'760	5'544'000	5'642'266
10	344'014	1'200'818	2'098'228	3'057'984	3'589'620	3'962'307	4'304'132	4'536'015	4'883'270	4'969'825

Hence the estimated expected incremental payments, $i + j > I$ are predicted by, see also (5.44),

$$\widehat{E}\left[X_{i,j} \middle| \mathcal{B}_J\right] = C_{i,j^*(i)}\ \widehat{f}_{j^*(i)} \cdots \widehat{f}_{j-2}\ \left(\widehat{f}_{j-1} - 1\right) = \widehat{C}_{i,j} - \widehat{C}_{i,j-1}: \qquad (5.60)$$

	0	1	2	3	4	5	6	7	8	9
1										
2										94'634
3									375'833	93'678
4								247'190	370'179	92'268
5							334'148	226'674	339'456	84'611
6						383'287	351'548	238'477	357'132	89'016
7					605'548	424'501	389'349	264'121	395'534	98'588
8				1'310'258	725'788	508'792	466'660	316'566	474'073	118'164
9			1'018'834	1'089'616	603'569	423'113	388'076	263'257	394'241	98'266
10		856'804	897'410	959'756	531'636	372'687	341'826	231'882	347'255	86'555

This leads to the following estimated expected payments $l_{t+k} = \widehat{E^{(t)}_{t+k}}$ in the accounting years (predictions for X_{t+k}):

$t+k$	$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	$t+7$	$t+8$	$t+9$
l_{t+k}	5'226'536	4'179'394	3'131'668	2'127'272	1'561'879	1'177'744	744'287	445'521	86'555

Table 5.1. Predicted payments X_{t+k} based in information $\mathcal{T}_t$

Hence we have estimated the valuation portfolio given by

$$\text{VaPo}_{(t)} = \sum_{k \geq 1} l_{t+k}\ Z^{(t+k)}. \qquad (5.61)$$

If we want to obtain the cash value, we need to apply an accounting principle $\mathcal{A}_t$ to our valuation portfolio. We choose three different examples: 1) nominal value, 2) constant interest rate $r = 1.5\%$, 3) risk-free rates used in the Swiss Solvency Field-Test 2005:

maturity	1	2	3	4	5	6	7	8	9
risk free rate	0.88%	1.14%	1.36%	1.57%	1.75%	1.91%	2.05%	2.18%	2.29%

Table 5.2. Swiss Solvency Test risk-free rates 2005

This gives the following values for $\mathcal{A}_t\left(\text{VaPo}_{(t)}\right)$ at time $t = I$:

$$\begin{aligned} \mathcal{A}_t\left(\text{VaPo}_{(t)}\right) &= \sum_{k \geq 1} l_{t+k}\ \mathcal{A}_t\left(Z^{(t+k)}\right) \\ &= \sum_{k \geq 1} l_{t+k}\ Z_t^{(t+k)}, \end{aligned} \qquad (5.62)$$

with prices $Z_t^{(t+k)}$ determined by 1) nominal values, i.e. equal to 1; 2) constant interest rates, i.e. equal to $(1+r)^{-k}$; 3) risk-free rates similar to (3.84). This provides the following results:

	reserves $\mathcal{A}_t$ $\left(\text{VaPo}_{(t)}\right)$	difference to $\widetilde{R}_t^{(t+1)}$	
1) $\widetilde{R}_t^{(t+1)}$(nominal)	18'680'856		
2) $r = 1.50\%$	17'873'967	806'888	4.32%
3) SST rates	17'847'512	833'344	4.46%

Table 5.3. Monetary value of the valuation portfolio for different accounting principles

□

5.5.3 Estimation of insurance technical risks in the chain-ladder model, single accident years

Let us, for the moment, fix one single accident year i. Hence, under the chain-ladder Model Assumptions 5.3 we have for $t = I$, $k \geq 1$

$$E_{t+k}^{(t)}(i) = E\left[X_{i,j^*(i)+k} \middle| \mathcal{T}_t \right] = E\left[X_{i,j^*(i)+k} \middle| C_{i,j^*(i)} \right], \tag{5.63}$$

which is estimated by (see also (5.60))

$$l_{t+k}(i) = \widehat{E_{t+k}^{(t)}}(i) = C_{i,j^*(i)} \prod_{l=0}^{k-2} \widehat{f}_{j^*(i)+l} \left(\widehat{f}_{j^*(i)+k-1} - 1 \right). \tag{5.64}$$

We use the bracket term (i) to indicate that we study one single accident year only.
This gives the following VaPo for accident year i:

period	instrument	cash flow	number of units
$t+1$	$Z^{(t+1)}$	X_{t+1}	$\rightarrow l_{t+1}(i) = C_{i,j^*(i)} \left(\widehat{f}_{j^*(i)} - 1 \right)$
$t+2$	$Z^{(t+2)}$	X_{t+2}	$\rightarrow l_{t+2}(i) = C_{i,j^*(i)} \, \widehat{f}_{j^*(i)} \left(\widehat{f}_{j^*(i)+1} - 1 \right)$
⋮	⋮	⋮	⋮
$t+k$	$Z^{(t+k)}$	X_{t+k}	$\rightarrow l_{t+k}(i) = C_{i,j^*(i)} \prod_{l=0}^{k-2} \widehat{f}_{j^*(i)+l} \left(\widehat{f}_{j^*(i)+k-1} - 1 \right)$
⋮	⋮	⋮	⋮

That is,

$$\text{VaPo}_{(t)}(i) = \sum_{k \geq 1} l_{t+k}(i) \; Z^{(t+k)}. \tag{5.65}$$

First approach to insurance technical risks

As in (5.11) we set for accident year i

$$V_{t+k}^{(t)}(i) = \text{Var}\left(X_{i,j^*(i)+k} \middle| \mathcal{T}_t \right) = \text{Var}\left(X_{i,j^*(i)+k} \middle| C_{i,j^*(i)} \right), \tag{5.66}$$

$$W_{t+k}^{(t)}(i) = \text{Var}\left(C_{i,j^*(i)+k} \middle| C_{i,j^*(i)} \right). \tag{5.67}$$

If we decompose the variance in its usual way and using the Markov property we obtain

$$\begin{aligned} W_{t+k}^{(t)}(i) &= E\left[\operatorname{Var}\left(C_{i,j^*(i)+k}\middle| C_{i,j^*(i)+k-1}\right)\middle| C_{i,j^*(i)}\right] \\ &\qquad\qquad +\operatorname{Var}\left(E\left[C_{i,j^*(i)+k}\middle| C_{i,j^*(i)+k-1}\right]\middle| C_{i,j^*(i)}\right) \\ &= E\left[\sigma_{j^*(i)+k-1}^2\; C_{i,j^*(i)+k-1}\middle| C_{i,j^*(i)}\right] \\ &\qquad\qquad +\operatorname{Var}\left(f_{j^*(i)+k-1}\; C_{i,j^*(i)+k-1}\middle| C_{i,j^*(i)}\right) \\ &= \sigma_{j^*(i)+k-1}^2\; C_{i,j^*(i)} \prod_{l=j^*(i)}^{j^*(i)+k-2} f_l \\ &\qquad + f_{j^*(i)+k-1}^2\; \operatorname{Var}\left(C_{i,j^*(i)+k-1}\middle| C_{i,j^*(i)}\right) \\ &= \sigma_{j^*(i)+k-1}^2\; C_{i,j^*(i)} \prod_{l=j^*(i)}^{j^*(i)+k-2} f_l + f_{j^*(i)+k-1}^2\; W_{t+k-1}^{(t)}(i). \end{aligned} \tag{5.68}$$

The first term on the r.h.s. of the equality above can be rewritten as

$$E\left[C_{i,j^*(i)+k-1}\middle| C_{i,j^*(i)}\right] = C_{i,j^*(i)} \prod_{l=j^*(i)}^{j^*(i)+k-2} f_l. \tag{5.69}$$

Hence we have found an iteration for the conditional variances, which immediately implies, see also Wüthrich-Merz [WM08] formula (3.10),

$$\begin{aligned} W_{t+k}^{(t)}(i) &= C_{i,j^*(i)} \sum_{m=j^*(i)}^{j^*(i)+k-1} \prod_{n=m+1}^{j^*(i)+k-1} f_n^2\; \sigma_m^2 \prod_{l=j^*(i)}^{m-1} f_l \\ &= \sum_{m=j^*(i)}^{j^*(i)+k-1} \prod_{n=m+1}^{j^*(i)+k-1} f_n^2\; \sigma_m^2\; E\left[C_{i,m}\left|C_{i,j^*(i)}\right.\right]. \end{aligned} \tag{5.70}$$

If we insert the estimators for f_l and σ_l^2 (estimated from $\mathcal{B}_J$) we immediately obtain an estimator for the conditional variances of the cumulative payments.

$$\begin{aligned} \widehat{W_{t+k}^{(t)}}(i) &= C_{i,j^*(i)} \sum_{m=j^*(i)}^{j^*(i)+k-1} \prod_{n=m+1}^{j^*(i)+k-1} \widehat{f}_n^2\; \widehat{\sigma}_m^2 \prod_{l=j^*(i)}^{m-1} \widehat{f}_l \\ &= \sum_{m=j^*(i)}^{j^*(i)+k-1} \prod_{n=m+1}^{j^*(i)+k-1} \widehat{f}_n^2\; \widehat{\sigma}_m^2\; \widehat{C}_{i,m} \\ &= \widehat{C}_{i,j^*(i)+k}^2 \sum_{m=j^*(i)}^{j^*(i)+k-1} \frac{\widehat{\sigma}_m^2/\widehat{f}_m^2}{\widehat{C}_{i,m}}, \end{aligned} \tag{5.71}$$

with

$$\widehat{C}_{i,m} = \widehat{E}\left[C_{i,m}\left|C_{i,j^*(i)}\right.\right] = C_{i,j^*(i)} \prod_{l=j^*(i)}^{m-1} \widehat{f}_l. \tag{5.72}$$

For the incremental payments we have

$$\begin{aligned} V_{t+k}^{(t)}(i) &= E\left[\operatorname{Var}\left(X_{i,j^*(i)+k}\middle| C_{i,j^*(i)+k-1}\right)\middle| C_{i,j^*(i)}\right] \\ &\quad + \operatorname{Var}\left(E\left[X_{i,j^*(i)+k}\middle| C_{i,j^*(i)+k-1}\right]\middle| C_{i,j^*(i)}\right) \\ &= E\left[\sigma^2_{j^*(i)+k-1}\; C_{i,j^*(i)+k-1}\middle| C_{i,j^*(i)}\right] \\ &\quad + \operatorname{Var}\left(\left(f_{j^*(i)+k-1}-1\right)\; C_{i,j^*(i)+k-1}\middle| C_{i,j^*(i)}\right). \end{aligned} \tag{5.73}$$

Hence we obtain the following estimator for the variance:

Estimator 5.10 (Process variance for single accident years)

$$\begin{aligned} \widehat{V_{t+k}^{(t)}}(i) &= \widehat{\sigma}^2_{j^*(i)+k-1}\; C_{i,j^*(i)} \prod_{l=j^*(i)}^{j^*(i)+k-2} \widehat{f}_l + \left(\widehat{f}_{j^*(i)+k-1}-1\right)^2\; \widehat{W_{t+k-1}^{(t)}}(i) \\ &= \widehat{C}^2_{i,j^*(i)+k}\; \frac{\widehat{\sigma}^2_{j^*(i)+k-1}/\widehat{f}^2_{j^*(i)+k-1}}{\widehat{C}_{i,j^*(i)+k-1}} + \left(\widehat{f}_{j^*(i)+k-1}-1\right)^2\; \widehat{W_{t+k-1}^{(t)}}(i) \\ &= \widehat{W_{t+k}^{(t)}}(i) + \left(1 - 2\; \widehat{f}_{j^*(i)+k-1}\right)\; \widehat{W_{t+k-1}^{(t)}}(i). \end{aligned} \tag{5.74}$$

Using the pragmatic approach (first approach with standard deviation loadings), we obtain for the VaPo protected against insurance technical risks (see (5.64) and (5.74)):

period	instrument	cash flow	number of units
$t+1$	$Z^{(t+1)}$	X_{t+1}	$\longrightarrow l^*_{t+1}(i) = \widehat{E_{t+1}^{(t)}}(i) + i\;\beta\; \widehat{V_{t+1}^{(t)}}(i)^{1/2}$
$t+2$	$Z^{(t+2)}$	X_{t+2}	$\longrightarrow l^*_{t+2}(i) = \widehat{E_{t+2}^{(t)}}(i) + i\;\beta\; \widehat{V_{t+2}^{(t)}}(i)^{1/2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$t+k$	$Z^{(t+k)}$	X_{t+k}	$\longrightarrow l^*_{t+k}(i) = \widehat{E_{t+k}^{(t)}}(i) + i\;\beta\; \widehat{V_{t+k}^{(t)}}(i)^{1/2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

That is,

$$\mathrm{VaPo}_{(t)}^{prot}(i) = \sum_{k\ge 1} l^*_{t+k}(i)\; Z^{(t+k)}, \tag{5.75}$$

with number of units determined with the standard deviation loading

$$l^*_{t+k}(i) = \widehat{E_{t+k}^{(t)}}(i) + i\;\beta\; \widehat{V_{t+k}^{(t)}}(i)^{1/2} \tag{5.76}$$

for $i, \beta > 0$ (the bracket term (i) denotes the accident year and i the cost-of-capital rate).

Example 5.1 (revisited).

We calculate the uncertainties $\widehat{V_{t+k}^{(t)}}(i)^{1/2}$ which correspond to predicted incremental payments $X_{i,j^*(i)+k}$ on page 111 (see also (5.60)).

	0	1	2	3	4	5	6	7	8	9
1										
2										48’832
3									75’052	48’603
4								45’268	74’566	48’243
5							178’062	44’398	72’835	46’336
6						225’149	183’669	47’407	77’269	47’781
7					229’965	238’145	194’528	52’055	84’306	50’590
8				346’712	258’879	264’121	216’465	62’761	100’344	56’248
9			226’818	332’762	242’972	244’204	200’828	62’482	98’906	52’140
10		234’816	275’881	364’358	250’614	240’498	199’983	74’127	115’011	51’779

The variational coefficients are estimated by

$$\widehat{\mathrm{Vco}}(X_{i,j^*(i)+k}|\mathcal{B}_J) = \frac{\widehat{V_{t+k}^{(t)}}(i)^{1/2}}{\widehat{E}\left[X_{i,j^*(i)+k}|\mathcal{B}_J\right]}, \tag{5.77}$$

and given by:

	0	1	2	3	4	5	6	7	8	9
1										
2										51.6%
3									20.0%	51.9%
4								18.3%	20.1%	52.3%
5							53.3%	19.6%	21.5%	54.8%
6						58.7%	52.2%	19.9%	21.6%	53.7%
7					38.0%	56.1%	50.0%	19.7%	21.3%	51.3%
8				26.5%	35.7%	51.9%	46.4%	19.8%	21.2%	47.6%
9			22.3%	30.5%	40.3%	57.7%	51.7%	23.7%	25.1%	53.1%
10		27.4%	30.7%	38.0%	47.1%	64.5%	58.5%	32.0%	33.1%	59.8%

Note that if we aggregate within accounting years, i.e.

$$\widehat{E_{t+k}^{(t)}} = \sum_{i+j=t+k} \widehat{X}_{i,j}, \tag{5.78}$$

we have a sum over different accident years i only. Since we have assumed that different accident years are independent, we can simply add the second moments to obtain the estimated variance of one accounting year (pay attention to the fact, that the accounting years are not independent). Hence the overall variance of one accounting year is estimated by

$$\widehat{V_{t+k}^{(t)}} = \widehat{\mathrm{Var}}\left(X_{t+k}\middle|\mathcal{T}_t\right) = \sum_i \widehat{\mathrm{Var}}\left(X_{i,j^*(i)+k}\middle|C_{i,j^*(i)}\right) = \sum_i \widehat{V_{t+k}^{(t)}}(i). \tag{5.79}$$

Hence the estimated standard deviations for accounting years $\widehat{V_{t+k}^{(t)}}^{1/2}$ and its estimated variational coefficients are given by:

$t+k$	$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	$t+7$	$t+8$	$t+9$
l_{t+k}	5'226'536	4'179'394	3'131'668	2'127'272	1'561'879	1'177'744	744'287	445'521	86'555
$\widehat{V_{t+k}^{(t)}}^{1/2}$	610'035	595'147	556'123	424'414	333'918	237'751	135'798	126'278	51'779
Vco	11.7%	14.2%	17.8%	20.0%	21.4%	20.2%	18.2%	28.3%	59.8%

We define the cost-of-capital charge for accounting year $t+k$ as follows

$$\mathrm{CoC}(k) = i\ \beta\ \widehat{V_{t+k}^{(t)}}^{1/2} \tag{5.80}$$

and the valuation portfolio protected against insurance technical risks is then given by

$$l_{t+k}^{*} = l_{t+k} + \mathrm{CoC}(k), \tag{5.81}$$

with $i = 8\%$ and $\beta = \Phi^{-1}(99\%)$. Observe that in (5.81) we consider the estimated expected claims payments l_{t+k} (this is only an estimate based on $\mathcal{B}_J$) and the cost-of-capital charge $\mathrm{CoC}(k)$ for the valuation portfolio protected against insurance technical risks. Then, we obtain the valuation portfolio protected against insurance technical risks given by

$$\mathrm{VaPo}_{(t)}^{prot} = \sum_{k\geq 1} l_{t+k}^{*}\ Z^{(t+k)}. \tag{5.82}$$

We obtain the numerical values provided in Table 5.4.

$t+k$	$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	$t+7$	$t+8$	$t+9$
CoC	113'532	110'761	103'499	78'987	62'145	44'247	25'273	23'501	9'636
l_{t+k}^{*}	5'340'068	4'290'156	3'235'166	2'206'259	1'624'024	1'221'991	769'560	469'023	96'191

Table 5.4. Valuation portfolio protected against insurance technical risk (process error)

Hence for the three different accounting principles (nominal, constant interest rate, risk-free rates SST (see Table 5.2)) we obtain Table 5.5. Table 5.5 should be compared to Table 5.3.

	VaPo	VaPo^{prot}	difference
1) nominal	18'680'856	19'252'438	571'582
2) $r = 1.50\%$	17'873'967	18'416'946	542'979
3) SST rates	17'847'512	18'387'990	540'479

Table 5.5. Monetary value of the valuation portfolio protected against insurance technical risks (process error) for different accounting principles

So we find that the cost-of-capital margin adds about 3% of monetary value to the value of the valuation portfolio. This has comparable size to the loadings in Salzmann-Wüthrich [SW10].

□

Question. As a regulator: are we satisfied with this solution?

Answer: **NO!**

In fact, we have replaced the certainty equivalent $E_{t+k}^{(t)} + i\ \beta\ {V_{t+k}^{(t)}}^{1/2}$ by an estimate $\widehat{E_{t+k}^{(t)}} + i\ \beta\ {\widehat{V_{t+k}^{(t)}}}^{1/2}$. This estimate covers the expected liabilities and gives a loading for the process variance. But it does not give a loading for the uncertainties in their parameter estimates (that is, we have replaced the chain-ladder factors f_k by the estimators $\widehat{f}_k$). Therefore, we also require a loading for the possible deviations

$$\widehat{E_{t+k}^{(t)}} - E_{t+k}^{(t)}, \tag{5.83}$$

which is called **parameter estimation uncertainty**.

Second approach for protection against insurance technical risks

We consider the (conditional) **mean square error of prediction** (MSEP) which is defined as follows, see Wüthrich-Merz [WM08], Section 3.1,

$$\begin{aligned} \operatorname{msep}_{X_{i,j^*(i)+k}|\mathcal{T}_t}\left(\widehat{E_{t+k}^{(t)}}(i)\right) &= E\left[\left(X_{i,j^*(i)+k} - \widehat{E_{t+k}^{(t)}}(i)\right)^2 \middle| \mathcal{T}_t\right] \\ &= \operatorname{Var}\left(X_{i,j^*(i)+k}\middle|\mathcal{T}_t\right) + \left(E_{t+k}^{(t)}(i) - \widehat{E_{t+k}^{(t)}}(i)\right)^2 \\ &= V_{t+k}^{(t)}(i) + \left(E_{t+k}^{(t)}(i) - \widehat{E_{t+k}^{(t)}}(i)\right)^2, \end{aligned} \tag{5.84}$$

note that we use that a $\widehat{E_{t+k}^{(t)}}(i)$ is $\mathcal{T}_t$-measurable predictor for $X_{i,j^*(i)+k}$.

So far we have only given an estimate for the **process errors** $W_{t+k}^{(t)}(i)$ and $V_{t+k}^{(t)}(i)$ of the insurance technical risks. Since we do not know the true parameters f_k and σ_k^2, we need to estimate them from the observations $\mathcal{T}_t$. Of course, in doing so, we have an additional potential error term, the so-called **parameter estimation error**. The parameter estimation error is reflected by the difference

$$\left(E_{t+k}^{(t)}(i) - \widehat{E_{t+k}^{(t)}}(i)\right)^2. \tag{5.85}$$

To calculate the parameter estimation error we would have to evaluate (5.85), but this requires that the true chain-ladder factors f_k are known (which, unfortunately, is not the case). Hence in the sequel we provide an estimate for (5.85). This estimator is based on an analysis on how much the estimators $\widehat{f}_k$ fluctuate around f_k.

To get an understanding for the parameter estimation error we introduce a time series version of the chain-ladder model (see Murphy [Mu94], Buchwalder et al. [BBMW05, BBMW06a])

Model Assumptions 5.11 *In addition to Model Assumptions 5.3 we assume that*

$$C_{i,j} = f_{j-1}\ C_{i,j-1} + \sigma_{j-1}\ \sqrt{C_{i,j-1}}\ \varepsilon_{i,j}, \tag{5.86}$$

with $\varepsilon_{i,j}$ independent, centered random variables with variance 1.

Remarks.

- We should also make sure that the $C_{i,j}$ stay positive $\mathcal{P}$-a.s. For the moment, we assume that this is a purely mathematical problem which is not further treated here. For a mathematically consistent treatment we refer to Model Assumptions 3.9 in Wüthrich-Merz [WM08].
- (5.86) defines an additive time series model. We could also define a multiplicative time series model in the spirit of Wüthrich [Wü10]. This way one may easily omit the difficulty with the positivity of $C_{i,j}$.
- (5.86) does not contradict the chain-ladder Model Assumptions 5.3.
- (5.86) defines an explicit stochastic model, which tells us what values our observations $\mathcal{B}_J$ could also have. In order to determine the parameter estimation error, we need to see, how much $\widehat{E_{t+k}^{(t)}(i)}$ fluctuates around its mean $E_{t+k}^{(t)}(i)$ for other realizations, i.e. how would $\widehat{E_{t+k}^{(t)}(i)}$ look like, if we would have different observations $\mathcal{B}_J$?

The chain-ladder factors are (with (5.86))

$$\widehat{f}_j = \frac{\sum_{i=1}^{i^*(j+1)} C_{i,j+1}}{\sum_{i=1}^{i^*(j+1)} C_{i,j}} = f_j + \frac{\sigma_j}{\sum_{i=1}^{i^*(j+1)} C_{i,j}} \sum_{i=1}^{i^*(j+1)} (C_{i,j})^{1/2}\ \varepsilon_{i,j+1}. \tag{5.87}$$

That is, (5.87) shows how the chain-ladder estimates $\widehat{f}_j$ fluctuate around the true f_j.

There are various different ways to resample the chain-ladder factors in this time series model, and there is an extended discussion in the literature on this issue (see Buchwalder et al. [BBMW06a], Mack et al. [MQB06] and Gisler [Gi06]). We will not further discuss this here. The only point that we would like to mention is that a Bayesian treatment of the parameter estimation error term leads to answers that are consistent within the defined (Bayesian) model, see Gisler [Gi06], Gisler-Wüthrich [GW08] and Bühlmann et al. [BFGMW09].

Our goal here is to conditionally resample these chain-ladder factors $\widehat{f}_j$ from a frequentist's point of view (this corresponds to Approach 3 in Buchwalder et al. [BBMW06a]), i.e. given the observation $\mathcal{B}_j$ we resample $\widehat{f}_j$ (this is the next step in the time series). This way we obtain resampled values $\widehat{f}_j^{\mathcal{B}_j}$ by

$$\widehat{f}_j^{\mathcal{B}_j} = f_j + \frac{\sigma_j}{\sum_{i=1}^{i^*(j+1)} C_{i,j}} \sum_{i=1}^{i^*(j+1)} (C_{i,j})^{1/2}\, \widetilde{\varepsilon}_{i,j+1}, \tag{5.88}$$

where $(\widetilde{\varepsilon}_{i,j})_{i,j}$ and $(\varepsilon_{i,j})_{i,j}$ are independent copies. Hence, we see that

$$\widehat{f}_j^{\mathcal{B}_j} \overset{(d)}{=} \widehat{f}_j \qquad \text{given } \mathcal{B}_j. \tag{5.89}$$

If we continue this procedure in an iterative way for every j we obtain a set of random variables $\widehat{f}_0^{\mathcal{B}_0}, \ldots, \widehat{f}_{J-1}^{\mathcal{B}_{J-1}}$ which are conditionally, given $\mathcal{B}_J$, independent. $\mathcal{B}_J$ plays the role of the (deterministic) volume measure in $\widehat{f}_j$ (denominator) and we only resample the numerator of $\widehat{f}_j$ which leads to $\widehat{f}_j^{\mathcal{B}_j}$ (next step in time series).

Observe that we do not claim that $\widehat{f}_j$ are independent (in fact their squares are correlated, see Mack et al. [MQB06] and Wüthrich-Merz [WM08], Lemma 3.8), but we use that the conditionally resampled values are independent given $\mathcal{B}_J$ (which gives a multiplicative structure).

For simplicity, we denoted this conditionally resampling measure by $P_{\mathcal{B}_J}$ and we drop the superscript $\mathcal{B}_j$ in the conditionally resampled observations $\widehat{f}_j^{\mathcal{B}_j}$. We have the following properties:

$$\text{1)}\quad \widehat{f}_0, \ldots, \widehat{f}_{J-1} \text{ are independent under the measure } P_{\mathcal{B}_J}, \tag{5.90}$$

$$\text{2)}\quad E_{\mathcal{B}_J}\left[\widehat{f}_j\right] = f_j, \tag{5.91}$$

$$\text{3)}\quad \mathrm{Var}_{P_{\mathcal{B}_J}}\left(\widehat{f}_j\right) = E_{\mathcal{B}_J}\left[\widehat{f}_j^2\right] - f_j^2 \;=\; \frac{\sigma_j^2}{\sum_{i=1}^{i^*(j+1)} C_{i,j}}, \tag{5.92}$$

see (5.56).

Remarks.

- In fact we do not need (5.86), all that we need for the derivation of the parameter estimation error is (5.87). For a Bayesian model we refer to Gisler-Wüthrich [GW08].
- (5.87) describes a possible model for the claims development factors. The fluctuation around f_j will be the crucial term do determine the quality of our estimate $\widehat{E_{t+k}^{(t)}}(i)$.
- In the sequel, we assume that $\mathcal{T}_t = \{C_{i,j};\ i + j \leq t\}$ is known and that we work with the conditionally resampled chain-ladder factor estimates (under the measure $P_{\mathcal{B}_J}$) as described above.

In the conditional resampling approach as presented in the paper of Buchwalder et al. [BBMW06a] the parameter estimation error (5.85) for a single accident year i is estimated by

$$
\begin{aligned}
E_{\mathcal{B}_J}&\left[\left(E_{t+k}^{(t)}(i)-\widehat{E_{t+k}^{(t)}}(i)\right)^2\right] && (5.93)\\
&= C_{i,j^*(i)}^2\; E_{\mathcal{B}_J}\left[\left(\prod_{l=0}^{k-2} f_{j^*(i)+l}\;\left(f_{j^*(i)+k-1}-1\right)\right.\right.\\
&\qquad\qquad\left.\left.-\prod_{l=0}^{k-2}\widehat{f}_{j^*(i)+l}\;\left(\widehat{f}_{j^*(i)+k-1}-1\right)\right)^2\right]\\
&= C_{i,j^*(i)}^2\;\mathrm{Var}_{P_{\mathcal{B}_J}}\left(\prod_{l=0}^{k-2}\widehat{f}_{j^*(i)+l}\;\left(\widehat{f}_{j^*(i)+k-1}-1\right)\right).
\end{aligned}
$$

Hence we need to study this last term. Due to the independence and the unbiasedness of the conditionally resampled $\widehat{f}_j$, given $\mathcal{B}_J$, we have

$$
\begin{aligned}
E_{\mathcal{B}_J}&\left[\left(\prod_{l=0}^{k-2} f_{j^*(i)+l}\;\left(f_{j^*(i)+k-1}-1\right)-\prod_{l=0}^{k-2}\widehat{f}_{j^*(i)+l}\;\left(\widehat{f}_{j^*(i)+k-1}-1\right)\right)^2\right]\\
&= \mathrm{Var}_{P_{\mathcal{B}_J}}\left(\prod_{l=0}^{k-2}\widehat{f}_{j^*(i)+l}\;\left(\widehat{f}_{j^*(i)+k-1}-1\right)\right) && (5.94)\\
&= E_{\mathcal{B}_J}\left[\prod_{l=0}^{k-2}\widehat{f}^2_{j^*(i)+l}\;\left(\widehat{f}_{j^*(i)+k-1}-1\right)^2\right]-\prod_{l=0}^{k-2} f^2_{j^*(i)+l}\;\left(f_{j^*(i)+k-1}-1\right)^2\\
&= \prod_{l=0}^{k-2}\left(\frac{\sigma^2_{j^*(i)+l}}{\sum_n C_{n,j^*(i)+l}}+f^2_{j^*(i)+l}\right)\left(\frac{\sigma^2_{j^*(i)+k-1}}{\sum_n C_{n,j^*(i)+k-1}}+\left(f_{j^*(i)+k-1}-1\right)^2\right)\\
&\qquad -\prod_{l=0}^{k-2} f^2_{j^*(i)+l}\;\left(f_{j^*(i)+k-1}-1\right)^2,
\end{aligned}
$$

where in the last step we have used (5.90) and (5.92).

Remark. If we would work in a Bayesian model similar to Bühlmann et al. [BFGMW09] we would obtain a similar result because the posterior distributions of the chain-ladder factors are independent, given the observations $\mathcal{B}_J$.

This last expression can be rewritten and approximated by

$$\prod_{l=0}^{k-2} f_{j^*(i)+l}^2 \ \left(f_{j^*(i)+k-1}-1\right)^2 \tag{5.95}$$

$$\left(\prod_{l=0}^{k-2}\left(\frac{\sigma_{j^*(i)+l}^2/f_{j^*(i)+l}^2}{\sum_n C_{n,j^*(i)+l}}+1\right)\left(\frac{\sigma_{j^*(i)+k-1}^2/\left(f_{j^*(i)+k-1}-1\right)^2}{\sum_n C_{n,j^*(i)+k-1}}+1\right)-1\right)$$

$$\approx \prod_{l=0}^{k-2} f_{j^*(i)+l}^2 \ \left(f_{j^*(i)+k-1}-1\right)^2$$

$$\left(\sum_{l=0}^{k-2}\frac{\sigma_{j^*(i)+l}^2/f_{j^*(i)+l}^2}{\sum_n C_{n,j^*(i)+l}}+\frac{\sigma_{j^*(i)+k-1}^2/\left(f_{j^*(i)+k-1}-1\right)^2}{\sum_n C_{n,j^*(i)+k-1}}\right).$$

In the last step we have made a linear approximation, see (A.1) in Merz-Wüthrich [MW08], which leads to the well-known Mack formula [Ma93]. Henceforth, the parameter estimation error for a single accident year i is estimated by

$$\widetilde{V_{t+k}^{(t)}}(i) \overset{def.}{=} \widehat{E}\left[\left(E_{t+k}^{(t)}(i)-\widehat{E_{t+k}^{(t)}}(i)\right)^2\middle|\mathcal{B}_J\right] \tag{5.96}$$

$$= \widehat{E_{t+k}^{(t)}}^2(i)\ \left(\sum_{l=0}^{k-2}\frac{\widehat{\sigma}_{j^*(i)+l}^2/\widehat{f}_{j^*(i)+l}^2}{\sum_n C_{n,j^*(i)+l}}+\frac{\widehat{\sigma}_{j^*(i)+k-1}^2/\left(\widehat{f}_{j^*(i)+k-1}-1\right)^2}{\sum_n C_{n,j^*(i)+k-1}}\right).$$

This gives the following estimator for the conditional mean square error of prediction:

Estimator 5.12 (Conditional MSEP)

$$\widehat{\text{msep}}_{X_{i,j^*(i)+k}|\mathcal{T}_t}\left(\widehat{E_{t+k}^{(t)}}(i)\right) = \widehat{E}\left[\left(X_{i,j^*(i)+k}-\widehat{E_{t+k}^{(t)}}(i)\right)^2\middle|\mathcal{T}_t\right]$$

$$= \widehat{V_{t+k}^{(t)}}(i)+\widetilde{V_{t+k}^{(t)}}(i), \tag{5.97}$$

where $\widehat{V_{t+k}^{(t)}}(i)$ *is given in* (5.74) *and* $\widetilde{V_{t+k}^{(t)}}(i)$ *is given in* (5.96).

Henceforth in the pragmatic approach (first version with standard deviation loadings) is the VaPo protected against insurance technical risks (for a single accident year i) given by:

period	instrument	cash flow		number of units
$t+1$	$Z^{(t+1)}$	X_{t+1}	$\longrightarrow$	$l^*_{t+1}(i)$
$t+2$	$Z^{(t+2)}$	X_{t+2}	$\longrightarrow$	$l^*_{t+2}(i)$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$t+k$	$Z^{(t+k)}$	X_{t+k}	$\longrightarrow$	$l^*_{t+k}(i)$
$\vdots$	$\vdots$	$\vdots$		$\vdots$

where for $k \geq 1$

$$l^*_{t+k}(i) = \widehat{E^{(t)}_{t+k}}(i) + i\ \beta\ \widehat{E}\left[\left(X_{i,j^*(i)+k} - \widehat{E^{(t)}_{t+k}}(i)\right)^2 \middle| \mathcal{T}_t\right]^{1/2}. \qquad (5.98)$$

Remarks.

- The i in front of the security loading β denotes the cost-of-capital rate, the i in the brackets of $\widehat{E^{(t)}_{t+k}}(\cdot)$ the accident year.
- Now $l^*_{t+k}(i)$ covers both, risks coming from the stochastic process (process error) and uncertainties coming from the fact that we have to estimate parameters, parameter estimation errors.
- **Pay attention** to the fact that we have not considered possible dependencies between the accounting years:
  Indeed, $E^{(t)}_{t+k}(i)$ is estimated by

$$\widehat{E^{(t)}_{t+k}}(i) = C_{i,j^*(i)} \prod_{l=0}^{k-2} \widehat{f}_{j^*(i)+l}\ \left(\widehat{f}_{j^*(i)+k-1} - 1\right) \qquad (5.99)$$

  and $E^{(t)}_{t+k+1}(i)$ is estimated by

$$\widehat{E^{(t)}_{t+k+1}}(i) = C_{i,j^*(i)} \prod_{l=0}^{k-1} \widehat{f}_{j^*(i)+l}\ \left(\widehat{f}_{j^*(i)+k} - 1\right). \qquad (5.100)$$

  Hence they use the same estimated age-to-age factors $\widehat{f}_j$, which means that the parameter estimation errors are correlated. We will not further investigate this problem here, since in the next section we have the same problem, when we aggregate different accident years.

Example 5.1 (revisited).

For our example we calculate both the parameter estimation error $\widetilde{V^{(t)}_{t+k}}(i)^{1/2}$ and the process error $\widehat{V^{(t)}_{t+k}}(i)^{1/2}$. For low development periods, the process error is the dominant term whereas for higher development periods they have about the same size (this comes from the fact that we have only little data to estimate late development factors which gives high uncertainties in these estimates).

Parameter error $\widetilde{V_{t+k}^{(t)}}(i)^{1/2}$.

	0	1	2	3	4	5	6	7	8	9
1										
2										57'628
3									56'970	57'055
4								27'163	56'151	56'199
5							87'733	25'353	51'975	51'565
6						102'068	92'721	27'334	55'408	54'296
7					99'925	113'519	103'131	30'954	62'123	60'182
8				151'271	122'620	137'303	124'760	38'833	76'414	72'259
9			82'715	131'364	104'103	115'123	104'622	33'562	65'004	60'188
10		75'503	92'152	130'499	97'599	104'075	94'627	32'962	61'279	53'294

Process error $\widehat{V_{t+k}^{(t)}}(i)^{1/2}$.

	0	1	2	3	4	5	6	7	8	9
1										
2										48'832
3									75'052	48'603
4								45'268	74'566	48'243
5							178'062	44'398	72'835	46'336
6						225'149	183'669	47'407	77'269	47'781
7					229'965	238'145	194'528	52'055	84'306	50'590
8				346'712	258'879	264'121	216'465	62'761	100'344	56'248
9			226'818	332'762	242'972	244'204	200'828	62'482	98'906	52'140
10		234'816	275'881	364'358	250'614	240'498	199'983	74'127	115'011	51'779

□

5.5.4 Aggregation of parameter estimation errors across different accident years

Now we consider the whole diagonal of our claims development trapezoids (see (5.41)). The expected accounting year payments

$$E_{t+k}^{(t)} = \sum_{i+j=t} C_{i,j}\ f_j \cdots f_{j+k-2}\ \left(f_{j+k-1} - 1\right) \tag{5.101}$$

are estimated by

$$\widehat{E_{t+k}^{(t)}} = \sum_{i+j=t} C_{i,j}\ \widehat{f}_j \cdots \widehat{f}_{j+k-2}\ \left(\widehat{f}_{j+k-1} - 1\right) = \sum_{i=0}^{I} \widehat{E_{t+k}^{(t)}}(i). \tag{5.102}$$

The conditional **mean square error of prediction** (MSEP) is now given by

$$\begin{aligned}
&\mathrm{msep}_{X_k|\mathcal{T}_t}\left(\sum_i \widehat{E_{t+k}^{(t)}}(i)\right)\\
&= E\left[\left(\sum_{i+j=t+k} X_{i,j} - \sum_i \widehat{E_{t+k}^{(t)}}(i)\right)^2 \middle| \mathcal{T}_t\right] \qquad (5.103)\\
&= \sum_i \mathrm{Var}\left(X_{i,j^*(i)+k}\middle|\mathcal{T}_t\right) + \left(\sum_i E_{t+k}^{(t)}(i) - \widehat{E_{t+k}^{(t)}}(i)\right)^2\\
&= \sum_i V_{t+k}^{(t)}(i) + \sum_i \left(E_{t+k}^{(t)}(i) - \widehat{E_{t+k}^{(t)}}(i)\right)^2\\
&\qquad + \sum_{i\neq m}\left(E_{t+k}^{(t)}(i) - \widehat{E_{t+k}^{(t)}}(i)\right)\left(E_{t+k}^{(t)}(m) - \widehat{E_{t+k}^{(t)}}(m)\right)\\
&= \mathrm{msep}_{X_{i,j^*(i)+k}|\mathcal{T}_t}\left(\widehat{E_{t+k}^{(t)}}(i)\right)\\
&\qquad + \sum_{i\neq m}\left(E_{t+k}^{(t)}(i) - \widehat{E_{t+k}^{(t)}}(i)\right)\left(E_{t+k}^{(t)}(m) - \widehat{E_{t+k}^{(t)}}(m)\right).
\end{aligned}$$

Hence the first term is estimated by (5.97), but now we obtain an additional (covariance) term for the parameter estimation error

$$\sum_{i\neq m}\left(E_{t+k}^{(t)}(i) - \widehat{E_{t+k}^{(t)}}(i)\right)\left(E_{t+k}^{(t)}(m) - \widehat{E_{t+k}^{(t)}}(m)\right). \qquad (5.104)$$

As above we resample $\widehat{f}_j$ in the conditional version. Hence we choose again the conditional resampling measure $P_{\mathcal{B}_J}$ and estimate the covariance term by

$$\begin{aligned}
&E_{\mathcal{B}_J}\left[\left(E_{t+k}^{(t)}(i) - \widehat{E_{t+k}^{(t)}}(i)\right)\left(E_{t+k}^{(t)}(m) - \widehat{E_{t+k}^{(t)}}(m)\right)\right] = C_{i,j^*(i)}\ C_{m,j^*(m)}\\
&\times E_{\mathcal{B}_J}\left[\left(\prod_{l=0}^{k-2} f_{j^*(i)+l}\ \left(f_{j^*(i)+k-1}-1\right) - \prod_{l=0}^{k-2}\widehat{f}_{j^*(i)+l}\ \left(\widehat{f}_{j^*(i)+k-1}-1\right)\right)\right.\\
&\times\left.\left(\prod_{l=0}^{k-2} f_{j^*(m)+l}\ \left(f_{j^*(m)+k-1}-1\right) - \prod_{l=0}^{k-2}\widehat{f}_{j^*(m)+l}\ \left(\widehat{f}_{j^*(m)+k-1}-1\right)\right)\right].
\end{aligned} \qquad (5.105)$$

It now depends on the choice of i, m, k whether the expression above is different from zero: W.l.o.g. we assume that $m > i$. If

$$\begin{aligned}
j^*(m)+k-1 < j^*(i) \quad &\Longleftrightarrow \quad m-(k-1) > i \qquad (5.106)\\
&\Longleftrightarrow \quad m > i+k-1,
\end{aligned}$$

then we only use different development factors for the estimation of $E_{t+k}^{(t)}(i)$ and $E_{t+k}^{(t)}(m)$. I.e. for this indices (5.105) is equal to zero because we have independence.

Choose m such that $i < m \le i+k-1$, and abbreviate

$$g_{i,k-2} = \prod_{l=0}^{k-2} f_{j^*(i)+l} \quad \text{and} \quad \widehat{g}_{i,k-2} = \prod_{l=0}^{k-2} \widehat{f}_{j^*(i)+l}. \tag{5.107}$$

Hence the last term in (5.105) is

$$\begin{aligned} E_{\mathcal{B}_J}\Bigg[\Big(g_{i,k-2}\ \left(f_{j^*(i)+k-1}-1\right) - \widehat{g}_{i,k-2}\ \left(\widehat{f}_{j^*(i)+k-1}-1\right)\Big) \\ \Big(g_{m,k-2}\ \left(f_{j^*(m)+k-1}-1\right) - \widehat{g}_{m,k-2}\ \left(\widehat{f}_{j^*(m)+k-1}-1\right)\Big)\Bigg] \\ = \mathrm{Cov}_{P_{\mathcal{B}_J}}\left(\widehat{g}_{i,k-2}\ \left(\widehat{f}_{j^*(i)+k-1}-1\right), \widehat{g}_{m,k-2}\ \left(\widehat{f}_{j^*(m)+k-1}-1\right)\right) \\ = E_{\mathcal{B}_J}\left[\widehat{g}_{i,k-2}\ \left(\widehat{f}_{j^*(i)+k-1}-1\right)\ \widehat{g}_{m,k-2}\ \left(\widehat{f}_{j^*(m)+k-1}-1\right)\right] \\ - g_{i,k-2}\ \left(f_{j^*(i)+k-1}-1\right)\ g_{m,k-2}\ \left(f_{j^*(m)+k-1}-1\right). \end{aligned} \tag{5.108}$$

The first term on the r.h.s. of (5.108) is equal to (we use the unbiasedness and the independence (5.90))

$$\begin{aligned} \prod_{l=j^*(m)}^{j^*(i)-1} f_l \prod_{l=j^*(i)}^{j^*(m)+k-2} E_{\mathcal{B}_J}\left[\widehat{f}_l^2\right] \\ E_{\mathcal{B}_J}\left[\left(\widehat{f}_{j^*(m)+k-1}-1\right)\ \widehat{f}_{j^*(m)+k-1}\right] \prod_{l=j^*(m)+k}^{j^*(i)+k-2} f_l\ \left(f_{j^*(i)+k-1}-1\right). \end{aligned} \tag{5.109}$$

This term is equal to (see (5.92))

$$\begin{aligned} \prod_{l=j^*(m)}^{j^*(i)-1} f_l \prod_{l=j^*(i)}^{j^*(m)+k-2} \left(\frac{\sigma_l^2}{\sum_n C_{n,l}} + f_l^2\right) \\ \left(\frac{\sigma_{j^*(m)+k-1}^2}{\sum_n C_{n,j^*(m)+k-1}} + \left(f_{j^*(m)+k-1}-1\right)\ f_{j^*(m)+k-1}\right) \\ \prod_{l=j^*(m)+k}^{j^*(i)+k-2} f_l\ \left(f_{j^*(i)+k-1}-1\right). \end{aligned} \tag{5.110}$$

Collecting all the term, we obtain for the r.h.s. of (5.108)

$$\prod_{l=j^*(m)}^{j^*(i)-1} f_l \left[\prod_{l=j^*(i)}^{j^*(m)+k-2} \left(\frac{\sigma_l^2}{\sum_n C_{n,l}} + f_l^2 \right) \right.$$

$$\left(\frac{\sigma_{j^*(m)+k-1}^2}{\sum_n C_{n,j^*(m)+k-1}} + \left(f_{j^*(m)+k-1} - 1\right) \; f_{j^*(m)+k-1} \right) \tag{5.111}$$

$$\left. - \prod_{l=j^*(i)}^{j^*(m)+k-2} f_l^2 \; \left(\left(f_{j^*(m)+k-1} - 1\right) \; f_{j^*(m)+k-1} \right) \right]$$

$$\prod_{l=j^*(m)+k}^{j^*(i)+k-2} f_l \; \left(f_{j^*(i)+k-1} - 1\right).$$

This leads to the following estimates of the covariance terms for $i < m \leq i + k - 1$

$$\widehat{E_{t+k}^{(t)}}(i) \; \widehat{E_{t+k}^{(t)}}(m) \left[\prod_{l=j^*(i)}^{j^*(m)+k-2} \left(\frac{\widehat{\sigma}_l^2 / \widehat{f}_l^2}{\sum_n C_{n,l}} + 1 \right) \right.$$

$$\left. \left(\frac{\widehat{\sigma}_{j^*(m)+k-1}^2 / \left(\left(\widehat{f}_{j^*(m)+k-1} - 1 \right) \; \widehat{f}_{j^*(m)+k-1} \right)}{\sum_n C_{n,j^*(m)+k-1}} + 1 \right) - 1 \right].$$

If we do a linear approximation (as above, see also (A.1) in Merz-Wüthrich [MW08]) we get for the correlation term if $i < m \leq i + k - 1$

$$\widetilde{V_{t+k}^{(t)}}(i, m) \stackrel{def.}{=} \widehat{E_{t+k}^{(t)}}(i) \; \widehat{E_{t+k}^{(t)}}(m) \tag{5.112}$$

$$\left[\sum_{l=j^*(i)}^{j^*(m)+k-2} \frac{\widehat{\sigma}_l^2 / \widehat{f}_l^2}{\sum_n C_{n,l}} + \frac{\widehat{\sigma}_{j^*(m)+k-1}^2 / \left(\left(\widehat{f}_{j^*(m)+k-1} - 1 \right) \; \widehat{f}_{j^*(m)+k-1} \right)}{\sum_n C_{n,j^*(m)+k-1}} \right].$$

Estimator 5.13 (Conditional MSEP) *Assume $J + 1 = I = t$. The conditional mean square error of prediction for the accounting year k is estimated by (see also (5.102))*

$$\widehat{\text{msep}}_{X_k|\mathcal{T}_t} \left(\sum_i \widehat{E_{t+k}^{(t)}}(i) \right)$$

$$= \widehat{E} \left[\left(\sum_{i+j=t+k} X_{i,j} - \widehat{E_{t+k}^{(t)}} \right)^2 \middle| \mathcal{T}_t \right]$$

$$= \sum_{i=2}^{I} \left(\widehat{V_{t+k}^{(t)}}(i) + \widetilde{V_{t+k}^{(t)}}(i) + 2 \sum_{m=i+1}^{i+k-1} \widetilde{V_{t+k}^{(t)}}(i, m) \right),$$

where $\widehat{V_{t+k}^{(t)}}(i)$ *is given in* (5.74), $\widetilde{V_{t+k}^{(t)}}(i)$ *is given in* (5.96) *and* $\widetilde{V_{t+k}^{(t)}}(i,m)$ *is given in* (5.112).

Henceforth in the pragmatic approach (first version with standard deviation loadings) is the VaPo protected against insurance technical risks (for aggregate accident years) given by

period	instrument	cash flow		number of units
$t+1$	$Z^{(t+1)}$	X_{t+1}	$\longrightarrow$	l_{t+1}^*
$t+2$	$Z^{(t+2)}$	X_{t+2}	$\longrightarrow$	l_{t+2}^*
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$t+k$	$Z^{(t+k)}$	X_{t+k}	$\longrightarrow$	l_{t+k}^*
$\vdots$	$\vdots$	$\vdots$		$\vdots$

where for $k \geq 1$ (and cost-of-capital rate i)

$$l_{t+k}^* = \widehat{E_{t+k}^{(t)}} + i\ \beta\ \widehat{E}\left[\left(\sum_{i+j=t+k} X_{i,j} - \widehat{E_{t+k}^{(t)}}\right)^2 \middle| \mathcal{T}_t\right]^{1/2}. \tag{5.113}$$

Remark. As above, our l_{t+k}^* gives now a protection against process and parameter estimation errors. But so far it doesn't take into account that parameter estimation errors for different accounting years are correlated. Hence in that sense, neglecting the dependencies between accounting years, we have still a simplified model.

Example 5.1 (revisited).

We obtain the following values for the parameter estimation errors: In the claims development triangle we give the individual parameter estimation errors $\widetilde{V_{t+k}^{(t)}}(i)^{1/2}$, whereas the last column illustrates the square root of the aggregate covariance terms within the accounting years

$$\widetilde{\mathrm{Cov}}^{1/2} = \left(2 \sum_{i<m} \widetilde{V_{t+k}^{(t)}}(i,m)\right)^{1/2}. \tag{5.114}$$

Hence we have:

	0	1	2	3	4	5	6	7	8	9	$\widetilde{\mathrm{Cov}}^{1/2}$
1											
2										57'628	0
3									56'970	57'055	133'023
4								27'163	56'151	56'199	151'288
5							87'733	25'353	51'975	51'565	135'535
6						102'068	92'721	27'334	55'408	54'296	112'891
7					99'925	113'519	103'131	30'954	62'123	60'182	77'669
8				151'271	122'620	137'303	124'760	38'833	76'414	72'259	38'802
9			82'715	131'364	104'103	115'123	104'622	33'562	65'004	60'188	19'970
10		75'503	92'152	130'499	97'599	104'075	94'627	32'962	61'279	53'294	0

Henceforth, the estimated square root of the conditional MSEP for accounting years $t+k \geq t+1$ is $\widehat{\text{msep}}_k^{1/2} = \widehat{E}\left[\left(\sum_{i+j=t+k} X_{i,j} - \widehat{E^{(t)}_{t+k}}\right)^2 \middle| \mathcal{T}_t\right]^{1/2}$, and its estimated variational coefficients are given by:

$t+k$	$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	$t+7$	$t+8$	$t+9$
l_{t+k}	5'226'536	4'179'394	3'131'668	2'127'272	1'561'879	1'177'744	744'287	445'521	86'555
$\widehat{\text{msep}}_k^{1/2}$	665'562	664'239	629'383	493'486	392'859	286'530	174'586	154'022	74'305
Vco	12.7%	15.9%	20.1%	23.2%	25.2%	24.3%	23.5%	34.6%	85.8%

We define the cost-of-capital charge for accounting year $t+k$ as follows

$$\text{CoC}(k) = i\ \beta\ \widehat{\text{msep}}_k^{1/2}, \tag{5.115}$$

for given cost-of-capital rate i and security loading β. Of course, both $\text{CoC}(k)$ and $\widehat{\text{msep}}_k^{1/2}$ depend on t because for the parameter estimation we have used information $\mathcal{T}_t$.

The valuation portfolio protected against insurance technical risks is defined as

$$l^*_{t+k} = l_{t+k} + \text{CoC}(k), \tag{5.116}$$

with $i = 8\%$ and $\beta = \Phi^{-1}(99\%)$. Then we obtain the following table.

$t+k$	$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	$t+7$	$t+8$	$t+9$
CoC	123'866	123'620	117'133	91'842	73'114	53'326	32'492	28'665	13'829
l^*_{t+k}	5'350'402	4'303'015	3'248'801	2'219'113	1'634'993	1'231'069	776'779	474'186	100'383

Table 5.6. Valuation portfolio protected against insurance technical risk (process error and parameter estimation error)

Hence for the three different accounting principles (nominal, constant interest rate, risk-free rates SST (see Table 5.2)) we obtain (see also Tables 5.3 and 5.5):

	VaPo	VaPo^{prot}	difference
1) nominal	18'680'856	19'338'741	657'886
2) $r = 1.50\%$	17'873'967	18'497'998	624'031
3) SST rates	17'847'512	18'468'169	620'657

Table 5.7. Monetary value of the valuation portfolio protected against insurance technical risks (process error and parameter estimation error) for different accounting principles

□

5.6 Unallocated loss adjustment expenses

5.6.1 Motivation

In this section we describe the "New York"-method for the estimation of unallocated loss adjustment expenses (ULAE). The "New York"-method for estimating ULAE is found in the literature e.g. as footnotes in Feldblum [Fe03] and CAS notes [CAS90] and in more detail in Buchwalder et al. [BBMW06b]. Sometimes this method is also called paid-to-paid method.

In non-life insurance there are usually two different kinds of claims handling costs, external ones and internal ones. External costs like costs for external lawyers or for an external expertise etc. are usually allocated to single claims and are therefore contained in the usual claims payments and loss development figures. These payments are called allocated loss adjustment expenses (ALAE). Typically, internal loss adjustment expenses (income of claims handling department, maintenance of claims handling system, internal layers, management fees, etc.) are not contained in the claims figures and therefore have to be estimated separately. These internal costs can usually not be allocated to single claims. We call these costs therefore unallocated loss adjustment expenses (ULAE). From a regulatory point of view, we should also build provisions for these costs/expenses because they are part of the claims handling process which guarantees that an insurance company is able to meet all its obligations. I.e. ULAE reserves should guarantee the smooth run-off of the old insurance liabilities without a "pay-as-you-go" system from new business/premium.

Concluding this means that ULAE reserves should also be part of the valuation portfolio, if we want to have a self-financing run-off of an insurance portfolio.

5.6.2 Pure claims payments

Usually, claims development figures only consist of "pure" claims payments not containing ULAE charges. They are usually studied in loss development triangles or trapezoids as described above (see Section 5.5).

In this section we denote by $X_{i,j}^{(pure)}$ the "pure" incremental payments for accident year $i \in \{1, \dots, I\}$ in development year $j \in \{0, \dots, J\}$. "Pure" always means, that these quantities do not contain ULAE (this is exactly the quantity studied in Section 5.5). The cumulative pure payments for accident year i after development period j are denoted by (see (5.29))

$$C_{i,j}^{(pure)} = \sum_{k=0}^{j} X_{i,k}{}^{(pure)}. \tag{5.117}$$

We assume that $X_{i,j}^{(pure)} = 0$ for all $j > J$, i.e. the ultimate pure cumulative loss is given by $C_{i,J}^{(pure)}$.

We have observations for $\mathcal{T}_t = \{X_{i,j}^{(pure)} : 1 \le i \le I \text{ and } 0 \le j \le \min\{J, t-i\}\}$ and the complement of $\mathcal{T}_t$ needs to be predicted.

For the New York-method we also need a second type of development trapezoids, namely a "reporting" trapezoid: For accident year i, $Z_{i,j}^{(pure)}$ denotes the pure cumulative ultimate claim amount for all those claims, which are reported up to (and including) development year j. Hence

$$\left(Z_{i,0}^{(pure)}, Z_{i,1}^{(pure)}, \ldots\right) \tag{5.118}$$

with $Z_{i,J}^{(pure)} = C_{i,J}^{(pure)}$ describes how the pure ultimate claim $C_{i,J}^{(pure)}$ is reported over time at the insurance company. Of course, this reporting pattern is much more delicate, because claims which are reported in the upper set $\widetilde{\mathcal{D}}_t = \{Z_{i,j}^{(pure)} : 1 \le i \le I \text{ and } 0 \le j \le \min\{J, t-i\}\}$ are still developing, since usually it takes quite some time between the reporting and the final settlement of a claim. Hence, the claim sizes/severities in $\widetilde{\mathcal{D}}_t$ are still random variables, however, they are already reported and therefore we already have some information on these reported claims. In general, the final value for $Z_{i,j}^{(pure)}$ is only known at time $i + J$.

Remark. The New York-method has to be understood as an algorithm used to estimate expected ULAE payments. This algorithm is not based on a stochastic model. Therefore, we assume in this section that all our variables are deterministic numbers. Stochastic variables are replaced by their best-estimate for its conditional mean at time t. We think that for the current presentation (to explain the New York-method) it is not helpful to work in a stochastic framework. Moreover, the volume of the ULAE is usually comparably small compared to the volume of pure payments. Therefore, often, the main risk drivers come from the pure payments.

5.6.3 ULAE charges

The cumulative ULAE payments for accident year i until development period j are denoted by $C_{i,j}^{(ULAE)}$. And finally, the total cumulative payments (pure and ULAE) are denoted by

$$C_{i,j} = C_{i,j}^{(pure)} + C_{i,j}^{(ULAE)}. \tag{5.119}$$

The cumulative ULAE payments $C_{i,j}^{(ULAE)}$ and the incremental ULAE charges

$$X_{i,j}^{(ULAE)} = C_{i,j}^{(ULAE)} - C_{i,j-1}^{(ULAE)} \tag{5.120}$$

need to be predicted: The main difficulty, now in practice, is that for each accounting year $t \le I$ we have only one aggregated observation

$$X_t^{(ULAE)} = \sum_{\substack{i+j=t \\ 0 \le j \le J}} X_{i,j}^{(ULAE)} \qquad \text{(sum over } t\text{-diagonal).} \tag{5.121}$$

That is, ULAE payments are usually not available for single accident years but rather we have a position "Total ULAE Expenses" for each accounting year t (in general ULAE charges are contained in the position "Administrative Expenses" in the annual profit-and-loss statement).

The reason for having only aggregated observations per accounting year is that in general the claims handling department treats several claims from different accident years simultaneously. Only an activity-based cost allocation split then allocates these expenses to different accident years. Hence, for the estimation of future ULAE payments we need first to define an appropriate model in order to split the aggregated observations $X_t^{(ULAE)}$ into the different accident years $X_{i,j}^{(ULAE)}$.

5.6.4 New York-method

The New York-method assumes that one part of the ULAE charge is proportional to the claims registration (denote this proportion by $r \in [0,1]$) and the other part is proportional to the settlement (payments) of the claims (proportion $1-r$).

Assumption 5.14 *There exist two (incremental) development patterns* $(\gamma_j)_{j=0,\ldots,J}$ *and* $(\alpha_j)_{j=0,\ldots,J}$ *with* $\gamma_j \ge 0$, $\alpha_j \ge 0$, *for all* j, *and* $\sum_{j=0}^J \gamma_j = \sum_{j=0}^J \alpha_j = 1$ *such that (***cash flow or payout pattern***)*

$$X_{i,j}^{(pure)} = \gamma_j \; C_{i,J}^{(pure)} \tag{5.122}$$

*and (***reporting pattern***)*

$$Z_{i,j}^{(pure)} = \sum_{l=0}^{j} \alpha_l \; C_{i,J}^{(pure)} \tag{5.123}$$

for all i *and* j.

Remarks.

- Equation (5.122) describes, how the pure ultimate claim $C_{i,J}^{(pure)}$ is paid over time. In fact γ_j gives the cash flow pattern for the pure ultimate claim $C_{i,J}^{(pure)}$. We assume that this payout model satisfies the classical deterministic chain-ladder assumptions for cumulative payments. Therefore we propose that γ_j is estimated by the classical chain-ladder factors f_j, see (5.43)

$$\widehat{\gamma}_j = \frac{1}{f_j \cdots f_{J-1}} \left(1 - \frac{1}{f_{j-1}}\right). \tag{5.124}$$

- The estimation of the claims reporting pattern α_j in (5.123) is more delicate. There are not many claims reserving methods which give a reporting pattern α_j. Such a pattern can only be obtained if one separates the claims estimates for reported claims and IBNyR claims (incurred but not yet reported).

Model 5.15 *Assume that there exists $r \in [0,1]$ such that the incremental ULAE payments satisfy for all i and all j*

$$X_{i,j}^{(ULAE)} = \big(r\ \alpha_j + (1-r)\ \gamma_j\big)\ C_{i,J}^{(ULAE)}. \tag{5.125}$$

Henceforth, we assume that one part (r) of the ULAE charge is proportional to the reporting pattern (one has loss adjustment expenses at the registration of the claim), and the other part $(1-r)$ of the ULAE charge is proportional to the claims settlement (measured by the payout pattern).

Definition 5.16 (Paid-to-paid ratio) *We define for all t*

$$\pi_t = \frac{X_t^{(ULAE)}}{X_t^{(pure)}} = \frac{\sum\limits_{\substack{i+j=t\\ 0\le j\le J}} X_{i,j}^{(ULAE)}}{\sum\limits_{\substack{i+j=t\\ 0\le j\le J}} X_{i,j}^{(pure)}}. \tag{5.126}$$

The paid-to-paid ratio measures the ULAE payments relative to the pure claims payments in each accounting year t.

Lemma 5.17 *Assume there exists $\pi > 0$ such that for all accident years i we have*

$$\frac{C_{i,J}^{(ULAE)}}{C_{i,J}^{(pure)}} = \pi. \tag{5.127}$$

Under Assumption 5.14 and Model 5.15 we have for all accounting years t

$$\pi_t = \pi, \tag{5.128}$$

whenever $C_{i,J}^{(pure)}$ is constant in i.

Proof of Lemma 5.17. We have

$$\pi_t = \frac{\sum\limits_{\substack{i+j=t\\ 0\le j\le J}} X_{i,j}^{(ULAE)}}{\sum\limits_{\substack{i+j=t\\ 0\le j\le J}} X_{i,j}^{(pure)}} = \frac{\sum_{j=0}^{J} \big(r\ \alpha_j + (1-r)\ \gamma_j\big)\ C_{t-j,J}^{(ULAE)}}{\sum_{j=0}^{J} \gamma_j\ C_{t-j,J}^{(pure)}}$$

$$= \pi\ \frac{\sum_{j=0}^{J} \big(r\ \alpha_j + (1-r)\ \gamma_j\big)\ C_{t-j,J}^{(pure)}}{\sum_{j=0}^{J} \gamma_j\ C_{t-j,J}^{(pure)}} = \pi. \tag{5.129}$$

This finishes the proof.

□

We define the following split of the claims reserves for accident year i at time j:

$$R_{i,j}^{(pure)} = \sum_{l>j} X_{i,l}^{(pure)} = \sum_{l>j} \gamma_l \; C_{i,J}^{(pure)} \text{(total reserv. for pure future paym.)},$$

$$R_{i,j}^{(IBNR)} = \sum_{l>j} \alpha_l \; C_{i,J}^{(pure)} \text{(IBNyR reserves, incurred but not yet reported)},$$

$$R_{i,j}^{(rep)} = R_{i,j}^{(pure)} - R_{i,j}^{(IBNR)} \quad \text{(reserves for reported claims)}.$$

Result 5.18 (New York-method) *Under the assumptions of Lemma 5.17 we can estimate π using the observations π_t (accounting year data). The reserves for ULAE charges for accident year i after development year j, $R_{i,j}^{(ULAE)} = \sum_{l>j} X_{i,l}^{(ULAE)}$, are estimated by*

$$\begin{aligned} \widehat{R}_{i,j}^{(ULAE)} &= \pi \; r \; R_{i,j}^{(IBNR)} + \pi \; (1-r) \; R_{i,j}^{(pure)} \\ &= \pi \; R_{i,j}^{(IBNR)} + \pi \; (1-r) \; R_{i,j}^{(rep)}. \end{aligned} \tag{5.130}$$

Explanation of Result 5.18.

We have under the assumptions of Lemma 5.17 for all i, j

$$\begin{aligned} R_{i,j}^{(ULAE)} &= \sum_{l>j} \big(r \; \alpha_l + (1-r) \; \gamma_l\big) \; C_{i,J}^{(ULAE)} \qquad (5.131) \\ &= \pi \; \sum_{l>j} \big(r \; \alpha_l + (1-r) \; \gamma_l\big) C_{i,J}^{(pure)} \\ &= \pi \; r \; R_{i,j}^{(IBNR)} + \pi \; (1-r) \; R_{i,j}^{(pure)}. \end{aligned}$$

Remarks.

- In practice one assumes the stationarity condition $\pi_t = \pi$ for all t. This implies that π can be estimated from the accounting data of the annual profit-and-loss statements. Pure claims payments are directly contained in the profit-and-loss statements, whereas ULAE payments are often contained in the administrative expenses. Hence one needs to divide this position into further subpositions (e.g. with the help of an activity-based cost allocation split).
- Result 5.18 gives an easy formula for estimating ULAE reserves. If we are interested into the total ULAE reserves after accounting year t we simply have

$$\widehat{R}_t^{(ULAE)} = \sum_{i+j=t} \widehat{R}_{i,j}^{(ULAE)} = \pi \sum_{i+j=t} R_{i,j}^{(IBNR)} + \pi \; (1-r) \sum_{i+j=t} R_{i,j}^{(rep)}, \tag{5.132}$$

i.e. all we need to know is, how to split the total pure claims reserves into reserves for IBNyR claims and reserves for reported claims.

- The assumptions for the New York-method are rather restrictive in the sense that the pure cumulative ultimate claim $C_{i,J}^{(pure)}$ must be constant in i (see Lemma 5.17). Otherwise the paid-to-paid ratio π_t for accounting years is not the same as the ratio $C_{i,J}^{(ULAE)}/C_{i,J}^{(pure)}$ even if the latter is assumed to be constant. Of course in practice the assumption of equal pure cumulative ultimate claim is never fulfilled. If we relax this condition we obtain the following lemma.

Lemma 5.19 *Assume there exists $\pi > 0$ such that for all accident years i we have*

$$\frac{C_{i,J}^{(ULAE)}}{C_{i,J}^{(pure)}} = \pi \left(r \, \frac{\overline{\alpha}}{\overline{\gamma}} + (1-r) \right)^{-1}, \tag{5.133}$$

with

$$\overline{\gamma} = \frac{\sum_{j=0}^{J} \gamma_j \; C_{t-j,J}^{(pure)}}{\sum_{j=0}^{J} C_{t-j,J}^{(pure)}} \qquad \text{and} \qquad \overline{\alpha} = \frac{\sum_{j=0}^{J} \alpha_j \; C_{t-j,J}^{(pure)}}{\sum_{j=0}^{J} C_{t-j,J}^{(pure)}}. \tag{5.134}$$

Under Assumption 5.14 and Model 5.15 we have for all accounting years t

$$\pi_t = \pi. \tag{5.135}$$

Proof of Lemma 5.19. As in Lemma 5.17 we obtain

$$\pi_t = \pi \left(r \, \frac{\overline{\alpha}}{\overline{\gamma}} + (1-r) \right)^{-1} \frac{\sum\limits_{j=0}^{J} \left(r \; \alpha_j + (1-r) \; \gamma_j \right) \; C_{t-j,J}^{(pure)}}{\sum\limits_{j=0}^{J} \gamma_j \; C_{t-j,J}^{(pure)}} = \pi. \tag{5.136}$$

This finishes the proof.

□

Remarks.

- If all pure cumulative ultimates are equal then $\overline{\gamma} = \overline{\alpha} = 1/J$ (apply Lemma 5.17).
- Assume that there exists a constant $i^{(p)} > 0$ such that for all $i \geq 1$ we have $C_{i+1,J}^{(pure)} = (1 + i^{(p)}) \; C_{i,J}^{(pure)}$, i.e. constant growth $i^{(p)}$. If we blindly apply (5.128) of Lemma 5.17 (i.e. we do not apply the correction factor in (5.133)) and estimate the incremental ULAE payments by (5.130) and (5.132) we obtain

$$
\begin{aligned}
&\sum_{i+j=t} \widehat{X}_{i,j}^{(ULAE)} \\
&\quad = \pi \sum_{j=0}^{J} \big(r\ \alpha_j + (1-r)\ \gamma_j\big)\ C_{t-j,J}^{(pure)} \\
&\quad = \frac{X_t^{(ULAE)}}{X_t^{(pure)}} \sum_{j=0}^{J} \big(r\ \alpha_j + (1-r)\ \gamma_j\big)\ C_{t-j,J}^{(pure)} \qquad (5.137)\\
&\quad = \sum_{i+j=t} X_{i,j}^{(ULAE)} \left(r\ \frac{\overline{\alpha}}{\overline{\gamma}} + (1-r) \right) \\
&\quad = \sum_{i+j=t} X_{i,j}^{(ULAE)} \left(r\ \frac{\sum_{j=0}^{J} \alpha_j\ \left(1+i^{(p)}\right)^{J-j}}{\sum_{j=0}^{J} \gamma_j\ \left(1+i^{(p)}\right)^{J-j}} + (1-r) \right) \\
&\quad > \sum_{i+j=t} X_{i,j}^{(ULAE)},
\end{aligned}
$$

where the last inequality in general holds true for $i^{(p)} > 0$, since usually $(\alpha_j)_j$ is more concentrated than $(\gamma_j)_j$, i.e. we usually have $J > 1$ and

$$\sum_{l=0}^{j} \alpha_l > \sum_{l=0}^{j} \gamma_l \quad \text{for } j = 0, \ldots, J-1. \qquad (5.138)$$

This comes from the fact that the claims are reported before they are paid. I.e. if we blindly apply the New York-method for constant positive growth then the ULAE reserves are too high (for constant negative growth we obtain the opposite sign). This implies that we have always a positive loss experience on ULAE reserves for constant positive growth.

5.6.5 Example

We assume that the observations for π_t are generated by i.i.d. random variables $\frac{X_t^{(ULAE)}}{X_t^{(pure)}}$. Hence we can estimate π from this sequence. Assume $\pi = 10\%$. Moreover $i^{(p)} = 0$ and set $r = 50\%$ (this is the usual choice often done in practice). Moreover we assume that we have the following reporting and cash flow patterns ($J = 4$):

$$(\alpha_0, \ldots, \alpha_4) = (90\%, 10\%, 0\%, 0\%, 0\%), \qquad (5.139)$$
$$(\gamma_0, \ldots, \gamma_4) = (30\%, 20\%, 20\%, 20\%, 10\%). \qquad (5.140)$$

Assume that $C_{i,J}^{(pure)} = 1'000$. Then the ULAE reserves for accident year i are given by

$$\left(\widehat{R}_{i,-1}^{(ULAE)}, \ldots, \widehat{R}_{i,3}^{(ULAE)}\right) = (100, 40, 25, 15, 5), \qquad (5.141)$$

which implies for the estimated incremental ULAE payments

$$\left(\widehat{X}_{i,0}^{(ULAE)}, \dots, \widehat{X}_{i,4}^{(ULAE)}\right) = (60, 15, 10, 10, 5). \tag{5.142}$$

Hence for the total estimated payments $\widehat{X}_{i,j} = X_{i,j}^{(pure)} + \widehat{X}_{i,j}^{(ULAE)}$ we have

$$\left(\widehat{X}_{i,0}, \dots, \widehat{X}_{i,4}\right) = (360, 215, 210, 210, 105). \tag{5.143}$$

5.7 Conclusions on the non-life VaPo

We have constructed both the Valuation Portfolio and the Valuation Portfolio protected against insurance technical risks for a run-off portfolio of a non-life insurance company. In fact our solution is only a first approach to the construction of an appropriate replicating portfolio for a non-life insurance portfolio.

Open problems for example are:

- Appropriate choice of the financial basis, such that we have an independent decoupling into insurance technical risks and financial risks. In fact, this is a rather difficult task because claims inflation and accounting year effects may substantially increase the uncertainties, see also Wüthrich [Wü10].
- Choice of an appropriate risk measure which also takes into account the dependencies between accounting years.
  Moreover, our valuation portfolio protected against insurance technical risks is purely cash flow based. In practice, however, one needs to consider:
  - Accounting rules will influence the loadings. For example, we did not treat the question about the time point when risk capital needs to be available. We therefore also refer to the next chapter and the so-called claims development result, see Merz-Wüthrich [MW08].
  - In our model, there is no diversification between financial risks and insurance technical risks. Insurance technical risks obtain a cost-of-capital margin, financial risks are treated by the Margrabe option. If both risks are treated by a risk margin then one also needs to quantify the diversification effect between these two risk classes, see Wüthrich-Bühlmann [WB08].
- Make an appropriate economic choice for the cost-of-capital rate i. All choices used in practice are rather ad-hoc.
- Choose an appropriate stochastic claims reserving model in order to determine claims reserves, cash flow patterns, uncertainties in the estimates and predictions, etc.
  In general, one has different sources of information, e.g. claims payments, claims incurred information, other internal information, expert knowledge,

etc. Most claims reserving methods are not able to cope with all of these different information channels simultaneously. An interesting new method, the so-called PIC method, treats claims payments, claims incurred and prior knowledge information simultaneously, see Merz-Wüthrich [MW10].

- Here we have only treated the run-off situation of a non-life insurance portfolio. The premium liability risk could, theoretically, also be put into our framework, by assuming that $C_{I+1,-1} = -\Pi_{I+1}$ and then applying the chain-ladder method also to this extended model. However, this approach does often not lead to good estimates for the premium liabilities and premium liability risks. We therefore recommend to rather treat the premium liability risk in a separate model (such as it is done in almost all risk-adjusted solvency calculations).
  In this separate model, premium liability risks are often split into two categories: i) small (single) claims, ii) large single claims, or cumulative events (such as hailstorms, floods, etc.) (see e.g. SST [SST06]).
  The main risk driver in i) is, that the prediction of future parameters may have large uncertainties. This can be modelled assuming that true future parameters are latent variables which we try to predict (see e.g. Wüthrich [Wü06b]).
  The risk drivers in class ii) are often modelled using a compound model (for low frequencies and high severities). One main difficulty in this class of risks is to estimate the parameters, because usually one has only little information. We propose to use internal data, external data and expert opinion for the estimation of the parameters. This can e.g. be done in a Bayesian or credibility framework, similarly as it is done for operational risks in the banking industry (see e.g. Bühlmann et al. [BSW07], Shevchenko-Wüthrich [SW06] and Lambrigger et al. [LSW07]).
  If one models premium liability risks and claims run-off risks separately one should, however, keep in mind that there might be inconsistencies over time because one switches from one model to another, this is also discussed in Ohlsson-Lauzeningks [OL09].

6

Selected Topics

We conclude these lecture notes with selected topics and remarks. These remarks are rather unstructured. They give some ideas that go beyond the presentation of the previous chapters. In general, we believe that still a lot of work, developments and research has to be done in order to come to a unified market-consistent valuation approach that respects economic intuition, financial mathematics and actuarial sciences.

6.1 Sources of losses and profits, profit sharing

We denote by $\mathbf{X} = \mathbf{X}_{(T+1)} = (X_{T+1}, X_{T+2}, \ldots)$ the cash flows after time T of **all contracts** which are in force at time T. Hence the valuation portfolio at time T of our business is given by (see Section 3.6)

$$\mathrm{VaPo}_{(T)}(\mathbf{X}) = \sum_{t>T} \mathrm{VaPo}_{(T)}(\mathbf{X}_t) = \sum_{t>T} \sum_{i=1}^{p} l_{i,t}^{(T)} \, \mathcal{U}_i, \tag{6.1}$$

with $\mathbf{X}_t = X_t \, \mathbf{Z}^{(t)}$, $(\mathcal{U}_i)_{i=1,\ldots p}$ are the units (basis, financial instruments) and

$$l_{i,t}^{(T)} = E\left[\left.\Lambda_i(\mathbf{X}_t)\right| \mathcal{T}_T\right] \tag{6.2}$$

denotes the expected number of units $\mathcal{U}_i$ generated by the stochastic cash flow $\mathbf{X}_t$ (seen from time T). Similarly, we consider one period later

$$\mathrm{VaPo}_{(T+1)}(\mathbf{X}) = \sum_{t>T} \mathrm{VaPo}_{(T+1)}(\mathbf{X}_t) = \sum_{t>T} \sum_{i=1}^{p} l_{i,t}^{(T+1)} \, \mathcal{U}_i. \tag{6.3}$$

Henceforth, the valuation portfolio $\mathrm{VaPo}_{(T)}(\mathbf{X})$ is $\mathcal{T}_T$-measurable and the valuation portfolio $\mathrm{VaPo}_{(T+1)}(\mathbf{X})$ is $\mathcal{T}_{T+1}$-measurable. Basically, what happens is that we update our information

M.V. Wüthrich et al., *Market-Consistent Actuarial Valuation*, EAA Series,
DOI 10.1007/978-3-642-14852-1_6,

$$\mathcal{T}_T \mapsto \mathcal{T}_{T+1} \tag{6.4}$$

in order to predict the outstanding loss liabilities $\mathbf{X}$ at times T and $T+1$, respectively.

Moreover, by applying conditional expectations iteratively,

$$\text{VaPo}_{(T)}(\mathbf{X}) = E\left[\left.\text{VaPo}_{(T+1)}(\mathbf{X})\right|\mathcal{T}_T\right] \tag{6.5}$$

is the valuation portfolio seen at time T. This means that the prediction is in the average correct (unbiased). Due to our construction we have the following recursion (linearity)

$$\text{VaPo}_{(T)}(\mathbf{X}) = E\left[\left.\text{VaPo}_{(T+1)}\left(\mathbf{X}_{(T+2)}\right)\right|\mathcal{T}_T\right] + E\left[\left.\text{VaPo}_{(T+1)}\left(\mathbf{X}_{T+1}\right)\right|\mathcal{T}_T\right]. \tag{6.6}$$

This is the self-financing property for stochastic cash flows (for deterministic cash flows see Section 3.4).

$\text{VaPo}_{(T+1)}(\mathbf{X}_{T+1})$ is simply cash value at time $T+1$ ($\mathcal{F}_{T+1}$-measurable). Hence it is replaced by X_{T+1}. This leads to the recursion (viewed from time $T+1$)

$$\text{VaPo}_{(T)}(\mathbf{X}) = E\left[\left.\text{VaPo}_{(T+1)}\left(\mathbf{X}_{(T+2)}\right)\right|\mathcal{T}_T\right] + E\left[\left.X_{T+1}\right|\mathcal{T}_T\right]. \tag{6.7}$$

Therefore the **insurance technical loss** in the interval $(T, T+1]$ at time $T+1$ is given by

$$\begin{aligned} &\text{VaPo}_{(T+1)}\left(\mathbf{X}_{(T+2)}\right) + X_{T+1} - \text{VaPo}_{(T)}(\mathbf{X}) \\ &\quad = \text{VaPo}_{(T+1)}\left(\mathbf{X}_{(T+2)}\right) + X_{T+1} \\ &\qquad\qquad - E\left[\left.\text{VaPo}_{(T+1)}\left(\mathbf{X}_{(T+2)}\right)\right|\mathcal{T}_T\right] - E\left[\left.X_{T+1}\right|\mathcal{T}_T\right] \\ &\quad = \text{VaPo}_{(T+1)}\left(\mathbf{X}_{(T+2)}\right) - E\left[\left.\text{VaPo}_{(T+1)}\left(\mathbf{X}_{(T+2)}\right)\right|\mathcal{T}_T\right] \\ &\qquad\qquad + X_{T+1} - E\left[\left.X_{T+1}\right|\mathcal{T}_T\right]. \end{aligned} \tag{6.8}$$

Hence the insurance technical loss has two parts:

1. prediction error in the next payment, which is given by

$$X_{T+1} - E\left[\left.X_{T+1}\right|\mathcal{T}_T\right], \tag{6.9}$$

2. prediction error in the valuation portfolio for cash flows after $T+1$, which is given by

$$\text{VaPo}_{(T+1)}\left(\mathbf{X}_{(T+2)}\right) - E\left[\left.\text{VaPo}_{(T+1)}\left(\mathbf{X}_{(T+2)}\right)\right|\mathcal{T}_T\right]. \tag{6.10}$$

Remark. In non-life insurance this insurance technical loss within $(T, T+1]$ is called claims development result (CDR), see Merz-Wüthrich [MW08]. It means that we predict the valuation portfolio at time T by $\text{VaPo}_{(T)}(\mathbf{X})$ and one period later by $\mathrm{VaPo}_{(T+1)}(\mathbf{X})$. Because this prediction is unbiased, see (6.5), we predict the insurance technical loss CDR for accounting year $T+1$ by 0 at time T. The risk driver of this profit-and-loss statement position then is that we face an insurance technical loss according to (6.8).

Note that we predict random variables by their (conditional) expectations. Henceforth, the error terms are called prediction errors. Moreover, if we estimate the conditional expectations

$$E\left[X_{T+1}\middle|\,\mathcal{T}_T\right] \quad \text{and} \quad E\left[\mathrm{VaPo}_{(T+1)}\left(\mathbf{X}_{(T+2)}\right)\middle|\,\mathcal{T}_T\right] \tag{6.11}$$

with the available data we obtain an additional error term known as the parameter estimation error term. It arises from the fact the true mean is not known and must be estimated from the data (this is completely analogous to the derivations in Chapter 5). Hence, in general, we have two different sources of uncertainty.

For both prediction error terms in (6.9)-(6.10) we have calculated a loading in the valuation portfolio protected against insurance technical risks. Observe that the two error terms are not necessarily independent (see Section 3.5). This loading was purely cash flow driven (independent of any accounting standard). In Section 6.3 we will give a slightly different view.

For the **financial loss** we proceed as follows: We have chosen an asset portfolio $\tilde{S}$ which fulfills the accounting condition at time T on the economic value scale

$$\mathcal{E}_T\left[\mathrm{VaPo}_{(T)}(\mathbf{X})\right] = \mathcal{E}_T\left[\widetilde{S}\right]. \tag{6.12}$$

We have a financial gain at time $T+1$ if

$$\mathcal{E}_{T+1}\left[\mathrm{VaPo}_{(T)}(\mathbf{X})\right] < \mathcal{E}_{T+1}\left[\widetilde{S}\right], \tag{6.13}$$

and a financial loss otherwise.

If the portfolio is protected against financial risks we have no financial loss (we exercise the Margrabe option in case of a loss) but a gain if (6.13) holds.

Both, insurance technical part and financial part may (and will) produce losses and gains:

1. If we have a protected portfolio, the gains should go to those who pay the protection fee.
2. If we have no protection, gains and losses should be shared in the same fixed proportion.

Other sources of risks: Model risk, credit risk, operational risk, etc. For a more detailed discussion of other risks we refer to Sandström [Sa06].

6.2 Remarks on the self-financing property of the insurance technical liabilities

a) If we have a cash flow $\mathbf{X} = (X_0, \ldots, X_n) \in L^2_{n+1}(P, \mathcal{G})$ with deterministic insurance technical risk we construct a valuation portfolio

$$\mathbf{X} \mapsto \mathrm{VaPo}(\mathbf{X}) = \sum_i \lambda_i(\mathbf{X})\, \mathcal{U}_i, \tag{6.14}$$

where the deterministic numbers $\lambda_i(\mathbf{X})$ are given in a natural way. As in Section 3.4 we then easily obtain the self-financing property (see Lemma 3.1)

$$\mathrm{VaPo}\left(\mathbf{X}_{(k)}\right) = \mathrm{VaPo}\left(\mathbf{X}_{(k+1)}\right) + \mathrm{VaPo}\left(\mathbf{X}_k\right). \tag{6.15}$$

Note that both sides of (6.15) should be read as portfolios.

b) If the cash flow $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$ has stochastic insurance technical risks, the situation is more complicated. In that case we have chosen for the valuation portfolio construction the expected number of units $\mathcal{U}_i$ (at time k)

$$l_i^{(k)}(\mathbf{X}) = E\left[\left.\Lambda_i(\mathbf{X})\right| \mathcal{T}_k\right] \tag{6.16}$$

and then the valuation portfolio at time k is given by

$$\mathbf{X} \mapsto \mathrm{VaPo}_{(k)}(\mathbf{X}) = \sum_i l_i^{(k)}(\mathbf{X})\, \mathcal{U}_i. \tag{6.17}$$

Observe that

$$l_i^{(k)}(\mathbf{X}) = E\left[\left. l_i^{(k+1)}(\mathbf{X}) \right| \mathcal{T}_k\right], \tag{6.18}$$

which implies

$$\begin{aligned} \mathrm{VaPo}_{(k)}\left(\mathbf{X}_{(k)}\right) &= \mathrm{VaPo}_{(k)}\left(\mathbf{X}_{(k+1)}\right) + \mathrm{VaPo}_{(k)}\left(\mathbf{X}_k\right) \\ &= E\left[\left.\mathrm{VaPo}_{(k+1)}\left(\mathbf{X}_{(k+1)}\right)\right| \mathcal{T}_k\right] + \mathrm{VaPo}_{(k)}\left(\mathbf{X}_k\right). \end{aligned} \tag{6.19}$$

The identity in (6.19) is called **self-financing property in the mean**, see also (6.5)–(6.6).

c) Assume we have a cash flow $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$ with stochastic insurance technical risks and we consider the valuation portfolio protected against insurance technical risks

$$\mathbf{X} \mapsto \mathrm{VaPo}^{prot}_{(k)}(\mathbf{X}) = \sum_i l_i^{*,k}(\mathbf{X})\, \mathcal{U}_i, \tag{6.20}$$

with the "distortion representation" (see also (3.65))

$$l_i^{*,k}(\mathbf{X}) = \frac{1}{\varphi_k^{(\mathcal{T})}}\, E\left[\left.\sum_t \varphi_t^{(\mathcal{T})}\; \Lambda_i(\mathbf{X}_t)\right| \mathcal{T}_k\right]. \tag{6.21}$$

The martingale property of Lemma 2.16 implies

$$l_i^{*,k}(\mathbf{X}) = \frac{1}{\varphi_k^{(\mathcal{T})}} E\left[\varphi_{k+1}^{(\mathcal{T})}\, l_i^{*,k+1}(\mathbf{X})\middle|\, \mathcal{T}_k\right] = E\left[\frac{\varphi_{k+1}^{(\mathcal{T})}}{\varphi_k^{(\mathcal{T})}}\, l_i^{*,k+1}(\mathbf{X})\middle|\, \mathcal{T}_k\right]. \quad (6.22)$$

The span probability distortion $\varphi_{k+1}^{(\mathcal{T})}/\varphi_k^{(\mathcal{T})}$ models the underlying risk aversion (risk margin) for the insurance technical risk when we increase the information from $\mathcal{T}_k$ to $\mathcal{T}_{k+1}$. Risk aversion then means that we have

$$l_i^{*,k}(\mathbf{X}) \geq E\left[l_i^{*,k+1}(\mathbf{X})\middle|\, \mathcal{T}_k\right]. \quad (6.23)$$

Hence, in general, we do not have the self-financing property in the mean as portfolio

$$\begin{aligned} \mathrm{VaPo}_{(k)}^{prot}\left(\mathbf{X}_{(k)}\right) &= \mathrm{VaPo}_{(k)}^{prot}\left(\mathbf{X}_{(k+1)}\right) + \mathrm{VaPo}_{(k)}^{prot}\left(\mathbf{X}_k\right) \\ &\neq E\left[\mathrm{VaPo}_{(k+1)}^{prot}\left(\mathbf{X}_{(k+1)}\right)\middle|\, \mathcal{T}_k\right] + \mathrm{VaPo}_{(k)}^{prot}\left(\mathbf{X}_k\right). \end{aligned} \quad (6.24)$$

However, under risk aversion we require for each unit $\mathcal{U}_i$

$$l_i^{*,k}(\mathbf{X}) - E\left[l_i^{*,k+1}(\mathbf{X})\middle|\, \mathcal{T}_k\right] \geq 0, \quad (6.25)$$

see also (6.23). This can be interpreted as the expected gain which compensates the risk taker for bearing the insurance technical risk. Another interpretation is that this is the risk margin for non-hedgeable risks in an incomplete market setting.

Distortion techniques are used quite often in actuarial practice, e.g. in life insurance when one replaces the best-estimate life table by a prudent life table. However, the loadings for insurance technical risks are often not naturally obtained via a distortion as in (6.21). An example was given in Chapter 5. In such cases we would at least expect under risk aversion

$$\sum_i l_i^{*,k}(\mathbf{X}_{(k)})\, \mathcal{U}_i \geq \sum_i E\left[l_i^{*,k+1}(\mathbf{X}_{(k+1)})\ \middle|\, \mathcal{T}_k\right]\, \mathcal{U}_i + X_k\, \mathcal{U}_0, \quad (6.26)$$

where $\mathcal{U}_0$ stands for the financial instrument representing cash value at time k. Note that the inequality (6.26) lacks of full mathematical rigour as it should be interpreted in a vector space.

This means that in general we assume under risk aversion

$$l_i^{*,k}(\mathbf{X}) \geq E\left[l_i^{*,k+1}(\mathbf{X})\middle|\, \mathcal{T}_k\right]. \quad (6.27)$$

Self-financing property as monetary value. Assume that the price process $(\mathcal{A}_s(\mathcal{U}_i))_s$ is independent of $\mathcal{T}$ and satisfies (4.9) with financial deflator $\boldsymbol{\varphi}^{(\mathcal{G})}$. The monetary value at time k of our protected VaPo is given by

$$\mathcal{A}_k\left(\mathrm{VaPo}_{(k)}^{prot}\left(\mathbf{X}_{(k)}\right)\right)=\sum_i l_i^{*,k}(\mathbf{X}_{(k)})\ \mathcal{A}_k\left(\mathcal{U}_i\right), \tag{6.28}$$

and due to the independence of insurance technical and financial risks, and because of (6.27)

$$\begin{aligned}
&E\left[\varphi_{k+1}^{(\mathcal{G})}\ \mathcal{A}_{k+1}\left(\mathrm{VaPo}_{(k+1)}^{prot}\left(\mathbf{X}_{(k+1)}\right)\right)\middle|\,\mathcal{F}_k\right]\\
&\qquad=E\left[\varphi_{k+1}^{(\mathcal{G})}\ \sum_i l_i^{*,k+1}(\mathbf{X}_{(k+1)})\ \mathcal{A}_{k+1}\left(\mathcal{U}_i\right)\middle|\,\mathcal{F}_k\right]\\
&\qquad=\sum_i E\left[l_i^{*,k+1}(\mathbf{X}_{(k+1)})\middle|\,\mathcal{T}_k\right]\ E\left[\varphi_{k+1}^{(\mathcal{G})}\ \mathcal{A}_{k+1}\left(\mathcal{U}_i\right)\middle|\,\mathcal{G}_k\right]\\
&\qquad\le\sum_i l_i^{*,k}(\mathbf{X}_{(k+1)})\ \varphi_k^{(\mathcal{G})}\ \mathcal{A}_k\left(\mathcal{U}_i\right),
\end{aligned} \tag{6.29}$$

where we have assumed the martingale property for the price processes of financial instruments, see Theorem 2.18.

This implies the “self-financing property” in monetary value

$$E\left[\frac{\varphi_{k+1}^{(\mathcal{G})}}{\varphi_k^{(\mathcal{G})}}\ \mathcal{A}_{k+1}\left(\mathrm{VaPo}_{(k+1)}^{prot}\left(\mathbf{X}_{(k+1)}\right)\right)\middle|\,\mathcal{F}_k\right]\le\mathcal{A}_k\left(\mathrm{VaPo}_{(k)}^{prot}\left(\mathbf{X}_{(k+1)}\right)\right). \tag{6.30}$$

This means that we obtain a super-martingale for the monetary value of the valuation portfolio protected against insurance technical risks. This super-martingale property (6.30) is explained by the two terms: (i) deflated price processes of financial instruments $\mathcal{U}_i$ are $(P,\mathbb{F})$-martingales, highlighting the fact that we can hedge financial risks by an appropriate asset allocation S; (2) probability distorted processes of the insurance technical variables $\Lambda_i(\mathbf{X})$ are super-martingales highlighting the fact that we ask for a margin for non-hedgeable risks.

Note that it is not straightforward that the super-martingale property is satisfied if we use ad-hoc methods for the calculation of the protection margin, e.g. the derivations in Chapter 5 do not necessarily lead to a self-financing valuation portfolio protected against insurance technical risks.

6.3 Claims development result in non-life insurance

When we have constructed the valuation portfolio protected against insurance technical risks we have taken a purely cash flow based point of view, that is we have considered

$$\mathrm{VaPo}_{(k)}(\mathbf{X})=\sum_i l_i^{(k)}(\mathbf{X})\ \mathcal{U}_i\ \mapsto\ \mathrm{VaPo}_{(k)}^{prot}(\mathbf{X})=\sum_i l_i^{*,k}(\mathbf{X})\ \mathcal{U}_i, \tag{6.31}$$

where

$$l_i^{*,k}(\mathbf{X}) = l_i^{(k)}(\mathbf{X}) + \text{ appropriate loading.} \tag{6.32}$$

The appropriate loading was determined by measuring the uncertainties between

$$\begin{aligned}&\text{the random variable } \Lambda_i(\mathbf{X}) \text{ and}\\ &\text{its predictor } l_i^{(k)}(\mathbf{X}) = E\left[\Lambda_i(\mathbf{X})\middle|\,\mathcal{T}_k\right] \text{ at time } k.\end{aligned}$$

This can be viewed as the so-called **long-term view** because it measures the uncertainty over the whole run-off of the liabilities. Under Solvency 2 one is rather interested in the **short-term view** that explains how predictions change over time, that is one studies the successive predictions

$$l_i^{(k)}(\mathbf{X}),\ l_i^{(k+1)}(\mathbf{X}),\ l_i^{(k+2)}(\mathbf{X}), \ldots. \tag{6.33}$$

The risk for accounting year k is then given by the difference

$$l_i^{(k+1)}(\mathbf{X}) - l_i^{(k)}(\mathbf{X}), \tag{6.34}$$

and the regulator asks for a protection of this difference (profit-and-loss statement position). The conclusion to this regulatory view is that the accounting principle becomes important in the study of the uncertainties and we will obtain accounting principle dependent risk margins.

For the chain-ladder method in non-life insurance reserving, this was studied in Merz-Wüthrich [MW08] and Bühlmann et al. [BFGMW09] for nominal payments. It turns out that the one-year risk measured for standard deviations makes about 2/3 of the total risk (depending on the line of business), which is similar to the findings in the AISAM-ACME study [AISAM07].

6.4 Legal quote in life insurance

In this section we consider participating life insurance contracts characterized by a guaranteed minimum rate of return and a specified rule of the distribution of annual excess investment returns above the guaranteed return. Often, the sharing of profits between the policyholder and the insurer is a legal requirement stipulating that bonuses shall be at least a certain percentage of the profits of the company (legal quote).

Suppose we have a guaranteed interest rate i (technical interest rate) for the minimum interest rate credited to the policyholder's account. We define the annual investment return of $\widetilde{S}$ in $[t, t+1]$ by

$$R_{t+1} = \frac{\widetilde{S}_{t+1} - \widetilde{S}_t}{\widetilde{S}_t}. \tag{6.35}$$

We assume that the policyholder receives a certain proportion $\beta \in [0,1]$ of the excess investment return above the guaranteed rate:

$$\rho_{t+1}^{(1)} = \frac{\max(\beta\, R_{t+1}, i) - i}{1+i}, \tag{6.36}$$

$$\rho_{t+1}^{(2)} = \beta\, \frac{\max(R_{t+1}, i) - i}{1+i}. \tag{6.37}$$

Definition (6.36) can be viewed as a net legal quote, whereas (6.37) is rather a gross legal quote.

Remark. There are different ways to think about profit sharing. The easiest ways probably are

- cash bonus paid to the policyholder,
- (in practice) transfer into final cash bonus (for single premium policies one increases the sum insured by $\rho_{t+1}^{(k)}$, $k = 1, 2$).

Example 6.1 (Purely theoretical).

Choose a pure endowment policy with a duration of 20 years and a single premium installment. A pure endowment contract of duration n years provides the payment of the sum insured only if the policyholder survives to the end of the contract period. However, for simplicity in this example, we assume that there is no mortality ($\Rightarrow$ only financial risk) and that the sum insured is 100. For an other example we refer to De Felice-Moriconi [dFM04].

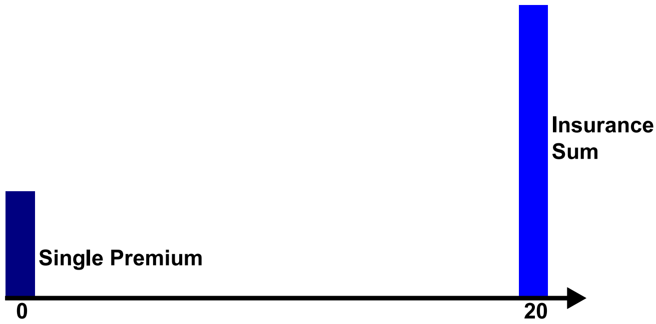


Fig. 6.1. Pure endowment policy

The valuation portfolio is given by (the basis is the zero coupon bond $Z^{(20)}$)

$$\text{VaPo} = 100\ Z^{(20)}. \tag{6.38}$$

The monetary actuarial value if we take constant interest rate $i = 3\%$ is given by

$$\mathcal{A}_0\,[\text{VaPo}] = 100\ (1+i)^{-20} = 55.37, \tag{6.39}$$

$$\mathcal{A}_t\,[\text{VaPo}] = 100\ (1+i)^{-(20-t)}, \qquad \text{for } t = 0,\ldots,20. \tag{6.40}$$

The economic value is given by

$$V_t = \mathcal{E}_t\,[\text{VaPo}] = 100\ \mathcal{E}_t\left[Z^{(20)}\right]. \tag{6.41}$$

On the other hand we have an investment portfolio $\widetilde{S}$ with price process

$$Y_t = \mathcal{E}_t\left[\widetilde{S}\right], \tag{6.42}$$

which determines the participation benefits. We consider an example with annual volatility of $\widetilde{Y}_t = Y_t/V_t$ of $\sigma = 4\%$ and participation rate $\beta = 100\%$ (see also Subsection 4.3.2).

The benefit is paid by increasing the sum insured each year by $\rho_s^{(k)}$. Hence the benefit paid at contract maturity (viewed from time 0) is

$$100\prod_{s=1}^{20}\left(1+\rho_s^{(k)}\right) = 100\ \phi_0^{(20)}, \tag{6.43}$$

where

$$\phi_0^{(20)} \quad \text{is a stochastic “zero coupon bond” including participations after time 0.} \tag{6.44}$$

If we think now of paying the legal quote to the insured we obtain the following valuation portfolio at time $t < 20$

$$\text{VaPo}_t^{lq} = 100\prod_{s=t+1}^{20}\left(1+\rho_s^{(k)}\right) = 100\ \phi_t^{(20)}. \tag{6.45}$$

Hence the non participating valuation portfolio and the valuation portfolio with legal quote participation after time $t < 20$ are given by

$$\text{VaPo} = 100\ Z^{(20)} \qquad \text{with} \qquad \mathcal{E}_t\,[\text{VaPo}] = 100\ \mathcal{E}_t\left[Z^{(20)}\right], \tag{6.46}$$

$$\text{VaPo}_t^{lq} = 100\ \phi_t^{(20)} \qquad \text{with} \qquad \mathcal{E}_t\left[\text{VaPo}_t^{lq}\right] = 100\ \mathcal{E}_t\left[\phi_t^{(20)}\right]. \tag{6.47}$$

One can also represent the values with a put option: Define

$$100\ \widetilde{\phi}_t^{(20)} = 100\prod_{s=t+1}^{20}\left(1+\frac{\beta\ R_s - i}{1+i}\right), \tag{6.48}$$

i.e. no guaranteed minimum interest rate i. Hence we can write

$$\mathcal{E}_t\left[\text{VaPo}_t^{lq}\right] = 100\ \mathcal{E}_t\left[\phi_t^{(20)}\right] = 100\ \left(\mathcal{E}_t\left[\widetilde{\phi}_t^{(20)}\right]\right) + \text{Put}_t, \tag{6.49}$$

where the put option Put_t adjusts for the minimum interest rate guarantee. With the call-put parity we also obtain

$$\mathcal{E}_t\left[\text{VaPo}_t^{lq}\right] = \mathcal{E}_t\left[\text{VaPo}_t\right] + \text{Call}_t. \tag{6.50}$$

For a specific example we use real Italian market data (which had very high yields initially). For the determination of the monetary values Monte Carlo simulation techniques were used. The results are shown in Table 6.1.

t	$\mathcal{A}_t$ [VaPo]	$\mathcal{E}_t$ [VaPo]	$\mathcal{E}_t\left[\text{VaPo}_t^{lq}\right]$	Call_t
0	55.37	5.60	33.00	27.40
1	57.03	6.81	35.13	28.32
2	58.74	11.49	39.63	28.14
3	60.50	12.09	41.20	29.11
4	62.32	24.71	49.59	24.88
5	64.19	16.63	46.82	30.19
6	66.11	23.02	50.88	27.86
7	68.10	36.90	59.23	22.33
8	70.14	50.25	67.35	17.10
9	72.24	61.82	76.33	14.51
10	74.41	56.58	71.99	15.41
11	76.64	61.62	73.56	11.94
12	78.94	66.71	76.06	9.35
13	81.31	75.25	81.31	6.06
14	83.75	79.30	84.38	5.08

Table 6.1. Calculations by courtesy of De Felice-Moriconi. For a description of their method we refer to [dFM04]

Remarks.

- The reserves $\mathcal{E}_t\left[\text{VaPo}_t^{lq}\right]$ in this example are **not** reserves in the actuarial sense and can **not** be used for solvency purposes.
- E.g. for going from $\mathcal{E}_8\left[\text{VaPo}_8^{lq}\right] = 67.35$ to $\mathcal{E}_9\left[\text{VaPo}_9^{lq}\right] = 76.33$ we need an investment return of 13.3%. But the legal quote caps our return at $i = 3\%$!
- Using $\mathcal{E}_t\left[\text{VaPo}_t^{lq}\right]$ leads to the wrong reasoning
  - The market value accounting principle $\mathcal{E}_t$ is here not applicable.
  - The participations given to the insured caps the returns available for reserve accumulation, hence we cannot apply $\mathcal{E}_t$.
- Correct reasoning: For policies with β-participation $\rho_t^{(2)}$ make two policies: Policy 1) for insured amount $\times\ \beta$; Policy 2) for insured amount $\times\ 1-\beta$.
  - For Policy 1) we need statutory reserves with discount rate i and a put option for the minimum interest rate guarantee.
  - For Policy 2) we have economic reserves with a put option for minimum interest rate guarantee.

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