Fuzzy Dynamic Programming Problem for Extremal Fuzzy Dynamic System

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Abstract. This work deals with the problems of the Expremal Fuzzy Continuous Dynamic System (EFCDS) optimization problems and briefly discuss the results developed by G. Sirbiladze [31]–[38]. The basic properties of extended extremal fuzzy measure are considered and several variants of their representation are given. In considering extremal fuzzy measures, several transformation theorems are represented for extended lower and upper Sugeno integrals. Values of extended extremal conditional fuzzy measures are defined as a levels of an expert knowledge reflections of EFCDS states in the fuzzy time intervals. The notions of extremal fuzzy time moments and intervals are introduced and their monotone algebraic structures that form the most important part of the fuzzy instrument of modeling extremal fuzzy dynamic systems are discussed. New approaches in modeling of EFCDS are developed. Applying the results of [31] and [32], fuzzy processes with possibilistic uncertainty, the source of which is extremal fuzzy time intervals, are constructed. The dynamics of EFCDS's is described. Questions of the ergodicity of EFCDS's are considered. Fuzzy-integral representations of controllable extremal fuzzy processes are given. Sufficient and necessary conditions are presented for the existence of an extremal fuzzy optimal control processes, for which we use R. Bellman's optimality principle and take into consideration the gain-loss fuzzy process. A separate consideration is given to the case where an extremal fuzzy control process acting on the EFCDS does not depend on an EFCDS state. Applying Bellman's optimality principle and assuming that the gain-loss process exists for the EFCDS, a variant of the fuzzy integral representation of an optimal control is given for the EFCDS. This variant employs the instrument of extended extremal fuzzy composition measures constructed in [32]. An example of constructing of the EFCDS optimal control is presented.

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1 Introduction

In recent years, both the dynamics of fuzzy system and the modeling issue received an increased attention. Dynamics is an obvious problem in control; moreover, its interest goes far beyond control applications. Applications of the dynamics of fuzzy systems and of the modeling of dynamic systems by fuzzy systems range from physics to biology to economics to pattern recognition and to time series prediction.

Evidence exists that fuzzy models can explain cooperative processes, such as in biology, chemistry, material sciences, or in economy. Relationships between dynamics of fuzzy systems and the performance of decision support systems were found, and chaotic processes in various classes of fuzzy systems were shown as a powerful tool in analyzing complex, weakly structurable systems, as anomal and extremal processes.

To make the decision-making effective in the framework of computer systems supporting this process, we must solve analytic problems of optimization, state evaluation, model identification, complex dynamic system control, optimal control, filtering and so on.

It is well recognized that optimization and other decision support technologies have been playing an important role in improving almost every aspect of human society. Intensive study over the past several years has resulted in significant progress in both the theory and applications of optimization and decision science.

Optimization and decision-making problems are traditionally handled by either the deterministic or probabilistic approaches. The former provides an approximate solution, completely ignoring uncertainty, while the latter assumes that any uncertainty can be represented as a probability distribution. Of course, both approaches only partially capture reality uncertainty (such as stock price, commodity, cost, natural resource availability and so on) that indeed exist but not in the form of known probability distributions.

In the Preface of the Journal of Fuzzy Optimization and Decision Making (vol. I, 2002, pp. 11–12) Professor L. A. Zadeh had said: “My 1970 paper with R.E. Bellman, “Decision-Making in a Fuzzy Environment” was intended to suggest a framework based on the theory of fuzzy sets for dealing with imprecision and partial truth in optimization and decision analysis. In the intervening years, a voluminous literature on applications of fuzzy logic to decision analysis has come into existence.”

In alternative classical approaches to modeling and when working with complex systems the main accent is placed on the assumption of fuzziness. As the complexity of systems increases, our ability to define exactly their behaviour drops to a certain level, below which such characteristics of information as exactness and uncertainty become mutually excluding. In such situations an exact quantitative analysis of real complex systems is apt to be not quite plausible. Hence, a conclusion comes to mind that problems of this kind should be solved by means of analytic-fundamental methods of fuzzy mathematics, while the system approach to constructing models of complex systems with fuzzy uncertainty guarantees the creation of computer-aided systems forming the instrumental basis of intelligent technology solutions of expert-analytic problems. It is obvious that the source of fuzzy-statistical samples is the population of fuzzy characteristics of expert knowledge. Fuzziness arises from observations of time moments as well as from other expert measurements.

Problems of making an optimal solution for systems with fuzzy uncertainty are difficult because it frequently happens that the controllable object possesses conflicting properties which might include:

1) imperfection of a control process due to information uncertainty;
2) unreliable elements of a control system;
3) nonuniqueness and the applicability of many criteria encountered in a control process;
4) restriction of possibilities (resources) of a control system;
5) loss of the ability of a control system to solve arising control problems.

Fuzzy programming problems have been discussed widely in literature ([1]–[3], [5], [7], [10], [11], [23], [25], [26], [35], [37], [39], [45]–[47] and so on) and applied in such various disciplines as operations research, economic management, business administration, engineering and so on. B. Liu [25] presents a brief review on fuzzy programming models, and classifies them into three broad classes: expected value models, chance-constrained programming and chance-dependent programming.

Our further study belongs to the first class, where we use the instrument of fuzzy measures and integrals ([8], [14]–[16], [31]–[33], [38], [40]–[42], [44] and so on) or, speaking more exactly, extremal fuzzy measures and Sugeno's type integrals along with extremal fuzzy expected values.

Therefore in the paper the new approach to the study of weakly structurable fuzzy dynamic systems optimization is presented (Extremal Fuzzy Continuous Dynamic System). This approach is based on the six papers published in the Int. Journal of General Systems (by G. Sirbiladze, "Modeling of Extremal Fuzzy Dynamic Systems". Parts I-VI: 34,2, 2005, 107-138; 139-167; 169-198; 35, 4, 2006, 435-459; 35, 5, 2006, 529-554; 36,1 2007, 19-58). Different from other approaches where the source of fuzzy uncertainty in dynamic systems is expert, this approach considers time as long as an expert to be the source of fuzzy uncertainty. This notably widens the area of studied problems. All these is connected to the incomplete, imprecise, anomal and extremal processes in nature and society, where connections between the system's objects are of subjective (expert) nature, which is caused by lack of objective information about the evolution of studied system, for example in 1) economics and business of developing countries, conflictology, sociology, medical diagnosis etc; 2) management of evacuation processes in catastrophe areas, estimation of disease spreading in epidemical regions; 3) research of complex systems of applied physics, etc. One of our purposes is to create scenarios describing possible evolution of EFCDS using methods of optimization developed in this paper by the framework of expert-possibilistic theory. This includes construction of algorithms of logical-possibilistic simulations of anomal and extremal process analysis.

Our attention is focused on the rapidly developing theory of fuzzy measures and integrals ([8], [14]–[16], [31]–[33], [38], [40]–[42], [44] and so on). The application of fuzzy measures and integrals as an instrument of constructing the intelligent decision-making systems is not a novel idea ([8], [13]–[16], [18], [20], [22], [25], [29], [30], [33]–[37], [39]–[42], [44], [46] and so on). We employ the part of the theory of fuzzy measures which concerns extremal fuzzy measures ([31]–[33], [38]) and which, in our opinion, is rather seldom used. We have constructed a new instrument of a fuzzy measure, the extension of which is based on Sugeno lower and upper integrals.

In the framework of this theory a new apparatus of extended fuzzy measures was constructed on the basis of Sugeno's upper and lower integrals ([31]–[33], [38]). Using this apparatus new fuzzy extremal models of weakly structurable dynamic system control were created, where fuzziness is represented in time. Here the structure of time is represented by monotone extremal classes of measurable sets. On such structures uncertainty is described by extremal fuzzy measures and problems of fuzzy analysis of extremal fuzzy processes: 1. Fuzzy Optimization, 2. Fuzzy Identification, 3. Fuzzy Filtration and so on. We will deal with the fuzzy control problems of fuzzy dynamic systems (EFCDS) ([33]–[36], [39]), where fuzzy uncertainty arises with time and time structures are monotone classes of measurable sets.

As known (Subsection 2.2 and [31]), in fuzzy dynamic processes where fuzziness participates as a time factor, an important role is assigned to the structures of extremal fuzzy time intervals $\{\widetilde{\mathscr{F}I}_*(T), \succeq \underset{*}{\otimes}\}\rangle)$, $(\langle\{\widetilde{\mathscr{F}I}^*(T), \preceq \overset{*}{\otimes}\}$. As the fuzzy time flows, the process of expert knowledge measurement on the system states with respect to time is affected by the incompleteness of the obtained information. The polar characteristics of this information manifest themselves as imprecision and uncertainty. A degree of information imprecision is defined by current fuzzy time moments $(\widetilde{t} \in \widetilde{\mathscr{B}}_1^*)$ and future fuzzy time moments $(\widetilde{t} \in \widetilde{\mathscr{B}}_{1*})$, while an uncertainty degree is defined by current fuzzy time intervals $(\widetilde{r} \in \widetilde{\mathscr{B}}_2^*)$ and future fuzzy time intervals $(\widetilde{r} \in \widetilde{\mathscr{B}}_{2*})$. We have constructed the corresponding fuzzy monotone structures

$$\{\widetilde{\mathscr{F}I}_*(T), \succeq, \underset{*}{\oplus}\} \quad \text{and} \quad \{\widetilde{\mathscr{F}I}^*(T), \preceq, \overset{*}{\oplus}\}, \tag{1}$$

in which sequential extremal fuzzy time intervals are calculated recurrently.

Here only note that when expert describes the dynamics of complex objects and "measure" system states, where the source of uncertainty is fuzzy measurements with respect to time, it is necessary to carry out "extremal" "dual" measurements, namely, measurements in extended current and compressed future fuzzy time intervals [31].

In the present paper, we represent the controllable extremal fuzzy processes defined in [35]–[37], [39]. The subject/matter of our investigation is the existence of an optimal control for EFCDS's. Section 2 contains some necessary preliminary concepts presented in [31]–[33], [38]. Sections 3 and 4 describe models of extremal fuzzy continuous dynamic system. Section 5 deals with problems of EFCDS optimization when the control parameter does not depend on a state in which an EFCDS is. Questions of the existence of an optimal control are studied, and variants of their fuzzy integral representation are proposed. Section 6 contains an example in which the EFCDS fuzzy optimal control process is constructed.

2 Preliminary Concepts

All definitions and results see in [31]–[33], [38].

2.1 On the Space of Extended Extremal Fuzzy Measures

Definition 1. Let X be some nonempty set.

a) We call some class $\mathscr{B}^* \subset 2^X$ of subsets of X an upper σ^*-monotone class if (i) $\varnothing, X \in \mathscr{B}^*$; (ii) $\forall A, B \in \mathscr{B}^* \Rightarrow A \cup B \in \mathscr{B}^*$; (iii) $\forall \{A_n\} \in \mathscr{B}^*$, $n = 1, 2, \ldots$, $A_n \uparrow A \Rightarrow A \in \mathscr{B}^*$.

b) We call some class $\mathscr{B}_* \subset 2^X$ of subsets of X a lower σ_*-monotone class if (i) $\varnothing, X \in \mathscr{B}_*$; (ii) $\forall A, B \in \mathscr{B}_* \Rightarrow A \cap B \in \mathscr{B}_*$; (iii) $\forall \{A_n\} \in \mathscr{B}_*$, $n = 1, 2, \ldots$, $A_n \downarrow A \Rightarrow A \in \mathscr{B}_*$.

Definition 2. We call the classes $\mathscr{B}^*$ and $\mathscr{B}_*$ extremal if and only if

$$\forall A \in \mathscr{B}^* \Leftrightarrow \overline{A} \in \mathscr{B}_*.$$

Remark 1. Let $\mathscr{B} \subseteq 2^X$ be some σ-algebra. Then B is both a σ^*-monotone class and a σ_*-monotone class.

Definition 3. 1) $(X, \mathscr{B}^*)$ is called an upper measurable space;

2) $(X, \mathscr{B}_*)$ is called a lower measurable space;

3) If $\mathscr{B}^*$ and $\mathscr{B}_*$ are extremal σ^*- and σ_*-monotone classes, then $(X, \mathscr{B}_*, \mathscr{B}^*)$ is called an extremal measurable space.

Example 1

$$\mathscr{B}_1^* \overset{\Delta}{=} \{A \subset \mathbb{R}_0^+ \mid A = (\alpha; +\infty),\ \alpha \in \mathbb{R}_0^+\} \cup \{\varnothing\} \cup \{\mathbb{R}_0^+\} \text{ is a } \sigma^*\text{-monotone class,}$$
$$\mathscr{B}_{1*} \overset{\Delta}{=} \{A \subset \mathbb{R}_0^+ \mid A = [0; \alpha],\ \alpha \in \mathbb{R}_0^+\} \cup \{\varnothing\} \cup \{\mathbb{R}_0^+\} \text{ is a } \sigma_*\text{-monotone class.}$$

$\mathscr{B}_1^*$ and $\mathscr{B}_{1*}$ are respectively called a Borel σ^*-monotone class and a Borel σ_*-monotone class of first kind. Clearly, $\mathscr{B}_1^*$ and $\mathscr{B}_{1*}$ are extremal.

Example 2

$$\mathscr{B}_2^* \overset{\Delta}{=} \{A \subset \mathbb{R}_0^+ \mid A = [0; \alpha),\ \alpha \in \mathbb{R}_0^+\} \cup \{\varnothing\} \cup \{\mathbb{R}_0^+\} \text{ is a } \sigma^*\text{-monotone class,}$$
$$\mathscr{B}_{2*} \overset{\Delta}{=} \{A \subset \mathbb{R}_0^+ \mid A = [\alpha; +\infty),\ \alpha \in \mathbb{R}_0^+\} \cup \{\varnothing\} \cup \{\mathbb{R}_0^+\} \text{ is a } \sigma_*\text{-monotone class.}$$

$\mathscr{B}_2^*$ and $\mathscr{B}_{2*}$ are respectively called a Borel σ^*- and a Borel σ_*-monotone class of second kind. It is obvious that $\mathscr{B}_2^*$ and $\mathscr{B}_{2*}$ are extremal.

Definition 4. Let $(X, \mathscr{B}^*)$ be some upper measurable space. A function $g^* : \mathscr{B}^* \to [0; 1]$ is called an upper fuzzy measure if: (i) $g^*(\varnothing) = 0$, $g^*(X) = 1$; (ii) $\forall A, B \in \mathscr{B}^*$,

$A \subset B \Rightarrow g^*(A) \leq g^*(B)$; (iii) $\forall \{A_n\} \in \mathscr{B}^*$, $n = 1,2,\ldots$, $A_n \uparrow A \Rightarrow g^*(A) = \lim\limits_{n\to\infty} g^*(A_n)$.

Definition 5. Let $(X, \mathscr{B}_*)$ be some lower measurable space. A function $g_* : \mathscr{B}_* \to [0;1]$ is called a lower fuzzy measure if: (i) $g_*(\varnothing) = 0$, $g_*(X) = 1$; (ii) $\forall A, B \in \mathscr{B}_*$, $A \subset B \Rightarrow g_*(A) \leq g_*(B)$; (iii) $\forall \{A_n\} \in \mathscr{B}_*$, $n = 1,2,\ldots$, $A_n \downarrow A \Rightarrow g_*(A) = \lim\limits_{n\to\infty} g_*(A_n)$.

Definition 6. Let $(X, \mathscr{B}_*, \mathscr{B}^*)$ be some extremal measurable space, g_* be a lower and g^* an upper fuzzy measure.

Then:

a) $g_* : \mathscr{B}_* \to [0;1]$ and $g^* : \mathscr{B}^* \to [0;1]$ is called extremal if and only if

$$\forall A \in \mathscr{B}_* \;:\; g_*(A) = 1 - g^*(\overline{A}).$$

b) $(X, \mathscr{B}_*, \mathscr{B}^*, g_*, g^*)$ is called a space of extremal fuzzy measures.

Definition 7. Let $(X_1, \mathscr{B}'_*, \mathscr{B}'^*)$ and $(X_2, \mathscr{B}''_*, \mathscr{B}''^*)$ be some extremal measurable spaces; $h : X_1 \to X_2$ is called measurable if

$$\forall A \in \mathscr{B}''^*,\; B \in \mathscr{B}''_* \;:\; h^{-1}(A) \in \mathscr{B}'^*,\; h^{-1}(B) \in \mathscr{B}'_*.$$

Definition 8. Let $(X, \mathscr{B}_*, \mathscr{B}^*)$ be some extremal measurable space. Then:

a) The function $h : X \to \mathbb{R}_0^*$ is called upper measurable if and only if h is measurable with respect to the spaces $(X, \mathscr{B}_*, \mathscr{B}^*)$ and $(\mathbb{R}_0^+, \mathscr{B}_{1*}, \mathscr{B}_1^*)$. Then

$$\forall \alpha \geq 0 \quad h^{-1}((\alpha;+\infty)) \in \mathscr{B}^*, \quad h^{-1}([0;\alpha]) \in \mathscr{B}_*.$$

b) The function $h : X \to \mathbb{R}_0^+$ is called lower measurable if and only if h is measurable with respect to the spaces $(X, \mathscr{B}_*, \mathscr{B}^*)$ and $(\mathbb{R}_0^+, \mathscr{B}_{2*}, \mathscr{B}_2^*)$. Then

$$\forall \alpha \geq 0 \quad h^{-1}([0;\alpha)) \in \mathscr{B}^*, \quad h^{-1}([\alpha;+\infty)) \in \mathscr{B}_*.$$

Definition 9. Let $(X, \mathscr{B}_*, \mathscr{B}^*)$ be some extremal measurable space.

a) The class of fuzzy subsets $\widetilde{A} \subset X$ with lower measurable compatibility functions

$$\begin{aligned}\widetilde{\mathscr{B}}_* &= \left\{\widetilde{A} \subset X \mid \mu_{\widetilde{A}} \text{ is lower measurable}\right\} \\ &= \left\{\widetilde{A} \in X \mid \forall\, 0 \leq \alpha \leq 1,\; \mu_{\widetilde{A}}^{-1}([0;\alpha)) \in \mathscr{B}^*,\; \mu_{\widetilde{A}}^{-1}([\alpha;+\infty)) \in \mathscr{B}^*\right\}\end{aligned}$$

is called an extension of the σ_*-monotone class $\mathscr{B}_*$.

b) The class of fuzzy subsets $\widetilde{A} \subset X$ with upper measurable compatibility functions

$$\widetilde{\mathscr{B}}^* = \left\{\widetilde{A} \subset X \mid \mu_{\widetilde{A}} \text{ is upper measurable}\right\}$$
$$= \left\{\widetilde{A} \in X \mid \forall\, 0 \le \alpha \le 1,\ \mu_{\widetilde{A}}^{-1}([0;\alpha]) \in \mathscr{B}_*,\ \mu_{\widetilde{A}}^{-1}((\alpha;+\infty)) \in \mathscr{B}^*\right\}$$

is called an extension of the σ^*-monotone class $\mathscr{B}^*$.

Definition 10. An extremal measurable space $(X, \widetilde{\mathscr{B}}_*, \widetilde{\mathscr{B}}^*)$ is called an extension of an extremal measurable space $(X, \mathscr{B}_*, \mathscr{B}^*)$.

Using the Sugeno integral, we next introduce the notion of extension of fuzzy extremal measures.

Definition 11. Let $(X, \mathscr{B}_*, \mathscr{B}^*, g_*, g^*)$ be some space of extremal fuzzy measures, and $(X, \widetilde{\mathscr{B}}_*, \widetilde{\mathscr{B}}^*)$ be an extension of the extremal measurable space $(X, \mathscr{B}_*, \mathscr{B}^*)$. Then:

a) the function

$$\widetilde{g}_*(\widetilde{A}) \equiv \int_{*X} \mu_{\widetilde{A}}(x) \circ g_*(\cdot) \overset{\Delta}{=} \bigvee_{0<\alpha\le 1} \left[\alpha \wedge g_*([\widetilde{A}]_{\bar{\alpha}})\right], \quad \forall \widetilde{A} \in \widetilde{\mathscr{B}}_*; \tag{2}$$

is called an extension of the fuzzy measure g_* on $\widetilde{\mathscr{B}}_*$;

b) the function

$$\widetilde{g}^*(\widetilde{A}) \equiv \int_X^* \mu_{\widetilde{A}}(x) \circ g^*(\cdot) \overset{\Delta}{=} \bigwedge_{0<\alpha\le 1} \left[\alpha \vee g^*([\widetilde{A}]_{\alpha})\right], \quad \forall \widetilde{A} \in \widetilde{\mathscr{B}}^*, \tag{3}$$

is called an extension of the fuzzy measure g^* on $\widetilde{\mathscr{B}}^*$.

Here $[\widetilde{A}]_\alpha = \{x \in X \mid \mu_{\widetilde{A}}(x) > \alpha\}$, $[\widetilde{A}]_{\bar{\alpha}} = \{x \in X \mid \mu_{\widetilde{A}}(x) \ge \alpha\}$, $0 < \alpha \le 1$.

Definition 12. A space of extremal fuzzy measures $(X, \widetilde{\mathscr{B}}_*, \widetilde{\mathscr{B}}^*, \widetilde{g}_*, \widetilde{g}^*)$ is called an extension of the space $(X, \mathscr{B}_*, \mathscr{B}^*, g_*, g^*)$.

Let $(X, \mathscr{B}_*, \mathscr{B}^*, g_*, g^*)$ be some space of extremal fuzzy measures and $(X, \widetilde{\mathscr{B}}_*, \widetilde{\mathscr{B}}^*, \widetilde{g}_*, \widetilde{g}^*)$ be its extension.

Definition 13. a) Let $\widetilde{A}, \widetilde{B} \in \widetilde{\mathscr{B}}_*$ be any fuzzy sets. Then the lower fuzzy Sugeno integral of the compatibility function $\mu_{\widetilde{B}}$ on the fuzzy set $\widetilde{A}$ is defined with respect to a lower fuzzy measure $\widetilde{g}_*$ by the formula

$$\int_{*\widetilde{A}} \mu_{\widetilde{B}}(x) \circ \widetilde{g}_*(\cdot) \overset{\Delta}{=} \bigvee_{0<\alpha\le 1} \left[\alpha \wedge \widetilde{g}_*(\widetilde{A} \cap [\widetilde{B}]_{\bar{\alpha}})\right]. \tag{4}$$

b) Let $\widetilde{A}, \widetilde{B} \in \widetilde{\mathscr{B}}^*$ be any fuzzy sets. Then the upper fuzzy Sugeno integral of the compatibility function $\mu_{\widetilde{B}}$ on the fuzzy set $\widetilde{A}$ is defined with respect to a upper fuzzy measure $\widetilde{g}^*$ by the formula

$$\int_{\widetilde{A}}^{*} \mu_{\widetilde{B}}(x) \circ \widetilde{g}^{*}(\cdot) \overset{\Delta}{\equiv} \bigwedge_{0<\alpha\leq 1} \left[\alpha \vee \widetilde{g}^{*}(\widetilde{A} \cup [\widetilde{B}]_{\alpha})\right]. \tag{5}$$

Definition 14. Let $(X, \mathscr{B}_*, \mathscr{B}^*, g_*, g^*)$ be some space of extremal fuzzy measures.

a) Let $h \in \widetilde{\mathscr{B}}_*$ be some fuzzy set. The measure

$$\forall A \in \widetilde{\mathscr{B}}_* \ : \ \widetilde{g}_{h*}(\widetilde{A}) \overset{\Delta}{=} \int_{*\widetilde{A}} \mu_h(x) \circ \widetilde{g}_*(\cdot) = \int_{*h} \mu_{\widetilde{A}}(x) \circ \widetilde{g}_*(\cdot) = \int_{*X} \mu_{h \cap \widetilde{A}}(x) \circ \widetilde{g}_*(\cdot) \tag{6}$$

is called the lower extension of g_* on $\widetilde{\mathscr{B}}_*$ with respect to h.

b) Let $h \in \widetilde{\mathscr{B}}^*$ be some fuzzy set. The measure

$$\forall A \in \widetilde{\mathscr{B}}^* \ : \ \widetilde{g}^*_h(\widetilde{A}) \overset{\Delta}{=} \int_{\widetilde{A}}^{*} \mu_h(x) \circ \widetilde{g}^*(\cdot) = \int_{h}^{*} \mu_{\widetilde{A}}(x) \circ \widetilde{g}^*(\cdot) = \int_{X}^{*} \mu_{h \cup \widetilde{A}}(x) \circ \widetilde{g}^*(\cdot) \tag{7}$$

is called the upper extension of g^* on $\widetilde{\mathscr{B}}^*$ with respect to h.

2.2 On the Composition Products of Spaces of Extremal Fuzzy Measures

Let $(X_1, \mathscr{B}'_*, \mathscr{B}'^*, g'_*, g'^*)$ and $(X_2, \mathscr{B}''_*, \mathscr{B}''^*, g''_*, g''^*)$ be any two spaces of extremal fuzzy measures.

Definition 15. Let some subset $H \subset X_1 \times X_2$ be a binary relation. We introduce the following mappings $\forall x_0 \in X_1$ and $\forall y_0 \in X_2$:

$$\begin{aligned} E_H(x_0, \cdot) &\overset{\Delta}{=} \{y \in X_2 \mid (x_0, y) \in H\}, \\ E_H(\cdot, y_0) &\overset{\Delta}{=} \{x \in X_1 \mid (x, y_0) \in H\}. \end{aligned} \tag{8}$$

a) A binary relation $H \subset X_1 \times X_2$ is called lower measurable if $\forall A \in \mathscr{B}''_*$ and $\forall B \in \mathscr{B}'_*$ there exist sequences $\{x_n\}_{n \in N} \subset B$, $\{y_n\}_{n \in N} \subset A$ such that $E_H(x_n, \cdot) \supset E_H(x_{n+1})$, $E_H(\cdot, y_n) \supset E_H(\cdot, y_{n+1})$, $n = 1, 2, \ldots$. We have

$$\Gamma_{H*}(A) \overset{\Delta}{=} \{x \in X_1 \mid \forall y \in A : (x,y) \in H\} \equiv \bigcap_{y \in A} E_H(\cdot, y) = \bigcap_{n=1}^{\infty} E_H(\cdot, y_n) \in \mathscr{B}'_* \tag{9}$$

and

$$\Gamma'_{H*}(B) \overset{\Delta}{=} \{y \in X_2 \mid \forall x \in B : (x,y) \in H\} \equiv \bigcap_{x \in B} E_H(x, \cdot) = \bigcap_{n=1}^{\infty} E_H(x_n, \cdot) \in \mathscr{B}''_*. \tag{10}$$

b) Denote by $\mathscr{B}'_* \otimes \mathscr{B}''_*$ the set of all binary lower measurable relations from $X_1 \times X_2$ and call it the composition product of measurable spaces $\mathscr{B}'_*$ and $\mathscr{B}''_*$.

a$'$) A binary relation $H \subset X_1 \times X_2$ is called upper measurable if $\forall A \in \mathscr{B}''^*$ and $\forall B \in \mathscr{B}'^*$ there exist sequences $\{x_n\}_{n\in N} \subset B$, $\{y_n\}_{n\in N} \subset A$ such that $E_H(x_n,\cdot) \subset E_H(x_{n+1})$, $E_H(\cdot,y_n) \subset E_H(\cdot,y_{n+1})$, $n = 1,2,\ldots$. We have

$$\Gamma_H^*(A) \overset{\Delta}{=} \{x \in X_1 \mid \exists y \in A : (x,y) \in H\} \equiv \bigcup_{y\in A} E_H(\cdot,y) = \bigcup_{n=1}^{\infty} E_H(\cdot,y_n) \in \mathscr{B}'^* \quad (11)$$

and

$$\Gamma_H'^*(B) \overset{\Delta}{=} \{y \in X_2 \mid \exists x \in B : (x,y) \in H\}$$
$$\equiv \bigcup_{x\in B} E_H(x,\cdot) = \bigcup_{n=1}^{\infty} E_H(x_n,\cdot) \in \mathscr{B}''^*. \quad (12)$$

b$'$) Denote by $\mathscr{B}'^* \otimes \mathscr{B}''^*$ the set of all binary upper measurable relations from $X_1 \times X_2$ and call it the composition product of measurable spaces $\mathscr{B}'^*$ and $\mathscr{B}''^*$.

It is not difficult to verify that $\mathscr{B}'_* \otimes \mathscr{B}''_*$ is a lower σ_*-monotone class and $\mathscr{B}'^* \otimes \mathscr{B}''^*$ is a upper σ^*-monotone class.

Theorem 1. *Let $(X_1, \mathscr{B}'_*, g'_*)$ and $(X_2, \mathscr{B}''_*, g''_*)$ be two spaces of lower fuzzy measures. Then on the composition lower measurable space $(X_1 \times X_2, \mathscr{B}'_* \otimes \mathscr{B}''_*)$ the measure $g_* : \forall H \in \mathscr{B}'_* \otimes \mathscr{B}'_*$ defined by*

$$g_*(H) \equiv g'_* \otimes g''_*(H) \overset{\Delta}{=} \bigvee_{E\in\mathscr{B}'_*} \{g'_*(E) \wedge g''_*(\Gamma'_{H*}(E))\}$$
$$\equiv \bigvee_{F\in\mathscr{B}''_*} \{g'_*(\Gamma_{H*}(F)) \wedge g''_*(F)\} \quad (13)$$

is a lower fuzzy measure.

Theorem 2. *Let $(X_1, \mathscr{B}'^*, g'^*)$ and $(X_2, \mathscr{B}''^*, g''^*)$ be two spaces of upper fuzzy measures. Then, on the composition upper measurable space $(X_1 \times X_2, \mathscr{B}'^* \otimes \mathscr{B}''^*)$, the measure $g^* : \forall H \in \mathscr{B}'^* \otimes \mathscr{B}''^*$ defined by*

$$g^*(H) \equiv g'^* \otimes g''^*(H) \overset{\Delta}{=} \bigwedge_{E\in\mathscr{B}'^*} \{g'^*(E) \vee g''^*(\Gamma_H'^*(E))\}$$
$$= \bigwedge_{F\in\mathscr{B}''^*} \{g'^*(\Gamma_H^*(F)) \vee g''^*(F)\} \quad (14)$$

is an upper fuzzy measure.

Theorem 3. a) *Let $H \in \mathscr{B}'_* \otimes \mathscr{B}''_*$ be some binary lower measurable relation $(H \subset X_1 \times X_2)$. Then the value of the measure $g'_* \otimes g''_*$ on H is represented through g'_* and g''_* as the following composition:*

$$g'_* \otimes g''_*(H) = \int_{X_2}^{*} g'_*(E_H(\cdot,y)) \circ g''_*(\cdot) = \int_{X_1}^{*} g''_*(E_H(x,\cdot)) \circ g'_*(\cdot); \quad (15)$$

b) *Let $H \in \mathscr{B}'^{*} \otimes \mathscr{B}''^{*}$ be some binary upper measurable relation. Then the value of the measure $g'^{*} \otimes g''^{*}$ on H is represented through g'^{*} and g''^{*} as the following composition:*

$$g'^{*} \otimes g''^{*}(H) = \int_{X_2}^{*} g'^{*}(E_H(\cdot, y)) \circ g''^{*}(\cdot) = \int_{X_1}^{*} g''^{*}(E_H(x, \cdot)) \circ g'^{*}(\cdot). \tag{16}$$

Now let us proceed to defining fuzzy binary relations on $X_1 \times X_2$.

Definition 16. a) A fuzzy set $\widetilde{H} \subset X_1 \times X_2$ is called a lower fuzzy binary relation if the compatibility function $\mu_{\widetilde{H}} : X_1 \times X_2 \to [0;1]$ is lower measurable;

b) A fuzzy set $\widetilde{H} \subset X_1 \times X_2$ is called an upper fuzzy binary relation if the compatibility function $\mu_{\widetilde{H}}$ is upper measurable.

We have constructed the compositional space of extremal extended fuzzy measures $(X_1 \times X_2, \widetilde{\mathscr{B}}'_{*} \otimes \widetilde{\mathscr{B}}''_{*}, \widetilde{\mathscr{B}}^{*\prime} \otimes \widetilde{\mathscr{B}}^{*\prime\prime}, \widetilde{g}'_{*} \otimes \widetilde{g}''_{*}, \widetilde{g}^{*\prime} \otimes \widetilde{g}^{*\prime\prime})$.

2.3 Extremal Fuzzy Time Moments and Intervals, and Their Structures

The questions investigated in the preceding paragraphs enable us to consider some extremal interval structures, in particular, extremal fuzzy time moments and intervals.

We would like to say just a few words about the origination of these structures and their importance in studying dynamic processes.

A person who makes a decision always gives an "incomplete" prognosis about a time moment for extremal, crisis, anomalous and other situations that may occur in the future. The person (expert) who makes a decision connects all such situations with future fuzzy time moments and intervals. Clearly, his/her prognosis is of fuzzy nature and the corresponding decisions should be obtained by possibilistic-statistical analysis or, speaking more exactly, by analysis of fuzzy time intervals, for which we need to construct a new mathematical fuzzy instrument.

When we make decisions on the basis of our past knowledge, we recall certain facts, reference data and the like. When doing so, we perform certain "expert measurements" ("expert reflections") of our knowledge. These measurements are connected with past time moments, which as a rule are fuzzy. Hence the results of such "measurements" may frequently be also fuzzy and these results of recollections are in the end reflected in experimental data (samples). It is understood that the source of such samples is the population of fuzzy characteristics of our knowledge. This can be explained mainly by the following two reasons: first, in terms of dynamics, moments of recollections of facts and moments of "expert measurements" are fuzzy moments; second, on frequent occasions the results of "measurements" are fuzzy. Let us illustrate this viewpoint by examples. Suppose that prior to diagnosing the disease the examining physician (expert) asked the patient to present data on his temperature distribution in time. If the patient measured his temperature but for various reasons did not record the time of measurements, then his replies would sound like this: "In the morning my temperature varied approximately from 38°C to 38.5°C, at noon it dropped to something like 37° and in the evening it was not higher than 39°". Clearly, the results of such "measurements" are fuzzy both in time and in numerical values. It might happen that the patient made measurements of his temperature during the whole day (measurement results are objective data with uncertainty of probabilistic-statistical nature), but he did not record the time moments at which his temperature was measured. Therefore, when asking the patient to present this information in dynamics, we deal with fuzzy time moments. In such situations objective data are characterized by possibilistic uncertainty.

It is clear that decisions (prognoses) made about a future state of the object (prognoses) on the basis of such data by means of the classical statistical methods are less plausible for one reason: the source from which data of this kind originate is the person. The nature of data uncertainty is dual. It is only statistical-and-possibilistic methods that can give us more or less plausible estimates and prognoses.

With this aim in view, we begin our study of fuzzy time moments and intervals and their structures. For convenience, the observation time is identified with the set of nonnegative real numbers: $T \overset{\Delta}{=} \mathbb{R}_0^+$. Any time moment $t \in T = \mathbb{R}_0^+$ is assumed to be a nonnegative number.

Our notion of a fuzzy time moment is based on the definition presented in [8].

Definition 17. A fuzzy nonnegative real number $\widetilde{t}$ with the compatibility function

$$\mu_{\widetilde{t}} : \mathbb{R}_0^+ \to [0;1] \tag{17}$$

with the following properties:

(i) $\mu_{\widetilde{t}}(0) = 0$;

(ii) $\bigvee_{\tau \geq 0} \mu_{\widetilde{t}}(\tau) = 1$ (normed);

(iii) $\forall \tau_0 \in \mathbb{R}_0^+$, $\mu_{\widetilde{t}}(\tau_0) = \bigvee_{\tau < \tau_0} \mu_{\widetilde{t}}(\tau)$ (left continuity);

(iv) $\mu_{\widetilde{t}}(\tau)$ is a nonincreasing function on $\mathbb{R}_0^+ \equiv T$,

is called a fuzzy time moment.

The set of all fuzzy time moments is denoted by $\widetilde{\mathscr{F}M}_0(\mathbb{R}_0^+) \equiv \widetilde{\mathscr{F}M}_0^*(T)$.

Now, let us consider the extremal measurable Borel space of first kind $(\mathbb{R}_0^+, \mathscr{B}_{1*}, \mathscr{B}_1^*)$ and its extension $(\mathbb{R}_0^+, \widetilde{\mathscr{B}}_{1*}, \widetilde{\mathscr{B}}_1^*)$. If $\widetilde{a} \in \widetilde{\mathscr{B}}_1^*$ is a fuzzy number, then $\forall \alpha \geq 0$, $\mu_{\widetilde{a}}^{-1}((\alpha;+\infty)) \equiv (\tau;+\infty) \in \mathscr{B}_1^*$ and $\mu_{\widetilde{a}}^{-1}([0;\alpha]) \equiv [0,\tau] \in \mathscr{B}_{1*}$, i.e., $\mu_{\widetilde{a}}$ is an upper measurable function (or $\mu_{\widetilde{a}} : \mathscr{B}_1^* \to \mathscr{B}_1^*$, $\mathscr{B}_{1*} \to \mathscr{B}_{1*}$ is measurable). It is not difficult to verify that the compatibility function of the fuzzy moment $\widetilde{t}$ is upper measurable, i.e., the fuzzy time moment $\widetilde{t}$ is an upper fuzzy number on $T = \mathbb{R}_0^+$. We obtain $\widetilde{\mathscr{F}M}_0(\mathbb{R}_0^+) \subset \widetilde{\mathscr{B}}_1^*$.

Let us consider the negation of the fuzzy moment $\widetilde{t}$. It clearly follows that $\overline{\widetilde{t}} \in \widetilde{\mathscr{B}}_{1*}$ or $\forall \alpha \geq 0$, $\mu_{\overline{\widetilde{t}}}^{-1}([\alpha;+\infty)) \equiv [0,\tau] \in \mathscr{B}_{1*}$ and $\mu_{\overline{\widetilde{t}}}([0;\alpha)) \equiv (\tau;+\infty) \in \mathscr{B}_1^*$, where $\mu_{\overline{\widetilde{t}}}$ is lower measurable.

In terms of information, the negation of the fuzzy time moment $\bar{\widetilde{t}}$ can be interpreted as follows: it describes a measurement time medium, where the fuzzy time moment $\widetilde{t}$ is excluded.

The relation between the time moment t and the time interval $[0;\tau)$ (and, accordingly, $[\tau;+\infty)$) is one-to-one:

$$t \in [0;\tau) \Longleftrightarrow t \notin [\tau;+\infty).$$

Therefore we may suppose that there exists a relation between the fuzzy time moment $\widetilde{t}$ and the intervals $[0;\tau)$ and $[\tau;+\infty)$. As indicated in [31], for the fuzzy time moment $\widetilde{t}$ its compatibility level $\mu_{\widetilde{t}}(\tau)$, $\tau \geq 0$, is understood as a level of belonging of the fuzzy time moment $\widetilde{t}$ to the time interval $[0;\tau)$ (a compatibility level). Our interpretation is as follows: $\mu_{\widetilde{t}}(\tau)$ is a level of "measurement" imprecision, a level of finding the fuzzy time moment $\widetilde{t}$ in the time interval $[0;\tau)$. A high compatibility level $\mu_{\widetilde{t}}(\tau)$ gives more plausibility that the fuzzy time moment $\widetilde{t}$ "is measured" up to the real moment τ in the time interval $[0;\tau)$. We call this interval the current time interval. Formally, it can be written as $\forall \tau \geq 0$

$$\mu_{\widetilde{t}}(\tau) := \langle \text{an imprecise measure of } (\widetilde{t} \in [0;\tau) := \text{the current time interval})\rangle. \quad (18)$$

Now let us consider the class of complements to fuzzy time moments $\bar{\widetilde{t}}$. Since $\widetilde{t} \in \widetilde{\mathscr{F}M}_0^*(T) \subset \mathscr{B}_1^*$, we denote this class by $\widetilde{\mathscr{F}M}_{0*}(T) \subset \widetilde{\mathscr{B}}_{1*}$. We call $\widetilde{\mathscr{F}M}_0^*(T)$ the class of upper fuzzy time moments, and $\widetilde{\mathscr{F}M}_{0*}(T)$ the class of lower fuzzy time moments.

Extending the above arguments to lower fuzzy time moments, we say that for a fuzzy time moment $\bar{\widetilde{t}}$ its compatibility level $\mu_{\widetilde{t}}(\tau)$ is understood as a level of belonging of the fuzzy time moment $\widetilde{t}$ to the interval $[\tau;+\infty)$, i.e., $\mu_{\bar{\widetilde{E}}}(\tau)$ is an imprecision level of measurement or, in other words, a level of finding a fuzzy time moment $\widetilde{t}$ in the time interval $[\tau;+\infty)$. A high compatibility level $\mu_{\widetilde{t}}(\tau)$ makes it more plausible that the fuzzy time moment $\widetilde{t}$ will be "measured" after the real moment τ in the time interval $[\tau;+\infty)$, which we call the future time interval. If $\widetilde{t} \in \widetilde{\mathscr{F}M}_{0*}(T)$, then $\forall \alpha \geq 0$, $\mu_{\widetilde{t}}^{-1}([\alpha;+\infty)) = [0;\tau] \in \mathscr{B}_{1*}$, $\mu_{\widetilde{t}}^{-1}([0;\alpha]) = [\tau;+\infty)$, i.e., $\mu_{\widetilde{t}}$ is a $\mathscr{B}_{1*} \to \mathscr{B}_{2*}$, $\mathscr{B}_1^* \to \mathscr{B}_2^*$-measurable function.

We call the moment $\widetilde{t} \in \widetilde{\mathscr{F}M}_{0*}(T) \subset \widetilde{\mathscr{B}}_{1*}$ a lower fuzzy time moment, while $\widetilde{t} \in \widetilde{\mathscr{F}M}_{0*}(T)$ and $\bar{\widetilde{t}} \in \widetilde{\mathscr{F}M}_0^*(T)$ extremal fuzzy time moments.

If $\widetilde{t} \in \widetilde{\mathscr{F}M}_{0*}(T)$, then, formally, this can be written as follows:

$$\begin{aligned} \mu_{\widetilde{t}}(\tau) := \langle & \text{ an imprecise measure of } (\widetilde{t} \in [\tau;+\infty) \\ & := \text{is the future time interval})\rangle, \quad \tau \geq 0. \end{aligned} \quad (19)$$

In the process of expert measurement with respect to time the values of the compatibility functions $\mu_{\widetilde{t}}(\tau)$ and $\mu_{\bar{\widetilde{t}}}(\tau)$, $\tau \geq 0$, are degrees of imprecision of finding the fuzzy time moment $\widetilde{t}$ in the future time interval $([\tau;+\infty))$ and the current time interval $([0;\tau))$, respectively.

When we discuss fuzzy time moments in the process of time flow, we should specially mention the pair of extremal fuzzy time moments $(\overline{t},\overline{\overline{t}})$. By the measurement of a fuzzy moment with respect to the real time τ we understand its measurement in the current time interval $[0;\tau)$ and in the future time interval $[\tau;+\infty)$ by (18) and (19). The extremal classes of fuzzy time moments $\widetilde{\mathscr{F}M}_{0*}(T)$ and $\widetilde{\mathscr{F}M}_0^*(T)$ are the classes of complementary fuzzy time moments

$$\widetilde{t} \in \widetilde{\mathscr{F}M}_{0*}(T) \Leftrightarrow \overline{\overline{\widetilde{t}}} \in \widetilde{\mathscr{F}M}_0^*(T).$$

Let us consider the structures of current and future fuzzy time intervals. By Definition 3 (Example 2) we know that

$$\mathscr{B}_2^* \overset{\Delta}{\equiv} \{[0;\tau),\ \tau \geq 0\} \quad \text{and} \quad \mathscr{B}_{2*} \overset{\Delta}{\equiv} \{\tau;+\infty),\ \tau \geq 0\}$$

are Borel σ_*- and σ^*-algebras of second kind. Clearly, the spaces of current and future time intervals are measurable or, speaking more exactly, coincide with extremal Borel spaces of second kind $(\mathbb{R}_0^+, \mathscr{B}_{2*}, \mathscr{B}_2^*)$.

Further, we introduce the notion of extremal fuzzy time interval in terms of extension $(\mathbb{R}_0^+, \widetilde{\mathscr{B}}_{2*}, \widetilde{\mathscr{B}}_2^*)$.

Definition 18. a) Any fuzzy positive number $\widetilde{r} \equiv \widetilde{[0,\tau)} \in \widetilde{\mathscr{B}}_2^*$ is called an extended fuzzy current time interval.

b) Any fuzzy positive number $\widetilde{r} \equiv \widetilde{[\tau;+\infty)} \in \widetilde{\mathscr{B}}_{2*}$ is called an extended fuzzy future time interval.

Obviously, if $\widetilde{r} \in \widetilde{\mathscr{B}}_2^*$, then $\forall \alpha \geq 0$ we have $\mu_{\widetilde{r}}^{-1}([\alpha;+\infty)) \equiv [0,t) \in \mathscr{B}_2^*$, $\mu_{\widetilde{r}}^{-1}([0;\alpha]) \equiv [t;+\infty) \in \mathscr{B}_{2*}$, i.e., $\mu_{\widetilde{r}}$ is the $\mathscr{B}_{2*} \to \mathscr{B}_{1*}$, $\mathscr{B}_2^* \to \mathscr{B}_1^*$-measurable function and if $\widetilde{r} \in \widetilde{\mathscr{B}}_{2*}$, then $\forall \alpha \geq 0$ we have $\mu_{\widetilde{r}}^{-1}([\alpha;+\infty)) \equiv [t;+\infty)$, $\mu_{\widetilde{r}}^{-1}([0;\alpha)) \equiv [0;t) \in \mathscr{B}_{2*}$, i.e., $\mu_{\widetilde{r}}$ is the $\mathscr{B}_{2*} \to \mathscr{B}_{2*}$, $\mathscr{B}_2^* \to \mathscr{B}_2^*$-measurable function. The fuzzy intervals $\widetilde{r} \in \widetilde{\mathscr{B}}_2^*$ and $\overline{\overline{\widetilde{r}}} \in \mathscr{B}_{2*}$ are called extremal.

Let us discuss the relation between fuzzy extremal time moments and intervals. Let $\widetilde{t} \in \widetilde{\mathscr{F}M}_0^*(T)$ and $\widetilde{r} \in \widetilde{\mathscr{B}}_2^*$ be respectively the fuzzy current time moment and the future fuzzy time interval. As has been mentioned above, $\mu_{\widetilde{t}}(\tau)$ is a degree of imprecision of finding the fuzzy moment $\widetilde{t}$ in the current time interval $[0;\tau)$ in the process of time flow. We think that the value $\mu_{\widetilde{r}}(\tau)$ defines the level of compatibility that the current fuzzy time interval $\widetilde{r}$ is not covered by the current time interval $[0;\tau)$. Moreover, $\mu_{\widetilde{r}}(\tau)$ is a degree of uncertainty that $\widetilde{r} \not\subset [0;\tau)$. $\forall \tau \geq 0$:

$$\begin{aligned}\mu_{\widetilde{r}}(\tau) := \langle &\text{ an uncertainty measure of } (\widetilde{r} \not\subset [0;\tau) \\ &:= \text{the current time interval})\rangle. \end{aligned} \tag{20}$$

Let $\widetilde{t} \in \widetilde{\mathscr{F}M}_{0*}(T)$ and $\widetilde{r} \in \widetilde{\mathscr{B}}_{2*}$ be the fuzzy future time moment and the fuzzy time interval, respectively. As has been mentioned above, $\mu_{\widetilde{t}}(\tau)$ is a degree of imprecision of finding the fuzzy moment $\widetilde{t}$ in the future time interval $[\tau;+\infty)$ in the process of time flow. We think that the value $\mu_{\widetilde{r}}(\tau)$ defines the level of compatibility that the fuzzy future time interval $\widetilde{r}$ is not covered by the future time interval $[\tau;+\infty)$. More exactly, $\mu_{\widetilde{r}}(\tau)$ is a degree of uncertainty that $\widetilde{r}\not\subset[\tau;+\infty)$ and $\forall\tau\geq 0$

$$\mu_{\widetilde{r}}(\tau) := \langle \text{ an uncertainty measure of } (\widetilde{r}\not\subset[\tau;+\infty) := \text{the future time interval})\rangle.$$

Note that in the time flow process, the values of the compatibility function of extended extremal fuzzy time intervals $\widetilde{r}\in\widetilde{\mathscr{B}}_2^*$ and $\overline{\widetilde{r}}\in\widetilde{\mathscr{B}}_{2*}$ are degrees of uncertainty that these intervals do not belong to the respective current and future time intervals $[0;\tau)$ and $[\tau;+\infty)$. When speaking of the calculus of fuzzy time intervals, we will mean the pair of extremal fuzzy time intervals $(\widetilde{r},\overline{\widetilde{r}})$, where $\widetilde{r}$ is the current fuzzy time interval $(\widetilde{r}\in\widetilde{\mathscr{B}}_2^*)$, and $\overline{\widetilde{r}}$ is the future fuzzy time interval $(\overline{\widetilde{r}}\in\widetilde{\mathscr{B}}_{2*})$.

In the sequel, we will make use of the following concrete subclass of extended extremal fuzzy time intervals.

Definition 19. The class of fuzzy nonnegative numbers $\widetilde{\mathscr{F}I}^*(T)$ with the properties $(\widetilde{r}\in\widetilde{\mathscr{F}I}^*(T))$:
(i) $\mu_{\widetilde{r}}(0)=1$;
(ii) $\forall\tau_0\geq 0$, $\mu_{\widetilde{r}}(\tau_0)=\bigvee_{\tau>\tau_0}\mu_{\widetilde{r}}(\tau)$ (right continuity);
(iii) $\mu_{\widetilde{r}}$ is nonincreasing on $T=\mathbb{R}_0^+$,
is called the class of current fuzzy time intervals $\widetilde{r}$.

It is not difficult to verify that $\mathscr{F}I^*(T)$ is a subclass of the space of extended fuzzy current time intervals $\widetilde{\mathscr{F}I}^*(T)\subset\widetilde{\mathscr{B}}_2^*$.

Analogously, we introduce the definition of the class $\widetilde{\mathscr{F}I}_*(T)$, which is a complement to $\widetilde{\mathscr{F}I}^*(T)$, i.e.,

$$\widetilde{r}\in\widetilde{\mathscr{F}I}_*(T)\subset\widetilde{\mathscr{B}}_{2*}\Leftrightarrow\overline{\widetilde{r}}\in\widetilde{\mathscr{F}I}^*(T)\subset\widetilde{\mathscr{B}}_2^*.$$

Now let us consider the algebraic structures of the classes of extremal fuzzy time intervals $\langle\mathscr{F}I^*(T),\mathscr{F}I_*(T)\rangle$.

First we will consider $\widetilde{\mathscr{F}I}^*(T)$. We introduce a partial ordering in $\widetilde{\mathscr{F}I}^*(T)$: If $\widetilde{r}_1,\widetilde{r}_2\in\widetilde{\mathscr{F}I}^*(T)$, then

$$\widetilde{r}_1\preceq\widetilde{r}_2\Leftrightarrow\forall\tau\in T\ \ \mu_{\widetilde{r}_1}(\tau)\leq\mu_{\widetilde{r}_2}(\tau). \tag{21}$$

On the semilattice $\{\widetilde{\mathscr{F}I}^*(T),\preceq\}$ we introduce the algebraic sum operation $\widetilde{r}_1\overset{*}{\oplus}\widetilde{r}_2$ [28]:

$$\mu_{\widetilde{r}_1\overset{*}{\oplus}\widetilde{r}_2}(\tau)\overset{\Delta}{=}\wedge\left\{\mu_{\widetilde{r}_1}(\tau_1)\vee\mu_{\widetilde{r}_2}(\tau_2)\mid\tau_1,\tau_2\in T,\ \ \tau_1+\tau_2=\tau\right\}. \tag{22}$$

It is not difficult to verify that the structure $\{\widetilde{\mathscr{F}I}^*(T),\preceq,\overset{*}{\oplus}\}$ is a partially ordered commutative semigroup.

Let us construct, in $\widetilde{\mathscr{F}I}^*(T)$, a monotonically increasing recurrent sequence of fuzzy time intervals

$$\widetilde{r}_n = \widetilde{r}_{n-1} \overset{*}{\oplus} \Delta\widetilde{r}, \quad n \geq 1, \tag{23}$$

where $\widetilde{r}_0$, $\Delta\widetilde{r} \in \mathscr{F}I^*(T)$ are respectively the initial and the stepwise fuzzy time interval. (Here $\widetilde{r}_0 \equiv \widetilde{\varnothing}$). We obtain

$$\widetilde{r}_1 \preceq \widetilde{r}_2 \preceq \cdots .$$

The partial ordering $\preceq$ in $\widetilde{\mathscr{F}I}^*(T)$ induces in $\widetilde{\mathscr{F}I}_*(T)$ another partial ordering $\succeq$ (conjugate to $\preceq$).

If $\widetilde{r}_1, \widetilde{r}_2 \in \widetilde{\mathscr{F}I}_*(T)$, then

$$\widetilde{r}_1 \succeq \widetilde{r}_2 \Leftrightarrow \overline{\overline{\widetilde{r}}}_1 \preceq \overline{\overline{\widetilde{r}}}_2 \Leftrightarrow \forall \tau \in T \; : \; \mu_{\widetilde{r}_1}(\tau) \geq \mu_{\widetilde{r}_2}(\tau). \tag{24}$$

The algebraic sum operation $\overset{*}{\oplus}$ in $\widetilde{\mathscr{F}I}^*(T)$ induces in $\widetilde{\mathscr{F}I}_*(T)$ another operation (conjugate to $\overset{*}{\oplus}$) $\underset{*}{\oplus}$:

$$\forall \widetilde{r}_1, \widetilde{r}_2 \in \widetilde{\mathscr{F}I}_*(T) \; : \; \widetilde{r}_1 \underset{*}{\oplus} \widetilde{r}_2 = \overline{\overline{\widetilde{r}_1}} \overset{*}{\oplus} \overline{\overline{\widetilde{r}}}_2 \tag{25}$$

or, $\forall \tau \in T$,

$$\mu_{\widetilde{r}_1 \underset{*}{\oplus} \widetilde{r}_2}(\tau) = 1 - \mu_{\overline{\overline{\widetilde{r}}} \overset{*}{\oplus} \overline{\overline{\widetilde{r}}}}(\tau) = \vee \left\{ \mu_{\widetilde{r}_1}(\tau_1) \wedge \mu_{\widetilde{r}_2}(\tau_2) \mid \tau_1, \tau_2 \in T, \;\; \tau_1 + \tau_2 = \tau \right\} \tag{26}$$

Then the monotonically increasing sequence of current fuzzy intervals from the class $\widetilde{\mathscr{F}I}^*(T)$

$$\widetilde{r}_1 \preceq \widetilde{r}_2 \preceq \cdots$$

induces, in $\widetilde{\mathscr{F}I}_*(T)$, a monotonically decreasing sequence of future fuzzy intervals

$$\overline{\overline{\widetilde{r}}}_1 \succeq \overline{\overline{\widetilde{r}}}_2 \succeq \cdots$$

defined recurrently as

$$\overline{\overline{\widetilde{r}}}_n = \overline{\overline{\widetilde{r}}}_{n-1} \overset{*}{\oplus} \overline{\overline{\Delta r}},$$

where $\overline{\overline{\widetilde{r}}}_0 = 1_T$ and $\overline{\overline{\Delta r}} \in \widetilde{\mathscr{F}I}_*(T)$ are respectively the initial fuzzy interval and the stepwise fuzzy time interval.

On $\widetilde{\mathscr{F}I}_*(T)$, the induced structure $\{\mathscr{F}I_*(T), \succeq, \underset{*}{\oplus}\}$ is a partially ordered commutative semigroup.

We call the pair of structures

$$\langle \{\widetilde{\mathscr{F}I}^*(T), \preceq, \overset{*}{\oplus}\}, \{\widetilde{\mathscr{F}I}_*(T), \succeq, \underset{*}{\oplus}\} \rangle \tag{27}$$

an extremal partially ordered commutative semigroup.

To conclude the subsection, we would like to note that

1) the extremal structure (27) of current and future fuzzy time intervals is the subject which will be used in next sections.

2) In the time flow process information (data) obtained by expert measurements is incomplete. The polar characteristics of such information are imprecision and uncertainty. The imprecision degree of the obtained information defines extremal fuzzy time moments, while the uncertainty degree defines algebraic structures represented in form (27).

2.4 Examples of Construction of Extremal Fuzzy Time Intervals

Example 1. Consider the extremal measurable Borel space of second kind $(T, \mathscr{B}_{2*}, \mathscr{B}_2^*)$. Let $f : T \to T$ be some monotonically nondecreasing, left continuous function such that $f(0) = 0$, $f(+\infty) = +\infty$. It is not difficult to verify that $\forall \tau \geq 0$

$$g_T^*([0;t)) \overset{\Delta}{=} \frac{f(t)}{1+f(t)} \tag{28}$$

is the upper fuzzy measure on $\mathscr{B}_2^*$, and its extremal fuzzy measure on $\mathscr{B}_{2*}$ is the lower fuzzy measure

$$g_{T*}([t;+\infty)) = \frac{1}{1+f(t)}\,. \tag{29}$$

Now, for the current fuzzy time interval we consider the extension $\widetilde{g}_T^* \ \forall \widetilde{r} \in \widetilde{\mathscr{F}I}^*(T) \subset \widetilde{\mathscr{B}}_2^*$:

$$\begin{aligned}
\widetilde{g}_T^*(\widetilde{r}) &= \int_T^* \mu_{\widetilde{r}}(t) \circ g_T^*(\cdot) = \bigwedge_{0\leq\alpha\leq 1}[\alpha \vee g_T^*([\widetilde{r}]_\alpha)] \\
&= \bigwedge_{0\leq\alpha\leq 1}[\alpha \vee g_T^*([0;t_\alpha))] = \bigwedge_{0\leq\alpha\leq 1}\left[\alpha \vee \frac{f(t_\alpha)}{1+f(t_\alpha)}\right],
\end{aligned}$$

where

$$t_\alpha = \vee\left\{t \geq 0 \mid \mu_{\widetilde{r}}(t) \leq \alpha \leq \mu_{\widetilde{r}}(t^+)\right\},$$

and calculate the extension $\widetilde{g}_{T*} \ \forall \widetilde{r} \in \widetilde{\mathscr{F}I}_*(T) \subset \widetilde{\mathscr{B}}_{2*}$ as follows:

$$\widetilde{g}_{T*}(\widetilde{r}) = \int_{*T} \mu_{\widetilde{r}}(t) \circ g_{T*}(\cdot) = \bigvee_{0\leq\alpha\leq 1}[\alpha \wedge g_{T*}([t_\alpha;+\infty))] = \bigvee_{0\leq\alpha\leq 1}\left[\alpha \wedge \frac{1}{1+f(t_\alpha)}\right],$$

where

$$t_\alpha = \wedge\{t \geq 0 \mid \mu_{\widetilde{r}}(t-) \leq \alpha \leq \mu_{\widetilde{r}}(t)\}.$$

Thus we have constructed the space of extended extremal fuzzy measures $(T, \widetilde{\mathscr{B}}_{2*}, \widetilde{\mathscr{B}}_2^*, \widetilde{g}_{T*}, \widetilde{g}_T^*)$.

Now, let us consider the problem of construction of extremal fuzzy time intervals. If $\widetilde{r} \in \widetilde{\mathscr{F}I}^{*}(T)$, then, by virtue of formula (20), g_T^* is assumed to be a fuzzy measure on $\mathscr{B}_1^*$, while the fuzzy interval $(\widetilde{r} \in \widetilde{\mathscr{B}}_1^*)$ is assumed to be known:

$$\mu_{\widetilde{r}}(t) = \int^{*}_{\overline{[0;t]}} \mu_{\widetilde{t}}(s) \circ g_T^*, \quad \forall t \geq 0.$$

Then

$$\begin{aligned}\mu_{\widetilde{r}}(t) &= \int^{*}_{T} I_{(t;+\infty)}(s) \vee \mu_{\widetilde{t}}(s) \circ g_T^*(\cdot) \\ &= \bigwedge_{0\leq\alpha\leq 1} [\alpha \vee g_T^*((t;+\infty) \cup (t_\alpha;+\infty))] = \bigwedge_{0\leq\alpha\leq 1} [\alpha \vee g_T^*(s_{t,\alpha};+\infty)],\end{aligned}$$

where

$$s_{t,\alpha} = t \wedge t_\alpha, \quad t_\alpha = \wedge\left\{t \geq 0 \mid \mu_{\widetilde{t}}(t) \leq \alpha \leq \mu_{\widetilde{t}}(t^+)\right\}.$$

If in the role of g_T^* we take $\forall t \geq 0$

$$g^*(t;+\infty)) = \frac{1}{1+f(t)},$$

where $f(t)$ is a monotonically nondecreasing, left continuous function $f : T \to T$, $f(0) = 0$, $f(+\infty) = +\infty$, then

$$\mu_{\widetilde{r}}(t) = \bigwedge_{0<\alpha\leq 1} \left[\alpha \vee \frac{1}{1+f(s_{t,\alpha})}\right].$$

If $\widetilde{r} \in \widetilde{\mathscr{F}I}_*(T)$, then, analogously, we construct

$$\mu_{\widetilde{r}}(t) = \int_{*\,\overline{(t;+\infty)}} \mu_{\widetilde{t}}(s) \circ g_{T*}(\cdot).$$

In that case

$$\mu_{\widetilde{r}}(t) = \bigvee_{0<\alpha\leq 1} [\alpha \wedge g_{T*}([0;s_{t,\alpha}])],$$

where

$$s_{t,\alpha} = t \wedge t_\alpha, \quad t_\alpha = \vee\left\{t \geq 0 \mid \mu_{\widetilde{t}}(t) \leq \alpha \leq \mu_{\widetilde{t}}(t^+)\right\}$$

or

$$\mu_{\widetilde{r}}(t) = 1 - \mu_{\overline{\widetilde{r}}}(t) = \bigvee_{0\leq\alpha\leq 1} \left[\alpha \wedge \frac{f(s_{t,\alpha})}{1+f(s_{t,\alpha})}\right].$$

Example 2. Let g_T^* be an upper possibilistic measure on $\mathscr{B}_2^*$, i.e., $\exists f^* : T \to [0;1]$ is a left continuous, monotonically nondecreasing function such that $f(0) = 0$, $f(+\infty) = 1$, and $\forall [0;t) \in \mathscr{B}_2^*$

$$g_T^*([0;t)) = \bigvee_{0<s<t} f(s) = f(t).$$

Then $\forall \widetilde{r} \in \widetilde{\mathscr{F}I}^*(T)$

$$\mu_{\widetilde{r}}(t) = \bigwedge_{0<\alpha\leq 1} [\alpha \vee f^*(s_{t,\alpha})],$$

where

$$s_{t,\alpha} = t \wedge t_\alpha, \quad t_\alpha = \wedge \left\{ t \geq 0 \mid \mu_{\widetilde{t}}(t) \leq \alpha \leq \mu_t(t^+) \right\}.$$

Example 3. Let g_T^* be an upper λ-fuzzy measure [44] on $\mathscr{B}_2^*$ ($g_T^* \equiv g_\lambda^*, -1 \leq \lambda \leq 0$), i.e., $\forall [0;t) \in \mathscr{B}_2^*$

$$g_\lambda^*([0;t)) = \frac{1 - f^*(t)}{1 + \lambda f^*(t)},$$

where f^* is a distribution function of the measure g_λ^*, $f^* : T \to [0;1]$ is a left continuous, monotonically nondecreasing function, $f^*(0) = 0$, $f^*(+\infty) = 1$. Then

$$g_{T*}([t;+\infty)) = g_{\lambda *}([t;+\infty)) = \frac{f^*(t)(1-\lambda)}{1 + \lambda f^*(t)}$$

and $\forall \widetilde{r} \in \widetilde{\mathscr{F}I}_*(T)$

$$\mu_{\widetilde{r}}(t) = \bigvee_{0\leq\alpha\leq 1} \left[\alpha \wedge \frac{f^*(s_{t,\alpha})(1-\lambda)}{1 + \lambda f^*(s_{t,\alpha})} \right],$$

while $\forall \widetilde{r} \in \widetilde{\mathscr{F}I}^*(T)$

$$\mu_{\widetilde{r}}(t) = \bigwedge_{0\leq\alpha\leq 1} \left[\alpha \vee \frac{1 - f^*(s_{t,\alpha})}{1 + \lambda f^*(s_{t,\alpha})} \right],$$

where $s_{t,\alpha}$ is defined from Example 2.

Example 4. It is natural to introduce a fuzzy time interval $\widetilde{r} \in \mathscr{B}_2^*$ such that the kernel of $\widetilde{r}$ would coincide with the interval $[0;\tau]$.

Let us define the upper fuzzy time interval as follows. For $\forall \tau \geq 0$, $\widetilde{r}_\tau \in \mathscr{B}_2^*$:

$$\mu_{\widetilde{r}_\tau}(t) = \begin{cases} 1, & 0 \leq t \leq \tau, \\ g_{T*}([t;+\infty)) \vee g_T^*([0;\tau)), & t \geq \tau. \end{cases}$$

If $g_T^* : \mathscr{B}_1^* \to [0;1]$:

$$g_T^*([0;t)) = \frac{f(t)}{1 + f(t)},$$

as in Example 1, then

$$\mu_{\widetilde{r}_\tau}(t) = \begin{cases} 1, & 0 \leq t \leq \tau, \\ \left(\frac{f(t)}{1+f(t)} \right) \wedge \left(\frac{f(\tau)}{1+f(\tau)} \right), & t > \tau. \end{cases}$$

If $0 \le \alpha \le 1$, then the solution of the equation

$$\alpha = \frac{1}{1+f(t)} \vee \frac{f(\tau)}{1+f(\tau)}$$

with respect to t is denoted by $t_{\tau,\alpha}$. If $t_{\tau,1} = \tau$, $t_{\tau,0} = \infty$, then

$$g_T^*(\widetilde{r}_\tau) = \int_T^* \mu_{\widetilde{r}_\tau}(t) \circ g_T^*(\cdot) = \bigwedge_{0<\alpha\le 1} \left[\alpha \vee \frac{t_{\tau,\alpha}}{1+t_{\tau,\alpha}}\right].$$

Note that if $\tau_1 < \tau_2$, then $\widetilde{r}_{\tau_1} \succeq \widetilde{r}_{\tau_2}$.

This example makes it possible to construct parametrically some sequence of extremal fuzzy intervals.

3 Description of a General Model of an Extremal Fuzzy Continuous Dynamic System (EFCDS)

Following the system approach of modeling complex systems [20] we propose the following: the time structure of fuzzy dynamic systems is represented by some space of extended extremal fuzzy measures

$$\langle T, \widetilde{\mathscr{F}I}_*(T), \widetilde{\mathscr{F}I}^*(T), \widetilde{g}_{T*}, \widetilde{g}_T^* \rangle, \quad T = \mathbb{R}_0^*, \tag{30}$$

and structure (1), where $\widetilde{g}_{T*}$ and $\widetilde{g}_T^*$ are some extremal fuzzy measures on $\widetilde{\mathscr{B}}_{T*} \equiv \widetilde{\mathscr{B}}_{2*}$ and $\widetilde{\mathscr{B}}_T^* \equiv \widetilde{\mathscr{B}}_2^*$, respectively (see Subsection 2.1).

Let us start describing objects of a fuzzy dynamic system. Let X $(X \neq \varnothing)$ be the set of states of some system to be investigated. Let $(X, \mathscr{B}, g)$ be the space of a fuzzy measure on the measurable space $(X, \mathscr{B})$, where $\mathscr{B}$ is a σ-algebra in X.

Let U $(U \neq \varnothing)$ be the set of all admissible controls (of external factors) acting on the system. Assume that controls are subjected to restrictions of uncertain character in the form of some space of a fuzzy measure $(U, \mathscr{B}_U, g_U)$, where $\mathscr{B}_U$ is the measurable space of controls, while the fuzzy measure g_U describes the restrictions imposed on controls.

Let Y $(Y \neq \varnothing)$ be the set of output states of the system under consideration, and $(Y, \mathscr{B}_Y, g_Y)$ be the space of a fuzzy measure, which describes a fuzzy distribution of output values of the system. Note that as usual Y is some transformation of the set of states of X.

Now let us consider the Cartesian product $X \times T$ and the space of extended composition extremal fuzzy measures (Subsection 2.2 and [32])

$$\left(X \times T, \widetilde{\mathscr{B} \otimes \mathscr{B}_{T*}}, \widetilde{\mathscr{B} \otimes \mathscr{B}_T^*}, \widetilde{g \otimes g_{T*}}, \widetilde{g \otimes g_T^*}\right),$$

which is induced by the spaces $(X, \mathscr{B}, \mathscr{B}, g, g)$ and $(T, \mathscr{B}_{T*}, \mathscr{B}_T^*, g_{T*}, g_T^*)$.

Definition 20. a) A lower measurable binary fuzzy relation $\widetilde{Q}_* \in \widetilde{\mathscr{B} \otimes \mathscr{B}_{T*}}$ is called a future fuzzy process on the measurable states of the system (i.e., $\mu_{\widetilde{Q}_*}(x,t)$ is a lower measurable function).

b) An upper measurable binary fuzzy relation $\widetilde{Q}^* \in \widetilde{\mathscr{B} \otimes \mathscr{B}_T^*}$ is called a current fuzzy process on the measurable states of the system (i.e., $\mu_{\widetilde{Q}^*}(x,t)$ is an upper measurable function).

c) A pair $(\widetilde{Q}_*, \widetilde{Q}^*)$ of lower and upper measurable binary fuzzy relations is called an extremal fuzzy process on the measurable states of the system (i.e., $Q^* \in \widetilde{\mathscr{B} \otimes \mathscr{B}_T^*}$ and $Q_* \in \widetilde{\mathscr{B} \otimes \mathscr{B}_{T*}}$).

d) An extremal fuzzy process (EFP) is said to be ergodic if there exist the limits $\forall x \in X$, $\lim\limits_{t \to \infty} \mu_{\widetilde{Q}^*}(x,t) \equiv \mu_{\widetilde{A}^*}(x)$, $\lim\limits_{t \to \infty} \mu_{\widetilde{Q}_*}(x,t) \equiv \mu_{\widetilde{A}_*}(x)$, and the limit fuzzy sets $\widetilde{A}^*$ and $\widetilde{A}_*$ are measurable $\widetilde{A}^*, \widetilde{A}_* \in \widetilde{\mathscr{B}}$.

Note that (see Subsection 2.2) $\forall \tau \in T$, $\forall x \in X$

$E_{\widetilde{Q}_*}(x,\cdot) \in \widetilde{\mathscr{B}}_{T*}$ is a future fuzzy time interval,

$E_{\widetilde{Q}^*}(x,\cdot) \in \widetilde{\mathscr{B}}_T^*$ is a current fuzzy time interval,

$E_{\widetilde{Q}_*}(\cdot,\tau) \in \widetilde{\mathscr{B}}$ is a fuzzy state of the system, which is "measurable" in the future fuzzy time interval $\widetilde{[\tau,+\infty)}$,

$E_{\widetilde{Q}^*}(\cdot,\tau) \in \widetilde{\mathscr{B}}$ is a fuzzy state of the system, which is "measurable" in the current fuzzy time interval $\widetilde{[0,\tau)}$.

It is obvious that model "measurements" of the states of the system at a real time moment $\tau > 0$ are understood as defining pairs of measurable fuzzy sets $E_{\widetilde{Q}_*}(\cdot,\tau)$, $E_{\widetilde{Q}^*}(\cdot,\tau) \in \widetilde{\mathscr{B}}$.

For all $x \in X$, $E_{\widetilde{Q}^*}(x,\cdot)$ and $E_{\widetilde{Q}_*}(x,\cdot)$ are a current fuzzy and a future fuzzy time intervals, respectively, in which the state $x \in X$ of the system is measured.

The family of fuzzy sets $\{E_{\widetilde{Q}_*}(\cdot,\tau)\}_{\tau \geq 0}$ from $\widetilde{\mathscr{B}}$ is called the trajectory of a future fuzzy process, and the family of fuzzy sets $\{E_{\widetilde{Q}^*}(\cdot,\tau)\}_{\tau \geq 0}$ from $\widetilde{\mathscr{B}}$ is called the trajectory of a current fuzzy process. The family of pairs of fuzzy sets $\{E_{\widetilde{Q}_*}(\cdot,\tau), E_{\widetilde{Q}^*}(\cdot,\tau)\}_{\tau \geq 0}$ is called the trajectory of an extremal fuzzy process $(\widetilde{Q}_*, \widetilde{Q}^*)$.

Let $\widetilde{\mathbb{R}}_* \subset X \times T \times Y$ be some lower measurable fuzzy relation $(\widetilde{\mathbb{R}}_* \in \widetilde{\mathscr{B} \otimes \mathscr{B}_{T*}} \otimes \widetilde{\mathscr{B}}_Y)$ describing expert knowledge reflections of fuzzy states of the system on the output values of the system in future fuzzy time intervals, and $\widetilde{\mathbb{R}}^* \subset X \times T \times Y$ be some upper measurable fuzzy relation $(\widetilde{\mathbb{R}}^* \in \widetilde{\mathscr{B} \otimes \mathscr{B}_T^*} \otimes \widetilde{\mathscr{B}}_Y)$ describing expert knowledge reflections of fuzzy states of the system on the output values of the system in current fuzzy time intervals.

Definition 21. a) A lower measurable relation $\widetilde{\mathbb{R}}_* \in \widetilde{\mathscr{B} \otimes \mathscr{B}_{T*}} \otimes \widetilde{\mathscr{B}}_Y$ is called a future fuzzy process of expert knowledge reflection of states of the system in future fuzzy time intervals.

b) An upper measurable relation $(\widetilde{\mathbb{R}}^* \in \widetilde{\mathscr{B} \otimes \mathscr{B}_T^*} \otimes \widetilde{\mathscr{B}_Y})$ is called a current fuzzy process of expert knowledge reflection of states of the system in current fuzzy time intervals.

c) A pair $(\widetilde{\mathbb{R}}_*, \widetilde{\mathbb{R}}^*)$ is called an extremal fuzzy process of expert knowledge reflection of states of the system in extremal fuzzy time intervals.

Let $\widetilde{\rho}_* \in (\widetilde{\mathscr{B} \otimes \mathscr{B}_{T*}}) \otimes (\widetilde{\mathscr{B}_U \otimes \mathscr{B}_{T*}}) \otimes \widetilde{\mathscr{B}}$ be some lower measurable fuzzy relation in the Cartesian product $(X \times T) \times (U \times T) \times X$, which describes system state transformations in time with control taken into account:

$$(X \times T) \times (U \times T) \to X.$$

This relation is a future fuzzy transition operator describing the dynamics of the system or, in other words, system state transformations in future fuzzy time intervals.

Let $\widetilde{\rho}^* \in (\widetilde{\mathscr{B} \otimes \mathscr{B}_T^*}) \otimes (\widetilde{\mathscr{B}_U \otimes \mathscr{B}_{T*}}) \otimes \widetilde{\mathscr{B}}$ be some upper measurable fuzzy relation in the Cartesian product $(X \times T) \times (U \times T) \times X$, which describes system state transformations in time with control taken into account:

$$(X \times T) \times (U \times T) \to X.$$

This relation is a current fuzzy transition operator describing the dynamics of the system or, in other words, system state transformations in current fuzzy time intervals.

We call $\widetilde{\rho}_*$ the fuzzy lower transition operator describing the system state dynamics, and $\widetilde{\rho}^*$ the fuzzy upper transition operator describing the system state dynamics. The pair $(\widetilde{\rho}_*, \widetilde{\rho}^*)$ is called the transition operator describing the system state dynamics in extremal fuzzy time intervals.

Let $\widetilde{u}^* \subset U \times T$ be some upper measurable fuzzy binary relation from $\widetilde{\mathscr{B}_U \otimes \mathscr{B}_T^*}$, which describes the action of external factors (controls) on the system in future fuzzy time intervals, and $\widetilde{u}_* \subset U \times T$ be some lower measurable fuzzy binary relation from $\widetilde{\mathscr{B}_U \otimes \mathscr{B}_T^*}$, which describes the action of external factors (controls) on the system in current fuzzy time intervals.

Definition 22. a) A fuzzy binary relation $\widetilde{u}^* \in \widetilde{\mathscr{B} \otimes \mathscr{B}_T^*}$ is called a current fuzzy control process.

b) A fuzzy binary relation $\widetilde{u}_* \in \widetilde{\mathscr{B} \otimes \mathscr{B}_{T*}}$ is called a future fuzzy control process.

c) A pair $(\widetilde{u}_*, \widetilde{u}^*)$ is called an extremal fuzzy control process.

Definition 23. a) The train

$$\left\{X, U, T, Y, \widetilde{\rho}_*, \widetilde{Q}_*, \widetilde{\mathbb{R}}_*\right\} \tag{31}$$

is called the future fuzzy dynamic system describing the dynamics of the system state in future fuzzy time intervals.

b) The train

$$\left\{X,U,T,Y,\widetilde{\rho}^*,\widetilde{Q}^*,\widetilde{\mathbb{R}}^*\right\} \tag{32}$$

is called the current fuzzy dynamic system describing the state dynamics of the system in current fuzzy time intervals.

c) The train

$$\left\{X,U,T,Y,(\widetilde{\rho}_*,\widetilde{\rho}^*),(\widetilde{Q}_*,\widetilde{Q}^*),(\widetilde{\mathbb{R}}_*,\widetilde{\mathbb{R}}^*)\right\} \tag{33}$$

is called the extremal fuzzy consitnuous dynamic system (EFCDS) describing the state dynamics of the system in extremal fuzzy time intervals.

In the sequel we will consider the case with $Y \equiv X$.

It is obvious that the EFCDS (33) describes the state dynamics of the system undergoing transformation with fuzzy uncertainty produced by observations at fuzzy time, while the extremality is due to the "measurement" of fuzzy states of the system in current and future fuzzy time intervals.

Definition 24. The system of composition equations

$$\begin{cases} \widetilde{\mathbb{R}}_* = \widetilde{\rho}_* \underset{*}{\bullet} \widetilde{Q}_*, \\ \widetilde{\mathbb{R}}^* = \widetilde{\rho}^* \overset{*}{\bullet} \widetilde{Q}^* \end{cases} \tag{34}$$

is called the system describing the state dynamics of the extremal fuzzy continuous dynamic system, where $\underset{*}{\bullet}$ and $\overset{*}{\bullet}$ are some composition operations over fuzzy relations.

Given $(\widetilde{\mathbb{R}}_*,\widetilde{\mathbb{R}}^*)$, $(\widetilde{\rho}_*,\widetilde{\rho}^*)$ and the initial fuzzy states of the system $\widetilde{A}_{0*},\widetilde{A}_0^* \in \mathscr{B}$ ($\mu_{\widetilde{A}_{0*}}(x) = \mu_{\widetilde{Q}_*}(x,0)$, $\mu_{\widetilde{A}_0^*}(x) = \mu_{\widetilde{Q}^*}(x,0)$, $\forall x \in X$), it is important to find a solution $(\widetilde{Q}_*,\widetilde{Q}^*)$ of (34), which we call an extremal fuzzy process of system state transformation on measurable states of the system.

Below we will consider a concrete controllable fuzzy system of form (34) for the continuous case. It is obvious that in concrete EFCDS's formulas (33) and (34) model concrete complex objects with fuzzy dynamics. The finding of a system state transformation process $(\widetilde{Q}_*,\widetilde{Q}^*)$ is important when we deal with problems pertaining to optimization problem (optimal control).

In recent years, the investigation of complex dynamic systems with fuzzy uncertainty by means of the theory of fuzzy sets has been developing mainly along the following two lines:

I. Lower dynamic systems are described by composition equations in the metric or normed spaces of system states, which can be formally written in terms of fuzzy integral equations if a fuzzy measure is assumed to be a possibilistic one ([9], [10], [24], [46], [47] and so on).

II. Quite a number of studies have been devoted to the development of fuzzy integro-differential calculus with an aim of describing fuzzy dynamic systems and their control. The main feature these approaches have in common is the assumption that the compatibility function is differentiable or integrable ([4]–[7], [10], [12], [17], [19], [23], [26], [27], [29], [30], [43] and so on), which to a certain extent facilitates the investigation of the definite class of fuzzy dynamic systems.

The instrument of extended composition fuzzy measures developed in [31] and [32], where some important properties of Sugeno lower and upper integrals and their extensions are investigated, makes it possible to study the so-called extremal fuzzy continuous dynamic systems for which:

1) a system of compositional equations for fuzzy dynamic systems is generalized in the form of system (34), where the extended Sugeno upper and lower integrals (see [31]) are used in the role of composition operations $\underset{*}{\bullet}$ and $\overset{*}{\bullet}$ (as a aggregation instrument for the EFCDS) describing the dynamics of the state of an EFCDS [33].

As known from the earlier sources of investigation of fuzzy statistics ([8], [13], [18], [22], [45] and so on) and also from our works ([31]–[36], [40]–[42]), the Sugeno integral most frequently estimates the most typical levels of compatibility of an integrable function. This is the reason for which we have chosen the Sugeno integral for the construction of extended fuzzy measures.

Systems of composition type equations [24] are a particular case of system (34), where equations are written with respect to possibility measure. The case we consider in this paper is more general since the equations are written in for any extremal fuzzy measure.

2) As different from the approach mentioned in Item II (where some processes are not integro-differentiable), in our proposed systems of equations any measurable compatibility function is integrable. However our consideration is not limited to this only class of dynamic systems.

To conclude the section, note that the compatibility functions, for which systems of equations can be written in form (34), are lower or upper measurable:

$$\begin{aligned}
&\mu_{\widetilde{Q}^*}(x,t) : X \times T \to [0;1] \text{ is } \mathscr{B}\otimes\mathscr{B}_T^*\text{-upper measurable;}\\
&\mu_{\widetilde{Q}_*}(x,t) : X \times T \to [0;1] \text{ is } \mathscr{B}\otimes\mathscr{B}_{T*}\text{-lower measurable;}\\
&\mu_{\widetilde{\mathbb{R}}^*}(x,t,y) : X \times T \times Y \to [0;1] \text{ is } \mathscr{B}\otimes\mathscr{B}_T^*\otimes\mathscr{B}\text{-upper measurable;}\\
&\mu_{\widetilde{\mathbb{R}}_*}(x,t,y) : X \times T \times Y \to [0;1] \text{ is} \mathscr{B}\otimes\mathscr{B}_{T*}\otimes\mathscr{B}\text{-lower measurable;}\\
&\mu_{\widetilde{\rho}_*}(x_0,t_0,u,t,x) : (X\times T)\times(U\times T)\times X \to [0;1] \text{ is}\\
&\qquad (\mathscr{B}\otimes\mathscr{B}_T^*)\otimes(\mathscr{B}_U\otimes\mathscr{B}_T^*)\otimes\mathscr{B}\text{-upper measurable;}\\
&\mu_{\widetilde{\rho}^*}(x_0,t_0,u,t,x) : (X\times T)\times(U\times T)\times X \to [0;1] \text{ is}\\
&\qquad (\mathscr{B}\otimes\mathscr{B}_{T*})\otimes(\mathscr{B}_U\otimes\mathscr{B}_{T*})\otimes\mathscr{B}\text{-lower measurable.}
\end{aligned} \tag{35}$$

4 Continuous Extremal Controllable Fuzzy Process

As has been mentioned above (Subsection 2.2), in [31] we have constructed monotone structures of current fuzzy time intervals $\{\widetilde{\mathscr{F}I}^*(T), \preceq, \overset{*}{\otimes}\}$ and future fuzzy time intervals $\{\widetilde{\mathscr{F}I}_*(T), \succeq, \underset{*}{\otimes}\}$. It is obvious that the flow process of a real time moment τ induces, in these structures, monotonically increasing and monotonically decreasing processes of current and future time intervals, respectively.

Definition 25. a) A family $\{\widetilde{r}_\tau^*\}_{\tau\geq 0}$, $\widetilde{r}_\tau^* \in \widetilde{\mathscr{B}}_T^*$, $\tau \geq 0$, of monotonically increasing sequences of upper fuzzy time intervals, i.e.,

$$\forall \tau_2 > \tau_1 \geq 0, \quad \widetilde{r}_{\tau_1}^* \preceq \widetilde{r}_{\tau_2}^*$$

is called a process of current fuzzy time intervals.

b) A family $\{\widetilde{r}_{\tau *}\}_{\tau\geq 0}$, $\widetilde{r}_{\tau *} \in \widetilde{\mathscr{B}}_{T*}$, $\tau \geq 0$, of monotonically decreasing sequences of upper fuzzy time intervals, i.e.,

$$\forall \tau_2 > \tau_1 \geq 0, \quad \widetilde{r}_{\tau_1 *} \succeq \widetilde{r}_{\tau_2 *}$$

is called a process of future fuzzy time intervals.

c) A pair of processes of future and current fuzzy time intervals $\{\widetilde{r}_{\tau *}, \widetilde{r}_\tau^*\}_{\tau\geq 0}$ is called a process of extremal fuzzy time intervals.

It obviously follows that
$\mu_{\widetilde{r}_{\tau *}}(t) : T \to [0;1]$ is $\mathscr{B}_{T*}$-lower measurable,
$\mu_{\widetilde{r}_\tau^*}(t) : T \to [0;1]$ is $\mathscr{B}_T^*$-upper measurable.
Note that a change of a real time moment $\tau > 0$ reflects model "measurements" of an extremal fuzzy process of system state transformation $(\widetilde{Q}_*, \widetilde{Q}^*)$ in extremal fuzzy time intervals $(\widetilde{r}_{\tau *}, \widetilde{r}_\tau^*)$.

Definition 26. A process of extremal fuzzy time intervals $(\widetilde{r}_{\tau *}, \widetilde{r}_\tau^*)$ is called ergodic if there exist the limits

$$\lim_{\tau\to+\infty} \widetilde{r}_{\tau *} = \widetilde{r}_{\infty *} \in \widetilde{\mathscr{B}}_{T*} \quad \left(\forall t \geq 0, \ \lim_{\tau\to+\infty} \mu_{\widetilde{r}_{\tau *}}(t) = \mu_{\widetilde{r}_{\infty *}}(t)\right),$$
$$\lim_{\tau\to+\infty} \widetilde{r}_\tau^* = \widetilde{r}_\infty^* \in \widetilde{\mathscr{B}}_T^* \quad \left(\forall t \geq 0, \ \lim_{\tau\to+\infty} \mu_{\widetilde{r}_\tau^*}(t) = \mu_{\widetilde{r}_\infty^*}(t)\right).$$

In what follows it will be assumed that there exists a relation between the measurable space of time $(T, \mathscr{B}_{T*}, \mathscr{B}_T^*)$ and the measurable space of system states $(X, \mathscr{B})$ in the form of conditional extremal fuzzy measures defined in [31]. In the considered case it is assumed that there exist conditional lower and upper fuzzy measures $g_{t*}(\cdot \mid x)$ and $g_t^*(\cdot \mid x)$, respectively, i.e., $\forall x \in X$

$$g_{t*}(\cdot \mid x) : \mathscr{B}_{T*} \to [0;1] \ \text{ is a lower fuzzy measure,}$$
$$g_t^*(\cdot \mid x) : \mathscr{B}_T^* \to [0;1] \ \text{ is an upper fuzzy measure.}$$

$g_{t*}(\cdot \mid x)$ and $g_t^*(\cdot \mid x)$ are extremal measures, while for a future fuzzy time interval $r \in \mathscr{B}_{T*}$

$$g_{t*}(r \mid \cdot) : X \to [0;1] \ \text{ is a } \mathscr{B}\text{-measurable function,}$$

and for a current time interval $r \in \mathscr{B}_T^*$

$$g_t^*(r \mid \cdot) : X \to [0;1] \ \text{ is a } \mathscr{B}\text{-measurable function.}$$

These properties also apply to extended conditional fuzzy measures $\widetilde{g}_{t*}(\cdot \mid x)$ and $\widetilde{g}_t^*(\cdot \mid x)$, i.e., $\forall x \in X$, $\widetilde{r}_* \in \widetilde{\mathscr{B}}_{T*}$, $\widetilde{r}^* \in \widetilde{\mathscr{B}}_T^*$

$$\begin{aligned}
&\widetilde{g}_{t*}(\cdot \mid x) : \widetilde{\mathscr{B}}_{T*} \to [0;1] \ \text{ is a lower fuzzy measure,}\\
&\widetilde{g}_t^*(\cdot \mid x) : \widetilde{\mathscr{B}}_T^* \to [0;1] \ \text{ is an upper fuzzy measure,}\\
&\widetilde{g}_{t*}(\widetilde{r}_* \mid \cdot) : X \to [0;1] \ \text{ is a } \mathscr{B}\text{-measurable function,}\\
&\widetilde{g}_t^*(\widetilde{r}^* \mid \cdot) : X \to [0;1] \ \text{ is a } \mathscr{B}\text{-measurable function.}
\end{aligned}$$

A relation between the spaces $(X, \mathscr{B}, g)$ and $(T, \mathscr{B}_{T*}, \mathscr{B}_T^*, g_{T*}, g_T^*)$ and their extensions through conditional measures can be represented as follows: $\forall r_* \in \mathscr{B}_{T*}$, $r^* \in \mathscr{B}_T^*$, $\widetilde{r}_* \in \widetilde{\mathscr{B}}_{T*}$, $\widetilde{r}^* \in \widetilde{\mathscr{B}}_T^*$

$$\begin{aligned}
g_{T*}(r_*) &= \int_X g_{t*}(r_* \mid x) \circ g(\cdot), \quad g_T^*(r^*) = \int_X g_t^*(r^* \mid x) \circ g(\cdot),\\
\widetilde{g}_{T*}(\widetilde{r}_*) &= \int_X \widetilde{g}_{t*}(\widetilde{r}_* \mid x) \circ g(\cdot), \quad \widetilde{g}_T^*(\widetilde{r}^*) = \int_X \widetilde{g}_t^*(\widetilde{r}^* \mid x) \circ g(\cdot),
\end{aligned} \tag{36}$$

Applying results from [31], we can write $\forall x \in X$, $\widetilde{r}_* \in \widetilde{\mathscr{B}}_{T*}$, $\widetilde{r}^* \in \widetilde{\mathscr{B}}_T^*$:

$$\begin{aligned}
\widetilde{g}_{t*}(\widetilde{r}_* \mid x) &= \int_{*T} \mu_{\widetilde{r}_*}(t) \circ g_{t*}(\cdot \mid x),\\
\widetilde{g}_t^*(\widetilde{r}^* \mid x) &= \int_T^* \mu_{\widetilde{r}^*}(t) \circ g_t^*(\cdot \mid x).
\end{aligned} \tag{37}$$

By the definition of $\widetilde{g}_{t*}(\cdot \mid x)$ and $\widetilde{g}_t^*(\cdot \mid x)$, for any lower and upper fuzzy time intervals $\widetilde{r}_* \in \widetilde{\mathscr{B}}_{T*}$ and $\widetilde{r}^* \in \widetilde{\mathscr{B}}_T^*$ there exist $\mathscr{B}$-measurable sets $\widetilde{A}_{\widetilde{r}_*} \in \widetilde{\mathscr{B}}$, $\widetilde{A}_{\widetilde{r}^*} \in \widetilde{\mathscr{B}}$ such that $\forall x \in X$

$$\mu_{\widetilde{A}_{\widetilde{r}_*}}(x) = \widetilde{g}_{t*}(\widetilde{r}_* \mid x), \quad \mu_{\widetilde{A}_{\widetilde{r}^*}}(x) = \widetilde{g}_t^*(\widetilde{r}^* \mid x). \tag{38}$$

Definition 27. The fuzzy sets $\widetilde{A}_{\widetilde{r}_*}$ and $\widetilde{A}_{\widetilde{r}^*} \in \widetilde{\mathscr{B}}$ from the extended measurable space of system states are called the expert reflections of an extremal fuzzy dynamic systems states in the extremal fuzzy time intervals $(\widetilde{r}_*, \widetilde{r}^*)$ with respect to extended extremal conditional fuzzy measures $\widetilde{g}_{t*}(\cdot \mid x)$ and $\widetilde{g}_t^*(\cdot \mid x)$.

Let us formulate a theorem that describes the ergodicity of an expert reflection process in an ergodic process of extremal fuzzy time intervals.

Theorem 4. *An ergodic process $(\widetilde{r}_{\tau*}, \widetilde{r}_\tau^*)_{\tau \geq 0}$ of extremal fuzzy time intervals on the measurable space of states of the system $(X, \widetilde{\mathscr{B}})$ induces an ergodic expert reflection process $(\widetilde{\mathbb{R}}_*, \widetilde{\mathbb{R}}^*) \equiv (\widetilde{A}_{\widetilde{r}_{\tau*}}, \widetilde{A}_{\widetilde{r}_\tau^*})_{\tau \geq 0}$.*

In this section, we consider problems of modeling EFCDS's when the control factor acts on the system or, speaking more exactly, on controllable extremal fuzzy processes.

As defined in Section 3, let U be the space of all admissible controls acting on an EFCDS in the course of its evolution. It is assumed that the restrictions on the space of control elements are of fuzzy nature: these restrictions exist in the form of a fuzzy measure on the measurable space $\mathscr{B}_U$ (the σ-algebra of subsets of U). Let $(U, \mathscr{B}_U, g_U)$ be some space of the fuzzy measure.

Let $\widetilde{u}_* \subset U \times T$ be some upper measurable binary fuzzy relation from $\widetilde{\mathscr{B}_U \otimes \mathscr{B}_T^*}$ that describes an external fuzzy action on the EFCDS in the course of current fuzzy time intervals, and $\widetilde{u}_* \subset U \times T$ be some lower measurable binary fuzzy relation from $\widetilde{\mathscr{B}_U \otimes \mathscr{B}_{T*}}$ that describes an external fuzzy action on the EFCDS in the course of future fuzzy time intervals. A pair $(\widetilde{u}_*, \widetilde{u}^*)$ is called an extremal fuzzy control (an extremal fuzzy control process), while $\widetilde{u}^*$ and $\widetilde{u}_*$ are respectively called a current fuzzy control and a future fuzzy control.

Let $\widetilde{\rho}_* \in \widetilde{\mathscr{B} \otimes \mathscr{B}} \otimes \widetilde{\mathscr{B}_U \otimes \mathscr{B}_{T*}}$ and $\widetilde{\rho}^* \in \widetilde{\mathscr{B} \otimes \mathscr{B}} \otimes \widetilde{\mathscr{B}_U \otimes \mathscr{B}_T^*}$, and $(\widetilde{\rho}_*, \widetilde{\rho}^*)$ be the operator of the EFCDS state change dynamics.

Definition 28. If $(\widetilde{r}_{\tau*}, \widetilde{r}_\tau^*)_{\tau \geq 0}$ is some process of extremal fuzzy time intervals, $(U, \mathscr{B}_U, g_U)$ is a space of a fuzzy measure (a space of fuzzy restrictions on controls), then a pair $(\widetilde{Q}'_*, \widetilde{Q}'^*)$ of lower and upper measurable binary fuzzy relations $(\widetilde{Q}'_* \in \widetilde{\mathscr{B}} \otimes \widetilde{\mathscr{B} \otimes \mathscr{B}_{T*}}, \widetilde{Q}'^* \in \widetilde{\mathscr{B}} \otimes \widetilde{\mathscr{B} \otimes \mathscr{B}_T^*})$ is called an extremal fuzzy process of measurable states of an EFCDS in the process $(\widetilde{r}_{\tau*}, \widetilde{r}_\tau^*)_{\tau \geq 0}$, taking into account the fuzzy restrictions on controls $(U, \mathscr{B}_U, g_U)$: $\forall x \in X$, $u \in U$, $\tau \in T$,

$$
\begin{aligned}
\mu_{\widetilde{Q}'_*}(x,u,\tau) &\stackrel{\Delta}{=} \fint_{*\,\widetilde{r}_{\tau*}} \left[\fint_{A_{0*}} \mu_{\widetilde{\rho}_*}(x,x',u,t) \circ g(\cdot) \right] \circ \widetilde{g}_{T*}(\cdot) \equiv \fint_{*\,\widetilde{r}_{\tau*}} \mu_{\widetilde{\rho}'_*}(x,u,t) \circ \widetilde{g}_{T*}(\cdot), \\
\mu_{\widetilde{Q}'^*}(x,u,\tau) &\stackrel{\Delta}{=} \fint^*_{\widetilde{r}_\tau^*} \left[\fint_{A_0^*} \mu_{\widetilde{\rho}^*}(x,x',u,t) \circ g(\cdot) \right] \circ \widetilde{g}_T^*(\cdot) \equiv \fint^*_{\widetilde{r}_\tau^*} \mu_{\widetilde{\rho}'^*}(x,u,t) \circ \widetilde{g}_{T*}(\cdot).
\end{aligned}
\tag{39}
$$

Definition 29. In the conditions of the action of an extremal fuzzy control process $(\widetilde{u}_*, \widetilde{u}^*)$ on an EFCDS, a pair $(\widetilde{Q}_*, \widetilde{Q}^*)$ $(\widetilde{Q}_* \in \widetilde{\mathscr{B} \otimes \mathscr{B}_{T*}}, \widetilde{Q}^* \in \widetilde{\mathscr{B} \otimes \mathscr{B}_T^*})$ defined as follows: $\forall (x, \tau) \in X \times T$

$$
\begin{aligned}
\mu_{\widetilde{Q}_*}(x,\tau) &\stackrel{\Delta}{=} \fint_{E_{\widetilde{u}_*}(\cdot,\tau)} \mu_{\widetilde{Q}'_*}(x,u,\tau) \circ g_U(\cdot), \\
\mu_{\widetilde{Q}^*}(x,\tau) &\stackrel{\Delta}{=} \fint_{E_{\widetilde{u}^*}(\cdot,\tau)} \mu_{\widetilde{Q}'^*}(x,u,\tau) \circ g_U(\cdot)
\end{aligned}
\tag{40}
$$

is called an extremal fuzzy process describing the system state dynamics.

Let us present the integral representation of the process $(\widetilde{Q}_*, \widetilde{Q}^*)$ [33].

Theorem 5. *In the conditions of the action of an extremal fuzzy control process* $(\widetilde{u}_*, \widetilde{u}^*)$ *on an EFCDS with the initial extremal fuzzy state* $\langle \widetilde{A}_{0*} \equiv E_{\widetilde{Q}_*}(\cdot, \tau_0), \widetilde{A}_0^* \equiv E_{\widetilde{Q}^*}(\cdot, \tau_0)\rangle$*, the system state change dynamics is described by the extremal fuzzy process* $(\widetilde{Q}_*, \widetilde{Q}^*)$*, the integral representation of which is as follows:* $\forall x \in X,\ \tau \in T$

$$\text{a)}\ \mu_{\widetilde{Q}_*}(x,\tau) = \int_{*\,U\times T} \left[\mu_{E_{\widetilde{u}_*}(\cdot,\tau)}(u) \wedge \mu_{E_{\widetilde{\rho}'_*}(x,\cdot,\cdot)}(u,t)\right] \circ g_U \otimes g_{E_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}(\cdot), \tag{41}$$

where $\widetilde{g_U} \otimes \widetilde{g_{E_{\widetilde{\mathbb{R}}_*(\cdot,\tau)}}}$ *is an extended composition lower fuzzy measure of the measures* g_U *and* $g_{E_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}$.

$$\text{b)}\ \mu_{\widetilde{Q}^*}(x,\tau) = \int^{*}_{U\times T} \left[\mu_{E_{\widetilde{u}^*}(\cdot,\tau)}(u) \vee \mu_{E_{\widetilde{\rho}'^*}(x,\cdot,\cdot)}(u,t)\right] \circ g_U \otimes g_{E_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}(\cdot), \tag{42}$$

where $\widetilde{g_U} \otimes \widetilde{g_{E_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}}$ *is an extended composition upper fuzzy measure of the measures* g_U *and* $g_{E_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}$.

Theorem 6. *Let* $(\widetilde{r}_{\tau*}, \widetilde{r}^*_\tau)_{\tau\geq 0}$ *be some ergodic process of extremal fuzzy time intervals,* $(\widetilde{\mathbb{R}}_*, \widetilde{\mathbb{R}}^*)$ *be an extremal fuzzy reflection process induced by the process* $(\widetilde{r}_{\tau*}, \widetilde{r}^*_\tau)_{\tau\geq 0}$*,* $(\widetilde{Q}_*, \widetilde{Q}^*)$ *be an extremal fuzzy process describing the EFCDS state dynamics, and* $(\widetilde{u}_*, \widetilde{u}^*)$ *be an extremal ergodic fuzzy control process acting on the EFCDS. Then the extremal fuzzy process* $(\widetilde{Q}_*, \widetilde{Q}^*)$ *is ergodic.*

Recalling the notion of lower and upper convergence of sequences of lower and upper measurable functions, respectively (see Subsection 2.1), and also the notion of lower and upper self-continuity of extremal fuzzy measures, we make the following statements on the ergodicity of extremal fuzzy processes.

Definition 30 (g_T-Ergodicity). We say that the fuzzy process of extremal fuzzy time intervals $(\widetilde{r}_{\tau*}, \widetilde{r}^*_\tau)_{\tau\geq 0}$ is g_T-ergodic on some extremal fuzzy time intervals $\widetilde{r}_* \in \widetilde{\mathscr{B}}_{T*}$ and $\widetilde{r}^* \in \widetilde{\mathscr{B}}^*_T$, if $\exists \widetilde{r}_{*\infty} \in \widetilde{\mathscr{B}}_{T*}$ and $\widetilde{r}^*_\infty \in \widetilde{\mathscr{B}}^*_T$ extremal fuzzy time intervals such that $\forall \varepsilon > 0$

$$\begin{aligned}
&\lim_{\tau\to+\infty} \widetilde{g}_{T*}\left(\widetilde{r}_* \cap \left\{t \in T \mid |\mu_{\widetilde{r}_{\tau*}}(t) - \mu_{\widetilde{r}_{*\infty}}(t)| \geq \varepsilon\right\}\right) = 0,\\
&\lim_{\tau\to+\infty} \widetilde{g}^*_T\left(\widetilde{r}^* \cup \left\{t \in T \mid |\mu_{\widetilde{r}^*_\tau}(t) - \mu_{\widetilde{r}^*_\infty}(t)| < \varepsilon\right\}\right) = 1.
\end{aligned} \tag{43}$$

Definition 31 (g_U-Ergodicity). We say that the extremal fuzzy control process $(\widetilde{u}_*\widetilde{u}^*)$ is g_U-ergodic on some fuzzy control $\widetilde{u} \in \widetilde{\mathscr{B}}_U$ if there exist extremal fuzzy controls $\widetilde{u}_\infty$ and $\widetilde{u}^\infty \in \widetilde{\mathscr{B}}_U$ such that $\forall \varepsilon > 0$ and $\forall t \in T$

$$\begin{aligned}
&\lim_{\tau\to+\infty} \widetilde{g}_U\left(\widetilde{u} \cap \left\{u \in U \mid |\mu_{E_{\widetilde{u}_*}(\cdot,\tau)}(u) - \mu_{\widetilde{u}_\infty}(u)| \geq \varepsilon\right\}\right) = 0,\\
&\lim_{\tau\to+\infty} \widetilde{g}_U\left(\widetilde{u} \cup \left\{u \in U \mid |\mu_{E_{\widetilde{u}^*}(\cdot,\tau)}(u) - \mu_{\widetilde{u}^\infty}(u)| < \varepsilon\right\}\right) = 1.
\end{aligned} \tag{44}$$

Analogously to Theorem 6, we formulate the statement that the extremal fuzzy process $(\widetilde{Q}_*, \widetilde{Q}^*)$ of describing the EFCDS state change dynamics is ergodic, where the ergodicity of the processes $(\widetilde{r}_{\tau *}, \widetilde{r}^*_\tau)_{\tau \geq 0}$ and $(\widetilde{u}_*, \widetilde{u}^*)$ is replaced by the g-ergodicity.

Theorem 7. *Let the process $(\widetilde{r}_{\tau *}, \widetilde{r}^*_\tau)_{\tau \geq 0}$ be g_T-ergodic on T with limit extremal fuzzy time intervals $(\widetilde{r}_{*\infty}, \widetilde{r}^*_\infty)$, and $(\widetilde{u}_*, \widetilde{u}^*)$ be g_U-ergodic on U with limit fuzzy controls $(\widetilde{u}_\infty, \widetilde{u}^\infty)$ so that the extended fuzzy measures $\widetilde{g}_{T*}$, $\widetilde{g}^*_T$ be self-continuous. Then the extremal fuzzy process $(\widetilde{Q}_*, \widetilde{Q}^*)$ of describing the EFCDS state change dynamics is ergodic.*

To conclude the section, we say that under the action of the extremal fuzzy control process $(\widetilde{u}_*, \widetilde{u}^*)$ on the EFCDS, the extremal fuzzy process $(\widetilde{Q}_*, \widetilde{Q}^*)$ of describing the EFCDS state change dynamics is ergodic if

a) the processes $(\widetilde{r}_{\tau *}, \widetilde{r}^*_\tau)_{\tau \geq 0}$ and $(\widetilde{u}_*, \widetilde{u}^*)$ are ergodic on T and U, respectively, or

b) the processes $(\widetilde{r}_{\tau *}, \widetilde{r}^*_\tau)_{\tau \geq 0}$ and $(\widetilde{u}_*, \widetilde{u}^*)$ are g-ergodic on T and U, respectively, the extended extremal fuzzy measures $\widetilde{g}_{T*}$ and $\widetilde{g}^*_T$ are respectively lower self-continuous and upper self-continuous, and the extended measure $\widetilde{g}_U$ is self-continuous on $\widetilde{\mathscr{B}}_U$.

Conclusions. Using the results obtained in [31]–[33] of this study, we have considered questions of fuzzy mathematical modeling of extremal fuzzy processes, where

a) we introduce the notion of an EFCDS with fuzzy uncertainty, the source of which is expert reflections on the states of EFCSD (“expert measurement”) in the so-called current and future fuzzy time intervals. The general EFCDS model is described;

b) the notion of processes of expert reflection and description of the EFCDS state change dynamics are introduced. With the aid of the conditional extremal fuzzy measures $g_{t*}(\cdot \mid x)$ and $g^*_t(\cdot, \mid x)$, the extremal fuzzy expert reflection process $(\widetilde{\mathbb{R}}_*, \widetilde{\mathbb{R}}^*)$ connects the fuzzy time interval measurement process $(\widetilde{r}_{\tau *}, \widetilde{r}^*_\tau)_{\tau \geq 0}$ with the space of measurable states of the system with fuzzy distribution $(X, \mathscr{B}, g)$, while the EFCDS state description process $(\widetilde{Q}_*, \widetilde{Q}^*)$ is defined through the extremal fuzzy expert reflection process $(\widetilde{\mathbb{R}}_*, \widetilde{\mathbb{R}}^*)$, using the extended upper and lower Sugeno integrals that are considered as extremal operators describing the EFCDS state dynamics;

c) questions of the ergodicity of extremal fuzzy processes are studied. The notion of g-ergodicity is introduced, which allows one to obtain a sufficient condition for the process $(\widetilde{Q}_*, \widetilde{Q}^*)$ to be ergodic;

d) the notion of an extremal fuzzy control process $(\widetilde{u}_*, \widetilde{u}^*)$ is introduced in the case of the action of control with fuzzy restrictions in the form of the space $(U, \mathscr{B}_U, g_U)$. Models of continuous extremal controllable fuzzy processes are constructed. Questions of the ergodicity of controllable extremal fuzzy processes are studied.

5 The Fuzzy Dynamic Programming Problem

All definitions and results see in [35], [37], [39].

In alternative classical approaches to modeling and when working with the EFCDS the main accent is often placed on the assumption of fuzzyness. We will deal with fuzzy dynamic systems, where fuzzy uncertainty arises with time and time structures are monotone classes of measurable sets.

We start describing objects of a fuzzy dynamic system. Let X $(X \neq \varnothing)$ be the set of states of some system (EFCDS) to be investigated. Let $(X, \mathscr{B}, g)$ be the space of a fuzzy measure on the measurable space $(X, \mathscr{B})$, where $\mathscr{B}$ is a σ-algebra in X (fuzzy restrictions on states).

Let the time structure of fuzzy dynamic system (EFCDS) be represented by (27) and some space of extended extremal fuzzy measures

$$(T, \widetilde{\mathscr{B}}_{T*}, \widetilde{\mathscr{B}}^*_T, \widetilde{g}_{T*}, \widetilde{g}^*_T), \quad T = \mathbb{R}^*_0,$$

where $\widetilde{g}_{T*}$ and $\widetilde{g}^*_T$ are some extremal fuzzy measures on $\widetilde{\mathscr{B}}_{T*} \equiv \widetilde{\mathscr{B}}_{2*}$ and $\widetilde{\mathscr{B}}^*_T \equiv \widetilde{\mathscr{B}}^*_2$, respectively.

Let U $(U \neq \varnothing)$ be the set of all admissible controls (of external factors) acting on the EFCDS. Assume that controls are subjected to restrictions of uncertain character in the form of some space of a fuzzy measure $(U, \mathscr{B}_U, g_U)$, where $\mathscr{B}_U$ is the measurable space of controls, while the fuzzy measure g_U describes the restrictions imposed on controls.

We consider the optimization problems of EFCDS when the model of the continuous extremal controllable fuzzy process is described by the system of fuzzy integral equations ([33] and Section 4):

$$\begin{cases} \mu_{\widetilde{Q}_*}(x,\tau) = \displaystyle\int_{*\,U\times T} \left\{ \mu_{\mathbb{E}_{\widetilde{u}_*}(\cdot,\tau)}(u) \wedge \mu_{\mathbb{E}_{\widetilde{\rho}'_*}(x,\cdot,\cdot)}(u,t) \right\} \circ \widetilde{g_U} \otimes \widetilde{g_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}}(\cdot), \\ \mu_{\widetilde{Q}^*}(x,\tau) = \displaystyle\int^*_{U\times T} \left\{ \mu_{\mathbb{E}_{\widetilde{u}^*}(\cdot,\tau)}(u) \vee \mu_{\mathbb{E}_{\widetilde{\rho}'^*}(x,\cdot,\cdot)}(u,t) \right\} \circ \widetilde{g^*_U} \otimes \widetilde{g_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}}(\cdot), \end{cases} \tag{45}$$

where $(\widetilde{Q}_*, \widetilde{Q}^*)$ is a fuzzy extremal process describing the system state dynamics; $(\widetilde{\mathbb{R}}_*, \widetilde{\mathbb{R}}^*)$ is an extremal fuzzy process of expert knowledge reflections in extremal fuzzy time intervals (the expert reflections on the states of EFCDS in the extremal fuzzy time intervals); $(\widetilde{\rho}_*, \widetilde{\rho}^*)$ is the transition operator of the EFCDS states; on right-hand sides of Sugeno extended lower and upper integrals the integration measures are the extremal compositional fuzzy measures extended with respect to the process $(\widetilde{\mathbb{R}}_*, \widetilde{\mathbb{R}}^*)$; μ is a symbol of a compatibility function of a fuzzy set; $\mathbb{E}$ is a symbol of projector of Galois indexing mapping.

We say that the effectiveness of EFCDS control is defined by some set of Criteria K, on which fuzzy restrictions are given for measurable subsets of K, i.e. the fuzzy measure space $(K, \mathscr{B}_K, g_K)$ (fuzzy restriction on the criteria) is defined on K [35].

Let $\widetilde{L} \in \widetilde{\mathscr{B}_K \otimes \mathscr{B}_U}$ be some fuzzy binary relation of "losses" with respect to each of the criteria $v \in K$ in the choice of control $u \in U$. Note that $\mu_{\widetilde{L}}$ is a $\mathscr{B}_K \otimes \mathscr{B}_U$-measurable compatibility function

$$\mu_{\widetilde{L}}(v,u) : K \times U \to [0,1]. \tag{46}$$

Then the complement $\overline{\widetilde{L}}$ is called the fuzzy relation of EFCDS "gain" and the values

$$\mu_{\overline{\widetilde{L}}}(v,u) = 1 - \mu_{\widetilde{L}}(v,u) \tag{47}$$

define the measure of gain in the choice of control $u \in U$ for a criterion $v \in K$.

Definition 32. a) Given all criteria, a $\mathscr{B}_U \otimes \mathscr{B}_T^*$-measurable function: $\forall (u,t) \in U \times T$

$$\mathbf{P}_{\widetilde{u}^*}^K(u,t) \stackrel{\Delta}{=} \int_K \left\{ \mu_{\mathbb{E}_{\widetilde{u}^*}(\cdot,t)}(u) \vee \mu_0^*(u) \vee \mu_{\overline{\widetilde{L}}}(v,u) \right\} \circ \widetilde{g}_K^*(\cdot), \tag{48}$$

where the extended fuzzy measure $\widetilde{g}_K^* : \mathscr{B}_K \to [0,1]$ is the dual fuzzy measure of $\widetilde{g}_K$ ($\forall \widetilde{S} \in \widetilde{\mathscr{B}}_K : \widetilde{g}_K^*(\widetilde{S}) = 1 - \widetilde{g}_K(\overline{\widetilde{S}})$), is called a gain with respect to a current (upper) fuzzy control process $\widetilde{u}^* \in \widetilde{\mathscr{B}_U \otimes \mathscr{B}_T^*}$ with respect to the initial fuzzy control $\mu_{\mathbb{E}_{\widetilde{u}^*}(\cdot,\tau_0)}(u) \equiv \mu_0^*(u)$.

b) Given all criteria, a $\mathscr{B}_U \otimes \mathscr{B}_{T*}$-measurable function: $\forall (u,t) \in U \times T$

$$\mathbf{q}_{\widetilde{u}_*}^K(u,t) \stackrel{\Delta}{=} \int_K \left\{ \mu_{\mathbb{E}_{\widetilde{u}_*}(\cdot,t)}(u) \wedge \mu_{0*}(u) \wedge \mu_{\widetilde{L}}(v,u) \right\} \circ \widetilde{g}_K(\cdot) \tag{49}$$

is called a loss with respect to a future (lower) fuzzy control process $\widetilde{u}_* \in \widetilde{\mathscr{B}_U \otimes \mathscr{B}_{T*}}$ with respect to the initial fuzzy control $\mu_{\mathbb{E}_{\widetilde{u}_*}(\cdot,\tau_0)}(u) \equiv \mu_{0*}(u)$.

Definition 33. a) A $\mathscr{B} \otimes \mathscr{B}_T^*$-measurable function: $\forall (u,\tau) \in U \times T$

$$I_{\widetilde{u}^*}(u,\tau) \stackrel{\Delta}{=} \int_T^* \mathbf{P}_{\widetilde{u}^*}^K(u,t) \circ \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}(\cdot) \tag{50}$$

is called an integral current gain with respect to a current (upper) fuzzy control process $\widetilde{u}^* \in \widetilde{\mathscr{B}_U \otimes \mathscr{B}_T^*}$ on a current fuzzy time interval $\widetilde{r}_\tau^* \in \widetilde{\mathscr{B}}_T^*$.

b) A $\mathscr{B}_U \otimes \mathscr{B}_{T*}$-measurable function: $\forall (u,\tau) \in U \times T$

$$J_{\widetilde{u}_*}(u,\tau) \stackrel{\Delta}{=} \int_{*T} \mathbf{q}_{\widetilde{u}_*}^K(u,t) \circ \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}(\cdot) \tag{51}$$

is called an integral future loss with respect to a future (upper) fuzzy control process $\widetilde{u}_* \in \widetilde{\mathscr{B}_U \otimes \mathscr{B}_{T*}}$ on a future fuzzy time interval $\widetilde{r}_{\tau*} \in \widetilde{\mathscr{B}}_{T*}$.

We have thus defined, on U, an extremal fuzzy "gain-loss" process $(I_{u^*}, \dot{J}_{u_*})$. Further, for model (45) we will consider, in terms of (50) and (51), the problem of formation of an optimal control (in the sense of minimization of the future loss and maximization of the current gain) of an extremal process: $\forall(u,t) \in U \times T$

$$\begin{aligned} &\int_T^{*} \mathbf{P}^K_{\widetilde{u}^*}(u,t) \circ \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}(\cdot) \Rightarrow \max_{\widetilde{u}^*}, \\ &\int_{*T} \mathbf{q}^K_{\widetilde{u}_*}(u,t) \circ \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}(\cdot) \Rightarrow \min_{\widetilde{u}_*}. \end{aligned} \tag{52}$$

Functional equations by means of which we can define an extremal fuzzy optimal control in the sense of extremalization of criteria (52) can be written in the following form, $\forall(u,\tau') \in U \times [\tau_0,\tau]$:

$$\begin{cases} J_{\overset{\circ}{\widetilde{u}}_*}(u,\tau') = \underset{\widetilde{u}_* \in \widetilde{\mathscr{B}_U \otimes \mathscr{B}_{T*}}}{\wedge} J_{\widetilde{u}_*}(u,\tau') = \underset{\widetilde{u}_* \in \widetilde{\mathscr{B}_U \otimes \mathscr{B}_{T*}}}{\wedge} \int_{*T} \mathbf{q}^K_{\widetilde{u}_*}(u,t) \circ \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau')}(\cdot), \\ I_{\overset{\circ}{\widetilde{u}}^*}(u,\tau') = \underset{\widetilde{u}_* \in \widetilde{\mathscr{B}_U \otimes \mathscr{B}^*_T}}{\vee} I_{\widetilde{u}^*}(u,\tau') = \underset{\widetilde{u}^* \in \widetilde{\mathscr{B}_U \otimes \mathscr{B}^*_T}}{\vee} \int_T^{*} \mathbf{P}^K_{\widetilde{u}^*}(u,t) \circ \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\tau')}(\cdot), \end{cases} \tag{53}$$

with the initial control conditions

$$\mathbb{E}_{\overset{\circ}{\widetilde{u}}_*}(\cdot,\tau_0) \equiv \widetilde{u}_{0*} \in \mathscr{B}_U, \quad \mathbb{E}_{\overset{\circ}{\widetilde{u}}^*}(\cdot,\tau_0) \equiv \widetilde{u}^*_0 \in \mathscr{B}_U \tag{54}$$

and the EFCDS initial states $\mathbb{E}_{\widetilde{Q}_*}(\cdot,\tau_0)$ and $\mathbb{E}_{\widetilde{Q}^*}(\cdot,\tau_0)$.

Definition 34. An extremal fuzzy control process $(\overset{\circ}{\widetilde{u}}_*, \overset{\circ}{\widetilde{u}}^*)$, $\tau_0 \le \tau' \le \tau$, with the initial conditions (54) is called an optimal for EFCDS (45) in the sense of Bellman's optimality principle if criterion (53) is satisfied.

The following theorem which gives the optimality condition (an analogue of Bellman's equation [1]) is valid.

Theorem 8. *Let a EFCDS be described by system* (45). *Then an extremal fuzzy control process* $(\overset{\circ}{\widetilde{u}}_*, \overset{\circ}{\widetilde{u}}^*)$, $\tau_0 \le \tau' \le \tau$, *is optimal in the sense of criterion* (53) *if and only if the following inequalities are fulfilled:* $\forall(u,\tau') \in U \times [\tau_0,\tau]$

$$\begin{cases} J_{\overset{\circ}{\widetilde{u}}_*}(u,\tau') \le \left(\int_K \mu_{\widetilde{\underline{L}}}(v,u) \circ \widetilde{g}_K(\cdot) \right) \wedge \mu_{\mathbb{E}_{\overset{\circ}{\widetilde{u}}_*}(\cdot,\tau_0)}(u), \\ I_{\overset{\circ}{\widetilde{u}}^*}(u,\tau') \ge \left(\int_K \mu_{\widetilde{\overline{L}}}(v,u) \circ \widetilde{g}^*_K(\cdot) \right) \vee \mu_{\mathbb{E}_{\overset{\circ}{\widetilde{u}}^*}(\cdot,\tau_0)}(u); \end{cases} \tag{55}$$

Theorem 9. *An extremal fuzzy optimal control process* $(\overset{\circ}{\widetilde{u}}_*, \overset{\circ}{\widetilde{u}}^*)$ *for the EFCDS* (45) *in the sense of criterion* (53) *not depending on a EFCDS state can be defined by the following system of fuzzy-integral equations:* $\forall(u,\tau') \in U \times [\tau_0,\tau]$

$$\begin{cases} \mu_{\overset{\circ}{\widetilde{u}}_*}(u,\tau') = \mu_{\overset{\circ}{\widetilde{u}}_*}(u,\tau_0) \wedge \left(\underset{K}{\int\!\!\!\!/} \mu_{\widetilde{L}}(v,u) \circ \widetilde{g}_K(\cdot) \right) \wedge \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\Delta(\tau_0,\tau'))}(T), \\ \mu_{\overset{\circ}{\widetilde{u}}^*}(u,\tau') = \mu_{\overset{\circ}{\widetilde{u}}^*}(u,\tau_0) \vee \left(\underset{K}{\int\!\!\!\!/} \mu_{\overline{\widetilde{L}}}(v,u) \circ \widetilde{g}_K^*(\cdot) \right) \vee \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\Delta(\tau_0,\tau'))}(T). \end{cases} \tag{56}$$

Remark 2. Expressions in (56) of an extremal optimal fuzzy control process $(\overset{\circ}{\widetilde{u}}_*, \overset{\circ}{\widetilde{u}}^*)$, $\tau_0 \le \tau' \le \tau$, are a variant of the solution of inequalities (55), but this fuzzy-integral representation of an optimal control gives a good analogue of the solution of the problem of stochastic dynamic programming, where the expression of an optimal control contains "direct" analogues to (56): $\underset{K}{\int\!\!\!\!/} \mu_{\widetilde{L}}(v,u) \circ g_K(\cdot)$ is the Bellman functional which is an analogue of the kernel in the representation of a stochastic optimal control or, more exactly, an analogue of the signal of a stochastic model or its deterministic part, while the values of the extended fuzzy measures $\widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\Delta(\tau_0,\tau'))}(T)$ and $\widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\Delta(\tau_0,\tau'))}(T)$ are analogues of stochastic measure in the representation of stochastic optimal controls.

The case where a fuzzy control of EFCDS depends not only on time $\tau' \in [0,\tau]$ but also on a EFCDS state $x \in X$ is also studied but is omitted here.

5.1 Example

Let the set of EFCDS states be finite, $X = \{1,2,3,4\}$; $g^* : 2^X \to [0,1]$ be the possibility measure with the possibility distribution on X

$$\Pi(i) \triangleq \frac{i}{4}, \quad i = 1,2,3,4 \quad \left(\forall B \in 2^X : g^*(A) = \bigvee_{i \in A} \pi(i) \right).$$

Let the EFCDS be subjected to the influence of an external control factor with the finite set $U = \{u_1, u_2\}$ (for example, $u_1 \triangleq$"+1", $u_2 \triangleq$"−1"). Let the uniform probability distribution play the role of the fuzzy measure $g_U : 2^U \to [0,1]$, i.e. $g_U(\{u_1\}) = g_U(\{u_2\}) = \frac{1}{2}$. The two-element set $K = \{v_1, v_2\}$ is taken as the set of chosen criteria, while the uniform probability distribution $g_K(\{v_1\}) = g_K(\{v_2\}) = \frac{1}{2}$ is considered as playing the role of the fuzzy measure $g_K : 2^K \to [0,1]$. Thus we have the fuzzy measure spaces $(X, 2^X, g)$, $(K, 2^K, g_K)$ and $(U, 2^U, g_U)$. The dual measure g^* on 2^X is the necessity measure $g(A) = 1 - \bigvee_{i \notin A} \pi(i)$. Since the fuzzy measures g_U and g_K are the probability ones, we know they are autodual and

$$g_U^* = g_U, \quad g_K^* = g_K.$$

It is assumed that the initial moment of EFCDS observation is $\tau_0 \equiv 0$. Let the initial extremal fuzzy distributions of an optimal control be

$$\mu_{\overset{\circ}{\widetilde{u}}_*}(u_1,0)=\frac{1}{2}=\mu_{\overset{\circ}{\widetilde{u}}^*}(u_1,0);\;\; \mu_{\overset{\circ}{\widetilde{u}}_*}(u_2,0)=\frac{1}{4}=\mu_{\overset{\circ}{\widetilde{u}}^*}(u_2,0).$$

Let the binary fuzzy loss relation $\widetilde{L}$ on $U\times K$ be defined as follows:

$$\mu_{\widetilde{L}}(u_1,v_1)=\mu_{\widetilde{L}}(u_2,v_2)=\frac{1}{2},\;\; \mu_{\widetilde{L}}(u_1,v_2)=\mu_{\widetilde{L}}(u_2,v_1)=\frac{1}{4}.$$

The distributions of extremal fuzzy time intervals are given as

$$\mu_{\widetilde{r}_{\tau *}}(t)=\begin{cases}0, & 0\le t\le \tau,\\ 1-\frac{\tau}{t}, & t>\tau,\end{cases}\qquad \mu_{\widetilde{r}^*_\tau}(t)=\begin{cases}1, & 0\le t<\tau,\\ \frac{\tau}{t}, & t\ge \tau.\end{cases} \tag{57}$$

Let the initial distribution ($\tau_0\equiv 0$) of the EFCDS state description process look like

$$\widetilde{A}_{0*}\sim\begin{pmatrix}1 & 2 & 3 & 4\\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2}\end{pmatrix},\quad \widetilde{A}^*_0\sim\begin{pmatrix}1 & 2 & 3 & 4\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{pmatrix}. \tag{58}$$

We consider the example of the space $(T,\widetilde{\mathscr{B}}_{T*},\widetilde{\mathscr{B}}^*_T,\widetilde{g}^*_T,\widetilde{g}^*_T)$ where

$$\begin{aligned} g_{T*}([t,+\infty))&\overset{\Delta}{=}\frac{1}{1+t},\quad [t,+\infty)\in\mathscr{B}_{T*},\\ g^*_T([0,t))&\overset{\Delta}{=}\frac{t}{1+t},\quad [0,t)\in\mathscr{B}^*_T,\quad t>0.\end{aligned} \tag{59}$$

Further, we introduce the conditional fuzzy measures on $\mathscr{B}_{T*}$ and $\mathscr{B}^*_T$ with respect to the set $X=\{1,2,3,4\}$:

$$\begin{aligned} g_{t*}(r_{\tau *}\mid i)&=\frac{1}{1+i\tau},\quad \text{where}\quad i\in X,\quad r_{\tau *}\in\mathscr{B}_{T*},\\ g^*_t(r^*_\tau\mid i)&=\frac{i\tau}{1+i\tau},\quad \text{where}\quad i\in X,\quad r^*_\tau\in\mathscr{B}^*_T.\end{aligned} \tag{60}$$

Thus the EFCDS state description process can be represented as follows:

$$\begin{cases}\mu_{\widetilde{Q}_*}(x,\tau)=\displaystyle\int_{*\,U\times T}\Big\{\mu_{\mathbb{E}_{\widetilde{u}_*}(\cdot,\tau)}(u)\wedge\mu_{\mathbb{E}_{\widetilde{\rho}'_*}(x,\cdot,\cdot)}(u,t)\Big\}\circ\widetilde{g_U}\otimes\widetilde{g_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}}(\cdot),\\ \mu_{\widetilde{Q}^*}(x,\tau)=\displaystyle\int^*_{U\times T}\Big\{\mu_{\mathbb{E}_{\widetilde{u}^*}(\cdot,\tau)}(u)\vee\mu_{\mathbb{E}_{\widetilde{\rho}'^*}(x,\cdot,\cdot)}(u,t)\Big\}\circ\widetilde{g_U}\otimes\widetilde{g_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}}(\cdot),\end{cases} \tag{61}$$

where $\widetilde{A}_{0*}\equiv\mathbb{E}_{\widetilde{Q}_*}(\cdot,0)$, $\widetilde{A}^*_0\equiv\mathbb{E}_{\widetilde{Q}^*}(\cdot,0)$, $(\widetilde{\mathbb{R}}_*,\widetilde{\mathbb{R}}^*)$ is the extremal fuzzy reflection process, $\forall(x,\tau)\in X\times T$, $\forall(x,t)\in U\times T$:

$$\begin{cases}\mu_{\widetilde{\mathbb{R}}_*}(x,\tau)\overset{\Delta}{=}\widetilde{g}_{t*}(\widetilde{r}_{\tau *}\mid x)=\mu_{\widetilde{A}_{\tau *}}(x),\\ \mu_{\widetilde{\mathbb{R}}^*}(x,\tau)\overset{\Delta}{=}\widetilde{g}^*_t(\widetilde{r}^*_\tau\mid x)=\mu_{\widetilde{A}^*_\tau}(x),\end{cases} \tag{62}$$

and

$$\begin{cases} \mu_{\widetilde{\rho}'_*}(x,u,t) \overset{\Delta}{=} \displaystyle\int\limits_{\widetilde{A}_{0*}} \mu_{\widetilde{\rho}_*}(x,u,x',t) \circ \widetilde{g}(\cdot), \\ \mu_{\widetilde{\rho}'^*}(x,u,t) \overset{\Delta}{=} \displaystyle\int\limits_{\widetilde{A}_0^*} \mu_{\widetilde{\rho}^*}(x,u,x',t) \circ \widetilde{g}^*(\cdot), \end{cases} \tag{63}$$

where $\widetilde{A}_{\tau*} \in \widetilde{\mathscr{B}}$ and $\widetilde{A}_\tau^* \in \mathscr{B}$ are expert reflections on the EFCDS states in the fuzzy extremal intervals $\widetilde{r}_{\tau*} \in \widetilde{\mathscr{B}}_{T*}$ and $\widetilde{r}_\tau^* \in \widetilde{\mathscr{B}}_T^*$, respectively; $(\widetilde{\rho}_*, \widetilde{\rho}^*)$ is the EFCDS transition operator (see [34]). As known the operator $(\widetilde{\rho}'_*, \widetilde{\rho}'^*)$ is restored from the experimental-expert knowledge base on the EFCDS so that if we fix some admissible extremal control process $(\widetilde{u}_*, \widetilde{u}^*)$ (including an optimal control too), then, using the calculation procedure for Sugeno extremal integrals [34], we can write expressions for the process $(\widetilde{Q}_*, \widetilde{Q}^*)$. However we pursue a different aim here: using EFCDS data, we are to construct the extremal optimal control process $(\widetilde{\overset{\circ}{u}}_*, \widetilde{\overset{\circ}{u}}^*)$.

Since the sets X, U, K are finite, it is not difficult to check that the conditions (55) of existence of an optimal extremal control process are satisfied. By virtue of the results of Theorems 8 and 9, we can write one of the variants for an extremal optimal fuzzy control process as follows: $\forall (u,\tau) \in (X,T)$

$$\begin{cases} \mu_{\widetilde{\overset{\circ}{u}}_*}(u,\tau) = \mu_{\widetilde{\overset{\circ}{u}}_*}(u,0) \wedge \left(\displaystyle\int\limits_K \mu_{\widetilde{L}}(u,v) \circ \widetilde{g}_K(\cdot) \right) \wedge \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}(T), \\ \mu_{\widetilde{\overset{\circ}{u}}^*}(u,\tau) = \mu_{\widetilde{\overset{\circ}{u}}^*}(u,0) \vee \left(\displaystyle\int\limits_K \mu_{\overline{\widetilde{L}}}(u,v) \circ \widetilde{g}_K^*(\cdot) \right) \vee \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}(T), \end{cases} \tag{64}$$

where $u \in \{\text{“}+1\text{”}, \text{“}-1\text{”}\}$, $v \in \{v_1, v_2\}$; $\mu_{\widetilde{\overset{\circ}{u}}_*}(u,0)$ and $\mu_{\widetilde{\overset{\circ}{u}}^*}(u,0)$ are already defined, while the extended extremal fuzzy measures are defined in the form:

$$\begin{cases} \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}(T) = \displaystyle\int\limits_{T}{}_{*} \mu_{\widetilde{r}_{\tau*}}(t) \circ \widetilde{g}_{T*}(\cdot) \overset{\Delta}{=} \displaystyle\int\limits_{T}{}_{*} \mu_{\widetilde{r}_{\tau*}}(t) \circ \displaystyle\int\limits_X g_{t*}(\cdot \mid x) \circ g(\cdot), \\ \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}(T) = \displaystyle\int\limits_{T}{}^{*} \mu_{\widetilde{r}_\tau^*}(t) \circ \widetilde{g}_T^*(\cdot) \overset{\Delta}{=} \displaystyle\int\limits_{T}{}^{*} \mu_{\widetilde{r}_\tau^*}(t) \circ \displaystyle\int\limits_X g_t^*(\cdot \mid x) \circ g^*(\cdot). \end{cases} \tag{65}$$

Now we are to calculate the Sugeno integrals in formulas (64) and the values of extremal fuzzy measures (65).

Let us calculate the values of $\displaystyle\int_K \mu_{\widetilde{L}}(u,v) \circ \widetilde{g}_K(\cdot)$:

1) $u = u_1 \equiv$"+1":

$$\int_K \mu_{\widetilde{L}}(u_1,v) \circ \widetilde{g}_K(\cdot) = \bigwedge_{0<\alpha\le 1} \left\{ \alpha \vee g_K(v \in K \mid \mu_{\widetilde{L}}(u_1,v) \ge \alpha \right\}$$

$$= \Big[\bigwedge_{0\le\alpha\le\frac{1}{4}} (\alpha \vee g_K(K)) \Big] \wedge \Big[\bigwedge_{\frac{1}{4}\le\alpha\le\frac{1}{2}} (\alpha \vee g_K(\{v_2\})) \Big]$$

$$\wedge \Big[\bigwedge_{\frac{1}{2}<\alpha\le 1} (\alpha \vee g_K(\varnothing)) \Big] = 1 \wedge \frac{1}{2} \wedge \frac{1}{2} = \frac{1}{2}.$$

2) $u = u_2 =$"-1":

$$\int_K \mu_{\widetilde{L}}(u_2,v) \circ \widetilde{g}_K(\cdot) = \bigwedge_{0<\alpha\le 1} \left\{ \alpha \vee g_K(v \in K \mid \mu_{\widetilde{L}}(u_2,v) \ge \alpha \right\}$$

$$= \Big[\bigwedge_{0\le\alpha\le\frac{1}{4}} (\alpha \vee g_K(K)) \Big] \wedge \Big[\bigwedge_{\frac{1}{4}\le\alpha\le\frac{1}{2}} (\alpha \vee g_K(\{v_1\})) \Big] \wedge \Big[\bigwedge_{\frac{1}{2}<\alpha\le 1} (\alpha \vee g_K(\varnothing)) \Big]$$

$$= 1 \wedge \Big[\bigwedge_{\frac{1}{4}\le\alpha\le\frac{1}{2}} \Big(\alpha \vee \frac{1}{2}\Big) \Big] \wedge \Big[\bigwedge_{\frac{1}{2}\le\alpha<1} (\alpha) \Big] = 1 \wedge \frac{1}{2} \wedge \frac{1}{2} = \frac{1}{2}.$$

Since

$$\int_K \mu_{\overline{\widetilde{L}}}(u,v) \circ \widetilde{g}^*_K(\cdot) = 1 - \int_K \mu_{\widetilde{L}}(u,v) \circ \widetilde{g}_K(\cdot),$$

we have

$$\int_K \mu_{\overline{\widetilde{L}}}(u_1,v) \circ \widetilde{g}^*_K(\cdot) = \int_K \mu_{\overline{\widetilde{L}}}(u_2,v) \circ \widetilde{g}_K(\cdot) = \frac{1}{2}\,.$$

Therefore $\forall \tau > 0$

$$\mu_{\overset{\circ}{\widetilde{u}}_*}(u_1,\tau) = \frac{1}{2} \wedge \frac{1}{2} \wedge \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}(T) = \frac{1}{2} \wedge \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}(T),$$
$$\mu_{\overset{\circ}{\widetilde{u}}_*}(u_2,\tau) = \frac{1}{4} \wedge \frac{1}{2} \wedge \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}(T) = \frac{1}{4} \wedge \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}(T),$$
$$\mu_{\overset{\circ}{\widetilde{u}}^*}(u_1,\tau) = \frac{1}{2} \vee \frac{1}{2} \vee \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}(T) = \frac{1}{2} \vee \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}(T),$$
$$\mu_{\overset{\circ}{\widetilde{u}}^*}(u_2,\tau) = \frac{1}{4} \vee \frac{1}{2} \vee \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}(T) = \frac{1}{2} \vee \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}(T).$$

Now we are to calculate the values of the so-called extremal fuzzy "white noise" (65):

$$\widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}(T) = \int_T^* \mu_{\widetilde{r}_{\tau*}}(t) \circ \int_X g_{t*}(\cdot \mid x) \circ g(\cdot)$$
$$= \bigvee_{0<\alpha\le 1} \{\alpha \wedge \widetilde{g}_{T*}([\widetilde{r}_{\tau*}]_{\overline{\alpha}})\} = \bigvee_{0<\alpha\le 1} \left\{ \alpha \wedge \int_X \widetilde{g}_{t*}([\widetilde{r}_{\tau*}]_{\overline{\alpha}} \mid x) \circ g(\cdot) \right\}.$$

From (57) we obtain the expression for an α-cut for $\widetilde{r}_{\tau *}$:

$$[\widetilde{r}_{\tau *}]_{\overline{\alpha}} = \left\{ \begin{array}{ll} T & \text{if } \alpha = 0, \\ \left[\frac{\tau}{1-\alpha}, +\infty\right) & \text{if } 0 < \alpha < 1, \\ \varnothing & \text{if } \alpha = 1, \end{array} \right\} \in \mathscr{B}_{T*}.$$

Now (60) implies

$$\widetilde{g}_{t*}([\widetilde{r}_{\tau *}]_{\overline{\alpha}} \mid i) = \begin{cases} 1 & \text{if } \alpha = 0, \\ \frac{1}{1+i\frac{\tau}{1-\alpha}} & \text{if } 0 < \alpha < 1, \quad \forall i \in X. \\ \varnothing & \text{if } \alpha = 1, \end{cases}$$

and

$$\int_X \widetilde{g}_t([\widetilde{r}_{\tau *}]_{\overline{\alpha}} \mid i) \circ g(\cdot) = \bigvee_{0<\beta\leq 1} \left\{ \beta \wedge g\left(\left\{ i \in X \mid \frac{1}{1+i\frac{\tau}{1-\alpha}} \geq \beta \right\} \right) \right\}.$$

It is not difficult to verify that $(0 < \alpha < 1,\ \tau > 0)$

$$\left\{ i \in X \mid \frac{1}{1+i\frac{\tau}{1-\alpha}} \geq \beta \right\} = \begin{cases} \varnothing & \text{if } 1 \geq \beta > \frac{1-\alpha}{1-\alpha+\tau}, \\ \{1\} & \text{if } \frac{1-\alpha}{1-\alpha+\tau} \geq \beta > \frac{1-\alpha}{1-\alpha+2\tau}, \\ \{1,2\} & \text{if } \frac{1-\alpha}{1-\alpha+2\tau} \geq \beta > \frac{1-\alpha}{1-\alpha+3\tau}, \\ \{1,2,3\} & \text{if } \frac{1-\alpha}{1-\alpha+3\tau} \geq \beta > \frac{1-\alpha}{1-\alpha+4\tau}, \\ X & \text{if } \frac{1-\alpha}{1-\alpha+4\tau} \geq \beta > 0. \end{cases}$$

Denote $B_0 \equiv \left(\frac{1-\alpha}{1-\alpha+\tau}; 1\right]$, $B_1 \equiv \left(\frac{1-\alpha}{1-\alpha+2\tau}; \frac{1-\alpha}{1-\alpha+\tau}\right]$, $B_2 \equiv \left(\frac{1-\alpha}{1-\alpha+3\tau}; \frac{1-\alpha}{1-\alpha+2\tau}\right]$, $B_3 \equiv \left(\frac{1-\alpha}{1-\alpha+4\tau}; \frac{1-\alpha}{1-\alpha+3\tau}\right]$, $B_4 \equiv \left(0; \frac{1-\alpha}{1-\alpha+4\tau}\right]$.

Then

$$\begin{aligned} \int_X \widetilde{g}_{t*}([\widetilde{r}_{\tau *}]_{\overline{\alpha}} \mid x) \circ g(\cdot) &= \left[\bigvee_{\beta \in B_0} (\beta \wedge g(\varnothing)) \right] \vee \left[\bigvee_{\beta \in B_1} (\beta \wedge g(\{1\}) \right] \\ &\vee \left[\bigvee_{\beta \in B_2} (\beta \vee g(\{1,2\})) \right] \vee \left[\bigvee_{\beta \in B_3} (\beta \vee g(\{1,2,3\})) \right] \vee \left[\bigvee_{\beta \in B_4} (\beta \wedge g(X)) \right] \\ &= 0 \vee \left[\bigvee_{\beta \in B_1} (\beta \wedge 0) \right] \vee \left[\bigvee_{\beta \in B_2} (\beta \wedge 0) \right] \vee \left[\bigvee_{\beta \in B_3} (\beta \wedge 0) \right] \\ &\vee \left[\bigvee_{\beta \in B_4} (\beta \wedge 1) \right] = \bigvee_{\beta \in B_4} \beta = \frac{1-\alpha}{1-\alpha+4\tau}. \end{aligned}$$

We finally obtain

$$\widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}(T) = \bigvee_{0<\alpha<1} \left\{ \alpha \wedge \int_X \widetilde{g}_{t*}([\widetilde{r}_{\tau *}]_{\overline{\alpha}} \mid x) \circ g(\cdot) \right\} = \bigvee_{0<\alpha<1} \left\{ \alpha \wedge \frac{1-\alpha}{1-\alpha+4\tau} \right\}.$$

After studying the function in the braces with respect to α, we can continue calculations:

$$\widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}(T) = \begin{cases} \bigvee_{0<\alpha<1}\{\alpha\} = 1 & \text{if } 0<\tau\leq 1, \\ \bigvee_{\alpha\in[1;2\tau-1-2\sqrt{\tau(\tau-1)}]}\{\alpha\} & \\ \quad = 2\tau-1-2\sqrt{\tau(\tau-1)} & \text{if } \tau>1. \end{cases}$$

Since $\widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}(\cdot)$ and $\widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}(\cdot)$ are extended extremal measures, we have

$$\widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}(T) = \begin{cases} 0 & \text{if } 0<\tau\leq 1, \\ 2+2\sqrt{\tau(\tau-1)} & \text{if } \tau>1. \end{cases}$$

For an optimal control we obtain the following expressions:

$$\mu_{\overset{\circ}{\widetilde{u}}_*}(u_1,\tau) = \begin{cases} \frac{1}{2}, & 0<\tau\leq 1, \\ \frac{1}{2}\wedge(2\tau-1-2\sqrt{\tau(\tau-1)}), & \tau>1, \end{cases}$$

$$\mu_{\overset{\circ}{\widetilde{u}}_*}(u_2,\tau) = \begin{cases} \frac{1}{4}, & 0<\tau\leq 1, \\ \frac{1}{4}\wedge(2\tau-1-2\sqrt{\tau(\tau-1)}), & \tau>1, \end{cases}$$

$$\mu_{\overset{\circ}{\widetilde{u}}^*}(u_1,\tau) = \begin{cases} \frac{1}{2}, & 0<\tau\leq 1, \\ \frac{1}{2}\vee(2+2\sqrt{\tau(\tau-1)}-2\tau), & \tau>1 \end{cases} = \mu_{\overset{\circ}{\widetilde{u}}^*}(u_2,\tau).$$

Note that when $\tau\to+\infty$ a current description process of fuzzy time intervals extends unlimitedly, while a future description process of fuzzy time intervals vanishes. The latter fact is reflected in the expressions for the fuzzy optimal extremal controls:

$$\begin{cases} \lim\limits_{\tau\to\infty}\mu_{\overset{\circ}{\widetilde{u}}^*}(u,\tau)\to 1, & u\in U=\{u_1,u_2\}, \\ \lim\limits_{\tau\to\infty}\mu_{\overset{\circ}{\widetilde{u}}_*}(u,\tau)\to 0, & u\in U=\{u_1,u_2\}. \end{cases}$$

i.e. the uncertainty for a current fuzzy control process vanishes, while a future fuzzy optimal control process is not considered.

We have thereby finished the consideration of the example.

6 Conclusion

Using the results presented in the papers [31]–[39], we have considered questions of the fuzzy optimization of extremal processes, where:

a) the basic properties of Sugeno's type extremal fuzzy measure and several variants of its representations are considered;

b) the notions of extremal fuzzy time moments and intervals are introduced and their monotone algebraic structures are defined. The dualization of a time structure forms the most important part of the fuzzy instrument of modeling and optimization of extremal fuzzy continuous dynamic systems;

c) we introduce the notion of an EFCDS with fuzzy uncertainty, the source of which is “fuzzy measurement” (“expert reflections” on the states of EFCDS) of the system state in the so-called current and future fuzzy time intervals. The general EFCDS model is described;

d) the notion of processes of expert reflection and description of the EFCDS state change dynamics are introduced. With the aid of the conditional extremal expert reflection measures $g_{t*}(\cdot \mid x)$ and $g_t^*(\cdot,\mid x)$, the extremal fuzzy reflection process $(\widetilde{\mathbb{R}}_*, \widetilde{\mathbb{R}}^*)$ connects the fuzzy time interval measurement process $(\widetilde{r}_{\tau*}, \widetilde{r}_\tau^*)_{\tau\geq 0}$ with the space of measurable states of the system with fuzzy distribution $(X, \mathscr{B}, g)$, while the EFCDS state description process $(\widetilde{Q}_*, \widetilde{Q}^*)$ is defined through the extremal fuzzy reflection process $(\widetilde{\mathbb{R}}_*, \widetilde{\mathbb{R}}^*)$, using the extended upper and lower Sugeno integrals that are considered as extremal operators describing the EFCDS state dynamics;

e) consideration is given to the continuous case of extremal fuzzy processes. Questions of the ergodicity of extremal fuzzy processes are studied. The notion of g-ergodicity is introduced, which allows one to obtain a sufficient condition for the process $(\widetilde{Q}_*, \widetilde{Q}^*)$ to be ergodic;

f) the notion of an extremal fuzzy control process $(\widetilde{u}_*, \widetilde{u}^*)$ is introduced in the case of the action of control with fuzzy restrictions in the form of the space $(U, \mathscr{B}_U, g_U)$. Models of continuous extremal controllable fuzzy processes are constructed. Questions of the ergodicity of controllable extremal fuzzy processes are studied;

g) problems of optimization of a continuous controllable extremal fuzzy process are considered using R. Bellman’s optimality principle. An extremal fuzzy “gain-loss” process is defined, which plays the role of Bellman’s function in the classical variant of the dynamic programming problem. Theorems 8 and 9 allow one to write variants of an optimal control for the EFCDS;

h) a practical example is given to illustrate the results obtained.

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