On Solving Optimization Problems with Ordered Average Criteria and Constraints

Włodzimierz Ogryczak and Tomasz Śliwiński

Abstract. The problem of aggregating multiple numerical attributes to form overall measure is of considerable importance in many disciplines. The ordered weighted averaging (OWA) aggregation, introduced by Yager, uses the weights assigned to the ordered values rather than to the specific attributes. This allows one to model various aggregation preferences, preserving simultaneously the impartiality (neutrality) with respect to the individual attributes. However, importance weighted averaging is a central task in multiattribute decision problems of many kinds. It can be achieved with the Weighted OWA (WOWA) aggregation though the importance weights make the WOWA concept much more complicated than the original OWA. In this paper we analyze solution procedures for optimization problems with the ordered average objective functions or constraints. We show that the WOWA aggregation with monotonic preferential weights can be reformulated in a way allowing to introduce linear programming optimization models, similar to the optimization models we developed earlier for the OWA aggregation. Computational efficiency of the proposed models is demonstrated.

Keywords: OWA, WOWA, Optimization, Linear Programming.

1 Introduction

Consider a decision problem defined characterized by m attribute functions $f_i(\mathbf{x})$. That means there is given a feasible set $\mathscr{F} \subset R^q$ of decision vectors $\mathbf{x}$ (vectors of decision variables). The feasible set is usually defined by some constraints on the

Włodzimierz Ogryczak · Tomasz Śliwiński
Warsaw University of Technology, Institute of Control & Computation Engineering, 00-665 Warsaw, Poland
e-mail: wogrycza@ia.pw.edu.pl, tsliwins@ia.pw.edu.pl

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decision variables. Further $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ is a vector function that maps the feasible set $\mathscr{F}$ into the outcome attribute space R^m. In order to make the multiple attribute model operational for the decision support process, one needs to assume some aggregation function for multiple attributes: $a : R^m \to R$. The aggregated attribute values can be either bounded or optimized (maximized or minimized).

The most commonly used aggregation is based on the weighted mean where positive importance weights p_i $(i = 1, \ldots, m)$ are allocated to several attributes

$$A_{\mathbf{p}}(\mathbf{y}) = \sum_{i=1}^{m} y_i p_i \tag{1}$$

The weights are typically normalized to the total 1 ($\sum_{i=1}^{m} p_i = 1$). However, the weighted mean allowing to define the importance of attributes does not allow to model the decision maker's preferences regarding distribution of outcomes. The latter is crucial when aggregating (normalized) uniform achievement criteria like those used in the fuzzy optimization methodologies [29] as well as in the goal programming and the reference point approaches to the multiple criteria decision support [13]. In the stochastic problems uniform objectives may represent various possible values of the same (uncertain) outcome under several scenarios [14].

The preference weights can be effectively introduced with the so-called Ordered Weighted Averaging (OWA) aggregation developed by Yager [25]. In the OWA aggregation the weights are assigned to the ordered values (i.e. to the smallest value, the second smallest and so on) rather than to the specific attributes. Since its introduction, the OWA aggregation has been successfully applied to many fields of decision making [29, 30, 2]. When applying the OWA aggregation to optimization problems with attributes modeled by variables the weighting of the ordered outcome values causes that the OWA operator is nonlinear even for linear programming (LP) formulation of the original constraints and criteria. Yager [26] has shown that the nature of the nonlinearity introduced by the ordering operations allows one to convert the OWA optimization into a mixed integer programming problem. We have shown [18] that the OWA optimization with monotonic weights can be formed as a standard linear program of higher dimension.

The OWA operator allows to model various aggregation functions from the maximum through the arithmetic mean to the minimum. Thus, it enables modeling of various preferences from the optimistic to the pessimistic one. On the other hand, the OWA does not allow to allocate any importance weights to specific attributes. Actually, the weighted mean (1) cannot be expressed in terms of the OWA aggregations.

Importance weighted averaging is a central task in multicriteria decision problems of many kinds, such as selection, classification, object recognition, and information retrieval. Therefore, several attempts have been made to incorporate importance weighting into the OWA operator [28, 6]. Finally, Torra [22] has introduced the Weighted OWA (WOWA) aggregation as a particular case of Choquet integral using a distorted probability as the measure. The WOWA averaging is defined by two weighting vectors: the preferential weights $\mathbf{w}$ and the importance weights $\mathbf{p}$. It covers both the weighted means (defined with $\mathbf{p}$) and the OWA averages (defined with $\mathbf{w}$) as special cases. Actually, the WOWA average is reduced to the weighted mean in the case of equal all the preference weights and it becomes the standard OWA average in the case of equal all the importance weights. Since its introduction, the WOWA operator has been successfully applied to many fields of decision making [24, 15, 19, 20] including metadata aggregation problems [1, 10].

In this paper we analyze solution procedures for optimization problems with the ordered average objective functions or constraints. Exactly we consider optimization problems

$$\max \{a(\mathbf{y}) \; : \; \mathbf{y} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathscr{F}\} \tag{2}$$

or

$$\max\{g(\mathbf{y}) \; : \; a(\mathbf{y}) \leq \rho, \quad \mathbf{y} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathscr{F}\} \tag{3}$$

As an aggregation we consider both OWA and WOWA, both with possible some generalization due to possible arbitrary density of grid of the preference weights. We show that the concepts of the LP formulations for the OWA optimization with monotonic preferential weights [18] can easily be extended to cover optimization problems with the WOWA bounds and objectives with arbitrary importance weights and arbitrary density of the preferential weights. A special attention will be paid to problems with linear attribute functions $f_i(\mathbf{x}) = \mathbf{c}_i\mathbf{x}$ and polyhedral feasible sets:

$$\mathbf{y} = \mathbf{f}(\mathbf{x}) = \mathbf{C}\mathbf{x} \quad \text{and} \quad \mathscr{F} = \{\mathbf{x} \in R^q \; : \; \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geqq \mathbf{0}\} \tag{4}$$

where $\mathbf{C}$ is an $m \times q$ matrix (consisting of rows $\mathbf{c}_i$), $\mathbf{A}$ is a given $r \times q$ matrix and $\mathbf{b} = (b_1, \ldots, b_r)^T$ is a given RHS vector.

The paper is organized as follows. In the next section we introduce formally the WOWA operator and its generalization on arbitrary density grids of the preferential weights. We derive some alternative computational formula based on the Lorenz curves and analyze the orness/andness properties of the WOWA operator with monotonic preferential weights. In Section 3 we introduce the LP formulations for bounds or minimization of the WOWA aggregations with decreasing preferential weights. Similarly in Section 4, the LP formulations for bounds or maximization of the WOWA aggregations with increasing weights are given. Finally, in Section 5 we demonstrate computational efficiency of the introduced models.

2 The Ordered Weighted Averages

2.1 OWA and WOWA Aggregations

Let $\mathbf{w} = (w_1, \ldots, w_m)$ be a weighting vector of dimension m such that $w_i \geq 0$ for $i = 1, \ldots, m$ and $\sum_{i=1}^{m} w_i = 1$. The corresponding OWA aggregation of attributes $\mathbf{y} = (y_1, \ldots, y_m)$ can be mathematically formalized as follows [25]. First, we introduce the ordering map $\Theta : R^m \rightarrow R^m$ such that $\Theta(\mathbf{y}) = (\theta_1(\mathbf{y}), \theta_2(\mathbf{y}), \ldots, \theta_m(\mathbf{y}))$, where $\theta_1(\mathbf{y}) \geq \theta_2(\mathbf{y}) \geq \cdots \geq \theta_m(\mathbf{y})$ and there exists a permutation τ of set I such that $\theta_i(\mathbf{y}) = y_{\tau(i)}$ for $i = 1, \ldots, m$. Further, we apply the weighted sum aggregation to ordered achievement vectors $\Theta(\mathbf{y})$, i.e. the OWA aggregation has the following form:

$$A_{\mathbf{w}}(\mathbf{y}) = \sum_{i=1}^{m} w_i \theta_i(\mathbf{y}) \tag{5}$$

The OWA aggregation (5) allows to model various aggregation functions from the maximum ($w_1 = 1$, $w_i = 0$ for $i = 2, \ldots, m$) through the arithmetic mean ($w_i = 1/m$ for $i = 1, \ldots, m$) to the minimum ($w_m = 1$, $w_i = 0$ for $i = 1, \ldots, m-1$). However, the weighted mean (1) cannot be expressed as an OWA aggregation. Actually, the OWA aggregations are symmetric (impartial, neutral) with respect to the individual attributes and it does not allow to represent any importance weights allocated to specific attributes.

Further, let $\mathbf{w} = (w_1, \ldots, w_m)$ and $\mathbf{p} = (p_1, \ldots, p_m)$ be weighting vectors of dimension m such that $w_i \geq 0$ and $p_i \geq 0$ for $i = 1, \ldots, m$ as well as $\sum_{i=1}^{m} w_i = 1$ and $\sum_{i=1}^{m} p_i = 1$. The corresponding Weighted OWA aggregation of outcomes $\mathbf{y} = (y_1, \ldots, y_m)$ is defined as follows [22]:

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{i=1}^{m} \omega_i \theta_i(\mathbf{y}) \tag{6}$$

where the weights ω_i are defined as

$$\omega_i = w^*(\sum_{k \leq i} p_{\tau(k)}) - w^*(\sum_{k < i} p_{\tau(k)}) \tag{7}$$

with w^* a monotone increasing function that interpolates points $(\frac{i}{m}, \sum_{k \leq i} w_k)$ together with the point (0.0) and τ representing the ordering permutation for $\mathbf{y}$ (i.e. $y_{\tau(i)} = \theta_i(\mathbf{y})$). Function w^* is required to be a straight line whenever the points can be interpolated in this way. Due to this requirement, the WOWA aggregation covers the standard weighted mean (1) with weights p_i as a special case of equal preference weights ($w_i = 1/m$ for $i = 1, \ldots, m$). Actually, the WOWA operator is a particular case of Choquet integral using a distorted probability as the measure [4].

Note that function w^* can be expressed as $w^*(\alpha) = \int_0^{\alpha} g(\xi)\, d\xi$ where g is a generation function. Let us introduce breakpoints $\beta_i = \sum_{k \leq i} p_{\tau(k)}$ and $\beta_0 = 0$. This allows one to express weights ω_i as

$$\omega_i = \int_0^{\beta_i} g(\xi)\, d\xi - \int_0^{\beta_{i-1}} g(\xi)\, d\xi = \int_{\beta_{i-1}}^{\beta_i} g(\xi)\, d\xi$$

and the entire WOWA aggregation as

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{i=1}^{m} \theta_i(\mathbf{y}) \int_{\beta_{i-1}}^{\beta_i} g(\xi)\, d\xi = \int_0^1 g(\xi) F_{\mathbf{y}}^{(-1)}(\xi)\, d\xi \tag{8}$$

where $F_{\mathbf{y}}^{(-1)}$ is the stepwise function $F_{\mathbf{y}}^{(-1)}(\xi) = \theta_i(\mathbf{y})$ for $\beta_{i-1} < \xi \leq \beta_i$. It can also be mathematically formalized as follows. First, we introduce the left-continuous right tail cumulative distribution function (cdf):

$$F_{\mathbf{y}}(d) = \sum_{i \in I} p_i \delta_i(d) \quad \text{where} \quad \delta_i(d) = \begin{cases} 1 & \text{if } y_i \geq d \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

which for any real (outcome) value d provides the measure of outcomes greater or equal to d. Next, we introduce the quantile function $F_{\mathbf{y}}^{(-1)}$ as the right-continuous inverse of the cumulative distribution function $F_{\mathbf{y}}$:

$$F_{\mathbf{y}}^{(-1)}(\xi) = \sup \{\eta : F_{\mathbf{y}}(\eta) \geq \xi\} \quad \text{for } 0 < \xi \leq 1.$$

Formula (8) provides the most general expression of the WOWA aggregation allowing for expansion to continuous case. The original definition of WOWA allows one to build various interpolation functions w^* [23] thus to use different generation functions g in formula (8). We focus our analysis on the simplest case of linear interpolation leading to the piecewise linear function w^*. Note, however, that the piecewise linear functions may be built with various number of breakpoints, not necessarily m. Thus, any nonlinear function can be well approximated by a piecewise linear function with appropriate number of breakpoints. Therefore, we will consider weights vectors $\mathbf{w}$ of dimension n not necessarily equal to m. Any such piecewise linear interpolation function w^* can be expressed with the stepwise generation function

$$g(\xi) = n w_k \quad \text{for } (k-1)/n < \xi \leq k/n, \quad k = 1, \ldots, n \tag{10}$$

This leads us to the following specification of formula (8):

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \int_0^1 g(\xi) F_{\mathbf{y}}^{(-1)}(\xi)\, d\xi = \sum_{k=1}^{n} w_k n \int_{(k-1)/n}^{k/n} F_{\mathbf{y}}^{(-1)}(\xi)\, d\xi \tag{11}$$

We will treat formula (11) as a formal definition of the WOWA aggregation of m-dimensional outcomes $\mathbf{y}$ defined by m-dimensional importance weights $\mathbf{p}$ and n-dimensional preferential weights $\mathbf{w}$. Note that quantities $n \int_{(k-1)/n}^{k/n} F_{\mathbf{y}}^{(-1)}(\xi)\, d\xi$ express the conditional means within the corresponding quantiles $(k-1)/n$ and k/n. In the case of $n = m$ and equal importance weights $p_i = 1/n$, formula (11) represents the standard definition of the OWA aggregation (5), since $F_{\mathbf{y}}^{(-1)}(\xi) = \theta_k(\mathbf{y})$ for $(k-1)/n \leq \xi < k/n$. Although formula (11) allows one to express general WOWA aggregations by using the preferential weights to redefine $F_{\mathbf{y}}^{(-1)}(\xi) = \theta_k(\mathbf{y})$ accordingly. Moreover, various values of n possibly different from the number of attributes m allows one to generalize both the WOWA aggregation as well as the OWA aggregation (with equal importance weights $p_i = 1/n$).

When in (8) using the integrals from the left end rather than those on intervals one gets

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{k=1}^{n} nw_k(L(\mathbf{y},\mathbf{p},\frac{k}{n}) - L(\mathbf{y},\mathbf{p},\frac{k-1}{n})) \tag{12}$$

where $L(\mathbf{y},\mathbf{p},\beta)$ is defined by left-tail integrating $F_{\mathbf{y}}^{(-1)}$, i.e.

$$L(\mathbf{y},\mathbf{p},0) = 0 \quad \text{and} \quad L(\mathbf{y},\mathbf{p},\beta) = \int_0^{\beta} F_{\mathbf{y}}^{(-1)}(\alpha)d\alpha \quad \text{for } 0 < \beta \leq 1 \tag{13}$$

In particular, $L(\mathbf{y},\mathbf{p},1) = \int_0^1 F_{\mathbf{y}}^{(-1)}(\alpha)d\alpha = A_{\mathbf{p}}(\mathbf{y})$. Graphs of functions $L(\mathbf{y},\mathbf{p},\beta)$ (with respect to β) take the form of concave curves, the so-called (upper) absolute Lorenz curves. In the case of $n = m$ and equal importance weights $p_i = 1/n$ thus representing the standard OWA aggregation, one gets $L(\mathbf{y},\mathbf{p},\frac{k}{n}) = \frac{1}{n}\sum_{i=1}^{k}\theta_i(\mathbf{y})$ and formula (12) reduces to (5). Although, for $n \neq m$ one gets more complicated formula for $L(\mathbf{y},\mathbf{p},\frac{k}{n})$ even in the case of equal importance weights $p_i = 1/n$ thus representing the generalized OWA aggregation.

Alternatively, one may refer in formula (11) to the integrals from the right end instead of intervals thus getting

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{k=1}^{n} nw_k(\overline{L}(\mathbf{y},\mathbf{p},1-\frac{k-1}{n}) - \overline{L}(\mathbf{y},\mathbf{p},1-\frac{k}{n})) \tag{14}$$

where $\overline{L}(\mathbf{y},\mathbf{p},\beta)$ is defined by right tail integrating $F_{\mathbf{y}}^{(-1)}$, i.e.

$$\overline{L}(\mathbf{y},\mathbf{p},0) = 0 \quad \text{and} \quad \overline{L}(\mathbf{y},\mathbf{p},\beta) = \int_0^{1-\beta} F_{\mathbf{y}}^{(-1)}(1-\alpha)d\alpha \quad \text{for } 0 < \beta \leq 1 \tag{15}$$

One may easily notice that for any $0 \leq \beta \leq 1$

$$L(\mathbf{y},\mathbf{p},\beta) + \overline{L}(\mathbf{y},\mathbf{p},1-\beta) = \int_0^1 F_{\mathbf{y}}^{(-1)}(\alpha)d\alpha = A_{\mathbf{p}}(\mathbf{y})$$

Hence, $\overline{L}(\mathbf{y},\mathbf{p},1) = A_{\mathbf{p}}(\mathbf{y})$. Graphs of functions $\overline{L}(\mathbf{y},\mathbf{p},\beta)$ (with respect to β) take the form of convex curves, the (lower) absolute Lorenz curves. In the case of the standard OWA aggregation represented by $n = m$ and equal importance weights $p_i = 1/n$, one gets $\overline{L}(\mathbf{y},\mathbf{p},1-\frac{k}{n}) = \frac{1}{n}\sum_{i=k}^{n}\theta_i(\mathbf{y})$ thus reducing formula (12) to (5).

2.2 *The Orness and Andness Properties*

The OWA aggregation may model various preferences from the optimistic (max) to the pessimistic (min). Yager [25] introduced a well appealing concept of the orness measure to characterize the OWA operators. The degree of orness associated with the OWA operator $A_{\mathbf{w}}(\mathbf{y})$ is defined as

$$\text{orness}(\mathbf{w}) = \sum_{i=1}^{m} \frac{m-i}{m-1} w_i \tag{16}$$

For the max aggregation representing the fuzzy 'or' operator with weights $\mathbf{w} = (1,0,\ldots,0)$ one gets $\text{orness}(\mathbf{w}) = 1$ while for the min aggregation representing the fuzzy 'and' operator with weights $\mathbf{w} = (0,\ldots,0,1)$ one has $\text{orness}(\mathbf{w}) = 0$. For the average (arithmetic mean) one gets $\text{orness}((1/m,1/m,\ldots,1/m)) = 1/2$. Actually, one may consider a complementary measure of andness defined as $\text{andness}(\mathbf{w}) = 1 - \text{orness}(\mathbf{w})$. OWA aggregations with orness greater or equal 1/2 are considered or-like whereas the aggregations with orness smaller or equal 1/2 are treated as and-like. The former correspond to rather optimistic preferences while the latter represents rather pessimistic preferences.

The OWA aggregations with monotonic weights are either or-like or and-like. Exactly, decreasing weights $w_1 \geq w_2 \geq \ldots \geq w_m$ define an or-like OWA operator, while increasing weights $w_1 \leq w_2 \leq \ldots \leq w_m$ define an and-like OWA operator. Actually, the orness and the andness properties of the OWA operators with monotonic weights are total in the sense that they remain valid for any subaggregations defined by subsequences of their weights. Namely, for any $2 \leq k \leq m$ and any k-dimensional normalized weights subvector $\mathbf{w}^k = \frac{1}{\bar{w}^k}(w_{i_1}, w_{i_2}, \ldots, w_{i_k})$ with $1 \leq i_1 < i_2 < \ldots < i_k \leq m$ and $\bar{w}^k = \sum_{j=1}^{k} w_{i_j}$, one gets $\text{orness}(\mathbf{w}^k) \geq 1/2$ for the OWA operators with decreasing or $\text{orness}(\mathbf{w}^k) \leq 1/2$ for the OWA operators with increasing weights, respectively. Moreover, appropriate weights monotonicity is necessary to achieve the above total orness or andness properties. Therefore, we will refer to the OWA aggregation with decreasing weights as the totally or-like OWA operator, and to the OWA aggregation with increasing weights as the totally and-like OWA operator.

Yager [27] proposed to define the OWA weighting vectors via the regular increasing monotone (RIM) quantifiers, which provide a dimension independent description of the aggregation. A fuzzy subset Q of the real line is called a RIM quantifier if Q is (weakly) increasing with $Q(0) = 0$ and $Q(1) = 1$. The OWA weights can be defined with a RIM quantifier Q as $w_i = Q(i/m) - Q((i-1)/m)$ and the orness measure can be extended to a RIM quantifier (according to $m \to \infty$) as follows [27]

$$\text{orness}(Q) = \int_0^1 Q(\alpha)\, d\alpha \tag{17}$$

Thus, the orness of a RIM quantifier is equal to the area under it. The measure takes the values between 0 (achieved for $Q(1) = 1$ and $Q(\alpha) = 0$ for all other α) and 1 (achieved for $Q(0) = 1$ and $Q(\alpha) = 0$ for all other α). In particular, $\text{orness}(Q) = 1/2$ for $Q(\alpha) = \alpha$ which is generated by equal weights $w_k = 1/n$. Formula (17) allows one to define the orness of the WOWA aggregation (6) which can be viewed with the RIM quantifier $Q(\alpha) = w^*(\alpha)$ [7]. Let us consider piecewise linear function $Q = w^*$ defined by weights vectors $\mathbf{w}$ of dimension n according to the stepwise generation function (10). One may easily notice that decreasing weights $w_1 \geq w_2 \geq \ldots \geq w_n$ generate a strictly increasing concave curve $Q(\alpha) \geq \alpha$ thus guaranteeing the or-likeness of the WOWA operator. Similarly, increasing weights

$w_1 \leq w_2 \leq \ldots \leq w_n$ generate a strictly increasing convex curve $Q(\alpha) \leq \alpha$ thus guaranteeing the and-likeness of the WOWA operator. Actually, the monotonic weights generate the totally or-like and and-like operators, respectively, in the sense that that they remain valid for any subaggregations defined with respect to subintervals of the interval $[0,1]$. Namely, for any interval $[a,b]$, where $0 \leq a < b \leq 1$, and the corresponding part of Q renormalized to represent a RIM quantifier

$$Q_a^b(\alpha) = \frac{Q(a+\alpha(b-a)) - Q(a)}{Q(b) - Q(a)}$$

one gets orness$(Q_a^b) \geq 1/2$ for the OWA operators with decreasing or orness$(Q_a^b) \leq 1/2$ for the OWA operators with increasing weights w_i, respectively. Moreover, in the case of piecewise linear function $Q = w^*$ defined by weights vectors $\mathbf{w}$ of dimension n according to the stepwise generation function (10), we consider, appropriate weights monotonicity is necessary to achieve the total orness or andness properties. Therefore, we will refer to the WOWA aggregation with decreasing preferential weights as the totally or-like WOWA operator, and to the WOWA aggregation with increasing preferential weights as the totally and-like WOWA operator.

3 Totally Or-Like Ordered Weighted Aggregations

Consider a totally or-like WOWA aggregation defined by decreasing weights $w_1 \geq w_2 \geq \ldots \geq w_n$. Following formula (12) the WOWA aggregation may be expressed as

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{k=1}^{n} n w_k (L(\mathbf{y},\mathbf{p},\frac{k}{n}) - L(\mathbf{y},\mathbf{p},\frac{k-1}{n})) = \sum_{k=1}^{n} w'_k L(\mathbf{y},\mathbf{p},\frac{k}{n}) \qquad (18)$$

where $w'_n = n w_n$, $w'_k = n(w_k - w_{k+1})$. Due to formula (13), values of function $L(\mathbf{y},\mathbf{p},\alpha)$ for any $0 \leq \alpha \leq 1$ can be found by optimization:

$$L(\mathbf{y},\mathbf{p},\alpha) = \max_{u_i} \{ \sum_{i=1}^{m} y_i u_i : \sum_{i=1}^{m} u_i = \alpha, \quad 0 \leq u_i \leq p_i \quad \forall\, i \} \qquad (19)$$

The above problem is an LP for a given outcome vector $\mathbf{y}$ while it becomes nonlinear for $\mathbf{y}$ being a vector of variables. This difficulty can be overcome by taking advantages of the LP dual to (19). Introducing dual variable t corresponding to the equation $\sum_{i=1}^{m} u_i = \alpha$ and variables d_i corresponding to upper bounds on u_i one gets the following LP dual of problem (19):

$$L(\mathbf{y},\mathbf{p},\alpha) = \min_{t,d_i} \{ \alpha t + \sum_{i=1}^{m} p_i d_i : t + d_i \geq y_i,\; d_i \geq 0 \quad \forall\, i \} \qquad (20)$$

Due the LP duality theory, the following assertion is valid.

Lemma 1. *For any value ρ, vector $\mathbf{y}$ fulfills inequality $L(\mathbf{y},\mathbf{p},\xi) \le \rho$ if and only if there exist t and d_i ($i = 1,\ldots,m$) such that*

$$\xi t + \sum_{i=1}^{m} p_i d_i \le \rho \quad \text{and} \quad t + d_i \ge y_i,\ d_i \ge 0 \quad \forall\, i$$

Note that following (18) the WOWA with increasing weights w_k takes the form

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{k=1}^{n} w'_k L(\mathbf{y},\mathbf{p},\frac{k}{n})$$

with positive weights w'_k. Therefore, the following assertions can be proven.

Theorem 1. *Any totally or-like WOWA aggregation $A_{\mathbf{w},\mathbf{p}}$ defined by decreasing weights $w_1 \ge w_2 \ge \ldots \ge w_n$ is a piecewise linear convex function of $\mathbf{y}$.*

Proof. Note that for any given $\mathbf{p}$ and ξ, due to formula (20), $L(\mathbf{y},\mathbf{p},\xi)$ is a piecewise linear convex function of $\mathbf{y}$. Hence, due to decreasing preferential weights, following formula (37) the entire WOWA aggregation is a piecewise linear convex function of $\mathbf{y}$ as a linear combination of functions $L(\mathbf{y},\mathbf{p},\xi)$ for $\xi = k/n$, $k = 1,2,\ldots,n$ with nonnegative weights w'_k.

Theorem 2. *For any totally or-like WOWA aggregation $A_{\mathbf{w},\mathbf{p}}$ defined by decreasing weights $w_1 \ge w_2 \ge \ldots \ge w_n$ and any constant ρ inequality $A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) \le \rho$ is valid if and only if there exist t_k and d_{ik} ($i = 1,\ldots,m; k = 1,2,\ldots,n$) such that*

$$\begin{aligned} &\sum_{k=1}^{n} w'_k [\frac{k}{n} t_k + \sum_{i=1}^{m} p_i d_{ik}] \le \rho \\ &t_k + d_{ik} \ge y_i,\ d_{ik} \ge 0 \qquad \text{for } i = 1,\ldots,m;\ k = 1,\ldots,n \end{aligned} \tag{21}$$

Proof. Assume that there exist t_k^0 and d_{ik}^0 ($i = 1,\ldots,m; k = 1,2,\ldots,n$) satisfying the requirements (21). Then, according to Lemma 1,

$$L(\mathbf{y},\mathbf{p},\frac{k}{n}) \le \frac{k}{n} t_k^0 + \sum_{i=1}^{m} p_i d_{ik}^0 \quad \forall\, k$$

Hence, due to nonnegative weights w'_k,

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{k=1}^{n} w'_k L(\mathbf{y},\mathbf{p},\frac{k}{n}) \le \sum_{k=1}^{n} w'_k [\frac{k}{n} t_k^0 + \sum_{i=1}^{m} p_i d_{ik}^0] \le \rho$$

which proves the required inequality.

Assume now that the inequality $A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) \le \rho$ holds. Define t_k^0 and d_{ik}^0 ($i = 1,\ldots,m; k = 1,2,\ldots,n$) as optimal solutions to problems (20) for $\xi = k/n$ ($k = 1,\ldots,n$), respectively. They obviously fulfill conditions (21).

Consider an optimization problem with an upper bound on a totally or-like WOWA aggregation

$$\max\{g(\mathbf{y}) \;:\; A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) \leq \rho, \quad \mathbf{y} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathscr{F}\} \tag{22}$$

Following Theorem 2 it can be reformulated as

$$\begin{aligned}
\max_{t_k,d_{ik},y_i,x_j} \quad & g(\mathbf{y}) \\
\text{s.t.} \quad & \sum_{k=1}^{n} w'_k[\frac{k}{n}t_k + \sum_{i=1}^{m} p_i d_{ik}] \leq \rho \\
& t_k + d_{ik} \geq y_i,\; d_{ik} \geq 0 \quad \text{for } i = 1,\ldots,m;\; k = 1,\ldots,n \\
& \mathbf{y} \leq \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathscr{F}
\end{aligned}$$

In the case of model (4) with linear function $g(\mathbf{y}) = \sum_{i=1}^{m} g_i y_i$ this leads us to the following LP formulation of the optimization problem (22):

$$\max \sum_{i=1}^{m} g_i y_i \tag{23}$$

$$\text{s.t.} \quad \mathbf{Ax} = \mathbf{b} \tag{24}$$

$$\mathbf{y} - \mathbf{Cx} = \mathbf{0} \tag{25}$$

$$\sum_{k=1}^{n} \frac{k}{n} w'_k t_k + \sum_{k=1}^{n}\sum_{i=1}^{m} w'_k p_i d_{ik} \leq \rho \tag{26}$$

$$d_{ik} \geq y_i - t_k \quad \text{for } i = 1,\ldots,m;\; k = 1,\ldots,n \tag{27}$$

$$d_{ik} \geq 0 \quad \text{for } i = 1,\ldots,m;\; k = 1,\ldots,n;\; x_j \geq 0\; \forall\, j \tag{28}$$

Model (23)–(28) is an LP problem with $mn+m+n+q$ variables and $mn+m+r+1$ constraints. In the case of multiple WOWA constraints one gets additional $mn+m$ variables and $mn+1$ inequalities per each constraint. Thus, for problems with not too large number of attributes (m) and preferential weights (n) it can be solved directly.

Consider now minimization of a totally or-like WOWA aggregation

$$\min\{A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) \;:\; \mathbf{y} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathscr{F}\} \tag{29}$$

Taking advantages of Theorem 2, minimization of the WOWA criterion may be expressed as the following problem with auxiliary linear inequalities:

$$\begin{aligned}
\min_{\zeta,t_k,d_{ik},y_i,x_j} \quad & \zeta \\
\text{s.t.} \quad & \sum_{k=1}^{n} w'_k[\frac{k}{n}t_k + \sum_{i=1}^{m} p_i d_{ik}] \leq \zeta \\
& t_k + d_{ik} \geq y_i,\; d_{ik} \geq 0 \quad \text{for } i = 1,\ldots,m;\; k = 1,\ldots,n \\
& \mathbf{y} \leq \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathscr{F}
\end{aligned}$$

While eliminating the ζ variable this leads us to the following LP formulation of the WOWA problem:

$$\min_{t_k,d_{ik},y_i,x_j} \sum_{k=1}^{n} w'_k[\frac{k}{n}t_k + \sum_{i=1}^{m} p_i d_{ik}]$$
$$\text{s.t. } t_k + d_{ik} \geq y_i,\ d_{ik} \geq 0 \quad \text{for } i=1,\dots,m;\ k=1,\dots,n$$
$$\mathbf{y} \leq \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathscr{F}$$

When taking into account the linear attributes and constraints (4) we get the following LP formulation of the WOWA optimization problem (29):

$$\min \sum_{k=1}^{n} \frac{k}{n} w'_k t_k + \sum_{k=1}^{n}\sum_{i=1}^{m} w'_k p_i d_{ik} \tag{30}$$
$$\text{s.t. } \mathbf{Ax} = \mathbf{b} \tag{31}$$
$$\mathbf{y} - \mathbf{Cx} = \mathbf{0} \tag{32}$$
$$d_{ik} \geq y_i - t_k \quad \text{for } i=1,\dots,m;\ k=1,\dots,n \tag{33}$$
$$d_{ik} \geq 0 \quad \text{for } i=1,\dots,m;\ k=1,\dots,n;\ x_j \geq 0\ \forall\ j \tag{34}$$

This LP problem contains $mn+m+n+q$ variables and $mn+m+r$ constraints. Thus, for not too large values of m and n it can be solved directly. Actually, the LP model is quite similar to that introduced in [18] for the OWA optimization (c.f., model (30)–(34)).

The number of constraints in problem (30)–(34) is similar to the number of variables. However, the crucial number of variables (mn variables d_{ik}) is associated with singleton columns. Therefore, it may be better to deal with the dual of (30)–(34) where the corresponding rows become simple upper bounds, thus reducing dramatically the LP problem size. While introducing the dual variables: $\mathbf{u} = (u_1,\dots,u_r)$, $\mathbf{v} = (v_1,\dots,v_m)$ and $\mathbf{z} = (z_{ik})_{i=1,\dots,m;\ k=1,\dots,n}$ corresponding to the constraints (31), (32) and (33), respectively, we get the following dual:

$$\begin{aligned} &\max\ \mathbf{ub} \\ &\text{s.t. } \mathbf{uA} - \mathbf{vC} \overset{\leq}{=} \mathbf{0} \\ &v_i - \sum_{k=1}^{n} z_{ik} = 0 \qquad \text{for } i=1,\dots,m \\ &\sum_{i=1}^{m} z_{ik} = \frac{k}{n} w'_k \qquad \text{for } k=1,\dots,n \\ &0 \leq z_{ik} \leq p_i w'_k \qquad \text{for } i=1,\dots,m;\ k=1,\dots,n \end{aligned} \tag{35}$$

The dual problem (35) is consisted of only $m+n+q$ structural constraints on $mn+r+m$ variables. Since the average complexity of the simplex method depends on the number of constraints, the dual model (35) can be directly solved for quite large values of m and n. Moreover, the columns corresponding to mn variables z_{ik} form the network (node-link incidence) matrix thus allowing one to employ special techniques of the network embedded simplex algorithm [3].

Similar to the case of minimization of the totally or-like WOWA, it may be also introduced the dual of (23)–(28) representing the WOWA constraints. Indeed, while introducing the dual variables: $\mathbf{u} = (u_1,\dots,u_r)$, $\mathbf{v} = (v_1,\dots,v_m)$, ξ and

$\mathbf{z} = (z_{ik})_{i=1,\dots,m;\ k=1,\dots,n}$ corresponding to the constraints (24), (25), (26) and (27), respectively, we get the following dual:

$$\begin{aligned} &\min\ \mathbf{ub} + \rho\xi \\ \text{s.t. }\ &\mathbf{uA} - \mathbf{vC} \geqq \mathbf{0} \\ &v_i + \sum_{k=1}^{n} z_{ik} = g_i && \text{for } i = 1,\dots,m \\ &\sum_{i=1}^{m} z_{ik} - \frac{k}{n} w'_k \xi = 0 && \text{for } k = 1,\dots,n \\ &z_{ik} \le w'_k p_i \xi && \text{for } i = 1,\dots,m;\ k = 1,\dots,n \\ &\xi \ge 0, \quad z_{ik} \ge 0 && \text{for } i = 1,\dots,m;\ k = 1,\dots,n \end{aligned} \tag{36}$$

However, the mn rows corresponding to variables d_{ik} represent variable upper bounds ([21, 11]) instead of simple upper bounds. Thus the model simplification is not so dramatic.

4 Totally And-Like Ordered Weighted Aggregations

Consider a totally and-like WOWA aggregation defined by increasing weights $w_1 \le w_2 \le \ldots \le w_n$. By consideration of $-\mathbf{y}$ instead of $\mathbf{y}$ such an aggregation may be viewed as a negative to the totally or-like WOWA aggregation defined by decreasing weights

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = -A_{\bar{\mathbf{w}},\mathbf{p}}(\mathbf{y}) \quad \text{where} \quad \bar{w}_k = w_{n-k+1} \quad \text{for } k = 1,\dots,n$$

Alternatively, taking advantages of formula (14) the WOWA aggregation may be expressed as

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{k=1}^{n} n w_k (\overline{L}(\mathbf{y},\mathbf{p},1-\frac{k-1}{n}) - \overline{L}(\mathbf{y},\mathbf{p},1-\frac{k}{n})) = \sum_{k=1}^{n} w''_k \overline{L}(\mathbf{y},\mathbf{p},\frac{k}{n}) \tag{37}$$

with weights $w''_k = -w'_{n-k} = n(w_{n-k+1} - w_{n-k})$ for $k = 1,\dots,n-1$ and $w''_n = nw_1$ while values of function $\overline{L}(\mathbf{y},\mathbf{p},\xi)$ for any $0 \le \xi \le 1$ are given by optimization:

$$\overline{L}(\mathbf{y},\mathbf{p},\xi) = \min_{u_i} \{\sum_{i=1}^{m} y_i u_i : \sum_{i=1}^{m} u_j = \xi, \quad 0 \le u_i \le p_i \quad \forall\, i\,\} \tag{38}$$

Introducing dual variable t corresponding to the equation $\sum_{i=1}^{m} u_i = \xi$ and variables d_i corresponding to upper bounds on u_i one gets the following LP dual expression of $\overline{L}(\mathbf{y},\mathbf{p},\xi)$

$$\overline{L}(\mathbf{y},\mathbf{p},\xi) = \max_{t,d_i} \{\xi t - \sum_{i=1}^{m} p_i d_i : t - d_i \le y_i,\ d_i \ge 0 \quad \forall\, i\} \tag{39}$$

Due the duality theory, for any given vector $\mathbf{y}$ the cumulated ordered coefficient $\overline{L}(\mathbf{y},\mathbf{p},\xi)$ can be found as the optimal value of the above LP problem. Actually, relation (39) can be expressed as the following assertion.

Lemma 2. *For any value ρ, vector $\mathbf{y}$ fulfills inequality $\overline{L}(\mathbf{y},\mathbf{p},\xi) \geq \rho$ if and only if there exist t and d_i ($i = 1,\ldots,m$) such that*

$$\xi t - \sum_{i=1}^{m} p_i d_i \geq \rho \quad \textit{and} \quad t - d_i \leq y_i,\ d_i \geq 0 \quad \forall\, i$$

Note that following (37) the WOWA with increasing weights w_k takes the form

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{k=1}^{n} w_k''\overline{L}(\mathbf{y},\mathbf{p},\frac{k}{n})$$

with positive weights w_k''. This enables the following statements.

Theorem 3. *Any totally and-like WOWA aggregation $A_{\mathbf{w},\mathbf{p}}(\mathbf{y})$ defined by increasing preferential weights $w_1 \leq w_2 \leq \ldots \leq w_n$ is a piecewise linear concave function of $\mathbf{y}$.*

Proof. Note that for any given $\mathbf{p}$ and ξ, due to formula (39), $\overline{L}(\mathbf{y},\mathbf{p},\xi)$ is a piecewise linear concave function of $\mathbf{y}$. Hence, due to increasing preferential weights, following formula (37) the entire WOWA aggregation is a piecewise linear concave function of $\mathbf{y}$ as a linear combination of functions $\overline{L}(\mathbf{y},\mathbf{p},\xi)$ for $\xi = k/n$, $k = 1,2,\ldots,n$ with nonnegative weights w_k'.

Theorem 4. *For any totally and-like WOWA aggregation $A_{\mathbf{w},\mathbf{p}}$ defined by increasing weights $w_1 \leq w_2 \leq \ldots \leq w_n$ and any constant ρ inequality $A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) \geq \rho$ is valid if and only if there exist t_k and d_{ik} ($i = 1,\ldots,m; k = 1,2,\ldots,n$) such that*

$$\begin{aligned} &\sum_{k=1}^{n} w_k''[\frac{k}{n}t_k - \sum_{i=1}^{m} p_i d_{ik}] \geq \rho \\ &t_k - d_{ik} \leq y_i,\ d_{ik} \geq 0 \qquad\qquad \textit{for}\ \ i = 1,\ldots,m;\ k = 1,\ldots,n \end{aligned} \tag{40}$$

Proof. Assume that there exist t_k^0 and d_{ik}^0 ($i = 1,\ldots,m; k = 1,2,\ldots,n$) satisfying the requirements (40). Then, according to Lemma 2,

$$\overline{L}(\mathbf{y},\mathbf{p},\frac{k}{n}) \geq \frac{k}{n}t_k^0 - \sum_{i=1}^{m} p_i d_{ik}^0 \quad \forall\, k$$

Hence, due to nonnegative weights w_k'',

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{k=1}^{n} w_k''\overline{L}(\mathbf{y},\mathbf{p},\frac{k}{n}) \geq \sum_{k=1}^{n} w_k''[\frac{k}{n}t_k^0 - \sum_{i=1}^{m} p_i d_{ik}^0] \geq \rho$$

which proves the required inequality.

Assume now that the inequality $A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) \geq \rho$ holds. Define t_k^0 and d_{ik}^0 ($i = 1,\dots,m; k = 1,2,\dots,n$) as optimal solutions to problems (39) for $\xi = k/n$ ($k = 1,\dots,n$), respectively. They obviously fulfill conditions (40).

Consider an optimization problem with a lower bound on a totally and-like WOWA aggregation

$$\max\{g(\mathbf{y}) \;:\; A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) \geq \rho, \quad \mathbf{y} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathscr{F}\} \tag{41}$$

Following Theorem 4 it can be reformulated as

$$\begin{aligned}
\max_{t_k, d_{ik}, y_i, x_j} \quad & g(\mathbf{y}) \\
\text{s.t.} \quad & \sum_{k=1}^{n} w_k''[\frac{k}{n} t_k - \sum_{i=1}^{m} p_i d_{ik}] \geq \rho \\
& t_k - d_{ik} \leq y_i, \; d_{ik} \geq 0 \quad \text{for } i = 1,\dots,m; \; k = 1,\dots,n \\
& \mathbf{y} \leq \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathscr{F}
\end{aligned}$$

For model (4) with linear function $g(\mathbf{y}) = \sum_{i=1}^{m} g_i y_i$ this leads us to the following LP formulation of the optimization problem (41):

$$\max \sum_{i=1}^{m} g_i y_i \tag{42}$$

$$\text{s.t.} \quad \mathbf{A}\mathbf{x} = \mathbf{b} \tag{43}$$

$$\mathbf{y} - \mathbf{C}\mathbf{x} = \mathbf{0} \tag{44}$$

$$\sum_{k=1}^{n} \frac{k}{n} w_k'' t_k - \sum_{k=1}^{n} \sum_{i=1}^{m} w_k'' p_i d_{ik} \geq \rho \tag{45}$$

$$d_{ik} \geq t_k - y_i \quad \text{for } i = 1,\dots,m; \; k = 1,\dots,n \tag{46}$$

$$d_{ik} \geq 0 \quad \text{for } i = 1,\dots,m; \; k = 1,\dots,n; \; x_j \geq 0 \; \forall \; j \tag{47}$$

Model (42)–(47) is an LP problem with $mn + m + n + q$ variables and $mn + m + r + 1$ constraints. In the case of multiple WOWA constraints one gets additional $mn + m$ variables and $mn + 1$ inequalities per each constraint. Thus, for problems with not too large number of attributes (m) and preferential weights (n) it can be solved directly.

Consider now maximization of a totally and-like WOWA aggregation

$$\max\{A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) \;:\; \mathbf{y} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathscr{F}\} \tag{48}$$

Maximization of the WOWA aggregation (48) can be expressed as follows

$$\begin{aligned}
\max_{\zeta, t_k, d_{ik}, y_i, x_j} \quad & \zeta \\
\text{s.t.} \quad & \zeta \leq \sum_{k=1}^{n} w''_k [\frac{k}{n} t_k - \sum_{i=1}^{m} p_i d_{ik}] \\
& t_k - d_{ik} \leq y_i, \; d_{ik} \geq 0 \quad \text{for } i = 1, \ldots, m; \; k = 1, \ldots, n \\
& \mathbf{y} \leq \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathscr{F}
\end{aligned}$$

While eliminating the ζ variable this leads us to the following LP formulation of the WOWA problem:

$$\begin{aligned}
\max_{t_k, d_{ik}, y_i, x_j} \quad & \sum_{k=1}^{n} w''_k [\frac{k}{n} t_k - \sum_{i=1}^{m} p_i d_{ik}] \\
\text{s.t.} \quad & t_k - d_{ik} \leq y_i, \; d_{ik} \geq 0 \quad \text{for } i = 1, \ldots, m; \; k = 1, \ldots, n \\
& \mathbf{y} \leq \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathscr{F}
\end{aligned}$$

In the case of model (4) this leads us to the following LP formulation of the WOWA maximization problem (48):

$$\max \sum_{k=1}^{n} \frac{k}{n} w''_k t_k - \sum_{k=1}^{n} \sum_{i=1}^{m} w''_k p_i d_{ik} \tag{49}$$

$$\text{s.t.} \quad \mathbf{Ax} = \mathbf{b} \tag{50}$$

$$\mathbf{y} - \mathbf{Cx} = \mathbf{0} \tag{51}$$

$$d_{ik} \geq t_k - y_i \quad \text{for } i = 1, \ldots, m; \; k = 1, \ldots, n \tag{52}$$

$$d_{ik} \geq 0 \quad \text{for } i = 1, \ldots, m; \; k = 1, \ldots, n; \; x_j \geq 0 \; \forall \; j \tag{53}$$

The problem has the identical structure as that of (30)–(34) differing only with some negative signs in the objective function (49) and the deviation variable definition (52). While in (30)–(34) variables d_{ik} represent the upperside deviations from the corresponding targets t_k, here they represent the downside deviations for those targets. Note that WOWA model (49)–(53) differs from the analogous deviational model for the OWA optimization [18] only due to coefficients within the objective function (49) and the possibility of different values of m and n. In other words, the OWA deviational model [18] can easily be expanded to accommodate the importance weighting of WOWA.

Model (49)–(53) is an LP problem with $mn + m + n + q$ variables and $mn + m + r$ constraints. Thus, for problems with not too large number of criteria (m) and preferential weights (n) it can be solved directly. However, similar to the case of minimization of the or-like WOWA, it may be better to deal with the dual of (49)–(53) where mn rows corresponding to variables d_{ik} represent only simple upper bounds. Indeed, while introducing the dual variables: $\mathbf{u} = (u_1, \ldots, u_r)$, $\mathbf{v} = (v_1, \ldots, v_m)$ and $\mathbf{z} = (z_{ik})_{i=1,\ldots,m;\; k=1,\ldots,n}$ corresponding to the constraints (50), (51) and (52), respectively, we get the following dual:

$$
\begin{aligned}
\min\ & \mathbf{ub} \\
\text{s.t. } & \mathbf{uA} - \mathbf{vC} \geqq \mathbf{0} \\
& v_i - \sum_{k=1}^{n} z_{ik} = 0 \quad && \text{for } i = 1,\dots,m \\
& \sum_{i=1}^{m} z_{ik} = \frac{k}{n} w''_k \quad && \text{for } k = 1,\dots,n \\
& 0 \le z_{ik} \le w''_k p_i \quad && \text{for } i = 1,\dots,m;\ k = 1,\dots,n
\end{aligned} \tag{54}
$$

The dual problem (54), similar to (35), contains $mn+r+m$ variables and $m+n+q$ structural constraints. Therefore, it can be directly solved for quite large values of m and n.

Similar to the case of maximization of the totally and-like WOWA, it may be also introduced the dual of (42)–(47) representing the WOWA constraints. Indeed, while introducing the dual variables: $\mathbf{u} = (u_1,\dots,u_r)$, $\mathbf{v} = (v_1,\dots,v_m)$, ξ and $\mathbf{z} = (z_{ik})_{i=1,\dots,m;\ k=1,\dots,n}$ corresponding to the constraints (43), (44), (45) and (46), respectively, we get the following dual:

$$
\begin{aligned}
\min\ & \mathbf{ub} - \rho\xi \\
\text{s.t. } & \mathbf{uA} - \mathbf{vC} \geqq \mathbf{0} \\
& v_i - \sum_{k=1}^{n} z_{ik} = g_i \quad && \text{for } i = 1,\dots,m \\
& \sum_{i=1}^{m} z_{ik} - \frac{k}{n} w''_k \xi = 0 \quad && \text{for } k = 1,\dots,n \\
& z_{ik} \le w''_k p_i \xi \quad && \text{for } i = 1,\dots,m;\ k = 1,\dots,n \\
& \xi \ge 0, \quad z_{ik} \ge 0 \quad && \text{for } i = 1,\dots,m;\ k = 1,\dots,n
\end{aligned} \tag{55}
$$

However, the mn rows corresponding to variables d_{ik} represent variable upper bounds ([21, 11]) instead of simple upper bounds. Thus the model simplification is not so dramatic.

5 Computational Tests

In order to examine computational performances of the LP models for the WOWA optimization we have solved randomly generated problems with varying number q of decision variables and number m of attributes. The core LP feasible set has been defined by a single knapsack-type constraint

$$
A\{\mathbf{y} = \mathbf{f}(\mathbf{x})\ :\ \sum_{j=1}^{q} x_j = 1, \quad x_j \ge 0 \quad \text{for } j = 1,\dots,q\} \tag{56}
$$

where $f_i(\mathbf{x}) = \mathbf{c}_i\mathbf{x} = \sum_{j=1}^{q} c_{ij}x_j$. Such problems may be interpreted as resource allocation decisions [17] as well as relocation ones [5] or portfolio selection

Table 1 WOWA criterion optimization times [s]: primal model (49)–(53)

Number of attributes (m)	Number of variables (q)							
	10	20	50	100	150	200	300	400
10	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
20	0.0	0.0	0.0	0.0	0.0	0.2	0.0	0.0
50	0.8	1.0	1.4	1.6	1.4	1.4	1.4	1.6
100	22.6	27.2	35.6	37.8	48.8	71.6	71.2	111.6
150	196.8	259.2	359.8	355.8	387.6	484.4	446.0	^{3}558.4

problem [12] when several attributes represent the unique scenario realizations under various scenarios. Assuming the attributes represent some desired quantities we have considered totally and-like WOWA aggregation defined by increasing weights $w_1 \leq w_2 \leq \ldots \leq w_n$. We have analyzed both the WOWA maximization problem

$$\max \{A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) \ : \ \mathbf{y} \in A\} \tag{57}$$

as well as the WOWA lower bounded problem of the weighted mean maximization

$$\max \{A_{\mathbf{p}}(\mathbf{y}) \ : \ A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) \geq \rho, \ \mathbf{y} \in A\} \tag{58}$$

The former correspond to the multiple conditional value-at-risk performance measure maximization [9] while the latter represents more traditional approach to the portfolio optimization where the expected return is maximized with some risk measure bound [8].

For our computational tests we have randomly generated problems (57) and (58). Coefficients c_{ij} were generated as follows. First, for each j the upper bound r_j was generated as a random number uniformly distributed in the interval $[0.05, 0.15]$. Next, individual coefficients c_{ij} were generated as uniformly distributed in the interval $[-0.75r_j, r_j]$. In order to generate strictly increasing and positive preference weights w_k, we generated randomly the corresponding increments $\delta_k = w_k - w_{k-1}$. The latter were generated as uniformly distributed random values in the range of 1.0 to 2.0, except from a few (5 on average) possibly larger increments ranged from 1.0 to $n/3$. Importance weights p_i were generated according to the exponential smoothing scheme, $p_i = \alpha(1-\alpha)^{i-1}$ for $i = 1, 2, \ldots, m$ and the parameter α is chosen for each test problem size separately to keep the smallest weight p_m around 0.001. The ρ value in (58) was always set to 90% of the objective of the corresponding problem (57).

For each number of decision variables q and number of attributes m we solved 5 randomly generated problems (57) and (58). All computations were performed on a PC with the Pentium 4 2.4GHz processor employing the CPLEX 9.1 package with standard settings and with the time limit of 600 seconds.

In Tables 1 and 2 we show the solution times for the primal (49)–(53) and the dual (54) forms of the computational model, being the averages of 5 randomly generated problems. Upper index in front of the time value indicates the number of tests

Table 2 WOWA criterion optimization times [s]: dual model (54)

Number of attributes (m)	Number of variables (q)							
	10	20	50	100	150	200	300	400
10	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
20	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
50	0.0	0.2	0.2	0.4	0.2	0.4	0.4	0.4
100	0.4	0.6	1.0	7.0	8.2	9.8	10.6	15.4
150	1.4	2.2	3.2	25.2	50.6	53.4	62.0	71.0
200	3.8	5.2	8.8	65.4	156.4	217.8	291.6	253.2
300	10.2	18.0	30.6	132.8	^{2}486.6	–	–	–
400	29.6	38.8	88.2	–	–	–	–	–

Table 3 WOWA bound optimization times [s]: primal model (42)–(47)

Number of attributes (m)	Number of variables (q)							
	10	20	50	100	150	200	300	400
10	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
20	0.0	0.0	0.0	0.0	0.2	0.0	0.2	0.0
50	1.0	1.6	1.8	2.4	2.4	2.4	2.8	3.0
100	30.0	52.2	111.4	97.6	88.0	81.4	76.4	82.2
150	189.6	309.8	^{1}539.6	^{1}513.8	^{1}499.6	^{2}528.0	^{1}532.6	^{4}234.6

among 5 that exceeded the time limit. The empty cell (minus sign) shows that this has occurred for all 5 instances. As one can see, the dual form of the model performs much better in each tested problem size. It behaves very well with increasing number of variables if the number of attributes does not exceed 150, and satisfactory if the number of attributes equals 200. Similarly, the model performs very well with increasing number of attributes if only the number of variables does not exceed 50.

Tables 3 and 4 contain solution times for the primal (42)–(47) and the dual (55) form of the model of the weighted mean maximization with the WOWA lower boundary.

As one can see the primal approach requires similar computation effort for the WOWA problem as well as for the weighted mean with the WOWA lower bound. The dual approach, however, is much better suited for the standard WOWA problem then for the weighted mean with the WOWA lower bound. The reason for that is the change of singleton columns to the doubleton ones resulting from the introduction of the WOWA constraint. But still, for the weighted mean with the WOWA lower bound the dual model is a better choice.

In order to examine how much importance weighting of the WOWA complicates our optimization models we have rerun all the tests assuming equal importance weights thus restricting the models to the standard OWA optimization according to [18]. Tables 5 to 8 show the solution times for the primal (49)–(53) and the dual (54) OWA models as well as for the primal (42)–(47) and the dual (55) OWA bounded

Table 4 WOWA bound optimization times [s]: dual model (55)

Number of attributes (m)	Number of variables (q) 10	20	50	100	150	200	300	400
10	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
20	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
50	1.2	1.0	0.4	0.6	0.8	0.8	0.8	1.0
100	20.4	42.2	41.2	26.4	20.8	22.0	22.4	23.8
150	138.8	186.4	319.6	149.6	171.8	175.4	178.6	^{1}221.2
200	422.2	^{3}529.4	–	^{4}578.2	^{3}554.6	^{3}589.4	–	–

Table 5 OWA criterion optimization times [s]: primal model (49)–(53)

Number of attributes (m)	Number of variables (q) 10	20	50	100	150	200	300	400
10	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
20	0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.0
50	0.6	0.8	1.0	1.2	1.2	1.2	1.2	1.4
100	18.6	26.4	37.2	40.6	40.2	50.6	49.4	72.4
150	170.8	246.4	355.8	305.0	330.2	365.2	380.0	387.8
200	^{2}537.4	–	–	–	–	–	–	–

Table 6 OWA criterion optimization times [s]: dual model

Number of attributes (m)	Number of variables (q) 10	20	50	100	150	200	300	400
10	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
20	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
50	0.2	0.0	0.4	0.2	0.4	0.4	0.6	0.8
100	0.4	0.8	2.4	7.2	7.8	12.2	13.4	18.4
150	1.4	2.2	5.8	26.8	44.6	48.8	73.6	102.2
200	3.4	4.8	9.8	62.6	107.2	179.2	246.0	197.6
300	8.6	15.8	29.2	223.0	^{2}503.0	^{4}592.4	–	–
400	21.6	35.6	67.4	^{2}315.0	–	–	–	–

weighted mean optimization models, respectively, with equal importance weights while all the other parameters remain the same.

One may notice that in the case of the primal model the WOWA optimization times (Table 1 and Table 3) are 10–30% longer than the corresponding OWA optimization times (Table 5 and Table 7). On the other hand, in the case of the dual model the WOWA optimization times (Table 2 and Table 4) turn out to be shorter than the corresponding OWA times (Table 6 and Table 8), and frequently even shorter.

The optimization times were analyzed for various size parameters m and q. The basic tests were performed for the standard WOWA model with $n = m$. However, we

Table 7 OWA bound optimization times [s]: primal model (42)–(47)

Number of attributes (m)	Number of variables (q) 10	20	50	100	150	200	300	400
10	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
20	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
50	0.8	1.2	1.6	2.0	1.8	1.8	2.0	2.2
100	29.4	51.0	85.6	70.2	73.4	58.4	64.6	59.2
150	162.4	303.0	[1]508.0	440.4	393.0	365.2	380.0	414.0
200	[2]321.0	–	–	–	–	–	–	–

Table 8 OWA bound optimization times [s]: dual model (55)

Number of attributes (m)	Number of variables (q) 10	20	50	100	150	200	300	400
10	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
20	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
50	1.2	1.2	0.6	1.0	1.0	1.0	1.4	1.4
100	23.2	40.0	50.2	27.6	25.0	26.0	48.4	64.2
150	131.4	244.6	411.0	236.6	[1]213.6	132.0	145.2	170.4
200	437.2	–	–	–	–	[3]535.6	–	–

Table 9 WOWA optimization times [s]: varying number of preferential weights ($m = 100$, $q = 50$)

Model	Number of preferential weights (n) 3	5	10	20	50	100	150	200	300	400
WOWA criterion	0.0	0.0	0.0	0.2	1.0	1.0	1.4	2.2	3.2	4.8
WOWA bound	0.2	0.0	0.2	0.8	7.8	42.2	259.8	185.6	[2]500.2	–

also analyzed the case of larger n for more detailed preferences modeling, as well as the case of smaller n thus representing a rough preferences model.

Table 9 presents solution times for the dual model with different numbers of the preferential weights for problems with 100 attributes and 50 variables. One may notice that the computational efficiency can be improved by reducing the number of preferential weights which can be reasonable in non-automated decision making support systems. On the other hand, in case of the WOWA optimization (but not WOWA bounded weighted mean optimization) increasing the number of preferential weights and thus the number of breakpoints in the interpolation function does not induce the massive increase in the computational complexity.

6 Concluding Remarks

The problem of aggregating multiple attributes to form overall functions is of considerable importance in many disciplines. The WOWA aggregation [22] represents a universal tool allowing to take into account both the preferential weights allocated to ordered attribute values and the importance weights allocated to several attributes. The ordered aggregation operators are generally hard to implement when applied to variables. We have shown that the WOWA aggregations with the monotonic weights can be modeled by introducing auxiliary linear constraints. Exactly, the OWA LP-solvable models introduced in [18] can be expanded to accommodate the importance weighting of the WOWA aggregation used within the inequalities or objective functions.

Our computational experiments have shown that the formulations enable to solve effectively medium size problems. While taking advantages of the dual model the WOWA problems with up to 100 attributes have been solved directly by general purpose LP code within less than half a minute. Although the problems with the WOWA constraints have required typically more time than similar problems with the WOWA objective function. Further research on efficient algorithm for LP problems with the WOWA bounds seems to be promising.

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