Functional Inequalities Characterizing the Frank Family of Copulas

Hans De Meyer and Bernard De Baets

Abstract. Given a random vector with components that are pairwisely coupled by means of a same commutative copula C, we analyze the transitivity of the reciprocal relation obtained from the pairwise comparison of these components. The transitivity of this reciprocal relation can be elegantly described within the cycle-transitivity framework if the commutative copula C satisfies a countably infinite family of (functional) inequalities. Each functional inequality uniquely characterizes the Frank family of copulas. Finally, we highlight the transitivity results for a random vector whose coupling structure is captured by an extended Frank m-copula.

Keywords: Copulas, Frank family, Functional inequality, Reciprocal relations, Transitivity.

1 Introduction

Many methods can be established for the comparison of the components (random variables, r.v.) of a random vector $(X_1,\ldots,X_n)$, as there are many ways to extract useful information from the joint cumulative distribution function (c.d.f.) $F_{X_1,\ldots,X_n}$ that characterizes the random vector.

A simplification consists in restricting the comparison strategy to methods that aim at comparing the r.v. two by two. We have recently put forward

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such a method [8]. We associate to a random vector a reciprocal relation Q, which can be regarded as a graded preference relation. The cornerstone for computing the reciprocal relation Q is the copula C_{ij} that joins the one-dimensional marginal c.d.f. F_{X_i} and F_{X_j} into the bivariate marginal c.d.f. F_{X_i,X_j}, i.e. $F_{X_i,X_j} = C_{ij}(F_{X_i}, F_{X_j})$. Note that the copulas should not be the same for all pairs of r.v. For a collection of independent r.v., however, they all coincide with the product $T_{\mathbf{P}}(x,y) = xy$.

We have analyzed the case where all copulas C_{ij} are the same but not necessarily equal to the product, neither to the greatest copula $T_{\mathbf{M}}(x,y) = \min(x,y)$, the minimum operator, nor to the smallest copula $T_{\mathbf{L}}(x,y) = \max(x+y-1,0)$, also known as the Łukasiewicz t-norm. Note that in the latter case the pairwise couplings should be considered as purely artificial, as no n-copula ($n \geq 3$) exists such that all 2-copulas contained in it are equal to $T_{\mathbf{L}}$. Our analysis has revealed that the reciprocal relations generated by these couplings possess transitivity properties that can be nicely characterized [4, 7, 8, 9, 10].

The concept of transitivity is unique for crisp relations, but for reciprocal relations there is a whole range of transitivity properties. Sometimes it is possible to capture the transitivity in the form of a type of stochastic transitivity or a type of T-transitivity, with T a t-norm, well known from the theory of fuzzy relations, but mostly these types prove insufficient to deal with the transitivity of reciprocal relations. Instead, we have developed a new framework, called the cycle-transitivity framework, that allows to characterize the types of transitivity that arise in the present investigation [3, 5].

As a by-product of our investigations, we have laid bare an infinite family of functional inequalities, each of which characterizes the Frank family of copulas [12]. In the past, many investigations were aimed at finding the solution(s) of functional equations in the space of uniform distribution functions [1]. The functional equation of Frank [11], the Frank equation for short, is perhaps the best known example. This equation, however, does not characterize uniquely the Frank family of copulas, as it also has as solutions the ordinal sums of Frank copulas. Note that Frank copulas and ordinal sums of Frank copulas are more often regarded as t-norms [13] and that in this context the Frank equation has even been studied for the more general class of uninorms [2]. The fact that a sharper characterization of a family of copulas can be acquired by means of a functional inequality, rather than by means of a functional equation, has to our knowledge, not been recognized before.

In the next section, we briefly summarize the concept of cycle-transitivity. In Section 3 we recall the method used to compare r.v. and the way a reciprocal relation is generated from it. We investigate its transitivity when the r.v. are coupled with the same copula and derive an infinite family of inequalities which the copula should satisfy. Section 4 emphasizes the role of the Frank family of copulas as unique solutions of the family of inequalities. Finally, Section 5 is concerned with the transitivity of the reciprocal relation generated by a random vector whose coupling structure is described by a Frank m-copula.

2 Cycle-Transitivity of Reciprocal Relations

Reciprocal ([0,1]-valued binary relations Q satisfying $Q(a,b)+Q(b,a)=1$, provide a convenient tool for expressing the result of the pairwise comparison of a set of alternatives. Recently, we have presented a general framework for studying the transitivity of reciprocal relations, encompassing various types of T-transitivity and stochastic transitivity [3, 5].

Recall that a fuzzy relation R on A is an $A^2 \to [0,1]$ mapping that expresses the degree of relationship between elements of A. For such relations, the concept of T-transitivity is very natural.

Definition 1. *Let T be a t-norm. A fuzzy relation R on A is called T-transitive if for any $(a,b,c) \in A^3$ it holds that $T(R(a,b),R(b,c)) \leq R(a,c)$.*

Though the semantics of reciprocal relations and fuzzy relations are different, the concept of T-transitivity is sometimes formally applied to reciprocal relations as well. However, more often the transitivity properties of reciprocal relations can be characterized as of one of various kinds of stochastic transitivity [3].

In the cycle-transitivity framework [5], for a reciprocal relation Q on A, the quantities

$$\alpha_{abc} = \min(Q(a,b),Q(b,c),Q(c,a))\,,\ \beta_{abc} = \text{med}(Q(a,b),Q(b,c),Q(c,a))\,,$$

$$\gamma_{abc} = \max(Q(a,b),Q(b,c),Q(c,a))\,,$$

are defined for all $(a,b,c) \in A^3$. Obviously, $\alpha_{abc} \leq \beta_{abc} \leq \gamma_{abc}$. Also, the notation $\Delta = \{(x,y,z) \in [0,1]^3 \,|\, x \leq y \leq z\}$ will be used.

Definition 2. *A function $U : \Delta \to \mathbb{R}$ is called an upper bound function if it satisfies:*

(i) $U(0,0,1) \geq 0$ and $U(0,1,1) \geq 1$;
(ii) for any $(\alpha,\beta,\gamma) \in \Delta$:

$$U(\alpha,\beta,\gamma) + U(1-\gamma,1-\beta,1-\alpha) \geq 1\,.$$

Definition 3. *A reciprocal relation Q on A is called cycle-transitive w.r.t. an upper bound function U if for any $(a,b,c) \in A^3$ it holds that*

$$\alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 \leq U(\alpha_{abc},\beta_{abc},\gamma_{abc})\,.$$

For two upper bound functions such that $U_1 \leq U_2$, it clearly holds that cycle-transitivity w.r.t. U_1 implies cycle-transitivity w.r.t. U_2. It is clear that $U_1 \leq U_2$ is not a necessary condition for the latter implication to hold. Two upper bound functions U_1 and U_2 will be called *equivalent* if for any $(\alpha,\beta,\gamma) \in \Delta$ it holds that $\alpha+\beta+\gamma-1 \leq U_1(\alpha,\beta,\gamma)$ is equivalent to $\alpha+\beta+\gamma-1 \leq U_2(\alpha,\beta,\gamma)$.

The different types of fuzzy and stochastic transitivity can be reformulated in the cycle-transitivity framework and are then characterized by an upper bound function $U(\alpha,\beta,\gamma)$.

Proposition 1. *A reciprocal relation Q is T-transitive, with T a 1-Lipschitz t-norm (or equivalently, an associative copula), if and only if Q is cycle-transitive w.r.t. to the upper bound function U_T given by*

$$U_T(\alpha,\beta,\gamma) = \alpha + \beta - T(\alpha,\beta)\,.$$

In fact, many examples of reciprocal relations we have encountered in our research on the comparison of random variables are neither fuzzy nor stochastic transitive, but have a type of transitivity that can be nicely expressed as an instance of cycle-transitivity. In the present study we will encounter the weaker counterparts of T-transitivity obtained by replacing in the expression for U_T α by β and β by γ.

Definition 4. *A reciprocal relation Q that is cycle transitive w.r.t. to the upper bound function U_{WT} defined by*

$$U_{WT}(\alpha,\beta,\gamma) = \beta + \gamma - T(\beta,\gamma)\,,$$

with T a 1-Lipschitz t-norm, is called weak T-transitive.

Weak $T_{\mathbf{M}}$-transitivity is also known as partial stochastic transitivity, weak $T_{\mathbf{P}}$-transitivity as dice-transitivity or weak product transitivity, whereas weak $T_{\mathbf{L}}$-transitivity is equivalent to $T_{\mathbf{L}}$-transitivity.

3 Generating Transitive Reciprocal Relations from Random Vectors

An immediate way of comparing two r.v. is to consider the probability that the first one takes a greater value than the second one. Proceeding along this line of thought, a random vector $(X_1,X_2,\ldots,X_n)$ generates a reciprocal relation.

Definition 5. *Given a random vector $(X_1,X_2,\ldots,X_n)$, the binary relation Q defined by*

$$Q(X_i,X_j) = \mathrm{Prob}\{X_i > X_j\} + \frac{1}{2}\mathrm{Prob}\{X_i = X_j\}$$

is a reciprocal relation.

Since the copulas C_{ij} that couple the univariate marginal c.d.f. into the bivariate marginal c.d.f. can be different from another, the analysis of the reciprocal relation and in particular the identification of its transitivity properties appear rather cumbersome. It is nonetheless possible to state in general, without making any assumptions on the bivariate c.d.f., that the reciprocal relation

Q generated by an arbitrary random vector always shows some minimal form of transitivity.

Proposition 2. *[4] The reciprocal relation Q generated by a random vector is $T_{\mathbf{L}}$-transitive.*

Our further interest is to study the situation where momentarily abstraction is made that the r.v. are components of a random vector, and all bivariate c.d.f. are enforced to depend in the same way upon the univariate c.d.f., in other words, we consider the situation of all copulas being the same.

To get insight in what kind of transitivity properties one might expect in general, the present authors have previously unravelled three particular cases, namely the case of the product copula $T_{\mathbf{P}}$, and the cases of the two extreme copulas, the minimum operator $T_{\mathbf{M}}$ and the Łukasiewicz t-norm $T_{\mathbf{L}}$, respectively related to a presumed but not-necessarily existing comonotonic and countermonotonic pairwise dependence of the r.v. [15]. From these studies the following results can be reported.

Proposition 3. *[8, 10] The reciprocal relation Q generated by a collection of independent random variables (i.e. pairwisely coupled by $T_{\mathbf{P}}$) is dice-transitive (weak $T_{\mathbf{P}}$-transitive).*

Proposition 4. *[7, 9] The reciprocal relation Q generated by a collection of random variables pairwisely coupled by $T_{\mathbf{M}}$ is $T_{\mathbf{L}}$-transitive.*

Proposition 5. *[7, 9] The reciprocal relation Q generated by a collection of random variables pairwisely coupled by $T_{\mathbf{L}}$ is partially stochastic transitive (weak $T_{\mathbf{M}}$-transitive).*

We further considered the case where all C_{ij} are the same copula C. It then turns out that the transitivity of the generated reciprocal relation Q can only be captured as a type of cycle-transitivity when the copula C fulfills a countably infinite family of conditions. These conditions are presented in the form of inequalities. Hence, we require C to be a solution of an infinite system of inequalities.

Proposition 6. *[4] Let C be a commutative copula such that for any $k > 1$ and for all $0 \le x_1 \le x_2 \le \cdots \le x_k \le 1$ and $0 \le y_1 \le y_2 \le \cdots \le y_k \le 1$, it holds that*

$$\begin{aligned}
&\sum_i C(x_i, y_i) - \sum_j C(x_{k-2j}, y_{k-2j-1}) - \sum_j C(x_{k-2j-1}, y_{k-2j}) \\
&\le C\Bigg(x_k + \sum_j C(x_{k-2j-2}, y_{k-2j-1}) - \sum_j C(x_{k-2j}, y_{k-2j-1}), \\
&\qquad y_k + \sum_j C(x_{k-2j-1}, y_{k-2j-2}) - \sum_j C(x_{k-2j-1}, y_{k-2j}) \Bigg), \qquad (1)
\end{aligned}$$

where the sums extend over all integer values that lead to meaningful indices of x and y. Then the reciprocal relation Q generated by a collection of random variables pairwisely coupled by C is cycle-transitive w.r.t. to the upper bound function U^C defined by:

$$U^C(\alpha,\beta,\gamma) = \max(\beta + C(1-\beta,\gamma), \gamma + C(\beta, 1-\gamma)) .$$

If C is stable, i.e. $C(x,y)+1-C(1-x,1-y) = x+y$ for all $(x,y) \in [0,1]^2$, then

$$U^C(\alpha,\beta,\gamma) = \beta + C(1-\beta,\gamma) = \gamma + C(\beta, 1-\gamma)) .$$

Note that symmetrical ordinal sums of Frank copulas are stable [14].

4 Solving the Family of Inequalities

It is natural to ask whether commutative copulas that fulfil (1) can be characterized in an alternative way. However, they are not necessarily satisfied for any stable commutative copula, as is illustrated by the following example of a symmetrical ordinal sum of two Frank copulas.

Example 1. Let C be the commutative copula defined by

$$C(x,y) = \begin{cases} \frac{1}{3} + \max(x+y-1,0) \ , & \text{if } (x,y) \in [1/3,2/3]^2 , \\ \min(x,y) & , \ \text{elsewhere} . \end{cases}$$

It is the ordinal sum $\langle 1/3, 2/3, T_{\mathbf{L}} \rangle$ with $T_{\mathbf{L}}$ linearly rescaled to the square $[1/3,2/3]^2$. It is easily verified that C is stable (as it is a symmetrical ordinal sum of Frank copulas [14]). Let $x_1 = y_1 = 1/4$ and $x_2 = y_2 = 3/4$. For $n = 2$, the left-hand side of (1) becomes $C(1/4,1/4) + C(3/4,3/4) - C(1/4,3/4) - C(3/4,1/4) = 1/4 + 3/4 - 1/4 - 1/4 = 1/2$, while the right-hand side evaluates to $C(x_2 - C(x_2,y_1), y_2 - C(x_1,y_2)) = C(1/2,1/2) = 1/3$, showing that (1) does not hold for $n = 2$ and for all $0 \le x_1 \le x_2 \le 1$ and $0 \le y_1 \le y_2 \le 1$.

In [4], we conjectured that for the Frank copulas themselves conditions (1) are always satisfied but only recently we were able to give a complete proof [6]. Moreover, the Frank copulas are the only associative copulas that are solution of all inequalities separately. Hence, each inequality uniquely characterizes the Frank family of copulas. Special attention is drawn on the first one ($k = 2$), which we call 'the Frank inequality'.

Proposition 7. *Let C be an associative copula. The following statements are equivalent:*

(i) For any $0 \le x \le x' \le 1$ and $0 \le y \le y' \le 1$, it holds that

$$C(x,y) + C(x',y') - C(x,y') - C(x',y) \le C(x' - C(x',y), y' - C(x,y'));$$

(ii) C is a member of the Frank family of copulas.

Note that the left-hand side of (i) is the C-volume of the rectangle $[x,x'] \times [y,y']$ and the condition states that this C-volume is bounded from above by some well-defined C-dependent quantity, i.e. the C-volume of the rectangle $[0,x'-C(x',y)] \times [0,y'-C(x,y')]$.

Proposition 7 can be extended to the other inequalities contained in 1.

Proposition 8. *Let C be an associative copula. The following statements are equivalent:*

(i) C satisfies all inequalities contained in (1);
(ii) C satisfies any one of the inequalities contained in (1);
(iii) C is a member of the Frank family of copulas.

The following result is now immediate.

Proposition 9. *The reciprocal relation Q generated by a collection of random variables pairwisely coupled by the Frank copula $T_\lambda^{\mathbf{F}}$ is cycle-transitive w.r.t. the upper bound function $U_\lambda^{\mathbf{F}}$ given by:*

$$U_\lambda^{\mathbf{F}}(\alpha,\beta,\gamma) = \beta + T_\lambda^{\mathbf{F}}(1-\beta,\gamma) = \beta + \gamma - T_{1/\lambda}^{\mathbf{F}}(\beta,\gamma),$$

otherwise stated, Q is weak $T_{1/\lambda}^{\mathbf{F}}$-transitive.

In the above transition, we have used the fact that $T_\lambda^{\mathbf{F}}(1-x,y) = y - T_{1/\lambda}^{\mathbf{F}}(x,y)$. Since for $\lambda \leq \lambda'$ it holds that $T_\lambda^{\mathbf{F}} \geq T_{\lambda'}^{\mathbf{F}}$, it also follows that $U_\lambda^{\mathbf{F}} \geq U_{\lambda'}^{\mathbf{F}}$. Therefore, the lower the value of λ when the r.v. are coupled by $T_\lambda^{\mathbf{F}}$, the weaker the type of transitivity exhibited by the probabilistic relation generated by these r.v. In particular, the strongest type of transitivity is encountered when coupling by $T_{\mathbf{L}}$ (i.e. partial stochastic transitivity), the weakest when coupling by $T_{\mathbf{M}}$ (i.e. $T_{\mathbf{L}}$-transitivity).

5 Reciprocal Relations Generated by Frank m-Copulas

So far, we have considered collections of r.v. that are pairwisely coupled by the same Frank copula. The obtained transitivity results can be easily extended to the case of random vectors with m components, such that the components are pairwisely coupled by the same Frank copula. It is known that such random vectors exist for certain values of the λ-parameter.

Definition 6. *For any $m \geq 2$ and any $\lambda \in]0,1]$, the m-ary function $C_\lambda^{m\mathbf{F}} : [0,1]^m \to [0,1]$, defined by*

$$C_\lambda^{m\mathbf{F}}(x_1,x_2,\ldots,x_m) = \log_\lambda \left[1 + \frac{(\lambda^{x_1}-1)(\lambda^{x_2}-1)\ldots(\lambda^{x_m}-1)}{(\lambda-1)^{m-1}}\right], \qquad (2)$$

is an m-copula, called Frank m-copula.

Note that the definition can be slightly extended to cover the parameter values $\lambda \in]0, s_m]$, where s_m is an m-dependent upper bound. For instance, $s_3 = 2$, $s_4 = 3 - \sqrt{3}$, $s_5 = 2(3 - \sqrt{6})$, ..., $s_9 = 1.00438$, $\lim_{k\to\infty} s_k = 1$.

Proposition 10. *The reciprocal relation Q generated by a random vector with m components and coupled by $C_\lambda^{m\mathbf{F}}$ is weak $T_{1/\lambda}^{\mathbf{F}}$-transitive.*

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