Representation of Exchangeable Sequences by Means of Copulas

Fabrizio Durante and Jan-Frederik Mai

Abstract. Given a sequence $\mathbf{X} = (X_n)_{n\in\mathbb{N}}$ of exchangeable continuous random variables, it is proved that the joint distribution function of every finite subset of random variables belonging to $\mathbf{X}$ is fully described by means of a suitable bivariate copula and a univariate distribution function.

Keywords: Copula, Exchangeability.

1 Introduction

Given a family $\mathbf{X} = \{X_i\}_{i\in\mathscr{J}}$ of real-valued random variables (=r.v.'s) defined on a probability space $(\Omega,\mathscr{A},\mathbb{P})$, it is well known that several properties of $\mathbf{X}$ can be expressed in terms of the class $\mathscr{H}$ that contains the joint distribution functions (=d.f.'s) of all the finite subfamilies of $\mathbf{X}$, $\mathscr{H} = \{H_A\}$, where A is a finite set of indices in $\mathscr{J}$, $A = \{i_1,i_2,\dots,i_k\}$, and $H_A : \mathbb{R}^{|A|} \to \mathbb{R}$ is the d.f. of $(X_{i_1},X_{i_2},\dots,X_{i_k})$, $H_A(x_1,\dots,x_k) = \mathbb{P}(X_{i_1} \le x_1,\dots,X_{i_k} \le x_k)$.

Moreover, since *Sklar's Theorem* [19], it is known that the d.f. H of every continuous random vector $(X_1,\dots,X_n)$ can be uniquely represented in terms of the univariate marginal d.f.'s F_i, $i \in \{1,\dots,n\}$, and a suitable $C_n\colon \mathbb{I}^n \to \mathbb{I}$ ($\mathbb{I} := [0,1]$), called *copula*, in the following way:

$$H(x_1,x_2,\dots,x_n) = C_n(F_1(x_1),F_2(x_2),\dots,F_n(x_n)), \tag{1}$$

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for every $x_1, x_2, \ldots, x_n$ in $\mathbb{R}$. We recall that a copula is an n–dimensional d.f. having univariate marginals uniformly distributed on $\mathbb{I}$. Basic examples of copulas are: the independence copula $\Pi_n(\mathbf{x}) = x_1 x_2 \cdots x_n$, and the comonotonicity copula $M_n(\mathbf{x}) = \min\{x_1, x_2, \ldots, x_n\}$. See, for example, [9, 10, 15].

Therefore, every family of continuous r.v.'s $\mathbf{X} = \{X_i\}_{i\in\mathscr{I}}$ can be uniquely expressed in terms of the couple $(\mathscr{F}_{\mathbf{X}}, \mathscr{C}_{\mathbf{X}})$, where $\mathscr{F}_{\mathbf{X}} = \{F_i\}_{i\in\mathscr{I}}$ is the family formed by the (univariate) d.f.'s associated with each X_i and $\mathscr{C}_{\mathbf{X}}$ contains the copulas that are associated with all finite subsets of $\{X_i\}_{i\in\mathscr{I}}$, in such a way that, if H is the joint d.f. of $(X_{i_1}, X_{i_2}, \ldots, X_{i_n})$, then H can be expressed in the form (1), where C_n is in $\mathscr{C}_{\mathbf{X}}$ and $F_{i_1}, F_{i_2}, \ldots, F_{i_n}$ are in $\mathscr{F}_{\mathbf{X}}$. This representation was adopted, for example, in [5] in order to describe a Markov process (see also [12]).

In this short note, we aim at giving a representation of the same type for an *exchangeable sequence* of continuous r.v.'s, i.e. for a sequence $\mathbf{X} = (X_n)_{n\in\mathbb{N}}$ of r.v.'s such that the d.f. of every finite subset of k $(k \geq 1)$ of these r.v.'s depends only upon k and not on the particular subset (see [11] for more details).

2 The Representation

Given a random vector $(X_1, \ldots, X_n)$ with joint d.f. H and continuous univariate marginals $F_1, \ldots, F_n$, its associated copula C_n is actually the d.f. of the random vector $(F_1(X_1), \ldots, F_n(X_n))$, and, hence, C_n can be recovered from the d.f. H of $(X_1, \ldots, X_n)$ by taking, for all $\mathbf{u} \in \mathbb{I}^n$,

$$C_n(u_1, \ldots, u_n) = H(F_1^{\leftarrow}(u_1), \ldots, F_n^{\leftarrow}(u_n)),$$

where $F^{\leftarrow}(y) = \inf\{x \in \mathbb{R} \mid F(x) \geq y\}$ denotes the quantile inverse of any univariate d.f. F.

Every copula C_n is a Lipschitz function (with constant 1) and admits partial derivatives $\frac{\partial C_n}{\partial u_i} = \partial_i C_n$ almost everywhere on $\mathbb{I}^n$. If C_n is the copula of the continuous random vector $(X_1, \ldots, X_n)$, then, similarly to [5], it can be proved that

$$\partial_j C_n(F(x_1), \ldots, F(x_{j-1}), F_j(X_j), F(x_{j+1}), \ldots, F(x_n))$$

is a version of $\mathbb{P}\left(\cap_{i\neq j}\{X_i \leq x_i\} \mid X_j\right) := \mathbb{E}\left(\mathbf{1}_{\{X_i \leq x_i, i\neq j\}} \mid X_j\right)$.

Following [16], $\mathbf{X} = (X_n)_{n\in\mathbb{N}}$ is an exchangeable sequence of real-valued r.v.'s if, and only if, there exists a real-valued r.v. Λ such that $X_1, X_2, \ldots$ are conditionally independent and identically distributed (briefly, i.i.d.) given Λ.

Now, starting with this last fact, we state our main result.

Theorem 1. *Let $(X_n)_{n\in\mathbb{N}}$ be an exchangeable sequence of continuous r.v.'s. Then there exist a one–dimensional d.f. F and a 2–copula A such that, the joint d.f. H_n of every subset of $n \geq 2$ r.v.'s from the sequence may be represented, for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, as*

$$H_n(x_1,x_2,\dots,x_n) = C_n(F(x_1),F(x_2),\dots,F(x_n)), \tag{2}$$

where the copula C_n *is given, for all* $(u_1,\dots,u_n) \in \mathbb{I}^n$*, by*

$$C_n(u_1,\dots,u_n) = \int_0^1 \frac{\partial A(u_1,t)}{\partial t} \cdot \frac{\partial A(u_2,t)}{\partial t} \cdots \frac{\partial A(u_n,t)}{\partial t}\, dt. \tag{3}$$

Proof. Given the exchangeable sequence $(X_n)_{n\in\mathbb{N}}$, as said before, there exists a r.v. Λ with d.f. L, such that the r.v.'s X_n are conditionally i.i.d. given Λ (see, e.g., [16]). Therefore, there is a family $(G_\lambda)_{\lambda\in\mathbb{R}}$ of d.f.'s such that, for all $n \in \mathbb{N}$ and for all $(x_1,\dots,x_n) \in \mathbb{R}^n$,

$$\mathbb{P}(X_1 \le x_1,\dots,X_n \le x_n \,|\, \Lambda) = G_\Lambda(x_1)\cdots G_\Lambda(x_n).$$

Without loss of generality (since we are only interested in statements in distribution) we may assume that the r.v.'s $(X_n)_{n\in\mathbb{N}}$ and Λ are defined on a probability space $(\Omega,\mathscr{A},\mathbb{P})$ in the following canonical manner, see [2, p. 12-13].

- Let U and $V_1,V_2,\dots,$ be i.i.d. r.v.'s on $(\Omega,\mathscr{A},\mathbb{P})$ that are uniformly distributed on $\mathbb{I}$.
- Define $\Lambda := L^{\leftarrow}(U)$, where $L^{\leftarrow}$ denotes the generalized inverse of L, i.e. Λ has distribution function L (see, for instance, [3, Theorem 2]).
- For each $n \in \mathbb{N}$ define the r.v. X_n as a function of Λ and V_n via $X_n := G_\Lambda^{\leftarrow}(V_n)$; i.e. conditioned on Λ, X_n has d.f. G_Λ, or, conditioned on U, X_n has d.f. $G_{L^{\leftarrow}(U)}$.

For each n, the copula C_n of $(X_1,\dots,X_n)$ coincides with the joint distribution function of $\big(F(X_1),\dots,F(X_n)\big)$ since F is continuous. Hence, using the canonical construction above as well as continuity of F (which implies that $F^{\leftarrow}$ is strictly increasing and $F^{\leftarrow}\circ F(X_n)$ is equal in distribution to X_n by [14, p. 495, Proposition A.3-4]), it holds that

$$\mathbb{P}\big(F(X_1) \le u_1, F(X_2) \le u_2,\dots,F(X_n) \le u_n \mid U\big) = \prod_{i=1}^n G_{L^{\leftarrow}(U)}\big(F^{\leftarrow}(u_i)\big).$$

Now let A be the joint distribution function of $\big(F(X_n),U\big)$. Notice that such A does not depend on n by conditional independence; in fact:

$$\begin{aligned}
\mathbb{P}\big(F(X_n) \le x, U \le u\big) &= \mathbb{E}[\mathbb{P}\big(F(X_n) \le x, U \le u \mid U\big)] \\
&= \mathbb{E}\Big[\mathbb{E}\Big[\mathbf{1}_{\left\{F\left(G^{\leftarrow}_{L^{\leftarrow}(U)}(V_n)\right)\le x\right\}}\,\Big|\, U\Big]\,\mathbf{1}_{\{U\le u\}}\Big] \\
&\overset{(*)}{=} \mathbb{E}\Big[\mathbb{E}\Big[\mathbf{1}_{\left\{F\left(G^{\leftarrow}_{L^{\leftarrow}(z)}(V_n)\right)\le x\right\}}\Big]\Big|_{z=U}\,\mathbf{1}_{\{U\le u\}}\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\Big[\mathbf{1}_{\left\{F\left(G^{\leftarrow}_{L^{\leftarrow}(z)}(V_1)\right)\le x\right\}}\Big]\Big|_{z=U}\,\mathbf{1}_{\{U\le u\}}\Big] \\
&= \mathbb{P}\big(F(X_1) \le x, U \le u\big),
\end{aligned}$$

where equality $(*)$ follows from [7, Example 1.5, page 224]. Then, for almost all $u_i \in \mathbb{I}$, one has

$$G_{L^{\leftarrow}(U)}\big(F^{\leftarrow}(u_i)\big) = \mathbb{P}\big(F(X_i) \le u_i \mid U\big) = \frac{\partial}{\partial t}A(u_i,t)\bigg|_{t=U}.$$

Thus, the copula C_n of any sequence of n r.v.'s in $(X_n)_{n\in\mathbb{N}}$ can be represented via (3). □

The copula C_n given by (3) is called the *n–product* of A. In [5], the authors considered an operation $*$ on the class of 2–copulas given, for any 2–copulas A and B, by

$$(A*B)(u_1,u_2) = \int_0^1 \frac{\partial A(u_1,t)}{\partial t}\cdot\frac{\partial B(t,u_2)}{\partial t}\,\mathrm{d}t.$$

It is easy to show that the 2–product of A coincides with the copula given by $A*A^T$, where $A^T(u_1,u_2) = A(u_2,u_1)$ for every $(u_1,u_2)\in\mathbb{I}^2$.

Theorem 1 can be used in order to construct a sequence of exchangeable r.v.'s by using only a univariate d.f. and a 2–copula. The procedure runs as follows:

1. assign in any manner a 2–copula A and a d.f. F;
2. for $n > 1$, set C_n the n–product of A given by (3);
3. set $H_1 = F$ and $H_n = C_n(F,F,\dots,F)$ for every $n \ge 2$;
4. apply Daniell-Kolmogorov Theorem [17] to $\mathscr{H} = \{H_n\}_{n\in\mathbb{N}}$ in order to obtain a sequence of exchangeable r.v.'s $\mathbf{X} = (X_n)_{n\in N}$ such that every joint d.f. of any finite subset of $\mathbf{X}$ is in $\mathscr{H}$.

A sequence of i.i.d. r.v.'s can be constructed, for example, by taking any univariate d.f. F and $A = \Pi_2$. Notice that different 2–copulas can produce the same sequence. For instance, the mapping $W_2\colon \mathbb{I}^2 \to \mathbb{I}$ given by $W_2(u_1,u_2) = \max\{u_1+u_2-1,0\}$ is a 2–copula such that $W_2 * W_2 = M_2 * M_2 = M_2$, therefore, the sequences generated by M_2 and by W_2 can be associated with the same family of finite-dimensional d.f.'s.

Some interesting consequences can be derived from Theorem 1.

We recall that a random vector $(X_1,\dots,X_n)$ is *infinitely extendible* if it is the first segment of a sequence of exchangeable r.v.'s (see [18, 20]). The following corollary gives a representation for any copula that is associated with an infinitely extendible random vector.

Corollary 1. *Let $(X_1,\dots,X_n)$ be an exchangeable random vector with (symmetric) copula C. Then $(X_1,\dots,X_n)$ is infinitely extendible if, and only if, C is the n–product of some copula A.*

If the 2–copula C_2 of (X_1,X_2) is symmetric and *idempotent* with respect to the operation $*$, i.e. $C_2 * C_2 = C_2$ (see [1, 5]), then (X_1,X_2) is infinitely extendible. Generally, if a family of symmetric bivariate copulas (like Fréchet family and FGM family) is closed with respect to the $*$-operation (and, hence, with respect to the 2-product operation), then its members can be used for constructing an infinitely extendible random pair.

Example 1. Corollary 1 is illustrated with bivariate Cuadras-Augé copulas, defined, for every $\alpha \in \mathbb{I}$, by $C_\alpha(u,v) = \min\{u,v\}\max\{u,v\}^{1-\alpha}$ (see [4]). Let (X_1,X_2) be a random vector with identical continuous marginals and 2–copula C_α. Given the bivariate Marshall-Olkin copula $A(u,v) = \min\{u, u^\alpha v\}$, it follows easily that

$$C_\alpha(u,v) = \int_0^1 \frac{\partial}{\partial t}A(u,t)\,\frac{\partial}{\partial t}A(v,t)\,dt.$$

Hence (X_1,X_2) is infinitely extendible. Moreover, for $n \geq 2$ it holds that

$$\int_0^1 \prod_{i=1}^{n} \frac{\partial}{\partial t}A(u_i,t)\,dt = u_{[1]} \prod_{i=2}^{n} u_{[i]}^{1-\alpha}, \tag{4}$$

where $u_{[1]} \leq u_{[2]} \leq \ldots \leq u_{[n]}$ denotes the components of $(u_1,\ldots,u_n) \in \mathbb{I}^n$, rearranged in increasing order. Copulas of type (4) have been considered in [6, 13].

The following result, instead, was proved in [2, Lemma 4.11] (see also [8]) and admits now another proof.

Corollary 2. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of exchangeable r.v.'s. If X_i and X_j are independent for every $i,j \in \mathbb{N}$, $i \neq j$, then $(X_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. r.v.'s.*

Proof. In view of Theorem 1, given a sequence $(X_n)_{n\in\mathbb{N}}$ of exchangeable r.v.'s, there exists a 2–copula A such that the 2–product of A, denoted by B, is the copula of the random pair (X_i,X_j) for every $i,j \in \mathbb{N}$, $i \neq j$. In particular, if X_i and X_j are independent, then $B = A * A^T = \Pi_2$. But, in general, $A * \Pi_2 = \Pi_2$, which yields that $(A * A^T) - (A * \Pi_2) = 0$. Thus, for every $(u_1,u_2) \in \mathbb{I}^2$, $\partial_t A(t,u_2) = u_2$ for almost all $t \in \mathbb{I}$, viz. A has linear section in the first component being the second fixed. Thus, $A = \Pi_2$ and the copula of every subset of n r.v.'s of the sequence is Π_n. □

Note that, as well known, for a finite vector of exchangeable r.v.'s, pairwise independence does not imply independence. Consider, for example, the trivariate vector (U_1,U_2,U_3) whose d.f. H is given on $\mathbb{I}^3$ by:

$$H(u_1,u_2,u_3) = u_1u_2u_3(1+\theta(1-u_1)(1-u_2)(1-u_3)),$$

for a suitable $\theta \in [-1,1]$ (see, e.g., [15, Example 3.31]).

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References

1. Albanese, A., Sempi, C.: Countably generated idempotent copulas. In: Soft methodology and random information systems, Adv. Soft Comput., pp. 197–204. Springer, Berlin (2004)
2. Aldous, D.J.: Exchangeability and related topics. In: École d'été de probabilités de Saint-Flour, XIII—1983. Lecture Notes in Math., vol. 1117, pp. 1–198. Springer, Berlin (1985)
3. Angus, J.E.: The probability integral transform and related results. SIAM Rev. 36(4), 652–654 (1994)
4. Cuadras, C.M., Augé, J.: A continuous general multivariate distribution and its properties. Comm. Statist. A—Theory Methods 10(4), 339–353 (1981)
5. Darsow, W.F., Nguyen, B., Olsen, E.T.: Copulas and Markov processes. Illinois J. Math. 36(4), 600–642 (1992)
6. Durante, F., Quesada-Molina, J.J., Úbeda-Flores, M.: On a family of multivariate copulas for aggregation processes. Inform. Sci. 177(24), 5715–5724 (2007)
7. Durrett, R.: Probability: theory and examples, 2nd edn. Duxbury Press, Belmont (1996)
8. Hu, T.C.: On pairwise independent and independent exchangeable random variables. Stochastic Anal. Appl. 15(1), 51–57 (1997)
9. Jaworski, P., Durante, F., Härdle, W., Rychlik, T. (eds.): Copula Theory and its Applications. Lecture Notes in Statistics - Proceedings. Springer, Dortrecht (2010)
10. Joe, H.: Multivariate models and dependence concepts. Monographs on Statistics and Applied Probability, vol. 73. Chapman & Hall, London (1997)
11. Kallenberg, O.: Probabilistic symmetries and invariance principles. In: Probability and its Applications. Springer, New York (2005)
12. Lagerås, A.N.: Copulas for Markovian dependence. Bernoulli (in press, 2010)
13. Mai, J.F., Scherer, M.: Lévy-Frailty copulas. J. Multivariate Anal. 100(7), 1567–1585 (2009)
14. McNeil, A.J., Frey, R., Embrechts, P.: Quantitative risk management. Concepts, techniques and tools. Princeton Series in Finance. Princeton University Press, Princeton (2005)
15. Nelsen, R.B.: An introduction to copulas, 2nd edn. Springer Series in Statistics. Springer, New York (2006)
16. Olshen, R.: A note on exchangable sequences. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 28, 317–321 (1973/1974)
17. Rogers, L.C.G., Williams, D.: Diffusions, Markov processes, and martingales. Cambridge Mathematical Library, vol. 1. Cambridge University Press, Cambridge (2000), (Foundations, Reprint of the second (1994) edition)
18. Scarsini, M., Verdicchio, L.: On the extendibility of partially exchangeable random vectors. Statist. Probab. Lett. 16(1), 43–46 (1993)
19. Sklar, A.: Fonctions de répartition à n dimensions et leurs marges. Publ. Inst. Statist. Univ. Paris 8, 229–231 (1959)
20. Spizzichino, F.: Extendibility of symmetric probability distributions and related bounds. In: Exchangeability in probability and statistics, Rome, pp. 313–320. North-Holland, Amsterdam (1981/1982)