Hadamard Majorants for the Convex Order and Applications

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Abstract. The problem of establishing Hadamard-type inequalities for convex functions on d-dimensional convex bodies ($d \geq 2$) translates into the problem of finding appropriate majorants of the involved random vector for the usual convex order. In this work, we use a stochastic approach based on the Brownian motion to establish a multidimensional version of the classical Hadamard inequality. The main result is closely related to the Dirichlet problem and is applied to obtain inequalities for harmonic functions on general convex bodies.

Keywords: Convex order, Hadamard inequalities, Convex functions, Brownian motion, Dirichlet problem, Harmonic functions.

1 Introduction

It is well known that, for every real convex function f on the interval $[a,b]$, we have

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$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{f(a)+f(b)}{2}. \tag{1}$$

This is the celebrated *Hadamard inequality.* In probabilistic words, it says that

$$\mathrm{E}f(\xi) \le \mathrm{E}f(\xi^*), \qquad f \in \mathscr{C}_{\mathrm{cx}}, \tag{2}$$

where E denotes mathematical expectation, ξ (respectively, ξ^*) is a random variable having the uniform distribution on the interval $[a,b]$ (respectively, on the set $\{a,b\}$), and $\mathscr{C}_{\mathrm{cx}}$ is the set of all real convex functions on $[a,b]$; another way of expressing (2) is

$$\xi \le_{\mathrm{cx}} \xi^*,$$

where $\le_{\mathrm{cx}}$ stands for the so called *convex order* of random variables (see [7] and [10]).

There is an extensive literature devoted to develop applications of the inequality (1), as well as to discuss its extensions, by considering other measures, other kinds of convexity, or higher dimensions. An account of many of such realizations is given in [4].

In this paper, which follows the spirit of [2] and [3], we consider multidimensional analogues of (2), where $[a,b]$ is replaced by a (nonempty) d-dimensional compact convex set $K \subset \mathbb{R}^d$ $(d \ge 2)$, ξ is an arbitrary integrable random vector taking values in K, and $\mathscr{C}_{\mathrm{cx}}$ is the set of all real continuous convex functions on K. In this multidimensional setting, we can distinguish two different problems, according to whether the role of $\{a,b\}$ is played by the set of extreme points of K, to be denoted by K^*, or by the boundary of K, to be denoted by K_*.

- *Strong problem*: Find an H*-majorant of ξ, that is, (the probability distribution of) a random vector ξ^* taking values in K^*, such that

$$\mathrm{E}f(\xi) \le \mathrm{E}f(\xi^*), \qquad f \in \mathscr{C}_{\mathrm{cx}}. \tag{3}$$

- *Weak problem*: Find an H$_*$-majorant of ξ, that is, (the probability distribution of) a random vector ξ_* taking values in K_*, such that

$$\mathrm{E}f(\xi) \le \mathrm{E}f(\xi_*), \qquad f \in \mathscr{C}_{\mathrm{cx}}. \tag{4}$$

Since $K^* \subset K_*$, each solution to the strong problem is also a solution to the weak one, and both problems coincide when $K^* = K_*$ (for instance, if K is a closed ball in l_p, with $1 < p < \infty$).

The following theorem is a specific version, in the setting of finite-dimensional spaces, of a more general result established by Niculescu [8] on the basis of Choquet's theory [9].

Theorem 1. *Every K-valued random vector ξ has at least one H^*-majorant.*

This interesting result leaves open the problem of finding such an H*-majorant, a necessary task in order to achieve concrete inequalities of the Hadamard type and other related results.

Remark 1. Observe that (3) (respectively, (4)) implies that $\mathrm{E}\xi = \mathrm{E}\xi^*$ (respectively, $\mathrm{E}\xi = \mathrm{E}\xi_*$), since affine functions are convex.

Remark 2. As it was pointed out in [2], the distributions of ξ_* and ξ^* are not necessarily unique. They depend on the geometric structure of K and on the probability distribution of ξ.

Example 1. In the 1-dimensional case, the strong problem coincides with the weak one. Further, taking into account Theorem 1 and Remark 1, given a random variable ξ defined in the interval $[a,b]$, it is easy to find the *unique* (in the sense of distribution) H*-majorant,

$$\xi^* = \begin{cases} a & \text{with probability } \frac{b-\mathrm{E}\xi}{b-a}, \\ b & \text{with probability } \frac{\mathrm{E}\xi-a}{b-a}. \end{cases}$$

This H*-majorant generates the Hadamard-type inequality

$$\mathrm{E}f(\xi) \le \frac{b-\mathrm{E}\xi}{b-a} f(a) + \frac{\mathrm{E}\xi - a}{b-a} f(b), \qquad f \in \mathscr{C}_{\mathrm{cx}},$$

which was first obtained by Fink [5] (see also [7, Example 1.10.5]).

In this paper, we deal with the weak problem described earlier. That is, given a K-valued random vector ξ, where K is a compact convex set of $\mathbb{R}^d$, we obtain a majorant for the convex order of ξ concentrated on the boundary of K, K_*. This is done in the next section. In Section 3, we show that the main result is closely related to the solution of the Dirichlet problem on K. Therefore, we derive some inequalities for the harmonic functions on K for which the restriction to the boundary is a convex function. Some multidimensional Hadamard-type inequalities are also obtained by using this new approach.

2 Main Result

Let $K \subset \mathbb{R}^d$ be a (nonempty) d-dimensional compact convex set and let ξ be a K-valued random vector. The main result of this section, Theorem 2 below, provides an explicit H$_*$-majorant of ξ. To find such a majorant, we note that given two integrable random vectors ξ and η such that $\xi \le_{\mathrm{cx}} \eta$, the Strassen's theorem (see for instance [10, Theorem 7.A.1., p. 324]) ensures that there exist two random vectors $\hat{\xi}$ and $\hat{\eta}$, defined on the same probability space, such that $\hat{\xi}$ and $\hat{\eta}$ have the same distribution as ξ and η, respectively, and $\{\hat{\xi}, \hat{\eta}\}$ is a martingale, that is,

$$\mathrm{E}[\,\hat{\eta}\,|\,\hat{\xi}\,] = \hat{\xi}, \quad \text{a.s.}$$

Therefore, to find an H_*-majorant of the vector ξ, we can construct a continuous time martingale $\{\xi_t\}_{t\geq 0}$ starting from ξ (i.e., $\xi_0 = \xi$) and stopped at a random time τ on the border of K (i.e., $\xi_\tau \in K_*$). We use the Brownian motion as a natural continuous time martingale connecting ξ with ξ_τ (see Figure 1).

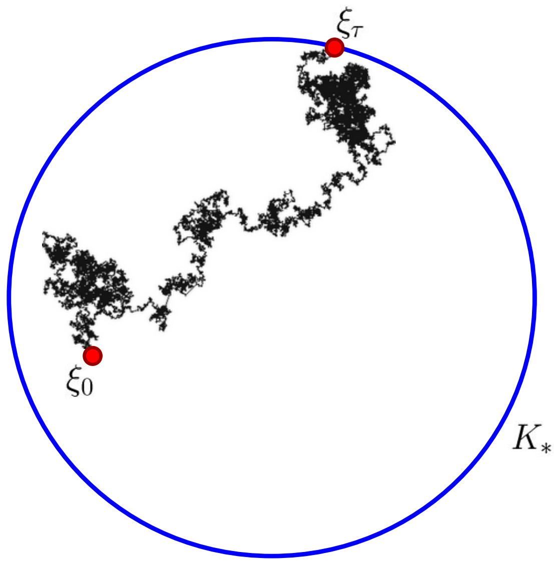

Fig. 1 Construction of an H_*-majorant through the Brownian motion on the disk

Therefore, to achieve our main result we need the Brownian motion. For general notions and results concerning this topic, we refer to [6].

We recall that a d-dimensional Brownian motion is an $\mathbb{R}^d$-valued stochastic process $\{\xi_t : t \geq 0\}$ having the following properties:

(a) it has stationary independent increments,
(b) for all $s \geq 0$ and $t > 0$, $\xi_{s+t} - \xi_s$ has the Gaussian distribution with density

$$g_t(x) = (2\pi t)^{-d/2} e^{-|x|^2/2t}, \qquad x \in \mathbb{R}^d$$

($|\cdot|$ being the Euclidean norm),
(c) with probability 1, it has continuous paths.

The random variable ξ_0 gives the (random) starting point of the process. The process $\{\xi_t - \xi_0 : t \geq 0\}$ is a Brownian motion starting at 0, which is called standard Brownian process. Such a process is defined on some probability space $(\Omega, \mathscr{F}, \mathrm{P})$, and it is well-known that it is a (continuous) martingale with respect to the right-continuous filtration $\{\mathscr{F}_t^+ : t \geq 0\}$ given by

$$\mathscr{F}_t^+ := \bigcap_{s>t} \mathscr{F}_s, \qquad t \geq 0,$$

where, for $s > 0$, $\mathscr{F}_s$ is the sub-σ-field of $\mathscr{F}$ generated by $\{\xi_t : 0 \leq t \leq s\}$.

Our main result is stated as follows. We use the standard convention that the infimum of the empty set is $+\infty$, and we denote by K° the interior of K. Also, $\xi =_{\mathrm{st}} \xi'$ stands for the fact that the random vectors ξ and ξ' have the same probability distribution.

Theorem 2. *Let $K \subset \mathbb{R}^d$ be a (nonempty) d-dimensional compact convex set, let ξ be a K-valued random vector, and let $\{\xi_t : t \geq 0\}$ be a d-dimensional Brownian motion such that $\xi_0 =_{\mathrm{st}} \xi$. Then, the random time τ given by*

$$\tau := \inf\{t \geq 0 : \xi_t \notin K^\circ\}$$

is a stopping time with respect to the filtration $\{\mathscr{F}_t^+ : t \geq 0\}$ that fulfills

$$\mathrm{P}(\tau < \infty) = 1, \tag{5}$$

and the random vector ξ_τ is an H_*-majorant of ξ.

3 Applications: Harmonic Functions and the Dirichlet Problem

It is well known that the stopped Brownian motion that appears in Theorem 2 is connected with the solution of the Dirichlet problem. Since K is convex, K° is obviously a regular domain, in the sense of [6, p. 394]. Then (see the last reference, or [1, p. 90]), for each $g \in \mathscr{C}(K_*)$, the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{on } K^\circ \\ u = g & \text{on } K_*, \end{cases}$$

has a unique solution in $\mathscr{C}^2(K^\circ) \cap \mathscr{C}(K)$, to be denoted by Hg, which is given by

$$Hg(x) = \mathrm{E}[g(\xi_\tau) \mid \xi_0 = x], \qquad x \in K, \tag{6}$$

(where $\mathrm{E}[\cdot \mid \cdot]$ denotes conditional expectation). We therefore have

$$\mathrm{E}g(\xi_\tau) = \mathrm{E}[Hg(\xi_0)] = \mathrm{E}[Hg(\xi)]$$

(the last equality because $\xi_0 =_{\mathrm{st}} \xi$), and this yields the following corollary (where we write Hf instead of $Hf_{|K_*}$).

Corollary 1. *For each $f \in \mathscr{C}_{cx}$, the upper Hermite-Hadamard inequality $\mathrm{E}f(\xi) \leq \mathrm{E}f(\xi_\tau)$ can be written in the form*

$$\mathrm{E}f(\xi) \leq \mathrm{E}[Hf(\xi)].$$

This result holds true for each K-valued random vector ξ. Therefore, on taking $\xi \equiv x$ ($x \in K$), we conclude the following.

Corollary 2. *For each $f \in \mathscr{C}_{cx}$, we have*

$$f \leq Hf.$$

Example 2. Let K be the closed unit Euclidean ball in $\mathbb{R}^d$. Using the previous ideas it is possible to generate a multidimensional version of the classical Hadamard inequality (1). When $f \in \mathscr{C}_{\rm cx}$ and ξ has the uniform distribution on K, we obtain

$$\frac{1}{\mathrm{Vol}(K)} \int_K f(x)\,\mathrm{d}x \leq \frac{1}{\sigma(K_*)} \int_{K_*} f(y)\,\sigma(\mathrm{d}y),$$

where σ is the surface measure on K_*. This result was already found in [2] by using a different approach.

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