On Concordance Measures and Copulas with Fractal Support

E. de Amo, M. Díaz Carrillo, and J. Fernández-Sánchez

Abstract. Copulas can be used to describe multivariate dependence structures. We explore the rôle of copulas with fractal support in the study of association measures.

1 General Introduction and Motivation

Copulas are of interest because they link joint distributions to their marginal distributions. Sklar [12] showed that, for any real-valued random variables X_1 and X_2 with joint distribution H, there exists a copula C such that $H(u,v) = C(F_1(u), F_2(v))$, where F_1 and F_2 denote the cumulative (or margin) distributions of X_1 and X_2, respectively. If the marginals are continuous, then the copula is unique. Notice that it is also true the converse implication of Sklar's Theorem. In fact, we may link any univariate distributions with any copula in order to obtain a valid joint distribution function. An implication of Sklar Theorem is that the dependence among X_1 and X_2 is fully described by the associated copula. Indeed, most conventional dependence measures can be explicitly expressed in terms of the copula, and they are designed to capture certain aspects of dependence or association between random variables.

On the other hand, all the examples of singular copulas we have found in the literature are supported by sets with Hausdorff dimension 1. However, it is implicit in some papers, for example in [11], that the well-known examples of Peano and Hilbert curves provide self-similar copulas with fractal support, since the Hausdorff dimension of their graphs is 3/2.

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Recently, Fredricks et al. [7], using an iterated function system, constructed families of copulas whose supports are fractals. In particular, they give sufficient conditions for the support of a self-similar copula to be a fractal whose Hausdorff dimension is between 1 and 2.

In [1], the authors prove that a necessary and sufficient condition for a copula to be the independence (or product) copula Π is that the pair of measure preserving transformations representing the copula be independent as random variables; and a general constructive method for representation of copulas in terms of measure preserving transformations is given. Specifically, we study the copulas introduced by Fredricks et al. in [7], through two representation number systems we construct *ad hoc*. This paper is devoted to study these copulas in depth.

Firstly, we give an example of copula with a support with a Lebesgue measure 0, and a Hausdorff dimension 2. Moreover, we study the coefficients of upper and lower tail dependence of these copulas. Finally, we explore some well-known measures of dependence, namely Kendall's τ, Spearman's ρ, and Gini's index γ. The results we prove coincide with those regard to the independence copula Π.

2 Preliminaries about Copulas with Fractal Support

This section contains background information and useful notation.

(1) Let $\mathbb{I}$ be the closed unit interval $[0,1]$ and let $\mathbb{I}^2 = \mathbb{I} \times \mathbb{I}$ be the unit square. For an introduction to copulas see, for example, [4] or [9].

(2) A *transformation matrix* is a matrix T with nonnegative entries, for which the sum of the entries is 1 and none row or column has zero as entry everywhere.

Following [7], we recall that each transformation matrix T determines a subdivision of $\mathbb{I}^2$ into subrectangles $R_{ij} = [p_{i-1}, p_i] \times \left[q_{j-1}, q_j\right]$, where p_i (respect. q_j) denotes the sum of the entries in the first i columns (respect. j rows) of T. For a transformation matrix T and a copula C, $T(C)$ denotes the copula that, for each (i,j), spreads mass on R_{ij} in the same way in which C spreads mass on $\mathbb{I}^2$.

Theorem 2 in [7] shows that for each transformation matrix $T \neq [1]$, there is an unique copula C_T such that $T(C_T) = C_T$.

(3) Let T be a transformation matrix, and let us consider the following conditions:

(i) T has at least one zero entry.
(ii) For each non-zero entry of T, the row and column sums through that entry are equal.
(iii) There is at least one row or column of T with two nonzero entries.

Theorem 3 in [7] shows that if T is a transformation matrix satisfying condition (i) , then C_T is singular (that is, its support has Lebesgue measure zero or $\mu_{C_T} \equiv \mu^s_{C_T}$, where $\mu^s_{C_T}$ is the singular measure given by the Lebesgue Decomposition Theorem). See, for example, [9] or [10].

We say that a copula C is *invariant* if $C = C_T$ for some transformation matrix T. An invariant copula C_T is said to be *self-similar* if T satisfies condition (ii).

Theorem 6 in [7] shows that the support of a self-similar copula C_T, with T satisfying (i) and (iii), is a fractal whose Hausdorff dimension is between 1 and 2.

(4) A mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is called a *contracting similarity* (or a similarity transformation) of ratio r $(0 < r < 1)$ if $\|F(x) - F(y)\| = r\|x - y\|$, for all $x, y \in \mathbb{R}^n$. A similarity transforms subsets of $\mathbb{R}^n$ into geometrically similar sets. The invariant set (or attractor) for a finite family of similitaries is said to be a *self-similar set*. Theorem 4 in [7] shows that the support of the copula C_T is the invariant set for a system of similarities obtained from partitions of $\mathbb{I}$ which are determined by T. (See (2) above.)

For an introduction to the techniques of fractal representation by iterated function systems (IFS) the reader can see [5] or [6].

(5) Finally, we recall that the notion of an IFS may be extended to define invariant measures supported by the attractor of the system. Explicitly, let $\{F_1, ..., F_m\}$ be an IFS on $K \subset \mathbb{R}^n$ and $p_1, \ldots, p_m$ be probabilities or mass ratios, with $p_i > 0$ for all i and $\sum_{i=1}^m p_i = 1$. A measure μ is said to be *self-similar* if for some p_i and F_i, $\mu(A) = \sum_{i=1}^m p_i \mu\left(F_i^{-1}(A)\right)$ and any Borel set A. The existence of such measure is ensured. (See [6, Th.2.8].)

3 A Singular Copula with Fractal Support and Hausdorff Dimension 2

Now, we consider the family of transformation matrices

$$T_r = \begin{pmatrix} r/2 & 0 & r/2 \\ 0 & 1-2r & 0 \\ r/2 & 0 & r/2 \end{pmatrix}$$

with $r \in \left]0, \frac{1}{2}\right[$. According to **(2)** and **(3)** above, $\left\{C_r = C_{T_r} : r \in \left]0, \frac{1}{2}\right[\right\}$ is a family of copulas whose support is a fractal with a Hausdorff dimension in the interval $]1, 2[$.

Precisely, Fredricks et al. [7, Th. 1] proved that if $s \in]1, 2[$, then there exists a copula $C = C_{r(s)}$ satisfying that the Hausdorff dimension of the support of C is s. We denote it by S_r. It is clear that there exist singular functions whose support is of Hausdorff dimension 1; and that there are no copulas whose support is of Hausdorff dimension smaller than 1, as well.

On the other hand, as far as we know, there is no a singular copula whose fractal support is, exactly, of Hausdorff dimension 2. We now show an example of copula with these properties.

Theorem 1. *Let* $J_i = \left[\frac{1}{i+1}, \frac{1}{i}\right]$, *with* $i \in \mathbb{Z}^+$, *and the copula* $C^i = C_{r\left(\frac{2i-1}{i}\right)}$. *Then, the ordinal sum of* $\left\{C^i\right\}_{i\in\mathbb{Z}^+}$ *with respect to* $\{J_i\}_{i\in\mathbb{Z}^+}$, *that is,*

$$C(u,v) = \begin{cases} \frac{1}{i+1} + \frac{1}{i(i+1)} C^i\left(i(i+1)\left(u - \frac{1}{i+1}\right), i(i+1)\left(v - \frac{1}{i+1}\right)\right), & (u,v) \in J_i^2 \\ \min(u,v), & \textit{otherwise,} \end{cases}$$

is a singular copula, and its support is of Hausdorff dimension 2.

Proof. The way we have defined C ensures that it is a copula (see [3] or [9]).

Out of the J_i^2 squares, the copula C does not have associated mass distribution. On each square J_i^2, the similarity

$$F_i(u,v) \to \left(\frac{1}{i+1} + \frac{u}{i(i+1)}, \frac{1}{i+1} + \frac{v}{i(i+1)}\right)$$

spreads the mass distribution in the support of C^i. It is straightforward (via the definition of C) that $\frac{\partial^2 C^i}{\partial u \partial v}(u,v) = 0$ out of that set.

The function F_i is a similarity, and hence, bi-Lipschitz (i.e., a bijective Lipschitz function whose inverse function is also Lipschitz). Therefore, it preserves the Hausdorff dimension (see [6, Cor. 2.4]). As a consequence,

$$\dim_{\mathscr{H}} F_i\left(S_{r\left(\frac{2i-1}{i}\right)}\right) = \frac{2i-1}{i}.$$

But, the Hausdorff dimension of the set $\cup_i F_i\left(S_{r((2i-1)/i)}\right)$ is the supremum of the above numbers; that is, 2. Therefore, C is a singular copula whose support is of Hausdorff dimension 2. □

4 Tail Dependence for C_r

Copulas are useful to model tail dependence, that is, the dependence that arises between random variables from extreme observations.

Let X and Y be continuous random variables with distribution functions F and G, respectively, and associated copula C. We study the coefficients of upper and lower tail dependence of X and Y (their definition can be found in [8] or [9]).

Theorem 2. *Let* $r \in \left]0, \frac{1}{2}\right[$. *Given the copula* C_r, *we have*

$$\lambda_U(C_r) = \lambda_L(C_r) = 0.$$

Proof. The symmetry of the mass distribution of the measure μ_{C_r} ensures that, if they exist, the two values are the same. We study the case λ_L.

The self-similarity of the measure provides that $C_r(r^n, r^n) = \left(\frac{r}{2}\right)^n$ for $n \in \mathbb{Z}^+$. Let $u \in]0,1[$. Then, there exists n satisfying $r^{n+1} < u \leq r^n$. Therefore,

$$\frac{C_r(r^{n+1},r^{n+1})}{r^n} \leq \frac{C_r(u,u)}{u} \leq \frac{C_r(r^n,r^n)}{r^{n+1}};$$

that is,

$$\frac{r}{2^{n+1}} \leq \frac{C_r(u,u)}{u} \leq \frac{1}{2^n r}.$$

But, in this case, if $u \to 0^+$, then $n \to \infty$; and there exists the limit

$$\lambda_L = \lim_{u \to 0^+} \frac{C_r(u,u)}{u} = 0. \qquad \square$$

5 Concordance Measures for C_r

We study several association measures that mesh the probability of concordance between random variables with a given copula. For a review of concordance measures and the rôle played by the copulas in the study of dependence or association between random variables, see [9, Chap. 5].

Let us recall that concordance or discordance are basic when introducing association measures. Formally, two ordered pairs of real numbers, (x_1,y_1) and (x_2,y_2), are concordant if $(x_1-x_2)(y_1-y_2) > 0$. They are discordant if they are not concordant.

Let (X_1,Y_1) and (X_2,Y_2) be two continuous random pairs with the same marginal distribution functions, and associated copulas C_1 and C_2, respectively. A *concordance function* is defined by

$$Q(C_1,C_2) = P[(X_1-X_2)(Y_1-Y_2) > 0] - P[(X_1-X_2)(Y_1-Y_2) < 0].$$

Moreover, if the above continuous random pairs (X_1,Y_1) and (X_2,Y_2) are independent then

$$Q(C_1,C_2) = 4\int_{\mathbb{I}^2} C_2(x,y)\,d\mu_{C_1}(x,y) - 1.$$

Definition 1. *Let (X,Y) be a continuous random pair with associated copula C. The value $Q(C,C)$ is a measure of association called the Kendall's τ of (X,Y). Moreover, the value $3Q(C,\Pi)$ is a measure of association called the Spearman's ρ of (X,Y). And, the value $Q(C,M)+Q(C,W)$ is another measure of association for (X,Y) called the Gini's γ.*

For computational purposes we have the following result (see **(5)** above):

Lemma 1. *Let $K \subset \mathbb{R}^n$ and let μ be a self-similar measure associated to the family of similarity transformations $\{F_1,\ldots,F_m\}$ with respective mass ratios $\{p_1,\ldots,p_m\}$. Then, for any continuous function $g: K \to \mathbb{R}$ and any $k, (1 \leq k \leq m)$, we have*

$$\int_{F_k(K)} g(x)\,d\mu(x) = p_k \int_K g(F_k(x))\,d\mu(x).$$

Proof. The map F_k is a self-similarity transformation, hence, it is an isomorphism between measurable spaces. As a consequence, there exists a natural bijection from the step functions on K to $F_k(K)$ (considering the induced σ-algebra, in both cases). The measures of the measurable sets A and $F_k(A)$ are proportional with ratio p_k, therefore, and the statement is true in the case that g is a step function. Density arguments establish that the statement is also true for all integrable functions. □

(6) An immediate consequence from the above lemma is the following useful expression:

$$\int_K g(x)d\mu(x) = \sum_{k=1}^{m} p_k \int_K g(F_k(x))d\mu(x).$$

Now, by applying **(6)** and using **(2)** and **(3)**, we can express the concordance in terms of the family of copulas C_{T_r}.

Proposition 1. *Given the copula $C_{T_r} = C_r$ of parameter $r \in \left]0, \frac{1}{2}\right[$, the following equalities hold:*

a) $\int_{[0,1]^2} \max(x+y-1,0)dC_r(x,y) = \frac{1-r}{8-10r}$
b) $\int_{[0,1]^2} \min(x,y)dC_r(x,y) = \frac{3-4r}{8-10r}$
c) $\int_{[0,1]^2} xydC_r(x,y) = 1/4$
d) $\int_{[0,1]^2} C_r(x,y)\ dC_r(x,y) = 1/4$

Proof. a) Let us decompose the integral as a sum on five regions in the unit square. The measure μ_{C_r} is self-similar; and therefore:

$$\begin{aligned}
\int_{\mathbb{I}^2} W(x,y)d\mu_{C_r}(x,y) &= \frac{r}{2}\int_{\mathbb{I}^2} W(rx,ry)d\mu_{C_r} + \frac{r}{2}\int_{\mathbb{I}^2} W(rx+1-r,ry)d\mu_{C_r} \\
&\quad + \frac{r}{2}\int_{\mathbb{I}^2} W(rx,ry+1-r)d\mu_{C_r} \\
&\quad + \frac{r}{2}\int_{\mathbb{I}^2} W(rx+1-r,ry+1-r)d\mu_{C_r} \\
&\quad + (1-2r)\int_{\mathbb{I}^2} W((1-2r)x+r,(1-2r)y+r)d\mu_{C_r} \\
&= \frac{r}{2}\int_{\mathbb{I}^2} 0\ d\mu_{C_r} + \frac{r}{2}\int_{\mathbb{I}^2} rW(x,y)d\mu_{C_r} + \frac{r}{2}\int_{\mathbb{I}^2} rW(x,y)d\mu_{C_r} \\
&\quad + \frac{r}{2}\int_{\mathbb{I}^2} (rx+ry+1-2r)d\mu_{C_r} + (1-2r)^2\int_{\mathbb{I}^2} W(x,y)d\mu_{C_r} \\
&= \left(r^2+(1-2r)^2\right)\int_{\mathbb{I}^2} W(x,y)d\mu_{C_r}(x,y) + \frac{r(1-r)}{2};
\end{aligned}$$

and working out the integral, we have the statement.

b) We proceed as in the above case.

c) Here,

$$
\begin{aligned}
\int_{\mathbb{I}^2} xy d\mu_{C_r}(x,y) &= \frac{r}{2}\int_{\mathbb{I}^2} rxryd\mu_{C_r} + \frac{r}{2}\int_{\mathbb{I}^2}(rx+1-r)ryd\mu_{C_r} \\
&\quad +\frac{r}{2}\int_{\mathbb{I}^2}(rx)(ry+1-r)d\mu_{C_r} \\
&\quad +\frac{r}{2}\int_{\mathbb{I}^2}(rx+1-r)(ry+1-r)d\mu_{C_r} \\
&\quad +(1-2r)\int_{\mathbb{I}^2}((1-2r)x+r)((1-2r)y+r)d\mu_{C_r} \\
&= 2r^3\int_{\mathbb{I}^2} xyd\mu_{C_r} + (1-2r)^3\int_{\mathbb{I}^2} xyd\mu_{C_r} \\
&\quad +\left(2r^2(1-r)+2(1-2r)^2 r\right)\int_{\mathbb{I}^2} yd\mu_{C_r} \\
&\quad +\frac{r(1-r)^2}{2}+(1-2r)r^2 \\
&= \left(2r^3+(1-2r)^3\right)\int_{\mathbb{I}^2} xyd\mu_{C_r}(x,y)+\frac{3r}{2}-3r^2+\frac{3r^3}{2};
\end{aligned}
$$

and the statement follows.

d) We decompose the integral as in the first case:

$$
\begin{aligned}
\int_{\mathbb{I}^2} C_r(x,y)d\mu_{C_r}(x,y) &= \frac{r}{2}\int_{\mathbb{I}^2} C_r(rx,ry)d\mu_{C_r} + \frac{r}{2}\int_{\mathbb{I}^2} C_r(rx+1-r,ry)d\mu_{C_r} \\
&\quad +\frac{r}{2}\int_{\mathbb{I}^2} C_r(rx,ry+1-r)d\mu_{C_r} \\
&\quad +\frac{r}{2}\int_{\mathbb{I}^2} C_r(rx+1-r,ry+1-r)d\mu_{C_r}+ \\
&\quad +(1-2r)\int_{\mathbb{I}^2} C_r((1-2r)x+r,(1-2r)y+r)d\mu_{C_r} \\
&= \frac{r}{2}\int_{\mathbb{I}^2}\frac{r}{2}C_r(x,y)d\mu_{C_r} + \frac{r}{2}\int_{\mathbb{I}^2}\frac{r}{2}y+\frac{r}{2}C_r(x,y)d\mu_{C_r} \\
&\quad +\frac{r}{2}\int_{\mathbb{I}^2}\frac{r}{2}x+\frac{r}{2}C_r(x,y)d\mu_{C_r}+\frac{r}{2}\int_{\mathbb{I}^2}\frac{r}{2}+1-2r+\frac{r}{2}(x+y) \\
&\quad +\frac{r}{2}C_r(x,y)d\mu_{C_r}+(1-2r)\int_{\mathbb{I}^2}\frac{r}{2}+(1-2r)C_r(x,y)d\mu_C \\
&= \left(r^2+(1-2r)^2\right)\int_{\mathbb{I}^2} C_r(x,y)d\mu_{C_r}(x,y)+\frac{3}{4}r^2+r(1-2r);
\end{aligned}
$$

and the result follows. □

Corollary 1. *Kendall's* τ, *Spearman's* ρ, *and Gini's* γ *are zero for all* $r \in \left]0,\frac{1}{2}\right[$.

The independence copula Π is the limit case when $r \to 1/2$. The values we obtain for these association measures and for the copula Π are the same. As a consequence, there is no monotone dependence in any degree for these measures, that is the same case for Π.

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