On the Hybrid Černý-Road Coloring Problem and Hamiltonian Paths*

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Abstract. The Hybrid Černý-Road coloring problem is investigated for graphs with Hamiltonian paths. We prove that if an aperiodic, strongly connected digraph of costant outdegree with n vertices has an Hamiltonian path, then it admits a synchronizing coloring with a reset word of length $2(n-2)(n-1)+1$. The proof is based upon some new results concerning locally strongly transitive automata.

Keywords: Černý conjecture, road coloring problem, synchronizing automaton, rational series.

1 Introduction

An important concept in Computer Science is that of *synchronizing automaton.* A deterministic automaton is called *synchronizing* if there exists an input-sequence, called *synchronizing* or *reset word*, such that the state attained by the automaton, when this sequence is read, does not depend on the initial state of the automaton itself. Two fundamental problems which have been intensively investigated in the last decades are based upon this concept: the *Černý conjecture* and the *Road coloring problem.*

The Černý conjecture [10] claims that a deterministic synchronizing n-state automaton has a reset word of length not larger than $(n-1)^2$. This conjecture has been shown to be true for several classes of automata (*cf.* [2,3,4,8,9,10,12,14,15,16,17,18,20,23]). The interested reader is refered to [23] for a historical survey of the Černý conjecture and to [7] for synchronizing unambiguous automata. In this theoretical setting, two results recently proven in [9] and [4] respectively, are relevant to us.

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In [9], the authors have introduced the notion of *local strong transitivity*. An n-state automaton $\mathcal{A}$ is said to be *locally strongly transitive* if it is equipped by a set W of k words and a set R of k distinct states such that, for all states s of $\mathcal{A}$ and all $r \in R$, there exists a word $w \in W$ taking the state s into r. The set W is called *independent* while R is called the *range* of W. The main result of [9] is that any synchronizing locally strongly transitive n-state automaton has a reset word of length not larger than $(k-1)(n+L)+\ell$, where k is the cardinality of an independent set W and L and ℓ denote respectively the maximal and the minimal length of the words of W.

In the case where all the states of the automaton are in the range, the automaton $\mathcal{A}$ is said to be *strongly transitive*. Strongly transitive automata have been studied in [8]. This notion is related with that of regular automata introduced in [18].

A remarkable example of locally strongly transitive automata is that of *1-cluster automata* introduced in [4]. An automaton is called 1-cluster if there exists a letter a such that the graph of the automaton has a unique cycle labelled by a power of a. Indeed, denoting by k the length of the cycle, one easily verifies that the words

$$a^{n-1}, a^{n-2}, \ldots, a^{n-k}$$

form an independent set of the automaton whose range is the set of vertices of the cycle. In [4] it is proven that every 1-cluster synchronizing n-state automaton has a reset word of length not larger than $2(n-1)(n-2)+1$.

The second problem we have mentioned above is the well-known *Road coloring problem*. This problem asks to determine whether any aperiodic and strongly connected digraph, with all vertices of the same outdegree (*AGW-graph*, for short) has a *synchronizing coloring*, that is, a labeling of its edges that turns it into a synchronizing deterministic automaton. The problem was formulated in the context of Symbolic Dynamics by Adler, Goodwyn and Weiss and it is explicitly stated in [1]. In 2007, Trahtman [21] has positively solved it. The solution by Trahtman has electrified the community of formal language theories and recently Volkov has raised in [22] the problem of evaluating, for any AGW-graph G, the minimal length of a reset word for a synchronizing coloring of G. This problem has been called *the Hybrid Černý–Road coloring problem.* It is worth to mention that Ananichev has found, for any $n \geq 2$, an AGW-graph of n vertices such that the length of the shortest reset word for any synchronizing coloring of the graph is $(n-1)(n-2)+1$ (see [22]). In [9], the authors have proven that, given an AGW-graph G of n vertices, without multiple edges, such that G has a simple cycle of prime length $p < n$, there exists a synchronizing coloring of G with a reset word of length $(2p-1)(n-1)$. Moreover, in the case $p = 2$, that is, if G contains a cycle of length 2, then, also in presence of multiple edges, there exists a synchronizing coloring with a reset word of length $5(n-1)$.

In this paper, we continue the investigation of the Hybrid Černý–Road coloring problem on a very natural class of digraphs, those having a *Hamiltonian path.* The main result of this paper states that any AGW-graph G of n vertices with a Hamiltonian path admits a synchronizing coloring with a reset word of length

$$2(n-1)(n-2)+1. \tag{1}$$

The proof of the theorem above is based upon some techniques on independent sets of words and on some results on synchronizing colorings of graphs.

A set K of states of an automaton $\mathcal{A}$ is *reducible* if there exists a word $w \in A^*$ taking all the states of K into a fixed state. The least congruence ρ of $\mathcal{A}$ such that any congruence class is reducible is called *stability.* This congruence, introduced in [11], plays a fundamental role in the solution [21] of the Road coloring problem. Even if $\mathcal{A}$ is not a synchronizing automaton, it is natural to ask for the minimal length of a word w taking all the states of a given stability class K into a single state. In Section 3, we prove that if $\mathcal{A}$ is a locally strongly transitive n-state automaton which is not synchronizing, then the minimal length of such a word w is at most

$$\left(\frac{k}{2}-1\right)(n+L-1)+L,$$

where k is the cardinality of any independent set W and L denotes the maximal length of the words of W. In the case where $\mathcal{A}$ is synchronizing, we obtain for the minimal length of a reset word of $\mathcal{A}$ the upper bound

$$(k-1)(n+L-1)+\ell \tag{2}$$

where k and L are defined as before and ℓ denotes the minimal length of the words of W. This bound refines the quoted bound of [9].

We close the introduction with the following remark. By using a more sophisticated and laborious technique similar to that of [19], the bound (2) can be lowered to $(k-1)(n+L-2)+\ell$. Moreover, a recent improvement [5] of the bound obtained in [4] should allow to slightly refine the bound (1).

2 Preliminaries

We assume that the reader is familiar with the theory of automata and rational series. In this section we shortly recall a vocabulary of few terms and we fix the corresponding notation used in the paper.

Let A be a finite alphabet and let A^* be the free monoid of words over the alphabet A. The identity of A^* is called the *empty word* and is denoted by ϵ. The *length* of a word $w \in A^*$ is the integer $|w|$ inductively defined by $|\epsilon| = 0$, $|wa| = |w| + 1$, $w \in A^*$, $a \in A$. For any positive integer n, we denote by $A^{<n}$ the set of words of length smaller than n.

For any finite set of words, W, we denote respectively by L_W and ℓ_W the maximal and minimal lengths of the words of W.

A finite automaton is a triple $\mathcal{A} = \langle Q, A, \delta \rangle$ where Q is a finite set of elements called *states* and δ is a map

$$\delta : Q \times A \longrightarrow Q.$$

The map δ is called the *transition function* of $\mathcal{A}$. The canonical extension of the map δ to the set $Q \times A^*$ is still denoted by δ.

If P is a subset of Q and u is a word of A^*, we denote by $\delta(P, u)$ and $\delta(P, u^{-1})$ the sets:

$$\delta(P, u) = \{\delta(s, u) \mid s \in P\}, \quad \delta(P, u^{-1}) = \{s \in Q \mid \delta(s, u) \in P\}.$$

With any automaton $\mathcal{A} = \langle Q, A, \delta \rangle$, we can associate a directed multigraph $G = (Q, E)$, where the multiplicity of the edge $(p, q) \in Q \times Q$ is given by $\mathrm{Card}(\{a \in A \mid \delta(p, a) = q\}$. If the automaton $\mathcal{A}$ is associated with G, we also say that $\mathcal{A}$ is a *coloring* of G. An automaton is *transitive* if the associated graph is strongly connected. If $n = \mathrm{Card}(Q)$, we will say that $\mathcal{A}$ is a n-state automaton. A *synchronizing* or *reset* word of $\mathcal{A}$ is any word $u \in A^*$ such that $\mathrm{Card}(\delta(Q, u)) = 1$. A *synchronizing* automaton is an automaton that has a reset word. The following conjecture has been raised in [10].

Černý Conjecture. *Each synchronizing n-state automaton has a reset word of length not larger than $(n-1)^2$.*

Let $\mathbb{K}$ be a field. We recall that a *formal power series* with coefficients in $\mathbb{K}$ and non-commuting variables in A is a mapping of the free monoid A^* into $\mathbb{K}$. According to Kleene-Schützenberger theorem on formal power series (see [6]), a series $\mathcal{S} : A^* \to \mathbb{K}$ is *rational* if there exists a triple (α, μ, β) where

- $\alpha \in \mathbb{K}^{1\times n}$, $\beta \in \mathbb{K}^{n\times 1}$ are a row vector and a column vector respectively,
- $\mu : A^* \to \mathbb{K}^{n\times n}$ is a morphism of the free monoid A^* in the multiplicative monoid $\mathbb{K}^{n\times n}$ of matrices with coefficients in $\mathbb{K}$,
- for every $u \in A^*$, $\mathcal{S}(u) = \alpha\mu(u)\beta$.

The triple (α, μ, β) is called a *linear representation* of $\mathcal{S}$ and the integer n is called its *dimension*. The minimal dimension of a linear representation of $\mathcal{S}$ is called the *dimension* of $\mathcal{S}$. Let $\mathcal{A} = \langle Q, A, \delta \rangle$ be any n-state automaton. One can associate with $\mathcal{A}$ a morphism

$$\varphi_{\mathcal{A}} : A^* \to \mathbb{Q}^{Q\times Q},$$

of the free monoid A^* in the multiplicative monoid $\mathbb{Q}^{Q\times Q}$ of matrices over the field $\mathbb{Q}$ of rational numbers, defined as: for any $u \in A^*$ and for any $s, t \in Q$,

$$\varphi_{\mathcal{A}}(u)_{st} = \begin{cases} 1 & \text{if } t = \delta(s, u) \\ 0 & \text{otherwise.} \end{cases}$$

Let R and K be subsets of Q and consider the rational series $\mathcal{S}$ with rational coefficients having the linear representation $(\alpha, \varphi_{\mathcal{A}}, \beta)$, where, for every $s \in Q$,

$$\alpha_s = \begin{cases} 1 & \text{if } s \in R, \\ 0 & \text{otherwise,} \end{cases} \qquad \beta_s = \begin{cases} 1 & \text{if } s \in K, \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that, for any $u \in A^*$, one has

$$\mathcal{S}(u) = \mathrm{Card}(\delta(K, u^{-1}) \cap R). \tag{3}$$

We say that a linear representation (α, μ, β) is *uniform* if there exists a vector $\gamma \in \mathbb{K}^{n\times 1}$ such that $\mu(a)\gamma = \gamma$ for all $a \in A$. For instance, the linear representation $(\alpha, \varphi_{\mathcal{A}}, \beta)$ above is uniform, since $\varphi_{\mathcal{A}}(a)\gamma = \gamma$ for all $a \in A$, with $\gamma = {}^{\mathrm{t}}(1\,1\,\cdots\,1)$.

The following result refines a fundamental theorem by Moore and Conway on automata equivalence (see [6,13]) in the case of uniform linear representations. The proof is inspired to some ideas of [3,4].

Theorem 1. *Let $\mathcal{S}_1$, $\mathcal{S}_2 : A^* \to \mathbb{K}$ be two rational series with coefficients in $\mathbb{K}$, having uniform linear representations of dimension n_1 and n_2 respectively. If, for every $u \in A^*$ such that $|u| \leq n_1 + n_2 - 2$, $\mathcal{S}_1(u) = \mathcal{S}_2(u)$, the series $\mathcal{S}_1$ and $\mathcal{S}_2$ are equal.*

Proof. It is sufficient to verify that the series $\mathcal{S} = \mathcal{S}_1 - \mathcal{S}_2$ has a linear representation of dimension $n = n_1 + n_2 - 1$. Indeed, in such a case, since $\mathcal{S}(u) = 0$ for all $u \in A^{<n}$, a classical result on rational formal power series (see [6]) ensures that $\mathcal{S}$ is the null series, that is, $\mathcal{S}_1 = \mathcal{S}_2$.

Let $(\alpha_i, \mu_i, \beta_i)$ be the uniform linear representations of $\mathcal{S}_i$ of dimension n_i, $i = 1, 2$. Then the series $\mathcal{S} = \mathcal{S}_1 - \mathcal{S}_2$ has the linear representation (α, μ, β) with

$$\alpha = (\alpha_1\ \alpha_2), \quad \beta = \begin{pmatrix} \beta_1 \\ -\beta_2 \end{pmatrix}, \quad \mu(a) = \begin{pmatrix} \mu_1(a) & 0 \\ 0 & \mu_2(a) \end{pmatrix}, \quad a \in A\,.$$

By a well known result on rational power series (see [6]), the dimension of $\mathcal{S}$ is upper bounded by the linear dimension of the space generated by the vectors $\alpha\mu(w)$, $w \in A^*$.

The matrices $\mu(a)$ have two common right eigenvectors $(\gamma_1\ 0)$ and $(0\ \gamma_2)$ both associated with the eigenvalue 1. Among the linear combinations of these eigenvectors, there is an eigenvector γ orthogonal to α. One obtains

$$\alpha\mu(w)\gamma = \alpha\gamma = 0\,, \quad w \in A^*\,.$$

This equation shows that the vectors $\alpha\mu(w)$ lie in a proper subspace of $\mathbb{K}^{n+1}$. Consequently, the dimension of $\mathcal{S}$ is not larger than n. The conclusion follows. □

We end this section by introducing the important notion of stability [11]. Let $\mathcal{A} = \langle Q, A, \delta\rangle$ be an automaton. Given two states p, q of $\mathcal{A}$, we say that the pair (p, q) is *stable* if, for all $u \in A^*$, there exists $v \in A^*$ such that $\delta(p, uv) = \delta(q, uv)$. The set ρ of stable pairs is a congruence of the automaton $\mathcal{A}$, which is called *stability relation*. It is easily seen that an automaton is synchronizing if and only if the stability relation is the universal equivalence.

3 Independent Systems of Words

In this section, we will present some results that can be obtained by using some techniques on independent systems of words. We begin by recalling a definition introduced in [9].

Definition 1. *Let $\mathcal{A} = \langle Q, A, \delta \rangle$ be an automaton. A set of k words $W = \{w_0, \ldots, w_{k-1}\}$ is called* independent *if there exist k distinct states $q_0, \ldots, q_{k-1}$ of $\mathcal{A}$ such that, for all $s \in Q$,*

$$\{\delta(s, w_0), \ldots, \delta(s, w_{k-1})\} = \{q_0, \ldots, q_{k-1}\}.$$

The set $R = \{q_0, \ldots, q_{k-1}\}$ will be called the range *of W.*

An automaton is called *locally strongly transitive* if it has an independent set of words. The following useful properties can be derived from Definition 1 (see [9], Section 3).

Lemma 1. *Let $\mathcal{A}$ be an automaton and let W be an independent set of $\mathcal{A}$ with range R. Then, for every $u \in A^*$, the set uW is an independent set of $\mathcal{A}$ with range R.*

Proposition 1. *Let $\mathcal{A} = \langle Q, A, \delta \rangle$ be an automaton and consider an independent set W of $\mathcal{A}$ with range R. Then, for every subset P of R,*

$$\sum_{w \in W} \mathrm{Card}(\delta(P, w^{-1}) \cap R) = \mathrm{Card}(W)\,\mathrm{Card}(P).$$

A remarkable example of locally transitive automata is that of *1-cluster automata*, recently investigated in [4]. A n-state automaton is called 1-cluster if there exists a letter a such that the graph of the automaton has a unique cycle labelled by a power of a. Indeed, denoting by k the length of the cycle, one easily verifies that the words

$$a^{n-1}, a^{n-2}, \ldots, a^{n-k}$$

form an independent set of the automaton whose range is the set of vertices of the cycle. We recall that the following result has been proven in [4].

Theorem 2. *Let $\mathcal{A}$ be a synchronizing n-state automaton. If $\mathcal{A}$ is 1-cluster, then it has a reset word of length $1 + 2(n-1)(n-2)$.*

Let $\mathcal{A} = \langle Q, A, \delta \rangle$ be a n-state automaton. We say that a set of states K of $\mathcal{A}$ is *reducible* if, for some word w, $\delta(K, w)$ is a singleton. A set $K \subseteq Q$ is *stable* if for any $p, q \in K$, the pair (p, q) is stable. Any stable set K is reducible. Thus, even if $\mathcal{A}$ is not synchronizing, one may want to evaluate the minimal length of a word w such that $\mathrm{Card}(\delta(K, w)) = 1$.

In the sequel, we assume that W is an independent set of $\mathcal{A}$ with range R and denote by M the maximal cardinality of reducible subsets of R. Moreover, for any set of states K and any $w \in A^*$, we denote by Kw^{-1} the set $\delta(K, w^{-1})$. The following lemma holds.

Lemma 2. *Let K be a non-empty reducible subset of R. The following conditions are equivalent:*

1. $\mathrm{Card}(K) = M$,
2. *for all $w \in W$, $v \in A^{<n}$,* $\mathrm{Card}(K(vw)^{-1} \cap R) \leq \mathrm{Card}(K)$,

3. *for all* $w \in W$, $v \in A^{<n}$, $\mathrm{Card}(K(vw)^{-1} \cap R) = \mathrm{Card}(K)$,
4. *for all* $w \in W$, $v \in A^*$, $\mathrm{Card}(K(vw)^{-1} \cap R) = \mathrm{Card}(K)$.

Proof. Implication 1. ⇒ 2. is trivial, since $K(vw)^{-1} \cap R$ is reducible. Let us verify 2. ⇒ 3. First we recall that, by Lemma 1, for any $v \in A^*$, the set vW is independent. By Proposition 1, one has:

$$\sum_{w \in W} \mathrm{Card}(K(vw)^{-1} \cap R) = \mathrm{Card}(W)\,\mathrm{Card}(K).$$

In view of Condition 2, one obtains Condition 3.

Now let us prove 3. ⇒ 4. Consider the series $\mathcal{S}_1$, $\mathcal{S}_2$ defined respectively by

$$\mathcal{S}_1(v) = \mathrm{Card}((Kw^{-1})v^{-1} \cap R), \quad \mathcal{S}_2(v) = \mathrm{Card}(K), \quad v \in A^*.$$

In view of (3), $\mathcal{S}_1$ has a uniform linear representation of dimension n. Moreover, $\mathcal{S}_2$ has a uniform linear representation of dimension 1. By Condition 3, $\mathcal{S}_1(v) = \mathcal{S}_2(v)$ for all $v \in A^*$ such that $|v| < n$. By Theorem 1, it follows that $\mathcal{S}_1 = \mathcal{S}_2$. Thus Condition 4 holds true.

Finally, let us prove implication 4. ⇒ 1. Let X be a reducible subset of R with $\mathrm{Card}(X) = M$. One has $\delta(X, v) = \{q\}$ and $\delta(q, w) \in K$ for some $q \in Q$, $w \in W$. Hence, $X \subseteq K(vw)^{-1} \cap R$ so that $\mathrm{Card}(K) = \mathrm{Card}(K(vw)^{-1} \cap R) \geq M$. □

Lemma 3. *There exist* $K \subseteq R$ *and* $v \in A^*$ *such that*

$$\mathrm{Card}(K) = M\,, \quad \mathrm{Card}(\delta(K, v)) = 1\,, \quad |v| \leq (M-1)(n + L_W - 1)\,.$$

Proof. Using Condition 2 of Lemma 2, one can prove the following claim by induction on m:

For $1 \leq m \leq M$, there exist $K \subseteq R$ and $v \in A^*$ such that $\mathrm{Card}(K) \geq m$, $\mathrm{Card}(\delta(K, v)) = 1$, $|v| \leq (m-1)(n + L_W - 1)$. □

Lemma 4. *Let* K *be a reducible subset of* R *of maximal cardinality. There is no stable pair in* $K \times (R \setminus K)$.

Proof. By contradiction, let $(p, q) \in K \times (R \setminus K)$ be a stable pair. Then, $\delta(K, v) = \{\delta(p, v)\}$ and $\delta(p, vu) = \delta(q, vu) = s$, $s \in Q$ for some $u, v \in A^*$. Thus $\delta(K \cup \{q\}, vu) = \{s\}$, contradicting the maximality of K. □

Proposition 2. *For any stable set* C *there exists a word* v *such that*

$$\mathrm{Card}(\delta(C, v)) = 1\,, \quad |v| \leq (M-1)(n + L_W - 1) + L_W\,.$$

Proof. By Lemma 3, there exist $K \subseteq R$ and $u \in A^*$ such that $\mathrm{Card}(K) = M$, $\mathrm{Card}(\delta(K, u)) = 1$, $|u| \leq (M-1)(n+L_W-1)$. Since W is an independent set with range R, there is $w \in W$ such that $\delta(C, w) \cap K \neq \emptyset$. Moreover, $\delta(C, w)$ is a stable subset of R. By Lemma 4, one derives $\delta(C, w) \subseteq K$, so that $\mathrm{Card}(\delta(C, wu)) = \mathrm{Card}(\delta(K, u)) = 1$. The statement is thus verified for $v = wu$. □

If $\mathcal{A}$ is synchronizing, then Q itself is a stable set. Thus, with some minor changes in the proof of Proposition 2, one obtains the following result which refines the bound of [8].

Proposition 3. *Any synchronizing n-state automaton with an independent set W has a reset word of length*

$$(\mathrm{Card}(W) - 1)(n + L_W - 1) + \ell_W .$$

Corollary 1. *If $\mathcal{A}$ is not synchronizing, then for any stable set C there exists a word v such that*

$$\mathrm{Card}(\delta(C, v)) = 1 , \quad |v| \leq \left(\frac{\mathrm{Card}(W)}{2} - 1\right)(n + L_W - 1) + L_W .$$

Proof. There are $K \subseteq R$, $v \in A^*$, $w \in W$, $q \in Q$ such that $\mathrm{Card}(K) = M$, $\delta(K, v) = \{q\}$, $\delta(q, w) \notin K$. In view of Lemma 2, K and $K(vw)^{-1} \cap R$ are disjoint subsets of R of cardinality M. One derives $M \leq \mathrm{Card}(W)/2$. The conclusion follows from Proposition 2. □

Corollary 2. *Let C be a stable set of a 1-cluster n-state automaton which is not synchronizing. There exists a word v such that* $\mathrm{Card}(\delta(C, v)) = 1$ *and* $|v| \leq (n - 1)^2$.

Proof. Any 1-cluster n-state automaton has an independent set W with $L_W = n - 1$. Taking into account that $\mathrm{Card}(W) \leq n$, the statement follows from Corollary 1. □

4 The Hybrid Černý-Road Coloring Problem

In the sequel, with the word graph, we will term a finite, directed multigraph with all vertices of the same outdegree. A graph is *aperiodic* if the greatest common divisor of the lengths of all cycles of the graph is 1. A graph is called an *AGW-graph* if it is aperiodic and strongly connected. A synchronizing automaton which is a coloring of a graph G will be called a *synchronizing coloring* of G. The *Road coloring problem* asks for the existence of a synchronizing coloring for every AGW-graph. This problem was formulated in the context of Symbolic Dynamics by Adler, Goodwyn and Weiss and it is explicitly stated in [1]. In 2007, Trahtman has positively solved this problem [21]. Recently Volkov has raised the following problem [22].

Hybrid Černý–Road coloring problem. *Let G be an AGW-graph. What is the minimum length of a reset word for a synchronizing coloring of G?*

4.1 Relabeling

In order to prove our main theorem, we need to recall some basic results concerning colorings of graphs. Let $\mathcal{A} = \langle Q, A, \delta \rangle$ be an automaton. A map

$\delta' : Q \times A \longrightarrow Q$ is a *relabeling* of $\mathcal{A}$ if, for each $q \in Q$, there exists a permutation π_q of A such that

$$\delta'(q,a) = \delta(q, \pi_q(a)),\ a \in A\,.$$

It is clear that δ' is a relabeling of $\mathcal{A}$ if and only if the automata $\mathcal{A}$ and $\mathcal{A}' = \langle Q, A, \delta' \rangle$ are associated with the same graph.

Let $\mathcal{A} = \langle Q, A, \delta \rangle$ be an automaton, α be a congruence on Q and δ' be a relabeling of $\mathcal{A}$. According to [11], δ' *respects* α if for each congruence class C there exists a permutation π_C of A such that

$$\delta'(q,a) = \delta(q, \pi_C(a))\,, \quad q \in C\,,\ a \in A\,.$$

In such a case, for all $v \in A^*$ there is a word $u \in A^*$ such that $|u| = |v|$ and $\delta'(q,u) = \delta(q,v)$ for all $q \in C$.

As α is a congruence, we may consider the quotient automaton $\mathcal{A}/\alpha$. Any relabeling $\widehat{\delta}$ of $\mathcal{A}/\alpha$ induces a relabeling δ' of $\mathcal{A}$ which respects α. This means that

1. α is a congruence of $\mathcal{A}' = \langle Q, A, \delta' \rangle$ and $\mathcal{A}'/\alpha = \langle Q/\alpha, A, \widehat{\delta} \rangle$,
2. for all α-class C and all $v \in A^*$, there exists $u \in A^*$ such that $|v| = |u|$ and $\delta'(C,u) = \delta(C,v)$.

We end this section by recalling the following important result proven in [11].

Proposition 4. *Let ρ be the stability congruence of an automaton $\mathcal{A}$ associated with an AGW-graph G. Then the graph G' associated with the quotient automaton $\mathcal{A}/\rho$ is an AGW-graph. Moreover, if G' has a synchronizing coloring, then G has a synchronizing coloring as well.*

4.2 Hamiltonian Paths

In this section we give a partial answer to the Hybrid Černý–Road coloring problem. Precisely we prove that an AGW-graph of n vertices with a Hamiltonian path admits a synchronizing coloring with a reset word of length not larger that $2(n-2)(n-1)+1$. In order to prove this result, we need to establish some properties concerning automata with a monochromatic Hamiltonian path.

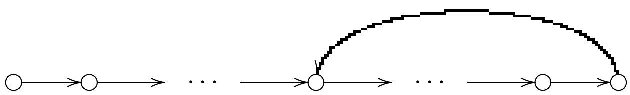
Let a be a letter. The graph of a-transitions of an automaton $\mathcal{A}$ consists of disjoint cycles and trees with root on the cycles. The *level* of a vertex in such a graph is its height in the tree to which it belongs. The following proposition was implicitly proved in [21, Theorem 3].

Proposition 5. *If in the graph of a-transitions of a transitive automaton $\mathcal{A}$ all the vertices of maximal positive level belong to the same tree, then $\mathcal{A}$ has a stable pair.*

As an application of the previous proposition, we obtain the following.

Proposition 6. *If an AGW-graph G with at least 2 vertices has a Hamiltonian path, then there is a coloring of G with a stable pair and a monochromatic Hamiltonian path.*

Proof. Let G be an AGW-graph with $n \geq 2$ vertices. Let us show that one can find in G a Hamiltonian path $(q_0, q_1, \dots, q_{n-1})$ and an edge (q_{n-1}, q) with $q \neq q_0$ (see fig.).



Indeed, if G has no Hamiltonian cycle, it is sufficient to take a Hamiltonian path $(q_0, q_1, \dots, q_{n-1})$ and any edge outgoing from q_{n-1}: such an edge exists because G has positive constant outdegree.

On the contrary, suppose that G has a Hamiltonian cycle $(q_0, q_1, \dots, q_{n-1}, q_0)$. Since G is aperiodic, there is an edge (p, q) of G which does not belong to the cycle. We may assume, with no loss of generality, $p = q_{n-1}$, so that $q \neq q_0$. Thus, $(q_0, q_1, \dots, q_{n-1})$ is a Hamiltonian path and (q_{n-1}, q) is an edge of G.

Choose a coloring $\mathcal{A}$ of G where the edges of the path $(q_0, q_1, \dots, q_{n-1}, q)$ are labeled by the same letter a. In such a way, there is a monochromatic Hamiltonian path. Moreover, the graph of a-transitions has a unique tree, so that, by Proposition 5, $\mathcal{A}$ has a stable pair. □

Lemma 5. *If an automaton $\mathcal{A}$ has a monochromatic Hamiltonian path, then any quotient automaton of $\mathcal{A}$ has the same property.*

Proof. With no loss of generality, we may reduce ourselves to the case that $\mathcal{A}$ is a 1-letter automaton. Now, a 1-letter automaton has a Hamiltonian path if and only if it has a state q from which all states are accessible. The conclusion follows from the fact that the latter property is inherited by the quotient automaton. □

We are ready to prove our main result.

Theorem 3. *Let G be an AGW-graph with $n > 1$ vertices. If G has a Hamiltonian path, then there is a synchronizing coloring of G with a reset word w of length*

$$|w| \leq 2(n-2)(n-1) + 1\,. \tag{4}$$

Proof. The proof is by induction on the number n of the vertices of G.

Let $n = 2$. Since G is aperiodic, G has an edge (q, q) which immediatly implies the statement. Suppose $n \geq 3$. By Proposition 6, among the colorings of G, there is an automaton $\mathcal{A} = \langle Q, A, \delta \rangle$ with a stable pair and a monochromatic Hamiltonian path. In particular, $\mathcal{A}$ is a transitive 1-cluster automaton. If $\mathcal{A}$ is synchronizing, then the statement follows from Theorem 2. Thus, we assume that $\mathcal{A}$ is not synchronizing.

Let ρ be the stability congruence of $\mathcal{A}$, k be its index and G_ρ be the graph of $\mathcal{A}/\rho$ respectively. Since $\mathcal{A}$ is not synchronizing, one has $k > 1$. By Proposition 4

G_ρ is an AGW-graph with k vertices and $k < n$. Moreover, by Lemma 5, G_ρ has a Hamiltonian path. By the induction hypothesis, we may assume that there is a relabeling $\widehat{\delta}$ of $\mathcal{A}/\rho$ such that the automaton $\widehat{\mathcal{A}} = \langle Q/\rho, A, \widehat{\delta}\rangle$ has a reset word u such that

$$|u| \leq 2(k-2)(k-1) + 1.$$

As viewed in Section 4.1, $\widehat{\delta}$ induces a relabeling δ' of $\mathcal{A}$ which respects ρ. Moreover, since u is a reset word of $\widehat{\mathcal{A}}$, $C = \delta'(Q, u)$ is a stable set of $\mathcal{A}$.

First, we consider the case $n \geq 2k$. By Corollary 2, there is a word v such that $|v| \leq (n-1)^2$ and $\mathrm{Card}(\delta(C, v)) = 1$. Since δ' respects ρ, there is a word v' such that $|v'| = |v|$ and $\delta'(C, v') = \delta(C, v)$. Set $w = uv'$. Then $\delta'(Q, w) = \delta'(Q, uv') = \delta'(C, v') = \delta(C, v)$ is reduced to a singleton. Hence, w is a reset word of $\mathcal{A}' = \langle Q, A, \delta'\rangle$ and

$$|w| \leq 2(k-2)(k-1) + (n-1)^2 + 1.$$

As $n \geq 2k$, one easily obtains (4).

Now, we consider the case $n < 2k$. In such a case, there is a ρ-class K of cardinality 1. Moreover, by the transitivity of $\widehat{\mathcal{A}}$, there is a word $v \in A^*$ such that $\delta'(C, v) = K$ and $|v| \leq k - 1$. Hence, $w = uv$ is a reset word of $\mathcal{A}'$ of length

$$|w| \leq 2(k-2)(k-1) + k.$$

As $n > k$, one easily obtains (4). This concludes the proof. □

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