Chapter 2
Constitutions, effectivity functions, and game forms

2.1 Motivation and summary

In this chapter we expound on Gärdenfors's (1981) theory of rights-systems or constitutions. Gärdenfors formalizes rights-systems as follows. If S is a coalition (a group of individuals, members, players,...) and B is a set of social states (outcomes, alternatives,...), then B is a 'right' of S in the sense of Gärdenfors if S is legally entitled to the final social state being in B. The set of all pairs (S, B) where S is a coalition and B is a right of S, is a rights-system. Under very mild conditions a rights-system is a so-called effectivity function. (Effectivity functions are formally introduced in Definition 2.3.1.) Under some additional intuitive conditions, implying the requirements of monotonicity and consistency as postulated in Gärdenfors (1981), a rights-system is a monotonic and superadditive effectivity function (see Section 2.3). Gärdenfors's definition of rights is somewhat indirect, as it is based on attainability of social states. Therefore, we first introduce Peleg's (1998) model of a constitution (Section 2.2). This model distinguishes between rights and social states and describes explicitly how rights of groups result in attainable sets of social states. Nevertheless, for a given assignment of rights the model can be reduced to a rights-system in the sense of Gärdenfors (see Section 2.3). Section 2.2 also contains some important examples – such as the example underlying Gibbard's Paradox (1974) – which are used throughout Part I of this book.

A society cannot function exclusively on the basis of a rights-system or constitution but additionally needs a collection of rules that delimits the actions available and permissible to individuals. For instance, freedom of speech is a basic right in most constitutions but in practice one needs a set of rules that distinguish between an individual's right to express his opinion on the one hand, and slander or discrimination on the other hand. Such rules are the legal means to reach social states that agree with the constitution. As another example, the right to a minimum subsistence level may be part of the

B. Peleg and H. Peters, *Strategic Social Choice*, Studies in Choice and Welfare, DOI 10.1007/978-3-642-13875-1_2,

constitution, but stealing is usually not regarded as a legal way to satisfy it. Such a collection of rules will be formalized by a game form (Definition 2.4.1). Thus, we search for a game form that 'represents' the effectivity function (constitution). This idea of representation will be given a precise meaning: the game form should endow each group in the society (including, of course, single individuals) with the same possibilities as intended by the constitution. This also implies, basically, that the 'legality' of the game form is judged in terms of the constitution itself. For instance, if stealing is forbidden and, thus, every group of individuals is entitled to a social state where nobody gets robbed, then no individual will have stealing available as a strategy in the game form, since this could lead to a social state where at least one other individual is the victim of theft. In Section 2.4 we prove the existence of a representation for every monotonic and superadditive effectivity function (Theorem 2.4.7). This theorem will be applied and extended throughout Part I.

In Section 2.5 we briefly comment on the possibility of the simultaneous exercising of rights by disjoint coalitions. Section 2.6 concludes with some further remarks.

Notations. The following notations are used throughout this book. For an arbitrary set D, $|D|$ denotes its cardinality (possibly infinite), $P(D)$ is the collection of all subsets of D, and $P_0(D)$ is the collection of all non-empty subsets of D. For a subset C of D, C^+ denotes the collection of all supersets of C, that is, $C^+ = \{C' \in P(D) \mid C \subseteq C'\}$.

2.2 Constitutions

In this section we present a precise definition of a constitution. Although our definition is based on Gärdenfors (1981), it is more general since we distinguish between rights on the one hand and attainable sets of social states on the other hand.[1] We start with the definition of a society.

Definition 2.2.1. A *society* is a list $\mathcal{S} = (N, A, \rho, \alpha, \gamma)$ where

(1) N is the (finite) set of *members* of $\mathcal{S}$.
(2) A is the (finite or infinite) set of *social states*.
(3) ρ is the (finite) set of *rights*.
(4) $\alpha : P(N) \to P(\rho)$, with $\alpha(\emptyset) = \emptyset$, is the (current) *assignment* of rights to groups of members of $\mathcal{S}$.
(5) $\gamma : P(N) \times P(\rho) \to P(P_0(A))$, with $\gamma(\emptyset, \theta) = \emptyset$ and $A \in \gamma(S, \theta)$ for all $\theta \subseteq \rho$ and $S \in P_0(N)$, is the *access correspondence*. Thus, γ determines the sets of attainable social states by groups of members of $\mathcal{S}$ as a function of their rights.

[1] A particular consequence is, that this approach allows the change of the constitution as a function of time. See also Remark 2.6.1. However, the possibility of changing constitutions is not further explored in this book.

Some comments on this definition of a society are in order. First, a social state is a complete description of all aspects relevant to the members of society of a possible social situation. Whether the number of social states is finite or infinite depends on the specific application. Sometimes it may be convenient and instructive to model the set of social states as an infinite set, possibly a continuum, with some topological or measure-theoretic structure. Instead of the term 'social state' we often also use the terms *outcome* and *alternative*.

In our definition of a society the set of rights ρ is an abstract set. Intuitively, however, rights are means to reach certain social states. They determine some major aspects of the 'distribution of power' in society $\mathcal{S}$. In our definition, this is reflected by the access correspondence γ. The definition of this correspondence needs a more detailed explanation. If S is a (non-empty) group of society members and $\theta \subseteq \rho$ is a set of rights, then $\gamma(S, \theta) = \{B_1, \ldots, B_m\}$ is interpreted as follows: if the group S has rights θ, then it is legally entitled to the final social state being in B_1 or in B_2 or ... or in B_m. More precisely, S could insist on the final social state being in B_1, or S could insist on the final social state being in B_2, etc. But it does not mean that S can insist on the final social state being in the intersection of these sets, assuming that this intersection is non-empty. We shall elaborate on this point in the examples below, and also in Section 2.5, when we discuss the simultaneous exercising of rights by disjoint coalitions. The additional conditions on the access correspondence in (5) above express that the empty coalition is not entitled to anything (this is merely a formal condition) and that any non-empty coalition is at least entitled to the set of all social states. If, in some case, we have $\gamma(S, \theta) = \{A\}$, then this means that S is essentially powerless. In particular, it is usually natural to have $\gamma(S, \emptyset) = \{A\}$ for any (non-empty) coalition S.

The first example we consider is a classical example, basically due to Gibbard (1974).

Example 2.2.2. Consider a society with two members. Each member has two shirts, a white one and a blue one, and must wear exactly one of the two. Denote w for white and b for blue. Then the set of members is $N = \{1, 2\}$ and the set of social states is $A = \{(w, w), (w, b), (b, w), (b, b)\}$, where for each state the first letter refers to the color of 1's shirt and the second letter to the color of 2's shirt. The set of rights is $\rho = \{r_1\}$, where r_1 is the right for each member of a group to which r_1 is assigned, to choose his own shirt.[2] The rights assignment α is given by $\alpha(\emptyset) = \emptyset$, and $\alpha(1) = \alpha(2) = \alpha(N) = r_1$.[3] The access correspondence γ is defined as follows. For all $\theta \subseteq \rho$, $\gamma(\emptyset, \theta) = \emptyset$. Further, $\gamma(S, \emptyset) = \{A\}$ for all non-empty $S \subseteq N$, $\gamma(1, r_1) = \{(w, w), (w, b)\}^+ \cup \{(b, w), (b, b)\}^+$,

[2] Thus, we formalize this right as one and the same right applicable to different groups. Alternatively, it could be modelled as three different rights for the three non-empty groups in this example.

[3] When no confusion is likely we will often denote a singleton $\{a\}$ by a.

$\gamma(2, r_1) = \{(w,w),(b,w)\}^+ \cup \{(w,b),(b,b)\}^+$, and $\gamma(N, r_1) = P_0(A)$. We shall return to this example more than once. It has played an important role in the literature, see also Gaertner, Pattanaik, and Suzumura (1992).

We proceed with a somewhat more elaborate example, which is related to another example in Gibbard (1974).

Example 2.2.3. Let $N = \{m_1, m_2, f\}$, where m_i is a man, $i = 1, 2$, and f is a woman. We let $A = \{w_1, w_2, s\}$, where w_i is the social state in which f marries m_i, $i = 1, 2$, and s denotes the state where f remains single. The set of rights is $\rho = \{r_1, r_2\}$, where r_1 is the right to remain single (which is not a vacuous right in some societies), and r_2 is the right of a mixed couple to marry (an orthodox society). The assignment of rights is given by $\alpha(m_1) = \alpha(m_2) = \alpha(f) = r_1$ and $\alpha(m_i, f) = r_2$ for $i = 1, 2$. The other groups have no rights as groups. The access correspondence is as follows. If m_1 has right r_1, then m_1 is entitled to the 'final' social state being in the set $\{w_2, s\}$, and, trivially, all supersets: so $\gamma(m_1, r_1) = \{w_2, s\}^+$. By a similar kind of reasoning we have $\gamma(m_1, r_2) = \{A\}$, since for m_1 having the right r_2 does not give any 'power' (legal entitlement): m_1 would need the consent of f to marry her and, moreover, m_2 might also have the right to marry f, and these rights cannot simultaneously be met if polyandry is prohibited. Table 2.1 presents the complete access correspondence for all non-empty groups.

rights	m_1	m_2	f	m_1m_2	m_1f	m_2f	m_1m_2f
$\emptyset$	A	A	A	A	A	A	A
r_1	$\{s, w_2\}$	$\{s, w_1\}$	$\{s\}$	$\{s\}$	$\{s\}$	$\{s\}$	$\{s\}$
r_2	A	A	A	A	$\{w_1\}$	$\{w_2\}$	$\{w_1\}, \{w_2\}$
ρ	$\{s, w_2\}$	$\{s, w_1\}$	$\{s\}$	$\{s\}$	$\{w_1\}, \{s\}$	$\{w_2\}, \{s\}$	$\{x\},\ x \in A$

Table 2.1 Description of the access correspondence of Example 2.2.3.

The entry in row ρ and column m_1f, for instance, must be read as $\gamma(\{m_1, f\}, \rho) = \{w_1\}^+ \cup \{s\}^+$. This means that the group consisting of man m_1 and woman f is entitled to the social state where the two get married and also to the social state where both remain single. Further, $\gamma(N, \rho) = \gamma(\{m_1, m_2, f\}, \rho) = P_0(A)$.

The constituents of a society which are directly connected with rights form the constitution.

Definition 2.2.4. Let $\mathcal{S} = (N, A, \rho, \alpha, \gamma)$ be a society. The triple (ρ, α, γ) is called a *constitution*.

Thus, a constitution consists of a set of rights, an assignment of rights to groups of members of the society, and a function that specifies for each group of members the attainable sets of social states as a function of the rights of the group.

Our definition of a constitution allows for personal rights: there are no *a priori* symmetry conditions imposed on α or γ. In existing constitutions the same rights are assigned to members of society with similar characteristics. This is of course not ruled out in applications of our definition of a society. For instance, in Example 2.2.3 the two men have the same characteristics and play symmetric roles, which is reflected by both the rights assignment α and the access correspondence γ.

Remark 2.2.5. The above consideration may be formalized by introducing a set of parameters π such that each member i of the society is completely specified, for the sake of the analysis of rights and power, by a non-empty subset $\pi_i \subseteq \pi$. Under this assumption, two members i and j can be called *symmetric* if $\pi_i = \pi_j$. Also, the constitution (ρ, α, γ) satisfies *equal treatment* if for every pair of symmetric members i and j the transposition (i, j) is a symmetry of the pair (α, γ): that is, if $\pi_i = \pi_j$ and $S \subseteq N \setminus \{i, j\}$, then $\alpha(S \cup i) = \alpha(S \cup j)$ and $\gamma(S \cup i, \theta) = \gamma(S \cup j, \theta)$ for every $\theta \subseteq \rho$. For instance, in Example 2.2.3 one could introduce the parameter set $\pi = \{\text{male, female}\}$ and check that indeed m_1 and m_2 are symmetric.

In a similar vein it is relevant to note that, although individuals may exercise the rights assigned to the groups of which they are members, that does not mean that these rights become individual. For instance, persons over 65 as well as disabled persons may be entitled to free public transportation. In the terminology of the preceding remark, parameters (dummy variables) stating whether a person is over 65 and whether a person is disabled would be included in the overall set of parameters. So a person i may be entitled to free public transportation because he is 65, or because he is disabled, or because he belongs to both groups, but not because he is person i. (Of course, it could happen that person i has special exemption from paying public transportation fares, in which case this right is strictly individual.) The same example also shows that there is no contradiction in one group having right r and another overlapping group not having right r.

It is also of interest to note that in our model rights should be interpreted in a broad sense: they may include obligations to society, e.g. paying taxes. The observation that a constitution may also contain obligations is not new, see e.g. Kanger and Kanger (1972).

The following example illustrates some of these considerations.

Example 2.2.6. Consider a society with $N = M \cup F$, $M \cap F = \emptyset$. The members in M are the males and the members in F the females. For a group S, m_S denotes the number of males in S and f_S the number of females. The total number of members is n, so $n = m_N + f_N$. There are two rights, $\rho = \{o, r\}$. If a group S has right o, then this means that all males in S are obliged to serve in the army. Thus, o is really an obligation. If a group S does not have right (obligation) o, then this means that the men in S are not allowed to serve in the army. If a group S has right r, then this means that every female in S has the right (but not the obligation) to serve in the army. If a group S does not have right r, this means that the women in S are not allowed to serve in the army. (Of course, different interpretations of o and r are possible, and these may lead to different expressions below.)

The set of social states is assumed to be $A = \{0, \ldots, n\}$, where $k \in A$ means that exactly k society members serve in the army.

The access correspondence is as follows. For every non-empty group S we have $\gamma(S, \emptyset) = \{A\}$, and $\gamma(\emptyset, \theta) = \emptyset$ for every $\theta \subseteq \rho$. Further, for any non-empty group S:

$$\gamma(S, o) = \{m_S, m_S + 1, \ldots, n - f_S\}^+ ,$$
$$\gamma(S, r) = \bigcup_{0 \leq x \leq f_S} \{x, x + 1, \ldots, x + n - |S|\}^+ ,$$
$$\gamma(S, \rho) = \bigcup_{0 \leq x \leq f_S} \{x + m_S, x + 1 + m_S, x + n - f_S\}^+ .$$

The first line reflects the fact that all men in group S have to serve in the army; the fact that the women in S do not have right to serve in the army means that they cannot serve in the army. So S is legally entitled to a social state where the number k of society members who serve in the army is between m_S and $n - f_S$, but cannot decide on the exact value of k. In the second expression, the group S can decide how many women x serve in the army; the men in S do not have the obligation to serve, which is interpreted as the impossibility to serve. The third equation reflects the fact that all men in S have to serve and all women in S can choose to serve.

This example illustrates that our definition of a constitution does not *formally* distinguish between rights and obligations. The access correspondence, however, shows that rights and obligations (in this case, r and o) play different roles and lead to essential differences in attainable sets of social states.

2.3 Constitutions and effectivity functions

Throughout this section let $\mathcal{S} = (N, A, \rho, \alpha, \gamma)$ be a society, with constitution (ρ, α, γ). Although the access correspondence γ is specified for any assignment of rights, all that matters to determine the actual attainable sets of social states is the assignment of rights α. Thus, all the relevant information inherent in the constitution (ρ, α, γ) can be summarized by a function $E : P(N) \to P(P_0(A))$ defined by

$$E(S) = E(S; \alpha, \gamma) = \bigcup_{T \subseteq S} \gamma(T, \alpha(T)) \tag{2.1}$$

for every $S \subseteq N$. Hence, according to E, group or *coalition* S is entitled to all sets of social states to which some subcoalition of S is legally entitled.

Since, by Definition 2.2.1, $\gamma(\emptyset, \cdot) = \emptyset$, we have $E(\emptyset) = \emptyset$. Also, (2.1) implies that E is *monotonic with respect to coalitions*:[4]

$$S \subseteq T \Rightarrow E(S) \subseteq E(T) \text{ for all } S, T \in P(N). \tag{2.2}$$

Since $A \in \gamma(S, \theta)$ for all $S \in P_0(N)$ and $\theta \subseteq \rho$, we have

$$A \in E(S) \text{ for every } S \neq \emptyset. \tag{2.3}$$

We call the constitution (ρ, α, γ) *non-imposed* if the *grand coalition* N has complete power in terms of E, that is:

$$E(N) = P_0(A) \ . \tag{2.4}$$

These conditions on the function E, except for monotonicity with respect to coalitions, are collected in the concept of an effectivity function. We will formally introduce effectivity functions in a more general framework where not all subsets of A are necessarily admitted as possible sets of social states. More precisely, a *structure* on A is a set $\mathcal{T} \subseteq P_0(A)$ such that (i) $A \in \mathcal{T}$; and (ii) $B_1 \cap B_2 \in \mathcal{T}$ for all $B_1, B_2 \in \mathcal{T}$ with $B_1 \cap B_2 \neq \emptyset$. Examples are situations where $(A, \mathcal{T})$ is a topological or measurable space. Of course, also $\mathcal{T} = P_0(A)$ is a structure. We use the notation $(A, \mathcal{T})$ to refer to a set of alternatives with structure and call this a *structured space*.

Definition 2.3.1. For a structured space $(A, \mathcal{T})$, an *effectivity function* (EF) is a function $E : P(N) \to P(\mathcal{T})$ that satisfies (i) $E(\emptyset) = \emptyset$, (ii) (2.3), and (iii) $E(N) = \mathcal{T}$.

Condition (iii) in this definition is equivalent to (2.4) if $\mathcal{T} = P_0(A)$.

The functions E associated according to (2.1) with the examples of Section 2.2 are described in the following example.

Example 2.3.2. (i) The function E associated with first Gibbard example concerning the choice of shirt color (Example 2.2.2) is given by $E(\emptyset) = \emptyset$ and:

$$E(1) = \{(w, w), (w, b)\}^+ \cup \{(b, w), (b, b)\}^+ \ ,$$
$$E(2) = \{(w, w), (b, w)\}^+ \cup \{(w, b), (b, b)\}^+ \ ,$$
$$E(N) = P_0(A) \ .$$

Clearly, (ρ, α, γ) is non-imposed and, consequently, E is an effectivity function.

(ii) The function E associated with the second Gibbard example concerning marriage within the trio $\{m_1, m_2, f\}$ (Example 2.2.3) is given by $E(\emptyset) = \emptyset$ and:

$$E(m_1) = \{s, w_2\}^+ \ , \ E(m_2) = \{s, w_1\}^+ \ , \ E(f) = \{s\}^+ \ ,$$

[4] This condition is also proposed in Gärdenfors (1981).

$$E(\{m_1, f\}) = \{s\}^+ \cup \{w_1\}^+ \ , E(\{m_2, f\}) = \{s\}^+ \cup \{w_2\}^+ \ ,$$

$$E(\{m_1, m_2\}) = \{s, w_2\}^+ \cup \{s, w_1\}^+ \ , \ E(N) = P_0(A) \ .$$

Again, (ρ, α, γ) is non-imposed, and consequently, E is an effectivity function.

(iii) For Example 2.2.6 concerning the right or obligation to serve in the army, we assume that every non-empty coalition is assigned $\rho = \{o, r\}$, that is: in every non-empty coalition the men have the obligation to serve in the army while each women has the right to serve in the army. Then $E(\emptyset) = \emptyset$. For every non-empty coalition S, every non-empty $T \subseteq S$ and every $0 \le x \le f_T$ we have

$$\{x + m_S, \ldots, x + n - f_S\} \subseteq \{x + m_T, \ldots, x + n - f_T\} \ .$$

This implies

$$E(S) = \gamma(S, \rho) = \bigcup_{0 \le x \le f_S} \{x + m_S, x + 1 + m_S, x + n - f_S\}^+$$

for every non-empty coalition S. In particular,

$$E(N) = \bigcup_{0 \le x \le f_N} \{x + m_N\}^+ \subsetneq P_0(A)$$

so (ρ, α, γ) violates non-imposition. Of course, this is obvious: since N has obligation o, states where not all men serve are not attainable. Thus, E is not an effectivity function. In the present example, if all men in all coalitions are obliged to serve in the army by assumption (rights assignment), then we could reformulate the set of social states by letting these indicate the number of women that serve in the army. Formulated this way, E would become an effectivity function. Alternatively, we can introduce the structure $\mathcal{T}$ consisting of the set A and all sets of social states where all men serve in the army. In that case,

$$E(S) = \bigcup_{0 \le x \le f_S} \{x + m_N, x + 1 + m_N, x + n - f_S\}^+, \text{ and}$$

$$E(N) = \bigcup_{0 \le x \le f_N} \{x + m_N\}^+ = \mathcal{T},$$

where the superscript '+' should be read as 'all supersets within $\mathcal{T}$'.

Gärdenfors introduced constitutions as effectivity functions. Independently, Moulin and Peleg (1982) introduced effectivity functions in the general context of game theory and especially in its relation to social choice. The concept of an effectivity function can be applied in many directions. For an early summary of the applications of effectivity functions to social choice see Abdou and Keiding (1991).

We continue with further properties of the access correspondence γ. We say that γ is *monotonic with respect to outcomes* if

$$[B \in \gamma(S,\theta),\ B \subseteq B^* \Rightarrow B^* \in \gamma(S,\theta)] \text{ for all } B, B^* \subseteq A,\ S \subseteq N,\ \theta \subseteq \rho. \quad (2.5)$$

Note that in the examples so far this condition is implicit since we always included all supersets in describing the access correspondences. Condition (2.5) simply means that if S can reach the set B of social states or outcomes when θ is its set of rights, then it can logically reach the larger set B^*.

Similarly, we say that $E : P(N) \to P(\mathcal{T})$ is *monotonic with respect to outcomes* if

$$[B \in E(S) \text{ and } B \subseteq B^* \Rightarrow B^* \in E(S)] \text{ for all } B, B^* \in \mathcal{T},\ S \subseteq N. \quad (2.6)$$

Remark 2.3.3. A function $E : P(N) \to P(P_0(A))$ that is monotonic with respect to outcomes, i.e. satisfies (2.6), also satisfies (2.4), i.e. $E(N) = P_0(A)$, as soon as $\{a\} \in E(N)$ for all $a \in A$. Thus, under monotonicity with respect to outcomes 'non-imposition' is equivalent to the grand coalition being effective for any single social state.

A function $E : P(N) \to P(\mathcal{T})$ is *monotonic* if it is monotonic both with respect to coalitions (cf. (2.2)) and with respect to outcomes.

A crucial property of a constitution is the property of coherence.[5]

Definition 2.3.4. The constitution (ρ, α, γ) is *coherent* if for all $S_1, S_2 \in P_0(N)$ with $S_1 \cap S_2 = \emptyset$ and all $B_1 \in \gamma(S_1, \alpha(S_1))$ and $B_2 \in \gamma(S_2, \alpha(S_2))$ we have: $B_1 \cap B_2 \neq \emptyset$.

The intuition behind coherence is straightforward: if, in this definition, $B_1 \cap B_2$ were empty then S_1 and S_2 could end up in an impossible situation since S_1 is entitled to the social state being in B_1 whereas S_2 is entitled to the social state being in B_2. It is easy to see that coherence of the constitution (ρ, α, γ) implies and is implied by the analogous condition on the function E defined by (2.1). In what follows, however, we usually need the following stronger condition on an effectivity function.

Definition 2.3.5. An effectivity function $E : P(N) \to P(\mathcal{T})$ is *superadditive* if for all $S_1, S_2 \in P_0(N)$ with $S_1 \cap S_2 = \emptyset$ and all $B_1 \in E(S_1)$ and $B_2 \in E(S_2)$ we have: $B_1 \cap B_2 \in E(S_1 \cup S_2)$.

Observe that, if $S \subsetneq T$ and $B \in E(S)$, then superadditivity of the EF E implies $B = B \cap A \in E(S \cup (T \setminus S)) = E(T)$, hence superadditivity implies monotonicity with respect to coalitions.

Also, if E defined by (2.1) is a superadditive EF, then the constitution (ρ, α, γ) is coherent. For let $B_1 \in \gamma(S_1, \alpha(S_1))$, $B_2 \in \gamma(S_2, \alpha(S_2))$, and $S_1 \cap S_2 = \emptyset$. Then clearly $B_1 \in E(S_1)$ and $B_2 \in E(S_2)$, so by superadditivity $B_1 \cap B_2 \in E(S_1 \cup S_2)$, which implies $B_1 \cap B_2 \neq \emptyset$.

Gärdenfors (1981) assumes monotonicity with respect to coalitions and coherence for constitutions modelled as effectivity functions.

[5] Called 'consistency' in Gärdenfors (1981).

2.4 Game forms and a representation theorem

Let $\mathcal{S}$ be a society, and suppose the constitution is described in a concise way by the function $E : P(N) \to P(\mathcal{T})$, as in the preceding section. Recall that E describes for any coalition the sets of social states to which this coalition is legally entitled. It does not tell us how the members of society can actually exercise their rights.

To this end, we assume now that every society member has at its disposal a set of 'legal' strategies. These strategies should be compatible with the constitution in the sense that they endow each coalition with the same legal power as the constitution does. To make this precise, we first introduce the concept of a game form.

Definition 2.4.1. A *game form* (GF) is a list $\Gamma = (N; \Sigma^1, \ldots, \Sigma^n; g; A)$ where N is the set of members of the society or *players*; Σ^i is the non-empty set of *strategies* of $i \in N$; $g : \Sigma = \Sigma^1 \times \ldots \times \Sigma^n \to A$ is the *outcome function*; and A is the set of social states or *outcomes*.

Example 2.4.2. Let $\Gamma = (\{1, 2, 3\}; \{2, 3\}, A, A; g; \{a, b, c\})$, with $g(2, x, y) = x$ and $g(3, x, y) = y$. This is the so-called 'kingmaker' game form: player 1 chooses the king (2 or 3), who in turn chooses an outcome from $A = \{a, b, c\}$. (Cf. Hurwicz and Schmeidler (1978).)

We mentioned that strategies should be 'legal'. We do not give a formal definition of this concept but – informally – call a game form 'legal' if the available strategies do not contradict the assignment of rights. For example, if Adam has the obligation to support his family and stealing is forbidden by law (so by the assignment of rights) then Adam cannot support his family by stealing. That is, stealing is not an available strategy. Moreover, we assume that also coalitions cannot break the law by coordination of their strategies.

We do formalize the preservation of legal power or entitlement resulting from a constitution through a game form by introducing the concept of representation below. First, we associate an effectivity function with each game form.

Definition 2.4.3. Let $\Gamma = (N; \Sigma^1, \ldots, \Sigma^n; g; A)$ be a game form, and let $\mathcal{T}$ be a structure on A. Let $S \in P_0(N)$, and let $B \in \mathcal{T}$. Then S is *effective* for B if there exists $\sigma_0^S \in \Sigma^S = \prod_{i \in S} \Sigma^i$ such that $g(\sigma_0^S, \sigma^{N \setminus S}) \in B$ for all $\sigma^{N \setminus S} \in \Sigma^{N \setminus S}$. If g is surjective, then $E^\Gamma : P(N) \to P(\mathcal{T})$ defined by $E^\Gamma(\emptyset) = \emptyset$ and

$$E^\Gamma(S) = \{B \in \mathcal{T} \mid S \text{ is effective for } B\} \text{ for all } S \in P_0(N)$$

is the *effectivity function for* $(A, \mathcal{T})$ *associated with* Γ.

Observe that E^Γ is indeed an effectivity function. In particular, surjectivity of g implies that $E^\Gamma(N) = \mathcal{T}$. Effectivity functions associated with game forms were introduced in Moulin and Peleg (1982) as so-called alpha-effectivity functions. It is straightforward to verify that E^Γ is monotonic and superadditive.

Example 2.4.4. For the kingmaker game form Γ of Example 2.4.2 the associated effectivity function $E = E^\Gamma$ is given by $E(i) = \{A\}$ for each $i \in N$ and $E(S) = P_0(A)$ for $|S| \geq 2$.

Example 2.4.5. Let $N = \{1, 2\}$, $A = \{a, b, c, d\}$, and consider the matrix

$$\begin{array}{c} \\ T \\ B \end{array}\begin{array}{c} \begin{array}{ccc} L & M & R \end{array} \\ \begin{pmatrix} a & d & c \\ c & b & d \end{pmatrix} \end{array}$$

where player 1 chooses rows and player 2 columns. This matrix defines a two-person game form $\Gamma = (N; \{T, B\}; \{L, M, R\}; g; A)$ in an obvious way. The associated effectivity function E^Γ is given by $E^\Gamma(1) = \{a, d, c\}^+ \cup \{c, b, d\}^+$, $E^\Gamma(2) = \{a, c\}^+ \cup \{d, b\}^+ \cup \{c, d\}^+$, and $E^\Gamma(N) = P_0(A)$. Such a game form is called a *bimatrix* game form since it results in a bimatrix game if utilities of the players on A are added.

The announced idea of representation is one of the main concepts of this part of the book. From now on, let $\mathcal{T}$ be a fixed structure on A.

Definition 2.4.6. Let $E : P(N) \to P(\mathcal{T})$ be an effectivity function. A game form Γ is a *representation* of E if $E^\Gamma = E$.

Thus, a game form represents an effectivity function E if its associated effectivity function is equal to E. In particular, if E is derived from a constitution as in (2.1), then a representing game form may be considered as a permissible mechanism that enables all the members of the society to exercise their rights simultaneously. Such an effectivity function E may be represented by many different game forms: each of these may be considered as a legal translation of the constitution into strategic behavior. Thus, similar societies and constitutions may be represented quite differently.

Since the effectivity function associated with a game form is monotonic and superadditive, these conditions are necessary for the existence of a representation of an effectivity function. We now show that they are also sufficient.

Theorem 2.4.7. *Let $E : P(N) \to P(\mathcal{T})$ be an effectivity function. Then E has a representation if and only if E is monotonic and superadditive.*

Proof. For the only-if direction, let $\Gamma = (N; \Sigma^1, \ldots, \Sigma^n; g; A)$ be a representation of E. We are done by observing that $E = E^\Gamma$ and E^Γ is monotonic and superadditive.

For the if-direction, assume that E is monotonic and superadditive. We construct a game form that represents E. For every $i \in N$ let $N^i = \{S \subseteq N \mid i \in S\}$ and

$$M^i = \{m^i : N^i \to N^i \times \mathcal{T} \mid m_1^i(S) \subseteq S,\ m_2^i(S) \in E(m_1^i(S))\} \tag{2.7}$$

where $m^i(\cdot) = \big(m_1^i(\cdot), m_2^i(\cdot)\big)$ and m^i is monotonic, that is

$$i \in S \subseteq T \Rightarrow m_1^i(S) \subseteq m_1^i(T) \text{ and } m_2^i(T) \subseteq m_2^i(S)\ . \tag{2.8}$$

Observe that $M^i \neq \emptyset$ since it contains the trivial function $S \mapsto (S, A)$. A selection from $\mathcal{T}$ is a function $\varphi : \mathcal{T} \to A$ such that $\varphi(B) \in B$ for every $B \in \mathcal{T}$. Denote by Φ the set of all selections from $\mathcal{T}$. We define a game form $\Gamma_0 = (N; \Sigma^1, \ldots, \Sigma^n; g_0; A)$ as follows. For each $i \in N$, the set of strategies of i is $\Sigma^i = M^i \times \Phi \times N$. Let $\sigma = (\sigma^1, \ldots, \sigma^n) \in \Sigma^1 \times \ldots \times \Sigma^n$, where $\sigma^i = (m^i, \varphi^i, t^i)$ for every $i \in N$. In order to define $g_0(\sigma)$ we introduce a sequence of partitions of N. First, for $S \in P_0(N)$ we define an equivalence relation $\sim_\sigma$ on S by

$$i \sim_\sigma j :\Leftrightarrow m^i(S) = m^j(S), \text{ for all } i, j \in S, \tag{2.9}$$

and denote by $D(S, \sigma)$ the partition of S with respect to $\sim_\sigma$. Now let the first partition of N be $H_0(\sigma) = \{N\}$. If $H_k = \{S_{k,1}, \ldots, S_{k,\ell}\}$ is the k-th partition, $k \geq 0$, then we define

$$H_{k+1}(\sigma) = \bigcup_{j=1}^{\ell} D(S_{k,j}, \sigma)\ .$$

Clearly, there exists a minimal r such that $H_k(\sigma) = H_r(\sigma)$ for all $k \geq r$. Write $H_r(\sigma) = \{S_1, \ldots, S_\ell\}$. Then $m_1^i(S_j) = S_j$ and $m_2^i(S_j) = B_j$ for some $B_j \in E(S_j)$, for all $i \in S_j$ and $j = 1, \ldots, \ell$. Since E is superadditive, $B := \bigcap_{j=1}^{\ell} B_j \neq \emptyset$ and $B \in \mathcal{T}$. Let $1 \leq i_0 \leq n$ be the player with $i_0 = (t^1 + \cdots + t^n) \bmod n$. Then we define $g_0(\sigma) = \varphi^{i_0}(B)$.

We prove that Γ_0 is a representation of E. Let $S \in P_0(N)$ and $B \in E(S)$. Choose $\sigma^i = (m^i, \varphi^i, t^i)$ for every $i \in S$ such that $m_1^i(S^*) = S$ and $m_2^i(S^*) = B$ for all $S^* \supseteq S$ and $i \in S$. Then S is an element of the partition $H_r(\sigma^S, \tau^{N \setminus S})$ for each $\tau^{N \setminus S} \in \Sigma^{N \setminus S}$, and $m_2^i(S) = B$ for all $i \in S$. Hence, by definition of g_0, $g_0(\sigma^S, \tau^{N \setminus S}) \in B$ for all $\tau^{N \setminus S} \in \Sigma^{N \setminus S}$. So $B \in E^{\Gamma_0}(S)$.

To prove the converse inclusion let $C \in \mathcal{T} \setminus E(S)$. Let $\sigma^S = (m^i, \varphi^i, t^i)_{i \in S} \in \Sigma^S$ and $i_0 \in N \setminus S$ (such an i_0 exists since $E(N) = \mathcal{T}$ and thus $S \neq N$). Choose strategies $\tau^i = (m^i, \varphi^i, t^i) \in \Sigma^i$ for every $i \in N \setminus S$, as follows. For every $T \supseteq N \setminus S$ and $i \in N \setminus S$, let $m_1^i(T) = N \setminus S$ and $m_2^i(T) = A$. Further, let $\left(\sum_{i \in S} t^i + \sum_{i \in N \setminus S} t^i\right) \bmod n = i_0$. Clearly, $H_r(\sigma^S, \tau^{N \setminus S}) = \{S_1, \ldots, S_\ell, N \setminus S\}$ for some partition $\{S_1, \ldots, S_\ell\}$ of S. Let $B_j = m_2^i(S_j)$, $i \in S_j$, $j = 1, \ldots, \ell$. Then $B_j \in E(S_j)$ for every $j = 1, \ldots, \ell$, so by superadditivity of E, $B = \bigcap_{j=1}^{\ell} B_j \in E(S)$. Thus, $B \setminus C \neq \emptyset$ since otherwise, by monotonicity of E, $C \in E(S)$, a contradiction. Since $m_2^i(N \setminus S) = A$ for all $i \in N \setminus S$, if $\varphi^{i_0}(B) \in B \setminus C$ then $g_0(\sigma^S, \tau^{N \setminus S}) \notin C$. Hence $C \notin E^{\Gamma_0}(S)$. □

The game form Γ_0 constructed in the proof of Theorem 2.4.7 is not the unique representation of E. Alternative game forms that can be used to prove the theorem can be found in Peleg (1998) and in Peleg, Peters, and Storcken (2002). For reasons that will become clear in the next chapter the game form Γ_0 will be called a *canonical* representation of E.

2.5 Representation and simultaneous exercising of rights

Recall from our earlier discussion that a constitution in the sense of Gärdenfors (1981) is simply an effectivity function $E : P(N) \to P(P_0(A))$ – assuming that $P_0(A)$ is the structure on A – and that a right for coalition S is simply any $B \in E(S)$. One may wonder whether rights are non-conflicting or, equivalently, whether they can be exercised simultaneously. For instance, in the first Gibbard (1974) example (Example 2.3.2(i)), the 'rights' $\{(w,w),(w,b)\} \in E(1)$ and $\{(b,w),(b,b)\} \in E(1)$ coincide with the right ρ_1 (Example 2.2.2) of player 1 to choose the color of his own shirt. Similarly, player 2 can choose the color of his own shirt or, equivalently, has rights $\{(w,w),(b,w)\}$ and $\{(w,b),(w,b)\}$. Clearly, these rights can be exercised simultaneously: any of the two rights of player 1 has non-empty intersection with any of the two rights of player 2. This follows from the coherence condition of Gärdenfors and, *a forteriori* from superadditivity of E. This implies that it also follows from the existence of a representation of E, a fact which is also easy to see directly.

Proposition 2.5.1. *Let $\Gamma = (N; \Sigma^1, \ldots, \Sigma^n; g; A)$ be a representation of E, and let $S_i \in P_0(N)$ and $B_i \in E(S_i)$ for $i = 1, 2$, such that $S_1 \cap S_2 = \emptyset$. Then $B_1 \cap B_2 \neq \emptyset$.*

Proof. Let, for $i = 1, 2$, $\sigma^{S_i} \in \Sigma^{S_i}$ satisfy $g(\sigma^{S_i}, \tau^{N\setminus S_i}) \in B_i$ for all $\tau^{N\setminus S_i} \in \Sigma^{N\setminus S_i}$. Then $g(\sigma^{S_1}, \sigma^{S_2}, \tau^{N\setminus(S_1\cup S_2)}) \in B_1 \cap B_2$ for all $\tau^{N\setminus(S_1\cup S_2)} \in \Sigma^{N\setminus(S_1\cup S_2)}$, so $B_1 \cap B_2 \neq \emptyset$. □

Note that the effectivity function of the second Gibbard example (Examples 2.2.3, 2.3.2(ii)) is superadditive and monotonic, and thus has a representation. The same holds for the adapted version of the effectivity function in Examples 2.2.6 and 2.3.2(iii) for the structure described at the end of Example 2.3.2(iii). In particular, in all these examples rights can be exercised simultaneously.

2.6 Notes and comments

This chapter is based mainly on Peleg (1998) and Gärdenfors (1981). The proof of Theorem 2.4.7 has benefited from Peleg, Peters, and Storcken (2002).

In the following remark we place the idea of a constitution as introduced in Section 2.2 in a dynamic perspective.

Remark 2.6.1. A constitution (ρ, α, γ) is at any given point of time a result of a past political process. In a democracy the constitution at a given time represents the status quo of the rights-system and the assignment of rights. Thus, it may be changed by the legislative institutions by procedures such as voting. So, implicitly, in our model rights are politically determined (cf. Sen, 1997). At each time t the members of society have a preference profile $R^N(t)$ that determines the direction of change. Thus, in our framework the problem of choosing the constitution does not arise since the constitution at a given period determines the possible legal constitutions at the next period. In particular, illegal changes such as *coups d'état* are not covered by our model.

Somewhat related to the previous remark is the concept of a *local effectivity function* (Abdou, 1995; Abdou and Keiding, 2003). In a local effectivity function the effectivity of a coalition depends on the current set of social states. Thus, the concept of a local effectivity function generalizes the concept of an effectivity function as used in this monograph.

The next remark concerns the idea of liberalism, a theme that will reoccur in the first part of this book.

Remark 2.6.2. Let $E : P(N) \to P(\mathcal{T})$ represent a constitution. E satisfies *liberalism* if each member $i \in N$ can veto some alternative, that is, for each $i \in N$ there exists some $B_i \in E(i) \setminus \{A\}$. For instance, in Example 2.2.3 each member can veto the possibility that he or she gets married and so the associated effectivity function E (Example 2.3.2(ii)) satisfies liberalism. The same is true for Example 2.2.2, where each member can choose the color of his own shirt. The adapted version of the army example (see the last part of Example 2.3.2(iii)) does not satisfy liberalism, since individual men have no say about the number of women in the army. A game form Γ satisfies *liberalism* if the associated effectivity function E^Γ satisfies liberalism.

Clearly, liberalism implies the existence of non-trivial rights in the sense of Gärdenfors (1981). E satisfies *minimal liberalism* if there are two different individuals i and j with non-trivial rights, i.e., there are $B_i \in E(i) \setminus \{A\}$ and $B_j \in E(j) \setminus \{A\}$. A game form Γ satisfies *minimal liberalism* if the associated effectivity function E^Γ does. The relationship between liberalism and Pareto optimality, as in Sen's Liberal Paradox (Sen, 1970), will be explored in Chapters 3 and 4.