Chapter 11
Consistent voting systems with a continuum of voters

11.1 Motivation and summary

In this chapter we extend the model of Chapters 9 and 10 to a classical voting system with still finitely many alternatives (candidates) but with very many voters. Such a system is representative of political elections on the local or national level. As an, in our view, best approximation we model voters as elements of a non-atomic measure space. In particular, this approach allows us to accommodate the fact that in such voting systems single voters have negligible influence on the final outcome, and to avoid potential combinatorial complexities of a model with a large but finite number of voters.

The focus of the chapter is again on strategic aspects. If we talk about strategic aspects in this model, we necessarily deal with strategic voting by groups of voters (coalitions). This does not have to imply that voters in coalitions actually meet to coordinate their voting behavior. Although single voters are negligible for the final outcome, they may nevertheless derive utility from voting and, thus, may also vote strategically, possibly resulting in strategic behavior of groups of equally-minded voters.

After introducing the basics of the model in Section 11.2, we continue in Section 11.3 by showing that in this model the result of Gibbard (1973) and Satterthwaite (1975) persists. In particular, the requirement of non-manipulability implies the (undesirable) existence of an 'invisible dictator' as in Kirman and Sondermann (1972). Since, therefore, we cannot hope to reach the sincere outcome since we cannot expect voters to reveal their true preferences, we ask whether this outcome is at least attainable in an equilibrium of the voting game. Specifically, like in the preceding chapters we consider social choice functions satisfying the weaker requirement of exact and strong consistency (ESC). This means that for every given profile of preferences there is another profile which (i) is a strong (Nash) equilibrium – no coalition can profitably deviate – in the strategic game in which each voter

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reports a preference and the outcome is evaluated according to given 'true' preferences, and (ii) results in the same alternative as the true preferences.

ESC social choice functions and associated effectivity functions are introduced in Section 11.4. We show that the main results of the model with finitely many voters go through: the effectivity function associated with an ESC social choice function is maximal, stable, and convex. This is no surprise: an ESC social choice function, seen as a game form, is a strong representation of the associated effectivity function, and the corresponding results of Chapter 5 continue to hold.

Next, we concentrate on anonymous ESC social choice functions, a natural restriction in large voting systems, and introduce blocking coefficients (Section 11.5) and feasible elimination procedures (Section 11.6). Here, our treatment deviates essentially from the case with finitely many voters. Sets of alternatives can be 'e-sets' or 'i-sets'. To block an i-set, a coalition needs to have size strictly larger than the blocking coefficient of that set, whereas for an e-set it can be larger or equal. Also, blocking coefficients constitute an additive function, contrary to the finitely many voters case (cf. Oren, 1981). As in Chapter 9, the main result is that any social choice function that selects maximal alternatives – that is, alternatives resulting from feasible elimination procedures – is exactly and strongly consistent. In Section 11.7 we establish equality of the core and the set of maximal alternatives for a collection of anonymous ESC social choice functions, and in Section 11.8 we show that this is actually a complete characterization of anonymous ESC social choice functions in case all blocking coefficients are required to be positive.

11.2 The basic model

Let $(\Omega, \Sigma, \lambda)$ be a non-atomic measure space. Here Ω is the set of *voters* or *players*; Σ is the σ-field of permissible *coalitions*; and λ is a nonnegative non-atomic measure on Σ, that is: $\lambda : \Sigma \to \mathbb{R}$ is a measure with $\lambda(S) \geq 0$ for all $S \in \Sigma$, and if $\lambda(S) > 0$ for some $S \in \Sigma$ then there is a $T \in \Sigma$ with $T \subseteq S$ and $0 < \lambda(T) < \lambda(S)$. The number $\lambda(S)$ for a coalition S is interpreted as the size of S. By $\Sigma_0 = \Sigma \setminus \{\emptyset\}$ we denote the set of all nonempty coalitions, and by Σ_+ we denote the set of all coalitions S with $\lambda(S) > 0$. Throughout we assume $\Omega \in \Sigma_+$ and $\lambda(\Omega) < \infty$.

Let A be a finite set of *alternatives*. We assume throughout that $|A| \geq 3$. As before, a linear ordering of A is a complete, reflexive, transitive, and antisymmetric binary relation on A, and the set of all linear orderings of A is denoted by L.

A *profile* (of preferences) is a measurable function $\mathbf{R} : \Omega \to L$, that is, for each $R \in L$, $\{t \in \Omega \mid \mathbf{R}(t) = R\}$ is in Σ. Two profiles $\mathbf{R}_1$ and $\mathbf{R}_2$ are *equivalent*, written $\mathbf{R}_1 \sim \mathbf{R}_2$, if they differ only for a coalition of zero measure, that is: $\lambda(\{t \in \Omega \mid \mathbf{R}_1(t) \neq \mathbf{R}_2(t)\}) = 0$. Let ρ denote the set of all profiles.

A *social choice function* (SCF) is a surjective function $F : \rho \to A$ that satisfies

$$\text{for all } \mathbf{R}_1, \mathbf{R}_2 \in \rho, \text{ if } \mathbf{R}_1 \sim \mathbf{R}_2, \text{ then } F(\mathbf{R}_1) = F(\mathbf{R}_2). \tag{11.1}$$

Condition (11.1) implies that social choice functions do not depend on the preferences of coalitions of measure 0. In particular, because of non-atomicity, single agents do not have any influence at all.

11.3 The Gibbard-Satterthwaite Theorem

In this section we show that the Gibbard-Satterthwaite Theorem continues to hold in our model with a continuum of voters, in the sense that any non-manipulable social choice function must exhibit a so-called *invisible dictator*. This is analogous to a similar result for Arrow's Impossibility Theorem in Kirman and Sondermann (1972). We start by formulating (non-)manipulability in the present context.

Let $\mathbf{R} \in \rho$ and $S \in \Sigma$. The social choice function F is *manipulable by S at $\mathbf{R}$* if there exists a $Q \in L$ with the following property: if $\mathbf{R}_1 \in \rho$ is a profile with $\mathbf{R}_1(t) = \mathbf{R}(t)$ for all $t \notin S$ and $\mathbf{R}_1(t) = Q$ for all $t \in S$, then $F(\mathbf{R}) \neq F(\mathbf{R}_1)$ and $F(\mathbf{R}_1)\mathbf{R}(t)F(\mathbf{R})$ for all $t \in S$.[1] Clearly, if F is manipulable by S at $\mathbf{R}$, then $\lambda(S) > 0$ by (11.1). We call F *non-manipulable* if there exist no $\mathbf{R} \in \rho$ and $S \in \Sigma$ such that F is manipulable by S at $\mathbf{R}$. In words, it can never happen that all members of a coalition obtain a preferred alternative if that coalition coordinates on an untruthful preference. Observe that this non-manipulability condition has necessarily the form of coalitional non-manipulability since in our model single voters have no influence. Nevertheless, it can be weakened to a condition that is a closer approximation of individual non-manipulability. This is elaborated in Remark 11.3.7 below.

In order to formulate and prove the analogue of the Gibbard-Satterthwaite Theorem in this model we need to introduce the following concepts. A collection $\mathcal{D} \subseteq \Sigma_+$ is called an *ultrafilter* if (i) $D \cap D' \in \mathcal{D}$ for all $D, D' \in \mathcal{D}$ and (ii) $D \in \mathcal{D}$ or $\Omega \setminus D \in \mathcal{D}$ for every $D \in \Sigma_+$.[2] A *partition* of Ω is a finite collection of pairwise disjoint sets in Σ_+ the union of which has measure equal to $\lambda(\Omega)$.

Let $\mathcal{P} = \{D_1, \ldots, D_k\}$ be a partition of Ω. Let $\mathcal{D}$ be an ultrafilter. We claim that there is at least one $i \in \{1, \ldots, k\}$ for which $D_i \in \mathcal{D}$. If not, then by property (ii) of $\mathcal{D}$, $D^i := \bigcup_{j=1,\ldots,k,\ j\neq i} D_j \in \mathcal{D}$ for every $i = 1, \ldots, k$, so by property (i), $\emptyset = \bigcap_{i=1,\ldots,k} D^i \in \mathcal{D}$, a contradiction since $\emptyset \notin \Sigma_+$. Hence,

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[1] Requiring the voters in S to coordinate on the same preference Q in this definition is without loss of generality, as is not difficult to show.

[2] Observe that by (i) exactly one of the two statements in (ii) must hold.

there is an i with $D_i \in \mathcal{D}$ and by property (i) again there is exactly one such i. Also, if a partition $\mathcal{P}'$ of Ω is coarser than $\mathcal{P}$ (i.e., each element of $\mathcal{P}$ is contained in an element of $\mathcal{P}'$; we also say that $\mathcal{P}$ is finer than $\mathcal{P}'$) then (i) implies $D \subseteq D'$, where D and D' are the elements of $\mathcal{P}$ and $\mathcal{P}'$ that are in $\mathcal{D}$, respectively. Therefore, there is a well defined mapping d that assigns to each partition its element in $\mathcal{D}$, and d satisfies:

$$\text{If } \mathcal{P}' \text{ is coarser than } \mathcal{P}, \text{ then } d(\mathcal{P}) \subseteq d(\mathcal{P}'). \tag{11.2}$$

The following lemma shows that also the converse holds.

Lemma 11.3.1. *Let d be a mapping that assigns to each partition of Ω exactly one element of its elements. Suppose d satisfies* (11.2)*. Then the collection*

$$\mathcal{D} = \{D \in \Sigma_+ \mid \textit{there is a partition } \mathcal{P} \textit{ of } \Omega \textit{ with } D = d(\mathcal{P})\}$$

is an ultrafilter.

Proof. Let $\mathcal{P}^1$ and $\mathcal{P}^2$ be partitions and $D^1 = d(\mathcal{P}^1)$, $D^2 = d(\mathcal{P}^2)$. We show that $D^1 \cap D^2 \in \mathcal{D}$. Consider the join $\mathcal{P}$ of $\mathcal{P}^1$ and $\mathcal{P}^2$, i.e., the partition

$$\mathcal{P} = \{D \cap E \mid D \in \mathcal{P}^1,\ E \in \mathcal{P}^2,\ D \cap E \in \Sigma_+\}.$$

Obviously, $\mathcal{P}$ is finer than both $\mathcal{P}^1$ and $\mathcal{P}^2$. Suppose $D^* = d(\mathcal{P})$. Then by (11.2), both $D^* \subseteq D^1$ and $D^* \subseteq D^2$, hence $D^* \subseteq D^1 \cap D^2$. By definition of $\mathcal{P}$ therefore, $D^* = D^1 \cap D^2$, which implies $D^1 \cap D^2 \in \mathcal{D}$.

Finally let $D \in \Sigma_+$. If $\lambda(D) = \lambda(\Omega)$ then $D = d(\{D\})$, so $D \in \mathcal{D}$. Otherwise, either $D = d(\{D, \Omega \setminus D\})$ or $\Omega \setminus D = d(\{D, \Omega \setminus D\})$, hence either $D \in \mathcal{D}$ or $\Omega \setminus D \in \mathcal{D}$.

Thus, $\mathcal{D}$ is an ultrafilter. □

Now let $R_1, \ldots, R_{|A|!}$ be an enumeration of the elements of L. Each profile $\mathbf{R} \in \rho$ results in a collection $\mathcal{P} = \{S_1, \ldots, S_{|A|!}\}$ of subsets of of Ω with $S_k = \{t \in \Omega \mid \mathbf{R}(t) = R_k\} \in \Sigma$ for each $1 \leq k \leq |A|!$. We denote by $\mathcal{P}(\mathbf{R})$ the collection obtained from $\mathcal{P}$ by omitting the sets of measure 0 and call this the *partition generated by* $\mathbf{R}$.

We associate with an ultrafilter $\mathcal{D}$ a social choice function $F^{\mathcal{D}}$, as follows. For a profile $\mathbf{R} \in \rho$ let D be the unique element of $\mathcal{P}(\mathbf{R})$ that is in $\mathcal{D}$. Define $F^{\mathcal{D}}(\mathbf{R}) := x$ where $x\,R\,y$ for all $y \in A$ and $R = \mathbf{R}(t)$ for (all) $t \in D$. We have:

Lemma 11.3.2. *Let $\mathcal{D}$ be an ultrafilter. Then the social choice function $F^{\mathcal{D}}$ is non-manipulable.*

Proof. Let $\mathbf{R} \in \rho$. Suppose that coalition S can manipulate at $\mathbf{R}$. Then $S \cap D = \emptyset$, where D is the element of $\mathcal{P}(\mathbf{R})$ in $\mathcal{D}$. Hence, a manipulation of S results in a profile $\mathbf{R}'$ such that $\mathcal{P}(\mathbf{R}')$ shares D with $\mathcal{P}(\mathbf{R})$. But then D is also the element of $\mathcal{P}(\mathbf{R}')$ that is in $\mathcal{D}$ by condition (i) of an ultrafilter. So $F^{\mathcal{D}}(\mathbf{R}') = F^{\mathcal{D}}(\mathbf{R})$, a contradiction. □

Conversely, let F be a non-manipulable social choice function. We will show that there is an ultrafilter $\mathcal{D}$ such that $F = F^{\mathcal{D}}$, by applying (the Gibbard-Satterthwaite) Theorem 8.2.1. In order to satisfy the range condition in the theorem, we fix profiles $\mathbf{R}_1, \ldots, \mathbf{R}_{|A|}$ in ρ such that $|\{F(\mathbf{R}_j) \mid j = 1, \ldots, |A|\}| = |A|$ – this is possible since F is surjective by assumption. For an arbitrary partition $\mathcal{P} \subseteq \Sigma_+$ of Ω let $\mathcal{P}^*$ be the coarsest common refinement of $\mathcal{P}$ and the generated partitions $\mathcal{P}(\mathbf{R}_j)$, $j = 1, \ldots, |A|$. Regard every element of $\mathcal{P}^*$ as a separate agent. By Theorem 8.2.1 there is a fixed element D^* of $\mathcal{P}^*$ such that, for every profile $\mathbf{R} \in \rho$ that is measurable with respect to $\mathcal{P}^*$, we have $F(\mathbf{R}) = x$ where x is the top element of $\mathbf{R}(t)$ for (all) $t \in D^*$. Denote by $d^F(\mathcal{P})$ the element of $\mathcal{P}$ that contains D^* and let

$$\mathcal{D}^F := \{d^F(\mathcal{P}) \mid \mathcal{P} \subseteq \Sigma_+ \text{ is a partition}\}.$$

Lemma 11.3.3. *(i) $\mathcal{D}^F$ is an ultrafilter. (ii) $F = F^{\mathcal{D}^F}$.*

Proof. (i) By Lemma 11.3.1 it is sufficient to prove that d^F satisfies (11.2). Let $\mathcal{P}$ and $\mathcal{P}'$ be partitions with $\mathcal{P}'$ coarser than $\mathcal{P}$. Let $D' \in \mathcal{P}'$ with $d^F(\mathcal{P}) \subseteq D'$. Let $R, Q \in L$ have different top elements. Take a profile $\mathbf{R} \in \rho$ that is measurable with respect to $\mathcal{P}'$, and hence with respect to $\mathcal{P}$, and with $\mathbf{R}(t) = R$ for all $t \in D'$ and with $\mathbf{R}(t) = Q$ otherwise. Then $F(\mathbf{R})$ is the top element of R since $R = \mathbf{R}(t)$ for (all) $t \in d^F(\mathcal{P})$. Hence, $d^F(\mathcal{P}') = D'$, so that $d^F(\mathcal{P}) \subseteq d^F(\mathcal{P}')$.

(ii) Let $\mathbf{R} \in \rho$ with generated partition $\mathcal{P}(\mathbf{R})$. Let D^* be the element of $\mathcal{P}(\mathbf{R})^*$ such that $F(\mathbf{R}) = x$, where x is the top element of $\mathbf{R}(t)$ for (all) $t \in D^*$. Let D be the element of $\mathcal{P}(\mathbf{R})$ with $D^* \subseteq D$. By definition, $F^{\mathcal{D}^F}(\mathbf{R})$ is the top element of $\mathbf{R}(t)$ for (all) $t \in D$, hence $F^{\mathcal{D}^F}(\mathbf{R}) = x = F(\mathbf{R})$. □

Lemmas 11.3.2 and 11.3.3 have the following corollary.

Corollary 11.3.4. *Let $F : \rho \to A$ be a social choice function. Then F is non-manipulable if and only if there is an ultrafilter $\mathcal{D}$ with $F = F^{\mathcal{D}}$.*

Corollary 11.3.4 is the form the Gibbard-Satterthwaite Theorem takes in our model with a continuum of voters and measurable profiles.[3] First, we show that the result is not vacuous.

Theorem 11.3.5. *There exists a non-manipulable social choice function.*

Proof. By Corollary 11.3.4 it is sufficient to show that there exists an ultrafilter of sets in Σ_+.

A *filter* in Σ_+ is a collection $\mathcal{F} \subseteq \Sigma_+$ satisfying

(i) for all $D, D' \in \mathcal{F}$, $D \cap D' \in \mathcal{F}$;

[3] For the case of finitely many voters the relation between the concepts of non-manipulability and ultrafilter has been examined before, see Batteau, Blin, and Monjardet (1981).

(ii) for all $D \in \mathcal{F}$ and $D' \in \Sigma_+$ with $D \subseteq D'$, $D' \in \mathcal{F}$.

(Clearly, an ultrafilter is a filter.) Let $\mathcal{U}$ be the collection of all filters $\mathcal{F}$ that satisfy, additionally,

(iii) for all $D \in \mathcal{F}$ and $D' \in \Sigma_+$ with $D' \subseteq D$ and $\lambda(D) = \lambda(D')$, $D' \in \mathcal{F}$.

Any set of positive measure together with all its subsets of the same measure and all measurable supersets of these form a filter, so $\mathcal{U}$ is non-empty. The inclusion relation is a partial ordering on $\mathcal{U}$ and each chain in $\mathcal{U}$ has an upper bound, namely the union of all filters in the chain. Hence, Zorn's Lemma implies that $\mathcal{U}$ has a maximal element, say $\mathcal{D}$. We claim that $\mathcal{D}$ is an ultrafilter. If not, then there is a $D \in \Sigma_+$ such that $D \notin \mathcal{D}$ and $\Omega \setminus D \notin \mathcal{D}$ (recall that $D \in \mathcal{D}$ and $\Omega \setminus D \in \mathcal{D}$ is not possible by (i)). By (ii), we have $D' \cap D \neq \emptyset$ and $D' \cap (\Omega \setminus D) \neq \emptyset$ for every $D' \in \mathcal{D}$ and by (iii), we have $\lambda(D' \cap D) > 0$ and $\lambda(D' \cap (\Omega \setminus D)) > 0$. Now consider the collection $\mathcal{D}'$ obtained by adding to $\mathcal{D}$ the collection $\{D' \cap D \mid D' \in \mathcal{D}\}$. Then it is easy to check that $\mathcal{D}'$ is a filter in $\mathcal{U}$ that is larger than $\mathcal{D}$, contradicting the maximality of $\mathcal{D}$. Hence, $\mathcal{D}$ is an ultrafilter. □

Since this existence proof is based on an application of Zorn's Lemma, it does not actually show how to construct a non-manipulable social choice function. If we require constructibility then it can be shown that a non-manipulable social choice function does not exist, so that Corollary 11.3.4 is truly an impossibility result. Observe that in our model a single voter cannot be a dictator in view of (11.1).

For a concrete illustration of Theorem 11.3.5 see the next example.

Example 11.3.6. Let $\Omega = [0, 1]$ and let λ be the Lebesgue measure. If $\mathcal{D}$ is an ultrafilter, then for any $t \in [0, 1]$ exactly one of the two intervals $[0, t]$ and $[t, 1]$ must be in $\mathcal{D}$. Suppose, for the sake of the argument, that this is always the lower one, $[0, t]$. Then for every positive ε, every element of $\mathcal{D}$ has an intersection of positive measure with $[0, \varepsilon]$. The point 0 is an *invisible dictator* in the sense of Kirman and Sondermann (1972). Of course, the singleton 0 does not have any power at all, but always needs, roughly, a coalition of positive measure in any arbitrarily small neighborhood to exercise its 'dictatorship'. In this sense, the social choice function $F^{\mathcal{D}}$ associated with $\mathcal{D}$ has an invisible dictator, namely voter 0.

We conclude this section by discussing a possible weakening of the non-manipulability condition.

Remark 11.3.7. Our non-manipulability condition can be weakened to a version that is a closer approximation of individual non-manipulability. Call F *ε-manipulable* if for every $\varepsilon > 0$ there is a profile $\mathbf{R} \in \rho$ and a coalition $S \in \Sigma$ with $\lambda(S) < \varepsilon$ such that F is manipulable by S at $\mathbf{R}$. Call F *non-ε-manipulable* if it is not ε-manipulable. This means that there is an $\varepsilon > 0$ such that at no profile coalitions with size smaller than ε can manipulate. Clearly, non-ε-manipulability is weaker than non-manipulability, hence for every ultrafilter $\mathcal{D}$ the social choice function $F^{\mathcal{D}}$ satisfies it. Conversely, suppose that F is non-ε-manipulable. Take $\varepsilon > 0$ so small that no coalition of size smaller than ε can ever manipulate, and take an arbitrary partition $\mathcal{P}_\varepsilon$ of Ω such that each element of $\mathcal{P}_\varepsilon$ has size smaller than ε. Modify the definition of $\mathcal{P}^*$ preceding Lemma 11.3.3 such that $\mathcal{P}^*$ is now the coarsest common refinement of $\mathcal{P}_\varepsilon$ and $\mathcal{P}(\mathbf{R}_j)$, $j = 1, \ldots, |A|$. Then Lemma 11.3.3 and Corollary 11.3.4 continue to hold if we replace non-manipulability by non-ε-manipulability.

11.4 Exactly and strongly consistent social choice functions

In the preceding section we have seen that a version of the Gibbard-Satterthwaite Theorem continues to hold in our model with a continuum of voters. Like in Chapters 9 and 10, as an answer to this we shall study exactly and strongly consistent social choice functions. We start with defining this concept within the present model.

Let F be a social choice function and observe that for every $\mathbf{R} \in \rho$ the pair $(F, \mathbf{R})$ defines a game in strategic form in the usual and natural way: each player $t \in \Omega$ has strategy set L and preference $\mathbf{R}(t)$ on A for evaluating any outcome $F(\mathbf{R}^*) \in A$, $\mathbf{R}^* \in \rho$. For $S \in \Sigma_0$, denote by ρ^S the set of all measurable functions $\mathbf{R}^S : S \to L$. Let $\mathbf{R} \in \rho$. The profile $\mathbf{Q} \in \rho$ is a *strong (Nash) equilibrium* of the game $(F, \mathbf{R})$ if for every $S \in \Sigma_+$ and every $\mathbf{V}^S \in \rho^S$, there exists $T \in \Sigma_+$ with $T \subseteq S$ and $F(\mathbf{Q})\, \mathbf{R}(t)\, F(\mathbf{Q}^{\Omega \setminus S}, \mathbf{V}^S)$ for every $t \in T$.

Definition 11.4.1. The social choice function F is *exactly and strongly consistent* (ESC) if for every $\mathbf{R} \in \rho$ there exists a strong equilibrium $\mathbf{Q}$ of $(F, \mathbf{R})$ such that $F(\mathbf{Q}) = F(\mathbf{R})$.

Thus, if F is an ESC social choice function, then for every profile there is a strong equilibrium profile that results in the same outcome, and therefore F is not necessarily distorted.

In the remainder of this chapter we shall concentrate on anonymous ESC social choice functions. In our model a social choice function $F : \rho \to A$ is *anonymous* if for all $\mathbf{R}_1, \mathbf{R}_2 \in \rho$ we have: if $\lambda(\{t \in \Omega \mid \mathbf{R}_1(t) = R\}) = \lambda(\{t \in \Omega \mid \mathbf{R}_2(t) = R\})$ for all $R \in L$, then $F(\mathbf{R}_1) = F(\mathbf{R}_2)$. Thus, a social choice function is anonymous if it only depends on the numbers of voters for each preference.

We first consider a simple example of an anonymous ESC social choice function.[4] For a profile $\mathbf{R}$ and $a, b \in A$, $a \neq b$, we say that *a Pareto dominates b* if $\lambda(\{t \in \Omega \mid b\, \mathbf{R}(t)\, a\}) = 0$. We call an alternative $a \in A$ *Pareto optimal*

[4] This example is similar to Example 5.2.2.

with respect to $\mathbf{R}$ if it is not Pareto dominated by some other element of A, and denote by $\mathrm{PAR}(\mathbf{R})$ the set of Pareto optimal alternatives with respect to $\mathbf{R}$.

Example 11.4.2. Let $\bar{a} \in A$ be a designated alternative, and let $R_0 \in L$ be fixed. Define a social choice function $F : \rho \to A$ by

$$F(\mathbf{R}) = \begin{cases} \bar{a} \text{ if } \bar{a} \in \mathrm{PAR}(\mathbf{R}) \\ a \text{ if } \bar{a} \notin \mathrm{PAR}(\mathbf{R}) \text{ and } a \text{ is the } R_0\text{-maximum} \\ \qquad \text{of } \{b \in \mathrm{PAR}(\mathbf{R}) \mid b \text{ Pareto dominates } \bar{a}\} \end{cases}$$

for all $\mathbf{R} \in \rho$. Note that $\bar{a}$ can be interpreted as the 'status quo'. Obviously, F is surjective and anonymous. We show that F is ESC. Let $\mathbf{R} \in \rho$. We distinguish the following possibilities.

(i) $\bar{a} \in \mathrm{PAR}(\mathbf{R})$.

Let $\mathbf{Q} \in \rho$ satisfy $\bar{a}\,\mathbf{Q}(t)\,a$ for all $t \in \Omega$ and $a \in A \setminus \{\bar{a}\}$. Then $\mathbf{Q}$ is a strong equilibrium of $(F, \mathbf{R})$ and $F(\mathbf{Q}) = F(\mathbf{R})$.

(ii) $\bar{a} \notin \mathrm{PAR}(\mathbf{R})$.

Let q be the R_0-maximum of $B = \{b \in \mathrm{PAR}(\mathbf{R}) \mid b \text{ Pareto dominates } \bar{a}\}$. Define $\mathbf{Q} \in \rho$ by $q\,\mathbf{Q}(t)\,\bar{a}\,\mathbf{Q}(t)\,a$ for all $t \in \Omega$ and $a \in A \setminus \{\bar{a}, q\}$. Then $F(\mathbf{Q}) = q = F(\mathbf{R})$ and $\mathbf{Q}$ is a strong equilibrium of $(F, \mathbf{R})$. Indeed, Ω does not have a profitable deviation from $\mathbf{Q}$ since q is Pareto optimal with respect to $\mathbf{R}$. Now let $S \in \Sigma_+$, $\lambda(S) < \lambda(\Omega)$, and $\mathbf{V}^S \in \rho^S$. Then $F(\mathbf{Q}^{\Omega \setminus S}, \mathbf{V}^S) \in \{\bar{a}, q\}$. Hence, $\mathbf{V}^S$ cannot be a profitable deviation for S.

11.4.1 Effectivity functions of ESC social choice functions

Before proceeding with our investigation of anonymous ESC social choice functions, we define the concept of an effectivity function in our model, and collect some properties of effectivity functions associated with ESC social choice functions – analogous to the case with finitely many voters in Chapter 10.

Definition 11.4.3. An *effectivity function* (EF) is a function $E : \Sigma \to P(P_0(A))$ that satisfies the following conditions: (i) $E(\Omega) = P_0(A)$; (ii) $E(\emptyset) = \emptyset$; (iii) $A \in E(S)$ for every $S \in \Sigma_0$; and (iv) if $S_1, S_2 \in \Sigma_0$ and $\lambda(S_1 \setminus S_2) + \lambda(S_2 \setminus S_1) = 0$, then $E(S_1) = E(S_2)$.

Condition (iv) in Definition 11.4.3 is specific for our model. It says that the effectivity function does not distinguish between coalitions that differ only in a set of measure 0.

All of the following definitions and statements are analogous to their counterparts in the finite case, but we nevertheless list them for the sake of completeness.

An effectivity function E is *superadditive* if for all $S_1, S_2 \in \Sigma$ with $S_1 \cap S_2 = \emptyset$ and all $B_1 \in E(S_1)$ and $B_2 \in E(S_2)$ we have: $B_1 \cap B_2 \in E(S_1 \cup S_2)$. The EF E is *monotonic* if for all $S, S^* \in \Sigma$ and $B, B^* \in P_0(A)$ with $B \in E(S)$, $S \subseteq S^*$ and $B \subseteq B^*$, we have $B^* \in E(S^*)$. An EF E is *maximal* if for all $S \in \Sigma_0$ and $B \in P_0(A)$ we have: if $B \notin E(S)$ then $A \setminus B \in E(\Omega \setminus S)$. An EF E is *convex* if for all $S_1, S_2 \in \Sigma$ and $B_1 \in E(S_1)$, $B_2 \in E(S_2)$ we have $B_1 \cap B_2 \in E(S_1 \cup S_2)$ or $B_1 \cup B_2 \in E(S_1 \cap S_2)$.

Also the core of an effectivity function is defined exactly as in the finite model. Let $E : \Sigma \to P(P_0(A))$ be an EF and let $\mathbf{R} \in \rho$. Let $B \in P_0(A)$, $x \in A \setminus B$, and $S \in \Sigma$. We say that B *dominates* x *via* S *at* $\mathbf{R}$ if $B \in E(S)$ and $b\,\mathbf{R}(t)\,x$ for all $b \in B$ and $t \in S$. Also, x is *dominated at* $\mathbf{R}$ if there exists $B \in P_0(A)$ and $S \in \Sigma$ such that B dominates x via S at $\mathbf{R}$. If b is not dominated at $\mathbf{R}$ then b is *undominated* at $\mathbf{R}$.

Definition 11.4.4. The *core* $C(E, \mathbf{R})$ is the set of all undominated alternatives at $\mathbf{R}$. The effectivity function E is *stable* if $C(E, \mathbf{R}) \neq \emptyset$ for all $\mathbf{R} \in \rho$.

Let $F : \rho \to A$ be a social choice function. We associate with F an effectivity function E^F as follows. Let $S \in \Sigma_0$ and let $B \in P_0(A)$. Call S effective for B if there exists an $\mathbf{R}^S \in \rho^S$ such that $F(\mathbf{R}^S, \mathbf{Q}^{\Omega\setminus S})$ is in B for every $\mathbf{Q}^{\Omega\setminus S} \in \rho^{\Omega\setminus S}$. Define $E^F(\emptyset) = \emptyset$, and for $S \in \Sigma \setminus \{\emptyset\}$

$$E^F(S) = \{B \in P_0(A) \mid S \text{ is effective for } B\}.$$

In the following theorem we collect some useful properties of E^F.

Theorem 11.4.5. *Let $F : \rho \to A$ be an ESC social choice function. Then E^F is superadditive, monotonic, maximal, stable, and convex. Moreover, $F(\mathbf{R}) \in C(E^F, \mathbf{R})$ for all $\mathbf{R} \in \rho$.*

Proof. Superadditivity and monotonicity are straightforward from the definition of E^F (ESC is not needed for this). Maximality and stability, as well as the last statement in the theorem, can be proved analogously to the case of finitely many voters, see Section 5.2. Finally, stability and maximality together imply convexity. A proof of this fact is analogous to the proof of Theorem 6.A.9 in Peleg (1984). □

11.5 Blocking coefficients of anonymous ESC SCFs

In the remainder of the chapter we concentrate on anonymous ESC social choice functions. Anonymity is a natural requirement for voting procedures. Moreover, imposing this condition will enable us to derive much more detailed results on both social choice functions and effectivity functions than Theorem 11.4.5 provides.

Let $F : \rho \to A$ be an anonymous ESC social choice function, with associated effectivity function E^F. Then E^F is superadditive, monotonic, maximal, stable and convex, cf. Theorem 11.4.5. A central concept is that of a blocking coefficient.

For $B \in P_0(A) \setminus \{A\}$, the *blocking coefficient* is the real number

$$\beta(B) = \inf\{\lambda(S) \mid A \setminus B \in E^F(S)\}. \tag{11.3}$$

The number $\beta(B)$ represents the minimum size of a 'blocking coalition' of B. It is useful since F is anonymous. We call B an *e-set* ('e' from 'equality') if $S \in \Sigma$ and $\lambda(S) = \beta(B)$ imply that $A \setminus B \in E^F(S)$; otherwise, B is called an *i-set* ('i' from 'inequality'). Thus, to block an e-set B we need a coalition of size at least $\beta(B)$ but to block an i-set B we need a coalition of size strictly larger than $\beta(B)$.

We formulate a first observation concerning the blocking coefficients $\beta(\cdot)$. Suppose $B \in P_0(A)$ is an e-set. If $\beta(B) = 0$ then $A \setminus B \in E^F(S)$ for some coalition $S \in \Sigma_0$ with $\lambda(S) = 0$. Since $B \in E^F(\Omega \setminus S)$ by conditions (i) and (iv) in the definition of an effectivity function, we have a violation of superadditivity of E^F. Thus, we have shown:

$$\text{If } B \text{ is an e-set, then } \beta(B) > 0. \tag{11.4}$$

We now derive a number of other properties of $\beta(\cdot)$, in particular Theorem 11.5.1 below, which says that $\beta(\cdot)$ is an additive function.

If $B_1, B_2 \in P_0(A)$ and $B_1 \cup B_2 \neq A$, then

$$\beta(B_1 \cup B_2) \leq \beta(B_1) + \beta(B_2). \tag{11.5}$$

To see this, note that if the right hand side of this inequality is greater than or equal to $\lambda(\Omega)$, then the inequality holds. Now assume it is smaller. Let $\varepsilon > 0$ be small and let $S_i \in \Sigma$ with $\lambda(S_i) = \beta(B_i) + \varepsilon$ and $A \setminus B_i \in E^F(S_i)$ for $i = 1, 2$, such that $S_1 \cap S_2 = \emptyset$. By superadditivity, $A \setminus (B_1 \cup B_2) \in E^F(S_1 \cup S_2)$, hence $\beta(B_1 \cup B_2) \leq \beta(B_1) + \beta(B_2) + 2\varepsilon$. By letting ε approach 0, (11.5) follows.

For every $B \in P_0(A) \setminus \{A\}$ we have

$$\beta(B) + \beta(A \setminus B) \geq \lambda(\Omega) \tag{11.6}$$

because otherwise there would be disjoint coalitions S and T with $B \in E^F(S)$ and $A \setminus B \in E^F(T)$, contradicting the superadditivity of E^F. We shall now show the reverse inequality. Assume $\beta(B) > 0$ otherwise there is nothing left to prove. For every $0 < \delta < \beta(B)$ and $S \in \Sigma$ with $\lambda(S) = \delta$ we have $A \setminus B \notin E^F(S)$. Hence by maximality of E^F, $B \in E^F(\Omega \setminus S)$, so $\beta(A \setminus B) \leq \lambda(\Omega) - \delta$. This implies the reverse inequality of (11.6), hence

$$\beta(B) + \beta(A \setminus B) = \lambda(\Omega) \tag{11.7}$$

for every $B \in P_0(A) \setminus \{A\}$.

Suppose that $B \in P_0(A) \setminus \{A\}$ is an e-set and let $S \in \Sigma$ such that $\beta(B) = \lambda(S)$ and $A \setminus B \in E^F(S)$. Then, by superadditivity, $B \notin E^F(\Omega \setminus S)$. Also, by (11.7), $\beta(A \setminus B) = \lambda(\Omega \setminus S)$, so that $A \setminus B$ is an i-set. Conversely, let $A \setminus B$ be an i-set and $S \in \Sigma$ with $\lambda(S) = \beta(A \setminus B)$. Then $B \notin E^F(S)$ so that, by maximality, $A \setminus B \in E^F(\Omega \setminus S)$. Since, by (11.7), $\beta(B) = \lambda(\Omega \setminus S)$, we have that B is an e-set. Summarizing,

$$B \text{ is an e-set} \Leftrightarrow A \setminus B \text{ is an i-set} \tag{11.8}$$

for every $B \in P_0(A) \setminus \{A\}$.

Moreover, monotonicity of E^F clearly implies monotonicity of the function $\beta(\cdot)$:

$$B_1 \subseteq B_2 \Rightarrow \beta(B_1) \leq \beta(B_2) \tag{11.9}$$

for all $B_1, B_2 \in P_0(A) \setminus \{A\}$.

We now show that blocking coefficients are actually additive, that is, $\beta(B_1 \cup B_2) = \beta(B_1) + \beta(B_2)$ for all $B_1, B_2 \in P_0(A)$ with $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 \neq A$.

Theorem 11.5.1. *$\beta(\cdot)$ is additive.*

Proof. Let $B_i \in P_0(A)$, $i = 1, 2$, with $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 \neq A$. In view of (11.5) it is sufficient to prove that $\beta(B_1 \cup B_2) \geq \beta(B_1) + \beta(B_2)$. By (11.9) we may assume $\beta(B_i) > 0$ for $i = 1, 2$. Let S and T satisfy $\lambda(S) < \beta(B_1)$, $\lambda(T) < \beta(B_2)$, and $S \cap T = \emptyset$. Then by (11.7) and (11.9), $B_1 \in E^F(\Omega \setminus S)$ and $B_2 \in E^F(\Omega \setminus T)$. By convexity of E^F, $B_1 \cup B_2 \in E^F(\Omega \setminus (S \cup T))$. Thus, by (11.7) and the definition of $\beta(\cdot)$,

$$\begin{aligned} \beta(B_1 \cup B_2) &= \lambda(\Omega) - \beta(A \setminus (B_1 \cup B_2)) \\ &\geq \lambda(\Omega) - \lambda(\Omega \setminus (S \cup T)) \\ &= \lambda(S) + \lambda(T). \end{aligned}$$

Since, by (11.7) and (11.9), $\beta(B_1) + \beta(B_2) \leq \lambda(\Omega)$, we can choose $\lambda(S)$ and $\lambda(T)$ as close to $\beta(B_1)$ and $\beta(B_2)$, respectively, as desired, which completes the proof. □

In view of Theorem 11.5.1 and (11.7) it is useful to define $\beta(A) = \lambda(\Omega)$ and let A be an i-set. Note that this is another deviation from the case of finitely many voters, where the analogous statement is $\sum_{a \in A} \beta(a) = n + 1$, cf. Section 9.2.

For e-sets we have the following theorem.

Theorem 11.5.2. *If B_1 and B_2 are e-sets, then $B_1 \cap B_2$ or $B_1 \cup B_2$ are e-sets.*

Proof. Let B_1 and B_2 be e-sets. If $B_1 \cap B_2 = \emptyset$ then take disjoint coalitions S_1 and S_2 of sizes $\beta(B_1)$ and $\beta(B_2)$, respectively. Then $A \setminus B_1 \in E^F(S_1)$ and $A \setminus B_2 \in E^F(S_2)$. By superadditivity, $A \setminus (B_1 \cup B_2) \in E^F(S_1 \cup S_2)$. Since $\lambda(S_1 \cup S_2) = \beta(B_1 \cup B_2)$ by Theorem 11.5.1, we conclude that $B_1 \cup B_2$ is an e-set.

Next, assume $B_1 \cap B_2 \neq \emptyset$. Choose pairwise disjoint sets S_1, S_2, and S_3 in Σ_0 such that $\lambda(S_1) = \beta(B_1) - \beta(B_1 \cap B_2)$, $\lambda(S_2) = \beta(B_2) - \beta(B_1 \cap B_2)$, and $\lambda(S_3) = \beta(B_1 \cap B_2)$. Define $T_1 = S_1 \cup S_3$ and $T_2 = S_2 \cup S_3$. Then $\lambda(T_1) = \beta(B_1)$, $\lambda(T_2) = \beta(B_2)$, $\lambda(T_1 \cap T_2) = \beta(B_1 \cap B_2)$, and $\lambda(T_1 \cup T_2) = \beta(B_1 \cup B_2)$. By assumption, $A \setminus B_1 \in E^F(T_1)$ and $A \setminus B_2 \in E^F(T_2)$. Since E^F is convex, $A \setminus (B_1 \cup B_2) \in E^F(T_1 \cup T_2)$ or $A \setminus (B_1 \cap B_2) \in E^F(T_1 \cap T_2)$. Thus, $B_1 \cup B_2$ or $B_1 \cap B_2$ are e-sets. □

Example 11.5.3. The effectivity function associated with the ESC social choice function of Example 11.4.2 is as follows. If $S \in \Sigma_+$ with $\lambda(S) < \lambda(\Omega)$ then $B \in E^F(S)$ if and only if $\bar{a} \in B$, for all $B \in P_0(A)$; and if $\lambda(S) = \lambda(\Omega)$ then $E^F(S) = P_0(A)$. This implies that for all $B \in P_0(A)$ we have $\beta(B) = 0$ if $\bar{a} \notin B$, and $\beta(B) = \lambda(\Omega)$ if $\bar{a} \in B$. Also, $B \neq A$ is an i-set if $\bar{a} \notin B$, and an e-set if $\bar{a} \in B$. In particular, $\beta(\{x\}) = 0$ and $\{x\}$ is an i-set for all $x \in A \setminus \{\bar{a}\}$, and $\beta(\{\bar{a}\}) = \lambda(\Omega)$ and $\{\bar{a}\}$ is an e-set.

We conclude this section by generalizing the concepts of e-sets and i-sets. Let $\beta : P_0(A) \to [0, \lambda(\Omega)]$ and let $\{\mathbf{i}, \mathbf{e}\}$ be a partition of $P_0(A)$ satisfying

$$\beta \text{ is additive, } \beta(A) = \lambda(\Omega), \text{ and } \beta(B) > 0 \text{ for all } B \in \mathbf{e}, \tag{11.10}$$

$$\text{for all } B \in P_0(A) \setminus \{A\},\ B \in \mathbf{e} \Leftrightarrow A \setminus B \in \mathbf{i}, \text{ and } A \in \mathbf{i}, \tag{11.11}$$

$$\text{for all } B_1, B_2 \in \mathbf{e}, \text{ we have } B_1 \cap B_2 \in \mathbf{e} \text{ or } B_1 \cup B_2 \in \mathbf{e}. \tag{11.12}$$

Properties (11.10)–(11.12) summarize exactly all the properties of the e-sets and i-sets of the effectivity function associated with an anonymous ESC social choice function established above.

Next, for a system $(\beta; \mathbf{e}, \mathbf{i})$ satisfying (11.10)–(11.12), we define an effectivity function E by $E(\Omega) = P_0(A)$, $E(\emptyset) = \emptyset$, $A \in E(S)$ for every $S \in \Sigma_0$, and

$$\text{for all } B \in \mathbf{e} \text{ and } S \in \Sigma, \text{ if } \lambda(S) \geq \beta(B) \text{ then } A \setminus B \in E(S), \tag{11.13}$$

$$\text{for all } B \in \mathbf{i} \text{ and } S \in \Sigma, \text{ if } \lambda(S) > \beta(B) \text{ then } A \setminus B \in E(S). \tag{11.14}$$

It is straightforward to check that E is an effectivity function according to Definition 11.4.3: the premise in condition (iv) implies in particular that $\lambda(S_1) = \lambda(S_2)$, so that $E(S_1) = E(S_2)$ according to the definition of E.

In the next sections we consider the following question. Given a system $(\beta; \mathbf{e}, \mathbf{i})$ satisfying (11.10)–(11.12) and associated effectivity function E, is there an (anonymous) ESC social choice function F such that $E = E^F$? By using feasible elimination procedures we will present a complete answer to this question for the case where there is exactly one i-alternative, i.e., there is exactly one $x \in A$ with $\{x\} \in \mathbf{i}$, in Corollary 11.7.3. This is restrictive since we already know that there are other cases: see Examples 11.4.2, 11.5.3. On the other hand, this case is the only possible one if we require all blocking coefficients to be positive: see Corollary 11.8.3.

11.6 Feasible elimination procedures

In this section we describe a procedure that will enable the construction of an anonymous exactly and strongly consistent social choice function. We start with the definition of a so-called *pseudo* feasible elimination procedure.

Throughout, $\beta : A \to \mathbb{R}$ is a function satisfying $\beta(a) \geq 0$ for all $a \in A$ and $\sum_{a\in A} \beta(a) = \lambda(\Omega)$.

Definition 11.6.1. Let $\mathbf{R} \in \rho$. A *pseudo feasible elimination procedure* (p.f.e.p.) is a sequence $(x_1, C_1; \ldots; x_{m-1}, C_{m-1}; x_m)$ that satisfies the following conditions:

$$A = \{x_1, \ldots, x_m\}. \tag{11.15}$$

$$\begin{aligned} &C_j \in \Sigma_0,\ C_j \cap C_k = \emptyset,\ \lambda(C_j) \geq \beta(x_j) \\ &\qquad \text{for all } j, k = 1, \ldots, m-1,\ j \neq k. \end{aligned} \tag{11.16}$$

$$y\,\mathbf{R}(t)\,x_j \text{ for all } j = 1, \ldots, m-1,\ y \in \{x_{j+1}, \ldots, x_m\},\ t \in C_j. \tag{11.17}$$

In a pseudo feasible elimination procedure, 'bottom' alternatives are eliminated consecutively. As $\sum_{a\in A} \beta(a) = \lambda(\Omega)$, it is obvious that for each profile there always exists at least one p.f.e.p., namely one with $\lambda(C_j) = \beta(x_j)$ for all $j = 1, \ldots, m-1$. In the following lemma we show that actually more is possible: if an alternative x is bottom for a coalition of size larger than $\beta(x)$, then there is a p.f.e.p. where this alternative is eliminated first and with strict inequality.

First we recall a notation: for a profile $\mathbf{R}$ and a subset B of A, denote by $\mathbf{R}_{|B}$ the profile of preferences restricted to the set B.

Lemma 11.6.2. *Let $\mathbf{R} \in \rho$ and let $x \in A$ satisfy*

$$\lambda(\{t \in \Omega \mid y\,\mathbf{R}(t)\,x \text{ for all } y \in A\}) > \beta(x).$$

Then there exists a p.f.e.p. $(x, C_x; x_1, C_1; \ldots; x_{m-1})$ with $\lambda(C_x) > \beta(x)$.

Proof. The proof is by induction on m. The case $m = 2$ is obvious. Let $m \geq 3$. We define

$$A^* = \{y \in A \mid \lambda(\{t \in \Omega \mid z\,\mathbf{R}(t)\,y \text{ for all } z \in A\}) > \beta(y)\}. \tag{11.18}$$

By assumption, $x \in A^*$. We distinguish the following cases.

(i) $|A^*| \geq 2$.

Let $y \in A^* \setminus \{x\}$ and choose $C_y \subseteq \Omega$ such that $\lambda(C_y) = \beta(y)$ and $C_y \subseteq \{t \in \Omega \mid z\,\mathbf{R}(t)\,y$ for all $z \in A\}$. Define the profile $\mathbf{Q} \in \rho$ as follows. If $t \in \Omega \setminus C_y$ with $z\,\mathbf{R}(t)\,y$ for all $z \in A$, then let $x\,\mathbf{Q}(t)\,A \setminus \{x, y\}\,\mathbf{Q}(t)\,y$; otherwise, $\mathbf{Q}(t) = \mathbf{R}(t)$. Consider the restricted profile $\mathbf{Q}_1 = \mathbf{Q}^{\Omega\setminus C_y}{}_{|A\setminus\{y\}}$ – observe that if x is a bottom alternative for a voter t in this restricted profile then it was a bottom element of $\mathbf{R}(t)$. By the induction hypothesis and by the construction of $\mathbf{Q}$ there exists a p.f.e.p. $(x, C_x; x_1, C_1; \ldots; x_{m-2})$ with respect to $\mathbf{Q}_1$ such that $\lambda(C_x) > \beta(x)$ and $C_x \subseteq \{t \in \Omega \mid z\,\mathbf{R}(t)\,x$ for all $z \in A\}$. Then the p.f.e.p. $(x, C_x; y, C_y; x_1, C_1; \ldots; x_{m-2})$ is as required.

(ii) $A^* = \{x\}$.

Let $\hat{C}_x$ satisfy $\hat{C}_x \subseteq \{t \in \Omega \mid y\,\mathbf{R}(t)\,x \text{ for all } y \in A\}$ and $\lambda(\hat{C}_x) = \beta(x)$. Consider the profile $\mathbf{R}_1 = \mathbf{R}^{\Omega\setminus\hat{C}_x}{}_{|A\setminus\{x\}}$. For all $y \neq x$ let $C_y = \{t \in \Omega \setminus \hat{C}_x \mid z\,\mathbf{R}(t)\,y \text{ for all } z \in A \setminus \{x\}\}$. We distinguish two subcases.

(ii.1) $\lambda(C_y) = \beta(y)$ for all $y \neq x$.
Choose $\bar{y} \in A \setminus \{x\}$ such that $\lambda(\hat{C}) > 0$, where

$$\hat{C} = \{t \in \Omega \setminus \hat{C}_x \mid z\,\mathbf{R}(t)\,\bar{y}\mathbf{R}(t)x \text{ for all } z \in A \setminus \{x\}\}.$$

(Observe that $\hat{C} \subseteq C_{\bar{y}}$, hence $\lambda(\hat{C}) \leq \beta(\bar{y})$.) Let $C_x = \hat{C}_x \cup \hat{C}$, and let $A \setminus \{x, \bar{y}\} = \{y_1, \ldots, y_{m-2}\}$. Then $(x, C_x; y_1, C_{y_1}; \ldots; y_{m-2}, C_{y_{m-2}}; \bar{y})$ is a p.f.e.p. as required.

(ii.2) There exists $\bar{y} \neq x$ such that $\lambda(C_{\bar{y}}) > \beta(\bar{y})$.
By the induction hypothesis there exists a p.f.e.p. $(\bar{y}, \hat{C}_{\bar{y}}; x_1, C_1; \ldots, x_{m-2})$ with respect to $\mathbf{R}_1$ such that $\lambda(\hat{C}_{\bar{y}}) > \beta(\bar{y})$. Note that $\lambda(\{t \in \hat{C}_{\bar{y}} \mid z\,\mathbf{R}(t)\,\bar{y}\,\mathbf{R}(t)\,x \text{ for all } z \in A \setminus \{x\}\}) > 0$ since $\bar{y} \notin A^*$. Choose $\hat{C} \subseteq \{t \in \hat{C}_{\bar{y}} \mid z\,\mathbf{R}(t)\,\bar{y}\,\mathbf{R}(t)\,x \text{ for all } z \in A \setminus \{x\}\}$ such that $0 < \lambda(\hat{C}) \leq \lambda(\hat{C}_{\bar{y}}) - \beta(\bar{y})$. Then $(x, \hat{C}_x \cup \hat{C}; \bar{y}, \hat{C}_{\bar{y}} \setminus \hat{C}; x_1, C_1; \ldots; x_{m-2})$ is a p.f.e.p. as required. □

Now note that if a procedure like p.f.e.p. should result in an anonymous ESC social choice function then clearly some of the alternatives might be i-alternatives and these should be blocked with inequality. The preceding lemma exhibits a case in which this is possible. If, however, there are two or more of such i-alternatives then it is not difficult to construct a profile where a p.f.e.p. does not exist if we require i-alternatives to be blocked with inequality. With this consideration and with observation (11.4) – which says that only i-alternatives can have zero blocking coefficients – in mind, all alternatives except at most one should have positive β-values. Therefore, in the rest of this section we make the following assumption.

Assumption 11.6.3 There is an alternative in A, denoted by s, such that $\beta(a) > 0$ for all $a \in A \setminus \{s\}$.

We next introduce the concept of a feasible elimination procedure within the model of this chapter. In this procedure, the designated alternative s of Assumption 11.6.3 can only be eliminated if, during the procedure, it becomes a bottom alternative for a coalition of size strictly larger than $\beta(s)$.

Definition 11.6.4. Let $\mathbf{R} \in \rho$. A p.f.e.p. $(x_1, C_1; \ldots; x_{m-1}, C_{m-1}; x_m)$ is a *feasible elimination procedure* (f.e.p.) if it satisfies the following condition:

$$x_m = s \text{ or } [x_j = s \text{ for some } j < m \text{ and } \lambda(C_j) > \beta(s)]. \tag{11.19}$$

We shall now prove the existence of f.e.p.'s in our model and then relate them to ESC social choice functions.

Theorem 11.6.5. *Let Assumption 11.6.3 hold. Then, for every* $\mathbf{R} \in \rho$ *there is an f.e.p. with respect to* $\mathbf{R}$.

Proof. Let $\mathbf{R} \in \rho$. The proof is by induction on m. The case $m = 2$ is obvious. Let $m \geq 3$. Define A^* as in (11.18). We distinguish the following possibilities.

(i) $A^* = \emptyset$.
For $a \in A$ let $C(a) = \{t \in \Omega \mid y\,\mathbf{R}(t)\,a \text{ for all } y \in A\}$. Then $\lambda(C(a)) = \beta(a)$ for all $a \in A$. Let $A \setminus \{s\} = \{a_1, \ldots, a_{m-1}\}$. Then $(a_1, C(a_1); \ldots; a_{m-1}, C(a_{m-1}); s)$ is an f.e.p.

(ii) $A^* \neq \emptyset$ and $s \notin A^*$.
Let $y \in A^*$ and let $C_y \subseteq \{t \in \Omega \mid z\,\mathbf{R}(t)\,y \text{ for all } z \in A\}$ satisfy $\lambda(C_y) = \beta(y)$. By the induction hypothesis for $\mathbf{R}^{\Omega \setminus C_y}{}_{|A \setminus \{y\}}$ there exists an f.e.p. $(x_1, C_1; \ldots; x_{m-1})$ for the restricted profile. Then $(y, C_y; x_1, C_1; \ldots; x_{m-1})$ is an f.e.p. for $\mathbf{R}$.

(iii) $s \in A^*$.
This case follows from Lemma 11.6.2. □

We shall use the existence of feasible elimination procedures established in Theorem 11.6.5 to derive the existence of an interesting class of ESC social choice functions, similarly as we did in Chapter 9. Let $\mathbf{R} \in \rho$. Call $x \in A$ $\mathbf{R}$-*maximal* if there exists an f.e.p. $(x_1, C_1; \ldots; x_m)$ with respect to $\mathbf{R}$ such that $x = x_m$. Further, denote

$$M(\mathbf{R}) = \{x \in A \mid x \text{ is } \mathbf{R}\text{-maximal}\}.$$

Thus, $M(\mathbf{R}) \neq \emptyset$ for all $\mathbf{R} \in \rho$ if Assumption 11.6.3 holds. The following observation concerning $M(\cdot)$ will be very useful below.

Remark 11.6.6. Let $\mathbf{R} \in \rho$ and let $x \in A \setminus \{s\}$ satisfy

$$\lambda(\{t \in \Omega \mid y\,\mathbf{R}(t)\,x \text{ for all } y \in A\}) \geq \beta(x).$$

Then $x \notin M(\mathbf{R})$. This is so since $\lambda(\bigcup_{y \in A \setminus \{x\}} \{t \in \Omega \mid A \setminus \{y\}\mathbf{R}(t)y\}) \leq \lambda(\Omega) - \beta(x)$ and s has to be eliminated strictly in an f.e.p.

Theorem 11.6.7. *Let Assumption 11.6.3 hold. Let the social choice function $F : \rho \to A$ be a selection from $M(\cdot)$, that is, $F(\mathbf{R}) \in M(\mathbf{R})$ for every $\mathbf{R} \in \rho$. Then F is exactly and strongly consistent.*

Proof. Let $\mathbf{R} \in \rho$ and $x = F(\mathbf{R})$. Then there exists an f.e.p. $(x_1, C_1; \ldots; x_{m-1}, C_{m-1}; x)$ with respect to $\mathbf{R}$. Choose $\mathbf{Q} \in \rho$ that satisfies $y\,\mathbf{Q}(t)\,x_j$ for all $t \in C_j$, $y \in A$, and $j = 1, \ldots, m-1$. We claim that $F(\mathbf{Q}) = F(\mathbf{R})$ and that $\mathbf{Q}$ is a strong equilibrium of the game $(F, \mathbf{R})$. We distinguish the following cases.

(i) $x = s$.
By Remark 11.6.6, $F(\mathbf{Q}) = s$. Now assume, on the contrary, that $\mathbf{Q}$ is not a strong equilibrium of $(F, \mathbf{R})$. Then there exist $S \in \Sigma_+$ and $\mathbf{V}^S \in \rho^S$ such that $F(\mathbf{Q}^{\Omega \setminus S}, \mathbf{V}^S) = y$, $y \neq s$, and $y\,\mathbf{R}(t)\,s$ for all $t \in S$. Let $y = x_j$ for some $1 \leq j \leq m-1$. Then $S \cap C_j = \emptyset$ because $s\,\mathbf{R}(t)\,x_j$ for all $t \in C_j$. Hence, by Remark 11.6.6, $F(\mathbf{Q}^{\Omega \setminus S}, \mathbf{V}^S) \neq x_j$, which is the desired contradiction.

(ii) $x \neq s$.

Then $s = x_{j_0}$ for some $j_0 \leq m - 1$. Thus, by definition of an f.e.p., $\lambda(C_{j_0}) > \beta(s)$. Hence, it is not possible to eliminate all $x' \neq s$ in an f.e.p. with respect to $\mathbf{Q}$, and therefore $F(\mathbf{Q}) \neq s$. By Remark 11.6.6 applied to all $x' \in A \setminus \{x, s\}$, $F(\mathbf{Q}) = x$. The proof that $\mathbf{Q}$ is a strong equilibrium of $(F, \mathbf{R})$ is analogous to that in case (i), observing that a profitable deviation from $\mathbf{Q}$ can never result in s since $\lambda(C_{j_0}) > \beta(s)$ and, therefore, it is not possible to eliminate all alternatives in $A \setminus \{s\}$ in an f.e.p. with respect to $\mathbf{Q}$. □

We conclude this section with some observations which relate Theorem 11.6.7 to the preceding sections.

Let $\hat{F}$ be an anonymous selection from $M(\cdot)$. For instance, for every $\mathbf{R} \in \rho$ select the maximal element in $M(\mathbf{R})$ according to a fixed order $R_0 \in L$. By Theorem 11.6.7, $\hat{F}$ is an anonymous ESC social choice function, and therefore its associated effectivity function $E^{\hat{F}}$ is characterized by blocking coefficients (say) $\hat{\beta}(B)$ for $B \in P_0(A)$. Since alternatives assigned by $\hat{F}$ result from feasible elimination procedures with weights $\beta(a)$ $(a \in A)$, it is easy to check that $\hat{\beta}(a) = \beta(a)$ for every $a \in A$, and that $\{s\}$ is an i-set whereas all other singleton sets are e-sets. By the results established in Section 11.5, it follows that a set $B \subseteq A$ is an i-set if and only if it contains s. Note that the effectivity function $E^{\hat{F}}$ is independent of the particular anonymous selection $\hat{F}$ chosen since it is completely determined by the weights $\beta(a)$ $(a \in A)$, and thus we can denote it by $\hat{E}$. Since, for all $\mathbf{R} \in \rho$ and for every element $x \in M(\mathbf{R})$ we can always find an anonymous selection choosing that particular element, Theorem 11.4.5 implies that $M(\mathbf{R}) \subseteq C(\hat{E}, \mathbf{R})$ for all $\mathbf{R} \in \rho$. We can also state this as $M(\mathbf{R}) \subseteq C(E, \mathbf{R})$ for all $\mathbf{R} \in \rho$, where E is the effectivity function associated with the system $(\beta, \mathbf{e}, \mathbf{i})$ as above (cf. Section 11.5). In the next section we shall establish the converse inclusion $C(E, \mathbf{R}) \subseteq M(\mathbf{R})$.

11.7 Core and feasible elimination procedures

In this section we prove that for any anonymous ESC social choice function that has exactly one i-alternative, every element in the core of the associated effectivity function can be obtained by a feasible elimination procedure.

Let $(\beta; \mathbf{e}, \mathbf{i})$ be a system satisfying (11.10)–(11.12) with $\mathbf{i}$ containing exactly one singleton $\{s\}$ for some designated $s \in A$. Hence, $\beta(y) > 0$ for all $y \in A \setminus \{s\}$. Let E be the associated effectivity function. Note that $\beta(\cdot)$ satisfies Assumption 11.6.3 and therefore $M(\mathbf{R}) \neq \emptyset$ for all $\mathbf{R} \in \rho$ by Theorem 11.6.5. As explained in the last paragraph of Section 11.6, we have $M(\mathbf{R}) \subseteq C(E, \mathbf{R})$ and in particular $C(E, \mathbf{R}) \neq \emptyset$ for every $\mathbf{R} \in \rho$.

Let $\mathbf{R} \in \rho$ and $x \in C(E, \mathbf{R})$. For every $y \in A \setminus \{x\}$ denote

$$S(y) = \{t \in \Omega \mid x\, \mathbf{R}(t)\, y\}.$$

As before for $B \in P_0(A)$ we denote $\beta(B) = \sum_{y \in B} \beta(y)$. Using (11.13), (11.14), and the definition of the core it is straightforward to derive, for all $z \in A$,

$$z \in C(E, \mathbf{R}) \Leftrightarrow \begin{cases} \lambda(\cup_{y \in B} S_z(y)) \geq \beta(B) \text{ for all } B \in P_0(A) \text{ with } s \notin B, z \notin B, \\ \lambda(\cup_{y \in B} S_z(y)) > \beta(B) \text{ for all } B \in P_0(A) \text{ with } s \in B, z \notin B, \end{cases} \tag{11.20}$$

where $S_z(y) = \{t \in \Omega \mid z\, \mathbf{R}(t)\, y\}$, so $S(y) = S_x(y)$. In particular, we have $\lambda(S(y)) \geq \beta(y)$ for all $y \neq x$, with strict inequality if $y = s$. Of course, the sets $S(y)$ need not be disjoint, but Theorem 11.7.1 below says that they can be shrunk in such a way that they become pairwise disjoint while maintaining the inequalities. This theorem is a continuous version of the discrete 'marriage theorem' (cf. Halmos and Vaughan, 1950), suitable for our context, in particular for deriving Theorem 11.7.2 below. The latter theorem says that core elements are maximal, and its proof follows the construction in the proof of Theorem 11.7.1.[5]

Theorem 11.7.1. *There exist pairwise disjoint measurable sets* $C(y)$, $y \in A \setminus \{x\}$, *such that* (*i*) $C(y) \subseteq S(y)$ *for every* $y \in A \setminus \{x\}$; (*ii*) $\lambda(C(y)) \geq \beta(y)$ *for all* $y \neq x$ *and* $\lambda(C(s)) > \beta(s)$.

Proof. We start by noting that, if $x \neq s$, we may increase $\beta(s)$ with a small $\varepsilon > 0$ and decrease $\beta(x)$ with the same amount (note that $\beta(x) > 0$). In this way, all inequalities in (11.20) still hold as weak inequalities and it is sufficient to prove (*ii*) in the theorem with only weak inequalities. Moreover, we may regard x as the i-alternative instead of s. For the rest of the proof we assume that this is the case.

We prove the theorem by induction on $|A| = m \geq 2$. The case $m = 2$ is obvious, so we concentrate on the induction step for $m \geq 3$. We first make the following observation.

Remark. Suppose there exists a set $B^* \subseteq A \setminus \{x\}$ with $\emptyset \neq B^* \neq A \setminus \{x\}$, such that $\lambda(\cup_{y \in B^*} S(y)) = \beta(B^*)$. In this case, we can decompose our problem into two smaller problems to which we can apply the induction hypothesis, as follows.

(i) The problem with set of alternatives $B^* \cup \{x\}$, set of voters $\cup_{y \in B^*} S(y)$, blocking coefficients $\hat{\beta}(x) = 0$ and $\hat{\beta}(y) = \beta(y)$ unchanged for $y \in B^*$, and preferences $\mathbf{R}(t)_{|B^* \cup \{x\}}$ for $t \in \cup_{y \in B^*} S(y)$. Note that all inequalities as in (11.20) restricted to voters in $\cup_{y \in B^*} S(y)$ and alternatives in $B^* \cup \{x\}$ still hold, and that $\lambda(\cup_{y \in B^*} S(y)) = \beta(B^*) = \hat{\beta}(B^* \cup \{x\})$.

(ii) The problem with set of alternatives $A \setminus B^*$, set of voters $\Omega \setminus \cup_{y \in B^*} S(y)$, blocking coefficients unchanged, and preferences $\mathbf{R}(t)$ restricted to $A \setminus B^*$.

[5] For a proof of a slightly less general version of the continuous 'marriage theorem' see Hart and Kohlberg (1974, p. 171).

Note that all inequalities still hold since for any set $B \subseteq A \setminus (\{x\} \cup B^*)$ we have

$$\begin{aligned}\lambda(\cup_{y\in B} S(y) \setminus \cup_{\hat{y}\in B^*} S(\hat{y})) &= \lambda(\cup_{y\in B\cup B^*} S(y)) - \lambda(\cup_{y\in B^*} S(y)) \\ &= \lambda(\cup_{y\in B\cup B^*} S(y)) - \beta(B^*) \\ &\geq \beta(B \cup B^*) - \beta(B^*) \\ &= \beta(B).\end{aligned}$$

Furthermore, $\lambda(\Omega \setminus \cup_{y\in B^*} S(y)) = \beta(A \setminus B^*)$.

The required sets $C(y)$, $y \in A \setminus \{x\}$ are now obtained by applying the induction hypothesis to each subproblem.

We now proceed to the induction step. Let $m \geq 3$. We are done if there is a decomposition possible as in the Remark, so suppose there is none. Let $b \in A \setminus \{x\}$ and consider the set $S = S(b) \setminus \cup_{y\in A\setminus\{x,b\}} S(y)$, i.e., $S = \{t \in \Omega \mid y\,\mathbf{R}(t)\,x\,\mathbf{R}(t)\,b$ for all $y \neq x, b\}$. We distinguish two cases.

Case 1: $\lambda(S) \geq \beta(b)$. Since $x \in C(E, \mathbf{R})$, $0 \leq \lambda(S) \leq \beta(x) + \beta(b)$. Now take $C(b)$ equal to S, and apply the induction hypothesis to the problem with set of alternatives $A \setminus \{b\}$, set of voters $\Omega \setminus S$, blocking weights β' unchanged except $\beta'(x) = \beta(x) - (\lambda(S) - \beta(b))$, and preferences equal to the original preferences restricted to $A \setminus \{b\}$.

Case 2: $\lambda(S) < \beta(b)$. We also know $\lambda(S(b)) > \beta(b)$ otherwise $\lambda(S(b)) = \beta(b)$ by (11.20), and we would have a decomposition as in the Remark with $B^* = \{b\}$. Now choose a measurable set S^* satisfying $S \subseteq S^* \subseteq S(b)$ and $\lambda(S^*) = \beta(b)$ (this is possible by Lyapunov's Theorem). Consider the set of vectors

$$\{(\lambda\,(S^* \cup T \cup (\cup_{y\in B} S(y))))_{B \subsetneq A\setminus\{b,x\}} \mid \emptyset \subseteq T \subseteq S(b) \setminus S^*\}. \tag{11.21}$$

For $T = S(b) \setminus S^*$ and $B = \emptyset$ we have

$$\lambda(S^* \cup T \cup (\cup_{y\in B} S(y))) = \lambda(S(b)) > \beta(b) \tag{11.22}$$

and for $T = S(b) \setminus S^*$ and $B \subseteq A \setminus \{x, b\}$ arbitrary we have

$$\lambda(S^* \cup T \cup (\cup_{y\in B} S(y))) = \lambda(\cup_{y\in B\cup\{b\}} S(y)) \geq \beta(b) + \beta(B) \tag{11.23}$$

by (11.20). For $T = \emptyset$ and $B = \emptyset$ we have

$$\lambda(S^* \cup T \cup (\cup_{y\in B} S(y))) = \lambda(S^*) = \beta(b). \tag{11.24}$$

Now for $B \subseteq A \setminus \{x, b\}$ with $B \neq A \setminus \{x, b\}$ and $T \subseteq S(b) \setminus S^*$ consider the expression

$$\begin{aligned}\lambda(S^* \cup T \cup (\cup_{y\in B} S(y))) = \lambda(T) &+ \lambda(S^* \cup (\cup_{y\in B} S(y))) \\ &-\lambda(T \cap (S^* \cup (\cup_{y\in B} S(y)))).\end{aligned}$$

This is an affine function, with variable T, of two measures $\lambda(T)$ and $\lambda(T \cap (S^* \cup (\cup_{y\in B} S(y))))$. As B varies on $\{B' \mid B' \subseteq A \setminus \{b, x\},\ B' \neq A \setminus \{b, x\}\}$ we obtain an affine combination of two vector measures. Hence, its range

(11.21) is compact and convex by Lyapunov's Theorem. By (11.22), (11.23), and (11.24), we can choose $T = T_0$ such that all inequalities in (11.23) are still valid but with at least one equality, say for B_0. Now set $S_0 = S^* \cup T_0$, and set $B^* = B_0 \cup \{b\}$. On $S(b) \setminus S_0$ change the preferences by shifting b up so that it becomes preferred to x. Use the notation $\bar{S}(\cdot)$ for the $S(\cdot)$-sets in the new profile. Then all sets $S(y)$, $y \neq b$, remain unchanged, i.e., $\bar{S}(y) = S(y)$, whereas $S(b)$ changes to $\bar{S}(b) = S_0$. Then, for this *new* profile, we have $\beta(B^*) = \beta(b) + \beta(B_0) = \lambda(S_0 \cup (\cup_{y \in B_0} S(y))) = \lambda(\cup_{y \in B^*} \bar{S}(y))$. The problem with the new profile is decomposable according to the Remark. Applying the Remark, we obtain the desired sets: in particular, the resulting set $C(b)$ is a subset of $\bar{S}(b) = S_0$ and therefore of $S(b)$. This concludes the proof of the theorem. □

Still under the assumptions made at the beginning of this section we proceed to show that x is a maximal alternative, i.e., $x \in M(\mathbf{R})$. We first attach a precise and formal meaning to the expression 'bottom alternative': we call $b \in A$ a *bottom alternative* of $\mathbf{R}$ if the set $\hat{S}(b) = \{t \in \Omega \mid y\,\mathbf{R}(t)\,b \text{ for all } y \in A\}$ has measure $\lambda(\hat{S}(b)) \geq \beta(b)$, with strict inequality sign for $b = s$. Observe that there is always a bottom alternative since $\sum_{a \in A} \beta(a) = \lambda(\Omega)$. Obviously, x is not a bottom alternative since it is in the core $C(E, \mathbf{R})$.

We have the following result.[6]

Theorem 11.7.2. *Alternative x is $\mathbf{R}$-maximal, that is, $x \in M(\mathbf{R})$. In particular, if b is a bottom alternative of $\mathbf{R}$, then there is an f.e.p.* $(b, C_b; y_1, C_1; \ldots; y_{m-2}, C_{m-2}; x)$.

Proof. Let b be a bottom alternative. If $b = s$ we slightly increase the blocking coefficient of b (as in the beginning of the proof of Theorem 11.7.1) so that we still have $\lambda(\hat{S}(b)) \geq \beta(b)$. (This has the advantage that in what follows it is sufficient to consider blocking with weak inequalities.)

The proof is by induction on $m = |A|$. For $m = 2$ the result is again obvious. Let $m \geq 3$.

(i) First suppose that the problem is decomposable into two subproblems with sets of alternatives $\{x\} \cup B^*$ and $A \setminus B^*$ as in the proof of Theorem 11.7.1, and with $b \in B^*$. Note that all voters in the problem with $A \setminus B^*$ rank B^* above x. By the induction hypothesis, each of the subproblems has an f.e.p. leading to x, with the one in the first subproblem starting with b. Let $|B^*| = k$, let $(b, C_b; y_1, C_1; \ldots; y_{k-1}, C_{k-1}; x)$ be an f.e.p. in the problem with $\{x\} \cup B^*$ and let $(x_1, \hat{C}_1; \ldots; x_{m-k-1}, \hat{C}_{m-k-1}; x)$ be an f.e.p. in the problem with $A \setminus B^*$. Then

$$(b, C_b; x_1, \hat{C}_1; \ldots; x_{m-k-1}, \hat{C}_{m-k-1}; y_1, C_1; \ldots; y_{k-1}, C_{k-1}; x)$$

is an f.e.p. for the original problem.

[6] The analogous result to Theorem 11.7.2 for the case with finitely many voters is Theorem 9.3.6. The proof of the latter theorem has benefitted from the analysis in this chapter.

(ii) Next, suppose the problem is not decomposable in this way. As in the proof of Theorem 11.7.1 let $S = S(b) \setminus \bigcup_{y \in A \setminus \{x,b\}} S(y)$ and distinguish two cases as there. In Case 1, $\lambda(S) \geq \beta(b)$, we take again $C(b) = S$, observing that $S \subseteq \hat{S}(b)$. Applying the induction hypothesis, we let $(y_1, C_1; \ldots; y_{m-2}, C_{m-2}; x)$ be an f.e.p. in the problem with set of alternatives $A \setminus \{b\}$, then $(b, C_b; y_1, C_1; \ldots; y_{m-2}, C_{m-2}; x)$ is as desired.

In Case 2, we proceed again as in the proof of Theorem 11.7.1 but we make sure that S_0 there is chosen in such a way that $\lambda(S_0 \cap \hat{S}(b)) \geq \beta(b)$. This is possible since $S \subseteq \hat{S}(b) \subseteq S(b)$ and so we can choose S^* (which is a subset of S_0 by construction) such that $S^* \subseteq \hat{S}(b)$. We have now again a decomposition as in (i) of this proof: since b is eliminated first, shifting b over x in the original preferences of voters in $S(b) \setminus S_0$ does not change the restriction of these preferences to $A \setminus B^*$. □

We conclude this section by summarizing the main results of Sections 11.6 and 11.7 in the following corollary.

Corollary 11.7.3.

(*i*) *Let F be an anonymous ESC social choice function. Suppose that the associated effectivity function E has exactly one i-alternative. Then $C(E, \cdot) = M(\cdot)$ and F is a selection from this set.*

(*ii*) *Let $(\beta; \mathbf{e}, \mathbf{i})$ be a system satisfying (11.10)–(11.12) such that $\mathbf{i}$ contains exactly one singleton. Then, for the associated effectivity function E, $C(E, \cdot) = M(\cdot)$, and any anonymous selection from this set is an anonymous ESC social choice function.*

11.8 Positive blocking coefficients

A natural question is whether Corollary 11.7.3 can be extended to general systems $(\beta; \mathbf{e}, \mathbf{i})$. We have already remarked that if there are two or more i-alternatives, then a feasible elimination procedure may fail to exist. On the other hand, Example 11.4.2 (or 11.5.3) shows that an anonymous ESC social choice function may generate more than one i-alternative. In this section we show that if an anonymous ESC social choice function generates only positive blocking coefficients, then there can be at most one i-alternative. In other words, Corollary 11.7.3 provides a complete characterization of anonymous ESC social choice functions if we require all blocking coefficients to be positive.

Let E be the effectivity function associated with a system $(\beta; \mathbf{e}, \mathbf{i})$, satisfying (11.10)–(11.12).

Definition 11.8.1. E satisfies $D(k)$, where $1 \leq k \leq m-2$, if there exist *no* partitions $\{x_1\}, \ldots, \{x_k\}, C_1, C_2$ of A and $S_1, \ldots, S_k, T_1, T_2$ of Ω, $S_1, \ldots, S_k, T_1, T_2 \in \Sigma_+$, such that[7]

(i) $\lambda(S_i) = \beta(x_i)$ for $i = 1, \ldots, k$, and $x_1, \ldots, x_k$ are e-alternatives;
(ii) $\lambda(T_i) = \beta(C_i)$ for $i = 1, 2$, and C_1 and C_2 are i-sets.

The following theorem is a counterpart of similar results for the case of finitely many voters, see Section 10.4 in particular. Its proof is deferred until the end of this section.

Theorem 11.8.2. *Let $F : \rho \to A$ be an anonymous ESC social choice function, and let $(\beta; \mathbf{e}, \mathbf{i})$ be the associated system. Suppose that $\beta(a) > 0$ for all $a \in \mathbf{i}$. Then $E = E^F$ satisfies $D(k)$ for all $1 \leq k \leq m-2$.*

We now have:

Corollary 11.8.3. *Let $F : \rho \to A$ be an anonymous ESC social choice function that generates only positive blocking coefficients. Then there is exactly one i-alternative.*

Proof. Clearly, by (11.11) and (11.12), there must be at least one i-alternative: if all alternatives were e-alternatives then repeated application of (11.12) would give a violation of (11.11). Also, there must be at least one e-alternative: if not, then $A \setminus \{x\}$ would be an e-set for each $x \in A$ by (11.11), hence $A \setminus \{x, y\} = A \setminus \{x\} \cap A \setminus \{y\}$ would be an e-set for all $x, y \in A$ by (11.12), and so on and so forth, implying that all singletons would be e-sets, a contradiction.

Suppose that there are two different i-alternatives x, y in the associated system. Let $\{x_1\}, \ldots, \{x_k\}$ be the e-singletons, hence $1 \leq k \leq m-2$. Define $C_1 = \{x\}$ and $C_2 = \{z \in A \mid z \neq x, \{z\} \in \mathbf{i}\}$. Then C_2 is an i-set, which can be seen as follows. Write $C_2 = \{y_1, \ldots, y_\ell\}$, where $\ell \geq 1$. If C_2 were an e-set, then also $C_2 \setminus \{y_\ell\} = C_2 \cap A \setminus \{y_\ell\}$ would be an e-set by (11.12). Hence, $C_2 \setminus \{y_\ell, y_{\ell-1}\} = C_2 \setminus \{y_\ell\} \cap A \setminus \{y_{\ell-1}\}$ is an e-set, and so on and so forth, until we obtain that $\{y_1\}$ is an e-set, which is a contradiction.

Now choose a partition of Ω as in Definition 11.8.1, so that $D(k)$ is violated. This contradicts Theorem 11.8.2. □

The combination of Corollaries 11.7.3 and 11.8.3 yields an almost complete characterization of anonymous ESC social choice functions. The case where there is more than one i-alternative – so that at least one i-alternative has zero blocking coefficient – is still open.

Proof of Theorem 11.8.2

We start with the following observation.

[7] Recall from Section 11.3 that elements of a partition have positive measure by definition.

Lemma 11.8.4. *Let $F : \rho \to A$ be an ESC SCF. Then there exist no partitions $S_1, \ldots, S_p$ of Ω and $B_1, \ldots, B_p$ of A (where $p \geq 2$) such that $A \setminus B_i \notin E^F(S_i)$ for all $i = 1, \ldots, p$.*

Proof. Assume, on the contrary, that there exist partitions as in the lemma. Consider the following profile:

S_1	S_2	$\cdots$	S_p
B_2	B_3	$\cdots$	B_1
B_3	B_4	$\cdots$	B_2
$\vdots$	$\vdots$		$\vdots$
B_p	B_1	$\cdots$	B_{p-1}
B_1	B_2	$\cdots$	B_p

By maximality (Theorem 11.4.5) of E^F, we have $B_i \in E^F(\Omega \setminus S_i)$ for every $i = 1, \ldots, p$. Hence, the alternatives in B_2 are blocked by $\Omega \setminus S_1$, the alternatives in B_3 by $\Omega \setminus S_2$, etc., so that $C(E^F, \mathbf{R}) = \emptyset$. But this contradicts stability (Theorem 11.4.5) of E^F. □

Proof of Theorem 11.8.2 The proof is by induction on k.

(1) $k = 1$. Assume, on the contrary, that there are partitions $\{x_1\}$, C_1, C_2 of A and S_1, T_1, T_2 of Ω, satisfying (i) and (ii) in Definition 11.8.1. Let $S_1 = S^1 \cup S^2$ with $S^1 \cap S^2 = \emptyset$ and $\lambda(S^1) = \lambda(S^2)$. Consider the following profile $\mathbf{R}$:

S^1	S^2	T_1	T_2
C_1	C_2	x_1	x_1
C_2	C_1	C_2	C_1
x_1	x_1	C_1	C_2

Since S_1 can block x_1 (i.e., $A \setminus \{x_1\} \in E(S_1)$), stability of E^F (Theorem 11.4.5) implies $F(\mathbf{R}) \neq x_1$. Without loss of generality $F(\mathbf{R}) \in C_1$. Let $\mathbf{Q}$ be a strong Nash equilibrium of $(F, \mathbf{R})$ with $F(\mathbf{R}) = F(\mathbf{Q})$. We distinguish the following possibilities.

(1.1) There exists $y \in C_2$ such that $\lambda(\{t \in S_1 \mid x_1 \, \mathbf{Q}(t) \, y\}) > 0$. Choose $S^3 \subseteq \{t \in S_1 \mid x_1 \, \mathbf{Q}(t) \, y\}$ such that $0 < \lambda(S^3) < \min_{a \in A} \beta(a)$. Define the $T_1 \cup T_2$-profile $\mathbf{P}$ by

$$x_1 \, \mathbf{P}(t) \, y \, \mathbf{P}(t) \, A \setminus \{x_1, y\} \quad \text{for all } t \in T_1 \cup T_2.$$

By considering the partitions $S_1 \setminus S^3$, S^3, $T_1 \cup T_2$, and $\{x_1\}$, $\{y\}$, $A \setminus \{x_1, y\}$, it follows from Lemma 11.8.4 that $T_1 \cup T_2$ blocks $A \setminus \{x_1, y\}$. Hence, $F(\mathbf{Q}^{S_1}, \mathbf{P}^{T_1 \cup T_2}) \in \{x_1, y\}$. As $T_1 \cup T_2 \cup S^3$ is effective for x_1, and $F(\mathbf{Q}^{S_1}, \mathbf{P}^{T_1 \cup T_2}) \in C(E, (\mathbf{Q}^{S_1}, \mathbf{P}^{T_1 \cup T_2}))$, we have $F(\mathbf{Q}^{S_1}, \mathbf{P}^{T_1 \cup T_2}) = x_1$. Thus, $T_1 \cup T_2$ has improved upon $F(\mathbf{Q})$, which is a contradiction.

(1.2) $C_2 \, \mathbf{Q}(t) \, x_1$ for all $t \in S_1$. Consider the $T_1 \cup S^2$-profile $\mathbf{P}$ defined by

$$C_2\,\mathbf{P}(t)\,x_1\,\mathbf{P}(t)\,C_1 \text{ for all } t \in T_1 \cup S^2.$$

Since $\lambda(T_1 \cup S^2) > \lambda(T_1) = \beta(C_1)$, $T_1 \cup S^2$ blocks C_1. Therefore, $F(\mathbf{Q}^{T_2 \cup S^1}, \mathbf{P}^{T_1 \cup S^2}) \notin C_1$. Suppose $F(\mathbf{Q}^{T_2 \cup S^1}, \mathbf{P}^{T_1 \cup S^2}) = x_1$. Note that, in the profile $(\mathbf{Q}^{T_2 \cup S^1}, \mathbf{P}^{T_1 \cup S^2})$, both T_1 and S_1 prefer C_2 over x_1. Moreover, $\lambda(T_1 \cup S_1) = \beta(C_1) + \beta(x_1)$ and $C_1 \cup \{x_1\}$ is an e-set, because C_2 is an i-set; therefore, $T_1 \cup S_1$ blocks $C_1 \cup \{x_1\}$. This contradicts $F(\mathbf{Q}^{T_2 \cup S^1}, \mathbf{P}^{T_1 \cup S^2}) = x_1$ and, hence, $F(\mathbf{Q}^{T_2 \cup S^1}, \mathbf{P}^{T_1 \cup S^2}) \in C_2$. But this contradicts the fact that $\mathbf{Q}$ is a strong Nash equilibrium in $(F, \mathbf{R})$.

(2) Let $1 < k \leq m-2$ and assume $D(1), \ldots, D(k-1)$. We shall prove $D(k)$.

Assume, on the contrary, that there exist partitions $\{x_1\}, \ldots, \{x_k\}, C_1, C_2$ of A and $S_1, \ldots, S_k$, T_1, T_2 of Ω, satisfying (i) and (ii) in Definition 11.8.1. If $C_1 \cup \{x_k\}$ is an i-set, then we obtain a contradiction to $D(k-1)$. Otherwise, $A \backslash (C_1 \cup \{x_k\})$ is an i-set. Then consider the partitions $\{x_k\}, C_1, A \backslash (C_1 \cup \{x_k\})$ of A, and S_k, T_1, $\Omega \setminus (S_k \cup T_1)$ of Ω: this implies a contradiction to $D(1)$. □

11.9 Notes and comments

Most of the results of this chapter first appeared in Peleg and Peters (2006). The extension of the Gibbard-Satterthwaite theorem to the continuum voter case first appeared in the working paper version of Peleg and Peters (2006). There, it is also shown that in this model an effectivity function is maximal and stable if and only if it can be represented by a strongly consistent game form. See Propositions 5.2.4 and 5.2.6 and Theorem 5.3.2, or Moulin and Peleg (1982), for the case with finitely many voters.