How to Avoid LEM Cycles in Mutual Rank Probability Relations

K. De Loof, B. De Baets, and H. De Meyer

Abstract. The mutual rank probability (MRP) relation of a poset of size $n \geq 9$ can contain linear extension majority (LEM) cycles. We experimentally derive minimum cutting levels for MRP relations of posets of size $n \leq 13$ such that the crisp cut relation is cycle-free.

1 Introduction

In the probability space consisting of the set of linear extensions of a given poset P equipped with the uniform probability measure, the concept of the mutual rank probability (MRP) relation of a poset appears naturally. It plays an important role from an application [7] as well as from a theoretical [4, 12, 17] point of view. Although the study of the type of transitivity exhibited by MRP relations has received considerable attention [4, 12, 19, 21], this transitivity remains far from characterized. It is, however, well known that MRP relations are in general not weakly stochastic transitive [4], allowing for the occurrence of linear extension majority (LEM) cycles in the MRP relation of posets of size $n \geq 9$.

Quite some attention has been given to LEM cycles in literature. Examples of posets with LEM cycles are given in [1, 11, 12, 13, 14, 16, 18], frequency estimates for LEM cycles have been reported in [15, 17], and the occurrence of LEM cycles in certain subclasses of posets has been studied in [2, 9]. Moreover, in previous work [8], the present authors have succeeded in counting the

K. De Loof and H. De Meyer
Department of Applied Mathematics and Computer Science,
Ghent University, Krijgslaan 281 S9, B-9000 Gent, Belgium
e-mail: karel.deloof@ugent.be

B. De Baets
Department of Applied Mathematics, Biometrics and Process Control,
Ghent University, Coupure links 653, B-9000 Gent, Belgium

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posets of size $n \leq 13$ with LEM cycles. Besides the fact that the existence of LEM cycles is an intriguing phenomenon in its own right, a better understanding of LEM cycles might help in the ongoing quest to characterize the transitivity of MRP relations. In the present paper we focus on the determination of minimum cutting levels at which the MRP relation becomes free of cycles.

2 Posets, MRP Relations and LEM Cycles

A binary relation $\leq_P$ on a set P is called an *order relation* if it is reflexive ($x \leq_P x$), antisymmetric ($x \leq_P y$ and $y \leq_P x$ imply $x =_P y$) and transitive ($x \leq_P y$ and $y \leq_P z$ imply $x \leq_P z$). A *linear order relation* $\leq_P$ is an order relation in which every two elements are comparable ($x \leq_P y$ or $y \leq_P x$). If $x \leq_P y$ and $x \neq y$, we write $x <_P y$. If neither $x \leq_P y$ nor $x \geq_P y$, we say that x and y are *incomparable* and write $x \,||_P\, y$. A couple $(P, \leq_P)$, where P is a set of objects and $\leq_P$ is an order relation on P, is called a partially ordered set or *poset* for short. The *size* of a poset $(P, \leq_P)$ is defined as the cardinality of P. A poset of size n will be called an n-element poset for short. The poset $(P, \leq_P^\top)$ for which $y \leq_P^\top x$ if and only if $x \leq_P y$ for all $x, y \in P$ is called the *dual poset* of $(P, \leq_P)$.

The binary relation $\prec_P$, for which it holds that $(x,y) \in \prec_P$ if and only if $x <_P y$ and there exists no $z \in P$ such that $x <_P z <_P y$, is called the *covering relation* of $(P, \leq_P)$. The covering relation $\prec_P$ of a poset $(P, \leq_P)$ can be conveniently represented by a so-called *Hasse diagram* where a sequence of connected lines upwards from x to y is present if and only if $x <_P y$. Examples of representations of posets by such Hasse diagrams can be found in the appendix of this paper.

Let Q be a set and R and S two binary relations on Q. If $R \subset S$, then (Q, S) is called an extension of (Q, R). A *linear extension* of a poset $(P, \leq_P)$ is an extension $(P, \leq_L)$ for which $\leq_L$ is a linear order relation. The *mutual rank probability* $\mathrm{p}(x > y)$ of two elements x and y of a poset $(P, \leq_P)$ is defined as the probability that $x >_L y$ in a linear extension $(P, \leq_L)$ that has been sampled uniformly at random from the set of linear extensions of $(P, \leq_P)$. Stated differently, it is the number of linear extensions of $(P, \leq_P)$ in which $x >_L y$, divided by the number of linear extensions of $(P, \leq_P)$. *The mutual rank probability (MRP) relation* M_P is the [0.1]-valued binary relation on P defined by $M_P(x,y) = \mathrm{p}(x > y)$ for all $x, y \in P$ where $x \neq y$ and $M_P(x,x) = 1/2$ for all $x \in P$. Note that M_P is a so-called reciprocal relation since $M_P(x,y) + M_P(y,x) = 1$.

The *linear extension majority (LEM) relation* [20] of a poset P is the binary relation $\succ_{\mathrm{LEM}}$ on P such that $x \succ_{\mathrm{LEM}} y$ if $\mathrm{p}(x > y) > \mathrm{p}(y > x)$. Due to the reciprocity of the MRP relation, it is equivalent to define $x \succ_{\mathrm{LEM}} y$ if $\mathrm{p}(x > y) > 1/2$. It is well known [10] that the LEM relation $\succ_{\mathrm{LEM}}$ can contain cycles, *i.e.* subsets $\{x_1, x_2, \ldots, x_m\}$ of elements of P such that $x_1 \succ_{\mathrm{LEM}} x_2 \succ_{\mathrm{LEM}} \cdots \succ_{\mathrm{LEM}} x_m \succ_{\mathrm{LEM}} x_1$, and thus is not transitive. These cycles are referred to as *LEM cycles* on m elements, or *m-cycles* for short.

The *strict* δ*-cut* with $\delta \in [1/2,1[$ of a reciprocal relation Q defined on a set A is the crisp relation Q^δ defined by

$$Q^\delta(x,y) = \begin{cases} 1\ , & \text{if } Q(x,y) > \delta\,, \\ 0\ , & \text{otherwise.} \end{cases}$$

We define the *minimum cutting level* δ_m as the smallest number such that for any finite poset the strict δ_m-cut of the corresponding MRP relation is free of l-cycles, with $l \leq m$.

3 Minimum Cutting Levels for Posets of Size $n \leq 13$

The present authors have shown in [6] that the MRP relation can be computed using the lattice of ideals representation of a poset without necessitating the enumeration of all linear extensions. This approach is ideally suited for obtaining the MRP relation of posets of size $n \leq 13$. A combination of the poset generation algorithm of Brinkmann and McKay [3] and the algorithm to compute the MRP relation for a given poset enabled us to obtain exact counts of LEM cycles for posets on up to 13 elements [8]. We adapted this algorithm to keep track of the minimum cutting level δ_m^n and of all posets requiring this cutting level δ_m^n such that all mutual rank probabilities in an m-cycle are greater than or equal to δ_m^n. In Table 1 these minimum cutting levels δ_m^n to avoid m-cycles in n-element posets are shown.

Since one can trivially construct a poset of size $n+1$ from a poset of size n with an equal minimum cutting level by adding an element which is either smaller than, larger than or incomparable to the given n elements, the minimum cutting levels δ_m^n are monotone in n. Therefore, a minimum cutting level to avoid m-cycles avoids all LEM cycles of length $l \leq m$. In Table 1 one can observe that for $n = 11$ no higher cutting level for avoiding 4-cycles is found than for $n = 10$ since $\delta_4^{11} = \delta_4^{10}$, and similarly it is found that $\delta_4^{13} = \delta_4^{12}$.

In Figures 1-14 the posets requiring the non-trivial minimum cutting levels indicated in boldface in Table 1 are depicted by their Hasse diagrams. Note that the dual of a poset has an equal minimum cutting level, and is therefore not shown. However, four depicted posets are identical to their dual posets (Figures 1, 5, 6 and 14). It is also interesting to mention that some posets

Table 1 Minimum cutting level δ_m^n to avoid m-cycles in posets of size $n = 9,\ldots,13$ for $m = 3,\ldots,7$.

$n\backslash m$	3	4	5	6	7
9	**0.50314465**	0.5	0.5	0.5	0.5
10	**0.50396825**	**0.50284900**	0.5	0.5	0.5
11	**0.50619469**	0.50284900	0.5	0.5	0.5
12	**0.50735039**	**0.50866575**	**0.50039788**	**0.50242592**	0.5
13	**0.50886687**	0.50866575	**0.50289997**	**0.50246440**	**0.50018080**

have multiple LEM cycles with an identical cutting level, while others have LEM cycles of different lengths. The 9-element poset in Figure 1, for example, has three 3-cycles with identical probabilities, while for the 12-element poset in Figure 5, aside from the cycle with length 3, a 4-cycle is present, since it holds that $\mathrm{p}(5 > 7) = \mathrm{p}(7 > 8) = 6184/12244$. The 12-element poset in Figure 7 is quite remarkable in this respect, since aside from the 5-cycle, cycles of length 3, 4 and 6 are present. The poset in Figure 14 even has cycles of length 3, 4, 5, 6 and 7. Furthermore, the poset in Figure 8 also has a 3-cycle, the poset in Figure 11 has cycles of length 3 and 4, and the posets in Figures 12 and 13 both have 3-cycles. For some cutting levels multiple posets, aside from their dual versions, are found. This is the case for the posets of size 13 in Figures 9 and 10 which attain the minimum cutting level for 3-cycles. The same is true for 6-cycles in Figures 12 and 13.

4 Conclusion

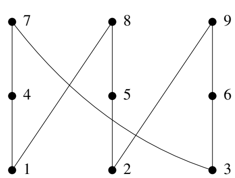
One of the aims of this experiment was to find common properties for posets with LEM cycles or to see a common structure emerging in the posets requiring the minimum cutting level. However, to our surprise the posets have little in common. Possibly due to the fact that the posets are still very limited in size no common (sub)structures can yet be observed for increasing size. The symmetrical and relatively simple structure of the 12-element poset in Figure 6 requiring the minimum cutting level δ_4 inspired us to try to generalize it and to find a lower bound for δ_4 as sharp as possible for increasing poset size. We have reported on these results elsewhere [5].

References

1. Aigner, M.: Combinatorial Search. Wiley-Teubner, Chichester (1988)
2. Brightwell, G., Fishburn, P., Winkler, P.: Interval orders and linear extension cycles. Ars Combin. 36, 283–288 (1993)
3. Brinkmann, G., McKay, B.: Posets on up to 16 points. Order 19, 147–179 (2002)
4. De Baets, B., De Meyer, H., De Loof, K.: On the cycle-transitivity of the mutual rank probability relation of a poset. Fuzzy Sets Syst., doi:10.1016/j.fss.2010.05.005
5. De Loof, K., De Baets, B., De Meyer, H.: Cycle-free cuts of mutual rank probability relations. Discrete Appl. Math. (submitted for publication, 2010)
6. De Loof, K., De Meyer, H., De Baets, B.: Exploiting the lattice of ideals representation of a poset. Fund. Inform. 71, 309–321 (2006)
7. De Loof, K., De Baets, B., De Meyer, H.: A hitchhiker's guide to poset ranking. Comb. Chem. High Throughput Screen. 11, 734–744 (2008)
8. De Loof, K., De Baets, B., De Meyer, H.: Counting linear extension majority cycles in partially ordered sets on up to 13 elements. Comput. Math. Appl. 59(4), 1541–1547 (2010)

9. Ewacha, K., Fishburn, P., Gehrlein, W.: Linear extension majority cycles in height-1 orders. Order 6, 313–318 (1990)
10. Fishburn, P.: On the family of linear extensions of a partial order. J. Combin. Theory Ser. B 17, 240–243 (1974)
11. Fishburn, P.: On linear extension majority graphs of partial orders. J. Combin. Theory Ser. B 21, 65–70 (1976)
12. Fishburn, P.: Proportional transitivity in linear extensions of ordered sets. J. Combin. Theory Ser. B 41, 48–60 (1986)
13. Fishburn, P., Gehrlein, W.: A comparative analysis of methods for constructing weak orders from partial orders. J. Math. Sociol. 4, 93–102 (1975)
14. Ganter, B., Hafner, G., Poguntke, W.: On linear extensions of ordered sets with a symmetry. Discrete Math 63, 153–156 (1987)
15. Gehrlein, W.: Frequency estimates for linear extension majority cycles on partial orders. RAIRO Oper. Res. 25, 359–364 (1991)
16. Gehrlein, W.: The effectiveness of weighted scoring rules when pairwise majority rule cycles exist. Math. Social Sci. 47, 69–85 (2004)
17. Gehrlein, W., Fishburn, P.: Linear extension majority cycles for partial orders. Ann. Oper. Res. 23, 311–322 (1990)
18. Gehrlein, W., Fishburn, P.: Linear extension majority cycles for small ($n \leq 9$) partial orders. Comput. Math. Appl. 20, 41–44 (1990)
19. Kahn, J., Yu, Y.: Log-concave functions and poset probabilities. Combinatorica 18, 85–99 (1998)
20. Kislitsyn, S.: Finite partially ordered sets and their associated sets of permutations. Matematicheskiye Zametki 4, 511–518 (1968)
21. Yu, Y.: On proportional transitivity of ordered sets. Order 15, 87–95 (1998)

Appendix: Posets Requiring Minimum Cutting Levels δ_m^n

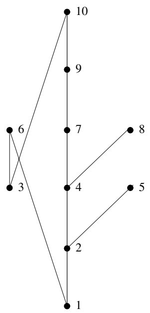


$$p(7>8)=p(8>9)=p(9>7)=\frac{720}{1431}\approx 0.50314465$$

$$p(4>5)=p(5>6)=p(6>4)=\frac{720}{1431}\approx 0.50314465$$

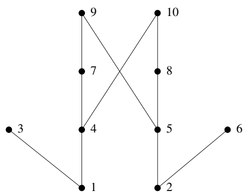
$$p(1>2)=p(2>3)=p(3>1)=\frac{720}{1431}\approx 0.50314465$$

Fig. 1 9-element poset with a LEM cycle requiring the minimal cutting level δ_3^9.



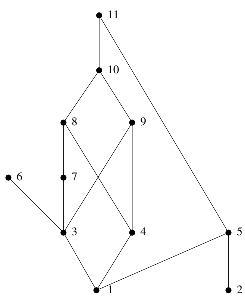
$$p(8>6)=p(6>9)=\frac{508}{1008}\approx 0.50396825$$
$$p(9>8)=\frac{512}{1008}$$

Fig. 2 10-element poset with a LEM cycle requiring the minimal cutting level δ_3^{10}.



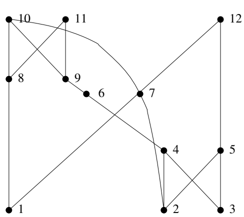
$$p(7>3)=p(3>8)=p(8>6)=p(6>7)$$
$$=\frac{1765}{3510}\approx 0.50284900$$

Fig. 3 10-element poset with a LEM cycle requiring the minimal cutting level δ_4^{10}.



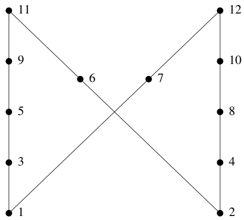
$$p(5>8)=\frac{1146}{2260}$$
$$p(8>6)=\frac{1144}{2260}\approx 0.50619469$$
$$p(6>5)=\frac{1145}{2260}$$

Fig. 4 11-element poset with a LEM cycle requiring the minimal cutting level δ_3^{11}.



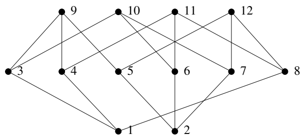
$$p(8>6)=p(6>5)=\frac{6214}{12244}$$
$$p(5>8)=\frac{6212}{12244}\approx 0.50735039$$

Fig. 5 12-element poset with a LEM cycle requiring the minimal cutting level δ_3^{12}.



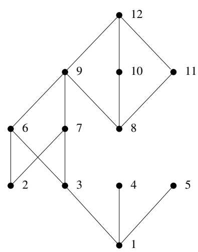
$$p(5>7)=p(7>8)=p(8>6)=p(6>5)$$
$$=\frac{7396}{14540}\approx 0.50866575$$

Fig. 6 12-element poset with a LEM cycle requiring the minimal cutting level δ_4^{12}.



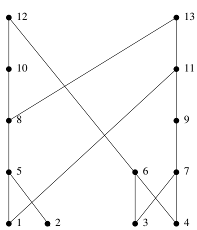
$$p(5>4)=p(4>3)=\frac{60400}{120640}$$
$$p(3>6)=p(6>8)=p(8>5)=\frac{60368}{120640}\approx 0.50039788$$

Fig. 7 12-element poset with a LEM cycle requiring the minimal cutting level δ_5^{12}.



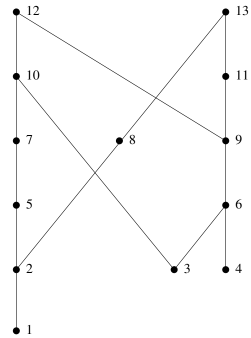
$$p(7>4)=p(6>5)=\frac{46392}{92336}\approx 0.50242592$$
$$p(4>10)=p(5>11)=\frac{46560}{92336}$$
$$p(10>6)=p(11>7)=\frac{46850}{92336}$$

Fig. 8 12-element poset with a LEM cycle requiring the minimal cutting level δ_6^{12}.



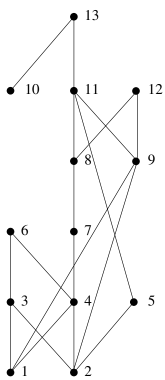
$$p(6>8)=\frac{12240}{24022}$$
$$p(8>9)=\frac{12262}{24022}$$
$$p(9>6)=\frac{12224}{24022}\approx 0.50886687$$

Fig. 9 First 13-element poset with a LEM cycle requiring the minimal cutting level δ_3^{13}.



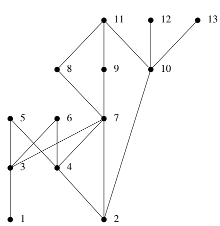
$$p(7 > 8) = \frac{6112}{12011} \approx 0.50886687$$
$$p(8 > 9) = \frac{6120}{12011}$$
$$p(9 > 7) = \frac{6131}{12011}$$

Fig. 10 Second 13-element poset with a LEM cycle requiring the minimal cutting level δ_3^{13}.



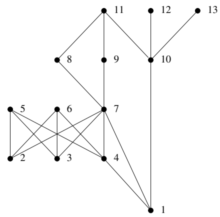
$$p(10 > 9) = \frac{33871}{67242}$$
$$p(9 > 5) = \frac{33916}{67242}$$
$$p(5 > 7) = \frac{33816}{67242} \approx 0.50289997$$
$$p(7 > 3) = \frac{33834}{67242}$$
$$p(3 > 10) = \frac{34151}{67242}$$

Fig. 11 13-element poset with a LEM cycle requiring the minimal cutting level δ_5^{13}.



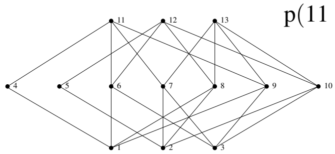
$$p(12 > 9) = p(13 > 8) = \frac{66354}{131472}$$
$$p(9 > 6) = p(8 > 5) = \frac{66060}{131472} \approx 0.50246440$$
$$p(6 > 13) = p(5 > 12) = \frac{66306}{131472}$$

Fig. 12 First 13-element poset with a LEM cycle requiring the minimal cutting level δ_6^{13}.



$$p(12 > 9) = p(13 > 8) = \frac{132708}{262944}$$
$$p(9 > 6) = p(8 > 5) = \frac{132120}{262944} \approx 0.50246440$$
$$p(6 > 13) = p(5 > 12) = \frac{132612}{262944}$$

Fig. 13 Second 13-element poset with a LEM cycle requiring the minimal cutting level δ_6^{13}.



$$p(11 > 6) = p(6 > 8) = p(4 > 11) = \frac{268352}{536510} \approx 0.50018080$$
$$p(8 > 10) = p(5 > 4) = \frac{268384}{536510}$$
$$p(10 > 12) = p(12 > 5) = \frac{268465}{536510}$$

Fig. 14 13-element poset with a LEM cycle requiring the minimal cutting level δ_7^{13}.