Mathematical Approaches for Fuzzy Portfolio Selection Problems with Normal Mixture Distributions

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Abstract. This chapter considers some versatile portfolio selection models with general normal mixture distributions and fuzzy or interval numbers. Then, these mathematical approaches to obtain the optimal portfolio are developed. Furthermore, in order to compare our proposed models with standard models and represent the advantage of our proposed models, a numerical example is provided.

1 Introduction

In recent rapid expansions of investment and financial instability such as the extreme ups and downs to future markets of stocks and commodities, the role of investment theory becomes more and more important. Furthermore, in current investment environment, with information science and computers, not only big companies and institutional investors but also individual investors perform investment in stock, currency, land and property. Therefore, it is time to review investment theory, particularly portfolio theory. Practical financial markets are affected by a lot of uncertainty to which has a great influence on the future returns such as randomness derived from statistical analysis of historical data and ambiguity such as the psychological aspect of investors and lack of received efficient information. Under such uncertain conditions, the investor needs to consider how to reduce risk, and receive the greatest future profit. Such a finance assets selection problem is generally called a portfolio selection problem, and various studies have been done. Markowitz [20] first proposed Mean-Variance model in the sense of the mathematical programming. Then, it has been central to research

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activity in the real financial field and numerous researchers have contributed to the development of the modern portfolio theory (for instance, Luenberger [19], Steinbach [23]). On the other hand, many researchers have proposed models of portfolio selection problems which extended Markowitz model; Mean-Absolute-Deviation model (Konno [16], Konno, et al. [17]), Semi-Variance model (Bawa and Lindenberg [1]), Safety-First model (Elton and Gruber [5]), and Value at Risk and conditional Value at Risk model (Rockafellar and Uryasev [22]).

In many previous studies in mathematical programming for investment, future returns are assumed to be continuous random variables according to normal distributions. By this assumption, they obtained useful mathematical knowledge and formulas for the portfolio theory. However, from recent experimental studies of investment markets, it is often shown that future returns do not occur according to normal distribution, but fat or heavy-tail distribution source. Then, in the case that investors predict future returns, since they need to consider much information derived from investment markets and some predictions of future returns based on subjectivity of veteran investors simultaneously, they usually assume not only one scenario but also several possibility scenarios of future returns. Therefore, in order to deal with these situations, we need to consider portfolio selection problems with more general random distribution with the heavy-tail.

Furthermore, we need to consider flexibly and ambiguously defined statistical distributions for the following cases that occur in practice: (1) financial information is incomplete, (2) expert investors are not estimating from historical data, (3) the need to mathematically deal with several marginal distributions considering higher or lower future returns simultaneously. In this paper, we propose a more extensional portfolio selection models including fuzzy factors. Until now, there is a body of research under various uncertainty conditions with respect to portfolio selection problems (Bilbao-Terol and Perez-Gladish [2], Carlsson et al. [3], Guo and Tanaka [6], Huang [10, 11], Inuiguchi et al. [12, 13], Katagiri et al. [14, 15], Tanaka et al. [24, 25], Watada [26]). We also proposed some portfolio models with both randomness and fuzziness [7, 8, 9]. However, there are few models considering both normal mixture distribution and ambiguity, simultaneously. Furthermore, there are no studies which are analytically extended and solved these types of portfolio selection problems. In this chapter, we propose more extensional portfolio selection models including the general random distribution with fuzzy factors and develop the efficient solution method.

On the other hand, in the sense of mathematical programming, these portfolio selection models with randomness and fuzziness are formulated as stochastic and fuzzy programming problems. Then, in order to solve them analytically, we need to use the stochastic and fuzzy optimization approaches. The stochastic optimization approach has been treated as a basic solution tool for portfolio selection problems. Recently, as well as the stochastic optimization approach, the fuzzy optimization approach has been used as one of useful tools in financial and investment fields because these approaches are dealt with the investor's subjectivity and the investment case under ineffective and linguistic received information. Considering recent complex investment markets involving investor's speculation and the mixture of reliable on unreliable information, it is obvious that fuzzy optimization approaches play an important role in the investment research field. Various types of fuzzy optimization models have been proposed; fuzzy max ranking method, possibility and necessity programming, etc.. However, there are not many studies to compare these fuzzy optimization models for portfolio selection problems under each case such as favorable, poor, or erratic ups and downs economic conditions, investor's subjectivities; optimistic, pessimistic or neutral. Furthermore, there are also few studies that analyze the suitability of various approaches for the differing market conditions (for example, favorable conditions, unfavorable conditions, erratic fluctuations) by comparing fuzzy, stochastic and fuzzy-stochastic optimization models for portfolio selection problems.

Thus, in order to represent some uncertain social conditions and compare various models for portfolio selection problems, it is important to construct the analytical solution method using the fuzzy and stochastic optimization method as well as to develop a versatile model for portfolio selection problems. Therefore, in this chapter, we consider some models based on the fuzzy optimization models for our proposed portfolio models including the general random distribution with fuzzy factors.

This paper is organized as follows. In Section 2, we introduce notations of parameters in this paper and introduce the basic Mean-Variance model proposed by Markowitz. Then, in Section 3, we formulate the proposed portfolio selection problems minimizing the total variance and maximizing the total future return with normal mixture distributions, respectively. Furthermore, we introduce the uncertainty sets for mean values, weights and probabilities as fuzzy numbers. With respect to several portfolio selection problems including randomness and fuzziness, we construct the solution method. In Section 4, in order to compare our proposed models with standard models, we provide a numerical example. Finally, in Section 5, we conclude this paper and discuss future research problems.

2 Basic Mean-Variance Model

Notations of parameters in this paper are as follows:

$\boldsymbol{r}_i$: random column vector ith scenario

$\boldsymbol{m}_i$: mean value column vector of random variable $\boldsymbol{r}_i$

$\mathbf{V}_i$: variance-covariance matrix of random variable $\boldsymbol{r}_i$

r_G : target value of the total future return

σ_G : target value of the total variance

b_j : Upper limited value of purchasing rate

$\boldsymbol{x}$: Purchasing volume (Decision variable)

In this chapter, we mainly deal with the standard Markowitz model for portfolio selection problem involving normal mixture distributions with respect to future returns. First, we introduce the following Markowitz model minimizing the total variance:

$$\begin{array}{ll} \text{Minimize} & \boldsymbol{x}^t \mathbf{V} \boldsymbol{x} \\ \text{subject to} & \boldsymbol{m}^t \boldsymbol{x} \geq r_G \\ & \boldsymbol{x} \in X \triangleq \left\{ \boldsymbol{x} \left| \sum_{j=1}^{n} x_j = 1,\ 0 \leq x_j \leq b_j,\ \left(j = 1, 2, ..., n \right) \right. \right\} \end{array} \tag{1}$$

This model has been the centre of investment fields until now, and so there are many studies by academic and practical researchers.

In the case that we obtain the strict value of parameters $\boldsymbol{m}$ and $\mathbf{V}$, problem (1) is equivalent to a quadratic programming problem in the mathematical programming and we find an optimal portfolio using standard convex programming approaches. Furthermore, while problem (1) considers minimizing the total variance, the case maximizing the total future return is formulated as the following form:

$$\begin{array}{ll} \text{Maximize} & \boldsymbol{m}^t \boldsymbol{x} \\ \text{subject to} & \boldsymbol{x}^t \mathbf{V} \boldsymbol{x} \leq \sigma_G,\ \boldsymbol{x} \in X \end{array} \tag{2}$$

This problem is also a quadratic programming problem and so we obtain an optimal portfolio using the basic convex programming approach.

3 Our Proposed Model with Normal Mixture Distributions

However, since it often happens that future returns occur according to the normal distribution in practical investment fields, we need to consider that future returns occur according to heavier tailed distributions than normal. Therefore, in this paper, we consider that each random variable r_{ij} occurs according to the following normal mixture distribution with a heavier tail than general normal distributions based on the study [21]:

$$\boldsymbol{r} \sim \sum_{i=1}^{m} w_i N\left(\boldsymbol{m}_i, \mathbf{V}_i\right) \tag{3}$$

where each parameter w_i is the non-negative scalar random variable. If parameter vector $\mathbf{W}$ of scalar random variable w_i is fixed, random variable vector $\boldsymbol{r}$ occurs according to the following normal distribution:

$$\begin{aligned} &\boldsymbol{r}\left|\mathbf{W}=\overline{\boldsymbol{w}} \sim N\left(\sum_{i=1}^{m} \overline{w}_i \overline{\boldsymbol{r}}_i, \sum_{i=1}^{m} \overline{w}_i \mathbf{V}_i\right)\right. \\ &\overline{\boldsymbol{w}}=\left(\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_m\right), \ \overline{w}_i : \text{fixed value} \end{aligned} \tag{4}$$

where we assume that random variable vector $\mathbf{W}$ is independent on each matrix $\mathbf{V}_i$. In case (4), since random variables r_{ij} are basic normal distributions, we analytically obtain an optimal portfolio of problems (1) and (2). However, w_i is also a random variable, and so random variables r_{ij} are not normal random distributions. In this paper, to simplify the discussion, we assume that $\mathbf{W}$ occurs according to the following discrete distribution introducing probabilities p_s:

$$\boldsymbol{w}=\begin{cases} \boldsymbol{w}^{(1)} & \Pr\left\{\boldsymbol{w}=\boldsymbol{w}^{(1)}\right\}=p_1 \\ \boldsymbol{w}^{(2)} & \Pr\left\{\boldsymbol{w}=\boldsymbol{w}^{(2)}\right\}=p_2 \\ \quad\vdots & \qquad\quad\vdots \\ \boldsymbol{w}^{(S)} & \Pr\left\{\boldsymbol{w}=\boldsymbol{w}^{(S)}\right\}=p_S \end{cases}, \ \sum_{s=1}^{S} p_s=1, \ \sum_{i=1}^{m} w_i^{(s)}=1 \tag{5}$$

where S is the total number of discrete random variables and each $w_i^{(s)}$ is the fixed weight. Using this discrete distribution, the expected value and covariance of random variable column vector to future return $\boldsymbol{r}$ are as follows:

$$\begin{aligned} &E(\boldsymbol{r})=E\left(E\left(\boldsymbol{r}\left|W\right.\right)\right)=\boldsymbol{w}_E \boldsymbol{m}_i \\ &\operatorname{cov}(\boldsymbol{r})=E\left(\operatorname{cov}\left(\boldsymbol{r}\left|W\right.\right)\right)+\operatorname{cov}\left(E\left(\boldsymbol{r}\left|W\right.\right)\right)=\boldsymbol{w}_E \mathbf{V}_i \\ &w_i^E=\sum_{s=1}^{S} p_s w_i^{(s)}, \ \boldsymbol{w}_E=\left(w_1^E, w_2^E, \ldots, w_m^E\right) \end{aligned} \tag{6}$$

Therefore, using this expression (6), the expected return and covariance of total profit $\boldsymbol{r}^t \boldsymbol{x}$ are as follows:

$$\begin{aligned} E\left(\boldsymbol{r}^t\boldsymbol{x}\right) &= E\left(E\left(\boldsymbol{r}^t\boldsymbol{x}\middle|W\right)\right) = \left(\boldsymbol{w}_E\boldsymbol{m}_i\right)^t\boldsymbol{x} \\ \operatorname{cov}\left(\boldsymbol{r}^t\boldsymbol{x}\right) &= E\left(\operatorname{cov}\left(\boldsymbol{r}^t\boldsymbol{x}\middle|W\right)\right) + \operatorname{cov}\left(E\left(\boldsymbol{r}^t\boldsymbol{x}\middle|W\right)\right) \\ &= \boldsymbol{x}^t\left(\boldsymbol{w}_E\mathbf{V}_i\right)\boldsymbol{x} \end{aligned} \tag{7}$$

3.1 Formulation of Our Proposed Model

Then, we equivalently transform main problems (1) and (2) into the following problems:

$$\begin{aligned} &\text{Minimize} && \boldsymbol{x}^t\left(\boldsymbol{w}_E\mathbf{V}_i\right)\boldsymbol{x} \\ &\text{subject to} && \left(\boldsymbol{w}_E\boldsymbol{m}_i\right)^t\boldsymbol{x} \geq r_G,\ \boldsymbol{x}\in X \end{aligned} \tag{8}$$

$$\begin{aligned} &\text{Maximize} && \left(\boldsymbol{w}_E\boldsymbol{m}_i\right)^t\boldsymbol{x} \\ &\text{subject to} && \boldsymbol{x}^t\left(\boldsymbol{w}_E\mathbf{V}_i\right)\boldsymbol{x} \leq \sigma_G,\ \boldsymbol{x}\in X \end{aligned} \tag{9}$$

With respect to problems (8) and (9), if random distribution of parameter $\mathbf{W}$ is obtained and each parameter is constant, we may solve these problems analytically using similar methods to problems (1) and (2). However, considering ambiguity conditions such as subjectivity of the decision maker and the lack of received reliable information, it is natural that each parameter such as the mean value and random variable W include fuzziness. Therefore, we consider the following cases where problems (8) and (9) include fuzziness.

3.2 Fuzzy Extension for Mean Values

First, we consider the case where each expected value $\boldsymbol{m}$ includes fuzziness and is assumed to be a fuzzy number. This case is considered that the decision maker is a veteran investor and performs the more aggressive or passive prediction than the statistical analysis derived from historical data.

In this subsection, since the random distribution of parameter $\mathbf{W}$ is obtained and discrete value $w_i^{(s)}$ and its probability p_s are constant, expected value vector $\boldsymbol{w}_i^E$ is also a constant. Then, the membership function of fuzzy numbers $\boldsymbol{m}$ is assumed to be a triangle fuzzy number and introduced by the following function:

$$\mu_{\tilde{r}_{ij}}(\omega)=\begin{cases}\dfrac{\omega-(m_{ij}-\gamma_{ij})}{\gamma_{ij}} & (m_{ij}-\gamma_{ij}\le\omega\le m_{ij})\\ \dfrac{(m_{ij}+\delta_{ij})-\omega}{\delta_{ij}} & (m_{ij}<\omega\le m_{ij}+\delta_{ij})\\ 0 & (otherwise)\end{cases} \tag{10}$$

where γ_{ij} and δ_{ij} are spreads of left and right side, respectively. In this paper, we assume that provided all fuzzy numbers are initially determined by the decision maker. Using these fuzzy numbers, the total future expected return $\tilde{R}_i=\tilde{\boldsymbol{m}}^t\boldsymbol{x}$ is also a fuzzy numbers characterized by the following membership function:

$$\mu_{\tilde{R}}(\omega)=\begin{cases}\dfrac{\omega-R_L}{\sum_{i=1}^{m}w_i^E\sum_{j=1}^{n}\gamma_{ij}x_j} & \left(R_L\le\omega\le\sum_{i=1}^{m}w_i^E\sum_{j=1}^{n}m_{ij}x_j\right)\\ \dfrac{R_U-\omega}{\sum_{i=1}^{m}w_i^E\sum_{j=1}^{n}\delta_{ij}x_j} & \left(\sum_{i=1}^{m}w_i^E\sum_{j=1}^{n}m_{ij}x_j<\omega\le R_U\right)\\ 0 & (\omega<R_L,\ R_U<\omega)\end{cases} \tag{11}$$

$$R_L=\sum_{i=1}^{m}w_i^E\sum_{j=1}^{n}m_{ij}x_j-\sum_{i=1}^{m}w_i^E\sum_{j=1}^{n}\gamma_{ij}x_j,\ R_U=\sum_{i=1}^{m}w_i^E\sum_{j=1}^{n}m_{ij}x_j+\sum_{i=1}^{m}w_i^E\sum_{j=1}^{n}\delta_{ij}x_j$$

Due to these fuzzy numbers, problems (8) and (9) are not well-defined problem in the sense of deterministic mathematical programming. Therefore, in order to solve the main problem analytically, we need to set some criterion for fuzzy variables. In this paper, we consider the case where the decision maker usually has a goal to earn the total profit more than the target value. Furthermore, taking account of the vagueness of human judgment and flexibility for the execution of a plan in many real decision cases, we give a fuzzy goal to the target future return as the fuzzy set characterized by a membership function. In this subsection, we consider the fuzzy goal of probability $\mu_{\tilde{G}}(\omega)$ which is represented by,

$$\mu_{\tilde{G}}(\omega)=\begin{cases}0 & \omega\le f_L\\ \dfrac{\omega-f_L}{f_U-f_L} & f_L\le\omega\le f_U\\ 1 & f_U\le\omega\end{cases} \tag{12}$$

Furthermore, using the concept of possibility measure and considering maximizing both membership functions, we introduce the degree of possibility as follows:

$$\prod_{\tilde{R}}\left(\tilde{G}\right)=\sup_{f}\min\left\{\mu_{\tilde{R}}\left(f\right),\ \mu_{\tilde{G}}\left(f\right)\right\} \tag{13}$$

Therefore, introducing this degree of possibility, we consider the following portfolio selection problems based on problems (8) and (9):

$$\begin{array}{ll}\text{Minimize} & \boldsymbol{x}^{t}\left(\boldsymbol{w}_{E}\mathbf{V}_{i}\right)\boldsymbol{x}\\ \text{subject to} & \prod_{\tilde{R}}\left(\tilde{G}\right)\geq h,\ \boldsymbol{x}\in X\end{array} \tag{14}$$

and

$$\begin{array}{ll}\text{Maximize} & \prod_{\tilde{R}}\left(\tilde{G}\right)\\ \text{subject to} & \boldsymbol{x}^{t}\left(\boldsymbol{w}_{E}\mathbf{V}_{i}\right)\boldsymbol{x}\leq\sigma_{G},\ \boldsymbol{x}\in X\\ & \text{Maximize} \quad h\\ \Leftrightarrow & \text{subject to} \quad \prod_{\tilde{R}}\left(\tilde{G}\right)\geq h,\\ & \qquad \boldsymbol{x}^{t}\left(\boldsymbol{w}_{E}\mathbf{V}_{i}\right)\boldsymbol{x}\leq\sigma_{G},\ \boldsymbol{x}\in X\end{array} \tag{15}$$

In this problem, each constraint $\Pi_{\tilde{P}_i}\left(G_i\right)\geq h$ is transformed into the following inequality:

$$\begin{aligned}&\prod_{\tilde{R}}\left(\tilde{G}\right)\geq h\\ \Leftrightarrow\ & \sup_{f}\min\left\{\mu_{\tilde{R}}\left(f\right),\ \mu_{\tilde{G}}\left(f\right)\right\}\geq h\\ \Leftrightarrow\ & \exists f:\ \mu_{\tilde{R}}\left(f\right)\geq h,\ \mu_{\tilde{G}}\left(f\right)\geq h\\ \Leftrightarrow\ & \sum_{i=1}^{m}w_i^E\sum_{j=1}^{n}m_{ij}x_j+\left(1-h\right)\delta_{ij}x_j\geq\left(1-h\right)f_L+f_U\end{aligned} \tag{16}$$

Therefore, problems (14) and (15) are equivalently transformed into the following problems:

$$\begin{array}{ll}\text{Minimize} & \boldsymbol{x}^{t}\left(\boldsymbol{w}_{E}\mathbf{V}_{i}\right)\boldsymbol{x}\\ \text{subject to} & \sum_{i=1}^{m}w_i^E\sum_{j=1}^{n}m_{ij}x_j+\left(1-h\right)\delta_{ij}x_j\geq\left(1-h\right)f_L+hf_U,\\ & \boldsymbol{x}\in X\end{array} \tag{17}$$

and

$$\begin{array}{ll}\text{Maximize} & \dfrac{\sum_{i=1}^{m}w_i^E\sum_{j=1}^{n}\left(m_{ij}x_j+\delta_{ij}x_j\right)-f_L}{\sum_{i=1}^{m}w_i^E\sum_{j=1}^{n}\delta_{ij}x_j+f_U-f_L}\\ \text{subject to} & \boldsymbol{x}^{t}\left(\boldsymbol{w}_{E}\mathbf{V}_{i}\right)\boldsymbol{x}\leq\sigma_{G},\ \boldsymbol{x}\in X\end{array} \tag{18}$$

Problem (17) is equivalent to problem (8), and so we analytically solve problem (17) using the same solution method to problem (8). Then, problem (18) is a standard fractional programming problem and the numerator and denominator of objective function are linear functions. Therefore, by performing the equivalent transformation using the fractional programming approach, problem (18) is a similar problem to problem (9), and so we also solve problem (18) analytically using the same solution method to problem (9).

3.3 Fuzzy Extension Model for Each Weight or Probability

In the previous subsection, we considered the case where each expected value is assumed to be a fuzzy number. However, in real-world decision cases, it is difficult to set not only the expected value but also the possible value of random variable w_i or the occurrence probability p_i strictly due to the ambiguity of decision maker's subjectivity. Therefore, in this subsection, we consider the case where random variable W also includes flexibility and is assumed to be a fuzzy number.

First, we assume that the possible value w_i includes the ambiguity and represents a fuzzy number. Then, the membership function of each value w_i is introduced by the following functions:

$$\boldsymbol{w}=\begin{cases}\tilde{\boldsymbol{w}}^{(1)} & \Pr\left\{\boldsymbol{w}=\tilde{\boldsymbol{w}}^{(1)}\right\}=p_1\\ \tilde{\boldsymbol{w}}^{(2)} & \Pr\left\{\boldsymbol{w}=\tilde{\boldsymbol{w}}^{(2)}\right\}=p_2\\ \vdots & \vdots\\ \tilde{\boldsymbol{w}}^{(S)} & \Pr\left\{\boldsymbol{w}=\tilde{\boldsymbol{w}}^{(S)}\right\}=p_S\end{cases},\quad \tilde{\boldsymbol{w}}^{(s)}=\left(\tilde{w}_1^{(s)},\tilde{w}_2^{(s)},\dots,\tilde{w}_m^{(s)}\right)$$

$$\mu_{\tilde{w}_i^{(s)}}(\omega)=\begin{cases}\dfrac{w_i^{(s)}-\omega}{\alpha_{si}} & \left(w_i^{(s)}-\alpha_{si}\le\omega\le w_i^{(s)}\right)\\ \dfrac{\omega-w_i^{(s)}}{\beta_{si}} & \left(w_i^{(s)}<\omega\le w_i^{(s)}+\beta_{si}\right)\\ 0 & \left(\omega<w_i^{(s)}-\alpha_{si}\ ,w_i^{(s)}+\beta_{si}<\omega\right)\end{cases}\tag{19}$$

Using these membership functions and the extension principle of fuzzy theory, expected value of $\tilde{w}_i^E$ is obtained as the following form:

$$\mu_{\tilde{w}_i^E}(\omega)=\begin{cases}\dfrac{\omega-E_L}{\sum_{s=1}^{S}p_s\alpha_{si}} & \left(E_L\le\omega\le\sum_{s=1}^{S}p_s w_i^{(s)}\right)\\ \dfrac{E_U-\omega}{\sum_{s=1}^{S}p_s\beta_{si}} & \left(\sum_{s=1}^{S}p_s w_i^{(s)}<\omega\le E_U\right)\\ 0 & \left(\omega<E_L\ ,E_U<\omega\right)\end{cases} \tag{20}$$

$$E_L\left(w_i^E\right)=\sum_{s=1}^{S}p_s w_i^{(s)}-\sum_{s=1}^{S}p_s\alpha_{si},$$
$$E_U\left(w_i^E\right)=\sum_{i=1}^{m}p_i\overline{w}_i+\sum_{s=1}^{S}p_s\beta_{si}$$

Therefore, we obtain the following membership function of the total variance:

$$\mu_{\tilde{V}}(\omega)=\begin{cases}\dfrac{\omega-V_L}{\boldsymbol{x}^t\left(\sum_{i=1}^{m}\left(\sum_{s=1}^{S}p_s\alpha_{si}\right)\mathbf{V}_i\right)\boldsymbol{x}}\\ \dfrac{V_U-\omega}{\boldsymbol{x}^t\left(\sum_{i=1}^{m}\left(\sum_{s=1}^{S}p_s\beta_{si}\right)\mathbf{V}_i\right)\boldsymbol{x}}\\ 0\end{cases} \tag{21}$$

$$V_L=\boldsymbol{x}^t\left(\sum_{i=1}^{m}\left(\sum_{s=1}^{S}p_s w_i^{(s)}-\sum_{s=1}^{S}p_s\alpha_{si}\right)\mathbf{V}_i\right)\boldsymbol{x},\ V_U=\boldsymbol{x}^t\left(\sum_{i=1}^{m}\left(\sum_{s=1}^{S}p_s w_i^{(s)}-\sum_{s=1}^{S}p_s\beta_{si}\right)\mathbf{V}_i\right)\boldsymbol{x}$$

In this case, we assume that each weight $w_i^{(s)}$ is represented as the h-cut set of the fuzzy number $\tilde{w}_i^{(s)}=\left[\underline{w}_i^{(s)}(h),\overline{w}_i^{(s)}(h)\right]$. Furthermore, in a way similar to subsection 3.2, using the concept of possibility measure to the total variance, we introduce the degree of possibility as follows:

$$\prod\nolimits_{\tilde{V}}\left(\tilde{G}\right)=\sup_{\sigma}\min\left\{\mu_{\tilde{V}}(\sigma),\ \mu_{\tilde{G}}(\sigma)\right\}$$
$$\mu_{\tilde{G}}(\omega)=\begin{cases}1 & \omega\le\sigma_L\\ \dfrac{\sigma_U-\omega}{\sigma_U-\sigma_L} & \sigma_L\le\omega\le\sigma_U\\ 0 & \sigma_U\le\omega\end{cases} \tag{22}$$

Therefore, from this degree of possibility, problems (8) and (9) are equivalently transformed into the following problems performing the method similar to (18):

$$\begin{aligned} &\text{Maximize} \quad \prod\nolimits_{\tilde{V}}\left(\tilde{G}\right) \\ &\text{subject to} \quad \sum_{i=1}^{m} w_i^E \sum_{j=1}^{n} m_{ij}x_j + (1-h)\delta_{ij}x_j \geq (1-h) f_L + hf_U, \\ &\qquad\qquad \boldsymbol{x} \in X \\ &\qquad \text{Minimize} \quad \frac{\boldsymbol{x}^t\left(\sum_{i=1}^{m}\left(\sum_{s=1}^{S} p_s w_i^{(s)} - \sum_{s=1}^{S} p_s \alpha_{si}\right)\mathbf{V}_i\right)\boldsymbol{x} - \sigma_U}{\boldsymbol{x}^t\left(\sum_{i=1}^{m}\left(\sum_{s=1}^{S} p_s \alpha_{si}\right)\mathbf{V}_i\right)\boldsymbol{x} + (\sigma_U - \sigma_L)} \\ \Leftrightarrow \quad &\text{subject to} \quad \sum_{i=1}^{m} w_i^E \sum_{j=1}^{n} m_{ij}x_j + (1-h)\delta_{ij}x_j \geq (1-h) f_L + hf_U, \\ &\qquad\qquad \boldsymbol{x} \in X,\ w_i^{(s)} \in \left[\underline{w}_i^{(s)}(h), \overline{w}_i^{(s)}(h)\right],\ \sum_{i=1}^{m} w_i^{(s)} = 1 \end{aligned} \tag{23}$$

and

$$\begin{aligned} &\text{Maximize} \quad \frac{\sum_{i=1}^{m} w_i^E \sum_{j=1}^{n}\left(m_{ij}x_j + \delta_{ij}x_j\right) - f_L}{\sum_{i=1}^{m} w_i^E \sum_{j=1}^{n} \delta_{ij}x_j + f_U - f_L} \\ &\text{subject to} \quad \prod\nolimits_{\tilde{V}}\left(\tilde{G}\right) \geq \overline{h},\ \boldsymbol{x} \in X \\ &\qquad \text{Maximize} \quad \frac{\sum_{i=1}^{m} w_i^E \sum_{j=1}^{n}\left(m_{ij}x_j + \delta_{ij}x_j\right) - f_L}{\sum_{i=1}^{m} w_i^E \sum_{j=1}^{n} \delta_{ij}x_j + f_U - f_L} \\ \Leftrightarrow \quad &\text{subject to} \quad \boldsymbol{x}^t\left(\sum_{i=1}^{m}\left(\sum_{s=1}^{S} p_s w_i^{(s)} - (1-h)\sum_{s=1}^{S} p_s \alpha_{si}\right)\mathbf{V}_i\right)\boldsymbol{x} \leq \overline{h}\sigma_L + \left(1-\overline{h}\right)\sigma_U, \\ &\qquad\qquad \boldsymbol{x} \in X,\ w_i^{(s)} \in \left[\underline{w}_i^{(s)}(h), \overline{w}_i^{(s)}(h)\right],\ \sum_{i=1}^{m} w_i^{(s)} = 1 \end{aligned} \tag{24}$$

Problem (23) is similar to problem (17), but it is a semi-infinite programming problem since it includes interval values $w_i^{(s)} \in \left[\underline{w}_i^{(s)}(h), \overline{w}_i^{(s)}(h)\right]$ and constraint $\sum_{i=1}^{m} w_i^{(s)} = 1$. Therefore, it is hard to solve it as the original form using the standard approaches. However, since these problems are equivalently transformed into the linear programming problems using the solution approach proposed by Lai and Wu [18] and Hasuike [7], we analytically and efficiently solve it. Then, with respect to problem (23), the numerator and denominator of objective function are convex functions. Therefore, we analytically solve it using the solution method proposed by Dinkelbach [4] and the semi-infinite solution approaches [7] and [18].

On the other hand, we also consider the case where each occurrence probability includes an ambiguity and is assumed to be the following membership function in a way similar to (19):

$$\boldsymbol{w}=\begin{cases}\boldsymbol{w}^{(1)} & \Pr\left\{\boldsymbol{w}=\boldsymbol{w}^{(1)}\right\}=\tilde{p}_1\\ \boldsymbol{w}^{(2)} & \Pr\left\{\boldsymbol{w}=\boldsymbol{w}^{(2)}\right\}=\tilde{p}_2\\ \vdots & \vdots\\ \boldsymbol{w}^{(S)} & \Pr\left\{\boldsymbol{w}=\boldsymbol{w}^{(S)}\right\}=\tilde{p}_S\end{cases}$$
$$\mu_{\tilde{p}_s}(\omega)=\begin{cases}\dfrac{\omega-(\overline{p}_s-\xi_s)}{\xi_s} & (\overline{p}_s-\xi_s\le\omega\le\overline{p}_s)\\ \dfrac{(\overline{p}_s+\zeta_s)-\omega}{\zeta_s} & (\overline{p}_s<\omega\le\overline{p}_s+\zeta_s)\\ 0 & (\omega<\overline{p}_s-\xi_s\ ,\overline{p}_s+\zeta_s<\omega)\end{cases} \tag{25}$$

In this case, we assume that each possible occurrence probability p_i includes the h-cut set of the fuzzy number $\tilde{p}_i$. Then, we also apply similar transformations and degree of possibility to this case, and we equivalently transform problems (8) and (9) into the following problems:

$$\begin{aligned}&\text{Minimize} \quad \boldsymbol{x}^t\left(\boldsymbol{w}_E\mathbf{V}_i\right)\boldsymbol{x}\\ &\text{subject to} \quad \sum_{j=1}^{n}m_jx_j+(1-h)\sum_{j=1}^{n}\delta_jx_j\ge(1-h)f_L+hf_U,\ \boldsymbol{x}\in X,\\ &\qquad p_s\in\left\{\overline{p}_s-\left(1-\overline{h}\right)\xi_s\le p_s\le\overline{p}_s+\left(1-\overline{h}\right)\zeta_s\right\},\ \sum_{s=1}^{S}p_s=1\end{aligned} \tag{26}$$

and

$$\begin{aligned}&\text{Maximize} \quad \frac{\sum_{j=1}^{n}m_jx_j+\sum_{j=1}^{n}\delta_jx_j-f_L}{\sum_{j=1}^{n}\delta_jx_j+f_U-f_L}\\ &\text{subject to} \quad \boldsymbol{x}^t\left(\boldsymbol{w}_E\mathbf{V}_i\right)\boldsymbol{x}\le\overline{h}\sigma_L+\left(1-\overline{h}\right)\sigma_U,\ \boldsymbol{x}\in X,\\ &\qquad p_s\in\left\{\overline{p}_s-\left(1-\overline{h}\right)\xi_s\le p_s\le\overline{p}_s+\left(1-\overline{h}\right)\zeta_s\right\},\ \sum_{s=1}^{S}p_s=1\end{aligned} \tag{27}$$

Consequently, problems (26) and (27) are similar to problems (23) and (24), respectively. Therefore, we obtain the optimal portfolio with respect to the various types of fuzzy portfolio selection problems with normal mixture random distributions. Furthermore, with respect to problems (23), (24), (26) and (27), these problems consider several weights and possible occurrence probability as their interval values, and so these problems are more versatile and robust portfolio models.

4 Numerical Example

In order to compare our proposed models and standard Markowitz models and to show the usefulness of our proposed model more clearly, we provide a brief numerical example based on historical data derived from Tokyo Stock Exchange. Let us consider four securities shown in Table 1, whose mean values and variances are based on historical data in the decade between 1995 and 2004.

Table 1 Sample data from Tokyo Exchange Market

	x_1	x_2	x_3	x_4
Mean	0.046	0.043	0.087	0.090
Variance	0.0836	0.0638	0.1507	0.1010

Subsequently, we introduce the asset allocation rate x_j to each security, and its upper value is assumed to be 0.4. Then, we consider the case where the fixed target profit and total variance for Markowitz model are 0.07 and 0.025, respectively. In this case, the optimal portfolio of Markowitz model minimizing the total variance is obtained as follows:

Table 2 Optimal portfolio of Markowitz model

	x_1	x_2	x_3	x_4
Problem (1)	0.192	0.231	0.226	0.351

Furthermore, we consider the more practical case where investors predict not only single scenario but also multi-scenario with respect to each future return. In this numerical example, based on the historical data of Tokyo Exchange Market, we consider three scenarios as the following Table 3.

Table 3 Mean values and variances for three scenarios with respect to future returns

		x_1	x_2	x_3	x_4
Scenario 1	Mean	0.046	0.043	0.087	0.090
	Variance	0.0836	0.0638	0.1507	0.1010
Scenario 2	Mean	0.068	0.114	0.130	0.236
	Variance	0.0936	0.0488	0.1007	0.1510
Scenario 3	Mean	0.023	0.015	0.012	0.032
	Variance	0.0736	0.0788	0.1807	0.0910

This example means that scenarios 1, 2 and 3 are neutral, positive and passive for the investor's subjectivity. Then, each weight of the scenario is provided as the following three forms:

Table 4 Weights and probabilities for scenarios

	w_1	w_2	w_3	*Prob.*
Case 1	0.5	0.25	0.25	1/3
Case 2	0.5	0.4	0.1	1/3
Case 3	0.5	0.1	0.4	1/3

Using Table 3 and each weight, we solve Case 1 of the weighted portfolio model and obtain the following optimal portfolio:

Table 5 Optimal portfolio for Case 1 of each weighted portfolio model

	x_1	x_2	x_3	x_4
Min. Var.	0.270	0.367	0.162	0.201

On the other hand, in order to consider our proposed models in Section 3, we set the fuzzy numbers and fuzzy goals. First, we assume mean values in all scenarios to be the following symmetric triangle fuzzy numbers:

Table 6 Triangle fuzzy numbers of mean values in all scenarios

	x_1	x_2	x_3	x_4
Scenario 1	<0.046,0.02>	<0.043,0.01>	<0.087,0.03>	<0.090,0.05>
Scenario 2	<0.068,0.03>	<0.114,0.04>	<0.130,0.08>	<0.236,0.03>
Scenario 3	<0.023,0.01>	<0.015,0.005>	<0.012,0.005>	<0.032,0.02>

Then, fuzzy numbers of weights and probabilities are assumed to be the following symmetric trapezoidal fuzzy numbers:

Table 7 Symmetric trapezoidal fuzzy numbers of weights and probabilities

	w_1	w_2	w_3	*Prob.*
Case 1	(0.42,0.58,0.1)	(0.21,0.29,0.05)	(0.21,0.29,0.05)	(0.3,0.35,0.1)
Case 2	(0.42,0.58,0.1)	(0.32,0.48,0.1)	(0.02,0.18,0.1)	(0.3,0.35.0.1)
Case 3	(0.42,0.58,0.1)	(0.02,0.18,0.1)	(0.32,0.48,0.1)	(0.3,0.35.0.1)

Furthermore, fuzzy goals for the total return and variance are provided as the following forms:

$$\mu_{\tilde{G}}(f)=\begin{cases} 0 \\ \dfrac{f-0.08}{0.06} \\ 1 \end{cases},\ \mu_{\tilde{G}}(\omega)=\begin{cases} 1 \\ \dfrac{0.025-\sigma}{0.005} \\ 0 \end{cases},$$

Using these parameters and fuzzy goals, we solve our proposed models (17), (23) and (26) minimizing the total variance, and obtain each optimal portfolio:

Table 8 Optimal portfolios for our proposed models

	x_1	x_2	x_3	x_4
Fuzzy mean (17)	0.270	0.367	0.162	0.201
Fuzzy weight (23): Case 1	0.064	0.394	0.188	0.354
Fuzzy weight (23): Case 2	0.041	0.400	0.227	0.332
Fuzzy weight (23): Case 3	0.039	0.400	0.187	0.374
Fuzzy probability (26)	0.051	0.400	0.216	0.333

Subsequently, we consider the case where an investor purchases securities at the end of 2004 according to each portfolio shown in Tables 2 and 8. Then, the total return of three models at term ends of 2005 and 2007 become the following values shown in Table 3.9, respectively.

Table 9 Total profit for the basic and our proposed models

	Term end of 2005	Term end of 2007
Markowitz model (1)	0.2042	0.2094
Fuzzy mean (17)	0.2329	0.3008
Fuzzy weight (23): Case 1	0.2489	0.2114
Fuzzy weight (23): Case 2	0.2465	0.2108
Fuzzy weight (23): Case 3	0.2519	0.2007
Fuzzy probability (26)	0.2473	0.2135

From the result in Table 9, we find that our proposed models earn more total profit than the basic Markowitz model. Particularly, we find that fuzzy weight and fuzzy probability models earns more profit in a semi-long term investment, and the fuzzy mean models earns more profit in a long-term investment. Surely, this numerical example is based on a little data, and so this conclusion may not hold true for the stock market in all conditions. However, this results show that it is possible to earn more profits by performing suitable investments according to subjectivity. Therefore, investors can deal with our model usefully according to their investment stiles.

5 Conclusion

In this paper, we have considered several portfolio selection models with normal mixture random distributions involving ambiguous factors extending Mean-Variance model. Since our proposed models are not well-defined problems due to randomness and fuzziness, we have set some criterion such as mean value and variance to stochastic aspect and possibility measure to fuzzy aspect. Then, by performing the equivalent transformations, we have constructed the efficient solution methods based on the standard mean-variance approaches. Therefore, we have developed more versatile portfolio models with randomness and fuzziness than previous standard portfolio models, and we may obtain more beneficial knowledge for the investment theory.

As the future studies, we are going to consider the case where random distributions are more general patterns not only based on the normal distribution but also ellipsoidal distributions including almost all statistical distributions. Then, we also need to consider the case that the optimal solutions are restricted to be integers.

References

[1] Bawa, V.S., Lindenberg, E.B.: Capital market equilibrium in a mean-lower partial moment framework. Journal of Financial Economics 5, 189–200 (1977)
[2] Bilbao-Terol, A., Perez-Gladish, B., Arenas-Parra, M., Rodriguez-Uria, M.V.: Fuzzy compromise programming for portfolio selection. Applied Mathematics and Computation 173, 251–264 (2006)
[3] Carlsson, C., Fuller, R., Majlender, P.: A possibilistic approach to selecting portfolios with highest utility score. Fuzzy Sets and Systems 131, 12–21 (2002)
[4] Dinkelbach, W.: On nonlinear fractional programming. Management Science 13, 492–498 (1967)
[5] Elton, E.J., Gruber, M.J.: Modern Portfolio Theory and Investment Analysis. Wiley, New York (1995)
[6] Guo, P., Tanaka, H.: Possibility data analysis and its application to portfolio selection problems. Fuzzy Economic Rev. 3, 3–23 (1998)
[7] Hasuike, T., Ishii, H.: Probability maximization model of portfolio selection problem under ambiguity. Central European Journal of Operational Research (to appear)

[8] Hasuike, T., Katagiri, H., Ishii, H.: Portfolio selection problems with random fuzzy variable returns. In: Proceedings of IEEE International Conference on Fuzzy Systems 2007, pp. 416–421 (2007)
[9] Hasuike, T., Katagiri, H., Ishii, H.: Probability Maximization Model of 0-1 Knapsack Problem with Random Fuzzy Variables. In: Proceedings of 2008 IEEE World Congress in Computing Intelligence (WCCI 2008), IEEE International Conference on Fuzzy Systems 2008, pp. 548–554 (2008)
[10] Huang, X.: Fuzzy chance-constrained portfolio selection. Applied Mathematics and Computation 177, 500–507 (2006)
[11] Hung, X.: Two new models for portfolio selection with stochastic returns taking fuzzy information. European Journal of Operational Research 180, 396–405 (2007)
[12] Inuiguchi, M., Ramik, J.: Possibilisitc linear programming: A brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem. Fuzzy Sets and Systems 111, 3–28 (2000)
[13] Inuiguchi, M., Tanino, T.: Portfolio selection under independent possibilistic information. Fuzzy Sets and Systems 115, 83–92 (2000)
[14] Katagiri, H., Ishii, H., Sakawa, M.: On fuzzy random linear knapsack problems. Central European Journal of Operations Research 12(1), 59–70 (2004)
[15] Katagiri, H., Sakawa, M., Ishii, H.: A study on fuzzy random portfolio selection problems using possibility and necessity measures. Scientiae Mathematicae Japonicae 65(2), 361–369 (2005)
[16] Konno, H.: Piecewise linear risk functions and portfolio optimization. Journal of Operations Research Society of Japan 33, 139–156 (1990)
[17] Konno, H., Shirakawa, H., Yamazaki, H.: A mean-absolute deviation-skewness portfolio optimization model. Annals of Operations Research 45, 205–220 (1993)
[18] Lai, H.C., Wu, S.Y.: On linear semi-infinite programming problems: an algorithm. Numerical Functional Analysis and Optimization 13, 287–304 (1992)
[19] Luenberger, D.G.: Investment Science. Oxford Univ. Press, Oxford (1997)
[20] Markowitz, H.: Portfolio Selection. Wiley, New York (1959)
[21] McNeil, A.J., Frey, R., Embrechts, P.: Quantitative Risk Management: Concepts, Techniques & Tools. Princeton University Press, Princeton (2005)
[22] Rockafellar, R.T., Uryasev, S.: Optimization of conditional value-at-risk. Journal of Risk 2(3), 1–21 (2000)
[23] Steinbach, M.C.: Markowitz revisited: Mean-variance models in financial portfolio analysis. SIAM Review 43(1), 31–85 (2001)
[24] Tanaka, H., Guo, P.: Portfolio selection based on upper and lower exponential possibility distributions. European Journal of Operational Researches 114, 115–126 (1999)
[25] Tanaka, H., Guo, P., Turksen, I.B.: Portfolio selection based on fuzzy probabilities and possibility distributions. Fuzzy Sets and Systems 111, 387–397 (2000)
[26] Watada, J.: Fuzzy portfolio selection and its applications to decision making. Tatra Mountains Math. Pub. 13, 219–248 (1997)