A Parametrized Model for Optimization with Mixed Fuzzy and Possibilistic Uncertainty

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Abstract. Fuzzy and possibilistic uncertainty are very closely related, and sometimes coexist in optimization under uncertainty problems. Fuzzy uncertainty in mathematical programming problems typically represents flexibility on the part of the decision maker. On the other hand, possibilistic uncertainty generally expresses a lack of information about the values the parameters will assume.

Several models for mixed fuzzy and possibilistic programming problems have previously been published. The semantic interpretation of these models, however, is of questionable value. The mixed models in the literature find solutions in which the fuzzy uncertainty (or flexibility) and the possibilistic uncertainty (or lack of confidence in the outcome) are held to the same levels.

This chapter proposes a new mixed model which allows a trade-off between fuzzy and possibilistic uncertainty. This trade-off corresponds to a semantic interpretations consistent with human decision-making. The new model shares characteristics with multi-objective programming and Markowitz models. Model structure, semantic justification, and solution approaches are covered.

1 Introduction

In the application of optimization theory, parameters are often not known with certainty. Fuzzy and possibilistic uncertainty are very closely related, and sometimes coexist in optimization under uncertainty problems. Fuzzy uncertainty in mathematical programming problems typically represents flexibility on the part of the decision maker. On the other hand, possibilistic uncertainty generally expresses a lack of information about the value a parameter will assume.

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however, is of questionable value. This chapter proposes a semantic interpretation of the mixed fuzzy and possibilistic linear programming problem that is not fully addressed by any of the existing models. The newe *parameterized model* shares characteristics with multi-objective problems and programming and Markowitz models.

In Section two of this chapter, we will provide background definitions, define the problem, and review the current state of the literature on fuzzy and possibilistic programming problems. Chepter 3 presents tha model, along with its derivation and justification. Chapter 4 examines problem structure, solution methods, and an industrial strength application.

2 Background

Fuzzy uncertainty describes vagueness, or a softening of the concept of belonging to a set. A fuzzy set $\tilde{X}$ is defined by its membership function, $\mu_{\tilde{X}}$. An element s has its degree of membership in $\tilde{X}$ described by $\mu_{\tilde{X}}(s)$, with 1 indicating full membership, 0 indicating full non-membership, and numbers between 0 and 1 indicating partial membership. In decision making applications, a fuzzy set may be used to indicate flexibility on the part of the decision maker. A goal or constraint may be softened via a fuzzy inequality.

Possibilistic uncertainty, on the other hand, describes ambiguity. A possibilistic variable is not random– its value is pre-determined, but is not known with certainty. The likelihood that the value of a possibilistic variable lies in a particular interval is described by a possibility distribution. Klir relates possibility distributions to fuzzy sets as follows [4].

> Given a universe Y, let y be a variable which takes on values in Y. Now let F be a fuzzy set on Y, then let $F(x)$ describe the extent to which x is a member of F. Then if we say, "y is F", $F(x)$ for each x in Y is the possibility that x is F.

Though they are related, possibilistic uncertainty and fuzzy sets are used to represent very different things. In optimization problems, a possibilistic variable is used to indicate a parameter whose value is fixed, but is not known with certainty, (for example: a measurement with a margin of error). A possibility distribution might also be used to reflect a decision maker's estimation of the distribution of a random variable when the exact distribution is too costly or impossible to determine.

2.1 Semantics of the Mixed Fuzzy and Possibilistic Linear Programming Problem

Occasionally, fuzzy and possibilistic uncertainty occur in the same optimization problem. Consider the linear program (LP):

$$\begin{aligned} \max\ & c^{\mathrm{T}}x \\ \text{subject to}\ & Ax \leq b \\ & x \geq 0. \end{aligned} \tag{1}$$

Possibilistic uncertainty (denoted in this chapter with a "hat") can occur in the parameters represented by $\hat{c}$, $\hat{b}$, and/or $\hat{A}$. (This happens when the value of the parameters is fixed, but the decision maker has incomplete information about its value.)

On the other hand, fuzzy uncertainty (denoted in this chapter with a "tilde") can occur in any of the parameters, $\tilde{A}$, $\tilde{b}$, and/or $\tilde{c}$. This happens when the values of the parameters are not sharp. Consider c_i, which represents the cost or value of x_i, in the problem. When there is a fuzzy interval of values which, to varying degrees, represent the cost, $\tilde{c_i}$ is fuzzy. Fuzzy parameters appear in optimization problems on occasion, but most fuzzy uncertainty occurs in the inequality. A fuzzy less than constraint ($\tilde{\leq}$) can be interpreted as "approximately less than." When this uncertainty represents a willingness on the part of the decision-maker to bend the constraints, optimization problem with fuzzy inequalities are sometimes called "flexible programs," and the constraints are called "soft constraints."

A common mixed fuzzy and possibilistic linear program, then, might assume the following form:

$$\begin{aligned} \max\ & \hat{c}^{\mathrm{T}}x \\ \text{subject to}\ & \hat{A}x \ \tilde{\leq}\ \hat{b} \\ & x \geq 0. \end{aligned} \tag{2}$$

An element's membership in a fuzzy set (or the degree to which a fuzzy inequality is satisfied) is quantified by the membership function, and represented by $\alpha \in [0, 1]$. The likelihood that an interval contains a possibilistic variable is quantified by the possibility distribution, but is also represented by an $\alpha \in [0, 1]$. The fuzzy α and the possibilistic α, however, mean very different things. The fuzzy α represents the level at which the decision-maker's requirements are satisfied. The possibilistic α, on the other hand, represents the likelihood that the parameters will take on values which will result in that level of satisfaction. With that in mind, let us examine some previously published approaches for mixed fuzzy and possibilistic linear programming problems.

2.2 Existing Models for Mixed Fuzzy and Possibilistic Programming

Delgado, Verdegay, and Villa [5] propose the following formulation for dealing with ambiguity in the constraint coefficients and right-hand sides, as well as vagueness in the inequality relationship:

$$\begin{aligned} \text{maximize} \quad & c^{\mathrm{T}}x \\ \text{subject to} \quad & \hat{A}x \mathrel{\tilde{\leq}} \hat{b} \\ & x \geq 0. \end{aligned} \tag{3}$$

In addition to (3), membership functions $\pi_{a_{ij}}$ are defined for the possible values of each possibilistic element of $\hat{A}$, membership functions π_{b_i} are defined for the possible values of each possibilistic element of $\hat{b}$, and membership function μ_i gives the degree to which the fuzzy constraint i is satisfied. Stated another way, μ_i is the membership function of the fuzzy inequality. Recall that the uncertainty in the $\tilde{a}_{ij}$s and the $\tilde{b}_i$s is due to ambiguity concerning the actual value of the parameter, while the uncertainty in the $\tilde{\leq}_i$s is due to the decision maker's flexibility regarding the necessity of satisfying the constraints in full.

Delgado, *et al.* solve the problem parametrically on α. For each $\alpha \in [0,1]$ (or practically speaking, for a finite subset of $\alpha \in [0,1]$), a set of constraints and objective function are produced. The resut is a fuzzy solution to the fuzzy problem. Since a fuzzy solution cannot be implemented, the decision maker must select an α and implement the corresponding solution. *This implementation has a likelihood α of satisfying the constraints of the problem at a level α.* The likelihood that the constraints will be satisfied and the level at which they are satisfied are two completely separate concepts, but this model holds both to the same α level.

Another mixed formulation is what Inuiguchi [3] refers to as the "fuzzy robust programming" problem [1, 8]. This is a mathematical program with possibilistic constraint coefficients $\hat{a}_{ij}$ that satisfy fuzzy constraints, $\tilde{b}_i$ as follows:

$$\max c^{\mathrm{T}}x \tag{4}$$

$$\text{subject to } \hat{a}'_i x' \subseteq \tilde{b}_i \tag{5}$$

$$x' = (1, x^t)t \geq 0.$$

Zadeh [9] defines the set-inclusion relation $\tilde{M} \subseteq \tilde{N}$ as $\mu_{\tilde{M}}(r) \leq \mu_{\tilde{N}}(r)$ for all r. Robust programming interprets the set-inclusive constraint to mean that the region in which $\tilde{a}'_i x'$ can possibly occur is restricted to $\tilde{b}_i$, a region which is tolerable to the decision maker. Therefore, the left side of (5) is possibilistic, and the right side is fuzzy.

Negoita [8] defines the fuzzy right hand side as follows:

$$\tilde{b}_i = \{r \in \mathcal{R} | r \geq b_i\}. \tag{6}$$

As a result, we can interpret $\hat{a}'_i x' \subseteq \tilde{b}_i$ as an extension of an inequality constraint. The set-inclusive constraint (5) is reduced to

$$a_i^+(\alpha)x \leq b_i^+(\alpha) \tag{7}$$

$$a_i^-(\alpha)x \geq b_i^-(\alpha)$$
$$\text{for all } \alpha \in (0, 1].$$

If the membership functions are linear, it suffices to satisfy the constraints for $\alpha = 1$ and for $\alpha = \epsilon$, where ϵ is close to zero, since all $\alpha \in (\epsilon, 1)$ will be satisfied by interpolation. If the membership functions are not linear, however, we have an infinitely constrained problem. If we abide by Negoita's definition of $\tilde{b}$ (6), $b_i^+ = \infty$ for all values of α, so we can drop the first constraint in (7). Nonetheless, we still have an infinitely constrained program, with a constraint for each value of $\alpha \in (0, 1]$.

Consider the semantics of the second inequality in [6]. It requires that the α level of a possibilistic $\hat{a}_i$ multiplied by x be greater than the α level of a fuzzy $\tilde{b}_i$. In other words, there is a likelihood greater than or equal to α that a_i has a value which leads to constraint satisfaction at a level α. Like Delgado's model, the fuzzy robust model holds two very different types of uncertainty to the same α level. That is to say, the optimal implementaion will have a likelihood α of satisfying the constraints at a level α. The decision maker might want to allow a lower level of constraint satisfaction in order to have a greater guarantee of his/her result, or vice versa.

This leads us to ask the following questions. What is it that makes this model "robust"? And how can robust optimization theory inform a more practical approach to solving the mixed optimization under uncertainty problem?

2.3 Robust Optimization

The goal of robust optimization, which has its roots in stochastic optimization, is to produce a solution whose quality will withstand a wide variety of parameter realizations. Robust optimization seeks to mitigate the effects of uncertainty rather than merely anticipating it. Hence, robustness reflects a tendency to hedge against uncertainty, sacrificing some performance in order to avoid excessive volatility [7]. Robust formulations are designed to yield solutions that are less sensitive to model data than classical mathematical programming formulations. Robust programs fall into two broad categories– *solution robust* programs seek to minimize variance in solution optimality, while *model robust* programs aim to decrease variance in feasibility.

The robust fuzzy optimization model in [7] is called "robust" (in the model robust sense) because it seeks a solution which guarantees compliance with constraints at every possibility level (every α level). Unfortunately, at lower possibility levels, it is held to lower standards of constraint compliance, bringing into question its "robust" designation. To shed light on this possible misnomer, let us examine a classic robust model– the Markowitz model, which is robust in the solution robust sense.

2.4 Markowitz Model

In 1952, Markowitz [6] proposed a novel approach to financial portfolio optimization. He makes the case that a traditional linear programming approach to portfolio optimization will never prefer a diversified portfolio to an undiversified portfolio. He observes that simply diversifying among top return solutions will not result in a reliable portfolio, since the returns are too interrelated for the law of large numbers to apply. He proposes that both maximizing the expected value of the return and minimizing the historical variance (risk) are valid objectives. An efficient combination, then, is one which has the minimum risk for a return greater than or equal to a given level; or one which has the maximum return for a risk less than or equal to a given level. The decision maker can move among these efficient combinations, or along the efficient frontier, according to hisor her degree of risk aversion.

3 Main Results

3.1 Model Concept

In the spirit of the Markowitz model, we wish to allow a trade-off between the potential reward of the outcome and the reliability of the outcome, with the weights of the two competing objectives determined by the decision maker's risk aversion. The desire is to obtain an objective function like the following:

$$\text{maximize}: \; \textit{reward} \; + \; (\lambda \times \textit{reliability}), \tag{8}$$

where λ is a parameter indicating risk aversion.

The reward variable is the α-level associated with the fuzzy constraints and goal(s). It tells the decision maker how satisfactory the solution is. The reliability variable is the α-level associated with the possibilistic parameters. It tells the decision maker how likely it is that the solution will actually be satisfactory. To avoid confusion, let us refer to the fuzzy constraint membership level as α and the possibilistic parameter membership level as β.

In addition, let $\lambda \in [0, 1]$ be an indicator of the decision maker's valuation of reward and risk-avoidance, with 0 indicating that the decision maker cares exclusively about the reward, and 1 indicating that only risk avoidance is important. Using this notation, the desired objective is

$$\max \; (1 - \lambda)\alpha \; + \; \lambda\beta. \tag{9}$$

There is a frontier, which we call *the efficient frontier* in accordance with the literature, along which a trade-off occurs. The choice of λ determines which solution along the frontier is chosen.

3.2 Model Implementation

Suppose we begin with the mixed problem:

$$\begin{aligned} \max\ & \hat{c}^{\mathrm{T}} x \\ \text{subject to}\ & \hat{A}x \ \tilde{\leq}\ b \\ & x \ \geq 0. \end{aligned} \tag{10}$$

Let us for the moment ignore the possibilistic parameters and deal with the fuzzy constraints according to Bellman and Zadeh. We first introduce a goal, g, for the objective function value and state the objective function and a fuzzy goal, $\hat{c}^{\mathrm{T}} x \ \tilde{\geq}\ g$. Now together the goal and the constraints form the decision space, and we wish to maximize the α-level at which the least satisfied constraint or goal is met. Allow (as a slight abuse of notation) $\tilde{\leq}_\alpha$ to denote the α-level at which a constraint or goal is met from a pessimistic point of view. The problem is now,

$$\begin{aligned} \max\ & \alpha \\ \text{subject to}\ & -\hat{c}^{\mathrm{T}} x \ \tilde{\leq}_\alpha\ -g \\ & \hat{A}x \ \tilde{\leq}_\alpha\ b \end{aligned} \tag{11}$$

$$x, \alpha \ \geq\ 0. \tag{12}$$

Now let $\hat{c}_\beta$ denote the right end-point of the $(1-\beta)$-cut of $-\hat{c}$, and $\hat{a}_{ij\beta}$ denote the right end-point of the $(1-\beta)$-cut of $\hat{a}_{ij}$ as illustrated in Figure 1.

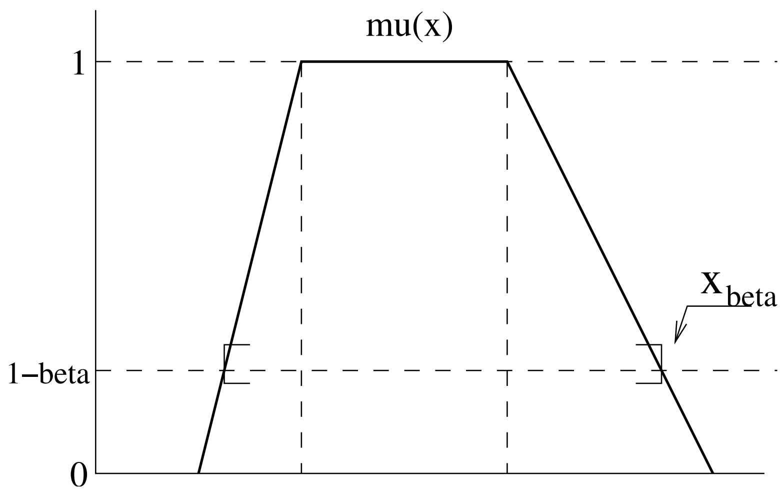

Fig. 1 The right end-point on the $1-\beta$-cut of the $\hat{c}$

We can now complete our formulation as follows:

$$\begin{array}{lll} \max\ (1-\lambda)\alpha\ +\ \lambda\beta & & \\ \text{subject to} \quad -\hat{c}_{\beta}^{\mathrm{T}}x & \tilde{\leq}_{\alpha} - g & \\ \hat{A}_{\beta}x & \tilde{\leq}_{\alpha}\ b & \\ x & \geq 0 & \\ \alpha, \beta & \in [0,1]. & \end{array} \tag{13}$$

The constraints are functions of x, α, and β, and although the objective function is linear (it does not have to be– some aggregation other than addition could have been chosen for the α and β terms), we shall soon see that the variables are quite entangled in the constraints, resulting in a non-linear formulation.

Let us suppose, for the sake of simplicity, that the possibility distribution for each a_{ij} (and b_i) is trapezoidal, with support (w_{ij}, z_{ij}) (or (w_i, z_i)), and core (u_{ij}, v_{ij}) (or (x_i, y_i)), as illustrated in Figure 2.

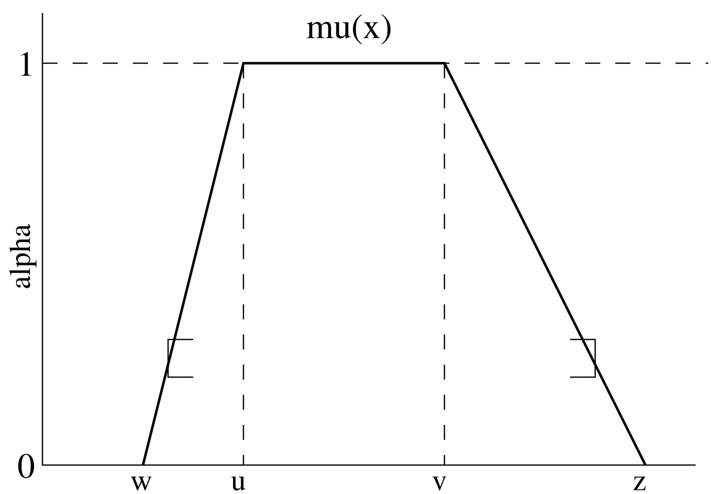

Fig. 2 Trapezoidal fuzzy number

Also, suppose that we have a single goal, and that the membership functions for the fuzzy constraint and goal are trapezoidal, with d_0 denoting the maximum acceptable deviation from the goal, and d_i denoting the maximum acceptable deviation from constraint i, as illustrated in Figure 3.

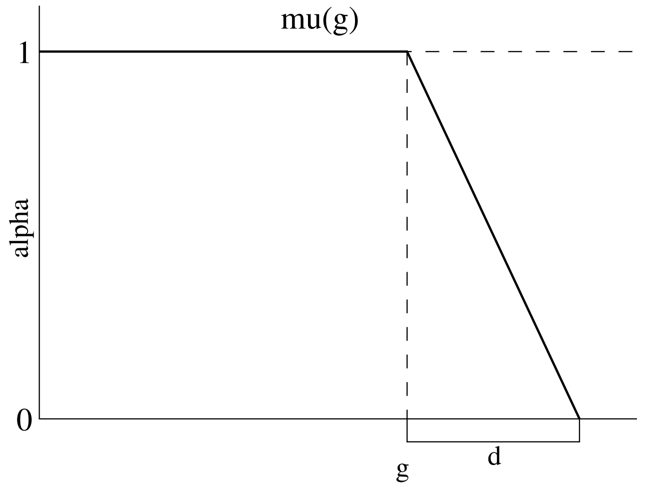

Fig. 3 Trapezoidal fuzzy goal

Then, incorporating fuzziness in the tradition of Zimmerman [10], we get

$$
\begin{aligned}
\max \quad & (1-\lambda)\alpha \; + \; \lambda\beta \\
\text{subject to} \quad \alpha & \le -\frac{g}{d_0} + \sum_j \frac{c_{j,\beta}}{d_0} x_j \\
\alpha & \le \frac{b_i}{d_i} - \sum_j \frac{a_{ij,\beta}}{d_i} x_j, \qquad \forall i \\
x & \ge 0 \\
\alpha, \beta & \in [0,1].
\end{aligned}
\tag{14}
$$

And incorporating a pessimistic view of possibility (if the problem has no solution, we return again with an optimistic point of view), we get

$$
\begin{aligned}
\max \quad & (1-\lambda)\alpha \; + \; \lambda\beta \\
\text{subject to} \quad \alpha & \le -\frac{g}{d_0} + \sum_j \frac{u_j}{d_0} x_j + \sum_j \frac{u_j - w_j}{d_0} x_j \beta \\
\alpha & \le \frac{b_i}{d_i} - \sum_j \frac{v_{ij}}{d_i} x_j - \sum_j \frac{z_{ij} - v_{ij}}{d_i} x_j \beta \\
x & \ge 0 \\
\alpha, \beta & \in [0,1].
\end{aligned}
\tag{15}
$$

3.3 A Simple Example

Let us examine a simple, two-variable toy problem as an example. For the sake of comparison, consider the numerical example treated by Delgado, et al.[5]:

$$\begin{aligned} \text{maximize } z = & \ 5x_1 + 6x_2 \\ \text{subject to} \quad & \hat{3}x_1 + \hat{4}x_2 \tilde{\leq} \hat{18}, \\ & \hat{2}x_1 + \hat{1}x_2 \tilde{\leq} \hat{7} \\ & x_1, x_2 \geq 0, \end{aligned} \tag{16}$$

where

$$\begin{aligned} &\hat{3} = (3, 2, 4), \hat{4} = (4, 2.5, 5.5), \hat{18} = (18, 16, 19) \\ &\hat{2} = (2, 1, 3), \ \ \hat{1} = (1, 0.5, 2), \ \ \hat{7} = (7, 6, 9), \end{aligned} \tag{17}$$

and the maximum violation of the first constraint (our d_1) is 3, while the maimum allowable violation of the second constraint (our d_2) is 1.

Our parametrized model is not yet developed to handle possibilistic right-hand sides, so let us modify Delgado, *et al*'s problem by making the right-hand sides crisp. Also, the parametrized mixed formulation requires that the objective function be reformulated as a goal. The solution to the associated crisp problem yields an objective function value of 28, so let the goal be 23, with maximum violation (d_0) of 5. Finally, introduce possibilistic uncertainty in the objective function coefficients.

The toy problem to solve is then:

$$\begin{aligned} \text{maximize } z = & \ \hat{5}x_1 + \hat{6}x_2 \\ \text{subject to} \quad & \hat{3}x_1 + \hat{4}x_2 \tilde{\leq} 18, \\ & \hat{2}x_1 + \hat{1}x_2 \tilde{\leq} 7 \\ & x_1, x_2 \geq 0, \end{aligned} \tag{18}$$

where

$$\begin{aligned} &\hat{5} = (5, 4, 6), \hat{6} = (6, 5, 7) \hat{3} = (3, 2, 4), \hat{4} = (4, 2.5, 5.5), \\ &\hat{2} = (2, 1, 3), \qquad \hat{1} = (1, 0.5, 2), \\ &\qquad d_1 = 3, \qquad\qquad d_2 = 1 \\ &\qquad g = 28, \qquad\qquad d_0 = 5. \end{aligned}$$

Reformulating (19) according to (15) yields:

$$
\begin{aligned}
\max \quad & (1-\lambda)\alpha \ + \ \lambda\beta && (19)\\
\text{subject to} \quad \alpha & \le -\frac{23}{5} + x_1 + \frac{6}{5}x_2 + \frac{1}{5}x_1\beta + \frac{1}{5}x_2\beta \\
\alpha & \le \frac{18}{3} - x_1 - \frac{4}{3}x_2 - \frac{1}{3}x_1\beta - \frac{1}{6}x_2\beta \\
\alpha & \le 7 - 2x_1 - 1x_2 - 1x_1\beta - 1x_2\beta \\
x & \ge 0 \\
\alpha, \beta & \in [0,1].
\end{aligned}
$$

This toy problem was formulated in GAMS for $\lambda = 0, \lambda = .5$ and $\lambda = 1$ and solved with the non-linear programming solver. The results are summarized in Table 1.

Table 1 GAMS results for toy problem. Risk aversion is represented by λ, the reformulated objective by $z = (1-\lambda)\alpha + \lambda\beta$, and the original (crisp) objective by o. Reliability, represented by β ranges between 0 and 1, and reward, represented by α also ranges between 0 and 1.

λ	z	o	α	β	x_1	x_2
1	1	19.0	0	1	.2	3.1
0.5	0.6	22.2	0.49	0.7	0	3.8
0	0.5	25.6	0.52	0	1.9	2.7

It is clear, from the results, that as risk aversion decreases, the objective function level that can possibly be attained (represented by o) increases. However, the certainty of attaining that level, represented by β, decreases. This is the desired and expected result for the model. In fact, in order for the model to make sense, we need to know that α and β are always inversely related. This is in contrast to other mixed models, in which α and β are equal.

4 Problem Structure and Its Relation to Solution Methods

Many optimization problems have special structures which are exploited by efficient solution algorithms. In the search for practical solution methods to (15), we first evaluate the structure of the problem.

The last terms in each of the constraints in (15) contain βx, so the constraints are bi-linear. If the possibility distributions were non-linear (i.e. not trapezoidal or triangular), the system would be non-linear rather than bi-linear. Most optimization models for fuzzy or possibilistic uncertainty (that have linear possibility distributions or membership functions, as we assume here) are linear programs. The fact that the fuzzy robust model results in a bi-linear program is a distinct disadvantage.

On the bright side, there are some simplicities to the model that may result in a specialized solution method. The objective function is linear. Also, the non-linearity in the constraints results from the product of a function of β taken independently with each component of x. There are no mixed terms, which would lead to quadratic constraints. Unfortunately, the bi-linear constraints (in the simplest case) form a non-convex feasible region, which makes for a very hard optimization problem.

One possibility is to try to convert the fuzzy robust model into a bi-linear program by adding the constraints to the objective function with penalties. The disadvantages of this approach are two-fold. First, the introduction of an auxiliary penalty does not really make sense in the scope of the problem, since our objective in the first place in to minimize a kind of penalty (uncertainty). Second, the problem with penalized constraint violations in the objective function would require sequential solutions, which may result in greater complexity than other non-linear programming methods. For these reasons the sequential solution of the fuzzy robust problem with penalties was not pursued, but may be an avenue for further research.

Convex Programming

We've observed that the fuzzy robust problem is particularly difficult because the βx term results in a non-convex feasible region. The shape of the feasible region is directly related to the trapezoidal shape of the possibility distribution for the possibilistic parameters in the constraint matrix.

However, if the left and right-hand sides of the possibility distributions were not linear, but were defined by sufficiently convex functions, the feasible region would be convex. Specifically, consider the special case in which the left- and right-hand sides of the possibility distributions are each bounded above by $\frac{c}{x}$ for some constant c. Then the constraints will be concave functions bounded above by the linear function cx.

For example, let the right hand side of the possibility function for a parameter a be defined as $a_v + (\frac{2}{\beta+1} - 1)(a_z - a_v)$ (see figure 4). Then a_β (the right end-point of the $(1-\beta)$-cut defined in section 3.2) will be $v + *(\frac{-2}{\beta} - 1)(z - v)$. This leads a problem with constraints of the form

$$Vx + (\frac{-2}{\beta+1} - 1)(Z - V)x \leq b + (\alpha - 1)d,$$

which are concave. In this situation, convex programming algorithms may be used to solve the problem.

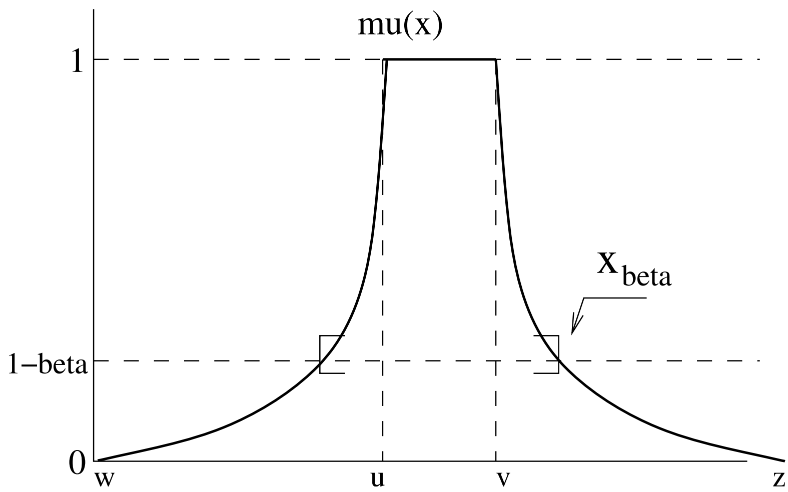

Fig. 4 The right end-point on the $1-\beta$-cut of the $\hat{c}$ when the right-hand-side of the possibility distribution is defined as $a_v + (\frac{2}{\beta+1} - 1)(a_z - a_v)$

Since membership functions are not typically determined empirically, but are chosen to represent the opinion of the decision maker, and often, the convenience of the modeler, this is a significant observation. Modelers often arbitrarily define linear membership functions for their models, but for the fuzzy robust model they could arbitrarily define sufficiently convex membership functions.

4.1 Testing the Model: The Radiation Therapy Problem

The fuzzy robust model with both non-convex and convex feasible regions was tested on the radiation therapy planning problem, using the MATLAB Optimization Toolbox function fmincon which minimizes non-linear constrained problems.

The use of particle beams to treat tumors is called the radiation therapy planning (RTP) problem [2]. Beams of particles are oriented at a variety of angles and with varying intensities to deposit radiation dose (measured as energy/unit mass) to the tumor. The goal is to deposit a tumorcidal dose to the tumor while minimizing damage to surrounding non-tumor tissue.

A *treatment plan* is the identification of a set of beam angles and weights that provides a lethal dose to the tumor cells while sparing healthy tissue, with a resulting dose distribution acceptable to the radiation oncologist. A *dose transfer matrix* A, specific to the patient's geometry, represents how a unit of radiation in beamlet j is deposited in body pixel i. The components of A are determined by the fraction of pixel i which intersects with beamlet j, attenuated by the distance of the pixel from the place where the beam enters the body. The dose transfer matrix A can be divided into the following: a matrix T which contains dose transfer information to tumor pixels only, matrices C_1 through C_K which contain does transfer information to pixel in critical organs 1 though K, and body matrix B which contains dose transfer information for all non-tumor and non-critical-organ pixels in the body. The variable vector x represents the beamlet intensities, and the right hand side vector b represents the dosage requirements.

The constraints for the crisp (non-fuzzy) formulation of the RTP is

$$\begin{aligned} \text{subject to} \quad B &\leq b_{body} \\ C_1 &\leq b_{C_1} \\ &\dots \\ C_K &\leq b_{C_K} \\ T &\leq b_{tumor} \\ -T &\leq -b_{tumor}. \end{aligned} \tag{20}$$

To test the fuzzy robust model, we interpret the radiation therapy problem in which the radiation oncologist is flexible regarding the dose limits (fuzziness in the inequality) and the components of the attenuation matrix are based on incomplete information (possibilistic parameters), as below.

$$\max \quad (1-\lambda)\alpha + \lambda\beta \tag{21}$$

$$\begin{aligned} \text{subject to} \quad (1-\beta)V_{body}x + \beta Z_{body}x &\leq b_{body} + (\alpha-1)d_{body} \\ (1-\beta)V_{C_1}x + \beta Z_{C_1}x &\leq b_{C_1} + (\alpha-1)d_{C_1} \\ &\dots \\ (1-\beta)V_{C_k}x + \beta Z_{C_k}x &\leq b_{C_k} + (\alpha-1)d_{C_k} \\ (1-\beta)V_{tumor}x + \beta Z_{tumor}x &\leq b_{tumor} + (\alpha-1)d_{tumor} \\ (\beta-1)V_{tumor}x - \beta Z_{tumor}x &\leq -b_{tumor} + (1-\alpha)d_{tumor} \\ x &\geq 0 \\ \alpha, \beta &\in [0,1]. \end{aligned} \tag{22}$$

The fuzzy robust model for the radiation therapy planning problem was solved in MATLAB. The starting point was found by solving a non-fuzzy version of the radiation therapy planning problem using linear programming. The code was tested on an image of one tumor and two critical organs with 64×64 pixel resolution the radiation beam discretized into 10 beamlets. In addition, the model was tested with concave constraints as in (20).

4.2 Testing the Model: Results

MATLAB found a feasible, acceptable solution to the Radiation Therapy problem for all values of λ between 0 and 1. The problem had 72 variables and 997 constraints, which made it a medium-to-large-scale problem. MATLAB prefers to use trust region methods for large scale problem, but because of the non-convex feasible region, it had to use a line-search method. MATLAB required over 20,000 function evaluations to converge to a solution, even though it was close after 2,000. The time taken to solve the fuzzy robust model was an order of magnitude larger than the time taken to solve other mixed fuzzy and possibilistic programming models, all of which had linear programming formulations. With concave constraints, MATLAB still used line-search methods to solve the problem, but required only 6,000 function evaluations to converge to a solution. The solutions found by the concave and non-concave formulations were not identical– but the resultant doses were equally satisfactory.

5 Conclusion

This chapter introduces a model for problems with both fuzzy and possibilistic variables. The model is semantically meaningful, and puts an additional fine-tuning parameter in the hands of the decision maker.

In addition, it introduces the idea of selecting a membership function that will facilitate the solution of the problem. The fact that a slight perturbation in the shape of the distribution of the possibilistic parameters improved model performance appears to be a novel observation.

Avenues for further research include:

- How can the possibilistic right hand side be incorporated into the mixed robust model? Is there a way to simultaneously represent constraint flexibility with α and right-hand-side imprecision with β?
- Both fuzzy and possibilistic intervals are upper-semi-continuous, so varying α-levels imply moving up or down either the left slope (or profile), or the right slope (or profile), but not over the entire interval. Because the decision maker is seeking to minimize risk, the current formulation selects whichever profile represents the pessimistic point of view. Is there a way to appropriately parametrize movement over the entire fuzzy or possibilistic interval so that an optimistic point of view can also be represented?

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