Bounds on Threshold of Regular Random k-SAT

Vishwambhar Rathi[1,2], Erik Aurell[1,3], Lars Rasmussen[1,2], and Mikael Skoglund[1,2]

[1] KTH Linnaeus Centre ACCESS, KTH-Royal Institute of Technology, Stockholm, Sweden

[2] School of Electrical Engineering, KTH-Royal Institute of Technology, Stockholm, Sweden

[3] Dept. Information and Computer Science, TKK-Helsinki University of Technology, Espoo, Finland

{vish,eaurell}@kth.se,
{lars.rasmussen,skoglund}@ee.kth.se

Abstract. We consider the regular model of formula generation in conjunctive normal form (CNF) introduced by Boufkhad et. al. in [6]. In [6], it was shown that the threshold for regular random 2-SAT is equal to unity. Also, upper and lower bound on the threshold for regular random 3-SAT were derived. Using the first moment method, we derive an upper bound on the threshold for regular random k-SAT for any $k \geq 3$ and show that for large k the threshold is upper bounded by $2^k \ln(2)$. We also derive upper bounds on the threshold for Not-All-Equal (NAE) satisfiability for $k \geq 3$ and show that for large k, the NAE-satisfiability threshold is upper bounded by $2^{k-1} \ln(2)$. For both satisfiability and NAE-satisfiability, the obtained upper bound matches with the corresponding bound for the uniform model of formula generation [9, 1].

For the uniform model, in a series of break through papers Achlioptas, Moore, and Peres showed that a careful application of the second moment method yields a significantly better lower bound on threshold as compared to any rigorously proven algorithmic bound [3, 1]. The second moment method shows the existence of a satisfying assignment with uniform positive probability (w.u.p.p.). Thanks to the result of Friedgut for uniform model [10], existence of a satisfying assignment w.u.p.p. translates to existence of a satisfying assignment with high probability (w.h.p.). Thus, the second moment method gives a lower bound on the threshold. As there is no known Friedgut type result for regular random model, we assume that for regular random model existence of a satisfying assignments w.u.p.p. translates to existence of a satisfying assignments w.h.p. We derive the second moment of the number of satisfying assignments for regular random k-SAT for $k \geq 3$. There are two aspects in deriving the lower bound using the second moment method. The first aspect is given any k, numerically evaluate the lower bound on the threshold. The second aspect is to derive the lower bound as a function of k for large enough k. We address the first aspect and evaluate the lower bound on threshold. The numerical evaluation suggests that as k increases the obtained lower bound on the satisfiability threshold of a regular random formula converges to the lower bound obtained for the uniform model. Similarly, we obtain lower bounds on the NAE-satisfiability threshold of the regular random formulas and observe that the obtained lower bound seems to converge to the corresponding lower bound for the uniform model as k increases.

O. Strichman and S. Szeider (Eds.): SAT 2010, LNCS 6175, pp. 264–277, 2010.

1 Regular Formulas and Motivation

A clause is a disjunction (OR) of k variables. A formula is a conjunction (AND) of a finite set of clauses. A k-SAT formula is a formula where each clause is a disjunction of k literals. A *legal* clause is one in which there are no repeated or complementary literals. Using the terminology of [6], we say that a formula is *simple* if it consists of only legal clauses. A *configuration* formula is not necessarily legal. A satisfying (SAT) assignment of a formula is a truth assignment of variables for which the formula evaluates to true. A Not-All-Equal (NAE) satisfying assignment is a truth assignment such that every clause is connected to at least one true literal and at least one false literal. We denote the number of variables by n, the number of clauses by m, and the clause density, i.e. the ratio of clauses to variables, by $\alpha = \frac{m}{n}$. We denote the binary entropy function by $h(\cdot)$, $h(x) \triangleq -x\ln(x) - (1-x)\ln(1-x)$, where the logarithm is the natural logarithm.

The popular, uniform k-SAT model generates a formula by selecting uniformly and independently m-clauses from the set of all $2^k\binom{n}{k}$ k-clauses. In this model, the literal degree can vary. We are interested in the model where the literal degree is almost constant, which was introduced in [6]. Suppose each literal has degree r. Then $2nr = km$, which gives $\alpha = 2r/k$. Hence α can only take values from a discrete set of possible values. To circumvent this, we allow each literal to take two possible values for a degree. For a given α, let $r = \frac{k\alpha}{2}$ and $\mathtt{r} = \lfloor r \rfloor$. Each literal has degree either $\mathtt{r}$ or $\mathtt{r}+1$. Also a literal and its negation have the same degree. Thus, we can speak of the degree of a variable which is the same as the degree of its literals. Let the number of variables with degree d be n_d, $d \in \{\mathtt{r}, \mathtt{r}+1\}$. Let $X_1, \ldots, X_{n_\mathtt{r}}$ be the variables which have degree $\mathtt{r}$ and $X_{n_\mathtt{r}+1}, \ldots, X_n$ be the variables with degree $\mathtt{r}+1$. Then,

$$n_\mathtt{r} = n + \mathtt{r}n - \left\lfloor \frac{k\alpha n}{2} \right\rfloor, \quad n_{\mathtt{r}+1} = \left\lfloor \frac{k\alpha n}{2} \right\rfloor - \mathtt{r}n$$

and $n_\mathtt{r} + n_{\mathtt{r}+1} = n$. As we are interested in the asymptotic setting, we will ignore the floor in the sequel. We denote the fraction of variables with degree $\mathtt{r}$ (resp. $\mathtt{r}+1$) by $\Lambda_\mathtt{r}$ (resp. $\Lambda_{\mathtt{r}+1}$) which is given by

$$\Lambda_\mathtt{r} = 1 + \mathtt{r} - \frac{k\alpha}{2}, \quad \Lambda_{\mathtt{r}+1} = \frac{k\alpha}{2} - \mathtt{r}. \tag{1}$$

When $\Lambda_\mathtt{r}$ or $\Lambda_{\mathtt{r}+1}$ is zero, we refer to such formulas as *strictly* regular random formulas. This implies that there is no variation in literal degree. If $\Lambda_\mathtt{r}, \Lambda_{\mathtt{r}+1} > 0$, then we say that the formulas are 2-regular random formulas.

A formula is represented by a bipartite graph. The left vertices represent the literals and right vertices represent the clauses. A literal is connected to a clause if it appears in the clause. There are $k\alpha n$ edges coming out from all the literals and $k\alpha n$ edges coming out from the clauses. We assign the labels from the set $\mathcal{E} = \{1, \ldots, k\alpha n\}$ to edges on both sides of the bipartite graph. In order to generate a formula, we generate a random permutation Π on $\mathcal{E}$. Now we connect an edge i on the literal node side to an edge $\Pi(i)$ on the clause node side. This gives rise to a regular random k-SAT formula. Note that not all the formulas generated by this procedure are simple. However, it was shown in [6] that the threshold is the same for this collection of formulas and the collection of simple formulas. Thus, we can work with the collection of configuration formulas generated by this procedure.

The regular random k-SAT formulas are of interest because such instances are computationally harder than the uniform k-SAT instances. This was experimentally observed in [6], where the authors also derived upper and lower bounds for regular random 3-SAT. The upper bound was derived using the first moment method. The lower bound was derived by analyzing a greedy algorithm proposed in [13]. To the best of our knowledge, there are no known upper and lower bounds on the thresholds for regular random formulas for $k > 3$.

Using the first moment method, we compute an upper bound α_u^* on the satisfiability threshold α^* for regular random formulas for $k \geq 3$. We show that $\alpha^* \leq 2^k \ln(2)$, which coincides with the upper bound for the uniform model. We also apply the first moment method to obtain an upper bound $\alpha_{u,\text{NAE}}^*$ on the NAE-satisfiability threshold α_{NAE}^* of regular random formulas. We show that $\alpha_{\text{NAE}}^* \leq 2^{k-1} \ln(2)$ which coincides with the corresponding bound for the uniform model.

In order to derive a lower bound α_l^* on the threshold, we apply the second moment method to the number of satisfying assignments. The second moment method shows the existence of a satisfying assignment with uniform positive probability (w.u.p.p.). Due to the result of Friedgut for uniform model [10], existence of a satisfying assignment w.u.p.p. translates to existence of a satisfying assignment with high probability (w.h.p.). Thus, the second moment method gives lower bound on the threshold for uniform model. As there is no known Friedgut type result for regular random model, we assume that for regular random model existence of a satisfying assignments w.u.p.p. translates to existence of a satisfying assignments w.h.p. This permits us to say that second moment method gives valid lower bound on the threshold. We compute the second moment of the number of satisfying assignments for regular random model. Similar to the case of the uniform model, we show that for the second moment method to succeed the term corresponding to overlap $n/2$ should dominate other overlap terms. We observe that the obtained lower bound α_l^* converges to the corresponding lower bound of the uniform model, which is $2^k \ln(2) - (k+1)\frac{\ln(2)}{2} - 1$ as k increases. Similarly, by computing the second moment of the number of NAE-satisfying assignments we obtain that $\alpha_{l,\text{NAE}}$ converges to the corresponding bound $2^{k-1} \ln(2) - O(1)$ for the uniform model. The lower bounds are not obtained explicitly as computing the second moment requires finding all the positive solutions of a system of polynomial equations. For small values of k, this can be done exactly. However, for large values of k we resort to a numerical approach. Our main contribution is that we obtain almost matching lower and upper bounds on the satisfiability (resp. NAE-satisfiability) threshold for the regular random formulas. Thus, we answer in affirmative the following question posed in [1]: *Does the second moment method perform well for problems that are symmetric "on average"? For example, does it perform well for regular random k-SAT where every literal appears an equal number of times?*.

In the next section, we obtain an upper bound on the satisfiability threshold and NAE satisfiability threshold.

2 Upper Bound on Threshold via First Moment

Let X be a non-negative integer-valued random variable and $E(X)$ be its expectation. Then the first moment method gives: $\mathrm{P}(X > 0) \leq E(X)$. Note that by choosing X to be the number of solutions of a random formula, we can obtain an upper bound on the threshold α^* beyond which no solution exists with probability one. This upper bound corresponds to the largest value of α at which the average number of solutions goes to zero as n tends to infinity. In the following lemma, we derive the first moment of the number of SAT solutions of the regular random k-SAT for $k \geq 3$.

Lemma 1. *Let $N(n,\alpha)$ (resp. $N_{\scriptscriptstyle NAE}(n,\alpha)$) be the number of satisfying (resp. NAE satisfying) assignments for a randomly generated regular k-SAT formula. Then*[1]*,*

$$E(N(n,\alpha)) = 2^n \frac{\left(\left(\frac{k\alpha n}{2}\right)!\right)^2}{(k\alpha n)!} \mathrm{coef}\left(\left(\frac{p(x)}{x}\right)^{\alpha n}, x^{\frac{k\alpha n}{2} - \alpha n}\right), \tag{2}$$

$$E(N_{\scriptscriptstyle NAE}(n,\alpha)) = 2^n \frac{\left(\left(\frac{k\alpha n}{2}\right)!\right)^2}{(k\alpha n)!} \mathrm{coef}\left(\left(\frac{p_{\scriptscriptstyle NAE}(x)}{x}\right)^{\alpha n}, x^{\frac{k\alpha n}{2} - \alpha n}\right), \tag{3}$$

where

$$p(x) = (1+x)^k - 1, \quad p_{\scriptscriptstyle NAE}(x) = (1+x)^k - 1 - x^k, \tag{4}$$

and $\mathrm{coef}\left(p(x)^{\alpha n}, x^{\frac{k\alpha n}{2}}\right)$ *denotes the coefficient of* $x^{\frac{k\alpha n}{2}}$ *in the expansion of* $p(x)^{\alpha n}$*.*

Proof. Due to symmetry of the formula generation, any assignment of variables has the same probability of being a solution. This implies

$$E(N(n,\alpha)) = 2^n \mathrm{P}\left(X = \{0,\ldots,0\} \text{ is a solution}\right).$$

The probability of the all-zero vector being a solution is given by

$$\mathrm{P}\left(X = \{0,\ldots,0\} \text{ is a solution}\right) = \frac{\text{Number of formulas for which } X = \{0,\ldots,0\} \text{ is a solution}}{\text{Total number of formulas}}.$$

The total number of formulas is given by $(k\alpha n)!$. The total number of formulas for which the all-zero assignment is a solution is given by

$$\left(\left(\frac{k\alpha n}{2}\right)!\right)^2 \mathrm{coef}\left(p(x)^m, x^{\frac{k\alpha n}{2}}\right).$$

The factorial terms correspond to permuting the edges among true and false literals. Note that there are equal numbers of true and false literals. The generating function $p(x)$ corresponds to placing at least one positive literal in a clause. With these results and observing that

$$\mathrm{coef}\left(p(x)^{\alpha n}, x^{\frac{k\alpha n}{2}}\right) = \mathrm{coef}\left(\left(\frac{p(x)}{x}\right)^{\alpha n}, x^{\frac{k\alpha n}{2} - \alpha n}\right),$$

[1] We assume that $k\alpha n$ is an even integer.

we obtain (2). The derivation for $E(N_{\text{NAE}}(n,\alpha))$ is identical except that the generating function for clauses is given by $p_{\text{NAE}}(x)$. □

We now state the Hayman method to approximate the coef-term which is asymptotically correct [11].

Lemma 2 (Hayman Method). *Let $q(y) = \sum_i q_i y^i$ be a polynomial with non-negative coefficients such that $q_0 \neq 0$ and $q_1 \neq 0$. Define*

$$a_q(y) = y\frac{dq(y)}{dy}\frac{1}{q(y)}, \quad b_q(y) = y\frac{da_q(y)}{dy}. \tag{5}$$

Then,

$$\operatorname{coef}(q(y)^n, y^{\omega n}) = \frac{q(y_\omega)^n}{(y_\omega)^{\omega n}\sqrt{2\pi n b_q(y_\omega)}}(1+o(1)), \tag{6}$$

where y_ω is the unique positive solution of the saddle point equation $a_q(y) = \omega$.

We now use Lemma 2 to compute the expectation of the total number of solutions.

Lemma 3. *Let $N(n,\alpha)$ (resp. $N_{NAE}(n,\alpha)$) denote the total number of satisfying (resp. NAE satisfying) assignments of a regular random k-SAT formula. Let $q(x) = \frac{p(x)}{x}$, $q_{NAE}(x) = \frac{p_{NAE}}{x}$, where $p(x)$ and $p_{NAE}(x)$ is defined in (4). Then,*

$$E(N(n,\alpha)) = \sqrt{\frac{k}{4b_q(x_k)}}e^{n\left(\ln(2)-k\alpha\ln(2)+\alpha\ln(q(x_k))-\left(\frac{k\alpha}{2}-\alpha\right)\ln(x_k)\right)}(1+o(1)), \tag{7}$$

$$E(N_{NAE}(n,\alpha)) = \frac{\sqrt{k}e^{n\left(\ln(2)(1-k\alpha)+\alpha\ln\left(q_{NAE}(x_{k,NAE})\right)-\left(\frac{k\alpha}{2}-\alpha\right)\ln\left(x_{k,NAE}\right)\right)}}{\sqrt{4b_{q_{NAE}}(x_{k,NAE})}}(1+o(1)), \tag{8}$$

where x_k (resp. $x_{k,NAE}$) is the positive solution of $a_q(x) = \frac{k}{2}-1$ (resp. $a_{q_{NAE}}(x) = \frac{k}{2}-1$). The quantity $a_q(x)$, $a_{q_{NAE}}(x)$, $b_q(x)$, and $b_{q_{NAE}}(x)$ are defined according to (5).

In the following lemma we derive explicit upper bounds on the satisfiability and NAE satisfiability thresholds for $k \geq 3$.

Lemma 4 (Upper bound). *Let α^* (resp. α^*_{NAE}) be the satisfiability (resp. NAE satisfiability) threshold for the regular random k-SAT formulas. Define α^*_u (resp. $\alpha^*_{u,NAE}$) to be the upper bound on α^* (resp. α^*_{NAE}) obtained by the first moment method. Then,*

$$\alpha^* \leq \alpha^*_u \leq 2^k\ln(2)(1+o_k(1)), \quad \alpha^*_{NAE} \leq \alpha^*_{u,NAE} = 2^{k-1}\ln(2) - \frac{\ln(2)}{2} - o_k(1). \tag{9}$$

Proof. We observe that the solution x_k of the saddle point equation $a_q(x) = \frac{k}{2}-1$ satisfies: $x_k = \operatorname{argmin}_{x>0}\frac{q(x)}{x^{\frac{k}{2}-1}}$, where $a_q(x)$ is defined according to (5). This implies that we obtain the following upper bound on the growth rate of $E(N(n,\alpha))$ for any $x > 0$,

$$\lim_{n\to\infty}\frac{\ln\left(E(N(n,\alpha))\right)}{n} \leq \ln(2) - k\alpha\ln(2) + \alpha\ln(q(x)) - \left(\frac{k\alpha}{2}-\alpha\right)\ln(x). \tag{10}$$

We substitute $x = 1 - \frac{1}{2^k}$ in (10). Then we use the series expansion of $\ln(1-x)$, $1/i \geq 1/2^i$, and $-1/i \geq -1$ to obtain the following upper bound on the threshold,

$$\alpha^* \leq \frac{2^k \ln(2)}{\frac{1}{\left(1-\frac{1}{2^{k+1}}\right)^k} + \frac{k}{2^{k+4}} + \frac{1}{2^{k+2}\left(1-\frac{1}{2^{k+1}}\right)^{2k}} - \frac{1}{2^{k+1}}}. \tag{11}$$

The summation of the last three terms in the denominator of (11) is positive. This can be easily seen for $k \geq 8$. For $3 \leq k < 8$, it can be verified by explicit calculation. Dropping this summation in (11), we obtain the desired upper bound on the threshold. To derive the bound for NAE satisfiability, we note that $x_{k,\text{NAE}} = 1$ for $k \geq 3$. By substituting this in the exponent of $E(N_{\text{NAE}})$ and equating it to zero, we obtain the desired expression for $\alpha^*_{u,\text{NAE}}$. □

In the next section we use the second moment method to obtain lower bounds on the satisfiability and NAE satisfiability thresholds of regular random k-SAT.

3 Second Moment

A lower bound on the threshold can be obtained by the second moment method. The second moment method is governed by the following equation

$$\mathrm{P}(X > 0) \geq \frac{E(X)^2}{E(X^2)}. \tag{12}$$

In this section we compute the second moment of $N(n,\alpha)$ and $N_{\text{NAE}}(n,\alpha)$. Our computation of the second moment is inspired by the computation of the second moment for the weight and stopping set distributions of regular LDPC codes in [14, 15] (see also [4]). We compute the second moment in the next lemma.

Lemma 5. *Let $N(n,\alpha)$ be the number of satisfying solutions to a regular random k-SAT formula. Define the function $f(x_1,x_2,x_3)$ by*

$$f(x_1,x_2,x_3) = (1+x_1+x_2+x_3)^k - (1+x_1)^k - (1+x_3)^k + 1. \tag{13}$$

If the regular random formulas are strictly regular, then

$$E\left(N(n,\alpha)^2\right) = \sum_{i=0}^{n} 2^n \binom{n}{i} \frac{((\mathtt{r}(n-i))!)^2\,((\mathtt{r}i)!)^2 \operatorname{coef}\left(f(x_1,x_2,x_3)^{\alpha n}, x_1^{\mathtt{r}(n-i)} x_2^{\mathtt{r}i} x_3^{\mathtt{r}(n-i)}\right)}{(k\alpha n)!}. \tag{14}$$

If the regular random formulas are 2-regular, then

$$E\left(N(n,\alpha)^2\right) = \sum_{i_{\mathtt{r}}=0}^{n_{\mathtt{r}}} \sum_{i_{\mathtt{r}+1}=0}^{n_{\mathtt{r}+1}} 2^n \binom{n_{\mathtt{r}}}{i_{\mathtt{r}}} \binom{n_{\mathtt{r}+1}}{i_{\mathtt{r}+1}} \left(\left(\frac{k\alpha n}{2} - \mathtt{r}i_{\mathtt{r}} - (\mathtt{r}+1)i_{\mathtt{r}+1}\right)!\right)^2$$
$$\frac{((\mathtt{r}i_{\mathtt{r}} + (\mathtt{r}+1)i_{\mathtt{r}+1})!)^2}{(k\alpha n)!} \operatorname{coef}\left(f(x_1,x_2,x_3)^{\alpha n}, (x_1x_3)^{\frac{k\alpha n}{2} - \mathtt{r}i_{\mathtt{r}} - (\mathtt{r}+1)i_{\mathtt{r}+1}} x_2^{\mathtt{r}i_{\mathtt{r}} + (\mathtt{r}+1)i_{\mathtt{r}+1}}\right). \tag{15}$$

For both the strictly regular and the 2-regular case, the expression for $E\left(N_{\scriptscriptstyle NAE}(n,\alpha)^2\right)$ *is the same as that for* $E\left(N(n,\alpha)^2\right)$ *except replacing the generating function* $f(x_1,x_2,x_3)$ *by* $f_{\scriptscriptstyle NAE}(x_1,x_2,x_3)$*, which is given by*

$$f_{\scriptscriptstyle NAE}(x_1,x_2,x_3) = (1+x_1+x_2+x_3)^k - \\ \left((1+x_1)^k + (1+x_3)^k - 1 + (x_1+x_2)^k - x_1^k + (x_2+x_3)^k - x_2^k - x_3^k\right). \quad (16)$$

Proof. Let $\mathbb{1}_{XY}$ be the indicator variable which evaluates to 1 if the truth assignments X and Y satisfy a randomly regular k-SAT formula. Then,

$$E(N(n,\alpha)^2) = \sum_{X,Y\in\{0,1\}^n} E\left(\mathbb{1}_{\mathbf{XY}}\right) = 2^n \sum_{Y\in\{0,1\}^n} \mathrm{P}\left(\mathbf{0} \text{ and } Y \text{ are solutions}\right).$$

The last simplification uses the fact that the number of formulas which are satisfied by both X and Y depends only on the number of variables on which X and Y agree. Thus, we fix X to be the all-zero vector.

We now consider the strictly regular case. The probability that the all-zero truth assignment and the truth assignment Y both are solutions of a randomly chosen regular formula depends only on the *overlap*, i.e., the number of variables where the two truth assignments agree. Thus for a given overlap i, we can fix Y to be equal to zero in the first i variables and equal to 1 in the remaining variables. This gives,

$$E(N(n,\alpha)^2) = \sum_{i=0}^{n} 2^n \binom{n}{i} \mathrm{P}\left(\mathbf{0} \text{ and } Y \text{ are solutions}\right). \quad (17)$$

In order to evaluate the probability that both $\mathbf{0}$ and Y are solutions for a given overlap i, we observe that there are four different types of edges connecting the literals and the clauses. There are $\mathtt{r}(n-i)$ **type 1** edges which are connected to true literals w.r.t. the $\mathbf{0}$ truth assignment and false w.r.t. to the Y truth assignment. The $\mathtt{r}i$ **type 2** edges are connected to true literals w.r.t. both the truth assignments. There are $\mathtt{r}(n-i)$ **type 3** edges which are connected to false literals w.r.t. the $\mathbf{0}$ truth assignment and true literals w.r.t. to the Y truth assignment. The $\mathtt{r}i$ **type 4** edges are connected to false literals w.r.t. both the truth assignments. Let $f(x_1,x_2,x_3)$ be the generating function counting the number of possible edge connections to a clause, where the power of x_i gives the number of edges of type i, $i \in \{1,2,3\}$. A clause is satisfied if it is connected to at least one type 2 edge. Otherwise, it is satisfied if it is connected to at least one type 1 and at least one type 3 edge. Then the generating function $f(x_1,x_2,x_3)$ is given as in (13). Using this, we obtain

$$\mathrm{P}\left(\mathbf{0} \text{ and } Y \text{ are solutions}\right) = \\ \frac{\left((\mathtt{r}(n-i))!\right)^2 \left((\mathtt{r}i)!\right)^2 \operatorname{coef}\left(f(x_1,x_2,x_3)^{\alpha n}, x_1^{\mathtt{r}(n-i)} x_2^{\mathtt{r}i} x_3^{\mathtt{r}(n-i)}\right)}{(k\alpha n)!}, \quad (18)$$

where $(k\alpha n)!$ is the total number of formulas. Consider a given formula which is satisfied by both truth assignments $\mathbf{0}$ and Y. If we permute the positions of type 1 edges on the clause side, we obtain another formula having $\mathbf{0}$ and Y as solutions. The argument holds true for the type i edges, $i \in \{2,3,4\}$. This explains the term $(\mathtt{r}(n-i))!$ in (18) which corresponds to permuting the type 1 edges (it is squared because of the same contribution from type 3 edges). Similarly, $(\mathtt{r}i)!^2$ corresponds to permuting type 2 and type 4 edges. Combining (17) and (18), we obtain the desired expression for the second moment of the number of solutions as given in (14).

We now consider the two regular case. Note that in this case the equivalent equation corresponding to (17) is

$$E(N(n,\alpha)^2) = \sum_{i_{\mathtt{r}}=0}^{n_{\mathtt{r}}} \sum_{i_{\mathtt{r}+1}=0}^{n_{\mathtt{r}+1}} 2^n \binom{n_{\mathtt{r}}}{i_{\mathtt{r}}} \binom{n_{\mathtt{r}+1}}{i_{\mathtt{r}+1}} \mathrm{P}\left((\mathbf{0},Y) \text{ is a solution}\right), \tag{19}$$

where $i_{\mathtt{r}}$ (resp. $i_{\mathtt{r}+1}$) is the variable corresponding to the overlap between truth assignments $\mathbf{0}$ and Y among variables with degree $\mathtt{r}$ (resp. $\mathtt{r}+1$). Similarly, the equivalent of (18) is given by

$$\begin{aligned} \mathrm{P}\left(\mathbf{0} \text{ and } Y \text{ are solutions}\right) &= \left((\mathtt{r}(n_{\mathtt{r}}-i_{\mathtt{r}}) + (\mathtt{r}+1)(n_{\mathtt{r}+1}-i_{\mathtt{r}+1}))!\right)^2 \\ &\times \frac{\left((\mathtt{r}i_{\mathtt{r}} + (\mathtt{r}+1)i_{\mathtt{r}+1})!\right)^2}{(k\alpha n)!} \\ &\times \operatorname{coef}\left(f(x_1,x_2,x_3)^{\alpha n}, (x_1x_3)^{\mathtt{r}(n_{\mathtt{r}}-i_{\mathtt{r}})+(\mathtt{r}+1)(n_{\mathtt{r}+1}-i_{\mathtt{r}+1})} x_2^{\mathtt{r}i_{\mathtt{r}}+(\mathtt{r}+1)i_{\mathtt{r}+1}}\right). \end{aligned} \tag{20}$$

Combining (19) and (20), and observing that $\mathtt{r}n_{\mathtt{r}} + (\mathtt{r}+1)n_{\mathtt{r}+1} = \frac{k\alpha n}{2}$, we obtain (15). The derivation of $E\left(N_{\text{NAE}}(n,\alpha)^2\right)$ is identical except the generating function for NAE-satisfiability of a clause is different. This can be easily derived by observing that a clause is not NAE-satisfied for the following edge connections. Consider the case when a clause is connected to only one type of edge, then it is not NAE-satisfied. Next consider the case when a clause is connected to two types of edges. Then the combinations of type 1 and type 4, type 3 and type 4, type 1 and type 2, or type 2 and type 3 do not NAE-satisfies a clause. This gives the generating function $f_{\text{NAE}}(x_1,x_2,x_3)$ defined in (16). □

In order to evaluate the second moment, we now present the multidimensional saddle point method in the next lemma [5]. A detailed technical exposition of the multidimensional saddle point method can be found in Appendix D of [18].

Theorem 1. *Let* $\underline{i} := (i_1,i_2,i_3)$, $\underline{j} := (j_1,j_2,j_3)$, *and* $\underline{x} = (x_1,x_2,x_3)$

$$0 < \lim_{n\to\infty} i_1/n, \quad 0 < \lim_{n\to\infty} i_2/n, \quad 0 < \lim_{n\to\infty} i_3/n.$$

Let further $f(\underline{x})$ *be as defined in (13) and* $\underline{t} = (t_1,t_2,t_3)$ *be a positive solution of the saddle point equations* $a_f(\underline{x}) \triangleq \left\{x_i \frac{\partial \ln(f(x_1,x_2,x_3))}{\partial x_i}\right\}_{i=1}^{3} = \frac{\underline{i}}{\alpha n}$. *Then* $\operatorname{coef}\left(f(\underline{x})^{\alpha n}, \underline{x}^{\underline{i}}\right)$ *can be approximated as ,*

$$\operatorname{coef}\left(f(\underline{x})^{\alpha n}, \underline{x}^{\underline{i}}\right) = \frac{f(\underline{t})^{\alpha n}}{(\underline{t})^{\underline{i}} \sqrt{(2\pi\alpha n)^3 \left|B(\underline{t})\right|}} (1+o(1)),$$

using the saddle point method for multivariate polynomials, where $B(\underline{x})$ *is a* 3×3 *matrix whose elements are given by* $B_{i,j} = x_j \frac{\partial a_{fi}(x_1,x_2,x_3)}{\partial x_j} = B_{j,i}$ *and* $a_{fi}(\underline{x})$ *is the* i^{th} *coordinate* *of* $a_f(\underline{x})$. *Also,* $\text{coef}\left(f(\underline{x})^{\alpha n}, \underline{x}^{\underline{j}}\right)$ *can be approximated in terms of* $\text{coef}\left(f(\underline{x})^{\alpha n}, \underline{x}^{\underline{i}}\right)$. *This approximation is called the* local limit theorem *of* $\underline{j}$ *around* $\underline{i}$. *Explicitly, if* $\underline{u} := \frac{1}{\sqrt{\alpha n}}(\underline{j} - \underline{i})$ *and* $\|\underline{u}\| = O((\ln n)^{\frac{1}{3}})$, *then*

$$\text{coef}\left(f(\underline{x})^{\alpha n}, \underline{x}^{\underline{j}}\right) = \underline{t}^{\underline{i}-\underline{j}} \exp\left(-\frac{1}{2}\underline{u} \cdot B(\underline{t})^{-1} \cdot \underline{u}^T\right) \text{coef}\left(f(\underline{x})^{\alpha n}, \underline{x}^{\underline{i}}\right)(1+o(1)).$$

Because of the relative simplicity of the expression for the second moment, we explain its computation in detail for the strictly regular case. Then we will show how the arguments can be easily extended to the 2-regular case. The derivation for the NAE-satisfiability is identical for both cases.

Theorem 2. *Consider the strictly regular random k-SAT model with literal degree* $\mathtt{r}$. *Let* $S(i)$ *denote the* i^{th} *summation term in (14), and* $\gamma = i/n$. *If* $S(n/2)$ *is the dominant term i.e.,*

$$\lim_{n\to\infty} \frac{\ln\left(S\left(\frac{n}{2}\right)\right)}{n} > \lim_{n\to\infty} \frac{\ln\left(S(\gamma n)\right)}{n}, \quad \gamma \in [0,1], \gamma \neq \frac{1}{2}, \tag{21}$$

then with positive probability a randomly chosen formula has a satisfying assignment, i.e.

$$\lim_{n\to\infty} \mathrm{P}\left(N(\alpha, n) > 0\right) \geq \frac{2\sqrt{|B_f(x_k, x_k^2, x_k)|}}{\sigma_s b_q(x_k)\sqrt{k}}, \tag{22}$$

where x_k *is the solution of the saddle point equation* $a_q(x) = \frac{k}{2} - 1$ *defined in Lemma 3,* $a_q(x)$ *and* $b_q(x)$ *are defined according to (5),* $B_f(x_k, x_k^2, x_k)$ *is defined as in Theorem 1, and the "normalized variance"* σ_s^2 *of the summation term around* $S\left(\frac{n}{2}\right)$ *is given by*

$$\sigma_s^2 = \frac{1}{4 + \frac{k\mathtt{r}}{2}([-1,1,-1] \cdot B_f(x_k, x_k^2, x_K)^{-1} \cdot [-1,1,-1]^T) - 8\mathtt{r}}. \tag{23}$$

Let $\mathtt{r}^*$ *be the largest literal degree for which* $S(n/2)$ *is the dominant term, i.e. (21) holds, then the threshold* α^* *is lower bounded by* $\alpha^* \geq \alpha_l^* \triangleq \frac{2\mathtt{r}^*}{k}$.

Proof. From (14) and Theorem 1, the growth rate of $S(\gamma n)$ is given by,

$$s(\gamma) \triangleq \lim_{n\to\infty} \frac{\ln\left(S(\gamma n)\right)}{n} = \\ (1-k\alpha)(\ln(2) + h(\gamma)) + \alpha \ln\left(f(t_1, t_2, t_3)\right) - \mathtt{r}(1-\gamma)(\ln(t_1) + \ln(t_3)) - \mathtt{r}\gamma\ln(t_2), \tag{24}$$

where t_1, t_2, t_3 is a positive solution of the saddle point equations as defined in Theorem 1,

$$a_f(\underline{t}) \triangleq \left\{t_1 \frac{\partial \ln\left(f(t_1,t_2,t_3)\right)}{\partial t_1}, \quad t_2 \frac{\partial \ln\left(f(t_1,t_2,t_3)\right)}{\partial t_2}, \quad t_3 \frac{\partial \ln\left(f(t_1,t_2,t_3)\right)}{\partial t_3}\right\} = \\ \left\{\frac{k}{2}(1-\gamma), \frac{k}{2}\gamma, \frac{k}{2}(1-\gamma)\right\}. \tag{25}$$

In order to compute the maximum exponent of the summation terms, we compute its derivative and equate it to zero,

$$\frac{ds(\gamma)}{d\gamma} = (1-k\alpha)\ln\left(\frac{1-\gamma}{\gamma}\right) + \mathtt{r}\ln(t_1) - \mathtt{r}\ln(t_2) + \mathtt{r}\ln(t_3) = 0. \tag{26}$$

Note that the derivatives of t_1, t_2 and t_3 w.r.t. γ vanish as they satisfy the saddle point equation. Every positive solution (t_1, t_2, t_3) of (25) satisfies $t_1 = t_3$ as (25) and $f(t_1, t_2, t_3)$ are symmetric in t_1 and t_3. If $\gamma = 1/2$ is a maximum, then the vanishing derivative in (26) and equality of t_1 and t_3 imply $t_2 = t_1^2$. We substitute $\gamma = 1/2$, $t_1 = t_3$, and $t_2 = t_1^2$ in (25). This reduces (25) to the saddle point equation corresponding to the polynomial $q(x)$ defined in Lemma 3 whose solution is denote by x_k. Then by observing $f(x_k, x_k^2, x_k) = p(x_k)^2$, we have

$$S(n/2) = \frac{k^{3/2}}{2^{7/2}\sqrt{\pi n}\sqrt{|B_f(x_k, x_k^2, x_k)|}} e^{n(2\ln(2)(1-k\alpha)+2\alpha\ln(p(x_k))-k\alpha\ln(x_k))}(1+o(1)). \tag{27}$$

Using the relation that $q(x) = \frac{p(x)}{x}$, we note that the exponent of $S(n/2)$ is twice the exponent of the first moment of the total number of solutions as given in (7). In order to compute the sum over $S(\gamma n)$, we now use Laplace's method, a detailed discussion of which can be found in [12,7,8]. We want to approximate the term $S(n/2 + \Delta i)$ in terms of $S(n/2)$. For the coef terms, we make use of the local limit theorem given in Theorem 1 and for the factorial terms we make use of Stirling's approximation. This gives,

$$S(n/2 + \Delta i) = S(n/2)e^{-\frac{\Delta i^2}{2n\sigma_s^2}}(1+o(1)), \quad \text{where } \Delta i = O(n^{1/2} ln(n)^{1/3}). \tag{28}$$

Note that in the exponent on the R.H.S. of (28), the linear terms in Δi are absent as the derivative of the exponent vanishes at $\gamma = 1/2$. As the deviation around the term $S(n/2)$ is $\Theta(\sqrt{n})$ and the approximation is valid for $\Delta i = O(\sqrt{n}\ln(n)^{1/3})$, the dominant contribution comes from $-\Theta(\sqrt{n}) \leq \Delta i \leq \Theta(\sqrt{(n)})$. We are now ready to obtain the estimate for the second moment.

$$\begin{aligned} E(N^2(\alpha, n)) &\overset{(28)}{=} \sum_{\Delta i = -c\sqrt{n}}^{c\sqrt{n}} S(n/2)e^{-\frac{\Delta i^2}{2n\sigma_s^2}}(1+o(1)), & (29)\\ &= S(n/2)\int_{\delta=-\infty}^{\infty} e^{-\frac{\delta^2}{2n\sigma_s^2}} d\delta(1+o(1)). & (30)\\ &= S(n/2)\sqrt{2\pi n\sigma_s^2}(1+o(1)). & (31) \end{aligned}$$

We can replace the sum by an integral by choosing sufficiently large c. Using the second moment method given in (12) and combining Lemma 3, (27), and (31), we obtain

$$\mathrm{P}(N(\alpha, n) > 0) \geq \frac{E(N(\alpha, n)^2)}{E(N(\alpha, n)^2)} = \frac{2\sqrt{|B_f(x_k, x_k^2, x_k)|}}{\sigma_s b_q(x_k)\sqrt{k}}(1+o(1)). \tag{32}$$

Letting n go to infinity, we obtain (22). Clearly, if the supremum of the growth rate of $S(\gamma n)$ is not achieved at $\gamma = 1/2$, then the lower bound given by the second moment method converges to zero. This gives the desired lower bound on the threshold. $\square$

We can easily extend this result to the 2-regular case. In the following theorem we accomplish this task. Due to space limitation, we omit explanation of some steps which can be found in [17].

Theorem 3. *Consider the 2-regular random k-SAT model where the number of variables with degree* $\mathtt{r}$ *(resp.* $\mathtt{r}+1$*) is* $n_{\mathtt{r}} = \Lambda_{\mathtt{r}} n$ *(resp.* $n_{\mathtt{r}+1} = \Lambda_{\mathtt{r}+1} n$*). Let* $S(i_{\mathtt{r}}, i_{\mathtt{r}+1}) \triangleq S(\gamma_{\mathtt{r}} n_{\mathtt{r}}, \gamma_{\mathtt{r}+1} n_{\mathtt{r}+1})$ *be the summation term on the R.H.S. of (15) corresponding to overlap* $i_{\mathtt{r}}$*(resp.* $i_{\mathtt{r}+1}$*) on the degree* $\mathtt{r}$*(resp.* $\mathtt{r}+1$*) literals. Let* $g(\gamma_{\mathtt{r}}, \gamma_{\mathtt{r}+1})$ *be the growth rate of* $S(\gamma_{\mathtt{r}} n_{\mathtt{r}}, \gamma_{\mathtt{r}+1} n_{\mathtt{r}+1})$ *i.e.* $g(\gamma_{\mathtt{r}}, \gamma_{\mathtt{r}+1}) \triangleq \lim_{n\to\infty} \frac{\ln(S(n_{\mathtt{r}}\gamma_{\mathtt{r}}, n_{\mathtt{r}+1}))}{n}$*. If*

$$g\left(\frac{1}{2},\frac{1}{2}\right) > g(\gamma_{\mathtt{r}}, \gamma_{\mathtt{r}+1}), \gamma_{\mathtt{r}} \in [0,1], \gamma_{\mathtt{r}+1} \in [0,1], \gamma_{\mathtt{r}} \neq \frac{1}{2}, \gamma_{\mathtt{r}+1} \neq \frac{1}{2},$$

then with positive probability a randomly chosen formula has a solution. More precisely,

$$\lim_{n\to\infty} \mathrm{P}\left(N(\alpha,n) > 0\right) \geq \frac{\sqrt{|B_f(x_k, x_k^2, x_k)|\Lambda_{\mathtt{r}}\Lambda_{\mathtt{r}+1}}}{b_q(x_k)\sqrt{k|\Sigma|}}. \tag{33}$$

The definition of x_k*,* $b_q(x_k)$*, and* $B_f(x_k, x_k^2, x_k)$ *is same as in the Theorem 2. The* 2×2 *matrix* Σ *is defined via,*

$$C_f = [-1,1,-1].(B_f(x_k,x_k^2,x_k))^{-1}.[-1,1,-1]^T, \quad A = \frac{4}{\Lambda_{\mathtt{r}}} + 2\mathtt{r}^2\left(\frac{C_f}{2\alpha} - \frac{4}{k\alpha}\right),$$
$$B = \frac{4}{\Lambda_{\mathtt{r}+1}} + \frac{2(\mathtt{r}+1)^2}{\alpha}\left(\frac{C_f}{2} - \frac{4}{k}\right), C = \frac{2\mathtt{r}(\mathtt{r}+1)}{\alpha}\left(\frac{C_f}{2} - \frac{4}{k}\right), \textit{then } \Sigma = \begin{bmatrix} A & B \\ B & C \end{bmatrix}^{-1}. \tag{34}$$

The threshold α^* *is lower bounded by* α_l^**, where* α_l^* *is defined by*

$$\alpha_l^* = \sup\left\{\alpha : g\left(\frac{1}{2},\frac{1}{2}\right) > g(\gamma_{\mathtt{r}}, \gamma_{\mathtt{r}+1}), \gamma_{\mathtt{r}} \in [0,1], \gamma_{\mathtt{r}+1} \in [0,1], \gamma_{\mathtt{r}} \neq \frac{1}{2}, \gamma_{\mathtt{r}+1} \neq \frac{1}{2}\right\}.$$

Proof. Define $\Gamma(\gamma_{\mathtt{r}}, \gamma_{\mathtt{r}+1}) = \mathtt{r}\Lambda_{\mathtt{r}}\gamma_{\mathtt{r}} + (\mathtt{r}+1)\Lambda_{\mathtt{r}+1}\gamma_{\mathtt{r}+1}$. Then by using Theorem 1 and Stirling's approximation, we obtain

$$\begin{aligned} g(\gamma_{\mathtt{r}}, \gamma_{\mathtt{r}+1}) = \ln(2) &+ \Lambda_{\mathtt{r}} h(\gamma_{\mathtt{r}}) + \Lambda_{\mathtt{r}+1} h(\gamma_{\mathtt{r}+1}) \\ &+ (k\alpha - 2\Gamma(\gamma_{\mathtt{r}}, \gamma_{\mathtt{r}+1}))\ln\left(\frac{k\alpha}{2} - \Gamma(\gamma_{\mathtt{r}}, \gamma_{\mathtt{r}+1})\right) \\ &+ 2\left(\Gamma(\gamma_{\mathtt{r}}, \gamma_{\mathtt{r}+1})\right)\ln\left(\Gamma(\gamma_{\mathtt{r}}, \gamma_{\mathtt{r}+1})\right) - k\alpha\ln(k\alpha) \\ + \alpha\ln\left(f(\underline{(t)})\right) &- \left(\frac{k\alpha}{2} - \Gamma(\gamma_{\mathtt{r}}, \gamma_{\mathtt{r}+1})\right)\ln(t_1 t_3) - \left(\Gamma(\gamma_{\mathtt{r}}, \gamma_{\mathtt{r}+1})\right)\ln(t_2), \end{aligned} \tag{35}$$

where $\underline{t} = \{t_1, t_2, t_3\}$ is a positive solution of the saddle point point equations as given in Theorem 1,

$$a_f(\underline{t}) = \left\{ \frac{k}{2} - \frac{\Gamma(\gamma_{\mathtt{r}}, \gamma_{\mathtt{r}+1})}{\alpha}, \frac{\Gamma(\gamma_{\mathtt{r}}, \gamma_{\mathtt{r}+1})}{\alpha}, \frac{k}{2} - \frac{\Gamma(\gamma_{\mathtt{r}}, \gamma_{\mathtt{r}+1})}{\alpha} \right\}, \tag{36}$$

corresponding to the coefficient term of power of $f(x_1, x_2, x_3)$. In order to obtain the maximum exponent, we take the partial derivatives of $g(\gamma_{\mathtt{r}}, \gamma_{\mathtt{r}+1})$ with respect to $\gamma_{\mathtt{r}}$ and $\gamma_{\mathtt{r}+1}$ and equate them to zero. This gives the following equations.

$$\ln\left(\frac{1-\gamma_{\mathtt{r}}}{\gamma_{\mathtt{r}}}\right) - 2\mathtt{r}\ln\left(\frac{k\alpha}{2} - \Gamma(\gamma_{\mathtt{r}}, \gamma_{\mathtt{r}+1})\right) + 2\mathtt{r}\ln\left(\Gamma(\gamma_{\mathtt{r}}, \gamma_{\mathtt{r}+1})\right) + \mathtt{r}\ln\left(\frac{t_1 t_3}{t_2}\right) = 0,$$
$$\ln\left(\frac{1-\gamma_{\mathtt{r}+1}}{\gamma_{\mathtt{r}+1}}\right) - 2(\mathtt{r}+1)\ln\left(\frac{k\alpha}{2} - \Gamma(\gamma_{\mathtt{r}}, \gamma_{\mathtt{r}+1})\right)$$
$$+ 2(\mathtt{r}+1)\ln(\Gamma(\gamma_{\mathtt{r}}, \gamma_{\mathtt{r}+1}) + (\mathtt{r}+1)\ln\left(\frac{t_1 t_3}{t_2}\right) = 0. \tag{37}$$

Note that $t_1 = t_3 = x_k$, $t_2 = x_k^2$, $\gamma_{\mathtt{r}} = \frac{1}{2}$, and $\gamma_{\mathtt{r}+1} = \frac{1}{2}$ is a solution of (36), (37), which corresponds to $S\left(\frac{n_{\mathtt{r}}}{2}, \frac{n_{\mathtt{r}+1}}{2}\right)$, where x_k is the solution of the saddle point equation corresponding to $q(x)$ defined in Lemma 3. We recall that for the second moment method to work, the maximum exponent should be equal to twice the exponent of the average number of solutions. Indeed by the proposed solution, the term $S\left(\frac{n_{\mathtt{r}}}{2}, \frac{n_{\mathtt{r}+1}}{2}\right)$ has an exponent which is twice that of the average number of solutions. If this is also the maximum, then we have the desired result. Assuming that $S\left(\frac{n_{\mathtt{r}}}{2}, \frac{n_{\mathtt{r}+1}}{2}\right)$ has the maximum exponent, we now compute the second moment of the total number of solutions. By using Stirling's approximation and the local limit result of Theorem 1, we obtain

$$\frac{S(i_{\mathtt{r}} + \Delta i_{\mathtt{r}}, i_{\mathtt{r}+1} + \Delta i_{\mathtt{r}+1})}{S(i_{\mathtt{r}}, i_{\mathtt{r}+1})} = e^{-\frac{1}{2n}[\Delta i_{\mathtt{r}}, \Delta i_{\mathtt{r}+1}]\cdot\Sigma^{-1}\cdot[\Delta i_{\mathtt{r}}, \Delta i_{\mathtt{r}+1}]^T}(1+o(1)), \tag{38}$$

where the matrix Σ is defined in (34). Using the same series of arguments as in Theorem 2, we obtain

$$E\left(N(\alpha,n)^2\right) = \sum_{\Delta i_{\mathtt{r}}, \Delta i_{\mathtt{r}+1}} S\left(\frac{n_{\mathtt{r}}}{2}, \frac{n_{\mathtt{r}+1}}{2}\right) e^{-\frac{1}{2n}[\Delta i_{\mathtt{r}}, \Delta i_{\mathtt{r}+1}]\cdot\Sigma^{-1}\cdot[\Delta i_{\mathtt{r}}, \Delta i_{\mathtt{r}+1}]^T}(1+o(1)), \tag{39}$$
$$= S\left(\frac{n_{\mathtt{r}}}{2}, \frac{n_{\mathtt{r}+1}}{2}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2n}[x_{\mathtt{r}}, x_{\mathtt{r}+1}]\cdot\Sigma^{-1}\cdot[x_{\mathtt{r}}, x_{\mathtt{r}+1}]^T} dx_{\mathtt{r}} dx_{\mathtt{r}+1}(1+o(1)), \tag{40}$$
$$= \frac{\sqrt{|\Sigma|}k^{\frac{3}{2}}}{4\sqrt{|B_f(x_k, x_k^2, x_k)|\Lambda_{\mathtt{r}}\Lambda_{\mathtt{r}+1}}} e^{2n\left(\ln(2) - k\alpha\ln(2) + \alpha\ln(p(x_k)) - \frac{k\alpha}{2}\ln(x_k)\right)}(1+o(1)). \tag{41}$$

By using the second moment method, we obtain the bound given in (22). Note that the second moment method fails if the term $S\left(\frac{n_{\mathtt{r}}}{2}, \frac{n_{\mathtt{r}+1}}{2}\right)$ is not the dominant term. This gives the lower bound α_l^* on α^*. □

In the next section we discuss the obtained lower and upper bounds on the satisfiability threshold and NAE-satisfiability threshold.

4 Bounds on Threshold

In Table 1, lower bounds and upper bounds for the satisfiability threshold are given. The upper bound is computed by the first moment method. As expected, we obtain the same upper bound for regular random 3-SAT as given in [6]. The lower bound is derived by the second moment method for strictly regular random k-SAT. In order to apply the second moment method, we have to verify that $s(\gamma)$, defined in (24), attains its maximum at $\gamma = \frac{1}{2}$ over the unit interval. This requires that $\gamma = \frac{1}{2}$ is a positive solution of the system of equations consisting of (25) and (26) and it corresponds to a global maximum over $\gamma \in [0,1]$. Also, σ_s^2 defined in (23) should be positive. The system of equations (25), (26) is equivalent to a system of polynomial equations. For small value of k, we can solve this system of polynomial equations and verify the desired conditions. In Table 1 this has been done for $k = 3,4$. The obtained lower bound for 3-SAT is 2.667 which is an improvement over the algorithmic lower bound 2.46 given in [6]. For larger values of k, the degree of monomials in (26) grows exponentially in k. Thus, solving (25) and (26) becomes computationally difficult. However, $s(\gamma)$ can be easily computed as its computation requires solving only (25), where the maximum monomial degree is only k. Thus, the desired condition for maximum of $s(\gamma)$ at $\gamma = \frac{1}{2}$ can be verified numerically in an efficient manner.

Note that the difference between the lower bound obtained by applying the second moment method to the strictly regular case can differ by at most $2/k$ from the corresponding lower bound for the 2-regular case. We observe that as k increases the lower bound seems to converge to $2^k \ln(2) - (k+1)\frac{\ln(2)}{2} - 1$, which is the lower bound for the uniform model.

We observe similar behavior for the NAE-satisfiability bounds. As expected, we observe that the upper bound on the NAE-satisfiability threshold for the regular random model converges to $2^{k-1}\ln(2) - \frac{\ln(2)}{2}$. The lower bound obtained by applying the second moment method to the regular random model seems to converge to the value obtained for the uniform model. Thus, the NAE-threshold for the regular random model is $2^{k-1}\ln(2) - O(1)$. This suggests that as k increases, the threshold of the regular model does not differ much from the uniform model.

Table 1. Bounds on the satisfiability threshold for strictly regular random k-SAT. α_l^* and $\mathtt{r}^*$ are defined in Theorem 2. The upper bound α_u^* is obtained by the first moment method. $\alpha_{l,\text{uni}}^* = 2^k \ln(2) - (k+1)\frac{\ln(2)}{2} - 1$ is lower bound for uniform model obtained in [3]. The quantities $\mathtt{r}_{\text{NAE}}^*, \alpha_{l,\text{NAE}}, \alpha_{u,\text{NAE}}$ are analogously defined for the NAE-satisfiability.

k	$\mathtt{r}^*$	α_l^*	α_u^*	$\alpha_{l,\text{uni}}^* - \alpha_l^*$	$\mathtt{r}_{\text{NAE}}^*$	$\alpha_{l,\text{NAE}}^*$	$\alpha_{u,\text{NAE}}^*$
3	4	2.667	3.78222	0.492216	3	2	2.40942
4	16	8	9.10776	0.357487	8	4	5.19089
7	296	84.571	85.8791	0.378822	152	43.4286	44.0139
10	3524	704.8	705.9533	0.170403	1770	354	354.545
15	170298	22706.4	22707.5	0.101635	85167	11355.6	11356.2
17	772182	90844.94	90845.9	0.007749	386114	45425.2	45425.7

Our immediate future work is to derive explicit lower bounds for the regular random k-SAT model for large values of k as was done for the uniform model in [3, 1]. The challenge is that the function $s(\gamma)$ depends on the solution of the system of polynomial equations given in (25). Thus determining the maximum requires determining the behavior of the positive solution of this system of polynomial equations. Another interesting direction is the maximum satisfiability of regular random formulas. For the uniform model, the maximum satisfiability problem was addressed in [2] using the second moment method. In [16], authors have derived lower and upper bounds on the maximum satisfiability threshold of regular random formulas.

References

1. AChlioptas, D., Moore, C.: Random k-SAT: Two moments suffice to cross a sharp threshold. SIAM J. COMPUT. 36, 740–762 (2006)
2. Achlioptas, D., Naor, A., Peres, Y.: On the maximum satisfiability of random formulas. Journal of the Association of Computing Machinary (JACM) 54 (2007)
3. AChlioptas, D., Peres, Y.: The threshold for random k-SAT is $2^k \ln(2) - O(k)$. Journal of the American Mathematical Society 17, 947–973 (2004)
4. Barak, O., Burshtein, D.: Lower bounds on the spectrum and error rate of LDPC code ensembles. In: International Symposium on Information Theory, Adelaide, Australia (2002)
5. Bender, E.A., Richmond, L.B.: Central and local limit theorems applied to asymptotic enumeration II: Multivariate generating functions. J. Combin. Theory, Ser. A 34, 255–265 (1983)
6. Boufkhad, Y., Dubois, O., Interian, Y., Selman, B.: Regular random k-SAT: Properties of balanced formulas. Journal of Automated Reasoning (2005)
7. Bruijn, N.G.D.: Asymptotic Methods in Analysis. North Holland, Amsterdam (1981)
8. Flajolet, P., Sedgewick, R.: Analytic Combinatorics. Cambridge University Press, Cambridge (2009)
9. Franco, J., Paull, M.: Probabilistic analysis of the Davis Putnam procedure for solving the satisfiability problem. Discrete Appl. Math. 5, 77–87 (1983)
10. Friedgut, E.: Sharp threshold for graph properties, and the k-SAT problem. Journal of the American Mathematical Society 17, 947–973 (2004)
11. Gardy, D.: Some results on the asymptotic behavior of coefficients of large powers of functions. Discrete Mathematics 139, 189–217 (1995)
12. Henrici, P.: Applied and Computation Complex Analysis, vol. 2. John Wiley, Chichester (1974)
13. Kaporis, A.C., Kirousis, L.M., Lalas, E.G.: The probabilistic analysis of a greedy satisfying algorithm. In: 10th Annual European Symposium on Algorithms, vol. Ser. A 34 (2002)
14. Rathi, V.: On the asymptotic weight and stopping set distributions of regular LDPC ensembles. IEEE Trans. Inform. Theory 52, 4212–4218 (2006)
15. Rathi, V.: Non-binary LDPC codes and EXIT like functions, PhD thesis, Swiss Federal Institute of Technology (EPFL), Lausanne (2008)
16. Rathi, V., Aurell, E., Rasmussen, L., Skoglund, M.: Bounds on maximum satisfiability threshold of regular random k-SAT. arXiv:1004.2425 (submitted)
17. Rathi, V., Aurell, E., Rasmussen, L., Skoglund, M.: Satisfiability and maximum satisfiability of regular random k-sat: Bounds on thresholds (in preparation for journal submission)
18. Richardson, T., Urbanke, R.: Modern Coding Theory. Cambridge University Press, Cambridge (2008)