Optimal Algorithms for the Economic Lot-Sizing Problem with Multi-supplier*

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Abstract. This paper considers the economic lot-sizing problem with multi-supplier in which the retailer may replenish his inventory from several suppliers. Each supplier is characterized by one of two types of order cost structures: incremental quantity discount cost structure and multiple set-ups cost structure. The problem is challenging due to the mix of different cost structures. By analyzing the optimal properties, we reduce the searching range of the optimal solutions and develop several optimal algorithms to solve all cases of this multi-supplier problem.

Keywords: Economic lot-sizing; optimal algorithm; dynamic programming.

1 Introduction

The classical Economic Lot-Sizing (ELS) problem was first introduced in [1] and it has been widely extended during recent years. The extended ELS problem becomes the focus of extensive studies and continues to receive considerable attention. Many versions of the ELS problem build on different cost structures. For example, the ELS problems with both all-unit quantity discount and incremental quantity discount cost structures were proposed and solved by dynamic programming(DP) algorithms in [2] with complexity $O(T^3)$ and $O(T^2)$, respectively. T is the length of the planning horizon. See also [3] for additional observations. Zhang et al.[4] presented the general model with multi-break point all-unit quantity discount cost structure, and designed a polynomial time DP algorithm. Indeed there are some other theoretical results for the ELS problem with other cost structures and assumptions (see[5],[6],[7],[8],[9] for example).

However, most of the literature discussed above assumes that products can be ordered from only one supplier. Sometimes this is not a valid assumption in real

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life. In order to replenish inventory with economical cost, the retailer usually faces not a single supplier but many suppliers. He should determine from which supplier and how many units to order in each period. Thus the ELS problem with multiple suppliers has more wider domain of applications.

In this paper, the ELS problem with multiple suppliers is called multi-supplier ELS problem. To the best of our knowledge, there are few results about the multi-supplier ELS problem. The first result about the multi-supplier problem we are aware of is the one proposed in [10]. Their ELS problem with multi-mode replenishment is equivalent to a multi-supplier ELS problem. They analyzed two structural properties for the $N(N > 2)$ suppliers problem, and presented a DP algorithm without a detail description for its calculation. Instead, they focused on a two-supplier problem. For the two-supplier problem, they discussed three special cases according to two types of cost structures (fixed set-up cost structure and multiple set-ups cost structure) and designed a polynomial time algorithm for each case. The ELS problem with multiple products and multiple suppliers was considered in [11]. In the model, the purchase cost and holding cost are stationary in each period, and they are simple linear functions about purchase and holding quantities. These linear cost functions help to prove two properties: there is no period where an order is made and inventory is carried into the period for each product, and no product is ordered from two (or more) suppliers in the same period. Based on the properties, enumerative and heuristic algorithms were given to solve the problem. However, the two properties are not true for piecewise linear cost functions, such as multiple set-ups cost function.

In this paper, we propose a multi-supplier ELS problem in which each supplier is characterized by different order cost structures including the incremental quantity discount cost structure and the multiple set-ups cost structure. In this problem, the purchase cost and holding cost vary from period to period, and are more general cost structures. This multi-supplier ELS problem can be divided into three cases according to the different combinations of the order cost structures. They are: (1) each supplier has an incremental quantity discount cost structure; (2) each supplier has a multiple set-ups cost structure; (3) some suppliers have incremental quantity discount cost structures, others have multiple set-ups cost structures. Since only two optimality properties are given for the case (2) in [10], this paper will continue discussing this case and give an optimal algorithm to solve it. We develop two polynomial time algorithms for the case (1) and a special case of case (3), two optimal algorithms for case (2) and case (3). Some previous literature results are the special cases of this multi-supplier ELS problem, such as [1], [10], [12], and so on.

2 Notations and Formulations

The multi-supplier ELS problem proposed in this paper consists of N suppliers and one retailer, $N \geq 2$. Each supplier is characterized by a different order cost structure. The retailer is the decision maker. He must determine: 1)from which supplier to order; 2)how many units to order and 3)when to order so as to minimize his total cost within a finite time planning horizon. Let T be the length of the planning horizon, and N be the total number of the suppliers. For each $t = 1, \dots, T$ and $n = 1, \dots, N$, we define the following notations.

- d_t = the demand in period t.
- x_{nt} = the order quantity from supplier-n in period t.
- x_t = the total order quantity in period t. It is easy to know that $x_t = \sum_{n=1}^{N} x_{nt}$.
- c_{nt} = the unit order cost from supplier-n in period t.
- I_t = the inventory level of the retailer at the end of period t. Without loss of generality, assume the initial and the final inventory during the planning horizon are zero. That is, $I_0 = 0$, $I_T = 0$.
- h_t = the unit inventory holding cost in period t.
- S_1 = the set of the suppliers who offer an incremental quantity discount cost structure.
- S_2 = the set of the suppliers who offer a multiple set-ups cost structure.
- K_{nt} = the fixed set-up cost when order from suppler-n in period t.
- A_{nt} = the fixed set-up cost per standard container when order from supplier-n in period t, $n \in S_2$. $A_{nt} = 0$ if $n \in S_1$.
- W_n = the standard container capacity when order from supplier-n, $n \in S_2$. $W_n = 0$ if $n \in S_1$.
- r_{nt} = the discount rate given by supplier-n in period t, $n \in S_1$. $r_{nt} = 0$ if $n \in S_2$. If the order quantity in period t is greater than the critical value Q_n (Q_n is a positive integer and is determined by supplier-n), the discounted unit order cost is implemented to the excess quantity $x_{nt} - Q_n$.
- $\lceil a \rceil$ = the smallest integer that is greater than or equal to a.
- $\lfloor a \rfloor$ = the largest integer that is less than or equal to a.
- $\delta(a) = 1$ if and only if $a > 0$; otherwise, $\delta(a) = 0$.
- $C_{nt}(x_{nt})$ = the cost of ordering x_{nt} units from supplier-n in period t. When $n \in S_1$, $C_{nt}(x_{nt})$ belongs to an incremental quantity discount cost structure,

$$C_{nt}(x_{nt}) = \begin{cases} K_{nt}\delta(x_{nt}) + c_{nt}x_{nt}, & x_{nt} \le Q_n \\ K_{nt} + c_{nt}Q_n + c_{nt}(1 - r_{nt})(x_{nt} - Q_n), & x_{nt} > Q_n \end{cases}. \tag{1}$$

When $n \in S_2$, $C_{nt}(x_{nt})$ belongs to a multiple set-ups cost structure,

$$C_{nt}(x_{nt}) = K_{nt}\delta(x_{nt}) + A_{nt}\lceil \frac{x_{nt}}{W_n} \rceil + c_{nt}x_{nt}. \tag{2}$$

With above notations, this multi-supplier ELS problem can be formulated as

$$\begin{aligned} \min \quad & \sum_{t=1}^{T} \Big[\sum_{n=1}^{N} C_{nt}(x_{nt}) + h_t I_t \Big] \\ s.t. \quad & I_{t-1} + x_t - d_t = I_t, \quad t = 1, \dots, T \\ & x_t = \textstyle\sum_{n=1}^{N} x_{nt}, \quad t = 1, \dots, T \\ & I_0 = 0, \quad I_T = 0 \\ & I_t \ge 0, \quad x_{nt} \ge 0 \end{aligned}$$

This paper will discuss three cases of the multi-supplier ELS problem:

(1) each supplier offers an incremental quantity discount cost structure, P_1 problem for short;

(2) each supplier offers a multiple set-ups cost structure, P_2 problem for short;

(3) some supplier offer incremental quantity discount cost structures, others offer multiple set-ups cost structures, P_3 problem for short.

For convenience, we use the traditional definitions of most ELS problems. Period t is called an **order period** if $x_t > 0$. If $I_t = 0$, period t is a **regeneration point**. x_{nt} is called a **Full-Truck Load(FTL) shipment** if $x_{nt} = lW_n$ for some positive integer l, $n \in S_2$, otherwise, it is a **Less-than-Truck Load(LTL) shipment**. $x_t > 0$ is called a **full order** if $\sum_{n\in S_1} x_{nt} = 0$, and x_{nt} is zero or an FTL shipment for all suppliers $n \in S_2$; otherwise, it is called a **partial order**.

For $1 \le i \le j \le T$, let

$$h(i,j) = \sum_{l=i}^{j} h_l, \quad d(i,j) = \sum_{l=i}^{j} d_l, \quad H(i,j) = \sum_{k=i+1}^{j} h(i,k-1)d_k.$$

If $i > j$, we define $d(i,j) = 0$, $h(i,j) = 0$ and $H(i,j) = 0$. By this definition, we can calculate $d(i,j)$, $h(i,j)$ and $H(i,j)$ in $O(T^2)$ time for all i and j with $1 \le i \le j \le T$.

In practical situation, the more frequently an item is ordered or dispatched, the more favorable its relevant cost is. Considering this situation, we assume that for each $n \in S_1 \cup S_2$, K_{nt}, A_{nt}, c_{nt} and h_t are non-increasing functions on t, and r_{nt} is a non-decreasing function on t. In other words, for $1 \le t \le T-1$, we have $K_{nt} \ge K_{n,t+1}$, $A_{nt} \ge A_{n,t+1}$, $c_{nt} \ge c_{n,t+1}$, $h_t \ge h_{t+1}$ and $r_{nt} \le r_{n,t+1}$.

Let $F(j)$ denote the minimum total cost of satisfying the demand from period 1 to period j and $C(i,j)$ denote the minimum cost of satisfying the demand from period i to period j, where $i-1$ and j are two consecutive regeneration points with $1 \le i \le j \le T$. Set $F(0) = 0$, the multi-supplier ELS problem can be solved by the following DP algorithm

$$F(j) = \min_{1\le i\le j} \{F(i-1) + C(i,j)\}, \quad 1 \le i \le j \le T \tag{3}$$

Obviously, the objective function of P_1 problem is $F(T)$. If the value of $C(i,j)$ for all $1 \le i \le j \le T$ is known, the value of $F(T)$ can be computed in no more than $O(T^2)$ time via the formula (3). Hence the remaining task is how to compute the value of $C(i,j)$ in an efficient time.

3 Optimality Properties and Algorithm for P_1 Problem

In this section, the first case of the multi-supplier ELS problem in which each supplier offers an incremental quantity discount cost structure is discussed, that is, $S_1 = \{1, \ldots, N\}$ and $S_2 = \emptyset$ hold in this section. To simplify the proof, we first analyze the optimality properties of P_1 problem with two-supplier. The optimality property of P_1 problem with N $(N > 2)$ suppliers can be proved via the induction on the number of suppliers.

Lemma 1. *There exists an optimal solution for the two-supplier P_1 problem such that if period t is an order period then the retailer orders products only from one supplier.*

Proof. Suppose that there is an optimal solution for P_1 problem such that $0 < x_{1t} < x_t$ and $0 < x_{2t} < x_t$. Order x_{2t} units in period t from supplier-1 instead of from supplier-2 if the unit order cost from supplier-1 is less than the one from supplier-2. Otherwise, order x_{1t} units from supplier-2 instead of from supplier-1. After this perturbation, we can obtain a solution with non-increasing total cost.

Using above Lemma, Theorem 1 can be proven by induction on the number of suppliers.

Theorem 1. *There exists an optimal solution for P_1 problem such that if period t is an order period then the retailer orders products only from one supplier.*

Theorem 2. *There exists an optimal solution for P_1 problem such that $I_{t-1}x_t = 0$, $t = 1, \ldots, T$.*

Proof. Suppose that there is an optimal solution for P_1 problem such that $x_t > 0$ and $I_{t-1} > 0$. Since $I_{t-1} > 0$, there exists an order before period t. Let s be latest order before period t. Then we have $I_k \geq I_{t-1} > 0$ for all $s \leq k \leq t-1$. Using Theorem 1, we assume that $x_s = x_{ms},\ x_t = x_{nt},\ m, n \in S_1$.

The proof can be completed via discussing four subcases: (1) $x_{nt} > Q_n$ and $x_{ms} > Q_m$; (2) $0 < x_{nt} \leq Q_n$ and $x_{ms} > Q_m$; (3) $x_{nt} > Q_n$ and $0 < x_{ms} \leq Q_m$; (4) $0 < x_{nt} \leq Q_n$ and $0 < x_{ms} \leq Q_m$. The discussion of these cases is similar, so we only discuss the first case in detail.

In case (1), if $c_{ms}(1 - r_{ms}) - c_{nt}(1 - r_{nt}) + h(s, t-1) \geq 0$, we decrease the value x_{ms} by Δ with $\Delta = \min\{I_{t-1}, x_{ms}\}$ and increase x_{nt} by the same amount. After the perturbation, we obtain a new solution with either $x_{ms} = 0$ or $I_{t-1} = 0$. The total cost of this new solution is either reduced by at least $[c_{ms}(1 - r_{ms}) - c_{nt}(1 - r_{nt}) + h(s, t-1)]\Delta \geq 0$, or not changed. Otherwise, we cancel the order from supplier n in period t and increase the value x_{ms} by x_{nt} units. The total cost of the new solution is reduced by $K_{nt} + c_{nt}r_{nt}Q_n + [c_{nt}(1 - r_{nt}) - c_{ms}(1 - r_{ms}) - h(s, t-1)]x_{nt} > 0$ after this perturbation.

For the other three cases, we can decrease the value x_{ms} by $\min\{I_{t-1}, x_{ms}\}$ and increase x_{nt} by the same amount if $c_{ms}(1 - r_{ms}) - c_{nt} + h(s, t-1) \geq 0$ in case (2), or $c_{ms} - c_{nt}(1 - r_{nt}) + h(s, t-1) \geq 0$ in case (3), or $c_{ms} - c_{nt} + h(s, t-1) \geq 0$ in case (4). Otherwise, we let $x_{nt} = 0$ and increase the value x_{ms} by x_{nt} units. The total cost of the new solution is not increased after the perturbations and we finish the proof.

Basing on the above theorems, we develop a polynomial time algorithm to calculate all $C(i, j)$. From Theorem 1, we know that the order cost in period t is $\min_{n \in S_1} C_{nt}(x_t)$. By Theorem 2, there exists only one order period between the two consecutive regeneration points $i - 1$ and j. This means that the order quantity in order period i is exactly $d(i, j)$. Thus for each pair of i and j with $1 \leq i \leq j \leq T$, the value of $C(i, j)$ can be computed by

$$C(i,j) = \min_{n \in S_1} C_{ni}\big(d(i,j)\big) + H(i,j), \quad 1 \le i \le j \le T \tag{4}$$

Obviously, the computational complexity of the formula (4) is no more than $O(NT^2)$ for all $1 \le i \le j \le T$. So the total computational complexity of the DP algorithm for the P_1 problem is $O(NT^2)$.

4 Optimality Properties and Algorithm for P_2 Problem

In this subsection, each supplier offers a multiple set-ups cost structure, that is, $S_2 = \{1, \ldots, N\}$, $S_1 = \emptyset$. We propose two optimality properties and design an optimal algorithm to solve it.

Theorem 3. *[10] There exists an optimal solution for P_2 problem such that there is at most one partial order during two consecutive regeneration points $i-1$ through j.*

Theorem 4. *There exists an optimal solution for P_2 problem such that if x_t is a partial order, then only one shipment is an LTL shipment in period t.*

Proof. Suppose the statement of this theorem is not true. That is, there exists an optimal solution of P_2 problem such that x_{mt} and x_{nt} are two LTL shipments in period t, $m, n \in S_2$. Without loss of generality, we suppose $c_{mt} \ge c_{nt}$. We extract $\min\{x_{mt}, \lceil \frac{x_{nt}}{W_n} \rceil W_n - x_{nt}\}$ units from x_{mt}, and add them to x_{nt}, then we can obtain a new solution with equal or more lower total cost.

Corollary 1. *There exists an optimal solution for P_2 problem such that if x_{nt} is an LTL shipment then $c_{nt} = \max_{k \in S_2} c_{kt}$.*

Suppose $c_{n_0 t} = \max_{n \in S_2} c_{nt}$, that is, $x_{n_0 t}$ is a potential LTL shipment in order period t. Let $x_{nt} = m_n W_n$, $m_n = 0, 1, \ldots, \lceil \frac{x_t}{W_n} \rceil$ for all $n \in S_2 \setminus \{n_0\}$. Then, for $i \le t \le j$ we have

$$x_t = \sum_{n \in S_2 \setminus \{n_0\}} m_n W_n + x_{n_0 t}.$$

Recall that $i-1$ and j are two consecutive regeneration points, we have

$$d(i,j) = x_i + \cdots + x_j.$$

The value of $C(i,j)$ is the minimal total cost to satisfying demand $d(i,j)$ among all combinations of x_{nt}, $n \in S_2$, $t = i, \ldots, j$. That is, we can calculate the value of $C(i,j)$ in $O(T(\lceil \frac{d(1,T)}{W^*} \rceil + 1)^{NT-1})$ time for each pair of i and j, where $W^* = \min_{n \in S_2} W_n$. After computing all value of $C(i,j)$, we can solve P_2 problem by the formula (3) in $O(T^2)$. So the total computational complexity of solving P_2 problem is $O\big(T^3(\lceil \frac{d(1,T)}{W^*} \rceil + 1)^{NT-1} + T^2\big)$. Obviously, it is not a polynomial time algorithm but an optimal one.

5 Optimality Properties and Algorithms for $\mathbf{P_3}$ Problem

In this section we propose some optimality properties for P_3 problem, in which some suppliers offer incremental quantity discount cost structures, others offer multiple set-ups cost structures. That is, $S_1 \neq \emptyset$, $S_2 \neq \emptyset$ and $S_1 \cup S_2 = \{1, \ldots, N\}$. After that we give an optimal algorithm for P_3 problem whose running time is non-polynomial. Then we show that there exists an polynomial time algorithm for a special case of the P_3 problem, we denoted it by SP_3 problem in this section.

Theorem 5. *There exists an optimal solution for P_3 problem such that if t is an order period then only one of the following two situations will happen:*
(1) There is at most one n_1 with $x_{n_1 t} > 0$ in set S_1, and x_{nt} is zero or an FTL shipment for all $n \in S_2$.
(2) $x_{nt} = 0$ for all $n \in S_1$, and there is at most one n_2 such that $x_{n_2 t}$ is an LTL shipment in set S_2.

Proof. Suppose that the statement is not true, that is, there exists an optimal solution for P_3 problem with $x_{n_1 t} > 0$ and $x_{n_2 t} \neq l W_{n_2}$. According to Theorem 1, we know $x_{n_1 t} = x_t$. According to Theorem 4, we know that x_{nt} is zero or an FTL shipment for all $n \in S_2 \setminus \{n_2\}$. Let $c'_{n_1 t}$ be the unit purchase cost, $c'_{n_1 t} = c_{n_1 t}$ when $x_{n_1 t} \leq Q_{n_1}$, $c'_{n_1 t} = c_{n_1 t}(1 - r_{n_1 t})$ when $x_{n_1 t} > Q_{n_1}$. If $c_{n_2 t} \geq c'_{n_1 t}$, we increase the value $x_{n_1 t}$ and decrease $x_{n_2 t}$ by $x_{n_2 t}$ units. After that, we can obtain a new solution with $x_{n_1 t} > 0$, $x_{n_2 t} = 0$ and non-increasing total cost. If $c_{n_2 t} < c'_{n_1 t}$, we increase $x_{n_2 t}$ and decrease $x_{n_1 t}$ by $\min\{x_{n_1 t}, \lceil \frac{x_{n_2 t}}{W_{n_2}} \rceil W_{n_2} - x_{n_2 t}\}$ units. After that, we can obtain a new solution in which $x_{n_1 t} = 0$ or $x_{n_2 t}$ is an FTL shipment. The total cost of the new solution is much lower.

Theorem 6. *There exists an optimal solution for P_3 problem such that there is at most one $n_1 \in S_1$ with $x_{n_1 s} > 0$ or at most one $n_2 \in S_2$ with an LTL shipment $x_{n_2 t}$ between two consecutive regeneration points $i - 1$ and j, $i \leq s, t \leq j$.*

Proof. When $s = t$, the Theorem reduces to Theorem 5. So we only prove the case where $s \neq t$. Without loss of generality, let $s < t$. Suppose that this Theorem is not true, that is, there exists an optimal solution for P_3 problem such that $x_{n_1 s} > 0$ and $x_{n_2 t}$ is an LTL shipment, $n_1 \in S_1$, $n_2 \in S_2$, $i \leq s < t \leq j$. If $c_{n_2 t} - c'_{n_1 s} - h(s, t-1) \leq 0$, decrease the value $x_{n_1 s}$ and increase $x_{n_2 t}$ by $\min\{x_{n_1 s}, \lceil \frac{x_{n_2 t}}{W_{n_2}} \rceil W_{n_2} - x_{n_2 t}\}$, we can obtain a new solution with a non-increasing total cost, in which $x_{n_1 s} = 0$ or $x_{n_2 t}$ is an FTL shipment. Otherwise, decrease the value $x_{n_2 t}$ and increase $x_{n_1 s}$ by $x_{n_2 t}$, we can obtain a new solution with a lower total cost, in which $x_{n_1 s} > 0$ and $x_{n_2 t} = 0$.

It is easy to verify that Theorem 3 and Theorem 4 are true for P_3 problem. Recall that periods $i - 1$ and j are two consecutive regeneration points, we have

$$d(i, j) = x_{1i} + \cdots + x_{Ni} + \cdots + x_{1j} + \cdots + x_{Nj} \tag{5}$$

According to Theorem 6, at most one period could be the potential partial order between two consecutive regeneration points $i-1$ and j. Let period t is the potential partial order, $t = i, \ldots, j$. For each such period t between two consecutive regeneration points $i-1$ and j, either there is at most one $n_1 \in S_1$ with $x_{n_1 t} > 0$, $x_{nu} = 0$ for all $n \in S_1 \setminus \{n_1\}$, and x_{nu} is zero or an FTL shipment for all $n \in S_2$, $u = i, \ldots, j$, or there is at most one $n_2 \in S_2$ with an LTL shipment $x_{n_2 t}$, $x_{nu} = 0$ for all $n \in S_1$, and x_{nu} is zero or an FTL shipment for all $n \in S_2 \setminus \{n_2\}$, $u = i, \ldots, j$. So the number of purchase policies between two consecutive regeneration points $i-1$ and j is not more than $O\big(T|S_1|(\lceil \frac{d(i,j)}{W^*} \rceil + 1)^{|S_2|T} + T|S_2|(\lceil \frac{d(i,j)}{W^*} \rceil + 1)^{|S_2|T-1}\big)$, where $W^* = \min_{n \in S_2} W_n$. In other words, it takes at most $O\big(T|S_1|(\lceil \frac{d(i,j)}{W^*} \rceil + 1)^{|S_2|T} + T|S_2|(\lceil \frac{d(i,j)}{W^*} \rceil + 1)^{|S_2|T-1}\big)$ time to calculate $C(i,j)$ for each pair of i and j. So the total complexity to calculate $C(i,j)$ for all i and j is at most $O\big(T^3|S_1|\lceil \frac{d(1,T)}{W^*} \rceil^{|S_2|T} + T^3|S_2|\lceil \frac{d(1,T)}{W^*} \rceil^{|S_2|T-1}\big)$. After that, we can use formula (3) to find the optimal value of P_3 problem in $O(T^2)$ time. Obviously, it is a non-polynomial time algorithm.

Fortunately, the P_3 problem have more optimality properties when $|S_2| = 1$. These optimality properties help to explore a polynomial time algorithm. Let SP_3 problem denote this special case of P_3 problem. In other words, in SP_3 problem only one supplier has a multiple set-ups cost structure, the rest of $N-1$ suppliers have incremental quantity discount cost structures. Without loss of generality, assume that supplier 1 has a multiple set-ups cost structure in the SP_3 problem. For this SP_3 problem, we obtain several optimality properties in addition.

Theorem 7. *There exists an optimal solution for* SP_3 *problem such that* $x_{1t} \in \{0, x_t, \lfloor \frac{x_t}{W_1} \rfloor W_1\}$ *for any* $t = 1, \ldots, T$.

Proof. Suppose that there exists an optimal solution for SP_3 problem such that $0 < x_{1t} < x_t$ and $x_{1t} \neq \lfloor \frac{x_t}{W_1} \rfloor W_1$. According to Theorem 6, if x_{1t} is an FTL shipment, there exists at most one $n_0 \in S_1$ with $x_{n_0 t} > 0$, and $x_{nt} = 0$ for all $n \in S_1 \setminus \{n_0\}$. If x_{1t} is an FTL shipment, and there is no $n \in S_1$ with $x_{nt} > 0$, then we have $x_{nt} = 0$ for all $n \in S_1$. If x_{1t} is an LTL shipment, we have $x_{nt} = 0$ for all $n \in S_1$. The above two cases mean that $x_{1t} = x_t$, which contradicts the fact $0 < x_{1t} < x_t$.

Now we assume that x_{1t} is an FTL shipment but $x_{1t} \neq \lfloor \frac{x_t}{W_1} \rfloor W_1$, and there is only one $n_0 \in S_1$ with $x_{n_0 t} > 0$. Without loss of generality, we let $n_0 = 2 \in S_1$, that is, $x_{2t} > 0$. Since x_{1t} is an FTL shipment and $x_{1t} \neq \lfloor \frac{x_t}{W_1} \rfloor W_1$, Let $x_{1t} = lW_1$, then we have $l \neq \lfloor \frac{x_t}{W_1} \rfloor$, and $x_{2t} \geq W_1$. If $\frac{A_{1t}}{W_1} + c_{1t} \geq c'_{2t}$ ($c'_{2t} = c_{2t}$ when $x_{2t} \leq Q_2$, $c'_{2t} = c_{2t}(1 - r_{2t})$ when $x_{2t} > Q_2$), we cancel the order from supplier 1 in period t, and increase x_{2t} by x_{1t} units, then we get a new solution with a non-increasing cost. Otherwise, let $\Delta = \lfloor \frac{x_{2t}}{W_1} \rfloor W_1$, increase x_{1t} by Δ units and decrease x_{2t} by the same amount. After this perturbation, the total cost is reduced.

Theorem 8. *There exists an optimal solution for* SP_3 *problem such that for any* $t = 1, \ldots, T$,

(1) $x_{1t} > 0$ *only if* $I_{t-1} < \min\{d_t, W_1\}$;

(2) For some supplier n with $n \in S_1$, $x_{nt} > 0$ only if $I_{t-1} < \min\{d_t, W_1\}$ or $d_t - W_1 < I_{t-1} < d_t$.

Proof. Since [10] provided the same property for their model, and the proof of result (1) in this theorem can be completed using the similar approach, so we only prove the proposition (2).

Following Theorem 1, the retailer orders products only from one supplier in set S_2 in period t. Here we suppose that there exists an optimal solution such that $x_{nt} > 0$ and $\min\{d_t, W_1\} \leq I_{t-1} \leq d_t - W_1$, $n \in S_1$, which means that $d_t \geq W_1$. Furthermore, we have $W_1 \leq I_{t-1} \leq d_t - W_1$. This means that there exists an order period before period t. Let s be the latest order period before period t. Then we have $x_s = x_{1s} \geq I_s \geq \cdots \geq I_{t-1}$. By proposition (1) of this theorem, $x_{1t} = 0$ holds if $I_{t-1} > W_1$, that is, $x_{nt} = x_t \geq W_1$ since $I_{t-1} \leq d_t - W_1$ and $I_t \geq 0$. If $c'_{nt}W_1 \geq A_{1s} + W_1 c_{1s} + h(s, t-1)W_1$, we increase x_{1s} by W_1 units and decrease x_{nt} by the same amount. $c'_{nt} = c_{nt}$ when $x_{nt} \leq Q_n$, $c'_{nt} = c_{nt}(1 - r_{nt})$ when $x_{nt} > Q_n$. The total cost will not increase after this perturbation. Otherwise, we increase x_{nt} by $\lfloor \frac{I_{t-1}}{W_1} \rfloor W_1$ units and decrease x_{1s} by the same amount. After this perturbation, the new solution has a lower cost which is a contradiction.

The above properties for SP$_3$ problem also hold for the two-supplier problem in which supplier 1 has a multiple set-ups and supplier 2 has a fixed set-up cost structure studied in [10], since the fixed set-up cost is a special case of incremental quantity discount cost with $Q_n = +\infty$. In other words, [10] becomes a special case of our problem. Using their optimality properties, they develop a polynomial time algorithm with complexity $O(T^4)$ based on the dynamic programming-based shortest-path-network approach to solve their problem. With the Theorems 3, 7 and 8, it is easy to show that SP$_3$ problem can be solved by their algorithm, except the calculation formula of order cost. Here we use the following formula to calculate the order cost.

$$\begin{aligned} C_t(x_t) = \min \Big\{ & C_{1t}(x_t), \min_{n \in S_1} C_{nt}(x_t), \\ & C_{1t}(\lfloor \frac{x_t}{W_1} \rfloor W_1) + \min_{n \in S_1} C_{nt}(x_t - \lfloor \frac{x_t}{W_1} \rfloor W_1) \Big\}. \end{aligned} \qquad (6)$$

So there exists a polynomial algorithm with running time $O(T^4 + NT)$ for SP$_3$ problem.

6 Numerical Example

In this section, we illustrate the optimal algorithm for P$_3$ problem with an example. The algorithm is written in the runtime environment MATLAB 7.0, and is achieved and executed on an Lenovo personal computer with a 2.16 GHz Intel Core 2 processor and 1 GB RAM. The running time of the algorithm is 39.15 seconds. The planning horizon of the considered example contains 4 periods, that is, $T = 4$. There are 3 suppliers in this example, that is, $N = 3$. Supplier 1

Table 1. The value of parameters

Parameter	Value	Parameter	Value
T	4	$\boldsymbol{K}_1$	(21,17,10,8)
N	3	$\boldsymbol{K}_2$	(20,16,12,7)
S_1	{1}	$\boldsymbol{K}_3$	(19,19,9,9)
S_2	{2,3}	$\boldsymbol{A}_1$	(0,0,0,0)
$\boldsymbol{Q}$	(20,0,0)	$\boldsymbol{A}_2$	(45,45,45,45)
$\boldsymbol{d}$	(14,9,28,13)	$\boldsymbol{A}_3$	(30,30,30,30)
$\boldsymbol{h}$	(3,3,3,2)	$\boldsymbol{c}_1$	(3,3,3,3)
$\boldsymbol{W}$	(0,20,15)	$\boldsymbol{c}_2$	(2,2,2,2)
$\boldsymbol{r}_1$	(0.2,0.3,0.3,0.4)	$\boldsymbol{c}_3$	(2,2,2,2)

Table 2. The results of $C(i, j)$

$i \backslash j$	1	2	3	4
1	63.0000	115.2000	276.6000	344.6000
2	0	44.0000	171.2000	237.5000
3	0	0	86.8000	153.1000
4	0	0	0	47.0000

Table 3. The results of $F(j)$

j	1	2	3	4
$F(j)$	63.0000	107.000	193.8000	240.8000

Table 4. The value of i correspond with $F(j)$

$F(j)$	$F(1)$	$F(2)$	$F(3)$	$F(4)$
i	1	2	3	4

is in set S_1, suppliers 2 and 3 are in set S_2, that is, $S_1 = \{1\}$, $S_2 = \{2, 3\}$. The other parameters are expressed by vectors (see Table 1).

The computation results of $C(i, j)$ and $F(j)$ are in Table 2, Table 3 and Table 4. From the computation results, we know that the optimal value of P_3 problem is $F(4) = F(3) + C(4, 4) = 240.8$, $F(3) = F(2) + C(3, 3) = 193.8$, $F(2) = F(1) + C(2, 2) = 107$, $F(1) = C(1, 1) = 63$. The optimal solution of P_3 problem is $x_{11} = 14$, $x_{12} = 9$, $x_{13} = 28$, $x_{14} = 13$, $x_{nt} = 0$ for all $n = 2, 3$, $t = 1, 2, 3, 4$.

7 Conclusion

This paper extends the classical economic lot-sizing problem to the multi-supplier ELS problem. Each supplier has one of the two types of order cost structures: incremental quantity discount cost structure and multiple set-ups cost structure. We analyzed all possible cases for the multi-supplier ELS problem. After proposing corresponding optimal properties for each case, we find that there exists a polynomial time algorithm for P_1 problem, in which each supplier offers an incremental quantity discount cost structure. It is difficult to find a polynomial time optimal algorithm for P_2 problem and P_3 problem. The optimal algorithm for P_2 and P_3 problems given in this paper can find an optimal solution in a short time for a small size of the problem. However, there exists a polynomial time algorithm for SP_3 problem, which is a special case of P_3 problem with $|S_2| = 1$. More future research includes multi-echelon economic lot-sizing problem with multi-delivery modes problem and economic lot-sizing problem with multi-supplier multi-item and multiple cost structures.

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