Chapter 6
Nash consistent representation through lottery models

6.1 Motivation and summary

In Chapter 3 we have seen that – under the usual assumptions of monotonicity and superadditivity, and for a finite set of alternatives (social states) – effectivity functions (constitutions) can be represented by Nash consistent game forms if and only if the intersection condition on individual polar sets (3.6) is satisfied. This condition is quite restrictive, for instance, it is not satisfied by the effectivity function derived from the familiar 2×2 bimatrix game form (cf. Example 3.3.11).

A well-known way to avoid this condition is offered by game theory. If we represent preferences by von Neumann-Morgenstern utilities and allow mixed strategies in the representing game form, e.g., the canonical game form Γ_0 constructed in the proof of Theorem 2.4.7, then there always exists a Nash equilibrium in mixed strategies (Nash, 1951). Mixed strategies, however, can be hard to interpret – this is a longstanding discussion in game theory that we do not want to enter into here. Moreover, also the representation issue is under discussion, since outcomes can be probability distributions over the alternatives resulting from the use of mixed strategies, rather than only the original pure alternatives. In other words, admitting mixed strategies implies admitting lotteries over outcomes.

In this chapter we follow a different route in order to avoid the intersection condition (3.6). Instead of allowing mixed strategies we allow for some objective uncertainty concerning the outcomes of the game form. That is, we allow (some) lotteries over outcomes but do not need to allow mixed strategies. Specifically, we add the (finite) set of equal chance lotteries over the alternatives to the set of outcomes. Of course, also in this approach we have to look closely at the representation issue: the effectivity function associated with such a game form assigns sets of lotteries to coalitions. We will handle this question by considering so called lottery models. If E is the original effectivity function and $\widetilde{E}$ is the effectivity function associated with the game

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form $\widetilde{\Gamma}$ augmented by lotteries, then we will say that $\widetilde{E}$ *is a lottery model for* E if the following holds: for each coalition S and each $B \in E(S)$ there is a $\widetilde{B} \in \widetilde{E}(S)$ such that the total support (i.e., union of the supports) of the lotteries in $\widetilde{B}$ is equal to B; and, conversely, if $\widetilde{B} \in \widetilde{E}(S)$, then $B \in E(S)$ where B is the total support of the lotteries in $\widetilde{B}$. This seems a natural way of extending the idea of representation to lotteries. If a coalition S is effective for a set B then this means that S is entitled to or can enforce the final outcome to be in B; the same holds if S is effective for a set of lotteries with total support equal to B.

In the augmented game form, we assume that players evaluate such lotteries by utility functions satisfying the minimal requirement of stochastic dominance: this means that utility increases by shifting probability to better (pure) alternatives. Thus, if lottery ℓ' can be obtained from lottery ℓ by shifting probability to more preferred alternatives, then ℓ' is preferred over ℓ. Expected utility, for instance, is a special case of this.

With these assumptions, we are able to prove that for any effectivity function (satisfying the usual necessary conditions of monotonicity and superadditivity, and for a finite set of alternatives) there exists a lottery model which has a Nash consistent representation, without imposing further conditions on the effectivity function. The representing game form is finite, and no mixed strategies are used. The players play pure strategies, but the outcome may be uncertain.

As a simple but illustrative example, consider the unanimity effectivity function, where the grand coalition is effective for every single alternative and all other coalitions are completely powerless, i.e., only effective for the set of all alternatives. Since in any representing game form of this effectivity function any individual can bring about any alternative, given the strategy profile of the coalition of all other players, it follows that for any profile of preferences in which at least two players have different top elements, a Nash equilibrium cannot exist. By extending the effectivity function with equal chance lotteries, for instance such that every coalition other than the grand coalition is effective for every set of lotteries containing the lottery that assigns equal probability to each alternative, we obtain an effectivity function that preserves power and does have a Nash consistent representing game form – see Example 6.3.4 below.

For a constitution modeled as a monotonic and superadditive effectivity function, the relevance of our main result (Theorem 6.3.2) is that such a constitution can always be 'decentralized' by a set of rules (a game form) that preserves the original rights and that is stable in the sense that for any preferences a Nash equilibrium exists, as long as we are willing to accept some uncertainty in the form of equal chance lotteries as outcomes of the game, evaluated by utility functions respecting stochastic dominance.

For a given finite game form our result implies that we can always find an alternative finite game form, preserving effectivity in the indicated sense, that has a pure Nash equilibrium for any profile of preferences, again evaluating lotteries by utility functions respecting stochastic dominance (e.g., expected utility). This also entails a solution to the Gibbard (1974) paradox – see Example 3.3.11 and Example 6.3.3 below.

In Section 6.2 we extend Theorem 3.3.10 to accommodate for cardinal utilities, which are used in lottery models. Section 6.3 introduces lottery models and presents our main result (Theorem 6.3.2). In Section 6.4 we consider the case of neutral effectivity functions, for which a natural and simple lottery model can be based on the so-called uniform core. Section 6.5 concludes.

6.2 Nash consistent representation: an extension

In this section we assume that the set of alternatives is some finite set Z, containing at least two alternatives. Choices for Z include our usual finite set of social states A, augmented with equal chance lotteries over A, to be introduced later.

Let U be a non-empty set of *utility functions* $u : Z \to \mathbb{R}$, and let $X \in P_0(Z)$. Call X *admissible with respect to* U if there is a $u \in U$ such that $u(x) > u(y)$ for every $x \in X$ and every $y \in Z \setminus X$. Hence, a player with utility function u strictly prefers every element of X to every element not in X.

The following theorem extends Theorem 3.3.10. In this theorem the intersection condition (3.6) is weakened to condition (6.1) by making it conditional on admissibility of the involved sets.

Theorem 6.2.1. *Let $E : P(N) \to P(P_0(Z))$ be a superadditive and monotonic effectivity function. Then E has a representation that is Nash consistent on U^N if and only if*

$$[X^i \in E^*(i) \text{ and } X^i \text{ admissible w.r.t. } U \text{ for all } i \in N] \Rightarrow \bigcap_{i=1}^{n} X^i \neq \emptyset. \quad (6.1)$$

Proof. First assume that E has a Nash consistent representation Γ on U^N. For each $i \in N$, let $X^i \in E^*(i)$ be an admissible set and $u^i \in U$ such that $u^i(z) > u^i(y)$ for all $z \in X^i$ and $y \in Z \setminus X^i$. Let $x \in Z$ be a Nash equilibrium outcome of (Γ, u^N). Then, by Proposition 3.2.1,

$$L(x, u^i) \in E^\Gamma(N \setminus \{i\}) = E(N \setminus \{i\}) \text{ for all } i \in N.$$

This implies $X^i \cap L(x, u^i) \neq \emptyset$ for every $i \in N$. By the choice of u^i, this implies $x \in X^i$ for every $i \in N$, hence $\bigcap_{i \in N} X^i \neq \emptyset$, so that (6.1) holds.

For the converse, assume (6.1). Let $u^N \in U^N$. For every $i \in N$ let $Y^i := \{y \in Z \mid Z \setminus L(y, u^i) \in E^*(i)\}$ and define

$$X^i = \begin{cases} \bigcap_{y \in Y^i} Z \setminus L(y, u^i) & \text{if } Y^i \neq \emptyset \\ Z & \text{otherwise.} \end{cases}$$

Then $X^i \in E^*(i)$ and X^i is admissible with respect to U for each $i \in N$, so by (6.1), $\bigcap_{i\in N} X^i \neq \emptyset$. Take $x \in \bigcap_{i\in N} X^i$. Then, by definition of X^i, $Z\backslash L(x, u^i) \notin E^*(i)$ for each $i \in N$. Hence, there is some set $Z^i \subseteq L(x, u^i)$ with $Z^i \in E(N\backslash\{i\}$, so by monotonicity of E, $L(x, u^i) \in E(N\backslash\{i\})$ for every $i \in N$. Theorem 3.2.3 now implies that E has a Nash consistent representation. □

6.3 Lottery models

We shall now be more specific about the set Z. Let A be a finite set of alternatives, $|A| \geq 2$. For each $B \in P_0(A)$, $\ell(B)$ denotes the lottery that assigns equal probability $1/|B|$ to each alternative in B. The set of all such equal chance lotteries with support in a set $B \in P_0(A)$ is denoted by $\widetilde{B}$, hence

$$\widetilde{B} = \{\ell(B') \mid B' \in P_0(B)\}.$$

By identifying each $x \in A$ with the degenerate lottery $\ell(\{x\})$, we have $B \subseteq \widetilde{B}$.

Typically, we shall consider the case $Z = \widetilde{A}$. Let $\widetilde{E} : P(N) \to P(P_0(\widetilde{A}))$ be an effectivity function. With $\widetilde{E}$ we associate an effectivity function $E : P(N) \to P(P_0(A))$ as follows. Let $E(\emptyset) = \emptyset$. For $S \in P_0(N)$ and $B \in P_0(A)$, we let $B \in E(S)$ if and only if there exists an $X \in \widetilde{E}(S)$ such that

$$B = \bigcup_{B' \in P_0(A):\ \ell(B') \in X} B'. \tag{6.2}$$

In other words, elements of $E(S)$ are obtained by taking the union of the supports of elements of $\widetilde{E}(S)$. It is straightforward to check that, indeed, E is an effectivity function. If E is derived from $\widetilde{E}$ in this way, then we call $\widetilde{E}$ a *lottery model for* E.

Remark 6.3.1. Monotonicity of $\widetilde{E}$ with respect to the players implies monotonicity of E with respect to the players, and monotonicity of $\widetilde{E}$ with respect to the alternatives implies monotonicity of E with respect to the alternatives. (To show the latter claim, if $B \in E(S)$ resulting from $\widetilde{B} \in \widetilde{E}(S)$, and $C \supseteq B$, then $C \in E(S)$ follows from considering $\widetilde{B} \cup C \in \widetilde{E}(S)$. Note that $A \subseteq \widetilde{A}$.) Hence, monotonicity of $\widetilde{E}$ is inherited by E. If $\widetilde{E}$ is monotonic and superadditive, then also E is superadditive. For let $\widetilde{E}$ be monotonic and superadditive, $S_1, S_2 \in P_0(N)$ with $S_1 \cap S_2 = \emptyset$, and let $B_1 \in E(S_1)$ and $B_2 \in E(S_2)$. Let $X_1 \in \widetilde{E}(S_1)$ and $X_2 \in \widetilde{E}(S_2)$ correspond to B_1 and B_2 as in the definition of E, i.e., as in (6.2). Then superadditivity of $\widetilde{E}$ implies $X := X_1 \cap X_2 \in \widetilde{E}(S_1 \cup S_2)$, hence

$$E(S_1 \cup S_2) \ni \bigcup_{B' \in P_0(A):\ell(B') \in X} B' \subseteq B_1 \cap B_2.$$

Monotonicity of E now implies $B_1 \cap B_2 \in E(S_1 \cup S_2)$. This shows that E is superadditive. The converse is not true: a lottery model $\widetilde{E}$ for a monotonic and superadditive effectivity function E is not itself necessarily monotonic and superadditive.

Let $u : \widetilde{A} \to \mathbb{R}$ and suppose that $u(x_1) \geq u(x_2) \geq \ldots \geq u(x_m)$, where $A = \{x_1, x_2, \ldots, x_m\}$. For $\ell \in \widetilde{A}$ and $i \in \{1, 2, \ldots, m\}$ let ℓ_i be the probability assigned by ℓ to x_i. We say that u *respects stochastic dominance* if $u(\ell) \geq u(\ell')$ whenever $\ell, \ell' \in \widetilde{A}$ satisfy

$$\sum_{i=k}^{m} \ell_i \leq \sum_{i=k}^{m} \ell'_i \text{ for all } k = 1, 2, \ldots, m.$$

We assume that lotteries are evaluated by utility functions satisfying this condition. Therefore, we define

$$U_{\text{sd}} := \left\{ u \in \mathbb{R}^{\widetilde{A}} \mid \; u \text{ respects stochastic dominance} \right\}.$$

The set U_{sd} contains in particular the set of expected utility functions

$$\left\{ u \in \mathbb{R}^{\widetilde{A}} \mid u(\ell(B)) = \sum_{a \in B} \frac{u(a)}{|B|} \text{ for all } B \in P_0(A) \right\}.$$

The main result of this chapter is that for every monotonic and superadditive effectivity function there exists a lottery model which has a Nash consistent representation on U_{sd}^N. Clearly, monotonicity and superadditivity cannot be left out here: a lottery model that has a representing game form must be monotonic and superadditive, and by Remark 6.3.1 the original 'deterministic' effectivity function must also be monotonic and superadditive. But, in contrast to Theorem 6.2.1, no additional condition is needed on E.

Theorem 6.3.2. *Let $E : P(N) \to P(P_0(A))$ be a monotonic and superadditive effectivity function. Then there exists an effectivity function $\widetilde{E} : P(N) \to P(P_0(\widetilde{A}))$ such that*

(i) $\widetilde{E}$ is a lottery model for E;
(ii) $\widetilde{E}$ has a representation which is Nash consistent on U_{sd}^N.

Proof. Define $\widetilde{E} : P(N) \to P(P_0(\widetilde{A}))$ as follows. Let $\widetilde{E}(\emptyset) = \emptyset$ and for every $S \in P_0(N)$ with $|S| \neq n - 1$ let

$$\widetilde{E}(S) = \left\{ X \in P_0(\widetilde{A}) \mid X \supseteq \widetilde{B} \text{ for some } B \in E(S) \right\}.$$

In order to define $\widetilde{E}$ for $(n-1)$-person coalitions we introduce a notation. For each $i \in N$, $C \in E(N \setminus \{i\})$ and $C' \in P_0(C)$ define the set $X(C, C') \in P_0(\widetilde{A})$ by

$$X(C, C') = \begin{cases} \{\ell(\{c\} \cup C') \mid c \in C \setminus C'\} & \text{if } C' \neq C \\ \{\ell(C)\} & \text{if } C' = C. \end{cases}$$

Note in particular that the union of the supports of the elements of the set $X(C, C')$ is equal to C.

Let now $S \in P_0(N)$ with $|S| = n - 1$, say $S = N \setminus \{i\}$ for some $i \in N$. Then we define $X \in \widetilde{E}(N \setminus \{i\})$ if and only if $X \supseteq \widetilde{B}$ for some $B \in E(N \setminus \{i\})$ or $X \supseteq X(C, C') \cup \widetilde{B}$ for some $C \in E(N \setminus \{i\})$, $C' \in P_0(C)$, and $B \in P_0(A)$ such that $B \cap B' \neq \emptyset$ for all $B' \in E(\{i\}$. This concludes the definition of $\widetilde{E}$. It is straightforward to verify that $\widetilde{E} : P(N) \to P(P_0(\widetilde{A}))$ is a monotonic and superadditive EF and that $\widetilde{E}$ is a lottery model for E.

It remains to prove that $\widetilde{E}$ has a Nash consistent representation on U_{sd}^N. For each $i \in N$, let $X^i \in \widetilde{E}^*(i)$ such that X^i is admissible with respect to U_{sd}. In view of Theorem 6.2.1 it is sufficient to prove $\bigcap_{i \in N} X^i \neq \emptyset$. For each $i \in N$ choose $u^i \in U_{\text{sd}}$ such that

$$u^i(x) > u^i(y) \text{ for all } x \in X^i \text{ and } y \in \widetilde{A} \setminus X^i \tag{6.3}$$

(this is possible since each X^i is admissible), and choose $B^i \in E(\{i\})$ and $b^i \in B^i$ such that both

$$u^i(b^i) = \min\{u^i(b) \mid b \in B^i\} \geq \min\{u^i(b) \mid b \in B\} \text{ for all } B \in E(\{i\})$$

and

$$B^i = \{b \in A \mid u^i(b^i) \leq u^i(b)\}.$$

(This is possible in view of monotonicity of E.)

Also, for each $i \in N$, define $C^i := \bigcap_{j \in N \setminus \{i\}} B^j$. Then $C^i \in E(N \setminus \{i\})$ by superadditivity of E. Choose $a^i \in C^i$ such that

$$u^i(a^i) = \max\{u^i(a) \mid a \in C^i\}.$$

Since, by superadditivity of E, $C^i \cap B^i \neq \emptyset$, we have $u^i(a^i) \geq u^i(b^i)$.

Now fix a player $i \in N$ and write $u^i(x_1) \geq u^i(x_2) \geq \ldots \geq u^i(x_m)$, where $A = \{x_1, x_2, \ldots, x_m\}$. Let $k \in \{1, 2, \ldots, m\}$ such that $a^i = x_k$. Choose p with $k \leq p \leq m$ such that

$$u^i(\ell(\{x_k, x_p, x_{p+1}, \ldots, x_m\})) \leq u^i(\ell(\{x_k, x_{p'}, x_{p'+1}, \ldots, x_m\}))$$

for all $k \leq p' \leq m$. Consider the set $D = \{x_k, \ldots, x_m\}$. Since $C^i \subseteq D$, we have $D \in E(N \setminus \{i\})$. Let $D' := \{x_p, x_{p+1}, \ldots, x_m\} \subseteq D$. Define $Y \in P_0(\widetilde{A})$ by

$$Y = X(D, D') \cup \widetilde{F}$$

where

$$F = \left\{b \in A \mid u^i(b^i) \geq u^i(b)\right\}.$$

Observe that, by definition of b^i, we have $F \cap B' \neq \emptyset$ for every $B' \in E(\{i\})$. By definition of $\widetilde{E}$ and Y, we have $Y \in \widetilde{E}(N \setminus \{i\})$. It follows, in particular, that the set X^i contains an element of Y, say y. Consider the lottery $\bar{\ell} = \ell(\bigcap_{j \in N} B^j)$. Then

$$a^i \in B^i \cap C^i = \bigcap_{j\in N} B^j \text{ and } u^i(a^i) \geq u^i(c) \geq u^i(b^i) \text{ for all } c \in \bigcap_{j\in N} B^j. \tag{6.4}$$

We show that $\bar{\ell} \in X^i$ by considering all the possible values for $y \in Y \cap X^i$.

If $y \in \widetilde{F}$, then $u^i(\bar{\ell}) \geq u^i(b^i) \geq u^i(y)$, where the first inequality follows from (6.4) and the last inequality by definition of F and the fact that u^i respects stochastic dominance. By (6.3), this implies $\bar{\ell} \in X^i$.

If $y \in X(D, D')$, then $y = \ell(\{x_{p'}, x_p, \ldots, x_m\})$ for some $p' \in \{k, k+1, \ldots, p-1\}$ if $k < m$ and $y = x_m$ if $k = m$. In that case, we argue as follows. Write $\bigcap_{j\in N} B^j = \{a^i, y_1, \ldots, y_r\}$ with $u^i(a^i) \geq u^i(y_1) \geq u^i(y_2) \geq \ldots \geq u^i(y_r)$. Then

$$\begin{aligned} u^i(\bar{\ell}) &\geq u^i(\ell(\{a^i, x_{m-r+1}, x_{m-r+2}, \ldots, x_m\})) \\ &\geq u^i(\ell(\{a^i, x_p, \ldots, x_m\})) \\ &\geq u^i(\ell(\{x_{p'}, x_p, \ldots, x_m\})) \\ &= u^i(y) \end{aligned}$$

where the second inequality follows from the choice of p, and the third follows since $u^i(a^i) \geq u^i(x_{p'})$. (Note that for the first and third inequalities the fact that u^i respects stochastic dominance is used.) Hence also in this case, (6.3) implies that $\bar{\ell} \in X^i$.

Since $i \in N$ was arbitrary, we conclude that $\bar{\ell} \in X^j$ for every $j \in N$, hence $\bigcap_{j\in N} X^j \neq \emptyset$. □

The following example illustrates the effectivity function $\widetilde{E}$, constructed in the proof of Theorem 6.3.2, for the effectivity function associated with a 2×2 bimatrix game form.

Example 6.3.3. Let $N = \{1, 2\}$, $A = \{a, b, c, d\}$, and consider the effectivity function E derived from the game form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where player 1 chooses a row and player 2 chooses a column. In particular, $E(\{1\})$ contains $\{a, b\}$, $\{c, d\}$, and all supersets; $E(\{2\})$ contains $\{a, c\}$, $\{b, d\}$, and all supersets. It is easy to see that condition (6.1) in Theorem 6.2.1 (with $Z = A$) is not fulfilled. For instance, $\{a, d\} \in E^*(\{1\})$, $\{b, c\} \in E^*(\{2\})$, both are (trivially) admissible with respect to $\mathbb{R}^A$, but $\{a, d\} \cap \{b, c\} = \emptyset$. [We recall the explanation (cf. Example 3.3.11) of why a Nash consistent representation cannot exist. Observe that $\{a, d\} \in E^*(\{1\})$ means that for any given strategy of player 2 in some representing game form, player 1 can make sure that the final outcome is in $\{a, d\}$. A similar statement holds for player 2 and the set $\{b, c\}$. Now suppose that preferences are such that player 1 prefers a and d over b and c and player 2 prefers b and c over a and d: then, clearly, a Nash equilibrium cannot exist. This illustrates, again, the necessity of the intersection condition (6.1).]

The effectivity function $\widetilde{E}$, constructed in the first two paragraphs of the proof of Theorem 6.3.2, assigns the following sets (where, e.g., ab is shorthand for $\ell(\{a,b\})$, the equal chance lottery on $\{a,b\}$):

$$\begin{array}{ll} \widetilde{E}(\{1\}) : & \{a,b,ab\}\ \{a,d,ab\}\ \{c,b,ab\}\ \{c,d,ab\} \\ & \{a,b,cd\}\ \{a,d,cd\}\ \{c,b,cd\}\ \{c,d,cd\} \\ \widetilde{E}(\{2\}) : & \{a,c,ac\}\ \{a,d,ac\}\ \{b,c,ac\}\ \{b,d,ac\} \\ & \{a,c,bd\}\ \{a,d,bd\}\ \{b,c,bd\}\ \{b,d,bd\} \end{array}$$

and all supersets of these sets within $\widetilde{A}$. It is easy to check that $\widetilde{E}$ is a lottery model for E. By the proof of Theorem 6.3.2, $\widetilde{E}$ has a Nash consistent representation. This can also be verified 'directly' by using Theorem 6.2.1 and showing that $X^1 \cap X^2 \neq \emptyset$ for all *admissible* $X^1 \in \widetilde{E}^*(\{1\})$ and $X^2 \in \widetilde{E}^*(\{2\})$, but this is a rather tedious task. Admissibility has strong implications. For instance, if $a, b \in X^1$, then also $ab \in X^1$, or if $ab \in X^1$, then also $a \in X^1$ or $b \in X^1$, etc.

The Gibbard Paradox (Example 3.3.11) is an instance of this example. Theorem 6.3.2 shows how it can be resolved by allowing lotteries, in the way as described above.

For particular cases, there may exist other lottery models that are less complex than the one constructed in the proof of Theorem 6.3.2 and in that sense more attractive. This is the case in the next example, where the unanimity effectivity function is considered.

Example 6.3.4. Let $E : P(N) \to P(P_0(A))$ be the *unanimity effectivity function*, i.e., $E(S) = \{A\}$ for all $S \in P_0(N)$, $S \neq N$. This effectivity function clearly fails to satisfy condition (6.1). It can be checked that here the lottery $\bar{\ell}$ in the proof of Theorem 6.3.2 is equal to $\ell(A)$, but in this case the lottery model $\widetilde{E}$ in that proof is overly complicated. It is straightforward to see that also the effectivity function $\widetilde{E}'$ is a lottery model for E, where for each $S \in P_0(N)$, $S \neq N$, $\widetilde{E}'(S)$ consists of $\{\ell(A)\}$ and all its supersets in $\widetilde{A}$, and $\widetilde{E}'(N) = P_0(\widetilde{A})$. The effectivity function $\widetilde{E}'$ is different from but simpler than $\widetilde{E}$. By applying Theorem 6.2.1 and checking condition (6.1) – for each player i each element of $\widetilde{E}^*(\{i\})$ must contain $\ell(A)$ – it follows that this lottery model has a Nash consistent representation.

Example 6.3.4 is a special case of a neutral effectivity function. These effectivity functions are studied in the next section.

6.4 Neutral effectivity functions

For convenience we recall from Section 3.5.1 some facts about veto functions. A veto function is a function $v : P(N) \to \{-1, 0, \ldots, |A| - 1\}$ such that

$v(\emptyset) = -1$, $v(S) \geq 0$ if $S \in P_0(N)$, and $v(N) = |A| - 1$. The interpretation is that coalition S can veto any subset of the alternatives with at most $v(S)$ elements. With v we can associate a neutral (i.e., not depending on the names of the alternatives) effectivity function E_v by

$$E_v(S) = \{B \in P_0(A) \mid v(S) \geq |A \setminus B|\} = \{B \in P_0(A) \mid v(S) \geq |A| - |B|)\}$$

for every $S \in P(N)$. Conversely, every neutral effectivity function is derived from some veto function. A veto function is monotonic if $v(S) \leq v(S^*)$ for all S, S^* with $S \subseteq S^*$, and superadditive if $v(S) + v(S^*) \leq v(S \cup S^*)$ for all $S, S^* \in P(N)$ with $S \cap S^* = \emptyset$. A veto function is monotonic [superadditive] if and only if the associated effectivity function is monotonic [superaditive].

We shall show that for neutral effectivity functions there exists a simple and quite natural lottery model that has a Nash consistent representation. To this end we need the concept of the uniform core.

Let $E : P(N) \to P(P_0(A))$ be a monotonic and superadditive effectivity function. Let $U = \mathbb{R}^A$, $u^N \in U^N$, and say that $x \in A$ is *uniformly dominated* by $B \in P_0(A)$ via $S \in P_0(N)$ if (i) $B \in E(S)$; (ii) $x \notin B$; and (iii) $u^i(b) > u^i(a)$ for all $b \in B$, $a \in A \setminus B$, and $i \in S$. We also say that S *blocks* x by B.

Observe that, if x is uniformly dominated by B via S, then x is also dominated (cf. Definition 5.2.5) by B via S. The converse is not true: for uniform domination we need that for every player in S the set of alternatives better than x is exactly the set B, for domination we only need that it contains B.

The set of all alternatives that are not uniformly dominated by some set B via some coalition S is called the *uniform core* and denoted $C_{\mathrm{uf}}(E, u^N)$. Obviously, by the above, the core $C(E, u^N)$ is a subset of the uniform core $C_{\mathrm{uf}}(E, u^N)$. While the core can be empty, the uniform core is never empty. This is proved in Abdou and Keiding (1991, Lemma 3.2). For completeness' sake we present a proof here.

Lemma 6.4.1. *Let $E : P(N) \to P(P_0(A))$ be a monotonic and superadditive effectivity function, and let $u^N \in U^N$. Then $C_{uf}(E, u^N) \neq \emptyset$.*

Proof. Suppose, to the contrary, that $C_{\mathrm{uf}}(E, u^N) = \emptyset$. Write $A = \{x_1, \ldots, x_m\}$ and for each $j = 1, \ldots, m$ let $S_j \in P_0(N)$ and $B_j \in E(S_j)$ such that x_j is uniformly dominated by B_j via S_j. Then, since for each j we have $x_j \notin B_j$, it follows that $\bigcap_{j=1}^m B_j = \emptyset$.

Now, without loss of generality, let $\{B_1, \ldots, B_r\}$, where $1 \leq r \leq m$, be those sets in $\{B_1, \ldots, B_m\}$ that are minimal under inclusion, where we take only one of two equal sets if any. We claim that the corresponding sets $S_1, \ldots, S_r$ are pairwise disjoint. Indeed, suppose for instance that $i \in S_1 \cap S_2$, and, say, $u^i(x_1) \geq u^i(x_2)$. Then, by definition of uniform domination it follows that $B_1 \subseteq B_2$, contradicting inclusion minimality of the sets in $\{B_1, \ldots, B_r\}$. By superadditivity we have $\bigcap_{j=1}^r B_j \in E(\bigcup_{j=1}^r S_j)$. Since, clearly, $\bigcap_{j=1}^m B_j = \bigcap_{j=1}^r B_j$, we obtain

$$\emptyset = \bigcap_{j=1}^{m} B_j = \bigcap_{j=1}^{r} B_j \in E(\bigcup_{j=1}^{r} S_j)\ ,$$

a contradiction. □

The uniform core represents E in the following sense. If S is effective for a set of alternatives B, then S has a utility profile such that the associated uniform core is a subset of B for every utility profile of the players outside S. Formally, we have the following lemma (cf. Keiding and Peleg, 2006a).

Lemma 6.4.2. *Let $E : P(N) \to P(P_0(A))$ be a monotonic and superadditive effectivity function. Then, for every $S \in P_0(N)$ and every $B \in P_0(A)$,*

$$B \in E(S) \Leftrightarrow \exists u^S \in U^S\ \forall u^{N\setminus S} \in U^{N\setminus S} : C_{uf}(E, (u^S, u^{N\setminus S})) \subseteq B.$$

Proof. Let $S \in P_0(N)$ and $B \in P_0(A)$.

First, suppose $B \in E(S)$. For each $i \in S$ let $u^i \in U$ be defined by $u^i(x) = 1$ for all $x \in B$ and $u^i(x) = 0$ for all $x \in A \setminus B$. Then all $x \in A \setminus B$ are blocked by S using B, so that $C_{\text{uf}}(E, (u^S, u^{N\setminus S})) \subseteq B$ for all $u^{N\setminus S} \in U^{N\setminus S}$.

Second, for the converse, let $u^S \in U^S$ such that $C_{\text{uf}}(E, (u^S, u^{N\setminus S})) \subseteq B$ for all $u^{N\setminus S} \in U^{N\setminus S}$. First observe that, if $x \in A \setminus B$, then x must be blocked via some coalition $S' \subseteq S$. Indeed, otherwise take, for each $i \notin S$, a preference u^i with $u^i(x) = 1$ and $u^i(y) = 0$ for all $y \neq x$: then no player outside S can participate in blocking x, so $x \in C_{\text{uf}}(E, (u^S, u^{N\setminus S})) \subseteq B$, a contradiction.

Now, for each player $i \in S$, take $x^i \in A$, $S^i \subseteq S$ with $i \in S^i$, and $B^i \in E(S^i)$ such that (i) x^i is blocked by B^i via S^i and (ii) for each $y \in B^i$, y is not blocked via any coalition $S' \subseteq S$ with $i \in S'$; if such a triple does not exist for some player i, then we take B^i equal to A. Without loss of generality let $S = \{1, \ldots, |S|\}$ and let $\{B^1, \ldots, B^k\}$ with $1 \leq k \leq |S|$ be the subset of those elements of $\{B^1, \ldots, B^{|S|}\}$ that are minimal under inclusion and all different (if two minimal sets B^i and B^j are equal then take only one of the two). By the same argument as in the proof of Lemma 6.4.1 the associated coalitions $S^1, \ldots, S^k$ are pairwise disjoint and thus, by superadditivity and monotonicity,

$$\bigcap_{i=1}^{|S|} B^i = \bigcap_{i=1}^{k} B^i \in E(\bigcup_{i=1}^{k} S^i) \subseteq E(S)\ .$$

Consider any $x \in \bigcap_{i=1}^{|S|} B^i$. If x were blocked by some coalition $S' \subseteq S$, then this would violate condition (ii) in the definition of the triple x^i, S^i, B^i for the players $i \in S'$, a contradiction. By the argument in the second paragraph of the proof, it follows that $x \in B$. Hence, $\bigcap_{i=1}^{|S|} B^i \subseteq B$, so that $B \in E(S)$ by monotonicity. □

We can now construct a lottery model for E based on the uniform core. We formulate this as a lemma: the proof is straightforward using Lemma 6.4.2.

Lemma 6.4.3. *Let $E : P(N) \to P(P_0(A))$ be a monotonic and superadditive effectivity function. Define $\widetilde{E}_{uf} : P(N) \to P(P_0(\widetilde{A}))$ by requiring for each $S \in P_0(N)$ and $X \in P_0(\widetilde{A})$:*

$$X \in \widetilde{E}_{uf}(S) \Leftrightarrow \exists u^S \in U^S \; \forall u^{N\setminus S} \in U^{N\setminus S} : \ell\left(C_{uf}(E, (u^S, u^{N\setminus S}))\right) \in X.$$

Then $\widetilde{E}_{uf}$ is a monotonic and superadditive lottery model for E.

The construction of $\widetilde{E}_{\text{uf}}$ implies, in fact, that it is the effectivity function associated with the game form $\Gamma_{\text{uf}} = (U, \ldots, U; g; \widetilde{A})$, defined by $g(u^N) = \ell(C_{\text{uf}}(E, u^N))$ for each $u^N \in U^N$. Hence, the game form Γ_{uf} represents the effectivity function $\widetilde{E}_{\text{uf}}$, which in turn is a lottery model for E. We will show that Γ_{uf} is Nash consistent. Observe that Γ_{uf} is in fact a social choice function, where each player just reports a weak ordering over the alternatives in the form of a utility function – in fact, it is crucial for the proof of Theorem 6.4.4 below that the reported ordering can be weak. Given such a profile of reports one computes the uniform core and the outcome of the game is the equal chance lottery over the elements of the uniform core.

Theorem 6.4.4. *Let $u^N \in U^N_{\text{sd}}$. Then the game (Γ_{uf}, u^N) has a Nash equilibrium.*

Proof. We construct a strategy profile $\hat{u}^N \in U^N$ inductively as follows. First, let $W(1) \subseteq A$ contain exactly $v(1)$ worst alternatives according to u^1, that is, $u^1(x) \le u^1(y)$ for all $x \in W(1)$ and $y \in A \setminus W(1)$. Define $\hat{u}^1(x) = 0$ and $\hat{u}^1(y) = 1$ for all $x \in W(1)$ and $y \in A \setminus W(1)$. Let $k \in \{2, \ldots, n\}$ and suppose that $\hat{u}^l$ has been defined for all $1 \le l \le k-1$. Then let $W(k) \subseteq A$ contain exactly $v(k)$ worst alternatives in $A \setminus \bigcup_{l=1}^{k-1} W(l)$ according to u^k, and define $\hat{u}^k(x) = 0$ and $\hat{u}^k(y) = 1$ for all $x \in W(k)$ and $y \in A \setminus W(k)$.

We claim that $\hat{u}^N$ is a Nash equilibrium in $(\Gamma_{\text{uf}}, u^N)$. Let $k \in N$ and assume that each player $l \in N \setminus \{k\}$ plays the strategy $\hat{u}^l$. Consider any coalition $S \subseteq N \setminus \{k\}$ with more than one player. Then, because of the strict inequality sign in condition (iii) of the definition of uniform domination, S could only possibly block some alternative by the set $A \setminus W(l)$ for some $l \in S$, but all these sets are different since all sets $W(l)$ are different. Hence, only singletons in $N \setminus \{k\}$ block: each $l \in N \setminus \{k\}$ blocks $W(l)$, so altogether the set $\bigcup_{l \in N\setminus\{k\}} W(l)$ is blocked by the single players in $N \setminus \{k\}$. Consider the decision problem for player k. By the same argument as before, a non-singleton coalition S containing player k can only possibly block some alternative if $S = \{k, j\}$ for some $j \neq k$ (since all sets $W(l)$, $l \in N \setminus \{k\}$ are different), but in that way S can only block the set $W(j)$, namely by player k playing some strategy u' such that $u'(x) < u'(y)$ for all $x \in W(j)$ and $y \in A \setminus W(j)$. Then the game would result in the equal chance lottery $\ell\left(A \setminus \bigcup_{l \in N\setminus\{k\}} W(l)\right)$. By only using his own blocking power, however, player k can make sure that the outcome of the game is $\ell\left(A \setminus \bigcup_{l \in N} W(l)\right)$ by playing $\hat{u}^k$. Since player k's utility function u^k respects stochastic dominance, this is clearly an improvement for player k, and also the best outcome attainable by using k's own blocking power. □

The Nash equilibrium exhibited in the proof of Theorem 6.4.4 is a very natural one, since it consists of successive sincere vetoing of alternatives: first, player 1 vetoes his $v(1)$ worst alternatives, next, player 2 vetoes his $v(2)$ worst alternatives of the remaining ones, etc. Of course, vetoing according to any other ordering of the players would also be a Nash equilibrium. These specific equilibria have the drawback that they need not be Pareto optimal. For instance, if all players have the same preference, with a unique top alternative, but $\sum_{i \in N} v(i) < |A| - 1$, then the resulting lottery does not put probability 1 on the common top alternative. Of course, in this example the profile in which every player reports his true preference is also a Nash equilibrium: the uniform core associated with this profile consists of the common top alternative and, thus, the degenerate lottery that puts all probability on this alternative results.

If E is non-neutral, then $\widetilde{E}_{\mathrm{uf}}$ is still a lottery model for E and $\widetilde{E}_{\mathrm{uf}} = E^{\Gamma_{\mathrm{uf}}}$, but it is not clear whether Γ_{uf} is still Nash consistent.

6.5 Notes and comments

In this chapter, which is based on Peleg and Peters (2009), we have proved that every (monotonic and superadditive) effectivity function can be augmented, by adding finitely many equal chance lotteries, to a new effectivity function (lottery model) which preserves the original effectivity and has a Nash consistent representation. This approach is based on two particular assumptions. We elaborate on these assumptions in the next two remarks.

Remark 6.5.1. First, we assume that in the lottery model the original effectiveness of a coalition S of players for a set B of alternatives is preserved if S is now effective for some set X of equal chance lotteries such that the union of the supports of the lotteries in X is equal to B. For instance, if $B = \{a, b, c\}$, then X could be the one-point set $\{\ell(\{a, b, c\})\}$ but also the two-point set $\{a, \ell(\{a, b, c\})\}$. This example shows that, in this set-up, we cannot really interpret effectiveness for B as the alternatives of B being equiprobable, even if we only add equal chance lotteries. Rather, players (or coalitions) evaluate effectiveness purely in terms of supports.

Remark 6.5.2. Second, we assume that equal chance lotteries resulting as outcomes of the representing game form are evaluated by utility functions respecting first order stochastic dominance. This is a minimal requirement and therefore hardly controversial. A special case of this is expected utility. For a justification of the use of expected utility see Fishburn (1972), where preferences on sets of alternatives are considered and the expected utility property for equal chance lotteries is derived from conditions on these preferences. The assumption of equal chance lotteries evaluated by expected utility has been made frequently in the social choice literature, such as in Barberà, Dutta, and Sen (2001), but also in earlier work, e.g., Feldman (1980). In these works, outcomes can be sets, which are evaluated as equal chance lotteries using expected utility. In fact, this was also done in Section 6.4, where we considered the uniform core and evaluated that set as an equal chance lottery.

The assumption of utility functions respecting first order stochastic dominance is called 'monotonicity' in Abreu and Sen (1991).

We next comment on Pareto optimality in relation to lottery models.

Remark 6.5.3. By using the game form Γ_0, constructed in the proof of Theorem 2.4.7, to represent a lottery model $\widetilde{E}$, we obtain again weak acceptability: for any profile of preferences there there is a Nash equilibrium with Pareto optimal outcome. This follows since Theorem 3.3.13 continues to hold in the extended framework of Theorem 6.2.1.

What does this mean? Suppose $u^N \in U^N_{\mathrm{sd}}$ is a profile of preferences and $a, b \in A$ such that $u^i(a) > u^i(b)$ for all $i \in N$. Then, clearly, a lottery that has b but not a in its support is not Pareto optimal, and so there is a Nash equilibrium where this lottery is not the associated outcome. On the other hand, it is not difficult to come up with an example of a Pareto optimal lottery containing both a and b, since we only allow equal-chance lotteries.[1] Thus, Pareto optimality of a lottery does not imply that only Pareto optimal pure alternatives occur in the support.

Our final comment is related to the avoidance of mixed strategies in the game form associated with a lottery model.

Remark 6.5.4. The main result in this chapter is also a contribution to the classical 'purification' problem – e.g., Harsanyi (1973). For any finite game form, it enables us to construct a new finite game form which preserves the strategic possibilities of players and coalitions in the sense that the associated effectivity function is a lottery model for the effectivity function associated with the original game form, and which has a pure Nash equilibrium for any profile of utility functions respecting first order stochastic dominance among equal chance lotteries.

[1] E.g., $N = \{1, 2\}$, $A = \{a, b, c\}$, $u^1(c) = 2$, $u^1(a) = 0.4$, $u^1(b) = 0$, $u^2(a) = 2$, $u^2(b) = 1.3$, $u^2(c) = 0$; assume expected utility. Then $\ell(A)$ is Pareto optimal, although both agents strictly prefer a to b.