Chebyshev Approximation of Inconsistent Fuzzy Relational Equations with Max-T Composition*

Pingke Li and Shu-Cherng Fang

Abstract. This paper considers resolving the inconsistency of a system of fuzzy relational equations with max-T composition by simultaneously modifying the coefficient matrix and the right hand side vector. We show that resolving the inconsistency of fuzzy relational equations with max-T composition by means of Chebyshev approximation is closely related to the generalized solvability of interval-valued fuzzy relational equations with max-T composition. An efficient procedure is proposed to obtain a consistent system with the smallest perturbation in the sense of Chebyshev distance.

Keywords: Fuzzy optimization, fuzzy relational equations, Chebyshev approximation.

1 Introduction

A system of fuzzy relational equations with max-T composition is of the form

$$\max_{j \in N} T(a_{ij}, x_j) = b_i, \quad \forall\ i \in M, \tag{1}$$

where $M = \{1, 2, \cdots, m\}$ and $N = \{1, 2, \cdots, n\}$ are two index sets, $A = (a_{ij})_{m\times n} \in [0,1]^{mn}$, $\boldsymbol{b} = (b_1, b_2, \cdots, b_m)^T \in [0,1]^m$, $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^T \in [0,1]^n$ and $T : [0,1]^2 \to [0,1]$ is a triangular norm (t-norm for short). A

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system of the above form in (1) is also called a system of max-T equations for short and denoted as $A \circ_T \boldsymbol{x} = \boldsymbol{b}$ in the matrix form where "$\circ_T$" stands for the max-T composition. Typically, the t-norm T involved in a system $A \circ_T \boldsymbol{x} = \boldsymbol{b}$ is required to be continuous, i.e., continuous as a function of two arguments.

The resolution of a system of max-T equations is to determine the unknown vector $\boldsymbol{x}$ for a given coefficient matrix A and a right hand side vector $\boldsymbol{b}$ such that $A \circ_T \boldsymbol{x} = \boldsymbol{b}$. The set of all solutions to a system of max-T equations $A \circ_T \boldsymbol{x} = \boldsymbol{b}$ is denoted by $S(A, \boldsymbol{b})$, i.e., $S(A, \boldsymbol{b}) = \{\boldsymbol{x} \in [0,1]^n \mid A \circ_T \boldsymbol{x} = \boldsymbol{b}\}$. A system $A \circ_T \boldsymbol{x} = \boldsymbol{b}$ is called consistent if $S(A, \boldsymbol{b}) \neq \emptyset$, otherwise, it is inconsistent.

Fuzzy relational equations were first investigated by Sanchez [26, 27] under the max-T_M composition where T_M is the *minimum* operator, i.e., $T_M(x, y) = \min(x, y)$. Since then, solving various types of fuzzy relational equations has become one of the most appealing issues in fuzzy set theory. It has been pointed out that fuzzy relational equations play an important role as a uniform platform in many applications of fuzzy sets and fuzzy systems. See, e.g., Pedrycz [22, 24], Mordeson and Malik [18] and Peeva and Kyosev [25].

The resolution of a system of max-T equations has been investigated by Pedrycz [20, 21], Miyakoshi and Shimbo [17], Di Nola *et al.* [6, 7, 8], Klir and Yuan [12], De Baets [5], etc. It is well known that the consistency of a system of max-T equations can be verified in polynomial time by constructing and checking a potential maximum solution. The set of all solutions, when it is nonempty, is a finitely generated root system which can be fully determined by a unique maximum solution and a finite number of minimal solutions. However, the detection of all minimal solutions is an NP-hard problem. Similar conclusions can be drawn for a system of max-T inequalities. The reader may refer to Li and Fang [14, 15] and references therein for more details.

Although the consistency of a system of max-T equations can be readily verified and its solution set can be well characterized, related investigations are meaningful only when the system under consideration is consistent. However, due to the inaccuracy and deficiency in data or the inappropriate choice of the t-norm, it happens quite often that the system of max-T equations obtained in modeling a real situation turns out to be inconsistent. Moreover, the consistency of a system of max-T equations could be very sensitive to the data, i.e., small perturbations in the data could lead a consistent system to become inconsistent.

To deal with the impreciseness of the data and resolve the inconsistency of the system, one possible approach is to consider the interval-valued max-T equations, i.e., each entry in the matrix A and the vector $\boldsymbol{b}$ is replaced by a closed interval of possible values in $[0, 1]$. A system of interval-valued max-T equations can be represented in the form $\tilde{A} \circ_T \boldsymbol{x} = \tilde{\boldsymbol{b}}$ where $\tilde{A} = (\tilde{a}_{ij})_{m \times n}$ is an interval-valued matrix with $\tilde{a}_{ij} = [\underline{a}_{ij}, \overline{a}_{ij}] \subseteq [0, 1]$ and $\tilde{\boldsymbol{b}} = (\tilde{b}_i)_{m \times 1}$ is an interval-valued vector with $\tilde{b}_i = [\underline{b}_i, \overline{b}_i] \subseteq [0,1]$. Denote $\underline{A} = (\underline{a}_{ij})_{m\times n}$, $\overline{A} = (\overline{a}_{ij})_{m\times n}$ and similarly, $\underline{\boldsymbol{b}} = (\underline{b}_i)_{m\times 1}$, $\overline{\boldsymbol{b}} = (\overline{b}_i)_{m\times 1}$. By extending the natural order in a componentwise manner, $\tilde{A}$ and $\tilde{\boldsymbol{b}}$ induce the sets $[\underline{A}, \overline{A}] \triangleq \{A \mid \underline{A} \le A \le \overline{A}\}$ and $[\underline{\boldsymbol{b}}, \overline{\boldsymbol{b}}] \triangleq \{\boldsymbol{b} \mid \underline{\boldsymbol{b}} \le \boldsymbol{b} \le \overline{\boldsymbol{b}}\}$, respectively. The matrices $\underline{A}$ and $\overline{A}$ are referred to as the lower and upper bounds of $\tilde{A}$, respectively, and similarly, the vectors $\underline{\boldsymbol{b}}$ and $\overline{\boldsymbol{b}}$ the lower and upper bounds of $\tilde{\boldsymbol{b}}$, respectively. A system of interval-valued max-T equations $\tilde{A} \circ_T \boldsymbol{x} = \tilde{\boldsymbol{b}}$ is understood as the family of all systems of max-T equations $A \circ_T \boldsymbol{x} = \boldsymbol{b}$ with $A \in [\underline{A}, \overline{A}]$ and $\boldsymbol{b} \in [\underline{\boldsymbol{b}}, \overline{\boldsymbol{b}}]$. Without loss of generality, we may always assume that $\underline{A} \le \overline{A}$ and $\underline{\boldsymbol{b}} \le \overline{\boldsymbol{b}}$ such that the system $\tilde{A} \circ \boldsymbol{x} = \tilde{\boldsymbol{b}}$ is properly defined. Interval-valued max-T equations have been investigated by Wagenknecht and Hartmann [28, 29], Wang and Chang [30], Li and Fang [13], Wang *et al.* [31] and Li and Fang [16].

Another approach to resolving the inconsistency of a system of max-T equations is to perturb as slightly as possible either the coefficient matrix, the right hand side vector or both to reach a consistent system. Based on the notion of "minimal distortions", Pedrycz [23] proposed a procedure to modify the right hand side vector of an inconsistent system of max-T_M equations. However, as indicated by Cuninghame-Green and Cechlárová [3], the procedure is not given in a precise algorithmic form and hence would be difficult to implement in a computer. Cuninghame-Green and Cechlárová [3] presented an algorithm to obtain a consistent system of max-T_M equations with the smallest perturbation of the right hand side vector in the sense of Chebyshev distance, whereas Cechlárová [2] proposed an analogous algorithm to resolve the inconsistency of a system of max-T_M equations by modifying the coefficient matrix. Both algorithms are essentially based on the idempotency property of T_M, i.e., $T_M(x,x) = \min(x,x) = x$, $\forall\ x \in [0,1]$. Hence, neither of them can be generalized for general max-T equations since T_M has been proved to be the unique t-norm which possesses the idempotency property. Moreover, no algorithm is known that resolves the inconsistency of a system of max-T_M equations by simultaneously modifying the coefficient matrix and the right hand side vector. To the best of our knowledge, resolving the inconsistency of a system of max-T equations by means of Chebyshev approximation remains to be an open problem.

In this paper, we show that resolving the inconsistency of a system of max-T equations by means of Chebyshev approximation is closely related to the generalized solvability of interval-valued max-T equations. A bisection method is proposed for an inconsistent system of max-T equations to obtain the smallest perturbation bound of both the coefficient matrix and the right hand side vector. The construction of a Chebyshev approximation is introduced thereafter and illustrated by numerical examples. The proposed procedure remains valid with necessary modifications if only the coefficient matrix or the right hand side vector, but not both, can be perturbed.

2 Preliminaries

In this section, we recall some basic concepts and results associated with fuzzy relational equations, which are indispensable for the introduction of the Chebyshev approximation approach in this context. All proofs in this section are omitted to make the paper succinct and readable. The reader may refer to the monograph of Klement *et al.* [11] for a comprehensive discussion on triangular norms, and Li and Fang [15, 16] and reference therein for a detailed discussion on max-T equations and interval-valued max-T equations.

2.1 Triangular Norms

Although originally introduced in the framework of probabilistic metric spaces, t-norms have been proposed as natural generalizations of the logical conjunction in fuzzy logic and played an important role in the construction of fuzzy systems which may be described by fuzzy relational equations.

Definition 2.1. *A t-norm is a binary operator $T : [0,1]^2 \to [0,1]$ such that for all $x, y, z \in [0,1]$ the following four axioms are satisfied:*

(T1) $T(x,y) = T(y,x)$. *(commutativity)*
(T2) $T(x,T(y,z)) = T(T(x,y),z)$. *(associativity)*
(T3) $T(x,y) \le T(x,z)$, *whenever* $y \le z$. *(monotonicity)*
(T4) $T(x,1) = x$. *(boundary condition)*

A t-norm is said to be continuous if it is continuous as a function of two arguments. Due to its commutativity and monotonicity properties, a t-norm is continuous if and only if it is continuous in one of its arguments. Analogously, a t-norm is said to be left- or right-continuous if it is left- or right-continuous, respectively, in one of its arguments.

The most frequently used continuous t-norm is the *minimum* operator $T_M(x,y) = \min(x,y)$. Other important continuous t-norms include the *product* operator $T_P(x,y) = xy$ and the *bounded difference* operator $T_L(x,y) = \max(x+y-1, 0)$, a.k.a., Łukasiewicz t-norm. Note that T_M is the largest t-norm while T_P and T_L are prototypical examples of two important classes of continuous t-norms, i.e., strict t-norms and nilpotent t-norms, respectively.

Definition 2.2. *Let $T : [0,1]^2 \to [0,1]$ be a left-continuous t-norm. The associated residual implicator is a binary operator $I_T : [0,1]^2 \to [0,1]$ such that*

$$I_T(x,y) = \sup\{z \in [0,1] \mid T(x,z) \le y\}, \quad \forall\ (x,y) \in [0,1]^2. \tag{2}$$

Residual implicators are also known as φ-operators which were introduced by Pedrycz [19, 20] in a different approach. The connection between a φ-operator and its corresponding t-norm has been investigated in full generality by Gottwald [9, 10], Miyakoshi and Shimbo [17] and Di Nola *et al.* [8].

The residual implicators with respect to the *minimum* operator T_M, the *product* operator T_P and the Łukasiewicz t-norm T_L are, respectively,

$$I_{T_M}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ y, & \text{otherwise,} \end{cases} \qquad \text{(Gödel implicator)}$$

$$I_{T_P}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ y/x, & \text{otherwise,} \end{cases} \qquad \text{(Goguen implicator)}$$

$$I_{T_L}(x,y) = \min(1 - x + y, 1). \qquad \text{(Łukasiewicz implicator)}$$

Lemma 2.3. *The residual implicator I_T with respect to a left-continuous t-norm T is left-continuous and decreasing in its first argument as well as right-continuous and increasing in its second argument.*

Lemma 2.4. *Let T be a left-continuous t-norm and I_T its residual implicator. The inequality $T(a, I_T(a,b)) \le b$ holds for all $a, b \in [0,1]$. Moreover, $T(a,x) \le b$ if and only if $x \le I_T(a,b)$.*

Lemma 2.4 plays a crucial role in the resolution of max-T equations, which is actually a special scenario of the general theory of Galois connections [1].

2.2 Fuzzy Relational Equations

Let $A \circ_T \boldsymbol{x} = \boldsymbol{b}$ be a system of max-T equations with T being a continuous t-norm. Due to the monotonicity of the t-norm T, we have $A \circ_T \boldsymbol{x}^1 \le A \circ_T \boldsymbol{x}^2$ whenever $\boldsymbol{x}^1 \le \boldsymbol{x}^2$. Hence, $\boldsymbol{x} \in S(A, \boldsymbol{b})$ if $\boldsymbol{x}^1,\ \boldsymbol{x}^2 \in S(A, \boldsymbol{b})$ and $\boldsymbol{x}^1 \le \boldsymbol{x} \le \boldsymbol{x}^2$. Therefore, we may focus on the extremal solutions as defined below.

Definition 2.5. *A solution $\check{\boldsymbol{x}} \in S(A, \boldsymbol{b})$ is called a minimal solution if for any $\boldsymbol{x} \in S(A, \boldsymbol{b})$, the relation $\boldsymbol{x} \le \check{\boldsymbol{x}}$ implies $\boldsymbol{x} = \check{\boldsymbol{x}}$. A solution $\hat{\boldsymbol{x}} \in S(A, \boldsymbol{b})$ is called a maximum solution if $\boldsymbol{x} \le \hat{\boldsymbol{x}}$, $\forall\ \boldsymbol{x} \in S(A, \boldsymbol{b})$.*

Lemma 2.6 *Let T be a left-continuous t-norm and I_T its residual implicator. For any $A \in [0,1]^{mn}$ and $\boldsymbol{b} \in [0,1]^m$, it holds that $A \circ_T (A^T \circ_\varphi \boldsymbol{b}) \le \boldsymbol{b}$ where "$\circ_\varphi$" stands for the min-I_T composition and $A^T \circ_\varphi \boldsymbol{b} \in [0,1]^n$ is the vector with its components being defined by*

$$(A^T \circ_\varphi \boldsymbol{b})_j = \min\{I_T(a_{ij}, b_i) \mid i \in M\}, \quad \forall\ j \in N. \tag{3}$$

Moreover, $A \circ_T \boldsymbol{x} \le \boldsymbol{b}$ if and only if $\boldsymbol{x} \le A^T \circ_\varphi \boldsymbol{b}$.

Theorem 2.7 *Let $A \circ_T \boldsymbol{x} = \boldsymbol{b}$ be a system of max-T equations with T being a left-continuous t-norm. The system is consistent if and only if $A^T \circ_\varphi \boldsymbol{b}$ is a solution to $A \circ_T \boldsymbol{x} = \boldsymbol{b}$. Moreover, if T is also right-continuous and hence* *continuous, the solution set $S(A,\boldsymbol{b})$, when it is nonempty, can be fully determined by one maximum solution and a finite number of minimal solutions, i.e.,*

$$S(A,\boldsymbol{b}) = \bigcup_{\check{\boldsymbol{x}}\in\check{S}(A,\boldsymbol{b})} \{\boldsymbol{x}\in[0,1]^n \mid \check{\boldsymbol{x}}\le\boldsymbol{x}\le\hat{\boldsymbol{x}}\}, \tag{4}$$

where $\check{S}(A,\boldsymbol{b})$ is the set of all minimal solutions of $A\circ_T\boldsymbol{x}=\boldsymbol{b}$ and $\hat{\boldsymbol{x}}=A^T\circ_\varphi\boldsymbol{b}$ is the maximum solution.

Lemma 2.6 is a direct result of Lemma 2.4. The solvability criteria of max-T equations were investigated by Sanchez [26], Pedrycz [20, 21] and Miyakoshi and Shimbo [17] while the structure of the solution set was characterized by Sanchez [27] and Di Nola *et al.* [6, 7, 8]. The particular structure of $S(A,\boldsymbol{b})$ is called a finitely generated root system by De Baets [4, 5]. Note that the intersection of two finitely generated root systems, when it is nonempty, remains to be a finitely generated root system.

According to Theorem 2.7, the consistency of a system of max-T equations $A\circ_T\boldsymbol{x}=\boldsymbol{b}$ can be verified by constructing and checking the potential maximum solution $\hat{\boldsymbol{x}}=A^T\circ_\varphi\boldsymbol{b}$ in a time complexity of $O(mn)$. However, the detection of all minimal solutions is a complicated and challenging issue for investigation. The reader may refer to Li and Fang [15] and references therein for more detailed discussion.

2.3 Interval-Valued Fuzzy Relational Equations

Let $\tilde{A}$ be an interval-valued matrix with the lower bound $\underline{A}\in[0,1]^{mn}$ and the upper bound $\overline{A}\in[0,1]^{mn}$, and $\tilde{\boldsymbol{b}}$ an interval-valued vector with the lower bound $\underline{\boldsymbol{b}}\in[0,1]^m$ and the upper bound $\overline{\boldsymbol{b}}\in[0,1]^m$. We now consider a system of interval-valued max-T equations $\tilde{A}\circ_T\boldsymbol{x}=\tilde{\boldsymbol{b}}$ with T being a continuous t-norm. The following two lemmas are crucial in dealing with the system $\tilde{A}\circ_T\boldsymbol{x}=\tilde{\boldsymbol{b}}$, both of which simply rely on the monotonicity and continuity properties of the t-norm T.

Lemma 2.8. *Let $\tilde{A}$ be an interval-valued matrix and T a continuous t-norm. Given a vector $\boldsymbol{x}\in[0,1]^n$, then for each vector $\boldsymbol{b}\in[\underline{A}\circ_T\boldsymbol{x},\overline{A}\circ_T\boldsymbol{x}]$ there exists $A\in[\underline{A},\overline{A}]$ such that $A\circ_T\boldsymbol{x}=\boldsymbol{b}$.*

Lemma 2.9. *Let $\tilde{A}$ be an interval-valued matrix and T a continuous t-norm. For any vector $\boldsymbol{x}\in[0,1]^n$, we have*

$$\{A\circ_T\boldsymbol{x} \mid A\in[\underline{A},\overline{A}]\} = [\underline{A}\circ_T\boldsymbol{x},\overline{A}\circ_T\boldsymbol{x}]. \tag{5}$$

Definition 2.10. *Let $\tilde{A}\circ_T\boldsymbol{x}=\tilde{\boldsymbol{b}}$ be a system of interval-valued max-T equations with T being a continuous t-norm. A vector $\boldsymbol{x}\in[0,1]^n$ is called a united solution of $\tilde{A}\circ_T\boldsymbol{x}=\tilde{\boldsymbol{b}}$ if there exist $A\in[\underline{A},\overline{A}]$ and $\boldsymbol{b}\in[\underline{\boldsymbol{b}},\overline{\boldsymbol{b}}]$ such that $A\circ_T\boldsymbol{x}=\boldsymbol{b}$.*

Denote $S_u(\tilde{A}, \tilde{\boldsymbol{b}})$ the set of united solutions for a system of interval-valued max-T equations $\tilde{A} \circ_T \boldsymbol{x} = \tilde{\boldsymbol{b}}$. By Lemma 2.9, we immediately have

$$S_u(\tilde{A}, \tilde{\boldsymbol{b}}) = \{\boldsymbol{x} \in [0,1]^n \mid [\underline{A} \circ_T \boldsymbol{x}, \overline{A} \circ_T \boldsymbol{x}] \cap [\underline{\boldsymbol{b}}, \overline{\boldsymbol{b}}] \neq \emptyset\} \tag{6}$$

$$= \{\boldsymbol{x} \in [0,1]^n \mid \underline{A} \circ_T \boldsymbol{x} \leq \overline{\boldsymbol{b}},\ \overline{A} \circ_T \boldsymbol{x} \geq \underline{\boldsymbol{b}}\} \tag{7}$$

and hence, by Lemma 2.6, the following straightforward result.

Theorem 2.11. *Let $\tilde{A} \circ_T \boldsymbol{x} = \tilde{\boldsymbol{b}}$ be a system of interval-valued max-T equations with T being a continuous t-norm. The set of united solutions $S_u(\tilde{A}, \tilde{\boldsymbol{b}}) \neq \emptyset$ if and only if $\overline{A} \circ_T (\underline{A}^T \circ_\varphi \overline{\boldsymbol{b}}) \geq \underline{\boldsymbol{b}}$.*

Definition 2.12. *Let $\tilde{A} \circ_T \boldsymbol{x} = \tilde{\boldsymbol{b}}$ be a system of interval-valued max-T equations with T being a continuous t-norm. A vector $\boldsymbol{x} \in [0,1]^n$ is called a tolerable solution of $\tilde{A} \circ_T \boldsymbol{x} = \tilde{\boldsymbol{b}}$ if for each $A \in [\underline{A}, \overline{A}]$ there exists $\boldsymbol{b} \in [\underline{\boldsymbol{b}}, \overline{\boldsymbol{b}}]$ such that $A \circ_T \boldsymbol{x} = \boldsymbol{b}$. Similarly, a vector $\boldsymbol{x} \in [0,1]^n$ is called a controllable solution of $\tilde{A} \circ_T \boldsymbol{x} = \tilde{\boldsymbol{b}}$ if for each $\boldsymbol{b} \in [\underline{\boldsymbol{b}}, \overline{\boldsymbol{b}}]$ there exists $A \in [\underline{A}, \overline{A}]$ such that $A \circ_T \boldsymbol{x} = \boldsymbol{b}$.*

Denote $S_t(\tilde{A}, \tilde{\boldsymbol{b}})$ and $S_c(\tilde{A}, \tilde{\boldsymbol{b}})$ the sets of tolerable solutions and controllable solutions, respectively, for a system of interval-valued max-T equations $\tilde{A} \circ_T \boldsymbol{x} = \tilde{\boldsymbol{b}}$. By Lemma 2.9, we have

$$S_t(\tilde{A}, \tilde{\boldsymbol{b}}) = \{\boldsymbol{x} \in [0,1]^n \mid [\underline{A} \circ_T \boldsymbol{x}, \overline{A} \circ_T \boldsymbol{x}] \subseteq [\underline{\boldsymbol{b}}, \overline{\boldsymbol{b}}]\} \tag{8}$$

$$= \{\boldsymbol{x} \in [0,1]^n \mid \overline{A} \circ_T \boldsymbol{x} \leq \overline{\boldsymbol{b}},\ \underline{A} \circ_T \boldsymbol{x} \geq \underline{\boldsymbol{b}}\} \tag{9}$$

and

$$S_c(\tilde{A}, \tilde{\boldsymbol{b}}) = \{\boldsymbol{x} \in [0,1]^n \mid [\underline{A} \circ_T \boldsymbol{x}, \overline{A} \circ_T \boldsymbol{x}] \supseteq [\underline{\boldsymbol{b}}, \overline{\boldsymbol{b}}]\} \tag{10}$$

$$= \{\boldsymbol{x} \in [0,1]^n \mid \underline{A} \circ_T \boldsymbol{x} \leq \underline{\boldsymbol{b}},\ \overline{A} \circ_T \boldsymbol{x} \geq \overline{\boldsymbol{b}}\}. \tag{11}$$

Theorem 2.13. *Let $\tilde{A} \circ_T \boldsymbol{x} = \tilde{\boldsymbol{b}}$ be a system of interval-valued max-T equations with T being a continuous t-norm. The set of tolerable solutions $S_t(\tilde{A}, \tilde{\boldsymbol{b}}) \neq \emptyset$ if and only if $\underline{A} \circ_T (\overline{A}^T \circ_\varphi \overline{\boldsymbol{b}}) \geq \underline{\boldsymbol{b}}$ while the set of controllable solutions $S_c(\tilde{A}, \tilde{\boldsymbol{b}}) \neq \emptyset$ if and only if $\overline{A} \circ_T (\underline{A}^T \circ_\varphi \underline{\boldsymbol{b}}) \geq \overline{\boldsymbol{b}}$.*

It is clear that $S_t(\tilde{A}, \tilde{\boldsymbol{b}}) \subseteq S_u(\tilde{A}, \tilde{\boldsymbol{b}})$ and $S_c(\tilde{A}, \tilde{\boldsymbol{b}}) \subseteq S_u(\tilde{A}, \tilde{\boldsymbol{b}})$. Moreover, $S_t(\tilde{A}, \tilde{\boldsymbol{b}}) \cap S_c(\tilde{A}, \tilde{\boldsymbol{b}}) = \{\boldsymbol{x} \in [0,1]^n \mid \underline{A} \circ_T \boldsymbol{x} = \underline{\boldsymbol{b}},\ \overline{A} \circ_T \boldsymbol{x} = \overline{\boldsymbol{b}}\}$.

By Theorems 2.11 and 2.13, the existence of a united solution can be verified in a time complexity of $O(mn)$ as well as the existence of a tolerable solution and controllable solution, respectively. Furthermore, as will be shown in Section 3, the notion of united solutions bridges the gap between an inconsistent system of max-T equations and a system of interval-valued max-T equations. The notions of tolerable solutions and controllable solutions are the key to the construction of a Chebyshev approximation of an inconsistent system of max-T equations.

3 The Chebyshev Approximation

In this section, we consider an inconsistent system of max-T equations $A \circ_T \boldsymbol{x} = \boldsymbol{b}$ with T being a continuous t-norm and resolve its inconsistency by means of Chebyshev approximation. Without loss of generality, we may always assume that the system $A \circ_T \boldsymbol{x} = \boldsymbol{b}$ is in its normal form, i.e., the equations are arranged in the way such that $b_1 \geq b_2 \geq \cdots \geq b_m \geq 0$. Note that the equations corresponding to the index set $M_0 = \{i \in M \mid b_i = 0\}$ should be taken into consideration whenever $S(A, \boldsymbol{b}) = \emptyset$ while they can be discarded with necessary modifications on the remaining equations in case of consistency.

For notational convenience, the infix notations "$\wedge$" and "$\vee$" are used to denote the *minimum* and *maximum* operators, respectively, i.e., $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$. Analogously, we denote $A^1 \wedge A^2 = (a_{ij}^1 \wedge a_{ij}^2)_{m \times n}$ and $A^1 \vee A^2 = (a_{ij}^1 \vee a_{ij}^2)_{m \times n}$ for any $A^1, A^2 \in [0,1]^{mn}$, and $\boldsymbol{b}^1 \wedge \boldsymbol{b}^2 = (b_i^1 \wedge b_i^2)_{m \times 1}$ and $\boldsymbol{b}^1 \vee \boldsymbol{b}^2 = (b_i^1 \vee b_i^2)_{m \times 1}$ for any $\boldsymbol{b}^1, \boldsymbol{b}^2 \in [0,1]^m$.

Denote $\mathscr{C}$ the set of all pairs of a coefficient matrix $A' \in [0,1]^{mn}$ and a right hand side vector $\boldsymbol{b}' \in [0,1]^m$ such that the corresponding system of max-T equations $A' \circ_T \boldsymbol{x} = \boldsymbol{b}'$ is consistent, i.e., $\mathscr{C} = \{(A', \boldsymbol{b}') \mid S(A', \boldsymbol{b}') \neq \emptyset\}$. It is clear that $(A, \boldsymbol{b}) \notin \mathscr{C}$ for the inconsistent system $A \circ_T \boldsymbol{x} = \boldsymbol{b}$. The Chebyshev distance between the pair $(A, \boldsymbol{b})$ and a pair $(A', \boldsymbol{b}') \in \mathscr{C}$ is defined as

$$\rho((A, \boldsymbol{b}), (A', \boldsymbol{b}')) = \max\left(\max_{i,j} |a_{ij} - a'_{ij}|,\ \max_i |b_i - b'_i|\right). \tag{12}$$

Definition 3.1. *A system of max-T equations $A' \circ_T \boldsymbol{x} = \boldsymbol{b}'$ is said to be a δ-approximation of the system $A \circ_T \boldsymbol{x} = \boldsymbol{b}$ if $(A', \boldsymbol{b}') \in \mathscr{C}$ and $\rho((A, \boldsymbol{b}), (A', \boldsymbol{b}')) \leq \delta$.*

Definition 3.2. *A system of max-T equations $A^\dagger \circ_T \boldsymbol{x} = \boldsymbol{b}^\dagger$ is said to be a Chebyshev approximation of the system $A \circ_T \boldsymbol{x} = \boldsymbol{b}$ if $(A^\dagger, \boldsymbol{b}^\dagger) \in \mathscr{C}$ and*

$$\rho((A, \boldsymbol{b}), (A^\dagger, \boldsymbol{b}^\dagger)) = \inf_{(A', \boldsymbol{b}') \in \mathscr{C}} \rho((A, \boldsymbol{b}), (A', \boldsymbol{b}')). \tag{13}$$

Clearly, it suffices to consider δ-approximations with $\delta \in [0,1]$ for the system $A \circ_T \boldsymbol{x} = \boldsymbol{b}$. A 1-approximation of the system $A \circ_T \boldsymbol{x} = \boldsymbol{b}$ always exists, while a Chebyshev approximation is a δ-approximation with the smallest possible value of δ. For a given $\delta \in [0,1]$, denote $\tilde{A}(\delta)$ the interval-valued matrix with the lower bound $\underline{A}(\delta)$ and the upper bound $\overline{A}(\delta)$ where

$$\underline{A}(\delta) = ((a_{ij} - \delta) \vee 0)_{m \times n} \quad \text{and} \quad \overline{A}(\delta) = ((a_{ij} + \delta) \wedge 1)_{m \times n}, \tag{14}$$

respectively. Similarly, denote $\tilde{\boldsymbol{b}}(\delta)$ the interval-valued vector with the lower bound $\underline{\boldsymbol{b}}(\delta)$ and the upper bound $\overline{\boldsymbol{b}}(\delta)$ where

$$\underline{\boldsymbol{b}}(\delta) = ((b_i - \delta) \vee 0)_{m \times 1} \quad \text{and} \quad \overline{\boldsymbol{b}}(\delta) = ((b_i + \delta) \wedge 1)_{m \times 1}, \tag{15}$$

respectively. Consequently, we obtain a properly defined system of interval-valued max-T equations $\tilde{A}(\delta) \circ_T \boldsymbol{x} = \tilde{\boldsymbol{b}}(\delta)$. It is clear that a matrix $A' \in [\underline{A}(\delta), \overline{A}(\delta)]$ if and only if $A' = (a'_{ij})_{m\times n} \in [0,1]^{mn}$ and $\max_{i,j} |a_{ij} - a'_{ij}| \le \delta$. Similarly, a vector $\boldsymbol{b}' \in [\underline{\boldsymbol{b}}(\delta), \overline{\boldsymbol{b}}(\delta)]$ if and only if $\boldsymbol{b}' = (b'_i)_{m\times 1} \in [0,1]^m$ and $\max_i |b_i - b'_i| \le \delta$.

Theorem 3.3 *The system of max-T equations $A \circ_T \boldsymbol{x} = \boldsymbol{b}$ has a δ-approximation if and only if the system of interval-valued max-T equations $\tilde{A}(\delta) \circ_T \boldsymbol{x} = \tilde{\boldsymbol{b}}(\delta)$ has a united solution.*

Proof: If the system $A \circ_T \boldsymbol{x} = \boldsymbol{b}$ has a δ-approximation $A' \circ_T \boldsymbol{x} = \boldsymbol{b}'$ such that $S(A', \boldsymbol{b}') \ne \emptyset$, it is clear that $\max_{i,j} |a_{ij} - a'_{ij}| \le \delta$ and $\max_i |b_i - b'_i| \le \delta$, respectively. Hence, $A' \in [\underline{A}(\delta), \overline{A}(\delta)]$, $\boldsymbol{b}' \in [\underline{\boldsymbol{b}}(\delta), \overline{\boldsymbol{b}}(\delta)]$ and consequently, $S_u(\tilde{A}(\delta), \tilde{\boldsymbol{b}}(\delta)) \ne \emptyset$.

Conversely, if $S_u(\tilde{A}(\delta), \tilde{\boldsymbol{b}}(\delta)) \ne \emptyset$, there exist $A' \in [\underline{A}(\delta), \overline{A}(\delta)]$ and $\boldsymbol{b}' \in [\underline{\boldsymbol{b}}(\delta), \overline{\boldsymbol{b}}(\delta)]$ such that the system $A' \circ_T \boldsymbol{x} = \boldsymbol{b}'$ is consistent. It is clear that $\rho((A, \boldsymbol{b}), (A', \boldsymbol{b}')) \le \delta$ and hence $A' \circ_T \boldsymbol{x} = \boldsymbol{b}'$ is a δ-approximation of the system $A \circ_T \boldsymbol{x} = \boldsymbol{b}$. $\square$

Theorem 3.4 *If the system of interval-valued max-T equations $\tilde{A}(\delta) \circ_T \boldsymbol{x} = \tilde{\boldsymbol{b}}(\delta)$ has a united solution for some $\delta \in [0,1]$, then the system $\tilde{A}(\delta') \circ_T \boldsymbol{x} = \tilde{\boldsymbol{b}}(\delta')$ has a united solution for any $\delta' \in [\delta, 1]$.*

Proof: It is straightforward from the observation of $[\underline{A}(\delta), \overline{A}(\delta)] \subseteq [\underline{A}(\delta'), \overline{A}(\delta')]$ and $[\underline{\boldsymbol{b}}(\delta), \overline{\boldsymbol{b}}(\delta)] \subseteq [\underline{\boldsymbol{b}}(\delta'), \overline{\boldsymbol{b}}(\delta')]$ for $\delta \le \delta'$. $\square$

Theorem 3.5 *If the system of interval-valued max-T equations $\tilde{A}(\delta) \circ_T \boldsymbol{x} = \tilde{\boldsymbol{b}}(\delta)$ has a united solution for all $\delta \in (\delta', 1]$, then the system $\tilde{A}(\delta') \circ_T \boldsymbol{x} = \tilde{\boldsymbol{b}}(\delta')$ also has a united solution.*

Proof: By Theorem 2.11, the system $\tilde{A}(\delta) \circ_T \boldsymbol{x} = \tilde{\boldsymbol{b}}(\delta)$ has a united solution if and only if $\overline{A}(\delta) \circ_T (\underline{A}^T(\delta) \circ_\varphi \overline{\boldsymbol{b}}(\delta)) \ge \underline{\boldsymbol{b}}(\delta)$. Notice that each component of the vector $\overline{A}(\delta) \circ_T (\underline{A}^T(\delta) \circ_\varphi \overline{\boldsymbol{b}}(\delta)) - \underline{\boldsymbol{b}}(\delta)$ is right-continuous with respect to δ since all involved operations are continuous except that the residual implicator I_T is left-continuous in its first argument and right-continuous in its second argument. Hence, the system $\tilde{A}(\delta') \circ_T \boldsymbol{x} = \tilde{\boldsymbol{b}}(\delta')$ has a united solution as long as the system $\tilde{A}(\delta) \circ_T \boldsymbol{x} = \tilde{\boldsymbol{b}}(\delta)$ has a united solution for all $\delta \in (\delta', 1]$. $\square$

Theorem 3.3 indicates that the existence of a δ-approximation of a system of max-T equations is equivalent to the existence of a united solution of a corresponding system of interval-valued max-T equations which, by Theorem 2.11, can be verified in polynomial time. Theorems 3.4 and 3.5 guarantee that a Chebyshev approximation does exist for a system of max-T equations and also suggest a bisection method to obtain the smallest perturbation bound δ^* with the existence of a δ^*-approximation.

Algorithm

Step 1. Specify ϵ as the required level of precision.

Step 2. Set $\underline{\delta} = 0$, $\overline{\delta} = 1$ and $\delta = (\underline{\delta} + \overline{\delta})/2$.

Step 3. Verify $\overline{A}(\delta) \circ_T (\underline{A}^T(\delta) \circ_\varphi \overline{\boldsymbol{b}}(\delta)) \geq \underline{\boldsymbol{b}}(\delta)$. If it holds, set $\overline{\delta} = \delta$, otherwise, set $\underline{\delta} = \delta$. Set $\delta = (\underline{\delta} + \overline{\delta})/2$.

Step 4. If $\overline{\delta} - \underline{\delta} \leq \epsilon$, output $\delta^* = \overline{\delta}$ and stop. Otherwise, go to Step 3.

The above algorithm offers an ϵ-optimal value for δ^* in a time complexity of $O(mn \log(1/\epsilon))$ where ϵ is the predetermined level of precision. Once we obtain the value of δ^*, the remaining problem is to construct a Chebyshev approximation, i.e., a δ^*-approximation for the system $A \circ_T \boldsymbol{x} = \boldsymbol{b}$.

Denote $\boldsymbol{x}(\delta^*) = \underline{A}^T(\delta^*) \circ_\varphi \overline{\boldsymbol{b}}(\delta^*)$. By Theorem 2.11, $\boldsymbol{x}(\delta^*)$ is a united solution of the system $\tilde{A}(\delta^*) \circ_T \boldsymbol{x} = \tilde{\boldsymbol{b}}(\delta^*)$ and hence $[\underline{A}(\delta^*) \circ_T \boldsymbol{x}(\delta^*), \overline{A}(\delta^*) \circ_T \boldsymbol{x}(\delta^*)] \cap [\underline{\boldsymbol{b}}(\delta^*), \overline{\boldsymbol{b}}(\delta^*)] \neq \emptyset$. Denote $\tilde{\boldsymbol{b}}^\dagger(\delta^*)$ the interval-valued vector with the lower bound $\underline{\boldsymbol{b}}^\dagger(\delta^*)$ and the upper bound $\overline{\boldsymbol{b}}^\dagger(\delta^*)$ where

$$\underline{\boldsymbol{b}}^\dagger(\delta^*) = \underline{A}(\delta^*) \circ_T \boldsymbol{x}(\delta^*) \vee \underline{\boldsymbol{b}}(\delta^*) \text{ and } \overline{\boldsymbol{b}}^\dagger(\delta^*) = \overline{A}(\delta^*) \circ_T \boldsymbol{x}(\delta^*) \wedge \overline{\boldsymbol{b}}(\delta^*), \quad (16)$$

respectively. Notice that $[\underline{\boldsymbol{b}}^\dagger(\delta^*), \overline{\boldsymbol{b}}^\dagger(\delta^*)] \subseteq [\underline{A}(\delta^*) \circ_T \boldsymbol{x}(\delta^*), \overline{A}(\delta^*) \circ_T \boldsymbol{x}(\delta^*)]$ and hence $\boldsymbol{x}(\delta^*)$ is also a controllable solution of the system $\tilde{A}(\delta^*) \circ_T \boldsymbol{x} = \tilde{\boldsymbol{b}}^\dagger(\delta^*)$ which means that for each $\boldsymbol{b}' \in [\underline{\boldsymbol{b}}^\dagger(\delta^*), \overline{\boldsymbol{b}}^\dagger(\delta^*)]$ there exists $A' \in [\underline{A}(\delta^*), \overline{A}(\delta^*)]$ such that $A' \circ_T \boldsymbol{x}(\delta^*) = \boldsymbol{b}'$.

Therefore, denote $\tilde{A}^\dagger(\delta^*)$ the interval-valued matrix with the lower bound $\underline{A}^\dagger(\delta^*)$ and the upper bound $\overline{A}^\dagger(\delta^*)$ where

$$\underline{A}^\dagger(\delta^*) = (\boldsymbol{x}(\delta^*) \circ_\varphi (\underline{\boldsymbol{b}}^\dagger(\delta^*))^T)^T \wedge \overline{A}(\delta^*) \text{ and } \overline{A}^\dagger(\delta^*) = (\boldsymbol{x}(\delta^*) \circ_\varphi (\overline{\boldsymbol{b}}^\dagger(\delta^*))^T)^T \wedge \overline{A}(\delta^*) \quad (17)$$

respectively. Since $\boldsymbol{x}(\delta^*)$ is a controllable solution of the system $\tilde{A}(\delta^*) \circ_T \boldsymbol{x} = \tilde{\boldsymbol{b}}^\dagger(\delta^*)$, by Lemma 2.6 and Theorem 2.7, we have $\underline{A}^\dagger(\delta^*) \circ_T \boldsymbol{x}(\delta^*) = \underline{\boldsymbol{b}}^\dagger(\delta^*)$ and $\overline{A}^\dagger(\delta^*) \circ_T \boldsymbol{x}(\delta^*) = \overline{\boldsymbol{b}}^\dagger(\delta^*)$, respectively. Therefore, $\boldsymbol{x}(\delta^*)$ is simultaneously a tolerable and controllable solution of the system $\tilde{A}^\dagger(\delta^*) \circ_T \boldsymbol{x} = \tilde{\boldsymbol{b}}^\dagger(\delta^*)$. Moreover, since $[\underline{A}^\dagger(\delta^*), \overline{A}^\dagger(\delta^*)] \subseteq [\underline{A}(\delta^*), \overline{A}(\delta^*)]$ and $[\underline{\boldsymbol{b}}^\dagger(\delta^*), \overline{\boldsymbol{b}}^\dagger(\delta^*)] \subseteq [\underline{\boldsymbol{b}}(\delta^*), \overline{\boldsymbol{b}}(\delta^*)]$, any pair of $A^\dagger \in [\underline{A}^\dagger(\delta^*), \overline{A}^\dagger(\delta^*)]$ and $\boldsymbol{b}^\dagger = A^\dagger \circ_T \boldsymbol{x}(\delta^*) \in [\underline{\boldsymbol{b}}^\dagger(\delta^*), \overline{\boldsymbol{b}}^\dagger(\delta^*)]$ defines a Chebyshev approximation $A^\dagger \circ_T \boldsymbol{x} = \boldsymbol{b}^\dagger$ of the system $A \circ_T \boldsymbol{x} = \boldsymbol{b}$.

The proposed procedure remains valid with necessary modifications if we are allowed to perturb the coefficient matrix only or the right hand side vector only. In the algorithm for determining the smallest perturbation bound, we can simply keep the matrix A unchanged, i.e., $\underline{A}(\delta) = \overline{A}(\delta) = A$ for any $\delta \in [0, 1]$, if only $\boldsymbol{b}$ can be modified. Once the smallest perturbation bound δ_1^* is obtained for $\boldsymbol{b}$, the system $A \circ_T \boldsymbol{x} = \boldsymbol{b}^\dagger$ with $\boldsymbol{b}^\dagger = A \circ_T (A^T \circ_\varphi \overline{\boldsymbol{b}}(\delta_1^*))$ is a Chebyshev approximation of the inconsistent system $A \circ_T \boldsymbol{x} = \boldsymbol{b}$. The situation is analogous if only A can be modified. We can keep the vector $\boldsymbol{b}$ unchanged, i.e., $\underline{\boldsymbol{b}}(\delta) = \overline{\boldsymbol{b}}(\delta) = \boldsymbol{b}$ for any $\delta \in [0,1]$, to obtain the smallest perturbation bound δ_2^* for A. Thereafter, the system $A^\dagger \circ_T \boldsymbol{x} = \boldsymbol{b}$ with $A^\dagger = ((\underline{A}^T(\delta_2^*) \circ_\varphi \boldsymbol{b}) \circ_\varphi \boldsymbol{b}^T)^T \wedge \overline{A}(\delta_2^*)$ is a Chebyshev approximation of the inconsistent system $A \circ_T \boldsymbol{x} = \boldsymbol{b}$. Besides, it is obvious that $\delta^* \leq \min(\delta_1^*, \delta_2^*)$ for an inconsistent system of max-T equations. Deeper relations among these perturbation bounds are subject to further investigation.

4 Numerical Examples

In this section, we provide a few numerical examples to illustrate the proposed procedure and compare with the known results in Cuninghame-Green and Cechlárová [3] and Cechlárová [2].

Example 1. Consider the system of max-T_M equations $A \circ_{T_M} \boldsymbol{x} = \boldsymbol{b}$ with

$$A = \begin{pmatrix} 0.7 & 0.5 & 0.3 & 0.5 \\ 1 & 0.4 & 0.5 & 0.7 \\ 0.2 & 1 & 1 & 0.6 \\ 0.4 & 0.5 & 0.5 & 0.8 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ 0.4 \\ 0.2 \\ 0 \end{pmatrix}.$$

This example was originally presented by Pedrycz [23] and also investigated by Cuninghame-Green and Cechlárová [3].

The system is inconsistent since the potential maximum solution $\hat{\boldsymbol{x}} = (0,0,0,0)^T$ is clearly not a solution. By perturbing A and $\boldsymbol{b}$ simultaneously, our algorithm obtains $\delta^* = 0.3$ and hence

$$\tilde{A}(\delta^*) = \begin{pmatrix} [0.4,1] & [0.2,0.8] & [0,0.6] & [0.2,0.8] \\ [0.7,1] & [0.1,0.7] & [0.2,0.8] & [0.4,1] \\ [0,0.5] & [0.7,1] & [0.7,1] & [0.3,0.9] \\ [0.1,0.7] & [0.2,0.8] & [0.2,0.8] & [0.5,1] \end{pmatrix}, \qquad \boldsymbol{b}(\delta^*) = \begin{pmatrix} [0.7,1] \\ [0.1,0.7] \\ [0,0.5] \\ [0,0.3] \end{pmatrix}.$$

Moreover, we have $\boldsymbol{x}(\delta^*) = (1, 0.5, 0.5, 0.3)^T$. Consequently,

$$\tilde{\boldsymbol{b}}^\dagger(\delta^*) = \begin{pmatrix} [0.4,1] \\ [0.7,1] \\ [0.5,0.5] \\ [0.3,0.7] \end{pmatrix} \bigcap \begin{pmatrix} [0.7,1] \\ [0.1,0.7] \\ [0,0.5] \\ [0,0.3] \end{pmatrix} = \begin{pmatrix} [0.7,1] \\ 0.7 \\ 0.5 \\ 0.3 \end{pmatrix},$$

$$\underline{A}^{\dagger}(\delta^*) = \left(\begin{pmatrix} 1 \\ 0.5 \\ 0.5 \\ 0.3 \end{pmatrix} \circ_{\varphi} (0.7, 0.7, 0.5, 0.3)\right)^T \bigwedge \begin{pmatrix} 1 & 0.8 & 0.6 & 0.8 \\ 1 & 0.7 & 0.8 & 1 \\ 0.5 & 1 & 1 & 0.9 \\ 0.7 & 0.8 & 0.8 & 1 \end{pmatrix} = \begin{pmatrix} 0.7 & 0.8 & 0.6 & 0.8 \\ 0.7 & 0.7 & 0.8 & 1 \\ 0.5 & 1 & 1 & 0.9 \\ 0.3 & 0.3 & 0.3 & 1 \end{pmatrix}$$

and

$$\overline{A}^{\dagger}(\delta^*) = \left(\begin{pmatrix} 1 \\ 0.5 \\ 0.5 \\ 0.3 \end{pmatrix} \circ_{\varphi} (1, 0.7, 0.5, 0.3)\right)^T \bigwedge \begin{pmatrix} 1 & 0.8 & 0.6 & 0.8 \\ 1 & 0.7 & 0.8 & 1 \\ 0.5 & 1 & 1 & 0.9 \\ 0.7 & 0.8 & 0.8 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0.8 & 0.6 & 0.8 \\ 0.7 & 0.7 & 0.8 & 1 \\ 0.5 & 1 & 1 & 0.9 \\ 0.3 & 0.3 & 0.3 & 1 \end{pmatrix}.$$

Hence, any $A^{\dagger} \in [\underline{A}^{\dagger}(\delta^*), \overline{A}^{\dagger}(\delta^*)]$ and $\boldsymbol{b}^{\dagger} = A^{\dagger} \circ_{T_M} \boldsymbol{x}(\delta^*)$ define a Chebyshev approximation $A^{\dagger} \circ_{T_M} \boldsymbol{x} = \boldsymbol{b}^{\dagger}$, for instance,

$$A^{\dagger} = \begin{pmatrix} 0.8 & 0.8 & 0.6 & 0.8 \\ 0.7 & 0.7 & 0.8 & 1 \\ 0.5 & 1 & 1 & 0.9 \\ 0.3 & 0.3 & 0.3 & 1 \end{pmatrix}, \qquad \boldsymbol{b}^{\dagger} = \begin{pmatrix} 0.8 \\ 0.7 \\ 0.5 \\ 0.3 \end{pmatrix}.$$

If we want to resolve the inconsistency by modifying the right hand side vector only, the corresponding smallest perturbation bound becomes $\delta_1^* = 0.4$. Hence we have

$$\tilde{\boldsymbol{b}}(\delta_1^*) = \begin{pmatrix} [0.6, 1] \\ [0, 0.8] \\ [0, 0.6] \\ [0, 0.4] \end{pmatrix}, \qquad \boldsymbol{b}^{\dagger} = A \circ_{T_M} (A^T \circ_{\varphi} \overline{\boldsymbol{b}}(\delta_1^*)) = \begin{pmatrix} 0.7 \\ 0.8 \\ 0.4 \\ 0.4 \end{pmatrix}.$$

Consequently, the system $A \circ_{T_M} \boldsymbol{x} = \boldsymbol{b}^{\dagger}$ is a Chebyshev approximation of $A \circ_{T_M} \boldsymbol{x} = \boldsymbol{b}$ in this case, which is exactly the same as that given by Cuninghame-Green and Cechlárová [3]. Note that $A \circ_{T_M} \boldsymbol{x} = \boldsymbol{b}^{\dagger}$ has a maximum solution $(0.8, 0.4, 0.4, 0.4)^T$.

On the other hand, the smallest perturbation bound for A becomes $\delta_2^* = 0.6$, if we keep $\boldsymbol{b}$ unchanged. Hence we have

$$\tilde{A}(\delta_2^*) = \begin{pmatrix} [0.1, 1] & [0, 1] & [0, 0.9] & [0, 1] \\ [0.4, 1] & [0, 1] & [0, 1] & [0.1, 1] \\ [0, 0.8] & [0.4, 1] & [0.4, 1] & [0, 1] \\ [0, 1] & [0, 1] & [0, 1] & [0.2, 1] \end{pmatrix},$$

$$A^{\dagger} = ((\underline{A}^T(\delta_2^*) \circ_{\varphi} \boldsymbol{b}) \circ_{\varphi} \boldsymbol{b}^T)^T \wedge \overline{A}(\delta_2^*) = \begin{pmatrix} 1 & 1 & 0.9 & 1 \\ 0.4 & 1 & 1 & 1 \\ 0.2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Consequently, the system $A^{\dagger} \circ_{T_M} \boldsymbol{x} = \boldsymbol{b}$ is a Chebyshev approximation of $A \circ_{T_M} \boldsymbol{x} = \boldsymbol{b}$ in this case. Note that $A^{\dagger} \circ_{T_M} \boldsymbol{x} = \boldsymbol{b}$ has a maximum solution $(1, 0.2, 0.2, 0)^T$.

Example 2. Consider the system of max-T_M equations $A \circ_{T_M} \boldsymbol{x} = \boldsymbol{b}$ with

$$A = \begin{pmatrix} 0.6 & 0.2 & 0.9 & 0.1 & 0.6 \\ 0.5 & 0.7 & 0.3 & 0.8 & 0.7 \\ 0.3 & 0.6 & 0.7 & 0.4 & 0.2 \\ 0.3 & 0.8 & 0.5 & 0.4 & 0.2 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 0.9 \\ 0.5 \\ 0.4 \\ 0.3 \end{pmatrix}.$$

This example was originally presented by Cechlárová [2].

The system is inconsistent since the potential maximum solution $\hat{\boldsymbol{x}} = (1, 0.3, 0.3, 0.3, 0.5)^T$ is not a solution. The smallest perturbation bound for A is $\delta_2^* = 0.3$, if we are required to resolve the inconsistency by modifying A only. Hence we have

$$\tilde{A}(\delta_2^*) = \begin{pmatrix} [0.3, 0.9] & [0, 0.5] & [0.6, 1] & [0, 0.4] & [0.3, 0.9] \\ [0.2, 0.8] & [0.4, 1] & [0, 0.6] & [0.5, 1] & [0.4, 1] \\ [0, 0.6] & [0.5, 1] & [0.2, 0.8] & [0.1, 0.7] & [0, 0.5] \\ [0, 0.6] & [0.3, 0.9] & [0.4, 1] & [0.1, 0.7] & [0, 0.5] \end{pmatrix},$$

$$A^{\dagger} = ((\underline{A}^T(\delta_2^*) \circ_{\varphi} \boldsymbol{b}) \circ_{\varphi} \boldsymbol{b}^T)^T \wedge \overline{A}(\delta_2^*) = \begin{pmatrix} 0.9 & 0.5 & 0.9 & 0.4 & 0.9 \\ 0.5 & 1 & 0.5 & 0.5 & 0.5 \\ 0.4 & 0.9 & 0.4 & 0.4 & 0.4 \\ 0.3 & 1 & 0.3 & 0.3 & 0.3 \end{pmatrix}.$$

Consequently, the system $A^{\dagger} \circ_{T_M} \boldsymbol{x} = \boldsymbol{b}$ is a Chebyshev approximation of $A \circ_{T_M} \boldsymbol{x} = \boldsymbol{b}$ in this case. Note that $A^{\dagger} \circ_{T_M} \boldsymbol{x} = \boldsymbol{b}$ has a maximum solution $(1, 0.3, 1, 1, 1)^T$. Since a different method is used to construct a Chebyshev approximation, the matrix $A^{\dagger}$ offered by our procedure is slightly different from that given by Cechlárová [2], but both matrices share the same Chebyshev distance.

Now we present an example to illustrate that the proposed procedure works for general max-T equations with T being a continuous t-norm.

Example 3. Consider the system of max-T_L equations $A \circ_{T_L} \boldsymbol{x} = \boldsymbol{b}$ with

$$A = \begin{pmatrix} 0.2 & 0.9 & 0.8 & 0.4 \\ 0.8 & 0.3 & 0.4 & 0.8 \\ 0.5 & 0.7 & 0.1 & 0.6 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 0.8 \\ 0.6 \\ 0.2 \end{pmatrix}.$$

The potential maximum solution is $\hat{\boldsymbol{x}} = (0.7, 0.5, 1, 0.6)^T$ and

$$\begin{pmatrix} 0.2 & 0.9 & 0.8 & 0.4 \\ 0.8 & 0.3 & 0.4 & 0.8 \\ 0.5 & 0.7 & 0.1 & 0.6 \end{pmatrix} \circ_{T_L} \begin{pmatrix} 0.7 \\ 0.5 \\ 0.1 \\ 0.6 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.5 \\ 0.2 \end{pmatrix} \neq \begin{pmatrix} 0.8 \\ 0.6 \\ 0.2 \end{pmatrix}.$$

Therefore, the system is inconsistent. By perturbing A and $\boldsymbol{b}$ simultaneously, our algorithm obtains $\boldsymbol{x}(\delta^*) = (0.7500, 0.5500, 1.0000, 0.6500)^T$ with $\delta^* = 0.0250$, and consequently,

$$\underline{\boldsymbol{b}}^\dagger(\delta^*) = \begin{pmatrix} 0.7750 \\ 0.5750 \\ 0.2250 \end{pmatrix}, \quad \overline{\boldsymbol{b}}^\dagger(\delta^*) = \begin{pmatrix} 0.8250 \\ 0.5750 \\ 0.2250 \end{pmatrix}$$

and

$$\underline{A}^\dagger(\delta^*) = \begin{pmatrix} 0.2250 & 0.9250 & 0.7750 & 0.4250 \\ 0.8250 & 0.3250 & 0.4250 & 0.8250 \\ 0.4750 & 0.6750 & 0.1250 & 0.5750 \end{pmatrix}, \ \overline{A}^\dagger(\delta^*) = \begin{pmatrix} 0.2250 & 0.9250 & 0.8250 & 0.4250 \\ 0.8250 & 0.3250 & 0.4250 & 0.8250 \\ 0.4750 & 0.6750 & 0.1250 & 0.5750 \end{pmatrix}.$$

Hence, any $A^\dagger \in [\underline{A}^\dagger(\delta^*), \overline{A}^\dagger(\delta^*)]$ and $\boldsymbol{b}^\dagger = A^\dagger \circ_{T_L} \boldsymbol{x}(\delta^*)$ define a Chebyshev approximation $A^\dagger \circ_{T_L} \boldsymbol{x} = \boldsymbol{b}^\dagger$, for instance,

$$A^\dagger = \begin{pmatrix} 0.2250 & 0.9250 & 0.8000 & 0.4250 \\ 0.8250 & 0.3250 & 0.4250 & 0.8250 \\ 0.4750 & 0.6750 & 0.1250 & 0.5750 \end{pmatrix}, \qquad \boldsymbol{b}^\dagger = \begin{pmatrix} 0.8000 \\ 0.5750 \\ 0.2250 \end{pmatrix}.$$

5 Concluding Remarks

We have shown that the existence of a δ-approximation of a system of max-T equations is equivalent to the existence of a united solution of a corresponding system of interval-valued max-T equations. Consequently, the smallest perturbation bound can be obtained by repeatedly constructing a system of interval-valued max-T equations and verifying its solvability condition. As illustrated by our numerical examples, a Chebyshev approximation can be constructed readily once the smallest perturbation bound is obtained. It is clear that the Chebyshev approximation may not necessarily be unique. In this case, we may be interested in obtaining a Chebyshev approximation of some special quality, for instance, the one with the smallest number of modifications in the coefficient matrix and the right hand side vector. This new challenge goes beyond the scope of this paper and subject to further investigation.

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