Option Pricing in Incomplete Markets Based on Partial Information

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Abstract. In this paper we describe a new approach for the valuation problem in incomplete markets with $m \geq 1$ stocks which can be used when the available information about the uncertainty model is only a partial conditional probability assessment $\mathbf{p}$. We select a risk neutral probability minimizing a discrepancy measure between $\mathbf{p}$ and the convex set of all possible risk neutral probabilities.

Keywords: Risk neutral valuation, Partial conditional probability assessments, Incomplete markets.

1 Introduction

In a viable single-period model with $m \geq 1$ stocks and $k \geq 2$ scenarios the completeness of the market is equivalent to the uniqueness of the risk neutral probability; this equivalence allows to price every derivative security with a unique fair price. In literature, different methods have been proposed in order to select a risk neutral probability when the market is incomplete (see for example [8], [9] and [10]). Contrary to the complete case, where the so called *"real world probability"* $\mathbf{p}$ expressing the agent behaviors is not involved in the valuation problem, in the methods for incomplete markets $\mathbf{p}$ is really used and its elicitation is a crucial point for the option pricing. In particular, $\mathbf{p}$ is supposed to be given over all the possible scenarios while usually the available information about the possible states of the world is partial, conditional or even incoherent. In this paper our purpose is to present a method for the

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risk neutral valuation in incomplete markets which can be used when **p** is a partial conditional probability assessment as well as when there are more partial conditional probability assessments given by different expert opinions. In the next Subsections 1.1 and 1.2 we introduce the valuation problem and a discrepancy measure which will be minimized in order to select a risk neutral probability among those characterizing the incomplete market. In Section 2 we describe such selection procedure and we give some illustrative examples.

1.1 Single-Period Models with m Stocks

In this section we describe the risk neutral valuation problem for a single-period financial model with m risky assets and a risk-free interest rate r (see [1], [7] and [12] for more details).

Let $\mathbf{S}_t = (S_t^1, \ldots, S_t^m)$ be the vector of the stock prices at time t with $t = 0, 1$ and let us suppose that the initial prices $S_0^1, \ldots, S_0^m$ are known at time 0 while the prices of each stock S_1^l, $l = 1, \ldots, m$, are finite random variables

$$S_1^l : \Omega \to \mathbb{R},$$

where $\Omega = \{\omega_1, \ldots, \omega_k\}$ is the set of possible scenarios (states of the world). If we denote with $\overline{\mathbf{S}}_t := \mathbf{S}_t/(1+r)^t$, $t = 0, 1$ the discounted stock price process, we can define a risk neutral probability as a probability distribution α over Ω under which the discounted stock price process is a martingale, that is

$$\overline{\mathbf{S}}_0 = \mathbb{E}_{\alpha}(\overline{\mathbf{S}}_1). \tag{1}$$

Notice that this means that $\mathbf{S}_0 = \mathbb{E}_{\alpha}(\mathbf{S}_1)/(1+r)$ and this expression explains why a probability measure α which verifies (1) is called risk neutral: α is a probability distribution such that the price S_0^l of each stock can be computed as the expected value with respect to α of S_1^l discounted with the risk free interest rate r. A market model is said to be *viable* if there are no arbitrage opportunities and it is said to be *complete* if every derivative security [1] admits a replicating portfolio (i.e. a portfolio with the same payoff). A single period model with a finite number of scenarios and an arbitrary number of stocks admits a risk neutral probability if and only if it is viable.

Let us consider the problem of completeness for a viable single-period model with m stocks and k scenarios. Since a market model is complete if and only if every derivative security D is attainable, the completeness is equivalent to have, for every derivative D, a portfolio (x, y) such that

$$\begin{cases} xB_1 + y\mathbf{S}_1(\omega_1) = D(\omega_1) \\ \cdots \\ xB_1 + y\mathbf{S}_1(\omega_k) = D(\omega_k) \end{cases}$$

where $B_1 = (1+r)B_0$ is the price at $t = 1$ of the risk-free asset.

[1] We recall that a derivative security is a security whose value at time 1 depends on the values of risky assets $S_1^1, \ldots, S_1^m$.

Thus a single-period model with k scenarios is complete if and only if

$$A := \begin{pmatrix} B_1 \; S_1^1(\omega_1) \dots S_1^m(\omega_1) \\ \vdots \qquad\qquad \vdots \\ B_1 \; S_1^1(\omega_k) \dots S_1^m(\omega_k) \end{pmatrix} \tag{2}$$

has rank k, that is it contains at least k independent assets. Notice that if a model is complete then we must have $m+1 \geq k$. It is easy to see that α is a risk neutral probability if and only if

$$A^T \alpha = (1+r) \begin{pmatrix} B_0 \\ \mathbf{S}_0 \end{pmatrix}$$

and $\alpha_j \geq 0,\; j = 1, \dots, k$. Therefore it follows that in a viable single period model, with $k \geq 2$ scenarios, a unique risk free asset and m risky assets, the completeness is equivalent to the uniqueness of the risk neutral probability.

When the market is viable and complete, the fair price π of any derivative security D is given by

$$\pi = \mathbb{E}_\alpha(\overline{D}) \tag{3}$$

where $\overline{D}$ is the discounted price of D and α is the risk neutral probability.

When the market is incomplete the set F of all possible fair prices for a derivative D is

$$F = [l, u]$$

where

$$l := \inf\{\mathbb{E}_\alpha(\overline{D}) \mid \alpha \text{ is a risk neutral probability}\},$$

$$u := \sup\{\mathbb{E}_\alpha(\overline{D}) \mid \alpha \text{ is a risk neutral probability}\}.$$

Obviously a derivative security is attainable if and only if $l = u$; otherwise $l < u$ and we have to consider the interval $[l, u]$. If we denote by $\mathcal{Q}$ the convex set of possible risk neutral probabilities, taken $\alpha^* \in \mathcal{Q}$, for every derivative security there will be a corresponding fair price π given by

$$\pi = \mathbb{E}_{\alpha^*}(\overline{D}) \in F.$$

In Section 2 we will describe how to select such a α^* that should be as close as possible to the agent behaviors expressed by partial conditional probability assessment $\mathbf{p}$. To express the closeness between α^* and $\mathbf{p}$ we will profit from a recently introduced ([2, 4]) discrepancy measure that, for the sake of completeness, we briefly describe in the following subsection.

1.2 Discrepancy Measure

Let $\mathbf{p} = (p_1, \dots, p_n) \in (0,1)^n$ be a conditional probability assessment given by an agent over a set of conditional events $\mathscr{E} = [E_1|H_1, \dots, E_n|H_n]$. $\mathscr{E}$ expresses events, the E_i's, considered under specific situations or hypothesis, the H_i's, over which the agent possess, or is able to express, probabilistic behaviors. In the following E_iH_i will denote the logical conjunction "$E_i \wedge H_i$", while E_i^c will denote the negation "$\neg E_i$". To be meaningful, the E_i's and the H_i's must be expressible through possible values of the assets present in the market, hence without loss of generality we consider the E_i's and the H_i's as subsets of the set of all possible states of the world $\Omega = \{\omega_1, \ldots, \omega_k\}$, with each ω_j representing a specific assets evaluation situation.

To properly define a pseudo-distance between a probabilistic evaluation over $\mathscr{E}$ and an other over Ω, we need to introduce the following hierarchy of probability mass function on Ω:
$\mathscr{A} := \left\{\alpha = [\alpha_1, \ldots, \alpha_k], \sum \alpha_j = 1, \alpha_j \geq 0, j = 1, \ldots, k\right\}$;
$\mathscr{A}_0 := \{\alpha \in \mathscr{A} | \alpha(\bigcup_{i=1}^n H_i) = 1\}$;
$\mathscr{A}_1 := \{\alpha \in \mathscr{A}_0 | \alpha(H_i) > 0, i = 1, \ldots, n\}$;
$\mathscr{A}_2 := \{\alpha \in \mathscr{A}_1 | 0 < \alpha(E_iH_i) < \alpha(H_i), i = 1, \ldots, n\}$.
Any $\alpha \in \mathscr{A}_1$ induces a coherent conditional assessment on $\mathscr{E}$ given by

$$\mathbf{q}_\alpha := [q_i = \frac{\sum_{j:\,\omega_j \subset E_iH_i} \alpha_j}{\sum_{j:\,\omega_j \subset H_i} \alpha_j}, i = 1, \ldots, n]. \tag{4}$$

Associated to any assessment $\mathbf{p} \in (0,1)^n$ over $\mathscr{E}$ we can define a scoring rule

$$S(\mathbf{p}) := \sum_{i=1}^n |E_iH_i| \ln p_i + \sum_{i=1}^n |E_i^cH_i| \ln(1 - p_i) \tag{5}$$

with $|\cdot|$ indicator function of unconditional events. This score $S(\mathbf{p})$ is an "adaptation" to partial and conditional probability assessments of the "proper scoring rule" for probability distributions proposed by Lad in [11]. By adopting the difference between the expected penalties suffered by the two evaluations $\mathbf{p}$ and $\mathbf{q}_\alpha$ as distance criterion, it is possible to define the "discrepancy" between a partial conditional assessment $\mathbf{p}$ over $\mathscr{E}$ and a distribution $\alpha \in \mathscr{A}_2$ through the expression

$$\Delta(\mathbf{p}, \alpha) := E_\alpha(S(\mathbf{q}_\alpha) - S(\mathbf{p})) = \sum_{j=1}^k \alpha_j [S_j(\mathbf{q}_\alpha) - S_j(\mathbf{p})]. \tag{6}$$

It is possible to extend by continuity the definition of $\Delta(\mathbf{p}, \alpha)$ in $\mathscr{A}_0$ as

$$\Delta(\mathbf{p}, \alpha) = \sum_{i|\alpha(H_i)>0} \alpha(H_i) \left(q_i \ln \frac{q_i}{p_i} + (1 - q_i) \ln \frac{(1 - q_i)}{(1 - p_i)} \right)$$

adopting the usual convention $0 \ln 0 = 0$.

In [4] is formally proved that $\Delta(\mathbf{p}, \alpha)$ is a non negative function on $\mathscr{A}_0$ and that $\Delta(\mathbf{p}, \alpha) = 0$ if and only if $\mathbf{p} = \mathbf{q}_\alpha$; moreover $\Delta(\mathbf{p}, \cdot)$ is a convex function on $\mathscr{A}_2$ and it admits a minimum on $\mathscr{A}_0$. Finally if $\alpha, \alpha^0 \in \mathscr{A}_0$ are distributions that minimize $\Delta(\mathbf{p}, \cdot)$, then for all $i \in \{1, \ldots, n\}$ such that $\alpha(H_i) > 0$ and $\alpha^0(H_i) > 0$

we have $(\mathbf{q}_\alpha)_i = (\mathbf{q}_{\alpha^0})_i$; in particular if $\Delta(\mathbf{p},\cdot)$ attains its minimum value on $\mathscr{A}_1$ then there is a unique coherent assessment $\mathbf{q}_{\underline{\alpha}}$ such that $\Delta(\mathbf{p},\underline{\alpha})$ is minimum. The discrepancy measure $\Delta(\mathbf{p},\alpha)$ can be used to correct incoherent assessments [2], to aggregate expert opinions [5] and it can be even applied with imprecise probabilities [3]. Here we propose a particular optimization problem involving $\Delta(\mathbf{p},\alpha)$ which will be used to select the risk neutral probability in the set of all possible martingale measures which better represents the agent's behaviors.

2 Selection of a Risk-Neutral Probability

In order to keep the market tractable, we start with a viable single period model like in Subsection 1.1, without transaction costs and with the following stock prices structure:

	ω_1	ω_2	...	ω_k
S_1^l	$a_1^l\ S_0^l$	$a_2^l\ S_0^l$	...	$a_k^l\ S_0^l$

l=1,...,m.

In this model a probability distribution α on Ω is risk neutral if and only if

$$\mathbf{S}_0 = \frac{1}{1+r}[\alpha_1\mathbf{a}_1\mathbf{S}_0 + \ldots + \alpha_k\mathbf{a}_k\mathbf{S}_0] \Leftrightarrow \mathbf{1} = \frac{1}{1+r}[\alpha_1\mathbf{a}_1 + \ldots + \alpha_k\mathbf{a}_k]$$

where $\mathbf{a}_j = (a_j^1,\ldots,a_j^m)$ for $j = 1,\ldots,k$. Thus we can define the set of all possible martingale measures as:

$$\mathscr{Q} := \{\alpha \in \mathbb{R}^k : \alpha\cdot\mathbf{1} = 1,\ \alpha \geq 0,\ \sum_{j=1}^{k} \alpha_j a_j^l = 1+r,\ l = 1,\ldots,m\}.$$

Finally we assume that $\mathbf{p} = (p_1,\ldots,p_n)$ is a partial conditional probability assessment given over the set of conditional events $\mathscr{E} = [E_1|H_1,\ldots,E_n|H_n]$. Notice that it is not required that the assessment $\mathbf{p}$ is coherent[2]; we can have an assessment which is inconsistent with all the distributions in $\mathscr{A}$ or we can have a coherent assessment that is anyhow inconsistent with the set of all martingale measures. Our purpose is to find the risk neutral probability which is the *closest* to the initial opinion with respect to the discrepancy measure Δ. Let $\mathscr{Q}_0$ be the convex set $\mathscr{Q}_0 := \mathscr{Q}\cap\mathscr{A}_0$; we propose to select a martingale measure in $\mathscr{Q}_0$ starting from the assessment $\mathbf{p}$. In fact, we suggest a selection procedure which is based on the following result:

Theorem 1. *Let* $\mathscr{M} := \arg\min\{\Delta(\mathbf{p},\alpha), \alpha \in \mathscr{Q}_0\}$ *be the set of all martingale measures minimizing* $\Delta(\mathbf{p},\alpha)$*; then* $\mathscr{M}$ *is a non-empty convex set.*

Proof. $\Delta(\mathbf{p},\alpha)$ is a convex function on $\mathscr{Q}_0$ and then there is at least one $\underline{\alpha}$ in $\mathscr{Q}_0$ such that

[2] For coherence notion of partial conditional assessments refer e.g. to [6]

$$\Delta(\mathbf{p},\underline{\alpha}) = \min_{\alpha\in\mathscr{Q}_0} \Delta(\mathbf{p},\alpha) \tag{7}$$

and then $\mathscr{M}$ is non empty. Notice that the convexity of $\Delta(\mathbf{p},\alpha)$ guarantees the existence of this minimum but it is possible that more than one distribution minimize $\Delta(\mathbf{p},\alpha)$ in $\mathscr{Q}_0$ and in this case $\mathscr{M}$ is not a singleton. However, since $\Delta(\mathbf{p},\alpha)$ is a convex function and $\mathscr{M}$ is the set of minimal points of $\Delta(\mathbf{p},\alpha)$ in $\mathscr{Q}_0$, $\mathscr{M}$ is a convex set. □

Thanks to this result we can select exactly one of such minimizer distributions $\underline{\alpha}$. Let us see how it works with first example.

Example 1. Let us consider a model with two risky assets S^1, S^2 with initial prices $S_0^1 = 200$, $S_0^2 = 150$ and final values

	ω_1	ω_2	ω_3	ω_4
S_1^l	220	210	200	180
S_1^2	180	150	150	120

Since we have $m = 2$ assets and $k = 4$ states of the world, the rank of matrix A in (2) is surely less than k, so that the market is incomplete. For $r = 0$ the set of all possible martingale measures is $\mathscr{Q}_\lambda = \{(\lambda, 0, 1-2\lambda, \lambda), \lambda \in [0, 1/2]\}$. Let us suppose that the agent assesses the probabilities $p_1 = P(\omega_4) = 1/3$ and $p_2 = P(\omega_1|\omega_1 \vee \omega_2) = 1/4$. Then

$$\Delta(\mathbf{p},\alpha) = \alpha_4 \ln 3\alpha_4 + (1-\alpha_4)\ln\frac{3}{2}(1-\alpha_4) + \alpha_1 \ln\frac{4\alpha_1}{(\alpha_1+\alpha_2)} + \alpha_2 \ln\frac{4\alpha_2}{3(\alpha_1+\alpha_2)}$$

that is $\Delta(\mathbf{p},\lambda) = \lambda\ln 3\lambda + (1-\lambda)\ln\frac{3}{2}(1-\lambda) + \lambda\ln 4$.

Since $\Delta'(\mathbf{p},\lambda) = \ln\lambda - ln(1-\lambda) + \ln 8 = 0 \Leftrightarrow \frac{\lambda}{1-\lambda} = \frac{1}{8} \Leftrightarrow \lambda = \frac{1}{9}$ we get $\underline{\alpha} = \left(\frac{1}{9}, 0, \frac{7}{9}, \frac{1}{9}\right)$.

Theorem 1 guarantees the existence of a solution $\underline{\alpha}$ for the optimization problem (7) but it does not assure its uniqueness; when the information that we have is not sufficient to give us a unique solution for (7) we need another criterion to choose, between the martingale measure minimizing $\Delta(\mathbf{p},\alpha)$, a unique α^* as risk-neutral probability. The idea is to select one distribution in $\mathscr{M}$ which in some sense minimizes the exogenous information. In fact, we will define α^* as

$$\alpha^* := arg \min_{\alpha\in\mathscr{M}} \sum_{j=1}^{k} \alpha_j \ln\alpha_j \tag{8}$$

that is the distribution which minimizes the relative entropy with respect to the uniform distribution (i.e. the distribution with maximum entropy).

Let us see how it can be operationally done with an exemplifying situation

Example 2. Let us consider again a model with two risky assets S^1, S^2 but now with initial prices $S_0^1 = 19$, $S_0^2 = 21$ and final values

	ω_1	ω_2	ω_3	ω_4	ω_5
S_1^l	22	21	20	19	18
S_1^2	25	24	21	21	20

so that the set of martingale measures is

$$\mathscr{Q}_{\lambda,\mu} = \left\{(\lambda, \mu-\lambda, \mu, 1-\lambda-5\mu, \lambda+3\mu), \lambda \le \frac{1}{6}, \mu \in [\lambda, \frac{1-\lambda}{5}]\right\}.$$

Let us suppose that the agent assesses the probabilities $p_1 = P(\omega_3) = 1/3$ and $p_2 = P(\omega_1 \vee \omega_2 | \omega_1 \vee \omega_2 \vee \omega_3) = 9/10$. Then

$$\begin{aligned} \Delta(\mathbf{p},\alpha) &= \alpha_3 \ln 3\alpha_3 + (1-\alpha_3)\ln\frac{3}{2}(1-\alpha_3) + \\ &+(\alpha_1+\alpha_2)\ln\frac{10(\alpha_1+\alpha_2)}{9\,(\alpha_1+\alpha_2+\alpha_3)} + \alpha_3 \ln\frac{10\alpha_3}{(\alpha_1+\alpha_2+\alpha_3)} \end{aligned}$$

that is $\Delta(\mathbf{p},\mu) = \mu\ln 3\mu + (1-\mu)\ln\frac{3}{2}(1-\mu) + \mu\ln\frac{5}{9} + \mu\ln 5$.

Since

$$\Delta'(\mathbf{p},\mu) = \ln\mu - ln(1-\mu) - \ln 4 - \ln(1/3) + \ln(2/3) - \ln(9/100)$$

we have

$$\Delta'(\mathbf{p},\mu) = 0 \Leftrightarrow \frac{\mu}{1-\mu} = \frac{18}{100} \Leftrightarrow \mu = \frac{9}{59}$$

so that the set of martingale measures which minimize $\Delta(\mathbf{p},\cdot)$ is

$$\mathscr{M} = \{(\lambda, 9/59-\lambda, 9/59, 14/59-\lambda, \lambda+27/29) : \lambda \le 9/59\}.$$

Among all such distributions we can select $\alpha^* \in \mathscr{M}$ maximizing the entropy

$$\begin{aligned} H(\lambda) &= -\lambda\ln\lambda - (\frac{9}{59}-\lambda)\ln(\frac{9}{59}-\lambda) - \frac{9}{59}\ln\frac{9}{59} + \\ &-(\frac{14}{59}-\lambda)\ln(\frac{14}{59}-\lambda) - (\lambda+\frac{27}{59})\ln(\lambda+\frac{27}{59}) \end{aligned}$$

obtaining $\alpha^* = (.043, .110, .153, .195, .5)$.

3 Conclusion

Thanks to the minimization of a pseudo-distance among a partial conditional probability assessments $\mathbf{p}$ and probability distributions α we have shown that it is possible to aggregate disparate fonts of information like those induced by market prices structures, which are usually extremely rich even in the context of incomplete markets, and those induced by human agents, which are typically non structured and partial.

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