Chapter 1

Uncertainty Theory

Some information and knowledge are usually represented by human language like "about 100km", "approximately 39 °C", "roughly 80kg", "low speed", "middle age", and "big size". How do we understand them? Perhaps some people think that they are subjective probability or they are fuzzy concepts. However, a lot of surveys showed that those imprecise quantities behave neither like randomness nor like fuzziness. This fact provides a motivation to invent another mathematical tool, namely uncertainty theory.

Uncertainty theory was founded by Liu [120] in 2007. Nowadays uncertainty theory has become a branch of mathematics based on normality, monotonicity, self-duality, countable subadditivity, and product measure axioms. The first fundamental concept in uncertainty theory is uncertain measure that is used to measure the belief degree of an uncertain event. The second one is uncertain variable that is used to represent imprecise quantities. The third one is uncertainty distribution that is used to describe uncertain variables in an incomplete but easy-to-use way. Uncertainty theory is thus deduced from those three foundation stones, and provides a mathematical model to deal with uncertain phenomena.

The emphasis in this chapter is mainly on uncertain measure, uncertain variable, uncertainty distribution, independence, operational law, expected value, variance, moments, critical values, entropy, distance, convergence almost surely, convergence in measure, convergence in mean, convergence in distribution, and conditional uncertainty.

1.1 Uncertain Measure

Let Γ be a nonempty set. A collection $\mathcal{L}$ of subsets of Γ is called a σ-algebra if (a) $\Gamma \in \mathcal{L}$; (b) if $\Lambda \in \mathcal{L}$, then $\Lambda^c \in \mathcal{L}$; and (c) if $\Lambda_1, \Lambda_2, \cdots \in \mathcal{L}$, then $\Lambda_1 \cup \Lambda_2 \cup \cdots \in \mathcal{L}$. Each element Λ in the σ-algebra $\mathcal{L}$ is called an event. Uncertain measure is a function from $\mathcal{L}$ to $[0,1]$. In order to present an axiomatic definition of uncertain measure, it is necessary to assign to each event Λ a number $\mathcal{M}\{\Lambda\}$ which indicates the belief degree that Λ will occur. In order to ensure that the number $\mathcal{M}\{\Lambda\}$ has certain mathematical properties, Liu [120] proposed the following four axioms:

Axiom 1. *(Normality Axiom)* $\mathcal{M}\{\Gamma\} = 1$ *for the universal set* Γ.

Axiom 2. *(Monotonicity Axiom)* $\mathcal{M}\{\Lambda_1\} \le \mathcal{M}\{\Lambda_2\}$ *whenever* $\Lambda_1 \subset \Lambda_2$.

B. Liu: Uncertainty Theory: A Branch of Mathematics, SCI 300, pp. 1–79.
springerlink.com

Axiom 3. *(Self-Duality Axiom)* $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ *for any event* Λ.

Axiom 4. *(Countable Subadditivity Axiom) For every countable sequence of events* $\{\Lambda_i\}$, *we have*

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}. \tag{1.1}$$

Remark 1.1: The law of contradiction tells us that a proposition cannot be both true and false at the same time, and the law of excluded middle tells us that a proposition is either true or false. The law of truth conservation is a generalization of the law of contradiction and the law of excluded middle, and says that the sum of truth values of a proposition and its negative proposition is identical to 1. Self-duality is in fact an application of the law of truth conservation in uncertainty theory. This is the main reason why self-duality axiom is assumed.

Remark 1.2: Pathology occurs if subadditivity is not assumed. For example, suppose that a universal set contains 3 elements. We define a set function that takes value 0 for each singleton, and 1 for each set with at least 2 elements. Then such a set function satisfies all axioms but subadditivity. Is it not strange if such a set function serves as a measure?

Remark 1.3: Pathology occurs if countable subadditivity axiom is replaced with finite subadditivity axiom. For example, assume the universal set consists of all real numbers. We define a set function that takes value 0 if the set is bounded, 0.5 if both the set and complement are unbounded, and 1 if the complement of the set is bounded. Then such a set function is finitely subadditive but not countably subadditive. Is it not strange if such a set function serves as a measure? This is the main reason why we accept the countable subadditivity axiom.

Remark 1.4: Although probability measure satisfies the above four axioms, probability theory is not a special case of uncertainty theory because the product probability measure does not satisfy the fifth axiom, namely product measure axiom on Page 7.

Definition 1.1 *(Liu [120]). The set function* $\mathcal{M}$ *is called an uncertain measure if it satisfies the normality, monotonicity, self-duality, and countable subadditivity axioms.*

Example 1.1: Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$. For this case, there are only 8 events. Define

$$\mathcal{M}\{\gamma_1\} = 0.6, \quad \mathcal{M}\{\gamma_2\} = 0.3, \quad \mathcal{M}\{\gamma_3\} = 0.2,$$

$$\mathcal{M}\{\gamma_1, \gamma_2\} = 0.8, \quad \mathcal{M}\{\gamma_1, \gamma_3\} = 0.7, \quad \mathcal{M}\{\gamma_2, \gamma_3\} = 0.4,$$

$$\mathcal{M}\{\emptyset\} = 0, \quad \mathcal{M}\{\Gamma\} = 1.$$

Then $\mathcal{M}$ is an uncertain measure because it satisfies the four axioms.

Example 1.2: Suppose that $\lambda(x)$ is a nonnegative function on $\Re$ satisfying

$$\sup_{x \neq y} (\lambda(x) + \lambda(y)) = 1. \tag{1.2}$$

Then for any set Λ of real numbers, the set function

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{x \in \Lambda} \lambda(x), & \text{if } \sup\limits_{x \in \Lambda} \lambda(x) < 0.5 \\ 1 - \sup\limits_{x \in \Lambda^c} \lambda(x), & \text{if } \sup\limits_{x \in \Lambda} \lambda(x) \geq 0.5 \end{cases} \tag{1.3}$$

is an uncertain measure on $\Re$.

Example 1.3: Suppose $\rho(x)$ is a nonnegative and integrable function on $\Re$ such that

$$\int_{\Re} \rho(x) \mathrm{d}x \geq 1. \tag{1.4}$$

Then for any Borel set Λ of real numbers, the set function

$$\mathcal{M}\{\Lambda\} = \begin{cases} \displaystyle\int_{\Lambda} \rho(x)\mathrm{d}x, & \text{if } \displaystyle\int_{\Lambda} \rho(x)\mathrm{d}x < 0.5 \\ 1 - \displaystyle\int_{\Lambda^c} \rho(x)\mathrm{d}x, & \text{if } \displaystyle\int_{\Lambda^c} \rho(x)\mathrm{d}x < 0.5 \\ 0.5, & \text{otherwise} \end{cases} \tag{1.5}$$

is an uncertain measure on $\Re$.

Example 1.4: Suppose $\lambda(x)$ is a nonnegative function and $\rho(x)$ is a nonnegative and integrable function on $\Re$ such that

$$\sup_{x \in \Lambda} \lambda(x) + \int_{\Lambda} \rho(x)\mathrm{d}x \geq 0.5 \quad \text{and/or} \quad \sup_{x \in \Lambda^c} \lambda(x) + \int_{\Lambda^c} \rho(x)\mathrm{d}x \geq 0.5 \tag{1.6}$$

for any Borel set Λ of real numbers. Then the set function

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{x \in \Lambda} \lambda(x) + \displaystyle\int_{\Lambda} \rho(x)\mathrm{d}x, & \text{if } \sup\limits_{x \in \Lambda} \lambda(x) + \displaystyle\int_{\Lambda} \rho(x)\mathrm{d}x < 0.5 \\ 1 - \sup\limits_{x \in \Lambda^c} \lambda(x) - \displaystyle\int_{\Lambda^c} \rho(x)\mathrm{d}x, & \text{if } \sup\limits_{x \in \Lambda^c} \lambda(x) + \displaystyle\int_{\Lambda^c} \rho(x)\mathrm{d}x < 0.5 \\ 0.5, & \text{otherwise} \end{cases}$$

is an uncertain measure on $\Re$.

Theorem 1.1. *Suppose that* $\mathcal{M}$ *is an uncertain measure. Then the empty set* $\emptyset$ *has an uncertain measure zero, i.e.,*

$$\mathcal{M}\{\emptyset\} = 0. \tag{1.7}$$

Proof: It follows from the normality that $\mathcal{M}\{\Gamma\} = 1$. Since $\emptyset = \Gamma^c$, the self-duality axioms yields $\mathcal{M}\{\emptyset\} = 1 - \mathcal{M}\{\Gamma\} = 1 - 1 = 0$.

Theorem 1.2. *Suppose that* $\mathcal{M}$ *is an uncertain measure. Then we have*

$$0 \le \mathcal{M}\{\Lambda\} \le 1 \tag{1.8}$$

for any event Λ*.*

Proof: It follows from the monotonicity axiom that $0 \le \mathcal{M}\{\Lambda\} \le 1$ because $\emptyset \subset \Lambda \subset \Gamma$ and $\mathcal{M}\{\emptyset\} = 0$, $\mathcal{M}\{\Gamma\} = 1$.

Theorem 1.3. *Suppose that* $\mathcal{M}$ *is an uncertain measure. Then for any events* Λ_1 *and* Λ_2*, we have*

$$\mathcal{M}\{\Lambda_1\} \vee \mathcal{M}\{\Lambda_2\} \le \mathcal{M}\{\Lambda_1 \cup \Lambda_2\} \le \mathcal{M}\{\Lambda_1\} + \mathcal{M}\{\Lambda_2\}. \tag{1.9}$$

Proof: The left-hand inequality follows from the monotonicity axiom and the right-hand inequality follows from the countable subadditivity axiom immediately.

Theorem 1.4. *Suppose that* $\mathcal{M}$ *is an uncertain measure. Then for any events* Λ_1 *and* Λ_2*, we have*

$$\mathcal{M}\{\Lambda_1\} + \mathcal{M}\{\Lambda_2\} - 1 \le \mathcal{M}\{\Lambda_1 \cap \Lambda_2\} \le \mathcal{M}\{\Lambda_1\} \wedge \mathcal{M}\{\Lambda_2\}. \tag{1.10}$$

Proof: The right-hand inequality follows from the monotonicity axiom and the left-hand inequality follows from the self-duality and countable subadditivity axioms, i.e.,

$$\begin{aligned}
\mathcal{M}\{\Lambda_1 \cap \Lambda_2\} &= 1 - \mathcal{M}\{(\Lambda_1 \cap \Lambda_2)^c\} = 1 - \mathcal{M}\{\Lambda_1^c \cup \Lambda_2^c\} \\
&\ge 1 - (\mathcal{M}\{\Lambda_1^c\} + \mathcal{M}\{\Lambda_2^c\}) \\
&= 1 - (1 - \mathcal{M}\{\Lambda_1\}) - (1 - \mathcal{M}\{\Lambda_2\}) \\
&= \mathcal{M}\{\Lambda_1\} + \mathcal{M}\{\Lambda_2\} - 1.
\end{aligned}$$

The inequalities are verified.

Null-Additivity Theorem

Null-additivity is a direct deduction from subadditivity. We first prove a more general theorem.

Theorem 1.5. *Let $\{\Lambda_i\}$ be a sequence of events with $\mathcal{M}\{\Lambda_i\} \to 0$ as $i \to \infty$. Then for any event Λ, we have*

$$\lim_{i\to\infty} \mathcal{M}\{\Lambda \cup \Lambda_i\} = \lim_{i\to\infty} \mathcal{M}\{\Lambda \backslash \Lambda_i\} = \mathcal{M}\{\Lambda\}. \tag{1.11}$$

Proof: It follows from the monotonicity and countable subadditivity axioms that

$$\mathcal{M}\{\Lambda\} \le \mathcal{M}\{\Lambda \cup \Lambda_i\} \le \mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda_i\}$$

for each i. Thus we get $\mathcal{M}\{\Lambda \cup \Lambda_i\} \to \mathcal{M}\{\Lambda\}$ by using $\mathcal{M}\{\Lambda_i\} \to 0$. Since $(\Lambda \backslash \Lambda_i) \subset \Lambda \subset ((\Lambda \backslash \Lambda_i) \cup \Lambda_i)$, we have

$$\mathcal{M}\{\Lambda \backslash \Lambda_i\} \le \mathcal{M}\{\Lambda\} \le \mathcal{M}\{\Lambda \backslash \Lambda_i\} + \mathcal{M}\{\Lambda_i\}.$$

Hence $\mathcal{M}\{\Lambda \backslash \Lambda_i\} \to \mathcal{M}\{\Lambda\}$ by using $\mathcal{M}\{\Lambda_i\} \to 0$.

Remark 1.5: It follows from the above theorem that the uncertain measure is null-additive, i.e., $\mathcal{M}\{\Lambda_1 \cup \Lambda_2\} = \mathcal{M}\{\Lambda_1\} + \mathcal{M}\{\Lambda_2\}$ if either $\mathcal{M}\{\Lambda_1\} = 0$ or $\mathcal{M}\{\Lambda_2\} = 0$. In other words, the uncertain measure remains unchanged if the event is enlarged or reduced by an event with uncertain measure zero.

Asymptotic Theorem

Theorem 1.6 *(Asymptotic Theorem). For any events $\Lambda_1, \Lambda_2, \cdots$, we have*

$$\lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} > 0, \quad \text{if } \Lambda_i \uparrow \Gamma, \tag{1.12}$$

$$\lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} < 1, \quad \text{if } \Lambda_i \downarrow \emptyset. \tag{1.13}$$

Proof: Assume $\Lambda_i \uparrow \Gamma$. Since $\Gamma = \cup_i \Lambda_i$, it follows from the countable subadditivity axiom that

$$1 = \mathcal{M}\{\Gamma\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Since $\mathcal{M}\{\Lambda_i\}$ is increasing with respect to i, we have $\lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} > 0$. If $\Lambda_i \downarrow \emptyset$, then $\Lambda_i^c \uparrow \Gamma$. It follows from the first inequality and self-duality axiom that

$$\lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} = 1 - \lim_{i\to\infty} \mathcal{M}\{\Lambda_i^c\} < 1.$$

The theorem is proved.

Example 1.5: Assume Γ is the set of real numbers. Let α be a number with $0 < \alpha \le 0.5$. Define a set function as follows,

$$\mathcal{M}\{\Lambda\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \\ \alpha, & \text{if } \Lambda \text{ is upper bounded} \\ 0.5, & \text{if both } \Lambda \text{ and } \Lambda^c \text{ are upper unbounded} \\ 1-\alpha, & \text{if } \Lambda^c \text{ is upper bounded} \\ 1, & \text{if } \Lambda = \Gamma. \end{cases} \tag{1.14}$$

It is easy to verify that $\mathcal{M}$ is an uncertain measure. Write $\Lambda_i = (-\infty, i]$ for $i = 1, 2, \cdots$ Then $\Lambda_i \uparrow \Gamma$ and $\lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} = \alpha$. Furthermore, we have $\Lambda_i^c \downarrow \emptyset$ and $\lim_{i\to\infty} \mathcal{M}\{\Lambda_i^c\} = 1 - \alpha$.

Independence of Events

Definition 1.2. *The events $\Lambda_1, \Lambda_2, \cdots, \Lambda_n$ are said to be independent if*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} \Lambda_i^*\right\} = \min_{1\le i\le n} \mathcal{M}\{\Lambda_i^*\} \tag{1.15}$$

where Λ_i^* are arbitrarily chosen from $\{\Lambda_i, \Lambda_i^c\}$, $i = 1, 2, \cdots, n$, respectively.

Note that (1.15) represents 2^n equations. For example, when $n = 2$, the four equations are

$$\begin{aligned}
\mathcal{M}\{\Lambda_1 \cap \Lambda_2\} &= \mathcal{M}\{\Lambda_1\} \wedge \mathcal{M}\{\Lambda_2\},\\
\mathcal{M}\{\Lambda_1^c \cap \Lambda_2\} &= \mathcal{M}\{\Lambda_1^c\} \wedge \mathcal{M}\{\Lambda_2\},\\
\mathcal{M}\{\Lambda_1 \cap \Lambda_2^c\} &= \mathcal{M}\{\Lambda_1\} \wedge \mathcal{M}\{\Lambda_2^c\},\\
\mathcal{M}\{\Lambda_1^c \cap \Lambda_2^c\} &= \mathcal{M}\{\Lambda_1^c\} \wedge \mathcal{M}\{\Lambda_2^c\}.
\end{aligned} \tag{1.16}$$

Theorem 1.7. *The events $\Lambda_1, \Lambda_2, \cdots, \Lambda_n$ are independent if and only if*

$$\mathcal{M}\left\{\bigcup_{i=1}^{n} \Lambda_i^*\right\} = \max_{1\le i\le n} \mathcal{M}\{\Lambda_i^*\} \tag{1.17}$$

where Λ_i^* are arbitrarily chosen from $\{\Lambda_i, \Lambda_i^c\}$, $i = 1, 2, \cdots, n$, respectively.

Proof: Assume $\Lambda_1, \Lambda_2, \cdots, \Lambda_n$ are independent events. It follows from the self-duality of uncertain measure that

$$\mathcal{M}\left\{\bigcup_{i=1}^{n} \Lambda_i^*\right\} = 1 - \mathcal{M}\left\{\bigcap_{i=1}^{n} (\Lambda_i^*)^c\right\} = 1 - \min_{1\le i\le n} \mathcal{M}\{(\Lambda_i^*)^c\} = \max_{1\le i\le n} \mathcal{M}\{\Lambda_i^*\}.$$

The equation (1.17) is proved. Conversely, assume (1.17). Then

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} \Lambda_i^*\right\} = 1 - \mathcal{M}\left\{\bigcup_{i=1}^{n} (\Lambda_i^*)^c\right\} = 1 - \max_{1\le i\le n} \mathcal{M}\{(\Lambda_i^*)^c\} = \min_{1\le i\le n} \mathcal{M}\{\Lambda_i^*\}.$$

The equation (1.15) is true. The theorem is proved.

Uncertainty Space

Definition 1.3 *(Liu [120]). Let Γ be a nonempty set, $\mathcal{L}$ a σ-algebra over Γ, and $\mathcal{M}$ an uncertain measure. Then the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.*

Product Measure Axiom and Product Uncertain Measure

Product uncertain measure was defined by Liu [123] in 2009, thus producing the fifth axiom of uncertainty theory called *product measure axiom*. Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots, n$. Write

$$\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n, \quad \mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n. \tag{1.18}$$

Then there is an uncertain measure $\mathcal{M}$ on the product σ-algebra $\mathcal{L}$ such that

$$\mathcal{M}\{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n\} = \mathcal{M}_1\{\Lambda_1\} \wedge \mathcal{M}_2\{\Lambda_2\} \wedge \cdots \wedge \mathcal{M}_n\{\Lambda_n\} \tag{1.19}$$

for any measurable rectangle $\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n$. Such an uncertain measure is called the product uncertain measure denoted by

$$\mathcal{M} = \mathcal{M}_1 \wedge \mathcal{M}_2 \wedge \cdots \wedge \mathcal{M}_n. \tag{1.20}$$

In fact, the extension from the class of rectangles to the product σ-algebra $\mathcal{L}$ may be represented as follows.

Axiom 5. *(Liu [123], Product Measure Axiom) Let Γ_k be nonempty sets on which $\mathcal{M}_k$ are uncertain measures, $k = 1, 2, \cdots, n$, respectively. Then the product uncertain measure $\mathcal{M}$ is an uncertain measure on the product σ-algebra $\mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$ satisfying*

$$\mathcal{M}\left\{\prod_{k=1}^{n} \Lambda_k\right\} = \min_{1 \le k \le n} \mathcal{M}_k\{\Lambda_k\}. \tag{1.21}$$

That is, for each event $\Lambda \in \mathcal{L}$, we have

$$\mathcal{M}\{\Lambda\} = \begin{cases} \displaystyle\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subset \Lambda} \min_{1 \le k \le n} \mathcal{M}_k\{\Lambda_k\}, \\ \qquad \text{if} \displaystyle\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subset \Lambda} \min_{1 \le k \le n} \mathcal{M}_k\{\Lambda_k\} > 0.5 \\ 1 - \displaystyle\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subset \Lambda^c} \min_{1 \le k \le n} \mathcal{M}_k\{\Lambda_k\}, \\ \qquad \text{if} \displaystyle\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subset \Lambda^c} \min_{1 \le k \le n} \mathcal{M}_k\{\Lambda_k\} > 0.5 \\ 0.5, \qquad \text{otherwise.} \end{cases} \tag{1.22}$$

Theorem 1.8 *(Peng [176]). The product uncertain measure (1.22) is an uncertain measure.*

Proof: In order to prove that the product uncertain measure (1.22) is indeed an uncertain measure, we should verify that the product uncertain measure

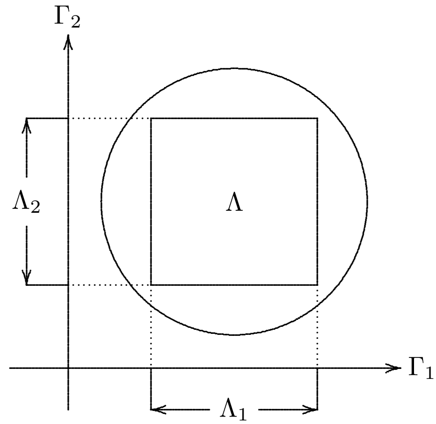

Figure 1.1: Graphical Illustration of Extension from the Class of Rectangles to the Product σ-Algebra. The uncertain measure of Λ (the disk) is essentially the acreage of its inscribed rectangle $\Lambda_1 \times \Lambda_2$ if it is greater than 0.5. Otherwise, we have to examine its complement Λ^c. If the inscribed rectangle of Λ^c is greater than 0.5, then $\mathcal{M}\{\Lambda^c\}$ is just its inscribed rectangle and $\mathcal{M}\{\Lambda\} = 1 - \mathcal{M}\{\Lambda^c\}$. If there does not exist an inscribed rectangle of Λ or Λ^c greater than 0.5, then we set $\mathcal{M}\{\Lambda\} = 0.5$.

satisfies the normality, monotonicity, self-duality and countable subadditivity axioms.

Step 1: At first, for any event $\Lambda \in \mathcal{L}$, it is easy to verify that

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda}\min_{1\le k\le n}\mathcal{M}_k\{\Lambda_k\} + \sup_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda^c}\min_{1\le k\le n}\mathcal{M}_k\{\Lambda_k\} \le 1.$$

This means that at most one of

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda}\min_{1\le k\le n}\mathcal{M}_k\{\Lambda_k\} \quad \text{and} \quad \sup_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda^c}\min_{1\le k\le n}\mathcal{M}_k\{\Lambda_k\}$$

is greater than 0.5. Thus the expression (1.22) is reasonable.

Step 2: The product uncertain measure is clearly normal, i.e., $\mathcal{M}\{\Gamma\} = 1$.

Step 3: We prove the self-duality, i.e., $\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Lambda^c\} = 1$. The argument breaks down into three cases. Case 1: Assume

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda}\min_{1\le k\le n}\mathcal{M}_k\{\Lambda_k\} > 0.5.$$

Then we immediately have

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda^c}\min_{1\le k\le n}\mathcal{M}_k\{\Lambda_k\} < 0.5.$$

It follows from (1.22) that

$$\mathcal{M}\{\Lambda\} = \sup_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda}\min_{1\le k\le n}\mathcal{M}_k\{\Lambda_k\},$$

$$\mathcal{M}\{\Lambda^c\} = 1 - \sup_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset(\Lambda^c)^c}\min_{1\le k\le n}\mathcal{M}_k\{\Lambda_k\} = 1-\mathcal{M}\{\Lambda\}.$$

The self-duality is proved. Case 2: Assume

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda^c}\min_{1\le k\le n}\mathcal{M}_k\{\Lambda_k\} > 0.5.$$

This case may be proved by a similar process. Case 3: Assume

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda}\min_{1\le k\le n}\mathcal{M}_k\{\Lambda_k\} \le 0.5$$

and

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda^c}\min_{1\le k\le n}\mathcal{M}_k\{\Lambda_k\} \le 0.5.$$

It follows from (1.22) that $\mathcal{M}\{\Lambda\} = \mathcal{M}\{\Lambda^c\} = 0.5$ which proves the self-duality.

Step 4: Let us prove that $\mathcal{M}$ is increasing. Suppose Λ and Δ are two events in $\mathcal{L}$ with $\Lambda\subset\Delta$. The argument breaks down into three cases. Case 1: Assume

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda}\min_{1\le k\le n}\mathcal{M}_k\{\Lambda_k\} > 0.5.$$

Then

$$\sup_{\Delta_1\times\Delta_2\times\cdots\times\Delta_n\subset\Delta}\min_{1\le k\le n}\mathcal{M}_k\{\Delta_k\} \ge \sup_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda}\min_{1\le k\le n}\mathcal{M}_k\{\Lambda_k\} > 0.5.$$

It follows from (1.22) that $\mathcal{M}\{\Lambda\}\le\mathcal{M}\{\Delta\}$. Case 2: Assume

$$\sup_{\Delta_1\times\Delta_2\times\cdots\times\Delta_n\subset\Delta^c}\min_{1\le k\le n}\mathcal{M}_k\{\Delta_k\} > 0.5.$$

Then

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda^c}\min_{1\le k\le n}\mathcal{M}_k\{\Lambda_k\} \ge \sup_{\Delta_1\times\Delta_2\times\cdots\times\Delta_n\subset\Delta^c}\min_{1\le k\le n}\mathcal{M}_k\{\Delta_k\} > 0.5.$$

Thus

$$\begin{aligned}\mathcal{M}\{\Lambda\} &= 1 - \sup_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda^c}\min_{1\le k\le n}\mathcal{M}_k\{\Lambda_k\}\\ &\le 1 - \sup_{\Delta_1\times\Delta_2\times\cdots\times\Delta_n\subset\Delta^c}\min_{1\le k\le n}\mathcal{M}_k\{\Delta_k\} = \mathcal{M}\{\Delta\}.\end{aligned}$$

Case 3: Assume

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda}\min_{1\le k\le n}\mathcal{M}_k\{\Lambda_k\} \le 0.5$$

and

$$\sup_{\Delta_1\times\Delta_2\times\cdots\times\Delta_n\subset\Delta^c}\min_{1\le k\le n}\mathcal{M}_k\{\Delta_k\}\le 0.5.$$

Then

$$\mathcal{M}\{\Lambda\}\le 0.5\le 1-\mathcal{M}\{\Delta^c\}=\mathcal{M}\{\Delta\}.$$

Step 5: Finally, we prove the countable subadditivity of $\mathcal{M}$. For simplicity, we only prove the case of two events Λ and Δ. The argument breaks down into three cases. Case 1: Assume $\mathcal{M}\{\Lambda\}<0.5$ and $\mathcal{M}\{\Delta\}<0.5$. For any given $\varepsilon>0$, there are two rectangles

$$\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda^c,\quad \Delta_1\times\Delta_2\times\cdots\times\Delta_n\subset\Delta^c$$

such that

$$1-\min_{1\le k\le n}\mathcal{M}_k\{\Lambda_k\}\le\mathcal{M}\{\Lambda\}+\varepsilon/2,$$
$$1-\min_{1\le k\le n}\mathcal{M}_k\{\Delta_k\}\le\mathcal{M}\{\Delta\}+\varepsilon/2.$$

Note that

$$(\Lambda_1\cap\Delta_1)\times(\Lambda_2\cap\Delta_2)\times\cdots\times(\Lambda_n\cap\Delta_n)\subset(\Lambda\cup\Delta)^c.$$

It follows from Theorem 1.4 that

$$\mathcal{M}_k\{\Lambda_k\cap\Delta_k\}\ge\mathcal{M}_k\{\Lambda_k\}+\mathcal{M}_k\{\Delta_k\}-1$$

for any k. Thus

$$\begin{aligned}\mathcal{M}\{\Lambda\cup\Delta\}&\le 1-\min_{1\le k\le n}\mathcal{M}_k\{\Lambda_k\cap\Delta_k\}\\&\le 1-\min_{1\le k\le n}\mathcal{M}_k\{\Lambda_k\}+1-\min_{1\le k\le n}\mathcal{M}_k\{\Delta_k\}\\&\le\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Delta\}+\varepsilon.\end{aligned}$$

Letting $\varepsilon\to 0$, we obtain

$$\mathcal{M}\{\Lambda\cup\Delta\}\le\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Delta\}.$$

Case 2: Assume $\mathcal{M}\{\Lambda\}\ge 0.5$ and $\mathcal{M}\{\Delta\}<0.5$. When $\mathcal{M}\{\Lambda\cup\Delta\}=0.5$, the subadditivity is obvious. Now we consider the case $\mathcal{M}\{\Lambda\cup\Delta\}>0.5$, i.e., $\mathcal{M}\{\Lambda^c\cap\Delta^c\}<0.5$. By using $\Lambda^c\cup\Delta=(\Lambda^c\cap\Delta^c)\cup\Delta$ and Case 1, we get

$$\mathcal{M}\{\Lambda^c\cup\Delta\}\le\mathcal{M}\{\Lambda^c\cap\Delta^c\}+\mathcal{M}\{\Delta\}.$$

Thus

$$\begin{aligned}\mathcal{M}\{\Lambda\cup\Delta\}&=1-\mathcal{M}\{\Lambda^c\cap\Delta^c\}\le 1-\mathcal{M}\{\Lambda^c\cup\Delta\}+\mathcal{M}\{\Delta\}\\&\le 1-\mathcal{M}\{\Lambda^c\}+\mathcal{M}\{\Delta\}=\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Delta\}.\end{aligned}$$

Case 3: If both $\mathcal{M}\{\Lambda\}\ge 0.5$ and $\mathcal{M}\{\Delta\}\ge 0.5$, then the subadditivity is obvious because $\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Delta\}\ge 1$. The theorem is proved.

Definition 1.4. *Let* $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k), k = 1, 2, \cdots, n$ *be uncertainty spaces,* $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$, $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$*, and* $\mathcal{M} = \mathcal{M}_1 \wedge \mathcal{M}_2 \wedge \cdots \wedge \mathcal{M}_n$. *Then* $(\Gamma, \mathcal{L}, \mathcal{M})$ *is called the product uncertainty space of* $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k), k = 1, 2, \cdots, n$.

1.2 Uncertain Variable

This section introduces a concept of uncertain variable (neither random variable nor fuzzy variable) in order to describe imprecise quantities in human systems.

Definition 1.5 *(Liu [120]). An uncertain variable is a measurable function* ξ *from an uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *to the set of real numbers, i.e., for any Borel set* B *of real numbers, the set*

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\} \tag{1.23}$$

is an event.

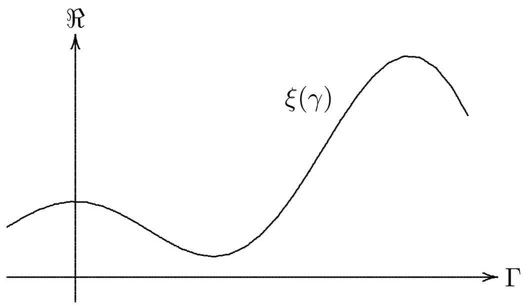

Figure 1.2: An Uncertain Variable

Example 1.6: Take $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with $\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = 0.5$. Then the function

$$\xi(\gamma) = \begin{cases} 0, \text{ if } \gamma = \gamma_1 \\ 1, \text{ if } \gamma = \gamma_2 \end{cases}$$

is an uncertain variable.

Example 1.7: A crisp number c may be regarded as a special uncertain variable. In fact, it is the constant function $\xi(\gamma) \equiv c$ on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$.

Definition 1.6. *Let* ξ *and* η *be uncertain variables defined on the uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$. *We say* $\xi = \eta$ *if* $\xi(\gamma) = \eta(\gamma)$ *for almost all* $\gamma \in \Gamma$.

Definition 1.7. *The uncertain variables ξ and η are identically distributed if*

$$\mathcal{M}\{\xi \in B\} = \mathcal{M}\{\eta \in B\} \tag{1.24}$$

for any Borel set B of real numbers.

It is clear that uncertain variables ξ and η are identically distributed if $\xi = \eta$. However, identical distribution does not imply $\xi = \eta$. For example, let $(\Gamma, \mathcal{L}, \mathcal{M})$ be $\{\gamma_1, \gamma_2\}$ with $\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = 0.5$. Define

$$\xi(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_1 \\ -1, & \text{if } \gamma = \gamma_2, \end{cases} \qquad \eta(\gamma) = \begin{cases} -1, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2. \end{cases}$$

The two uncertain variables ξ and η are identically distributed but $\xi \neq \eta$.

Uncertain Vector

Definition 1.8. *An n-dimensional uncertain vector is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of n-dimensional real vectors, i.e., for any Borel set B of $\Re^n$, the set*

$$\{\boldsymbol{\xi} \in B\} = \left\{\gamma \in \Gamma \mid \boldsymbol{\xi}(\gamma) \in B\right\} \tag{1.25}$$

is an event.

Theorem 1.9. *The vector $(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain vector if and only if $\xi_1, \xi_2, \cdots, \xi_n$ are uncertain variables.*

Proof: Write $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$. Suppose that $\boldsymbol{\xi}$ is an uncertain vector on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. For any Borel set B of $\Re$, the set $B \times \Re^{n-1}$ is a Borel set of $\Re^n$. Thus the set

$$\{\xi_1 \in B\} = \{\xi_1 \in B, \xi_2 \in \Re, \cdots, \xi_n \in \Re\} = \{\boldsymbol{\xi} \in B \times \Re^{n-1}\}$$

is an event. Hence ξ_1 is an uncertain variable. A similar process may prove that $\xi_2, \xi_3, \cdots, \xi_n$ are uncertain variables. Conversely, suppose that all $\xi_1, \xi_2, \cdots, \xi_n$ are uncertain variables on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. We define

$$\mathcal{B} = \left\{B \subset \Re^n \mid \{\boldsymbol{\xi} \in B\} \text{ is an event}\right\}.$$

The vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$ is proved to be an uncertain vector if we can prove that $\mathcal{B}$ contains all Borel sets of $\Re^n$. First, the class $\mathcal{B}$ contains all open intervals of $\Re^n$ because

$$\left\{\boldsymbol{\xi} \in \prod_{i=1}^{n}(a_i, b_i)\right\} = \bigcap_{i=1}^{n}\{\xi_i \in (a_i, b_i)\}$$

is an event. Next, the class $\mathcal{B}$ is a σ-algebra of $\Re^n$ because (i) we have $\Re^n \in \mathcal{B}$ since $\{\boldsymbol{\xi} \in \Re^n\} = \Gamma$; (ii) if $B \in \mathcal{B}$, then $\{\boldsymbol{\xi} \in B\}$ is an event, and

$$\{\boldsymbol{\xi} \in B^c\} = \{\boldsymbol{\xi} \in B\}^c$$

is an event. This means that $B^c \in \mathcal{B}$; (iii) if $B_i \in \mathcal{B}$ for $i = 1, 2, \cdots$, then $\{\boldsymbol{\xi} \in B_i\}$ are events and

$$\left\{\boldsymbol{\xi} \in \bigcup_{i=1}^{\infty} B_i\right\} = \bigcup_{i=1}^{\infty}\{\boldsymbol{\xi} \in B_i\}$$

is an event. This means that $\cup_i B_i \in \mathcal{B}$. Since the smallest σ-algebra containing all open intervals of $\Re^n$ is just the Borel algebra of $\Re^n$, the class $\mathcal{B}$ contains all Borel sets of $\Re^n$. The theorem is proved.

Uncertain Arithmetic

Definition 1.9. *Suppose that $f : \Re^n \to \Re$ is a measurable function, and $\xi_1, \xi_2, \cdots, \xi_n$ uncertain variables on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. Then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable defined as*

$$\xi(\gamma) = f(\xi_1(\gamma), \xi_2(\gamma), \cdots, \xi_n(\gamma)), \quad \forall \gamma \in \Gamma. \tag{1.26}$$

Example 1.8: Let ξ_1 and ξ_2 be two uncertain variables. Then the sum $\xi = \xi_1 + \xi_2$ is an uncertain variable defined by

$$\xi(\gamma) = \xi_1(\gamma) + \xi_2(\gamma), \quad \forall \gamma \in \Gamma.$$

The product $\xi = \xi_1 \xi_2$ is also an uncertain variable defined by

$$\xi(\gamma) = \xi_1(\gamma) \cdot \xi_2(\gamma), \quad \forall \gamma \in \Gamma.$$

The reader may wonder whether $\xi(\gamma_1, \gamma_2, \cdots, \gamma_n)$ defined by (1.26) is an uncertain variable. The following theorem answers this question.

Theorem 1.10. *Let $\boldsymbol{\xi}$ be an n-dimensional uncertain vector, and $f : \Re^n \to \Re$ a measurable function. Then $f(\boldsymbol{\xi})$ is an uncertain variable such that*

$$\mathcal{M}\{f(\boldsymbol{\xi}) \in B\} = \mathcal{M}\{\boldsymbol{\xi} \in f^{-1}(B)\} \tag{1.27}$$

for any Borel set B of real numbers.

Proof: Assume that $\boldsymbol{\xi}$ is an uncertain vector on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. For any Borel set B of $\Re$, since f is a measurable function, the $f^{-1}(B)$ is a Borel set of $\Re^n$. Thus the set $\{f(\boldsymbol{\xi}) \in B\} = \{\boldsymbol{\xi} \in f^{-1}(B)\}$ is an event for any Borel set B. Hence $f(\boldsymbol{\xi})$ is an uncertain variable.

1.3 Uncertainty Distribution

This section introduces a concept of uncertainty distribution in order to describe uncertain variables. In many cases, it is sufficient to know the uncertainty distribution rather than the uncertain variable itself.

Definition 1.10 *(Liu [120]). The uncertainty distribution* Φ*:* $\Re \to [0,1]$ *of an uncertain variable* ξ *is defined by*

$$\Phi(x) = \mathcal{M}\{\xi \le x\} \tag{1.28}$$

for any real number x*.*

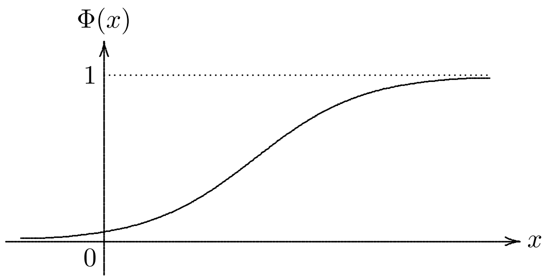

Figure 1.3: An Uncertainty Distribution

Theorem 1.11 *(Peng and Iwamura [177], Sufficient and Necessary Condition for Uncertainty Distribution). A function* $\Phi : \Re \to [0,1]$ *is an uncertainty distribution if and only if it is an increasing function except* $\Phi(x) \equiv 0$ *and* $\Phi(x) \equiv 1$*.*

Proof: It is obvious that an uncertainty distribution Φ is an increasing function. In addition, both $\Phi(x) \not\equiv 0$ and $\Phi(x) \not\equiv 1$ follow from the asymptotic theorem immediately. Conversely, suppose that Φ is an increasing function but $\Phi(x) \not\equiv 0$ and $\Phi(x) \not\equiv 1$. We will prove that there is an uncertain variable whose uncertainty distribution is just Φ. Let $\mathcal{C}$ be a collection of all intervals of the form $(-\infty, a]$, (b, ∞), $\emptyset$ and $\Re$. We define a set function on $\Re$ as follows,

$$\mathcal{M}\{(-\infty, a]\} = \Phi(a),$$

$$\mathcal{M}\{(b, +\infty)\} = 1 - \Phi(b),$$

$$\mathcal{M}\{\emptyset\} = 0, \quad \mathcal{M}\{\Re\} = 1.$$

For an arbitrary Borel set B of real numbers, there exists a sequence $\{A_i\}$ in $\mathcal{C}$ such that

$$B \subset \bigcup_{i=1}^{\infty} A_i.$$

Note that such a sequence is not unique. Thus the set function $\mathcal{M}\{B\}$ is defined by

$$\mathcal{M}\{B\} = \begin{cases} \inf\limits_{B\subset\cup A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\}, & \text{if } \inf\limits_{B\subset\cup A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\} < 0.5 \\ 1 - \inf\limits_{B^c\subset\cup A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\}, & \text{if } \inf\limits_{B^c\subset\cup A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\} < 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

We may prove that the set function $\mathcal{M}$ is indeed an uncertain measure on $\Re$, and the uncertain variable defined by the identity function $\xi(\gamma) = \gamma$ from the uncertainty space $(\Re, \mathcal{L}, \mathcal{M})$ to $\Re$ has the uncertainty distribution Φ.

Example 1.9: Let c be a number with $0 < c < 1$. Then $\Phi(x) \equiv c$ is an uncertainty distribution. When $c \le 0.5$, we define a set function over $\Re$ as follows,

$$\mathcal{M}\{\Lambda\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \\ c, & \text{if } \Lambda \text{ is upper bounded} \\ 0.5, & \text{if both } \Lambda \text{ and } \Lambda^c \text{ are upper unbounded} \\ 1-c, & \text{if } \Lambda^c \text{ is upper bounded} \\ 1, & \text{if } \Lambda = \Gamma. \end{cases}$$

Then $(\Re, \mathcal{L}, \mathcal{M})$ is an uncertainty space. It is easy to verify that the identity function $\xi(\gamma) = \gamma$ is an uncertain variable whose uncertainty distribution is just $\Phi(x) \equiv c$. When $c > 0.5$, we define

$$\mathcal{M}\{\Lambda\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \\ 1-c, & \text{if } \Lambda \text{ is upper bounded} \\ 0.5, & \text{if both } \Lambda \text{ and } \Lambda^c \text{ are upper unbounded} \\ c, & \text{if } \Lambda^c \text{ is upper bounded} \\ 1, & \text{if } \Lambda = \Gamma. \end{cases}$$

Then the function $\xi(\gamma) = -\gamma$ is an uncertain variable whose uncertainty distribution is just $\Phi(x) \equiv c$.

Example 1.10: Assume that two uncertain variables ξ and η have the same uncertainty distribution. One question is whether $\xi = \eta$ or not. Generally speaking, it is not true. Take $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with

$$\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = 0.5.$$

We now define two uncertain variables as follows,

$$\xi(\gamma) = \begin{cases} -1, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2, \end{cases} \qquad \eta(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_1 \\ -1, & \text{if } \gamma = \gamma_2. \end{cases}$$

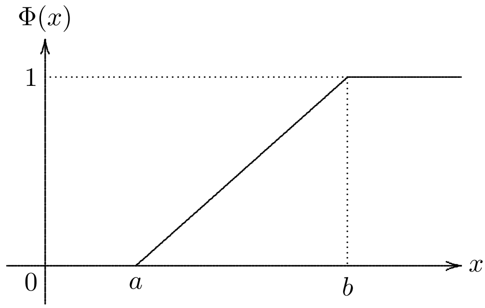

Figure 1.4: Linear Uncertainty Distribution

Then ξ and η have the same uncertainty distribution,

$$\Phi(x) = \begin{cases} 0, & \text{if } x < -1 \\ 0.5, & \text{if } -1 \le x < 1 \\ 1, & \text{if } x \ge 1. \end{cases}$$

However, it is clear that $\xi \ne \eta$ in the sense of Definition 1.6.

Definition 1.11. *An uncertain variable ξ is called linear if it has a linear uncertainty distribution*

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le a \\ (x-a)/(b-a), & \text{if } a \le x \le b \\ 1, & \text{if } x \ge b \end{cases} \tag{1.29}$$

denoted by $\mathcal{L}(a,b)$ where a and b are real numbers with $a < b$.

Definition 1.12. *An uncertain variable ξ is called zigzag if it has a zigzag uncertainty distribution*

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le a \\ (x-a)/2(b-a), & \text{if } a \le x \le b \\ (x+c-2b)/2(c-b), & \text{if } b \le x \le c \\ 1, & \text{if } x \ge c \end{cases} \tag{1.30}$$

denoted by $\mathcal{Z}(a,b,c)$ where a, b, c are real numbers with $a < b < c$.

Definition 1.13. *An uncertain variable ξ is called normal if it has a normal uncertainty distribution*

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \Re \tag{1.31}$$

denoted by $\mathcal{N}(e,\sigma)$ where e and σ are real numbers with $\sigma > 0$.

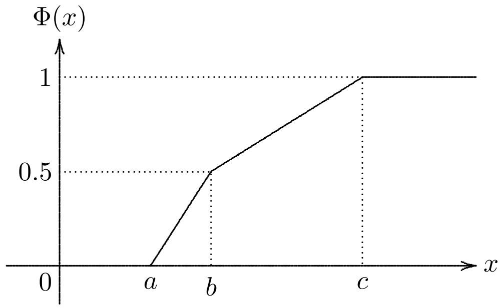

Figure 1.5: Zigzag Uncertainty Distribution

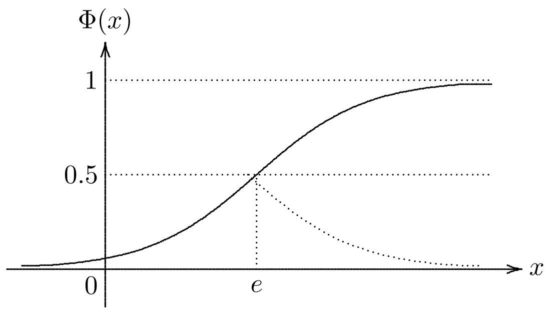

Figure 1.6: Normal Uncertainty Distribution

Definition 1.14. *An uncertain variable ξ is called lognormal if $\ln \xi$ is a normal uncertain variable $\mathcal{N}(e, \sigma)$. In other words, a lognormal uncertain variable has an uncertainty distribution*

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e - \ln x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \geq 0 \tag{1.32}$$

denoted by $\mathcal{LOGN}(e, \sigma)$, where e and σ are real numbers with $\sigma > 0$.

Definition 1.15. *An uncertain variable ξ is called discrete if it takes values in $\{x_1, x_2, \cdots, x_m\}$ and*

$$\Phi(x_i) = \alpha_i, \quad i = 1, 2, \cdots, m \tag{1.33}$$

where $x_1 < x_2 < \cdots < x_m$ and $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m = 1$. For simplicity, the discrete uncertain variable will be denoted by

$$\xi = \begin{array}{|c|c|c|c|}\hline \alpha_1 & \alpha_2 & \cdots & \alpha_m \\ \hline x_1 & x_2 & \cdots & x_m \\ \hline \end{array} \tag{1.34}$$

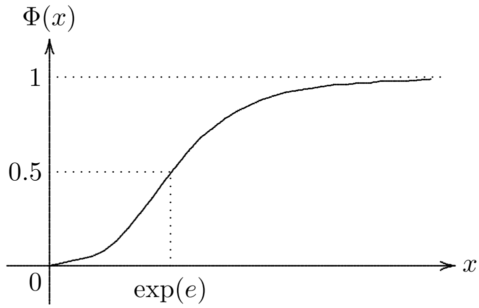

Figure 1.7: Lognormal Uncertainty Distribution

The uncertainty distribution Φ of the discrete uncertain variable (1.34) is a step function jumping only at $x_1, x_2, \cdots, x_m$, i.e.,

$$\Phi(x) = \begin{cases} \alpha_0, & \text{if } x < x_1 \\ \alpha_i, & \text{if } x_i \le x < x_{i+1},\ i = 1, 2, \cdots, m \\ \alpha_m, & \text{if } x \ge x_m \end{cases} \tag{1.35}$$

where $\alpha_0 \equiv 0$ and $\alpha_m \equiv 1$.

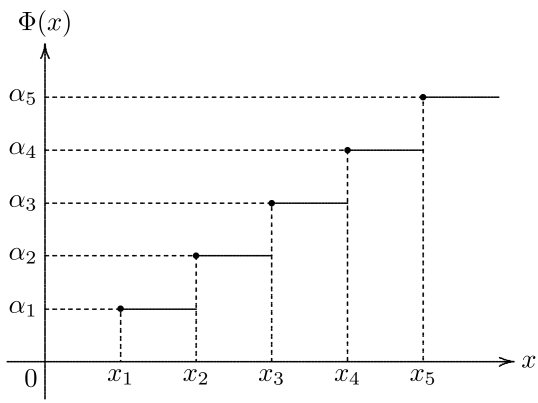

Figure 1.8: Discrete Uncertainty Distribution

Measure Inversion Theorem

Theorem 1.12 *(Measure Inversion Theorem). Let ξ be an uncertain variable with continuous uncertainty distribution Φ. Then for any real number x, we have*

$$\mathcal{M}\{\xi \le x\} = \Phi(x), \quad \mathcal{M}\{\xi \ge x\} = 1 - \Phi(x). \tag{1.36}$$

Proof: The equation $\mathcal{M}\{\xi \le x\} = \Phi(x)$ follows from the definition of uncertainty distribution immediately. By using the self-duality of uncertain measure and continuity of uncertainty distribution, we get $\mathcal{M}\{\xi \ge x\} = 1 - \mathcal{M}\{\xi < x\} = 1 - \Phi(x)$.

Theorem 1.13. *Let ξ be an uncertain variable with continuous uncertainty distribution Φ. Then for any interval $[a, b]$, we have*

$$\Phi(b) - \Phi(a) \le \mathcal{M}\{a \le \xi \le b\} \le \Phi(b) \wedge (1 - \Phi(a)). \tag{1.37}$$

Proof: It follows from the subadditivity of uncertain measure and the measure inversion theorem that

$$\mathcal{M}\{a \le \xi \le b\} + \mathcal{M}\{\xi \le a\} \ge \mathcal{M}\{\xi \le b\}.$$

That is,

$$\mathcal{M}\{a \le \xi \le b\} + \Phi(a) \ge \Phi(b).$$

Thus the inequality on the left hand side is verified. It follows from the monotonicity of uncertain measure and the measure inversion theorem that

$$\mathcal{M}\{a \le \xi \le b\} \le \mathcal{M}\{\xi \in (-\infty, b]\} = \Phi(b).$$

On the other hand,

$$\mathcal{M}\{a \le x \le b\} \le \mathcal{M}\{\xi \in [a, +\infty)\} = 1 - \Phi(a).$$

Hence the inequality on the right hand side is proved.

Perhaps some readers would like to get an exactly scalar value of the uncertain measure $\mathcal{M}\{a \le x \le b\}$. Generally speaking, it is an impossible job (except $a = -\infty$ or $b = +\infty$) if only an uncertainty distribution is available. I would like to ask if there is a need to know it. In fact, it is not a must for practical purpose. Would you believe?

Regular Uncertainty Distribution

Definition 1.16. *An uncertainty distribution Φ is said to be regular if its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0, 1)$.*

It is easy to verify that a regular uncertainty distribution Φ is a continuous function. In addition, Φ is strictly increasing at each point x with $0 < \Phi(x) < 1$. Furthermore,

$$\lim_{x \to -\infty} \Phi(x) = 0, \qquad \lim_{x \to +\infty} \Phi(x) = 1. \tag{1.38}$$

For example, linear uncertainty distribution, zigzag uncertainty distribution, normal uncertainty distribution, and lognormal uncertainty distribution are all regular.

In this book we will assume all uncertainty distributions are regular. Otherwise, we may give the uncertainty distribution a small perturbation such that it becomes regular.

Inverse Uncertainty Distribution

Definition 1.17. *Let ξ be an uncertain variable with uncertainty distribution Φ. Then the inverse function Φ^{-1} is called the inverse uncertainty distribution of ξ.*

Note that the inverse uncertainty distribution $\Phi^{-1}(\alpha)$ is well defined on the open interval $(0, 1)$. If needed, we may extend the domain via

$$\Phi^{-1}(0) = \lim_{\alpha\to 0} \Phi^{-1}(\alpha), \quad \Phi^{-1}(1) = \lim_{\alpha\to 1} \Phi^{-1}(\alpha). \tag{1.39}$$

It is easy to verify that inverse uncertainty distribution is a monotone increasing function on $[0, 1]$.

Example 1.11: The inverse uncertainty distribution of linear uncertain variable $\mathcal{L}(a, b)$ is

$$\Phi^{-1}(\alpha) = (1 - \alpha)a + \alpha b. \tag{1.40}$$

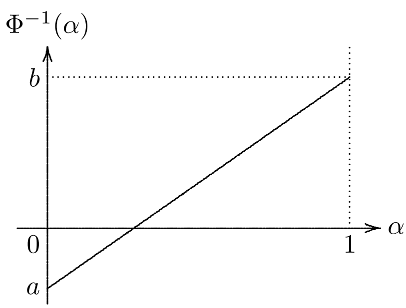

Figure 1.9: Inverse Linear Uncertainty Distribution

Example 1.12: The inverse uncertainty distribution of zigzag uncertain variable $\mathcal{Z}(a, b, c)$ is

$$\Phi^{-1}(\alpha) = \begin{cases} (1 - 2\alpha)a + 2\alpha b, & \text{if } \alpha < 0.5 \\ (2 - 2\alpha)b + (2\alpha - 1)c, & \text{if } \alpha \ge 0.5. \end{cases} \tag{1.41}$$

Example 1.13: The inverse uncertainty distribution of normal uncertain variable $\mathcal{N}(e, \sigma)$ is

$$\Phi^{-1}(\alpha) = e + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}. \tag{1.42}$$

Example 1.14: The inverse uncertainty distribution of lognormal uncertain variable $\mathcal{LOGN}(e, \sigma)$ is

$$\Phi^{-1}(\alpha) = \exp(e) \left(\frac{\alpha}{1 - \alpha} \right)^{\sqrt{3}\sigma/\pi}. \tag{1.43}$$

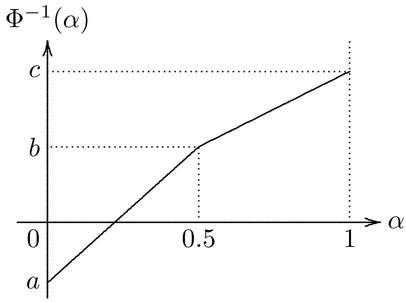

Figure 1.10: Inverse Zigzag Uncertainty Distribution

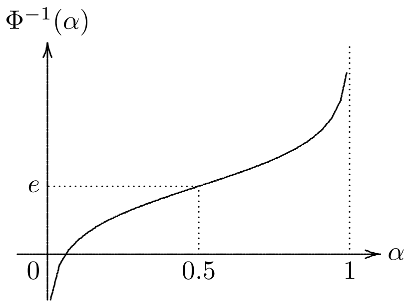

Figure 1.11: Inverse Normal Uncertainty Distribution

Joint Uncertainty Distribution

Definition 1.18 *Let* $(\xi_1, \xi_2, \cdots, \xi_n)$ *be an uncertain vector. Then the joint uncertainty distribution* $\Phi : \Re^n \to [0, 1]$ *is defined by*

$$\Phi(x_1, x_2, \cdots, x_n) = \mathcal{M}\{\xi_1 \le x_1, \xi_2 \le x_2, \cdots, \xi_n \le x_n\} \tag{1.44}$$

for any real numbers $x_1, x_2, \cdots, x_n$.

1.4 Independence

Independence has been explained in many ways. However, the essential feature is that those uncertain variables may be separately defined on different uncertainty spaces. In order to ensure that we are able to do so, we may define independence in the following mathematical form.

Definition 1.19 *(Liu [123]). The uncertain variables* $\xi_1, \xi_2, \cdots, \xi_m$ *are said to be independent if*

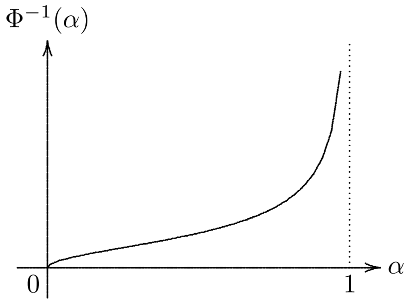

Figure 1.12: Inverse Lognormal Uncertainty Distribution

$$\mathcal{M}\left\{\bigcap_{i=1}^{m}(\xi_i \in B_i)\right\} = \min_{1\le i\le m} \mathcal{M}\left\{\xi_i \in B_i\right\} \tag{1.45}$$

for any Borel sets $B_1, B_2, \cdots, B_m$ of real numbers.

Example 1.15: Let ξ_1 be an uncertain variable and let ξ_2 be a constant c. For any Borel sets B_1 and B_2, if $c \in B_2$, then $\mathcal{M}\{\xi_2 \in B_2\} = 1$ and

$$\mathcal{M}\left\{(\xi_1 \in B_1) \cap (\xi_2 \in B_2)\right\} = \mathcal{M}\{\xi_1 \in B_1\} = \mathcal{M}\{\xi_1 \in B_1\} \wedge \mathcal{M}\{\xi_2 \in B_2\}.$$

If $c \notin B_2$, then $\mathcal{M}\{\xi_2 \in B_2\} = 0$ and

$$\mathcal{M}\left\{(\xi_1 \in B_1) \cap (\xi_2 \in B_2)\right\} = \mathcal{M}\{\emptyset\} = 0 = \mathcal{M}\{\xi_1 \in B_1\} \wedge \mathcal{M}\{\xi_2 \in B_2\}.$$

It follows from the definition of independence that an uncertain variable is always independent of a constant.

Theorem 1.14. *The uncertain variables $\xi_1, \xi_2, \cdots, \xi_m$ are independent if and only if*

$$\mathcal{M}\left\{\bigcup_{i=1}^{m}(\xi_i \in B_i)\right\} = \max_{1\le i\le m} \mathcal{M}\left\{\xi_i \in B_i\right\} \tag{1.46}$$

for any Borel sets $B_1, B_2, \cdots, B_m$ of real numbers.

Proof: It follows from the self-duality of uncertain measure that $\xi_1, \xi_2, \cdots,$ ξ_m are independent if and only if

$$\mathcal{M}\left\{\bigcup_{i=1}^{m}(\xi_i \in B_i)\right\} = 1 - \mathcal{M}\left\{\bigcap_{i=1}^{m}(\xi_i \in B_i^c)\right\}$$
$$= 1 - \min_{1\le i\le m} \mathcal{M}\{\xi_i \in B_i^c\} = \max_{1\le i\le m} \mathcal{M}\left\{\xi_i \in B_i\right\}.$$

Thus the proof is complete.

Theorem 1.15. *Let Φ_i be uncertainty distributions of uncertain variables ξ_i, $i = 1, 2, \cdots, m$, respectively, and Φ the joint uncertainty distribution of uncertain vector $(\xi_1, \xi_2, \cdots, \xi_m)$. If $\xi_1, \xi_2, \cdots, \xi_m$ are independent, then we have*

$$\Phi(x_1, x_2, \cdots, x_m) = \min_{1 \le i \le m} \Phi_i(x_i) \tag{1.47}$$

for any real numbers $x_1, x_2, \cdots, x_m$.

Proof: Since $\xi_1, \xi_2, \cdots, \xi_m$ are independent uncertain variables, we have

$$\Phi(x_1, x_2, \cdots, x_m) = \mathcal{M}\left\{\bigcap_{i=1}^{m}(\xi_i \le x_i)\right\} = \min_{1 \le i \le m} \mathcal{M}\{\xi_i \le x_i\} = \min_{1 \le i \le m} \Phi_i(x_i)$$

for any real numbers $x_1, x_2, \cdots, x_m$. The theorem is proved.

Example 1.16: However, the equation (1.47) does not imply that the uncertain variables are independent. For example, let ξ be an uncertain variable with uncertainty distribution Φ. Then the joint uncertainty distribution Ψ of uncertain vector (ξ, ξ) is

$$\Psi(x_1, x_2) = \mathcal{M}\{\xi \le x_1, \xi \le x_2\} = \mathcal{M}\{\xi \le x_1\} \wedge \mathcal{M}\{\xi \le x_2\} = \Phi(x_1) \wedge \Phi(x_2)$$

for any real numbers x_1 and x_2. But, generally speaking, an uncertain variable is not independent with itself.

Theorem 1.16. *Let $\xi_1, \xi_2, \cdots, \xi_m$ be independent uncertain variables, and $f_1, f_2, \cdots, f_n$ measurable functions. Then $f_1(\xi_1), f_2(\xi_2), \cdots, f_m(\xi_m)$ are independent uncertain variables.*

Proof: For any Borel sets $B_1, B_2, \cdots, B_m$ of $\Re$, it follows from the definition of independence that

$$\begin{aligned}
\mathcal{M}\left\{\bigcap_{i=1}^{m}(f_i(\xi_i) \in B_i)\right\} &= \mathcal{M}\left\{\bigcap_{i=1}^{m}(\xi_i \in f_i^{-1}(B_i))\right\} \\
&= \min_{1 \le i \le m} \mathcal{M}\{\xi_i \in f_i^{-1}(B_i)\} = \min_{1 \le i \le m} \mathcal{M}\{f_i(\xi_i) \in B_i\}.
\end{aligned}$$

Thus $f_1(\xi_1), f_2(\xi_2), \cdots, f_m(\xi_m)$ are independent uncertain variables.

1.5 Operational Law

This section will introduce an operational law of independent uncertain variables and present a 99-method for calculating the uncertainty distribution of monotone function of uncertain variables.

Theorem 1.17 *(Liu [123], Operational Law). Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables, and $f : \Re^n \to \Re$ a measurable function. Then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable such that*

$$\mathcal{M}\{\xi \in B\} = \begin{cases} \sup\limits_{f(B_1,B_2,\cdots,B_n)\subset B} \min\limits_{1\le k\le n} \mathcal{M}_k\{\xi_k \in B_k\}, \\ \qquad \text{if} \sup\limits_{f(B_1,B_2,\cdots,B_n)\subset B} \min\limits_{1\le k\le n} \mathcal{M}_k\{\xi_k \in B_k\} > 0.5 \\ 1 - \sup\limits_{f(B_1,B_2,\cdots,B_n)\subset B^c} \min\limits_{1\le k\le n} \mathcal{M}_k\{\xi_k \in B_k\}, \\ \qquad \text{if} \sup\limits_{f(B_1,B_2,\cdots,B_n)\subset B^c} \min\limits_{1\le k\le n} \mathcal{M}_k\{\xi_k \in B_k\} > 0.5 \\ 0.5, \qquad \text{otherwise} \end{cases}$$

where $B, B_1, B_2, \cdots, B_n$ are Borel sets, and $f(B_1, B_2, \cdots, B_n) \subset B$ means $f(x_1, x_2, \cdots, x_n) \in B$ for any $x_1 \in B_1, x_2 \in B_2, \cdots, x_n \in B_n$.

Proof: Write $\Lambda = \{\xi \in B\}$ and $\Lambda_k = \{\xi_k \in B_k\}$ for $k = 1, 2, \cdots, n$. It is easy to verify that

$$\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subset \Lambda \text{ if and only if } f(B_1, B_2, \cdots, B_n) \subset B,$$

$$\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subset \Lambda^c \text{ if and only if } f(B_1, B_2, \cdots, B_n) \subset B^c.$$

Thus the operational law follows from the product measure axiom immediately.

Increasing Function of Single Uncertain Variable

Theorem 1.18. *Let ξ be an uncertain variable with uncertainty distribution Φ, and let f be a strictly increasing function. Then $f(\xi)$ is an uncertain variable with inverse uncertainty distribution*

$$\Psi^{-1}(\alpha) = f(\Phi^{-1}(\alpha)). \tag{1.48}$$

Proof: Since f is a strictly increasing function, we have, for each $\alpha \in (0, 1)$,

$$\mathcal{M}\{f(\xi) \le f(\Phi^{-1}(\alpha))\} = \mathcal{M}\{\xi \le \Phi^{-1}(\alpha)\} = \alpha.$$

Thus we have $\Psi^{-1}(\alpha) = f(\Phi^{-1}(\alpha))$. In fact, the uncertainty distribution of $f(\xi)$ is

$$\Psi(x) = \Phi(f^{-1}(x)).$$

The theorem is proved.

99-Method 1.1. *It is suggested that an uncertain variable ξ with uncertainty distribution Φ is represented by a 99-table,*

0.01	0.02	0.03	$\cdots$	0.99
x_1	x_2	x_3	$\cdots$	x_{99}

(1.49)

where $0.01, 0.02, 0.03, \cdots, 0.99$ *in the first row are the values of uncertainty distribution* Φ*, and* $x_1, x_2, x_3, \cdots, x_{99}$ *in the second row are the corresponding values of* $\Phi^{-1}(0.01), \Phi^{-1}(0.02), \Phi^{-1}(0.03), \cdots, \Phi^{-1}(0.99)$*. Essentially, the 99-table is a discrete representation of uncertainty distribution* Φ*. Then for any strictly increasing function* $f(x)$*, the uncertain variable* $f(\xi)$ *has a 99-table,*

0.01	0.02	0.03	$\cdots$	0.99
$f(x_1)$	$f(x_2)$	$f(x_3)$	$\cdots$	$f(x_{99})$

(1.50)

The 99-method may be extended to the 999-method if a more precise result is needed.

Example 1.17: Let ξ be an uncertain variable with uncertainty distribution Φ. Then for any number $k > 0$, the inverse uncertainty distribution of $k\xi$ is

$$\Psi^{-1}(\alpha) = k\Phi^{-1}(\alpha). \tag{1.51}$$

If ξ is represented by a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
x_1	x_2	x_3	$\cdots$	x_{99}

(1.52)

then the 99-method yields that $k\xi$ has a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
kx_1	kx_2	kx_3	$\cdots$	kx_{99}

(1.53)

Example 1.18: If ξ is an uncertain variable with uncertainty distribution Φ and k is a constant, then $\xi+k$ is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \Phi^{-1}(\alpha) + k. \tag{1.54}$$

If ξ is represented by a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
x_1	x_2	x_3	$\cdots$	x_{99}

(1.55)

then the 99-method yields that $\xi + k$ has a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
$x_1 + k$	$x_2 + k$	$x_3 + k$	$\cdots$	$x_{99} + k$

(1.56)

Example 1.19: Let ξ be a nonnegative uncertain variable with uncertainty distribution Φ. Since x^2 is a strictly increasing function on $[0, +\infty)$, the square ξ^2 is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \left(\Phi^{-1}(\alpha)\right)^2. \tag{1.57}$$

If ξ is represented by a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
x_1	x_2	x_3	$\cdots$	x_{99}

(1.58)

then the 99-method yields that the uncertain variable ξ^2 has a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
x_1^2	x_2^2	x_3^2	$\cdots$	x_{99}^2

(1.59)

Example 1.20: Let ξ be an uncertain variable with uncertainty distribution Φ. Since $\exp(x)$ is a strictly increasing function, $\exp(\xi)$ is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \exp\left(\Phi^{-1}(\alpha)\right). \tag{1.60}$$

If ξ is represented by a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
x_1	x_2	x_3	$\cdots$	x_{99}

(1.61)

then the 99-method yields that the uncertain variable $\exp(\xi)$ has a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
$\exp(x_1)$	$\exp(x_2)$	$\exp(x_3)$	$\cdots$	$\exp(x_{99})$

(1.62)

Decreasing Function of Single Uncertain Variable

Theorem 1.19. *Let ξ be an uncertain variable with uncertainty distribution Φ, and let f be a strictly decreasing function. Then $f(\xi)$ is an uncertainty distribution with inverse uncertainty distribution*

$$\Psi^{-1}(\alpha) = f(\Phi^{-1}(1-\alpha)). \tag{1.63}$$

Proof: Since f is a strictly decreasing function, we have, for each $\alpha \in (0,1)$,

$$\mathcal{M}\{f(\xi) \le f(\Phi^{-1}(1-\alpha))\} = \mathcal{M}\{\xi \ge \Phi^{-1}(1-\alpha)\} = \alpha.$$

Thus we have $\Psi^{-1}(\alpha) = f(\Phi^{-1}(1-\alpha))$. In fact, the uncertainty distribution of $f(\xi)$ is

$$\Psi(x) = 1 - \Phi(f^{-1}(x)).$$

The theorem is proved.

99-Method 1.2. *Let ξ be an uncertain variable represented by a 99-table,*

0.01	0.02	0.03	$\cdots$	0.99
x_1	x_2	x_3	$\cdots$	x_{99}

(1.64)

Then for any strictly decreasing function $f(x)$, the uncertain variable $f(\xi)$ has a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
$f(x_{99})$	$f(x_{98})$	$f(x_{97})$	$\cdots$	$f(x_1)$

(1.65)

Example 1.21: Let ξ be an uncertain variable with uncertainty distribution Φ. Then $-\xi$ has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = -\Phi^{-1}(1-\alpha). \tag{1.66}$$

If ξ is represented by a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
x_1	x_2	x_3	$\cdots$	x_{99}

(1.67)

then the 99-method yields that the uncertain variable $-\xi$ has a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
$-x_{99}$	$-x_{98}$	$-x_{97}$	$\cdots$	$-x_1$

(1.68)

Example 1.22: Let ξ be a positive uncertain variable with uncertainty distribution Φ. Since $1/x$ is a strictly decreasing function on $(0,+\infty)$, the reciprocal $1/\xi$ is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \frac{1}{\Phi^{-1}(1-\alpha)}. \tag{1.69}$$

If ξ is represented by a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
x_1	x_2	x_3	$\cdots$	x_{99}

(1.70)

then the 99-method yields that the uncertain variable $1/\xi$ has a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
$1/x_{99}$	$1/x_{98}$	$1/x_{97}$	$\cdots$	$1/x_1$

(1.71)

Example 1.23: Let ξ be an uncertain variable with uncertainty distribution Φ. Since $\exp(-x)$ is a strictly decreasing function, $\exp(-\xi)$ is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \exp\left(-\Phi^{-1}(1-\alpha)\right). \tag{1.72}$$

If ξ is represented by a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
x_1	x_2	x_3	$\cdots$	x_{99}

(1.73)

then the 99-method yields that the uncertain variable $\exp(-\xi)$ has a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
$\exp(-x_{99})$	$\exp(-x_{98})$	$\exp(-x_{97})$	$\cdots$	$\exp(-x_1)$

(1.74)

Increasing Function of Multiple Uncertain Variables

A real-valued function $f(x_1, x_2, \cdots, x_n)$ is said to be strictly increasing if

$$f(x_1, x_2, \cdots, x_n) < f(y_1, y_2, \cdots, y_n) \tag{1.75}$$

whenever $x_i \le y_i$ for $i = 1, 2, \cdots, n$ and $x_j < y_j$ for at least one index j.

Theorem 1.20. *Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f : \Re^n \to \Re$ is a strictly increasing function, then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{1.76}$$

is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)). \tag{1.77}$$

Proof: Since $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables and f is a strictly increasing function, we have

$$\begin{aligned} & \mathcal{M}\{\xi \le \Psi^{-1}(\alpha)\} \\ = \; & \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) \le f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha))\} \\ \ge \; & \mathcal{M}\{(\xi_1 \le \Phi_1^{-1}(\alpha)) \cap (\xi_2 \le \Phi_2^{-1}(\alpha)) \cap \cdots \cap (\xi_n \le \Phi_n^{-1}(\alpha))\} \\ = \; & \mathcal{M}\{\xi_1 \le \Phi_1^{-1}(\alpha)\} \wedge \mathcal{M}\{\xi_2 \le \Phi_2^{-1}(\alpha)\} \wedge \cdots \wedge \mathcal{M}\{\xi_n \le \Phi_n^{-1}(\alpha)\} \\ = \; & \alpha \wedge \alpha \wedge \cdots \wedge \alpha = \alpha. \end{aligned}$$

On the other hand, there exists some index i such that

$$\{f(\xi_1, \xi_2, \cdots, \xi_n) \le f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha))\} \subset \{\xi_i \le \Phi_i^{-1}(\alpha)\}.$$

Thus

$$\mathcal{M}\{\xi \le \Psi^{-1}(\alpha)\} \le \mathcal{M}\{\xi_i \le \Phi_i^{-1}(\alpha)\} = \alpha.$$

It follows that $\mathcal{M}\{\xi \le \Psi^{-1}(\alpha)\} = \alpha$. In other words, Ψ is just the uncertainty distribution of ξ. In fact, we also have

$$\Psi(x) = \sup_{f(x_1, x_2, \cdots, x_n) = x} \; \min_{1 \le i \le n} \Phi_i(x_i). \tag{1.78}$$

The theorem is proved.

99-Method 1.3. *Assume $\xi_1, \xi_2, \cdots, \xi_n$ are uncertain variables, and each ξ_i is represented by a 99-table,*

0.01	0.02	0.03	$\cdots$	0.99
x_1^i	x_2^i	x_3^i	$\cdots$	x_{99}^i

(1.79)

Then for any strictly increasing function $f(x_1, x_2, \cdots, x_n)$, the uncertain variable $f(\xi_1, \xi_2, \cdots, \xi_n)$ has a 99-table,

0.01	0.02	$\cdots$	0.99
$f(x_1^1, x_1^2, \cdots, x_1^n)$	$f(x_2^1, x_2^2, \cdots, x_2^n)$	$\cdots$	$f(x_{99}^1, x_{99}^2, \cdots, x_{99}^n)$

(1.80)

Example 1.24: Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Then the sum

$$\xi = \xi_1 + \xi_2 + \cdots + \xi_n \tag{1.81}$$

is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) + \cdots + \Phi_n^{-1}(\alpha). \tag{1.82}$$

If each ξ_i $(1 \le i \le n)$ is represented by a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
x_1^i	x_2^i	x_3^i	$\cdots$	x_{99}^i

(1.83)

then the 99-method yields that the sum $\xi_1 + \xi_2 + \cdots + \xi_n$ has a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
$\sum_{i=1}^n x_1^i$	$\sum_{i=1}^n x_2^i$	$\sum_{i=1}^n x_3^i$	$\cdots$	$\sum_{i=1}^n x_{99}^i$

(1.84)

Example 1.25: Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent and nonnegative uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Then the product

$$\xi = \xi_1 \times \xi_2 \times \cdots \times \xi_n \tag{1.85}$$

is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) \times \Phi_2^{-1}(\alpha) \times \cdots \times \Phi_n^{-1}(\alpha). \tag{1.86}$$

If each ξ_i $(1 \le i \le n)$ is represented by a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
x_1^i	x_2^i	x_3^i	$\cdots$	x_{99}^i

(1.87)

then the 99-method yields that the product $\xi_1 \times \xi_2 \times \cdots \times \xi_n$ has a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
$\prod_{i=1}^n x_1^i$	$\prod_{i=1}^n x_2^i$	$\prod_{i=1}^n x_3^i$	$\cdots$	$\prod_{i=1}^n x_{99}^i$

(1.88)

Example 1.26: Assume ξ_1, ξ_2, ξ_3 are independent and nonnegative uncertain variables with uncertainty distributions Φ_1, Φ_2, Φ_3, respectively. Then the inverse uncertainty distribution of $(\xi_1 + \xi_2)\xi_3$ is

$$\Psi^{-1}(\alpha) = \left(\Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha)\right) \Phi_3^{-1}(\alpha). \tag{1.89}$$

If ξ_1, ξ_2, ξ_3 are respectively represented by 99-tables,

0.01	0.02	0.03	$\cdots$	0.99
x_1^1	x_2^1	x_3^1	$\cdots$	x_{99}^1

0.01	0.02	0.03	$\cdots$	0.99
x_1^2	x_2^2	x_3^2	$\cdots$	x_{99}^2

0.01	0.02	0.03	$\cdots$	0.99
x_1^3	x_2^3	x_3^3	$\cdots$	x_{99}^3

(1.90)

then the 99-method yields that the uncertain variable $(\xi_1 + \xi_2)\xi_3$ has a 99-table,

0.01	0.02	$\cdots$	0.99
$(x_1^1 + x_1^2)x_1^3$	$(x_2^1 + x_2^2)x_2^3$	$\cdots$	$(x_{99}^1 + x_{99}^2)x_{99}^3$

(1.91)

Theorem 1.21. *Assume that ξ_1 and ξ_2 are independent linear uncertain variables $\mathcal{L}(a_1, b_1)$ and $\mathcal{L}(a_2, b_2)$, respectively. Then the sum $\xi_1 + \xi_2$ is also a linear uncertain variable $\mathcal{L}(a_1 + a_2, b_1 + b_2)$, i.e.,*

$$\mathcal{L}(a_1, b_1) + \mathcal{L}(a_2, b_2) = \mathcal{L}(a_1 + a_2, b_1 + b_2). \tag{1.92}$$

The product of a linear uncertain variable $\mathcal{L}(a, b)$ and a scalar number $k > 0$ is also a linear uncertain variable $\mathcal{L}(ka, kb)$, i.e.,

$$k \cdot \mathcal{L}(a, b) = \mathcal{L}(ka, kb). \tag{1.93}$$

Proof: Assume that the uncertain variables ξ_1 and ξ_2 have uncertainty distributions Φ_1 and Φ_2, respectively. Then

$$\Phi_1^{-1}(\alpha) = (1 - \alpha)a_1 + \alpha b_1,$$

$$\Phi_2^{-1}(\alpha) = (1 - \alpha)a_2 + \alpha b_2.$$

It follows from the operational law that the inverse uncertainty distribution of $\xi_1 + \xi_2$ is

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) = (1 - \alpha)(a_1 + a_2) + \alpha(b_1 + b_2).$$

Hence the sum is also a linear uncertain variable $\mathcal{L}(a_1 + a_2, b_1 + b_2)$. The first part is verified. Next, suppose that the uncertainty distribution of the uncertain variable $\xi \sim \mathcal{L}(a, b)$ is Φ. It follows from the operational law that when $k > 0$, the inverse uncertainty distribution of $k\xi$ is

$$\Psi^{-1}(\alpha) = k\Phi^{-1}(\alpha) = (1 - \alpha)(ka) + \alpha(kb).$$

Hence $k\xi$ is just a linear uncertain variable $\mathcal{L}(ka, kb)$.

Theorem 1.22. *Assume that ξ_1 and ξ_2 are independent zigzag uncertain variables $\mathcal{Z}(a_1, b_1, c_1)$ and $\mathcal{Z}(a_2, b_2, c_3)$, respectively. Then the sum $\xi_1 + \xi_2$ is also a zigzag uncertain variable $\mathcal{Z}(a_1 + a_2, b_1 + b_2, c_1 + c_2)$, i.e.,*

$$\mathcal{Z}(a_1, b_1, c_1) + \mathcal{Z}(a_2, b_2, c_2) = \mathcal{Z}(a_1 + a_2, b_1 + b_2, c_1 + c_2). \tag{1.94}$$

The product of a zigzag uncertain variable $\mathcal{Z}(a, b, c)$ and a scalar number $k > 0$ is also a zigzag uncertain variable $\mathcal{Z}(ka, kb, kc)$, i.e.,

$$k \cdot \mathcal{Z}(a, b, c) = \mathcal{Z}(ka, kb, kc). \tag{1.95}$$

Proof: Assume that the uncertain variables ξ_1 and ξ_2 have uncertainty distributions Φ_1 and Φ_2, respectively. Then

$$\Phi_1^{-1}(\alpha) = \begin{cases} (1 - 2\alpha)a_1 + 2\alpha b_1, & \text{if } \alpha < 0.5 \\ (2 - 2\alpha)b_1 + (2\alpha - 1)c_1, & \text{if } \alpha \geq 0.5, \end{cases}$$

$$\Phi_2^{-1}(\alpha) = \begin{cases} (1 - 2\alpha)a_2 + 2\alpha b_2, & \text{if } \alpha < 0.5 \\ (2 - 2\alpha)b_2 + (2\alpha - 1)c_2, & \text{if } \alpha \geq 0.5. \end{cases}$$

It follows from the operational law that the inverse uncertainty distribution of $\xi_1 + \xi_2$ is

$$\Psi^{-1}(\alpha) = \begin{cases} (1 - 2\alpha)(a_1 + a_2) + 2\alpha(b_1 + b_2), & \text{if } \alpha < 0.5 \\ (2 - 2\alpha)(b_1 + b_2) + (2\alpha - 1)(c_1 + c_2), & \text{if } \alpha \geq 0.5. \end{cases}$$

Hence the sum is also a zigzag uncertain variable $\mathcal{Z}(a_1 + a_2, b_1 + b_2, c_1 + c_2)$. The first part is verified. Next, suppose that the uncertainty distribution of the uncertain variable $\xi \sim \mathcal{Z}(a, b, c)$ is Φ. It follows from the operational law that when $k > 0$, the inverse uncertainty distribution of $k\xi$ is

$$\Psi^{-1}(\alpha) = k\Phi^{-1}(\alpha) = \begin{cases} (1 - 2\alpha)(ka) + 2\alpha(kb), & \text{if } \alpha < 0.5 \\ (2 - 2\alpha)(kb) + (2\alpha - 1)(kc), & \text{if } \alpha \geq 0.5. \end{cases}$$

Hence $k\xi$ is just a zigzag uncertain variable $\mathcal{Z}(ka, kb, kc)$.

Theorem 1.23. *Let ξ_1 and ξ_2 be independent normal uncertain variables $\mathcal{N}(e_1, \sigma_1)$ and $\mathcal{N}(e_2, \sigma_2)$, respectively. Then the sum $\xi_1 + \xi_2$ is also a normal uncertain variable $\mathcal{N}(e_1 + e_2, \sigma_1 + \sigma_2)$, i.e.,*

$$\mathcal{N}(e_1, \sigma_1) + \mathcal{N}(e_2, \sigma_2) = \mathcal{N}(e_1 + e_2, \sigma_1 + \sigma_2). \tag{1.96}$$

The product of a normal uncertain variable $\mathcal{N}(e, \sigma)$ and a scalar number $k > 0$ is also a normal uncertain variable $\mathcal{N}(ke, k\sigma)$, i.e.,

$$k \cdot \mathcal{N}(e, \sigma) = \mathcal{N}(ke, k\sigma). \tag{1.97}$$

Proof: Assume that the uncertain variables ξ_1 and ξ_2 have uncertainty distributions Φ_1 and Φ_2, respectively. Then

$$\Phi_1^{-1}(\alpha) = e_1 + \frac{\sigma_1\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha},$$

$$\Phi_2^{-1}(\alpha) = e_2 + \frac{\sigma_2\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}.$$

It follows from the operational law that the inverse uncertainty distribution of $\xi_1 + \xi_2$ is

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) = (e_1 + e_2) + \frac{(\sigma_1+\sigma_2)\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}.$$

Hence the sum is also a normal uncertain variable $\mathcal{N}(e_1 + e_2, \sigma_1 + \sigma_2)$. The first part is verified. Next, suppose that the uncertainty distribution of the uncertain variable $\xi \sim \mathcal{N}(e, \sigma)$ is Φ. It follows from the operational law that, when $k > 0$, the inverse uncertainty distribution of $k\xi$ is

$$\Psi^{-1}(\alpha) = k\Phi^{-1}(\alpha) = (ke) + \frac{(k\sigma)\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}.$$

Hence $k\xi$ is just a normal uncertain variable $\mathcal{N}(ke, k\sigma)$.

Theorem 1.24. *Assume that ξ_1 and ξ_2 are independent lognormal uncertain variables $\mathcal{LOGN}(e_1, \sigma_1)$ and $\mathcal{LOGN}(e_2, \sigma_2)$, respectively. Then the product $\xi_1 \cdot \xi_2$ is also a lognormal uncertain variable $\mathcal{LOGN}(e_1 + e_2, \sigma_1 + \sigma_2)$, i.e.,*

$$\mathcal{LOGN}(e_1, \sigma_1) \cdot \mathcal{LOGN}(e_2, \sigma_2) = \mathcal{LOGN}(e_1 + e_2, \sigma_1 + \sigma_2). \tag{1.98}$$

The product of a lognormal uncertain variable $\mathcal{LOGN}(e, \sigma)$ and a scalar number $k > 0$ is also a lognormal uncertain variable $\mathcal{LOGN}(e + \ln k, \sigma)$, i.e.,

$$k \cdot \mathcal{LOGN}(e, \sigma) = \mathcal{LOGN}(e + \ln k, \sigma). \tag{1.99}$$

Proof: Assume that the uncertain variables ξ_1 and ξ_2 have uncertainty distributions Φ_1 and Φ_2, respectively. Then

$$\Phi_1^{-1}(\alpha) = \exp(e_1)\left(\frac{\alpha}{1-\alpha}\right)^{\sqrt{3}\sigma_1/\pi},$$

$$\Phi_2^{-1}(\alpha) = \exp(e_2)\left(\frac{\alpha}{1-\alpha}\right)^{\sqrt{3}\sigma_2/\pi}.$$

It follows from the operational law that the inverse uncertainty distribution of $\xi_1 \cdot \xi_2$ is

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) \cdot \Phi_2^{-1}(\alpha) = \exp(e_1 + e_2)\left(\frac{\alpha}{1-\alpha}\right)^{\sqrt{3}(\sigma_1+\sigma_2)/\pi}.$$

Hence the product is a lognormal uncertain variable $\mathcal{LOGN}(e_1+e_2,\sigma_1+\sigma_2)$. The first part is verified. Next, suppose that the uncertainty distribution of the uncertain variable $\xi \sim \mathcal{LOGN}(e,\sigma)$ is Φ. It follows from the operational law that, when $k>0$, the inverse uncertainty distribution of $k\xi$ is

$$\Psi^{-1}(\alpha)=k\Phi^{-1}(\alpha)=\exp(e+\ln k)\left(\frac{\alpha}{1-\alpha}\right)^{\sqrt{3}\sigma/\pi}.$$

Hence $k\xi$ is just a lognormal uncertain variable $\mathcal{LOGN}(e+\ln k,\sigma)$.

Example 1.27: Let $\xi_1,\xi_2,\cdots,\xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1,\Phi_2,\cdots,\Phi_n$, respectively. Then the maximum

$$\xi=\xi_1\vee\xi_2\vee\cdots\vee\xi_n \tag{1.100}$$

is an uncertain variable with uncertainty distribution

$$\Psi(x)=\Phi_1(x)\wedge\Phi_2(x)\wedge\cdots\wedge\Phi_n(x) \tag{1.101}$$

whose inverse function is

$$\Psi^{-1}(\alpha)=\Phi_1^{-1}(\alpha)\vee\Phi_2^{-1}(\alpha)\vee\cdots\vee\Phi_n^{-1}(\alpha). \tag{1.102}$$

If each ξ_i $(1\le i\le n)$ is represented by a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
x_1^i	x_2^i	x_3^i	$\cdots$	x_{99}^i

(1.103)

then the 99-method yields that the maximum $\xi_1\vee\xi_2\vee\cdots\vee\xi_n$ has a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
$\bigvee_{i=1}^n x_1^i$	$\bigvee_{i=1}^n x_2^i$	$\bigvee_{i=1}^n x_3^i$	$\cdots$	$\bigvee_{i=1}^n x_{99}^i$

(1.104)

Example 1.28: Let $\xi_1,\xi_2,\cdots,\xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1,\Phi_2,\cdots,\Phi_n$, respectively. Then the minimum

$$\xi=\xi_1\wedge\xi_2\wedge\cdots\wedge\xi_n \tag{1.105}$$

is an uncertain variable with uncertainty distribution

$$\Psi(x)=\Phi_1(x)\vee\Phi_2(x)\vee\cdots\vee\Phi_n(x) \tag{1.106}$$

whose inverse function is

$$\Psi^{-1}(\alpha)=\Phi_1^{-1}(\alpha)\wedge\Phi_2^{-1}(\alpha)\wedge\cdots\wedge\Phi_n^{-1}(\alpha). \tag{1.107}$$

If each ξ_i $(1 \le i \le n)$ is represented by a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
x_1^i	x_2^i	x_3^i	$\cdots$	x_{99}^i

(1.108)

then the 99-method yields that the minimum $\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n$ has a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
$\bigwedge_{i=1}^{n} x_1^i$	$\bigwedge_{i=1}^{n} x_2^i$	$\bigwedge_{i=1}^{n} x_3^i$	$\cdots$	$\bigwedge_{i=1}^{n} x_{99}^i$

(1.109)

Example 1.29: If ξ is an uncertain variable with uncertainty distribution Φ and k is a constant, then $\xi \vee k$ is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \Phi^{-1}(\alpha) \vee k \tag{1.110}$$

and has a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
$x_1 \vee k$	$x_2 \vee k$	$x_3 \vee k$	$\cdots$	$x_{99} \vee k$

(1.111)

In addition, $\xi \wedge k$ is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \Phi^{-1}(\alpha) \wedge k \tag{1.112}$$

and has a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
$x_1 \wedge k$	$x_2 \wedge k$	$x_3 \wedge k$	$\cdots$	$x_{99} \wedge k$

(1.113)

Decreasing Function of Multiple Uncertain Variables

A real-valued function $f(x_1, x_2, \cdots, x_n)$ is said to be strictly decreasing if

$$f(x_1, x_2, \cdots, x_n) > f(y_1, y_2, \cdots, y_n) \tag{1.114}$$

whenever $x_i \le y_i$ for $i = 1, 2, \cdots, n$ and $x_j < y_j$ for at least one index j.

Theorem 1.25. *Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f : \Re^n \to \Re$ is a strictly decreasing function, then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{1.115}$$

is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)). \tag{1.116}$$

Proof: Since $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables and f is a strictly decreasing function, we have

$$\begin{aligned} & \mathcal{M}\{\xi \le \Psi^{-1}(\alpha)\} \\ = & \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) \le f(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))\} \\ \ge & \mathcal{M}\{(\xi_1 \ge \Phi_1^{-1}(1-\alpha)) \cap (\xi_2 \ge \Phi_2^{-1}(1-\alpha)) \cap \cdots \cap (\xi_n \ge \Phi_n^{-1}(1-\alpha))\} \\ = & \mathcal{M}\{\xi_1 \ge \Phi_1^{-1}(1-\alpha)\} \wedge \mathcal{M}\{\xi_2 \ge \Phi_2^{-1}(1-\alpha)\} \wedge \cdots \wedge \mathcal{M}\{\xi_n \ge \Phi_n^{-1}(1-\alpha)\} \\ = & \alpha \wedge \alpha \wedge \cdots \wedge \alpha = \alpha. \quad \text{(By the continuity of } \Phi_i\text{'s)} \end{aligned}$$

On the other hand, there exists some index i such that

$$\{f(\xi_1, \cdots, \xi_n) \le f(\Phi_1^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))\} \subset \{\xi_i \ge \Phi_i^{-1}(1-\alpha)\}.$$

Thus

$$\mathcal{M}\{\xi \le \Psi^{-1}(\alpha)\} \le \mathcal{M}\{\xi_i \ge \Phi_i^{-1}(1-\alpha)\} = \alpha.$$

It follows that $\mathcal{M}\{\xi \le \Psi^{-1}(\alpha)\} = \alpha$. In other words, Ψ is just the uncertainty distribution of ξ. In fact, we also have

$$\Psi(x) = \sup_{f(x_1, x_2, \cdots, x_n) = x} \min_{1 \le i \le n} (1 - \Phi_i(x_i)). \tag{1.117}$$

The theorem is proved.

99-Method 1.4. *Assume $\xi_1, \xi_2, \cdots, \xi_n$ are uncertain variables, and each ξ_i is represented by a 99-table,*

0.01	0.02	0.03	$\cdots$	0.99
x_1^i	x_2^i	x_3^i	$\cdots$	x_{99}^i

(1.118)

Then for any strictly decreasing function $f(x_1, x_2, \cdots, x_n)$, the uncertain variable $f(\xi_1, \xi_2, \cdots, \xi_n)$ has a 99-table,

0.01	0.02	$\cdots$	0.99
$f(x_{99}^1, x_{99}^2, \cdots, x_{99}^n)$	$f(x_{98}^1, x_{98}^2, \cdots, x_{98}^n)$	$\cdots$	$f(x_1^1, x_1^2, \cdots, x_1^n)$

Alternating Monotone Function of Multiple Uncertain Variables

A real-valued function $f(x_1, x_2, \cdots, x_n)$ is said to be alternating monotone if it is increasing with respect to some variables and decreasing with respect to other variables.

Theorem 1.26. *Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$$

is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)). \quad (1.119)$$

Proof: We only prove the case of $m = 1$ and $n = 2$. Since ξ_1 and ξ_2 are independent uncertain variables and the function $f(x_1, x_2)$ is strictly increasing with respect to x_1 and strictly decreasing with x_2, we have

$$\begin{aligned}\mathcal{M}\{\xi \le \Psi^{-1}(\alpha)\} &= \mathcal{M}\{f(\xi_1, \xi_2) \le f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(1-\alpha))\}\\ &\ge \mathcal{M}\{(\xi_1 \le \Phi_1^{-1}(\alpha)) \cap (\xi_2 \ge \Phi_2^{-1}(1-\alpha))\}\\ &= \mathcal{M}\{\xi_1 \le \Phi_1^{-1}(\alpha)\} \wedge \mathcal{M}\{\xi_2 \ge \Phi_2^{-1}(1-\alpha)\}\\ &= \alpha \wedge \alpha = \alpha.\end{aligned}$$

On the other hand, the event $\{\xi \le \Psi^{-1}(\alpha)\}$ is a subset of either $\{\xi_1 \le \Phi_1^{-1}(\alpha)\}$ or $\{\xi_2 \ge \Phi_2^{-1}(1-\alpha)\}$. Thus $\mathcal{M}\{\xi \le \Psi^{-1}(\alpha)\} \le \alpha$. It follows that

$$\mathcal{M}\{\xi \le \Psi^{-1}(\alpha)\} = \alpha.$$

In other words, Ψ is just the uncertainty distribution of ξ. In fact, we also have

$$\Psi(x) = \sup_{f(x_1, x_2, \cdots, x_n) = x} \left(\min_{1 \le i \le m} \Phi_i(x_i) \wedge \min_{m+1 \le i \le n} (1 - \Phi_i(x_i)) \right). \quad (1.120)$$

The theorem is proved.

99-Method 1.5. *Assume $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables, and each ξ_i is represented by a 99-table,*

0.01	0.02	0.03	$\cdots$	0.99
x_1^i	x_2^i	x_3^i	$\cdots$	x_{99}^i

(1.121)

If the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with $x_{m+1}, x_{m+2}, \cdots, x_n$, then the uncertain variable $f(\xi_1, \xi_2, \cdots, \xi_n)$ has a 99-table,

0.01	$\cdots$	0.99
$f(x_1^1, \cdots, x_1^m, x_{99}^{m+1}, \cdots, x_{99}^n)$	$\cdots$	$f(x_{99}^1, \cdots, x_{99}^m, x_1^{m+1}, \cdots, x_1^n)$

Example 1.30: Let ξ_1 and ξ_2 be independent uncertain variables with uncertainty distributions Φ_1 and Φ_2, respectively. Then the inverse uncertainty distribution of the difference $\xi_1 - \xi_2$ is

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) - \Phi_2^{-1}(1-\alpha). \quad (1.122)$$

If ξ_1 and x_2 are respectively represented by 99-tables,

0.01	0.02	0.03	$\cdots$	0.99
x_1^1	x_2^1	x_3^1	$\cdots$	x_{99}^1

(1.123)

0.01	0.02	0.03	$\cdots$	0.99
x_1^2	x_2^2	x_3^2	$\cdots$	x_{99}^2

(1.124)

then the 99-method yields that $\xi_1 - \xi_2$ has a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
$x_1^1 - x_{99}^2$	$x_2^1 - x_{98}^2$	$x_3^1 - x_{97}^2$	$\cdots$	$x_{99}^1 - x_1^2$

(1.125)

Example 1.31: Let ξ_1 and ξ_2 be independent and positive uncertain variables with uncertainty distributions Φ_1 and Φ_2, respectively. Then the inverse uncertainty distribution of the quotient ξ_1/ξ_2 is

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha)/\Phi_2^{-1}(1-\alpha). \tag{1.126}$$

If ξ_1 and ξ_2 are respectively represented by 99-tables,

0.01	0.02	0.03	$\cdots$	0.99
x_1^1	x_2^1	x_3^1	$\cdots$	x_{99}^1

(1.127)

0.01	0.02	0.03	$\cdots$	0.99
x_1^2	x_2^2	x_3^2	$\cdots$	x_{99}^2

(1.128)

then the 99-method yields that ξ_1/ξ_2 has a 99-table,

0.01	0.02	0.03	$\cdots$	0.99
x_1^1/x_{99}^2	x_2^1/x_{98}^2	x_3^1/x_{97}^2	$\cdots$	x_{99}^1/x_1^2

(1.129)

Example 1.32: Assume ξ_1, ξ_2, ξ_3 are independent and positive uncertain variables with uncertainty distributions Φ_1, Φ_2, Φ_3, respectively. Then the inverse uncertainty distribution of $\xi_1/(\xi_2 + \xi_3)$ is

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha)/(\Phi_2^{-1}(1-\alpha) + \Phi_3^{-1}(1-\alpha)). \tag{1.130}$$

If ξ_1, ξ_2, ξ_3 are respectively represented by 99-tables,

0.01	0.02	0.03	$\cdots$	0.99
x_1^1	x_2^1	x_3^1	$\cdots$	x_{99}^1

(1.131)

0.01	0.02	0.03	$\cdots$	0.99
x_1^2	x_2^2	x_3^2	$\cdots$	x_{99}^2

(1.132)

0.01	0.02	0.03	$\cdots$	0.99
x_1^3	x_2^3	x_3^3	$\cdots$	x_{99}^3

(1.133)

then the 99-method yields that $\xi_1/(\xi_2 + \xi_3)$ has a 99-table,

0.01	0.02	$\cdots$	0.99
$x_1^1/(x_{99}^2 + x_{99}^3)$	$x_2^1/(x_{98}^2 + x_{98}^3)$	$\cdots$	$x_{99}^1/(x_1^2 + x_1^3)$

(1.134)

Operational Law for Boolean Uncertain Variables

A function is said to be Boolean if it is a mapping from $\{0,1\}^n$ to $\{0,1\}$. For example, the following are Boolean functions,

$$f(x_1, x_2, \cdots, x_n) = x_1 \vee x_2 \vee \cdots \vee x_n, \tag{1.135}$$

$$f(x_1, x_2, \cdots, x_n) = x_1 \wedge x_2 \wedge \cdots \wedge x_n. \tag{1.136}$$

An uncertain variable is said to be Boolean if it takes values either 0 or 1. For example, the following is a Boolean uncertain variable,

$$\xi = \begin{cases} 1 \text{ with uncertain measure } a \\ 0 \text{ with uncertain measure } 1-a \end{cases} \tag{1.137}$$

where a is a number between 0 and 1. This subsection introduces an operational law for this type of uncertain variables.

Theorem 1.27. *Assume that $\xi_1, \xi_2, \cdots, \xi_n$ are independent Boolean uncertain variables, i.e.,*

$$\xi_i = \begin{cases} \textit{1 with uncertain measure } a_i \\ \textit{0 with uncertain measure } 1-a_i \end{cases} \tag{1.138}$$

for $i = 1, 2, \cdots, n$. If f is a Boolean function (not necessarily monotone), then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is a Boolean uncertain variable such that

$$\mathcal{M}\{\xi = 1\} = \begin{cases} \displaystyle\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i), \\ \qquad \textit{if} \displaystyle\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) < 0.5 \\ 1 - \displaystyle\sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i), \\ \qquad \textit{if} \displaystyle\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) \ge 0.5 \end{cases} \tag{1.139}$$

and

$$\mathcal{M}\{\xi = 0\} = \begin{cases} \displaystyle\sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i), \\ \qquad \textit{if} \displaystyle\sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i) < 0.5 \\ 1 - \displaystyle\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i), \\ \qquad \textit{if} \displaystyle\sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i) \ge 0.5 \end{cases} \tag{1.140}$$

where x_i take values either 0 or 1, and ν_i are defined by

$$\nu_i(x_i) = \begin{cases} a_i, & \textit{if } x_i = 1 \\ 1 - a_i, & \textit{if } x_i = 0 \end{cases} \tag{1.141}$$

for $i = 1, 2, \cdots, n$, respectively.

Proof: It follows from the operational law and independence of uncertain variables that

$$\mathcal{M}\{\xi = 1\} = \begin{cases} \displaystyle\sup_{f(B_1,B_2,\cdots,B_n)=1} \min_{1\le i\le n} \mathcal{M}\{\xi_i \in B_i\}, \\ \quad \text{if} \displaystyle\sup_{f(B_1,B_2,\cdots,B_n)=1} \min_{1\le i\le n} \mathcal{M}\{\xi_i \in B_i\} > 0.5 \\ 1 - \displaystyle\sup_{f(B_1,B_2,\cdots,B_n)=0} \min_{1\le i\le n} \mathcal{M}\{\xi_i \in B_i\}, \\ \quad \text{if} \displaystyle\sup_{f(B_1,B_2,\cdots,B_n)=0} \min_{1\le i\le n} \mathcal{M}\{\xi_i \in B_i\} > 0.5 \\ 0.5, \text{ otherwise} \end{cases} \tag{1.142}$$

where $B_1, B_2, \cdots, B_n$ are subsets of $\{0, 1\}$, and $f(B_1, B_2, \cdots, B_n) = 1$ means $f(x_1, x_2, \cdots, x_n) = 1$ for any $x_1 \in B_1, x_2 \in B_2, \cdots, x_n \in B_n$. Please also note that

$$\nu_i(1) = \mathcal{M}\{\xi_i = 1\}, \quad \nu_i(0) = \mathcal{M}\{\xi_i = 0\}$$

for $i = 1, 2, \cdots, n$. The argument breaks down into four cases. Case 1: Assume

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) < 0.5.$$

Then we have

$$\sup_{f(B_1,B_2,\cdots,B_n)=0} \min_{1\le i\le n} \mathcal{M}\{\xi_i \in B_i\} = 1 - \sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) > 0.5.$$

It follows from (1.142) that

$$\mathcal{M}\{\xi = 1\} = \sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i).$$

Case 2: Assume

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) > 0.5.$$

Then we have

$$\sup_{f(B_1,B_2,\cdots,B_n)=1} \min_{1\le i\le n} \mathcal{M}\{\xi_i \in B_i\} = 1 - \sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i) > 0.5.$$

It follows from (1.142) that

$$\mathcal{M}\{\xi = 1\} = 1 - \sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i).$$

Case 3: Assume

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) = 0.5,$$

$$\sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i) = 0.5.$$

Then we have

$$\sup_{f(B_1,B_2,\cdots,B_n)=1} \min_{1\le i\le n} \mathcal{M}\{\xi_i \in B_i\} = 0.5,$$

$$\sup_{f(B_1,B_2,\cdots,B_n)=0} \min_{1\le i\le n} \mathcal{M}\{\xi_i \in B_i\} = 0.5.$$

It follows from (1.142) that

$$\mathcal{M}\{\xi = 1\} = 0.5 = 1 - \sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i).$$

Case 4: Assume

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) = 0.5,$$

$$\sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i) < 0.5.$$

Then we have

$$\sup_{f(B_1,B_2,\cdots,B_n)=1} \min_{1\le i\le n} \mathcal{M}\{\xi_i \in B_i\} = 1 - \sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i) > 0.5.$$

It follows from (1.142) that

$$\mathcal{M}\{\xi = 1\} = 1 - \sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i).$$

Hence the equation (1.139) is proved for the four cases. Similarly, we may verify the equation (1.140).

Theorem 1.28. *Assume that $\xi_1, \xi_2, \cdots, \xi_n$ are independent Boolean uncertain variables, i.e.,*

$$\xi_i = \begin{cases} \text{1 with uncertain measure } a_i \\ \text{0 with uncertain measure } 1 - a_i \end{cases} \tag{1.143}$$

for $i = 1, 2, \cdots, n$. Then the minimum

$$\xi = \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n \tag{1.144}$$

is a Boolean uncertain variable such that

$$\mathcal{M}\{\xi = 1\} = a_1 \wedge a_2 \wedge \cdots \wedge a_n, \tag{1.145}$$

$$\mathcal{M}\{\xi = 0\} = (1 - a_1) \vee (1 - a_2) \vee \cdots \vee (1 - a_n). \tag{1.146}$$

Proof: Since ξ is the minimum of Boolean uncertain variables, the corresponding Boolean function is

$$f(x_1, x_2, \cdots, x_n) = x_1 \wedge x_2 \wedge \cdots \wedge x_n. \tag{1.147}$$

Without loss of generality, we assume $a_1 \geq a_2 \geq \cdots \geq a_n$. Then we have

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\leq i\leq n} \nu_i(x_i) = \min_{1\leq i\leq n} \nu_i(1) = a_n,$$

$$\sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\leq i\leq n} \nu_i(x_i) = (1-a_n) \wedge \min_{1\leq i<n} (a_i \vee (1-a_i))$$

where $\nu_i(x_i)$ are defined by (1.141) for $i = 1, 2, \cdots, n$, respectively. When $a_n < 0.5$, we have

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\leq i\leq n} \nu_i(x_i) = a_n < 0.5.$$

It follows from Theorem 1.27 that

$$\mathcal{M}\{\xi = 1\} = \sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\leq i\leq n} \nu_i(x_i) = a_n.$$

When $a_n \geq 0.5$, we have

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\leq i\leq n} \nu_i(x_i) = a_n \geq 0.5.$$

It follows from Theorem 1.27 that

$$\mathcal{M}\{\xi = 1\} = 1 - \sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\leq i\leq n} \nu_i(x_i) = 1 - (1-a_n) = a_n.$$

Thus $\mathcal{M}\{\xi = 1\}$ is always a_n, i.e., the minimum value of $a_1, a_2, \cdots, a_n$. Thus the equation (1.145) is proved. The equation (1.146) may be verified by the self-duality of uncertain measure.

Theorem 1.29. *Assume that $\xi_1, \xi_2, \cdots, \xi_n$ are independent Boolean uncertain variables, i.e.,*

$$\xi_i = \begin{cases} 1 \text{ with uncertain measure } a_i \\ 0 \text{ with uncertain measure } 1 - a_i \end{cases} \tag{1.148}$$

for $i = 1, 2, \cdots, n$. Then the maximum

$$\xi = \xi_1 \vee \xi_2 \vee \cdots \vee \xi_n \tag{1.149}$$

is a Boolean uncertain variable such that

$$\mathcal{M}\{\xi = 1\} = a_1 \vee a_2 \vee \cdots \vee a_n, \tag{1.150}$$

$$\mathcal{M}\{\xi = 0\} = (1-a_1) \wedge (1-a_2) \wedge \cdots \wedge (1-a_n). \tag{1.151}$$

Proof: Since ξ is the maximum of Boolean uncertain variables, the corresponding Boolean function is

$$f(x_1, x_2, \cdots, x_n) = x_1 \vee x_2 \vee \cdots \vee x_n. \tag{1.152}$$

Without loss of generality, we assume $a_1 \geq a_2 \geq \cdots \geq a_n$. Then we have

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\leq i\leq n} \nu_i(x_i) = a_1 \wedge \min_{1<i\leq n} (a_i \vee (1-a_i)),$$

$$\sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\leq i\leq n} \nu_i(x_i) = \min_{1\leq i\leq n} \nu_i(0) = 1 - a_1$$

where $\nu_i(x_i)$ are defined by (1.141) for $i = 1, 2, \cdots, n$, respectively. When $a_1 \geq 0.5$, we have

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\leq i\leq n} \nu_i(x_i) \geq 0.5.$$

It follows from Theorem 1.27 that

$$\mathcal{M}\{\xi = 1\} = 1 - \sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\leq i\leq n} \nu_i(x_i) = 1 - (1 - a_1) = a_1.$$

When $a_1 < 0.5$, we have

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\leq i\leq n} \nu_i(x_i) = a_1 < 0.5.$$

It follows from Theorem 1.27 that

$$\mathcal{M}\{\xi = 1\} = \sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\leq i\leq n} \nu_i(x_i) = a_1.$$

Thus $\mathcal{M}\{\xi = 1\}$ is always a_1, i.e., the maximum value of $a_1, a_2, \cdots, a_n$. Thus the equation (1.150) is proved. The equation (1.151) may be verified by the self-duality of uncertain measure.

Theorem 1.30. *Assume that $\xi_1, \xi_2, \cdots, \xi_n$ are independent Boolean uncertain variables, i.e.,*

$$\xi_i = \begin{cases} \text{1 with uncertain measure } a_i \\ \text{0 with uncertain measure } 1 - a_i \end{cases} \tag{1.153}$$

for $i = 1, 2, \cdots, n$. Then (k-out-of-n)

$$\xi = \begin{cases} 1, & \text{if } \xi_1 + \xi_2 + \cdots + \xi_n \geq k \\ 0, & \text{if } \xi_1 + \xi_2 + \cdots + \xi_n < k \end{cases} \tag{1.154}$$

is a Boolean uncertain variable such that

$$\mathcal{M}\{\xi = 1\} = \text{the } k\text{th largest value of } a_1, a_2, \cdots, a_n, \tag{1.155}$$

$$\mathcal{M}\{\xi = 0\} = \text{the } k\text{th smallest value of } 1 - a_1, 1 - a_2, \cdots, 1 - a_n. \tag{1.156}$$

Proof: This is the so-called k-out-of-n system. The corresponding Boolean function is

$$f(x_1, x_2, \cdots, x_n) = \begin{cases} 1, & \text{if } x_1 + x_2 + \cdots + x_n \geq k \\ 0, & \text{if } x_1 + x_2 + \cdots + x_n < k. \end{cases} \tag{1.157}$$

Without loss of generality, we assume $a_1 \geq a_2 \geq \cdots \geq a_n$. Then we have

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\leq i\leq n} \nu_i(x_i) = a_k \wedge \min_{k<i\leq n} (a_i \vee (1 - a_i)),$$

$$\sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\leq i\leq n} \nu_i(x_i) = (1 - a_k) \wedge \min_{k<i\leq n} (a_i \vee (1 - a_i))$$

where $\nu_i(x_i)$ are defined by (1.141) for $i = 1, 2, \cdots, n$, respectively. When $a_k \geq 0.5$, we have

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\leq i\leq n} \nu_i(x_i) \geq 0.5.$$

It follows from Theorem 1.27 that

$$\mathcal{M}\{\xi = 1\} = 1 - \sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\leq i\leq n} \nu_i(x_i) = 1 - (1 - a_k) = a_k.$$

When $a_k < 0.5$, we have

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\leq i\leq n} \nu_i(x_i) = a_k < 0.5.$$

It follows from Theorem 1.27 that

$$\mathcal{M}\{\xi = 1\} = \sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\leq i\leq n} \nu_i(x_i) = a_k.$$

Thus $\mathcal{M}\{\xi = 1\}$ is always a_k, i.e., the kth largest value of $a_1, a_2, \cdots, a_n$. Thus the equation (1.155) is proved. The equation (1.156) may be verified by the self-duality of uncertain measure.

Operational Law with Joint Uncertainty Distribution

Let $\xi_1, \xi_2, \cdots, \xi_n$ be uncertain variables with joint uncertainty distribution Φ. It is clear that $\Phi^{-1}(\alpha)$ is a set of $\Re^n$ rather than a single point. Assume $f : \Re^n \to \Re$ is an increasing function. It follows from the operational law and maximum uncertainty principle that $f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \begin{cases} \displaystyle\min_{(x_1,x_2,\cdots,x_n)\in\Phi^{-1}(\alpha)} f(x_1, x_2, \cdots, x_n), & \text{if } \alpha \leq 0.5 \\ \displaystyle\max_{(x_1,x_2,\cdots,x_n)\in\Phi^{-1}(\alpha)} f(x_1, x_2, \cdots, x_n), & \text{if } \alpha > 0.5. \end{cases} \tag{1.158}$$

If $f : \Re^n \to \Re$ is a decreasing function, then $f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \begin{cases} \min\limits_{(x_1,x_2,\cdots,x_n)\in\Phi^{-1}(1-\alpha)} f(x_1, x_2, \cdots, x_n), & \text{if } \alpha \le 0.5 \\ \max\limits_{(x_1,x_2,\cdots,x_n)\in\Phi^{-1}(1-\alpha)} f(x_1, x_2, \cdots, x_n), & \text{if } \alpha > 0.5. \end{cases} \tag{1.159}$$

1.6 Expected Value

Expected value is the average value of uncertain variable in the sense of uncertain measure, and represents the size of uncertain variable.

Definition 1.20 *(Liu [120]). Let ξ be an uncertain variable. Then the expected value of ξ is defined by*

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \ge r\}\mathrm{d}r - \int_{-\infty}^0 \mathcal{M}\{\xi \le r\}\mathrm{d}r \tag{1.160}$$

provided that at least one of the two integrals is finite.

Theorem 1.31. *Let ξ be an uncertain variable with uncertainty distribution Φ. If the expected value exists, then*

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x. \tag{1.161}$$

Proof: It follows from the definitions of expected value operator and uncertainty distribution that

$$\begin{aligned} E[\xi] &= \int_0^{+\infty} \mathcal{M}\{\xi \ge r\}\mathrm{d}r - \int_{-\infty}^0 \mathcal{M}\{\xi \le r\}\mathrm{d}r \\ &= \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x. \end{aligned}$$

See Figure 1.13. The theorem is proved.

Theorem 1.32. *Let ξ be an uncertain variable with uncertainty distribution Φ. If the expected value exists, then*

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha. \tag{1.162}$$

Proof: It follows from the definitions of expected value operator and uncertainty distribution that

$$\begin{aligned} E[\xi] &= \int_0^{+\infty} \mathcal{M}\{\xi \ge r\}\mathrm{d}r - \int_{-\infty}^0 \mathcal{M}\{\xi \le r\}\mathrm{d}r \\ &= \int_{\Phi(0)}^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha + \int_0^{\Phi(0)} \Phi^{-1}(\alpha)\mathrm{d}\alpha = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha. \end{aligned}$$

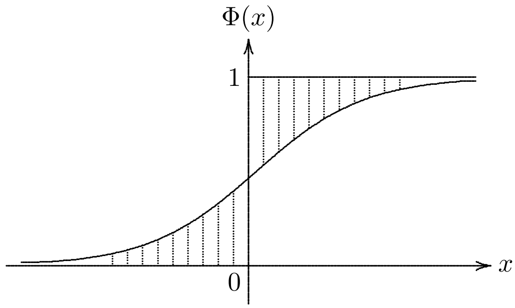

Figure 1.13: $E[\xi] = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x$

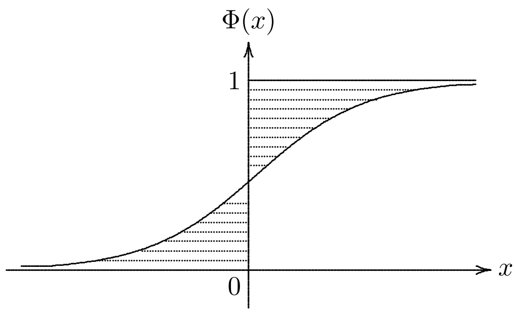

Figure 1.14: $E[\xi] = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha$

See Figure 1.14. The theorem is proved.

Theorem 1.33. *Let ξ be an uncertain variable with uncertainty distribution Φ. If the expected value exists, then*

$$E[\xi] = \int_{-\infty}^{+\infty} x\mathrm{d}\Phi(x). \tag{1.163}$$

Proof: It follows from Theorem 1.32 that

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha.$$

Now write $\Phi^{-1}(\alpha) = x$. Then we immediately have $\alpha = \Phi(x)$. The change of variable of integral produces (1.163). The theorem is verified.

Example 1.33: Suppose that ξ is a discrete uncertain variable represented by

$$\xi = \begin{array}{|c|c|c|c|}\hline \alpha_1 & \alpha_2 & \cdots & \alpha_m \\ \hline x_1 & x_2 & \cdots & x_m \\ \hline\end{array} \tag{1.164}$$

where $x_1 < x_2 < \cdots < x_m$ and $0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_m = 1$. The uncertainty distribution Φ of ξ is a step function shown in (1.35). Write $\alpha_0 \equiv 0$. If $x_1 \ge 0$, then the expected value is

$$\begin{aligned} E[\xi] &= \int_0^{x_1} 1\mathrm{d}x + \sum_{i=1}^{m-1} \int_{x_i}^{x_{i+1}} (1-\alpha_i)\mathrm{d}x + \int_{x_m}^{+\infty} 0\mathrm{d}x \\ &= x_1 + \sum_{i=1}^{m-1} (1-\alpha_i)(x_{i+1}-x_i) + 0 \\ &= \sum_{i=1}^{m} (\alpha_i - \alpha_{i-1}) x_i. \end{aligned}$$

If $x_m \le 0$, then the expected value is

$$\begin{aligned} E[\xi] &= -\int_{-\infty}^{x_1} 0\mathrm{d}x - \sum_{i=1}^{m-1} \int_{x_i}^{x_{i+1}} \alpha_i \mathrm{d}x - \int_{x_m}^{0} 1\mathrm{d}x \\ &= 0 - \sum_{i=1}^{m-1} \alpha_i (x_{i+1}-x_i) + x_m \\ &= \sum_{i=1}^{m} (\alpha_i - \alpha_{i-1}) x_i. \end{aligned}$$

If there exists an index k such that $x_k \le 0 \le x_{k+1}$, then the expected value is

$$\begin{aligned} E[\xi] &= \int_0^{x_{k+1}} (1-\alpha_k)\mathrm{d}x + \sum_{i=k+1}^{m-1} \int_{x_i}^{x_{i+1}} (1-\alpha_i)\mathrm{d}x \\ &\qquad - \sum_{i=1}^{k-1} \int_{x_i}^{x_{i+1}} \alpha_i \mathrm{d}x - \int_{x_k}^{0} \alpha_k \mathrm{d}x \\ &= x_{k+1}(1-\alpha_k) + \sum_{i=k+1}^{m-1} (1-\alpha_i)(x_{i+1}-x_i) \\ &\qquad - \sum_{i=1}^{k-1} \alpha_i (x_{i+1}-x_i) + x_k \alpha_k \\ &= \sum_{i=1}^{m} (\alpha_i - \alpha_{i-1}) x_i. \end{aligned}$$

Thus we always have the expected value

$$E[\xi] = \sum_{i=1}^{m} (\alpha_i - \alpha_{i-1}) x_i \tag{1.165}$$

where $\alpha_0 \equiv 0$ and $\alpha_m \equiv 1$.

Example 1.34: Let $\xi \sim \mathcal{L}(a,b)$ be a linear uncertain variable. If $a \ge 0$, then the expected value is

$$E[\xi] = \left(\int_0^a 1\mathrm{d}x + \int_a^b \left(1 - \frac{x-a}{b-a}\right)\mathrm{d}x + \int_b^{+\infty} 0\mathrm{d}x\right) - \int_{-\infty}^0 0\mathrm{d}x = \frac{a+b}{2}.$$

If $b \le 0$, then the expected value is

$$E[\xi] = \int_0^{+\infty} 0\mathrm{d}x - \left(\int_{-\infty}^a 0\mathrm{d}x + \int_a^b \frac{x-a}{b-a}\mathrm{d}x + \int_b^0 1\mathrm{d}x\right) = \frac{a+b}{2}.$$

If $a < 0 < b$, then the expected value is

$$E[\xi] = \int_0^b \left(1 - \frac{x-a}{b-a}\right)\mathrm{d}x - \int_a^0 \frac{x-a}{b-a}\mathrm{d}x = \frac{a+b}{2}.$$

Thus we always have the expected value

$$E[\xi] = \frac{a+b}{2}. \tag{1.166}$$

Example 1.35: The zigzag uncertain variable $\xi \sim \mathcal{Z}(a,b,c)$ has an expected value

$$E[\xi] = \frac{a+2b+c}{4}. \tag{1.167}$$

Example 1.36: The normal uncertain variable $\xi \sim \mathcal{N}(e,\sigma)$ has an expected value e, i.e.,

$$E[\xi] = e. \tag{1.168}$$

Example 1.37: If $\sigma < \pi/\sqrt{3}$, then the lognormal uncertain variable $\xi \sim \mathcal{LOGN}(e,\sigma)$ has an expected value

$$E[\xi] = \sqrt{3}\sigma \exp(e) \csc(\sqrt{3}\sigma). \tag{1.169}$$

Otherwise, $E[\xi] = +\infty$.

Linearity of Expected Value Operator

Theorem 1.34. *Let ξ and η be independent uncertain variables with finite expected values. Then for any real numbers a and b, we have*

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]. \tag{1.170}$$

Proof: Suppose that ξ and η have uncertainty distributions Φ and Ψ, respectively.

STEP 1: We first prove $E[a\xi] = aE[\xi]$. If $a = 0$, then the equation holds trivially. If $a > 0$, then the inverse uncertainty distribution of $a\xi$ is

$$\Upsilon^{-1}(\alpha) = a\Phi^{-1}(\alpha).$$

It follows from Theorem 1.32 that

$$E[a\xi] = \int_0^1 a\Phi^{-1}(\alpha)\mathrm{d}\alpha = a\int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha = aE[\xi].$$

If $a < 0$, then the inverse uncertainty distribution of $a\xi$ is

$$\Upsilon^{-1}(\alpha) = a\Phi^{-1}(1-\alpha).$$

It follows from Theorem 1.32 that

$$E[a\xi] = \int_0^1 a\Phi^{-1}(1-\alpha)\mathrm{d}\alpha = a\int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha = aE[\xi].$$

Thus we always have $E[a\xi] = aE[\xi]$.

STEP 2: We prove $E[\xi + \eta] = E[\xi] + E[\eta]$. The inverse uncertainty distribution of the sum $\xi + \eta$ is

$$\Upsilon^{-1}(\alpha) = \Phi^{-1}(\alpha) + \Psi^{-1}(\alpha).$$

It follows from Theorem 1.32 that

$$E[\xi+\eta] = \int_0^1 \Upsilon^{-1}(\alpha)\mathrm{d}\alpha = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha + \int_0^1 \Psi^{-1}(\alpha)\mathrm{d}\alpha = E[\xi] + E[\eta].$$

STEP 3: Finally, for any real numbers a and b, it follows from Steps 1 and 2 that

$$E[a\xi + b\eta] = E[a\xi] + E[b\eta] = aE[\xi] + bE[\eta].$$

The theorem is proved.

Example 1.38: Generally speaking, the expected value operator is not necessarily linear if ξ and η are not independent. For example, take $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \gamma_3\}$ with $\mathcal{M}\{\gamma_1\} = 0.7$, $\mathcal{M}\{\gamma_2\} = 0.3$, $\mathcal{M}\{\gamma_3\} = 0.2$, $\mathcal{M}\{\gamma_1, \gamma_2\} = 0.8$, $\mathcal{M}\{\gamma_1, \gamma_3\} = 0.7$, $\mathcal{M}\{\gamma_2, \gamma_3\} = 0.3$. The uncertain variables are defined by

$$\xi_1(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_1 \\ 0, & \text{if } \gamma = \gamma_2 \\ 2, & \text{if } \gamma = \gamma_3, \end{cases} \qquad \xi_2(\gamma) = \begin{cases} 0, & \text{if } \gamma = \gamma_1 \\ 2, & \text{if } \gamma = \gamma_2 \\ 3, & \text{if } \gamma = \gamma_3. \end{cases}$$

Note that ξ_1 and ξ_2 are not independent, and their sum is

$$(\xi_1 + \xi_2)(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_1 \\ 2, & \text{if } \gamma = \gamma_2 \\ 5, & \text{if } \gamma = \gamma_3. \end{cases}$$

Thus $E[\xi_1]=0.9$, $E[\xi_2]=0.8$, and $E[\xi_1+\xi_2]=1.9$. This fact implies that

$$E[\xi_1+\xi_2]>E[\xi_1]+E[\xi_2].$$

If the uncertain variables are defined by

$$\eta_1(\gamma)=\begin{cases}0, & \text{if } \gamma=\gamma_1\\ 1, & \text{if } \gamma=\gamma_2\\ 2, & \text{if } \gamma=\gamma_3,\end{cases}\qquad \eta_2(\gamma)=\begin{cases}0, & \text{if } \gamma=\gamma_1\\ 3, & \text{if } \gamma=\gamma_2\\ 1, & \text{if } \gamma=\gamma_3.\end{cases}$$

Then we have

$$(\eta_1+\eta_2)(\gamma)=\begin{cases}0, & \text{if } \gamma=\gamma_1\\ 4, & \text{if } \gamma=\gamma_2\\ 3, & \text{if } \gamma=\gamma_3.\end{cases}$$

Thus $E[\eta_1]=0.5$, $E[\eta_2]=0.9$, and $E[\eta_1+\eta_2]=1.2$. This fact implies that

$$E[\eta_1+\eta_2]<E[\eta_1]+E[\eta_2].$$

Expected Value of Function of Single Uncertain Variable

Let ξ be an uncertain variable, and $f:\Re\to\Re$ a function. Then the expected value of $f(\xi)$ is

$$E[f(\xi)]=\int_0^{+\infty}\mathcal{M}\{f(\xi)\ge r\}\mathrm{d}r-\int_{-\infty}^0\mathcal{M}\{f(\xi)\le r\}\mathrm{d}r.$$

For random case, it has been proved that the expected value $E[f(\xi)]$ is the Lebesgue-Stieltjes integral of $f(x)$ with respect to the uncertainty distribution Φ of ξ if the integral exists. However, generally speaking, it is not true for uncertain case.

Example 1.39: We consider an uncertain variable ξ whose first identification function is given by

$$\lambda(x)=\begin{cases}0.3, & \text{if } -1\le x<0\\ 0.5, & \text{if } 0\le x\le 1.\end{cases}$$

Then the expected value $E[\xi^2]=0.5$. However, the uncertainty distribution of ξ is

$$\Phi(x)=\begin{cases}0, & \text{if } x<-1\\ 0.3, & \text{if } -1\le x<0\\ 0.5, & \text{if } 0\le x<1\\ 1, & \text{if } x\ge 1\end{cases}$$

and the Lebesgue-Stieltjes integral

$$\int_{-\infty}^{+\infty}x^2\mathrm{d}\Phi(x)=(-1)^2\times 0.3+0^2\times 0.2+1^2\times 0.5=0.8\ne E[\xi^2].$$

Theorem 1.35 *(Liu and Ha [132]). Let ξ be an uncertain variable whose uncertainty distribution Φ. If $f(x)$ is a strictly monotone function such that the expected value $E[f(\xi)]$ exists, then*

$$E[f(\xi)] = \int_{-\infty}^{+\infty} f(x)\mathrm{d}\Phi(x). \tag{1.171}$$

Proof: We first suppose that $f(x)$ is a strictly increasing function. Then $f(\xi)$ has an uncertainty distribution $\Phi(f^{-1}(x))$. It follows from the change of variable of integral that

$$E[f(\xi)] = \int_{-\infty}^{+\infty} x\mathrm{d}\Phi(f^{-1}(x)) = \int_{-\infty}^{+\infty} f(x)\mathrm{d}\Phi(x).$$

If $f(x)$ is a strictly decreasing function, then $-f(x)$ is a strictly increasing function. Hence

$$E[f(\xi)] = -E[-f(\xi)] = -\int_{-\infty}^{+\infty} -f(x)\mathrm{d}\Phi(x) = \int_{-\infty}^{+\infty} f(y)\mathrm{d}\Phi(y).$$

The theorem is verified.

Example 1.40: Let ξ be a positive linear uncertain variable $\mathcal{L}(a,b)$. Then its uncertainty distribution is $\Phi(x) = (x-a)/(b-a)$. Thus

$$E[\xi^2] = \int_a^b x^2\mathrm{d}\Phi(x) = \frac{a^2+b^2+ab}{3}.$$

Example 1.41: Let ξ be a positive linear uncertain variable $\mathcal{L}(a,b)$. Then its uncertainty distribution is $\Phi(x) = (x-a)/(b-a)$. Thus

$$E[\exp(\xi)] = \int_a^b \exp(x)\mathrm{d}\Phi(x) = \frac{\exp(b)-\exp(a)}{b-a}.$$

Theorem 1.36 *(Liu and Ha [132]). Assume ξ is an uncertain variable with uncertainty distribution Φ. If $f(x)$ is a strictly monotone function such that the expected value $E[f(\xi)]$ exists, then*

$$E[f(\xi)] = \int_0^1 f(\Phi^{-1}(\alpha))\mathrm{d}\alpha. \tag{1.172}$$

Proof: Suppose that f is a strictly increasing function. It follows from Theorem 1.20 that the inverse uncertainty distribution of $f(\xi)$ is

$$\Psi^{-1}(\alpha) = f(\Phi^{-1}(\alpha)).$$

By using Theorem 1.32, the equation (1.172) is proved. When f is a strictly decreasing function, it follows from Theorem 1.25 that the inverse uncertainty distribution of $f(\xi)$ is

$$\Psi^{-1}(\alpha) = f(\Phi^{-1}(1-\alpha)).$$

By using Theorem 1.32 and the change of variable of integral, we get the equation (1.172). The theorem is verified.

Example 1.42: Let ξ be a nonnegative uncertain variable with uncertainty distribution Φ. Then

$$E[\sqrt{\xi}] = \int_0^1 \sqrt{\Phi^{-1}(\alpha)}\mathrm{d}\alpha. \tag{1.173}$$

Example 1.43: Let ξ be a positive uncertain variable with uncertainty distribution Φ. Then

$$E\left[\frac{1}{\xi}\right] = \int_0^1 \frac{1}{\Phi^{-1}(1-\alpha)}\mathrm{d}\alpha = \int_0^1 \frac{1}{\Phi^{-1}(\alpha)}\mathrm{d}\alpha. \tag{1.174}$$

Expected Value of Function of Multiple Uncertain Variables

Theorem 1.37 *(Liu and Ha [132]). Assume $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f : \Re^n \to \Re$ is a strictly monotone function, then the uncertain variable $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has an expected value*

$$E[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha))\mathrm{d}\alpha \tag{1.175}$$

provided that the expected value $E[\xi]$ exists.

Proof: Suppose that f is a strictly increasing function. It follows from Theorem 1.20 that the inverse uncertainty distribution of ξ is

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)).$$

By using Theorem 1.32, we obtain (1.175). When f is a strictly decreasing function, it follows from Theorem 1.25 that the inverse uncertainty distribution of ξ is

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

By using Theorem 1.32 and the change of variable of integral, we obtain (1.175). The theorem is proved.

Example 1.44: Let ξ and η be independent and nonnegative uncertain variables with uncertainty distributions Φ and Ψ, respectively. Then

$$E[\xi\eta] = \int_0^1 \Phi^{-1}(\alpha)\Psi^{-1}(\alpha)\mathrm{d}\alpha. \tag{1.176}$$

Exercise 1.1: What is the expected value of an alternating monotone function of uncertain variables?

Exercise 1.2: Let ξ and η be independent and positive uncertain variables with uncertainty distributions Φ and Ψ, respectively. Prove

$$E\left[\frac{\xi}{\eta}\right] = \int_0^1 \frac{\Phi^{-1}(\alpha)}{\Psi^{-1}(1-\alpha)}\mathrm{d}\alpha. \tag{1.177}$$

1.7 Variance

The variance of uncertain variable provides a degree of the spread of the distribution around its expected value. A small value of variance indicates that the uncertain variable is tightly concentrated around its expected value; and a large value of variance indicates that the uncertain variable has a wide spread around its expected value.

Definition 1.21 *(Liu [120]). Let ξ be an uncertain variable with finite expected value e. Then the variance of ξ is defined by $V[\xi] = E[(\xi - e)^2]$.*

Let ξ be an uncertain variable with expected value e. If we only know its uncertainty distribution Φ, then the variance

$$\begin{aligned}
V[\xi] &= \int_0^{+\infty} \mathcal{M}\{(\xi - e)^2 \ge r\}\mathrm{d}r \\
&= \int_0^{+\infty} \mathcal{M}\{(\xi \ge e + \sqrt{r}) \cup (\xi \le e - \sqrt{r})\}\mathrm{d}r \\
&\le \int_0^{+\infty} (\mathcal{M}\{\xi \ge e + \sqrt{r}\} + \mathcal{M}\{\xi \le e - \sqrt{r}\})\mathrm{d}r \\
&= \int_0^{+\infty} (1 - \Phi(e + \sqrt{r}) + \Phi(e - \sqrt{r}))\mathrm{d}r \\
&= \int_e^{+\infty} 2(r - e)(1 - \Phi(r) + \Phi(2e - r))\mathrm{d}r.
\end{aligned}$$

For this case, we will stipulate that the variance is

$$V[\xi] = 2\int_e^{+\infty} (r - e)(1 - \Phi(r) + \Phi(2e - r))\mathrm{d}r. \tag{1.178}$$

Mention that this is a stipulation rather than a precise formula!

Example 1.45: It has been verified that the linear uncertain variable $\xi \sim \mathcal{L}(a, b)$ has an expected value $(a+b)/2$. Note that the uncertainty distribution is $\Phi(x) = (x-a)/(b-a)$ when $a \le x \le b$. It follows from the stipulation (1.178) that the variance is

$$V[\xi] = 2\int_{(a+b)/2}^{b} \left(r - \frac{a+b}{2}\right)\left(1 - \frac{r-a}{b-a} + \frac{b-r}{b-a}\right)\mathrm{d}r = \frac{(b-a)^2}{12}.$$

In fact, a precise conclusion is $(b-a)^2/24 \le V[\xi] \le (b-a)^2/12$.

Example 1.46: It has been verified that the normal uncertain variable $\xi \sim \mathcal{N}(e,\sigma)$ has expected value e. It follows from the stipulation (1.178) that the variance is

$$V[\xi] = \sigma^2. \tag{1.179}$$

In fact, a precise conclusion is $\sigma^2/2 \le V[\xi] \le \sigma^2$.

Theorem 1.38. *If ξ is an uncertain variable with finite expected value, a and b are real numbers, then $V[a\xi+b] = a^2V[\xi]$.*

Proof: It follows from the definition of variance that

$$V[a\xi+b] = E\left[(a\xi+b-aE[\xi]-b)^2\right] = a^2E[(\xi-E[\xi])^2] = a^2V[\xi].$$

Theorem 1.39. *Let ξ be an uncertain variable with expected value e. Then $V[\xi]=0$ if and only if $\mathcal{M}\{\xi = e\} = 1$.*

Proof: If $V[\xi]=0$, then $E[(\xi-e)^2]=0$. Note that

$$E[(\xi-e)^2] = \int_0^{+\infty} \mathcal{M}\{(\xi-e)^2 \ge r\}\mathrm{d}r$$

which implies $\mathcal{M}\{(\xi-e)^2 \ge r\} = 0$ for any $r > 0$. Hence we have

$$\mathcal{M}\{(\xi-e)^2 = 0\} = 1.$$

That is, $\mathcal{M}\{\xi = e\} = 1$. Conversely, if $\mathcal{M}\{\xi = e\} = 1$, then we have $\mathcal{M}\{(\xi - e)^2 = 0\} = 1$ and $\mathcal{M}\{(\xi-e)^2 \ge r\} = 0$ for any $r > 0$. Thus

$$V[\xi] = \int_0^{+\infty} \mathcal{M}\{(\xi-e)^2 \ge r\}\mathrm{d}r = 0.$$

The theorem is proved.

Maximum Variance Theorem

Let ξ be an uncertain variable that takes values in $[a,b]$, but whose uncertainty distribution is otherwise arbitrary. If its expected value is given, what is the possible maximum variance? The maximum variance theorem will answer this question, thus playing an important role in treating games against nature.

Theorem 1.40. *Let f be a convex function on $[a,b]$, and ξ an uncertain variable that takes values in $[a,b]$ and has expected value e. Then*

$$E[f(\xi)] \le \frac{b-e}{b-a}f(a) + \frac{e-a}{b-a}f(b). \tag{1.180}$$

Proof: For each $\gamma \in \Gamma$, we have $a \le \xi(\gamma) \le b$ and

$$\xi(\gamma) = \frac{b-\xi(\gamma)}{b-a}a + \frac{\xi(\gamma)-a}{b-a}b.$$

It follows from the convexity of f that

$$f(\xi(\gamma)) \le \frac{b-\xi(\gamma)}{b-a}f(a) + \frac{\xi(\gamma)-a}{b-a}f(b).$$

Taking expected values on both sides, we obtain the inequality.

Theorem 1.41 *(Maximum Variance Theorem). Let ξ be an uncertain variable that takes values in $[a,b]$ and has expected value e. Then*

$$V[\xi] \le (e-a)(b-e) \tag{1.181}$$

and equality holds if the uncertain variable ξ is determined by

$$\mathcal{M}\{\xi = x\} = \begin{cases} \dfrac{b-e}{b-a}, & \text{if } x = a \\ \dfrac{e-a}{b-a}, & \text{if } x = b. \end{cases} \tag{1.182}$$

Proof: It follows from Theorem 1.40 immediately by defining $f(x) = (x-e)^2$. It is also easy to verify that the uncertain variable determined by (1.182) has variance $(e-a)(b-e)$. The theorem is proved.

1.8 Moments

Definition 1.22 *(Liu [120]). Let ξ be an uncertain variable. Then for any positive integer k,*
(a) the expected value $E[\xi^k]$ is called the kth moment;
(b) the expected value $E[|\xi|^k]$ is called the kth absolute moment;
(c) the expected value $E[(\xi - E[\xi])^k]$ is called the kth central moment;
(d) the expected value $E[|\xi - E[\xi]|^k]$ is called the kth absolute central moment.

Note that the first central moment is always 0, the first moment is just the expected value, and the second central moment is just the variance.

Theorem 1.42. *Let ξ be a nonnegative uncertain variable, and k a positive number. Then the k-th moment*

$$E[\xi^k] = k\int_0^{+\infty} r^{k-1}\mathcal{M}\{\xi \ge r\}\mathrm{d}r. \tag{1.183}$$

Proof: It follows from the nonnegativity of ξ that

$$E[\xi^k] = \int_0^\infty \mathcal{M}\{\xi^k \geq x\}\mathrm{d}x = \int_0^\infty \mathcal{M}\{\xi \geq r\}\mathrm{d}r^k = k\int_0^\infty r^{k-1}\mathcal{M}\{\xi \geq r\}\mathrm{d}r.$$

The theorem is proved.

Theorem 1.43. *Let ξ be an uncertain variable, and t a positive number. If $E[|\xi|^t] < \infty$, then*

$$\lim_{x\to\infty} x^t\mathcal{M}\{|\xi| \geq x\} = 0. \tag{1.184}$$

Conversely, if (1.184) holds for some positive number t, then $E[|\xi|^s] < \infty$ for any $0 \leq s < t$.

Proof: It follows from the definition of expected value operator that

$$E[|\xi|^t] = \int_0^{+\infty} \mathcal{M}\{|\xi|^t \geq r\}\mathrm{d}r < \infty.$$

Thus we have

$$\lim_{x\to\infty}\int_{x^t/2}^{+\infty} \mathcal{M}\{|\xi|^t \geq r\}\mathrm{d}r = 0.$$

The equation (1.184) is proved by the following relation,

$$\int_{x^t/2}^{+\infty} \mathcal{M}\{|\xi|^t \geq r\}\mathrm{d}r \geq \int_{x^t/2}^{x^t} \mathcal{M}\{|\xi|^t \geq r\}\mathrm{d}r \geq \frac{1}{2}x^t\mathcal{M}\{|\xi| \geq x\}.$$

Conversely, if (1.184) holds, then there exists a number $a > 0$ such that

$$x^t\mathcal{M}\{|\xi| \geq x\} \leq 1, \ \forall x \geq a.$$

Thus we have

$$\begin{aligned}
E[|\xi|^s] &= \int_0^a \mathcal{M}\{|\xi|^s \geq r\}\mathrm{d}r + \int_a^{+\infty} \mathcal{M}\{|\xi|^s \geq r\}\mathrm{d}r \\
&= \int_0^a \mathcal{M}\{|\xi|^s \geq r\}\mathrm{d}r + \int_a^{+\infty} sr^{s-1}\mathcal{M}\{|\xi| \geq r\}\mathrm{d}r \\
&\leq \int_0^a \mathcal{M}\{|\xi|^s \geq r\}\mathrm{d}r + s\int_a^{+\infty} r^{s-t-1}\mathrm{d}r \\
&< +\infty. \quad \left(\text{by} \int_a^{+\infty} r^p\mathrm{d}r < \infty \text{ for any } p < -1\right)
\end{aligned}$$

The theorem is proved.

Theorem 1.44. *Let ξ be an uncertain variable that takes values in $[a,b]$ and has expected value e. Then for any positive integer k, the kth absolute moment and kth absolute central moment satisfy the following inequalities,*

$$E[|\xi|^k] \le \frac{b-e}{b-a}|a|^k + \frac{e-a}{b-a}|b|^k, \tag{1.185}$$

$$E[|\xi - e|^k] \le \frac{b-e}{b-a}(e-a)^k + \frac{e-a}{b-a}(b-e)^k. \tag{1.186}$$

Proof: It follows from Theorem 1.40 immediately by defining $f(x) = |x|^k$ and $f(x) = |x-e|^k$.

1.9 Critical Values

In order to rank uncertain variables, we may use two critical values: optimistic value and pessimistic value.

Definition 1.23 *(Liu [120]). Let ξ be an uncertain variable, and $\alpha \in (0,1]$. Then*

$$\xi_{\sup}(\alpha) = \sup\left\{r \;\middle|\; \mathcal{M}\{\xi \ge r\} \ge \alpha\right\} \tag{1.187}$$

is called the α-optimistic value to ξ, and

$$\xi_{\inf}(\alpha) = \inf\left\{r \;\middle|\; \mathcal{M}\{\xi \le r\} \ge \alpha\right\} \tag{1.188}$$

is called the α-pessimistic value to ξ.

This means that the uncertain variable ξ will reach upwards of the α-optimistic value $\xi_{\sup}(\alpha)$ with uncertain measure α, and will be below the α-pessimistic value $\xi_{\inf}(\alpha)$ with uncertain measure α.

Theorem 1.45. *Let ξ be an uncertain variable with uncertainty distribution Φ. Then its α-optimistic value and α-pessimistic value are*

$$\xi_{\sup}(\alpha) = \Phi^{-1}(1-\alpha), \tag{1.189}$$

$$\xi_{\inf}(\alpha) = \Phi^{-1}(\alpha). \tag{1.190}$$

Proof: It follows from the definition of α-optimistic value and α-pessimistic value immediately.

Example 1.47: Let ξ be a linear uncertain variable $\mathcal{L}(a,b)$. Then its α-optimistic and α-pessimistic values are

$$\xi_{\sup}(\alpha) = \alpha a + (1-\alpha)b,$$

$$\xi_{\inf}(\alpha) = (1-\alpha)a + \alpha b.$$

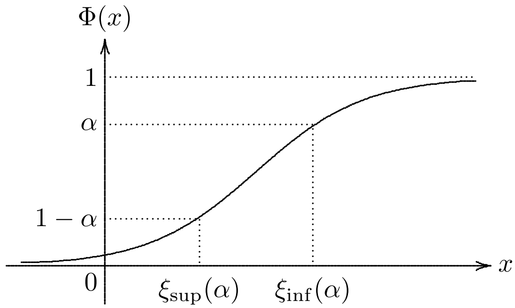

Figure 1.15: Optimistic Value and Pessimistic Value

Example 1.48: Let ξ be a zigzag uncertain variable $\mathcal{Z}(a,b,c)$. Then its α-optimistic and α-pessimistic values are

$$\xi_{\sup}(\alpha) = \begin{cases} 2\alpha b + (1-2\alpha)c, & \text{if } \alpha < 0.5 \\ (2\alpha-1)a + (2-2\alpha)b, & \text{if } \alpha \ge 0.5, \end{cases}$$

$$\xi_{\inf}(\alpha) = \begin{cases} (1-2\alpha)a + 2\alpha b, & \text{if } \alpha < 0.5 \\ (2-2\alpha)b + (2\alpha-1)c, & \text{if } \alpha \ge 0.5. \end{cases}$$

Example 1.49: Let ξ be a normal uncertain variable $\mathcal{N}(e,\sigma)$. Then its α-optimistic and α-pessimistic values are

$$\xi_{\sup}(\alpha) = e - \frac{\sigma\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha},$$

$$\xi_{\inf}(\alpha) = e + \frac{\sigma\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}.$$

Example 1.50: Let ξ be a lognormal uncertain variable $\mathcal{LOGN}(e,\sigma)$. Then its α-optimistic and α-pessimistic values are

$$\xi_{\sup}(\alpha) = \exp(e)\left(\frac{1-\alpha}{\alpha}\right)^{\sqrt{3}\sigma/\pi},$$

$$\xi_{\inf}(\alpha) = \exp(e)\left(\frac{\alpha}{1-\alpha}\right)^{\sqrt{3}\sigma/\pi}.$$

Theorem 1.46. *Let ξ be an uncertain variable, and $\alpha \in (0,1]$. Then for any positive number ε, we have*

$$\mathcal{M}\{\xi \le \xi_{\inf}(\alpha) + \varepsilon\} \ge \alpha, \quad \mathcal{M}\{\xi \ge \xi_{\sup}(\alpha) - \varepsilon\} \ge \alpha. \tag{1.191}$$

Proof: It follows from the definition of α-pessimistic value that there exists a decreasing sequence $\{x_i\}$ such that $\mathcal{M}\{\xi \le x_i\} \ge \alpha$ and $x_i \downarrow \xi_{\inf}(\alpha)$ as $i \to \infty$. Thus for any positive number ε, there exists an index i such that $x_i < \xi_{\inf}(\alpha) + \varepsilon$. Hence

$$\mathcal{M}\{\xi \le \xi_{\inf}(\alpha) + \varepsilon\} \ge \mathcal{M}\{\xi \le x_i\} \ge \alpha.$$

Similarly, there exists an increasing sequence $\{x_i\}$ such that $\mathcal{M}\{\xi \ge x_i\} \ge \alpha$ and $x_i \uparrow \xi_{\sup}(\alpha)$ as $i \to \infty$. Thus for any positive number ε, there exists an index i such that $x_i > \xi_{\sup}(\alpha) - \varepsilon$. Hence

$$\mathcal{M}\{\xi \ge \xi_{\sup}(\alpha) - \varepsilon\} \ge \mathcal{M}\{\xi \ge x_i\} \ge \alpha.$$

The theorem is proved.

Theorem 1.47. *Let ξ be an uncertain variable, and $\alpha \in (0, 1]$. Then we have*
(a) $\xi_{\inf}(\alpha)$ is an increasing and left-continuous function of α;
(b) $\xi_{\sup}(\alpha)$ is a decreasing and left-continuous function of α.

Proof: (a) Let α_1 and α_2 be two numbers with $0 < \alpha_1 < \alpha_2 \le 1$. Then for any number $r < \xi_{\sup}(\alpha_2)$, we have

$$\mathcal{M}\{\xi \ge r\} \ge \alpha_2 > \alpha_1.$$

Thus, by the definition of optimistic value, we obtain $\xi_{\sup}(\alpha_1) \ge \xi_{\sup}(\alpha_2)$. That is, the value $\xi_{\sup}(\alpha)$ is a decreasing function of α. Next, we prove the left-continuity of $\xi_{\inf}(\alpha)$ with respect to α. Let $\{\alpha_i\}$ be an arbitrary sequence of positive numbers such that $\alpha_i \uparrow \alpha$. Then $\{\xi_{\inf}(\alpha_i)\}$ is an increasing sequence. If the limitation is equal to $\xi_{\inf}(\alpha)$, then the left-continuity is proved. Otherwise, there exists a number z^* such that

$$\lim_{i\to\infty} \xi_{\inf}(\alpha_i) < z^* < \xi_{\inf}(\alpha).$$

Thus $\mathcal{M}\{\xi \le z^*\} \ge \alpha_i$ for each i. Letting $i \to \infty$, we get $\mathcal{M}\{\xi \le z^*\} \ge \alpha$. Hence $z^* \ge \xi_{\inf}(\alpha)$. A contradiction proves the left-continuity of $\xi_{\inf}(\alpha)$ with respect to α. The part (b) may be proved similarly.

Theorem 1.48. *Let ξ be an uncertain variable, and $\alpha \in (0, 1]$. Then we have*
(a) if $\alpha > 0.5$, then $\xi_{\inf}(\alpha) \ge \xi_{\sup}(\alpha)$;
(b) if $\alpha \le 0.5$, then $\xi_{\inf}(\alpha) \le \xi_{\sup}(\alpha)$.

Proof: Part (a): Write $\bar{\xi}(\alpha) = (\xi_{\inf}(\alpha) + \xi_{\sup}(\alpha))/2$. If $\xi_{\inf}(\alpha) < \xi_{\sup}(\alpha)$, then we have

$$1 \ge \mathcal{M}\{\xi < \bar{\xi}(\alpha)\} + \mathcal{M}\{\xi > \bar{\xi}(\alpha)\} \ge \alpha + \alpha > 1.$$

A contradiction proves $\xi_{\inf}(\alpha) \ge \xi_{\sup}(\alpha)$. Part (b): Assume that $\xi_{\inf}(\alpha) > \xi_{\sup}(\alpha)$. It follows from the definition of $\xi_{\inf}(\alpha)$ that $\mathcal{M}\{\xi \le \bar{\xi}(\alpha)\} < \alpha$.

Similarly, it follows from the definition of $\xi_{\sup}(\alpha)$ that $\mathcal{M}\{\xi \geq \bar{\xi}(\alpha)\} < \alpha$. Thus

$$1 \leq \mathcal{M}\{\xi \leq \bar{\xi}(\alpha)\} + \mathcal{M}\{\xi \geq \bar{\xi}(\alpha)\} < \alpha + \alpha \leq 1.$$

A contradiction proves $\xi_{\inf}(\alpha) \leq \xi_{\sup}(\alpha)$. The theorem is verified.

Theorem 1.49 *(Zuo [247]). Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions. If $f : \Re^n \to \Re$ is a continuous and strictly increasing function, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable, and*

$$\xi_{\sup}(\alpha) = f(\xi_{1\sup}(\alpha), \xi_{2\sup}(\alpha), \cdots, \xi_{n\sup}(\alpha)), \tag{1.192}$$

$$\xi_{\inf}(\alpha) = f(\xi_{1\inf}(\alpha), \xi_{2\inf}(\alpha), \cdots, \xi_{n\inf}(\alpha)). \tag{1.193}$$

Proof: Since f is a strictly increasing function, it follows from Theorem 1.20 that the inverse uncertainty distribution of ξ is

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha))$$

where $\Phi_1, \Phi_2, \cdots, \Phi_n$ are uncertainty distributions of $\xi_1, \xi_2, \cdots, \xi_n$, respectively. By using Theorem 1.45, we get (1.192) and (1.193). The theorem is proved.

Example 1.51: Let ξ be an uncertain variable, and $\alpha \in (0, 1]$. If $c \geq 0$, then

$$(c\xi)_{\sup}(\alpha) = c\xi_{\sup}(\alpha), \quad (c\xi)_{\inf}(\alpha) = c\xi_{\inf}(\alpha).$$

Example 1.52: Suppose that ξ and η are independent uncertain variables, and $\alpha \in (0, 1]$. Then we have

$$(\xi + \eta)_{\sup}(\alpha) = \xi_{\sup}(\alpha) + \eta_{\sup}(\alpha), \quad (\xi + \eta)_{\inf}(\alpha) = \xi_{\inf}(\alpha) + \eta_{\inf}(\alpha),$$

$$(\xi \vee \eta)_{\sup}(\alpha) = \xi_{\sup}(\alpha) \vee \eta_{\sup}(\alpha), \quad (\xi \vee \eta)_{\inf}(\alpha) = \xi_{\inf}(\alpha) \vee \eta_{\inf}(\alpha),$$

$$(\xi \wedge \eta)_{\sup}(\alpha) = \xi_{\sup}(\alpha) \wedge \eta_{\sup}(\alpha), \quad (\xi \wedge \eta)_{\inf}(\alpha) = \xi_{\inf}(\alpha) \wedge \eta_{\inf}(\alpha).$$

Example 1.53: Let ξ and η be independent and positive uncertain variables. Since $f(x, y) = xy$ is a strictly increasing function when $x > 0$ and $y > 0$, we immediately have

$$(\xi\eta)_{\sup}(\alpha) = \xi_{\sup}(\alpha)\eta_{\sup}(\alpha), \quad (\xi\eta)_{\inf}(\alpha) = \xi_{\inf}(\alpha)\eta_{\inf}(\alpha). \tag{1.194}$$

Theorem 1.50 *(Zuo [247]). Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions. If f is a continuous and strictly decreasing function, then*

$$\xi_{\sup}(\alpha) = f(\xi_{1\inf}(\alpha), \xi_{2\inf}(\alpha), \cdots, \xi_{n\inf}(\alpha)), \tag{1.195}$$

$$\xi_{\inf}(\alpha) = f(\xi_{1\sup}(\alpha), \xi_{2\sup}(\alpha), \cdots, \xi_{n\sup}(\alpha)). \tag{1.196}$$

Proof: Since f is a strictly decreasing function, it follows from Theorem 1.25 that the inverse uncertainty distribution of ξ is

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

By using Theorem 1.45, we get (1.195) and (1.196). The theorem is proved.

Example 1.54: Let ξ be an uncertain variable, and $\alpha \in (0,1]$. If $c < 0$, then

$$(c\xi)_{\sup}(\alpha) = c\xi_{\inf}(\alpha), \quad (c\xi)_{\inf}(\alpha) = c\xi_{\sup}(\alpha).$$

Exercise 1.3: What are the critical values to an alternating monotone function of uncertain variables?

Exercise 1.4: Let ξ and η be independent and positive uncertain variables. Prove

$$\left(\frac{\xi}{\eta}\right)_{\sup}(\alpha) = \frac{\xi_{\sup}(\alpha)}{\eta_{\inf}(\alpha)}, \qquad \left(\frac{\xi}{\eta}\right)_{\inf}(\alpha) = \frac{\xi_{\inf}(\alpha)}{\eta_{\sup}(\alpha)}. \tag{1.197}$$

1.10 Entropy

This section provides a definition of entropy to characterize the uncertainty of uncertain variables resulting from information deficiency.

Definition 1.24 *(Liu [123]). Suppose that ξ is an uncertain variable with uncertainty distribution Φ. Then its entropy is defined by*

$$H[\xi] = \int_{-\infty}^{+\infty} S(\Phi(x))\mathrm{d}x \tag{1.198}$$

where $S(t) = -t\ln t - (1-t)\ln(1-t)$.

Example 1.55: Let ξ be an uncertain variable with uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x < a \\ 1, & \text{if } x \ge a. \end{cases} \tag{1.199}$$

Essentially, ξ is a constant a. It follows from the definition of entropy that

$$H[\xi] = -\int_{-\infty}^{a} (0\ln 0 + 1\ln 1)\,\mathrm{d}x - \int_{a}^{+\infty} (1\ln 1 + 0\ln 0)\,\mathrm{d}x = 0.$$

This means a constant has no uncertainty.

Example 1.56: Let ξ be a linear uncertain variable $\mathcal{L}(a,b)$. Then its entropy is

$$H[\xi] = -\int_a^b \left(\frac{x-a}{b-a}\ln\frac{x-a}{b-a} + \frac{b-x}{b-a}\ln\frac{b-x}{b-a}\right)\mathrm{d}x = \frac{b-a}{2}. \tag{1.200}$$

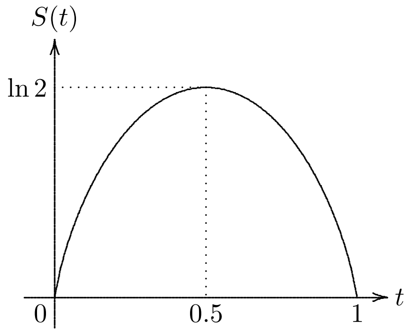

Figure 1.16: Function $S(t) = -t \ln t - (1-t)\ln(1-t)$. It is easy to verify that $S(t)$ is a symmetric function about $t = 0.5$, strictly increases on the interval $[0, 0.5]$, strictly decreases on the interval $[0.5, 1]$, and reaches its unique maximum $\ln 2$ at $t = 0.5$.

Example 1.57: Let ξ be a zigzag uncertain variable $\mathcal{Z}(a, b, c)$. Then its entropy is

$$H[\xi] = \frac{c-a}{2}. \tag{1.201}$$

Example 1.58: Let ξ be a normal uncertain variable $\mathcal{N}(e, \sigma)$. Then its entropy is

$$H[\xi] = \frac{\pi\sigma}{\sqrt{3}}. \tag{1.202}$$

Theorem 1.51. *Let ξ be an uncertain variable. Then $H[\xi] \ge 0$ and equality holds if ξ is essentially a constant.*

Proof: The positivity is clear. In addition, when an uncertain variable tends to a constant, its entropy tends to the minimum 0.

Theorem 1.52. *Let ξ be an uncertain variable taking values on the interval $[a, b]$. Then*

$$H[\xi] \le (b-a)\ln 2 \tag{1.203}$$

and equality holds if ξ has an uncertainty distribution $\Phi(x) = 0.5$ on $[a, b]$.

Proof: The theorem follows from the fact that the function $S(t)$ reaches its maximum $\ln 2$ at $t = 0.5$.

Theorem 1.53. *Let ξ be an uncertain variable, and let c be a real number. Then*

$$H[\xi + c] = H[\xi]. \tag{1.204}$$

That is, the entropy is invariant under arbitrary translations.

Proof: Write the uncertainty distribution of ξ by Φ. Then the uncertain variable $\xi + c$ has an uncertainty distribution $\Phi(x - c)$. It follows from the definition of entropy that

$$H[\xi + c] = \int_{-\infty}^{+\infty} S\left(\Phi(x - c)\right) \mathrm{d}x = \int_{-\infty}^{+\infty} S(\Phi(x))\mathrm{d}x = H[\xi].$$

The theorem is proved.

Theorem 1.54 *(Dai and Chen [24]). Let ξ be an uncertain variable with uncertainty distribution Φ. Then*

$$H[\xi] = \int_0^1 \Phi^{-1}(\alpha) \ln \frac{\alpha}{1 - \alpha} \mathrm{d}\alpha. \tag{1.205}$$

Proof: It is clear that $S(\alpha)$ is a derivable function with $S'(\alpha) = -\ln \alpha/(1 - \alpha)$. Since

$$S(\Phi(x)) = \int_0^{\Phi(x)} S'(\alpha)\mathrm{d}\alpha = -\int_{\Phi(x)}^1 S'(\alpha)\mathrm{d}\alpha,$$

we have

$$H[\xi] = \int_{-\infty}^{+\infty} S(\Phi(x))\mathrm{d}x = \int_{-\infty}^{0} \int_0^{\Phi(x)} S'(\alpha)\mathrm{d}\alpha\mathrm{d}x - \int_0^{+\infty} \int_{\Phi(x)}^1 S'(\alpha)\mathrm{d}\alpha\mathrm{d}x.$$

It follows from Fubini theorem that

$$\begin{aligned}
H[\xi] &= \int_0^{\Phi(0)} \int_{\Phi^{-1}(\alpha)}^0 S'(\alpha)\mathrm{d}x\mathrm{d}\alpha - \int_{\Phi(0)}^1 \int_0^{\Phi^{-1}(\alpha)} S'(\alpha)\mathrm{d}x\mathrm{d}\alpha \\
&= -\int_0^{\Phi(0)} \Phi^{-1}(\alpha)S'(\alpha)\mathrm{d}\alpha - \int_{\Phi(0)}^1 \Phi^{-1}(\alpha)S'(\alpha)\mathrm{d}\alpha \\
&= -\int_0^1 \Phi^{-1}(\alpha)S'(\alpha)\mathrm{d}\alpha = \int_0^1 \Phi^{-1}(\alpha) \ln \frac{\alpha}{1 - \alpha} \mathrm{d}\alpha.
\end{aligned}$$

The theorem is verified.

Theorem 1.55 *(Dai and Chen [24]). Let ξ and η be independent uncertain variables. Then for any real numbers a and b, we have*

$$H[a\xi + b\eta] = |a|H[\xi] + |b|H[\eta]. \tag{1.206}$$

Proof: Suppose that ξ and η have uncertainty distributions Φ and Ψ, respectively.

STEP 1: We prove $H[a\xi] = |a|H[\xi]$. If $a > 0$, then the inverse uncertainty distribution of $a\xi$ is

$$\Upsilon^{-1}(\alpha) = a\Phi^{-1}(\alpha).$$

It follows from Theorem 1.54 that

$$H[a\xi] = \int_0^1 a\Phi^{-1}(\alpha)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha = a\int_0^1 \Phi^{-1}(\alpha)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha = |a|H[\xi].$$

If $a = 0$, we immediately have $H[a\xi] = 0 = |a|H[\xi]$. If $a < 0$, then the inverse uncertainty distribution of $a\xi$ is

$$\Upsilon^{-1}(\alpha) = a\Phi^{-1}(1-\alpha).$$

It follows from Theorem 1.54 that

$$H[a\xi] = \int_0^1 a\Phi^{-1}(1-\alpha)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha = (-a)\int_0^1 \Phi^{-1}(\alpha)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha = |a|H[\xi].$$

Thus we always have $H[a\xi] = |a|H[\xi]$.

STEP 2: We prove $H[\xi+\eta] = H[\xi] + H[\eta]$. Note that the inverse uncertainty distribution of $\xi+\eta$ is

$$\Upsilon^{-1}(\alpha) = \Phi^{-1}(\alpha) + \Psi^{-1}(\alpha).$$

It follows from Theorem 1.54 that

$$H[\xi+\eta] = \int_0^1 (\Phi^{-1}(\alpha)+\Psi^{-1}(\alpha))\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha = H[\xi] + H[\eta].$$

STEP 3: Finally, for any real numbers a and b, it follows from Steps 1 and 2 that

$$H[a\xi + b\eta] = H[a\xi] + H[b\eta] = |a|H[\xi] + |b|H[\eta].$$

The theorem is proved.

Entropy of Function of Uncertain Variables

Theorem 1.56 *(Dai and Chen [24]). Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f : \Re^n \to \Re$ is a strictly increasing function, then the uncertain variable $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has an entropy*

$$H[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha))\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha. \tag{1.207}$$

Proof: Since f is a strictly increasing function, it follows from Theorem 1.20 that the inverse uncertainty distribution of ξ is

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)).$$

By using Theorem 1.54, we get (1.207). The theorem is thus verified.

Exercise 1.5: Let ξ and η be independent and nonnegative uncertain variables with uncertainty distributions Φ and Ψ, respectively. Then

$$H[\xi\eta] = \int_0^1 \Phi^{-1}(\alpha)\Psi^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha. \tag{1.208}$$

Theorem 1.57 *(Dai and Chen [24]). Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If f is a strictly decreasing function, then*

$$H[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)) \ln \frac{1-\alpha}{\alpha} \mathrm{d}\alpha. \tag{1.209}$$

Proof: Since f is a strictly decreasing function, it follows from Theorem 1.25 that the inverse uncertainty distribution of ξ is

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

By using Theorem 1.54, we get (1.209). The theorem is thus verified.

Exercise 1.6: What is the entropy of an alternating monotone function of uncertain variables?

Exercise 1.7: Let ξ and η be independent and positive uncertain variables with uncertainty distributions Φ and Ψ, respectively. Prove

$$H\left[\frac{\xi}{\eta}\right] = \int_0^1 \frac{\Phi^{-1}(\alpha)}{\Psi^{-1}(1-\alpha)} \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha. \tag{1.210}$$

Maximum Entropy Principle

Given some constraints, for example, expected value and variance, there are usually multiple compatible uncertainty distributions. Which uncertainty distribution shall we take? The *maximum entropy principle* attempts to select the uncertainty distribution that maximizes the value of entropy and satisfies the prescribed constraints.

Theorem 1.58 *(Chen and Dai [19]). Let ξ be an uncertain variable whose uncertainty distribution is arbitrary but the expected value e and variance σ^2. Then*

$$H[\xi] \le \frac{\pi\sigma}{\sqrt{3}} \tag{1.211}$$

and the equality holds if ξ is a normal uncertain variable $\mathcal{N}(e, \sigma)$.

Proof: Let $\Phi(x)$ be the uncertainty distribution of ξ and write $\Psi(x) = \Phi(2e-x)$ for $x \ge e$. It follows from the stipulation (1.178) that the variance is

$$V[\xi] = 2\int_e^{+\infty} (x-e)(1-\Phi(x))\mathrm{d}x + 2\int_e^{+\infty} (x-e)\Psi(x)\mathrm{d}x = \sigma^2.$$

Thus there exists a real number κ such that

$$2\int_e^{+\infty}(x-e)(1-\Phi(x))\mathrm{d}x = \kappa\sigma^2,$$

$$2\int_e^{+\infty}(x-e)\Psi(x)\mathrm{d}x = (1-\kappa)\sigma^2.$$

The maximum entropy distribution Φ should maximize the entropy

$$H[\xi] = \int_{-\infty}^{+\infty} S(\Phi(x))\mathrm{d}x = \int_e^{+\infty} S(\Phi(x))\mathrm{d}x + \int_e^{+\infty} S(\Psi(x))\mathrm{d}x$$

subject to the above two constraints. The Lagrangian is

$$\begin{aligned} L = & \int_e^{+\infty} S(\Phi(x))\mathrm{d}x + \int_e^{+\infty} S(\Psi(x))\mathrm{d}x \\ & -\alpha\left(2\int_e^{+\infty}(x-e)(1-\Phi(x))\mathrm{d}x - \kappa\sigma^2\right) \\ & -\beta\left(2\int_e^{+\infty}(x-e)\Psi(x)\mathrm{d}x - (1-\kappa)\sigma^2\right). \end{aligned}$$

The maximum entropy distribution meets Euler-Lagrange equations

$$\ln\Phi(x) - \ln(1-\Phi(x)) = 2\alpha(x-e),$$

$$\ln\Psi(x) - \ln(1-\Psi(x)) = 2\beta(e-x).$$

Thus Φ and Ψ have the forms

$$\Phi(x) = (1+\exp(2\alpha(e-x)))^{-1},$$

$$\Psi(x) = (1+\exp(2\beta(x-e)))^{-1}.$$

Substituting them into the variance constraints, we get

$$\Phi(x) = \left(1+\exp\left(\frac{\pi(e-x)}{\sqrt{6\kappa}\sigma}\right)\right)^{-1},$$

$$\Psi(x) = \left(1+\exp\left(\frac{\pi(x-e)}{\sqrt{6(1-\kappa)}\sigma}\right)\right)^{-1}.$$

Then the entropy is

$$H[\xi] = \frac{\pi\sigma\sqrt{\kappa}}{\sqrt{6}} + \frac{\pi\sigma\sqrt{1-\kappa}}{\sqrt{6}}$$

which achieves the maximum when $\kappa = 1/2$. Thus the maximum entropy distribution is just the normal uncertainty distribution $\mathcal{N}(e, \sigma)$.

1.11 Distance

Definition 1.25 *(Liu [120]). The distance between uncertain variables ξ and η is defined as*

$$d(\xi, \eta) = E[|\xi - \eta|]. \tag{1.212}$$

Theorem 1.59. *Let ξ, η, τ be uncertain variables, and let $d(\cdot, \cdot)$ be the distance. Then we have*
(a) (Nonnegativity) $d(\xi, \eta) \ge 0$;
(b) (Identification) $d(\xi, \eta) = 0$ if and only if $\xi = \eta$;
(c) (Symmetry) $d(\xi, \eta) = d(\eta, \xi)$;
(d) (Triangle Inequality) $d(\xi, \eta) \le 2d(\xi, \tau) + 2d(\eta, \tau)$.

Proof: The parts (a), (b) and (c) follow immediately from the definition. Now we prove the part (d). It follows from the countable subadditivity axiom that

$$\begin{aligned}
d(\xi, \eta) &= \int_0^{+\infty} \mathcal{M}\{|\xi - \eta| \ge r\} \mathrm{d}r \\
&\le \int_0^{+\infty} \mathcal{M}\{|\xi - \tau| + |\tau - \eta| \ge r\} \mathrm{d}r \\
&\le \int_0^{+\infty} \mathcal{M}\{(|\xi - \tau| \ge r/2) \cup (|\tau - \eta| \ge r/2)\} \mathrm{d}r \\
&\le \int_0^{+\infty} (\mathcal{M}\{|\xi - \tau| \ge r/2\} + \mathcal{M}\{|\tau - \eta| \ge r/2\}) \mathrm{d}r \\
&= 2E[|\xi - \tau|] + 2E[|\tau - \eta|] = 2d(\xi, \tau) + 2d(\tau, \eta).
\end{aligned}$$

Example 1.59: Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$. Define $\mathcal{M}\{\emptyset\} = 0$, $\mathcal{M}\{\Gamma\} = 1$ and $\mathcal{M}\{\Lambda\} = 1/2$ for any subset Λ (excluding $\emptyset$ and Γ). We set uncertain variables ξ, η and τ as follows,

$$\xi(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2 \\ 0, & \text{if } \gamma = \gamma_3, \end{cases} \qquad \eta(\gamma) = \begin{cases} 0, & \text{if } \gamma = \gamma_1 \\ -1, & \text{if } \gamma = \gamma_2 \\ -1, & \text{if } \gamma = \gamma_3, \end{cases} \qquad \tau(\gamma) \equiv 0.$$

It is easy to verify that $d(\xi, \tau) = d(\tau, \eta) = 1/2$ and $d(\xi, \eta) = 3/2$. Thus

$$d(\xi, \eta) = \frac{3}{2}(d(\xi, \tau) + d(\tau, \eta)).$$

A conjecture is $d(\xi, \eta) \le 1.5(d(\xi, \tau) + d(\tau, \eta))$ for arbitrary uncertain variables ξ, η and τ. This is an open problem.

1.12 Inequalities

Theorem 1.60 *(Liu [120]). Let ξ be an uncertain variable, and f a nonnegative function. If f is even and increasing on $[0,\infty)$, then for any given number $t>0$, we have*

$$\mathcal{M}\{|\xi|\ge t\}\le\frac{E[f(\xi)]}{f(t)}. \tag{1.213}$$

Proof: It is clear that $\mathcal{M}\{|\xi|\ge f^{-1}(r)\}$ is a monotone decreasing function of r on $[0,\infty)$. It follows from the nonnegativity of $f(\xi)$ that

$$\begin{aligned} E[f(\xi)] &= \int_0^{+\infty}\mathcal{M}\{f(\xi)\ge r\}\mathrm{d}r=\int_0^{+\infty}\mathcal{M}\{|\xi|\ge f^{-1}(r)\}\mathrm{d}r \\ &\ge\int_0^{f(t)}\mathcal{M}\{|\xi|\ge f^{-1}(r)\}\mathrm{d}r\ge\int_0^{f(t)}\mathrm{d}r\cdot\mathcal{M}\{|\xi|\ge f^{-1}(f(t))\} \\ &= f(t)\cdot\mathcal{M}\{|\xi|\ge t\} \end{aligned}$$

which proves the inequality.

Theorem 1.61 *(Liu [120], Markov Inequality). Let ξ be an uncertain variable. Then for any given numbers $t>0$ and $p>0$, we have*

$$\mathcal{M}\{|\xi|\ge t\}\le\frac{E[|\xi|^p]}{t^p}. \tag{1.214}$$

Proof: It is a special case of Theorem 1.60 when $f(x)=|x|^p$.

Example 1.60: For any given positive number t, we define an uncertain variable as follows,

$$\xi=\begin{cases}0 \text{ with uncertain measure } 1/2\\ t \text{ with uncertain measure } 1/2.\end{cases}$$

Then $E[\xi^p]=t^p/2$ and $\mathcal{M}\{\xi\ge t\}=1/2=E[\xi^p]/t^p$.

Theorem 1.62 *(Liu [120], Chebyshev Inequality). Let ξ be an uncertain variable whose variance $V[\xi]$ exists. Then for any given number $t>0$, we have*

$$\mathcal{M}\{|\xi-E[\xi]|\ge t\}\le\frac{V[\xi]}{t^2}. \tag{1.215}$$

Proof: It is a special case of Theorem 1.60 when the uncertain variable ξ is replaced with $\xi-E[\xi]$, and $f(x)=x^2$.

Example 1.61: For any given positive number t, we define an uncertain variable as follows,

$$\xi=\begin{cases}-t \text{ with uncertain measure } 1/2\\ t \text{ with uncertain measure } 1/2.\end{cases}$$

Then $V[\xi] = t^2$ and $\mathcal{M}\{|\xi - E[\xi]| \ge t\} = 1 = V[\xi]/t^2$.

Theorem 1.63 *(Liu [120], Hölder's Inequality). Let p and q be positive numbers with $1/p + 1/q = 1$, and let ξ and η be independent uncertain variables with $E[|\xi|^p] < \infty$ and $E[|\eta|^q] < \infty$. Then we have*

$$E[|\xi\eta|] \le \sqrt[p]{E[|\xi|^p]}\sqrt[q]{E[|\eta|^q]}. \tag{1.216}$$

Proof: The inequality holds trivially if at least one of ξ and η is zero a.s. Now we assume $E[|\xi|^p] > 0$ and $E[|\eta|^q] > 0$. It is easy to prove that the function $f(x, y) = \sqrt[p]{x}\sqrt[q]{y}$ is a concave function on $\{(x, y) : x \ge 0, y \ge 0\}$. Thus for any point (x_0, y_0) with $x_0 > 0$ and $y_0 > 0$, there exist two real numbers a and b such that

$$f(x, y) - f(x_0, y_0) \le a(x - x_0) + b(y - y_0), \quad \forall x \ge 0, y \ge 0.$$

Letting $x_0 = E[|\xi|^p]$, $y_0 = E[|\eta|^q]$, $x = |\xi|^p$ and $y = |\eta|^q$, we have

$$f(|\xi|^p, |\eta|^q) - f(E[|\xi|^p], E[|\eta|^q]) \le a(|\xi|^p - E[|\xi|^p]) + b(|\eta|^q - E[|\eta|^q]).$$

Taking the expected values on both sides, we obtain

$$E[f(|\xi|^p, |\eta|^q)] \le f(E[|\xi|^p], E[|\eta|^q]).$$

Hence the inequality (1.216) holds.

Theorem 1.64 *(Liu [120], Minkowski Inequality). Let p be a real number with $p \ge 1$, and let ξ and η be independent uncertain variables with $E[|\xi|^p] < \infty$ and $E[|\eta|^p] < \infty$. Then we have*

$$\sqrt[p]{E[|\xi + \eta|^p]} \le \sqrt[p]{E[|\xi|^p]} + \sqrt[p]{E[|\eta|^p]}. \tag{1.217}$$

Proof: The inequality holds trivially if at least one of ξ and η is zero a.s. Now we assume $E[|\xi|^p] > 0$ and $E[|\eta|^p] > 0$. It is easy to prove that the function $f(x, y) = (\sqrt[p]{x} + \sqrt[p]{y})^p$ is a concave function on $\{(x, y) : x \ge 0, y \ge 0\}$. Thus for any point (x_0, y_0) with $x_0 > 0$ and $y_0 > 0$, there exist two real numbers a and b such that

$$f(x, y) - f(x_0, y_0) \le a(x - x_0) + b(y - y_0), \quad \forall x \ge 0, y \ge 0.$$

Letting $x_0 = E[|\xi|^p]$, $y_0 = E[|\eta|^p]$, $x = |\xi|^p$ and $y = |\eta|^p$, we have

$$f(|\xi|^p, |\eta|^p) - f(E[|\xi|^p], E[|\eta|^p]) \le a(|\xi|^p - E[|\xi|^p]) + b(|\eta|^p - E[|\eta|^p]).$$

Taking the expected values on both sides, we obtain

$$E[f(|\xi|^p, |\eta|^p)] \le f(E[|\xi|^p], E[|\eta|^p]).$$

Hence the inequality (1.217) holds.

Theorem 1.65 *(Liu [120], Jensen's Inequality). Let ξ be an uncertain variable, and $f:\Re\to\Re$ a convex function. If $E[\xi]$ and $E[f(\xi)]$ are finite, then*

$$f(E[\xi]) \le E[f(\xi)]. \tag{1.218}$$

Especially, when $f(x)=|x|^p$ and $p\ge 1$, we have $|E[\xi]|^p \le E[|\xi|^p]$.

Proof: Since f is a convex function, for each y, there exists a number k such that $f(x)-f(y)\ge k\cdot(x-y)$. Replacing x with ξ and y with $E[\xi]$, we obtain

$$f(\xi)-f(E[\xi]) \ge k\cdot(\xi-E[\xi]).$$

Taking the expected values on both sides, we have

$$E[f(\xi)]-f(E[\xi]) \ge k\cdot(E[\xi]-E[\xi]) = 0$$

which proves the inequality.

1.13 Convergence Concepts

We have the following four convergence concepts of uncertain sequence: convergence almost surely (a.s.), convergence in measure, convergence in mean, and convergence in distribution.

Table 1.1: Relationship among Convergence Concepts

Convergence in Mean	$\Rightarrow$	Convergence in Measure	$\Rightarrow$	Convergence in Distribution

Definition 1.26 *(Liu [120]). Suppose that $\xi,\xi_1,\xi_2,\cdots$ are uncertain variables defined on the uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$. The sequence $\{\xi_i\}$ is said to be convergent a.s. to ξ if there exists an event Λ with $\mathcal{M}\{\Lambda\}=1$ such that*

$$\lim_{i\to\infty}|\xi_i(\gamma)-\xi(\gamma)| = 0 \tag{1.219}$$

for every $\gamma\in\Lambda$. In that case we write $\xi_i\to\xi$, a.s.

Definition 1.27 *(Liu [120]). Suppose that $\xi,\xi_1,\xi_2,\cdots$ are uncertain variables. We say that the sequence $\{\xi_i\}$ converges in measure to ξ if*

$$\lim_{i\to\infty}\mathcal{M}\{|\xi_i-\xi|\ge\varepsilon\} = 0 \tag{1.220}$$

for every $\varepsilon>0$.

Definition 1.28 *(Liu [120]). Suppose that $\xi, \xi_1, \xi_2, \cdots$ are uncertain variables with finite expected values. We say that the sequence $\{\xi_i\}$ converges in mean to ξ if*

$$\lim_{i \to \infty} E[|\xi_i - \xi|] = 0. \tag{1.221}$$

In addition, the sequence $\{\xi_i\}$ is said to converge in mean square to ξ if

$$\lim_{i \to \infty} E[|\xi_i - \xi|^2] = 0. \tag{1.222}$$

Definition 1.29 *(Liu [120]). Suppose that $\Phi, \Phi_1, \Phi_2, \cdots$ are the uncertainty distributions of uncertain variables $\xi, \xi_1, \xi_2, \cdots$, respectively. We say that $\{\xi_i\}$ converges in distribution to ξ if*

$$\lim_{i \to \infty} \Phi_i(x) = \Phi(x) \tag{1.223}$$

at any continuity point x of Φ.

Convergence in Mean vs. Convergence in Measure

Theorem 1.30 *(Liu [120]). Suppose that $\xi, \xi_1, \xi_2, \cdots$ are uncertain variables. If $\{\xi_i\}$ converges in mean to ξ, then $\{\xi_i\}$ converges in measure to ξ.*

Proof: It follows from the Markov inequality that for any given number $\varepsilon > 0$, we have

$$\mathcal{M}\{|\xi_i - \xi| \geq \varepsilon\} \leq \frac{E[|\xi_i - \xi|]}{\varepsilon} \to 0$$

as $i \to \infty$. Thus $\{\xi_i\}$ converges in measure to ξ. The theorem is proved.

Example 1.62: Convergence in measure does not imply convergence in mean. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \cdots\}$ with

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_i \in \Lambda} 1/i, & \text{if } \sup\limits_{\gamma_i \in \Lambda} 1/i < 0.5 \\ 1 - \sup\limits_{\gamma_i \notin \Lambda} 1/i, & \text{if } \sup\limits_{\gamma_i \notin \Lambda} 1/i < 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

The uncertain variables are defined by

$$\xi_i(\gamma_j) = \begin{cases} i, & \text{if } j = i \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \cdots$ and $\xi \equiv 0$. For some small number $\varepsilon > 0$, we have

$$\mathcal{M}\{|\xi_i - \xi| \geq \varepsilon\} = \mathcal{M}\{|\xi_i - \xi| \geq \varepsilon\} = \frac{1}{i} \to 0.$$

That is, the sequence $\{\xi_i\}$ converges in measure to ξ. However, for each i, we have

$$E[|\xi_i - \xi|] = 1.$$

That is, the sequence $\{\xi_i\}$ does not converge in mean to ξ.

Convergence in Measure vs. Convergence in Distribution

Theorem 1.31 *(Liu [120]). Suppose* $\xi, \xi_1, \xi_2, \cdots$ *are uncertain variables. If* $\{\xi_i\}$ *converges in measure to* ξ*, then* $\{\xi_i\}$ *converges in distribution to* ξ*.*

Proof: Let x be a given continuity point of the uncertainty distribution Φ. On the one hand, for any $y > x$, we have

$$\{\xi_i \le x\} = \{\xi_i \le x, \xi \le y\} \cup \{\xi_i \le x, \xi > y\} \subset \{\xi \le y\} \cup \{|\xi_i - \xi| \ge y - x\}.$$

It follows from the countable subadditivity axiom that

$$\Phi_i(x) \le \Phi(y) + \mathcal{M}\{|\xi_i - \xi| \ge y - x\}.$$

Since $\{\xi_i\}$ converges in measure to ξ, we have $\mathcal{M}\{|\xi_i - \xi| \ge y - x\} \to 0$ as $i \to \infty$. Thus we obtain $\limsup_{i\to\infty} \Phi_i(x) \le \Phi(y)$ for any $y > x$. Letting $y \to x$, we get

$$\limsup_{i\to\infty} \Phi_i(x) \le \Phi(x). \tag{1.224}$$

On the other hand, for any $z < x$, we have

$$\{\xi \le z\} = \{\xi_i \le x, \xi \le z\} \cup \{\xi_i > x, \xi \le z\} \subset \{\xi_i \le x\} \cup \{|\xi_i - \xi| \ge x - z\}$$

which implies that

$$\Phi(z) \le \Phi_i(x) + \mathcal{M}\{|\xi_i - \xi| \ge x - z\}.$$

Since $\mathcal{M}\{|\xi_i - \xi| \ge x - z\} \to 0$, we obtain $\Phi(z) \le \liminf_{i\to\infty} \Phi_i(x)$ for any $z < x$. Letting $z \to x$, we get

$$\Phi(x) \le \liminf_{i\to\infty} \Phi_i(x). \tag{1.225}$$

It follows from (1.224) and (1.225) that $\Phi_i(x) \to \Phi(x)$. The theorem is proved.

Example 1.63: Convergence in distribution does not imply convergence in measure. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with $\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = 1/2$. We define an uncertain variables as

$$\xi(\gamma) = \begin{cases} -1, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2. \end{cases}$$

We also define $\xi_i = -\xi$ for $i = 1, 2, \cdots$ Then ξ_i and ξ have the same chance distribution. Thus $\{\xi_i\}$ converges in distribution to ξ. However, for some small number $\varepsilon > 0$, we have

$$\mathcal{M}\{|\xi_i - \xi| \ge \varepsilon\} = \mathcal{M}\{|\xi_i - \xi| \ge \varepsilon\} = 1.$$

That is, the sequence $\{\xi_i\}$ does not converge in measure to ξ.

Convergence Almost Surely vs. Convergence in Measure

Example 1.64: Convergence a.s. does not imply convergence in measure. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \cdots\}$ with

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_i \in \Lambda} i/(2i+1), & \text{if } \sup\limits_{\gamma_i \in \Lambda} i/(2i+1) < 0.5 \\ 1 - \sup\limits_{\gamma_i \notin \Lambda} i/(2i+1), & \text{if } \sup\limits_{\gamma_i \notin \Lambda} i/(2i+1) < 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

Then we define uncertain variables as

$$\xi_i(\gamma_j) = \begin{cases} i, & \text{if } j = i \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \cdots$ and $\xi \equiv 0$. The sequence $\{\xi_i\}$ converges a.s. to ξ. However, for some small number $\varepsilon > 0$, we have

$$\mathcal{M}\{|\xi_i - \xi| \geq \varepsilon\} = \mathcal{M}\{|\xi_i - \xi| \geq \varepsilon\} = \frac{i}{2i+1} \to \frac{1}{2}.$$

That is, the sequence $\{\xi_i\}$ does not converge in measure to ξ.

Example 1.65: Convergence in measure does not imply convergence a.s. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. For any positive integer i, there is an integer j such that $i = 2^j + k$, where k is an integer between 0 and $2^j - 1$. Then we define uncertain variables as

$$\xi_i(\gamma) = \begin{cases} 1, & \text{if } k/2^j \leq \gamma \leq (k+1)/2^j \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \cdots$ and $\xi \equiv 0$. For some small number $\varepsilon > 0$, we have

$$\mathcal{M}\{|\xi_i - \xi| \geq \varepsilon\} = \mathcal{M}\{|\xi_i - \xi| \geq \varepsilon\} = \frac{1}{2^j} \to 0$$

as $i \to \infty$. That is, the sequence $\{\xi_i\}$ converges in measure to ξ. However, for any $\gamma \in [0, 1]$, there is an infinite number of intervals of the form $[k/2^j, (k+1)/2^j]$ containing γ. Thus $\xi_i(\gamma)$ does not converge to 0. In other words, the sequence $\{\xi_i\}$ does not converge a.s. to ξ.

Convergence Almost Surely vs. Convergence in Mean

Example 1.66: Convergence a.s. does not imply convergence in mean. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \cdots\}$ with

$$\mathcal{M}\{\Lambda\} = \sum_{\gamma_i \in \Lambda} \frac{1}{2^i}.$$

The uncertain variables are defined by

$$\xi_i(\gamma_j) = \begin{cases} 2^i, & \text{if } j = i \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \cdots$ and $\xi \equiv 0$. Then ξ_i converges a.s. to ξ. However, the sequence $\{\xi_i\}$ does not converge in mean to ξ because $E[|\xi_i - \xi|] \equiv 1$.

Example 1.67: Convergence in mean does not imply convergence a.s. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0,1]$ with Borel algebra and Lebesgue measure. For any positive integer i, there is an integer j such that $i = 2^j + k$, where k is an integer between 0 and $2^j - 1$. The uncertain variables are defined by

$$\xi_i(\gamma) = \begin{cases} 1, & \text{if } k/2^j \le \gamma \le (k+1)/2^j \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \cdots$ and $\xi \equiv 0$. Then

$$E[|\xi_i - \xi|] = \frac{1}{2^j} \to 0.$$

That is, the sequence $\{\xi_i\}$ converges in mean to ξ. However, for any $\gamma \in [0,1]$, there is an infinite number of intervals of the form $[k/2^j, (k+1)/2^j]$ containing γ. Thus $\xi_i(\gamma)$ does not converge to 0. In other words, the sequence $\{\xi_i\}$ does not converge a.s. to ξ.

Convergence Almost Surely vs. Convergence in Distribution

Example 1.68: Convergence in distribution does not imply convergence a.s. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with $\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = 1/2$. We define an uncertain variable ξ as

$$\xi(\gamma) = \begin{cases} -1, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2. \end{cases}$$

We also define $\xi_i = -\xi$ for $i = 1, 2, \cdots$ Then ξ_i and ξ have the same uncertainty distribution. Thus $\{\xi_i\}$ converges in distribution to ξ. However, the sequence $\{\xi_i\}$ does not converge a.s. to ξ.

Example 1.69: Convergence a.s. does not imply convergence in distribution. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \cdots\}$ with

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_i \in \Lambda} i/(2i+1), & \text{if } \sup\limits_{\gamma_i \in \Lambda} i/(2i+1) < 0.5 \\ 1 - \sup\limits_{\gamma_i \notin \Lambda} i/(2i+1), & \text{if } \sup\limits_{\gamma_i \notin \Lambda} i/(2i+1) < 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

The uncertain variables are defined by

$$\xi_i(\gamma_j) = \begin{cases} i, & \text{if } j = i \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \cdots$ and $\xi \equiv 0$. Then the sequence $\{\xi_i\}$ converges a.s. to ξ. However, the uncertainty distributions of ξ_i are

$$\Phi_i(x) = \begin{cases} 0, & \text{if } x < 0 \\ (i+1)/(2i+1), & \text{if } 0 \le x < i \\ 1, & \text{if } x \ge i \end{cases}$$

for $i = 1, 2, \cdots$, respectively. The uncertainty distribution of ξ is

$$\Phi(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \ge 0. \end{cases}$$

It is clear that $\Phi_i(x)$ does not converge to $\Phi(x)$ at $x > 0$. That is, the sequence $\{\xi_i\}$ does not converge in distribution to ξ.

1.14 Conditional Uncertainty

We consider the uncertain measure of an event A after it has been learned that some other event B has occurred. This new uncertain measure of A is called the conditional uncertain measure of A given B.

In order to define a conditional uncertain measure $\mathcal{M}\{A|B\}$, at first we have to enlarge $\mathcal{M}\{A \cap B\}$ because $\mathcal{M}\{A \cap B\} < 1$ for all events whenever $\mathcal{M}\{B\} < 1$. It seems that we have no alternative but to divide $\mathcal{M}\{A \cap B\}$ by $\mathcal{M}\{B\}$. Unfortunately, $\mathcal{M}\{A \cap B\}/\mathcal{M}\{B\}$ is not always an uncertain measure. However, the value $\mathcal{M}\{A|B\}$ should not be greater than $\mathcal{M}\{A \cap B\}/\mathcal{M}\{B\}$ (otherwise the normality will be lost), i.e.,

$$\mathcal{M}\{A|B\} \le \frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}}. \tag{1.226}$$

On the other hand, in order to preserve the self-duality, we should have

$$\mathcal{M}\{A|B\} = 1 - \mathcal{M}\{A^c|B\} \ge 1 - \frac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}}. \tag{1.227}$$

Furthermore, since $(A \cap B) \cup (A^c \cap B) = B$, we have $\mathcal{M}\{B\} \le \mathcal{M}\{A \cap B\} + \mathcal{M}\{A^c \cap B\}$ by using the countable subadditivity axiom. Thus

$$0 \le 1 - \frac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}} \le \frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}} \le 1. \tag{1.228}$$

Hence any numbers between $1 - \mathcal{M}\{A^c \cap B\}/\mathcal{M}\{B\}$ and $\mathcal{M}\{A \cap B\}/\mathcal{M}\{B\}$ are reasonable values that the conditional uncertain measure may take. Based on the maximum uncertainty principle, we have the following conditional uncertain measure.

Definition 1.32 *(Liu [120]). Let* $(\Gamma, \mathcal{L}, \mathcal{M})$ *be an uncertainty space, and* A, $B \in \mathcal{L}$. *Then the conditional uncertain measure of* A *given* B *is defined by*

$$\mathcal{M}\{A|B\} = \begin{cases} \dfrac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}}, & \text{if } \dfrac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}} < 0.5 \\ 1 - \dfrac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}}, & \text{if } \dfrac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}} < 0.5 \\ 0.5, & \text{otherwise} \end{cases} \tag{1.229}$$

provided that $\mathcal{M}\{B\} > 0$.

It follows immediately from the definition of conditional uncertain measure that

$$1 - \frac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}} \le \mathcal{M}\{A|B\} \le \frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}}. \tag{1.230}$$

Furthermore, the conditional uncertain measure obeys the maximum uncertainty principle, and takes values as close to 0.5 as possible.

Remark 1.6: Assume that we know the *prior* uncertain measures $\mathcal{M}\{B\}$, $\mathcal{M}\{A \cap B\}$ and $\mathcal{M}\{A^c \cap B\}$. Then the conditional uncertain measure $\mathcal{M}\{A|B\}$ yields the *posterior* uncertain measure of A after the occurrence of event B.

Theorem 1.66. *Let* $(\Gamma, \mathcal{L}, \mathcal{M})$ *be an uncertainty space, and* B *an event with* $\mathcal{M}\{B\} > 0$. *Then* $\mathcal{M}\{\cdot|B\}$ *defined by (1.229) is an uncertain measure, and* $(\Gamma, \mathcal{L}, \mathcal{M}\{\cdot|B\})$ *is an uncertainty space.*

Proof: It is sufficient to prove that $\mathcal{M}\{\cdot|B\}$ satisfies the normality, monotonicity, self-duality and countable subadditivity axioms. At first, it satisfies the normality axiom, i.e.,

$$\mathcal{M}\{\Gamma|B\} = 1 - \frac{\mathcal{M}\{\Gamma^c \cap B\}}{\mathcal{M}\{B\}} = 1 - \frac{\mathcal{M}\{\emptyset\}}{\mathcal{M}\{B\}} = 1.$$

For any events A_1 and A_2 with $A_1 \subset A_2$, if

$$\frac{\mathcal{M}\{A_1 \cap B\}}{\mathcal{M}\{B\}} \le \frac{\mathcal{M}\{A_2 \cap B\}}{\mathcal{M}\{B\}} < 0.5,$$

then

$$\mathcal{M}\{A_1|B\} = \frac{\mathcal{M}\{A_1 \cap B\}}{\mathcal{M}\{B\}} \le \frac{\mathcal{M}\{A_2 \cap B\}}{\mathcal{M}\{B\}} = \mathcal{M}\{A_2|B\}.$$

If

$$\frac{\mathcal{M}\{A_1 \cap B\}}{\mathcal{M}\{B\}} \le 0.5 \le \frac{\mathcal{M}\{A_2 \cap B\}}{\mathcal{M}\{B\}},$$

then $\mathcal{M}\{A_1|B\} \le 0.5 \le \mathcal{M}\{A_2|B\}$. If

$$0.5 < \frac{\mathcal{M}\{A_1 \cap B\}}{\mathcal{M}\{B\}} \le \frac{\mathcal{M}\{A_2 \cap B\}}{\mathcal{M}\{B\}},$$

then we have

$$\mathcal{M}\{A_1|B\} = \left(1 - \frac{\mathcal{M}\{A_1^c \cap B\}}{\mathcal{M}\{B\}}\right) \vee 0.5 \le \left(1 - \frac{\mathcal{M}\{A_2^c \cap B\}}{\mathcal{M}\{B\}}\right) \vee 0.5 = \mathcal{M}\{A_2|B\}.$$

This means that $\mathcal{M}\{\cdot|B\}$ satisfies the monotonicity axiom. For any event A, if

$$\frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}} \ge 0.5, \quad \frac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}} \ge 0.5,$$

then we have $\mathcal{M}\{A|B\} + \mathcal{M}\{A^c|B\} = 0.5 + 0.5 = 1$ immediately. Otherwise, without loss of generality, suppose

$$\frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}} < 0.5 < \frac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}},$$

then we have

$$\mathcal{M}\{A|B\} + \mathcal{M}\{A^c|B\} = \frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}} + \left(1 - \frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}}\right) = 1.$$

That is, $\mathcal{M}\{\cdot|B\}$ satisfies the self-duality axiom. Finally, for any countable sequence $\{A_i\}$ of events, if $\mathcal{M}\{A_i|B\} < 0.5$ for all i, it follows from the countable subadditivity axiom that

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} A_i \cap B\right\} \le \frac{\mathcal{M}\left\{\bigcup_{i=1}^{\infty} A_i \cap B\right\}}{\mathcal{M}\{B\}} \le \frac{\sum_{i=1}^{\infty}\mathcal{M}\{A_i \cap B\}}{\mathcal{M}\{B\}} = \sum_{i=1}^{\infty}\mathcal{M}\{A_i|B\}.$$

Suppose there is one term greater than 0.5, say

$$\mathcal{M}\{A_1|B\} \ge 0.5, \quad \mathcal{M}\{A_i|B\} < 0.5, \quad i = 2, 3, \cdots$$

If $\mathcal{M}\{\cup_i A_i|B\} = 0.5$, then we immediately have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} A_i \cap B\right\} \le \sum_{i=1}^{\infty}\mathcal{M}\{A_i|B\}.$$

If $\mathcal{M}\{\cup_i A_i|B\} > 0.5$, we may prove the above inequality by the following facts:

$$A_1^c \cap B \subset \bigcup_{i=2}^{\infty}(A_i \cap B) \cup \left(\bigcap_{i=1}^{\infty} A_i^c \cap B\right),$$

$$\mathcal{M}\{A_1^c \cap B\} \le \sum_{i=2}^{\infty}\mathcal{M}\{A_i \cap B\} + \mathcal{M}\left\{\bigcap_{i=1}^{\infty} A_i^c \cap B\right\},$$

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} A_i|B\right\} = 1 - \frac{\mathcal{M}\left\{\bigcap_{i=1}^{\infty} A_i^c \cap B\right\}}{\mathcal{M}\{B\}},$$

$$\sum_{i=1}^{\infty}\mathcal{M}\{A_i|B\} \ge 1 - \frac{\mathcal{M}\{A_1^c \cap B\}}{\mathcal{M}\{B\}} + \frac{\sum_{i=2}^{\infty}\mathcal{M}\{A_i \cap B\}}{\mathcal{M}\{B\}}.$$

If there are at least two terms greater than 0.5, then the countable subadditivity is clearly true. Thus $\mathcal{M}\{\cdot|B\}$ satisfies the countable subadditivity axiom. Hence $\mathcal{M}\{\cdot|B\}$ is an uncertain measure. Furthermore, $(\Gamma, \mathcal{L}, \mathcal{M}\{\cdot|B\})$ is an uncertainty space.

Definition 1.33 *(Liu [120]). The conditional uncertainty distribution Φ: $\Re \to [0,1]$ of an uncertain variable ξ given B is defined by*

$$\Phi(x|B) = \mathcal{M}\left\{\xi \le x|B\right\} \tag{1.231}$$

provided that $\mathcal{M}\{B\} > 0$.

Theorem 1.67. *Let ξ be an uncertain variable with uncertainty distribution $\Phi(x)$, and t a real number with $\Phi(t) < 1$. Then the conditional uncertainty distribution of ξ given $\xi > t$ is*

$$\Phi(x|(t,+\infty)) = \begin{cases} 0, & \text{if } \Phi(x) \le \Phi(t) \\ \dfrac{\Phi(x)}{1-\Phi(t)} \wedge 0.5, & \text{if } \Phi(t) < \Phi(x) \le (1+\Phi(t))/2 \\ \dfrac{\Phi(x)-\Phi(t)}{1-\Phi(t)}, & \text{if } (1+\Phi(t))/2 \le \Phi(x). \end{cases}$$

Proof: It follows from $\Phi(x|(t,+\infty)) = \mathcal{M}\left\{\xi \le x|\xi > t\right\}$ and the definition of conditional uncertainty that

$$\Phi(x|(t,+\infty)) = \begin{cases} \dfrac{\mathcal{M}\{(\xi \le x) \cap (\xi > t)\}}{\mathcal{M}\{\xi > t\}}, & \text{if } \dfrac{\mathcal{M}\{(\xi \le x) \cap (\xi > t)\}}{\mathcal{M}\{\xi > t\}} < 0.5 \\ 1 - \dfrac{\mathcal{M}\{(\xi > x) \cap (\xi > t)\}}{\mathcal{M}\{\xi > t\}}, & \text{if } \dfrac{\mathcal{M}\{(\xi > x) \cap (\xi > t)\}}{\mathcal{M}\{\xi > t\}} < 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

When $\Phi(x) \le \Phi(t)$, we have $x \le t$, and

$$\frac{\mathcal{M}\{(\xi \le x) \cap (\xi > t)\}}{\mathcal{M}\{\xi > t\}} = \frac{\mathcal{M}\{\emptyset\}}{1-\Phi(t)} = 0 < 0.5.$$

Thus

$$\Phi(x|(t,+\infty)) = \frac{\mathcal{M}\{(\xi \le x) \cap (\xi > t)\}}{\mathcal{M}\{\xi > t\}} = 0.$$

When $\Phi(t) < \Phi(x) \le (1+\Phi(t))/2$, we have $x > t$, and

$$\frac{\mathcal{M}\{(\xi > x) \cap (\xi > t)\}}{\mathcal{M}\{\xi > t\}} = \frac{1-\Phi(x)}{1-\Phi(t)} \ge \frac{1-(1+\Phi(t))/2}{1-\Phi(t)} = 0.5$$

and

$$\frac{\mathcal{M}\{(\xi \le x) \cap (\xi > t)\}}{\mathcal{M}\{\xi > t\}} \le \frac{\Phi(x)}{1-\Phi(t)}.$$

It follows from the maximum uncertainty principle that

$$\Phi(x|(t,+\infty)) = \frac{\Phi(x)}{1-\Phi(t)} \wedge 0.5.$$

When $(1+\Phi(t))/2 \le \Phi(x)$, we have $x \ge t$, and

$$\frac{\mathcal{M}\{(\xi > x) \cap (\xi > t)\}}{\mathcal{M}\{\xi > t\}} = \frac{1-\Phi(x)}{1-\Phi(t)} \le \frac{1-(1+\Phi(t))/2}{1-\Phi(t)} \le 0.5.$$

Thus

$$\Phi(x|(t,+\infty)) = 1 - \frac{\mathcal{M}\{(\xi > x) \cap (\xi > t)\}}{\mathcal{M}\{\xi > t\}} = 1 - \frac{1-\Phi(x)}{1-\Phi(t)} = \frac{\Phi(x)-\Phi(t)}{1-\Phi(t)}.$$

The theorem is proved.

Example 1.70: Let ξ be a linear uncertain variable $\mathcal{L}(a,b)$, and t a real number with $a < t < b$. Then the conditional uncertainty distribution of ξ given $\xi > t$ is

$$\Phi(x|(t,+\infty)) = \begin{cases} 0, & \text{if } x \le t \\ \dfrac{x-a}{b-t} \wedge 0.5, & \text{if } t < x \le (b+t)/2 \\ \dfrac{x-t}{b-t} \wedge 1, & \text{if } (b+t)/2 \le x. \end{cases}$$

Theorem 1.68. *Let ξ be an uncertain variable with uncertainty distribution $\Phi(x)$, and t a real number with $\Phi(t) > 0$. Then the conditional uncertainty distribution of ξ given $\xi \le t$ is*

$$\Phi(x|(-\infty,t]) = \begin{cases} \dfrac{\Phi(x)}{\Phi(t)}, & \text{if } \Phi(x) \le \Phi(t)/2 \\ \dfrac{\Phi(x)+\Phi(t)-1}{\Phi(t)} \vee 0.5, & \text{if } \Phi(t)/2 \le \Phi(x) < \Phi(t) \\ 1, & \text{if } \Phi(t) \le \Phi(x). \end{cases}$$

Proof: It follows from $\Phi(x|(-\infty,t]) = \mathcal{M}\{\xi \le x | \xi \le t\}$ and the definition of conditional uncertainty that

$$\Phi(x|(-\infty,t]) = \begin{cases} \dfrac{\mathcal{M}\{(\xi \le x) \cap (\xi \le t)\}}{\mathcal{M}\{\xi \le t\}}, & \text{if } \dfrac{\mathcal{M}\{(\xi \le x) \cap (\xi \le t)\}}{\mathcal{M}\{\xi \le t\}} < 0.5 \\ 1 - \dfrac{\mathcal{M}\{(\xi > x) \cap (\xi \le t)\}}{\mathcal{M}\{\xi \le t\}}, & \text{if } \dfrac{\mathcal{M}\{(\xi > x) \cap (\xi \le t)\}}{\mathcal{M}\{\xi \le t\}} < 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

When $\Phi(x) \le \Phi(t)/2$, we have $x < t$, and

$$\frac{\mathcal{M}\{(\xi \le x) \cap (\xi \le t)\}}{\mathcal{M}\{\xi \le t\}} = \frac{\Phi(x)}{\Phi(t)} \le \frac{\Phi(t)/2}{\Phi(t)} = 0.5.$$

Thus

$$\Phi(x|(-\infty, t]) = \frac{\mathcal{M}\{(\xi \le x) \cap (\xi \le t)\}}{\mathcal{M}\{\xi \le t\}} = \frac{\Phi(x)}{\Phi(t)}.$$

When $\Phi(t)/2 \le \Phi(x) < \Phi(t)$, we have $x < t$, and

$$\frac{\mathcal{M}\{(\xi \le x) \cap (\xi \le t)\}}{\mathcal{M}\{\xi \le t\}} = \frac{\Phi(x)}{\Phi(t)} \ge \frac{\Phi(t)/2}{\Phi(t)} = 0.5$$

and

$$\frac{\mathcal{M}\{(\xi > x) \cap (\xi \le t)\}}{\mathcal{M}\{\xi \le t\}} \le \frac{1 - \Phi(x)}{\Phi(t)},$$

i.e.,

$$1 - \frac{\mathcal{M}\{(\xi > x) \cap (\xi \le t)\}}{\mathcal{M}\{\xi \le t\}} \ge \frac{\Phi(x) + \Phi(t) - 1}{\Phi(t)}.$$

It follows from the maximum uncertainty principle that

$$\Phi(x|(-\infty, t]) = \frac{\Phi(x) + \Phi(t) - 1}{\Phi(t)} \vee 0.5.$$

When $\Phi(t) \le \Phi(x)$, we have $x \ge t$, and

$$\frac{\mathcal{M}\{(\xi > x) \cap (\xi \le t)\}}{\mathcal{M}\{\xi \le t\}} = \frac{\mathcal{M}\{\emptyset\}}{\Phi(t)} = 0 < 0.5.$$

Thus

$$\Phi(x|(-\infty, t]) = 1 - \frac{\mathcal{M}\{(\xi > x) \cap (\xi \le t)\}}{\mathcal{M}\{\xi \le t\}} = 1 - 0 = 1.$$

The theorem is proved.

Example 1.71: Let ξ be a linear uncertain variable $\mathcal{L}(a, b)$, and t a real number with $a < t < b$. Then the conditional uncertainty distribution of ξ given $\xi \le t$ is

$$\Phi(x|(-\infty, t]) = \begin{cases} \dfrac{x-a}{t-a} \vee 0, & \text{if } x \le (a+t)/2 \\ \left(1 - \dfrac{b-x}{t-a}\right) \vee 0.5, & \text{if } (a+t)/2 \le x < t \\ 1, & \text{if } x \le t. \end{cases}$$

Definition 1.34 *(Liu [120]). Let ξ be an uncertain variable. Then the conditional expected value of ξ given B is defined by*

$$E[\xi|B] = \int_0^{+\infty} \mathcal{M}\{\xi \ge r|B\}\mathrm{d}r - \int_{-\infty}^0 \mathcal{M}\{\xi \le r|B\}\mathrm{d}r \qquad (1.232)$$

provided that at least one of the two integrals is finite.