Chapter 2
Examples of applications

We now consider some examples in which various aspects of the Π-Theorem are explained.

2.1 Period of rotation of a body in a circular orbit (laws of similarity)

A body of mass m is kept in a circular orbit of radius r by a central force F. It is required to find the period of rotation

$$P = f(r,m,F).$$

Here and in what follows we fix the basis of fundamental physical units that is standard in mechanics, namely, length, mass and time, which, following Maxwell, we denote by $\{L,M,T\}$. (In thermodynamics the symbol T is used to denote absolute temperature, but unless otherwise stated, we shall meanwhile use this notation for the unit of time.)

Let us find the dimension vector of the quantities P, r, m, F in the basis $\{L,M,T\}$. We write them as the columns of the following table:

	P	r	m	F
L	0	1	0	1
M	0	0	1	1
T	1	0	0	-2

Since, as we have shown, the dimension function always has degree form, multiplication of such functions corresponds to addition of the degree exponents, in other words, they correspond to linear operations on the dimension vectors of the corresponding physical quantities.

V. Zorich, *Mathematical Analysis of Problems in the Natural Sciences*,
DOI 10.1007/978-3-642-14813-2_2, © Springer-Verlag Berlin Heidelberg 2011

Hence, using standard linear algebra, one can find a system of independent quantities from the matrix formed by their dimension vectors; also, by decomposing the dimension vector of some quantity into the dimension vectors of the selected independent quantities, one can find a formula for the dimension of this quantity in the system of independent quantities of the concrete problem.

Thus, in our case the quantities r, m, F, are independent because the matrix formed by the vectors $[r]$, $[m]$, $[F]$ is non-singular. Finding the expansion $[P] = \frac{1}{2}[r] + \frac{1}{2}[m,] - \frac{1}{2}[F]$ on the basis of formula (1.6) of Chapter I we immediately see that

$$P = \left(\frac{mr}{F}\right)^{1/2} \cdot f(1,1,1).$$

Thus, to within the positive factor $c = f(1,1,1)$ (which can be found by a single laboratory experiment) we have found the dependence of P on r,m,F. Of course, knowing Newton's law $F = m \cdot a$, in the present instance we could easily have found the final formula, where $c = 2\pi$. However, everything that we have used is just a general indication of the existence of the dependence $P = f(r,m,F)$.

2.2 The gravitional constant. Kepler's third law and the degree exponent in Newton's law of universal gravitation.

After Newton we find the degree exponent α in the law of universal gravitation

$$F = G\frac{m_1 m_2}{r^\alpha}.$$

We use the previous problem and Kepler's third law (which was known to Newton) which, for circular orbits, implies that the square of the periods of rotation of the planets (with respect to a central body of mass M) are proportional to the cubes of the radii of their orbits. In view of the result of the previous problem and the law of universal gravitation (with the exponent α not yet found) we have

$$\left(\frac{P_1}{P_2}\right)^2 = \left(\frac{m_1 r_1}{F_1}\right)^{1/2} / \left(\frac{m_2 r_2}{F_2}\right)^{1/2} = \left(\frac{m_1 r_1}{\frac{m_1 M}{r^\alpha}}\right)^{1/2} / \left(\frac{m_2 r_2}{\frac{m_2 M}{r^\alpha}}\right)^{1/2} = \left(\frac{r_1}{r_2}\right)^{\alpha+1}.$$

But by Kepler's law $\left(\frac{P_1}{P_2}\right)^2 = \left(\frac{r_1}{r_2}\right)^3$. Hence $\alpha = 2$.

2.3 Period of oscillation of a heavy pendulum (inclusion of g).

After the detailed explanations involved in the solution of the first problem we can now allow ourselves a more compact account, pausing only at certain new circumstances.

We shall find the period of oscillation of a pendulum. More precisely, a load of mass m is fixed at the end of a weightless suspension of length l inclined from the equilibrium position at some initial angle φ_0, is let go and under the action of the force of gravity starts to perform a periodic oscillation. We shall find the period P of these oscillations.

To write $P = f(l, m, \varphi_0)$ would be wrong because a pendulum has different periods of oscillation on the Earth and the Moon in view of the difference in the forces of gravity at the surfaces of these two bodies. The force of gravity at the surface of a body, for example, the Earth, is characterized by the quantity g which is the acceleration of free fall at the surface of this body. Therefore, instead of the impossible relation $P = f(l, m, \varphi_0)$, one must assume that $P = f(l, m, g, \varphi_0)$.

We write the dimension vectors of all these quantities in the basis $\{L, M, T\}$:

	P	l	m	g	φ_0
L	0	1	0	1	0
M	0	0	1	0	0
T	1	0	0	-2	0

Clearly the vectors $[l]$, $[m]$, $[g]$ are independent and $[P] = \frac{1}{2}[l] - \frac{1}{2}[g]$.

In view of the Π-Theorem in the form of relation 1.6 of Chapter I, it follows that

$$P = \left(\frac{l}{g}\right)^{\frac{1}{2}} \cdot f(1, 1, 1, \varphi_0).$$

We have found that $P = c(\varphi_0) \cdot \sqrt{\frac{l}{g}}$, where the dimensionless factor $c(\varphi_0)$ depends only on the dimensionless angle φ_0 of the initial inclination (measured in radians).

The precise value of $c(\varphi_0)$ can also be found, although this time this is no longer all that easy. It can be done by solving the equation of motion of an oscillating heavy pendulum and invoking the elliptic integral

$$F(k, \varphi) := \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$

Namely, $c(\varphi_0) = 4K(\sin(\frac{1}{2}\varphi_0))$, where $K(k) := F(k, \frac{1}{2}\pi)$.

2.4 Outflow of volume and mass in a waterfall

On a broad shelf having the form of a step on the upper platform, water falling under the action of gravity forms a waterfall. The depth of the water on the upper platform is known and is equal to h. It is required to find the specific volumetric outflow V (per unit of time on a unit of width of the step) of the water. If we look at the mechanism of the phenomenon in the right way, then we see that $V = f(g,h)$.

Since this phenomenon is determined by gravitation, along with the dimensional constant g (free-fall acceleration), we could, as a precaution, introduce the density ϱ of the fluid, that is, we suppose that

$$V = f(\varrho, g, h).$$

We now carry out the standard procedure of finding the dimension vectors:

	V	ϱ	g	h
L	2	−3	1	1
M	0	1	0	0
T	−1	0	−2	0

Clearly the vectors $[\varrho], [g], [h]$ are independent and $[V] = \frac{1}{2}[g] + \frac{3}{2}[h]$.

In view of the Π-Theorem we now obtain that

$$V = g^{\frac{1}{2}} h^{\frac{3}{2}} \cdot f(1,1,1).$$

Thus, $V = c \cdot g^{\frac{1}{2}} h^{\frac{3}{2}}$, where c is a constant to be determined, for example, in a laboratory experiment. Here the specific outflow Q of the mass is clearly equal to ϱV. One could also have arrived at the same formula by applying the method of dimensions to the relation $Q = f(\varrho, g, h)$.

2.5 Drag force for the motion of a ball in a non-viscous medium

A ball of radius r moves with velocity v in a non-viscous medium of density ϱ. It is required to find the drag force acting on the ball. (One could, of course, assume that there is a flow moving with velocity v past a ball at rest, which is a typical situation in wind-tunnel tests.)

We write down the general formula $F = f(\varrho, v, r)$ and analyse it in terms of dimensions:

	F	ϱ	v	r
L	1	−3	1	1
M	1	1	0	0
T	−2	0	−1	0

Clearly the vectors $[\varrho], [v], [r]$ are independent and $[F] = [\varrho] + 2[v] + 2[r]$. In view of the Π-Theorem we now obtain

$$F = \varrho v^2 r^2 \cdot f(1,1,1). \tag{2.1}$$

Thus, $F = c \cdot \varrho v^2 r^2$, where c is a dimensionless constant coefficient.

2.6 Drag force for the motion of a ball in a viscous medium

Before we turn to the formulation of this problem we recall the notion of viscosity of a medium and find the dimension of viscosity.

If one places a sheet of paper on the surface of thick honey, then in order to move the sheet along the surface one needs to apply certain forces. In first approximation the force F applied to the sheet stuck on the surface of the honey will be proportional to the area S of the sheet, the speed v of its motion and inversely proportional to the distance h from the surface to the bottom where the honey is also stuck and stays motionless in spite of the motion at the top (like a river).

Thus, $F = \eta \cdot Sv/h$. The coefficient η in this formula depends on the medium (honey, water, air, and so on) and is called the *coefficient of viscosity* of the medium or simply the *viscosity*.

The ratio $\nu = \eta/\varrho$, where, as always, ϱ is the density of the medium, is frequently encountered in problems of hydrodynamics and is called the *kinematic viscosity* of the medium.

We now find the dimensions of these quantities in the standard $\{L, M, T\}$ basis. Since $[\eta] = [FhS^{-1}v^{-1}]$, the dimension function corresponding to the viscosity in this basis has the form $\varphi_\eta = L^{-1}M^1T^{-1}$, and the dimension vector is $[\eta] = (-1, 1, -1)$. For the kinematic viscosity $[\nu] = [\eta/\varrho]$, therefore $\varphi_\nu = L^2M^0T^{-1}$ and $[\nu] = (2, 0, -1)$.

We now try to solve the previous problem on the drag force arising in the motion of the same ball, but now in a viscous medium. The initial dependence now looks like this: $F = f(\eta, \varrho, v, r)$. We analyse it in terms of dimensions:

	F	η	ϱ	v	r
L	1	−1	−3	1	1
M	1	1	1	0	0
T	−2	−1	0	−1	0

Clearly the vectors $[\varrho], [v], [r]$ are independent; $[F] = [\varrho] + 2[v] + 2[r]$ and $[\eta] = [\varrho] + [v] + [r]$. In view of the Π-Theorem we have

$$F = \varrho v^2 r^2 \cdot f(\mathrm{Re}^{-1}, 1, 1, 1), \tag{2.2}$$

where the function $f(\mathrm{Re}^{-1}, 1, 1, 1)$ remains unknown. This last function depends on the dimensionless parameter

$$\mathrm{Re} = \varrho v r / \eta = v r / \nu, \tag{2.3}$$

which plays a key role in questions of hydrodynamics.

This dimensionless quantity Re (the indication of the ratio of the force of inertia and the viscosity) is called the *Reynolds number* after the English physicist and engineer Osborn Reynolds, who first drew attention to it in his papers on turbulence in 1883. It turns out that as the Reynolds number increases, for example, as the speed of the flow increases or as the viscosity of the medium decreases, the character of the flow undergoes structural transformations (called bifurcations) evolving from a calm stable laminar flow to turbulence and chaos.

It is very instructive to pause at this juncture and wonder why the results (2.1) and (2.2) of the last two problems appear to be essentially the same. The wonder disappears if one considers more closely the variable quantity $f(\mathrm{Re}^{-1}, 1, 1, 1)$. Under the assumptions that, in modern terminology, are equivalent to the relative smallness of the Reynolds number, Stokes as long ago as 1851 found that $F = 6\pi\eta v r$. This does not contradict formula (2.2) but merely states that for small Reynolds numbers the function $f(\mathrm{Re}^{-1}, 1, 1, 1)$ behaves asymptotically like $6\pi \mathrm{Re}^{-1}$. In fact, substituting this value in formula (2.2) and recalling the definition (2.3) we obtain Stokes's formula.

2.7 Exercises

1. Since orchestras exist, it is natural to suggest that the speed of sound is weakly dependent (or not dependent) on the wave length ?

(Recall the nature of a sound wave, introduce the modulus of elasticity E of the medium and, starting from the dependence $v = f(\varrho, E, \lambda)$, prove that $v = c \cdot (E/\varrho)^{1/2}$.)

2. What is the law of the change of speed of propagation of a shock wave resulting from a very strong explosion in the atmosphere ?

(Introduce the energy E_0 of the explosion. The pressure in front of the shock wave can be ignored; the elasticity of the air no longer plays a role. Start by finding the law $r = f(\varrho, E_0, t)$ of propagation of the shock wave.)

3. Obtain the formula $v = c \cdot (\lambda g)^{1/2}$ for the speed of propagation of a wave in a deep reservoir under the action of the force of gravity. (Here c is a numerical coefficient, g is the acceleration of free fall and λ is the wave length.)

4. The speed of propagation of a wave in shallow water does not depend on the wave length. Accepting this observation as a fact show that it is proportional to the square root of the depth of the reservoir.

5. The formula used for determining the quantity of liquid flowing along a cylindrical tube (for example, along an artery) has the form

$$v = \frac{\pi \varrho P r^4}{8 \eta l},$$

where v is the speed of the flow, ϱ is the density of the liquid, P is the difference in pressure at the ends of the tube, r is the radius of cross-section of the tube, η is the viscosity of the liquid and l is the length of the tube. Derive this formula (to within a numerical factor) by verifying the agreement of the dimensions on both sides of the formula.

6. a) In a desert inhabited by animals it is required to overcome the large distances between the sources of water. How does the maximal time that the animal can run depend on the size L of the animal ? (Assume that evaporation only occurs from the surface, the size of which is proportional to L^2.)

b) How does the speed of running (on the level and uphill) depend on the size of the animal ? (Assume that the power developed and the corresponding intensity of heat loss (say, through evaporation) are proportional to each other, and the resistive force against horizontal motion (for example, air resistance) is proportional to the square of the speed and the area of the frontal surface.)

c) How does the distance that an animal can run depend on its size ? (Compare with the answers to the previous two questions.)

d) How does the height of the jump of an animal depend on its size ? (The critical load that can be borne by a column that is not too high is proportional to the area of cross-section of the column. Assume that the answer to the question depends only on the strength of the bones and the "capability" of the muscles (corresponding to the strengths of the bones).

Here we are dealing throughout with animals of size on the human scale, such as camels, horses, dogs, hares, kangaroos, jerboas, in their customary habitats. In this connection see the books by Arnold and Schmidt cited below.

7. After Lord Rayleigh, find the period of small oscillations of drops of liquid under the action of their surface tension, assuming that everything happens outside a gravitational field (in the cosmos).

(Answer: $c \cdot (\varrho r^3/s)^{1/2}$, where ϱ is the density of the liquid, r is the radius of the drop and s is the surface tension, $[s] = (0, 1, -2)$.)

8. Find the period of rotation of a double star. We have in mind that two bodies with masses m_1 and m_2 rotate in circular orbits about their common centre of mass. The system occurs in empty space and is maintained by the forces of mutual attraction between these bodies. (If you are puzzled, recall the gravitational constant and its dimension.)

9. "Discover" Wien's displacement law $\varepsilon(\nu, T) = \nu^3 F(\nu/T)$ and also the Rayleigh-Jeans law $\varepsilon(\nu, T) = \nu^2 T\ G(\nu/T)$ for the distribution of the intensity of black-body radiation as a function of the frequency and the absolute temperature.

[Wien's fundamental law (not the displacement law given above) has the form $\varepsilon(\nu, T) = \nu^3 \exp(-a\nu/T)$ and is valid for $\nu/T \gg 1$, while the Rayleigh-Jeans law $\varepsilon(\nu, T) = 8\pi\nu^2 kT/c^3$ is valid for relatively small values of ν/T.

Both these laws (the specific intensity of radiation in the frequency interval from ν to $\nu + d\nu$) are united by Planck's formula (1900) launching the ground-breaking epoch of quantum theory:

$$\varepsilon(\nu, T) = \frac{8\pi}{c^3}\nu^2 \frac{h\nu}{e^{h\nu/kT} - 1}.$$

Here c is the velocity of light, h is Planck's constant, k is the Boltzmann constant ($k = R/N$, where R is the universal gas constant and N is Avogadro's number). Wien's law and the Rayleigh-Jeans law are obtained from Planck's formula for $h\nu \gg kT$ and $kT \gg h\nu$, respectively.]

Let ν_T be the frequency at which the function $\varepsilon(\nu, T) = \nu^3 F(\nu/T)$ attains its maximum for a fixed value of the temperature T. Verify (after Wien) that we have the remarkable displacement law $\nu_T/T = \text{const}$. Find this constant using Planck's formula.

10. Taking the gravitational constant G, the speed of light c and Planck's constant h as the basic units, find the universal Planck units of length $L^* = (hG/c^3)^{1/2}$, time $T^* = (hG/c^5)^{1/2}$ and mass $M^* = (hc/G)^{1/2}$.

(The values $G = 6.67 \cdot 10^{-11} \mathrm{H} \cdot \mathrm{m}^2/\mathrm{kg}^2$, $c = 2.997925 \cdot 10^8 \mathrm{m/s}$ and $h = 6.625 \cdot 10^{-34}$J·s, other physical constants, as well as other information on units of measurement can be found in the books [4a], [4b], [4c].)

Many problems, analysed examples, instructive discussions and warnings relating to the analysis of dimensionality and principles of similarity can be found in the books [1], [2], [3], [4].

2.8 Concluding remarks

The little that has been said about the analysis of dimension and its applications already enables us to make the following observations.

The effectiveness of the use of the method mainly depends on a proper understanding of the nature of the phenomenon to which it is being applied. (By the way, in an early stage of analysis only people at the level of Newton, the brothers Bernoulli and Euler knew how to apply the analysis of infinitesimals without getting embroiled in paradoxes, which was required for extra intuition).[1]

Dimension analysis is particularly useful when the laws of the phenomenon have not yet been described. Namely, in this situation it sometimes reveals connections (albeit very general), which are useful for an understanding of the mechanism of the phenomenon and the choice of the direction of further investigations and refinements. We shall then demonstrate this by the example of Kolmogorov's approach to the description of the (still mysterious) fundamental phenomenon of turbulence.

The main postulate of dimension theory relates to the linear theory of similarity transformations, the theory of measurements, the notion of a rigid body and the homogeneity of a space, among other things. In Lobachevskii's hyperbolic geometry there are no similar figures at all, as is well known. Even so, locally this geometry admits a Euclidean approximation. Hence, as in all laws, the postulate of dimension theory is itself applicable in certain scales, depending on the problem. These scales were rarely known in advance and were most often discovered when incongruities arose.

The method shows that the larger the number of dimensionally independent quantities are, the simpler and more concrete the functional dependence of the quantities under study becomes. On the other hand, the more physical relations are discovered the less remains of the dimensionally independent quantities. (For example, distance can now be measured in light years.) So we see that the more we know, the less general dimension analysis gives us. Counterbalancing this, the penetration into essentially new areas is usually accompanied by the appearance of new dimensionally independent quantities (the algebraic aspect of dimensions and many other matters can be found, for example, in the book [14].)

Disregarding Problems 9 and 10 we restrict ourselves here to the discussion of phenomena described within the framework of the quantities of classical mechanics. This will suffice to begin with. But true enjoyment can only be obtained by reading the discussions of scholars, thinkers and, in general, professionals capable of a large-scale multischeme and unique view of the

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[1] I quote the justified misgivings of V.I. Arnol'd concerning the possible overestimation of the Π-theorem: "Such an approach is extremely dangerous because it opens up the possibility of irresponsible speculation (under the name of dimension theory) in those places where the corresponding laws of similarity should be verified experimentally, since they do not at all follow from the dimensions of the quantities describing the phenomenon under study, and they are deep subtle facts". Rather the same relates to a clumsy use of multiplication tables, statistics or catastrophe theory.

Using these new publications of the present book I add that in his recent book "Mathematical understanding of nature" (MCCME, Moscow, 2009) discussing such a theory of adiabatic invariants V.I. (on p.117) observes that " The theory of adiabatic invariants is a strange example of a physical theory seemingly contradicting the purport of easily verifiable mathematical facts. In spite of such an undesirable property of this "theory" it provided remarkable physical discoveries to those who were not afraid to use its conclusions (even though they were mathematically unjustified)". In a word: "Think it out for yourself, solve it yourself, take it or leave it".

world or the subject matter. And this is in connection with various areas. It is like a symphony and it captivates!

[If you resign yourself to the "obscurantism" of dimension analysis but what has been set out still does not seem rather crazy, then it amuses me to quote the following excerpt from a well-known physicist (whom I shall not name so as not to accidentally subject his good name to attacks by less free-thinking people).

"Physicists begin the study of a phenomenon by introducing suitable units of measurement. It is unreasonable to measure the radius of an atom in metres or the speed of an electron in kilometres per hour; one needs to find appropriate units. There are already important immediate consequences of one such choice of units. Thus, from the charge e of an electron and its mass m one cannot form a quantity having the dimensions of length. This means that in classical mechanics the atom is impossible — an electron cannot move in a stationary orbit. The situation changed with the appearance of Planck's constant $\hbar$ ($\hbar = h/2\pi$). As is clear from the definition, $\hbar = 1{,}054 \cdot 10^{-34}$ J·s has dimensions of energy times time.[2] We can now form the quantity of the dimension of length: $a_0 = \hbar/me^2$.

If in this relation we substitute the values of the constants occurring therein, then we should get a quantity of the order of the dimensions of the atom; one obtains $0{,}5 \cdot 10^{-10}$m. Thus from a simple dimensional estimate one has found the size of the atom.

It is easy to see that $e^2/\hbar$ has the dimension of velocity, it is roughly 100 times smaller than the speed of light. If one divides this quantity by the speed of light c, then one obtains the dimensionless quantity $\alpha = e^2/\hbar c = 1/137$, characterizing the interaction of the electron with an electric field. This quantity is called the *fine structure constant*.

We have given estimates for the hydrogen atom. It is easy to obtain them for an atom with nuclear charge Ze. The motion of an electron in an atom is determined by its interaction with the nucleus, which is proportional to the product of the charge on the nucleus and the charge on the electron. Therefore for a nucleus with charge Ze, in the formulae for α and a_0 we must replace e^2 by Ze^2. In heavy elements with $Z \sim 100$ the velocity of the electrons is close to the speed of light."]

Finally we make some practical observations.

Dimension analysis is a good means of double checking:

a) if the dimensions of the left- and right-hand sides of an equation are not equal, then one must look for the error;

b) if under a sign that is not a degree function (for example, under a logarithm or exponential sign) there is a quantity that is not dimensionless, then one must look for the error (or one must look for a transformation getting rid of this situation);

[2] *Author's comment*: The dimension of h can be worked out from Planck's formula in Problem 9.

c) only quantities of the same dimension can be added.

(If $v = at$ is the velocity and $s = \frac{1}{2}at^2$ is the distance passed under uniform acceleration, then formally it is, of course, true that $v + s = at + \frac{1}{2}at^2$. However, from a physical point of view this equality reduces to two: $v = at$ and $s = \frac{1}{2}at^2$. Bridgeman, in whose cited book we gave this example, indicates a complete analogy with the equality of vectors , which gives rise to equalities of coordinates with the same name.)