Chapter 3
LDA Principles and Laser Optics

The technique of Laser Doppler Anemometry (LDA), as the name stands for, is a technique of using the laser light and the Doppler effect for velocity measurements. It is an optical method and hence tightly related to both the physical and geometrical optics. In order to depict the functionality of the LDA method, some physical properties of the light and the light wave propagation in medium will firstly be considered.

3.1 Light Wave and Its Propagation

The light is the electromagnetic wave which is specified by its amplitude, the polarization and the wavelength λ. In the LDA technology regarding special laser light properties, the polarization of the laser light is less interesting, at least for LDA users. It will only be considered by splitting the laser beam into parts and transmitting them through single-mode fibers to the LDA head. In addition, it will also be taken into account if the change in the light intensity due to refraction has to be concerned based on Fresnel equations. For laser light refractions at not too large refraction angles, the effect of laser light polarization on the change in the laser light intensity has usually been neglected. By disregarding the light polarization the spatial propagation of a plane light wave of the amplitude E_0 in the positive x-direction can be expressed by

$$E = E_0 \cos(\omega t - kx) \tag{3.1}$$

The parameters $\omega = 2\pi/T$ and $k = 2\pi/\lambda$, respectively, represent the angular frequency and the angular wavenumber, or simply wavenumber of the light wave. They are coupled by the wave propagation speed i.e. the light speed in the medium in which the light propagates. As known, the wave propagation speed is expressed by the phase velocity that is determined by

$$\frac{\mathrm{d}(\omega t - kx)}{\mathrm{d}t} = 0 \tag{3.2}$$

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The light speed is then obtained as

$$c = \frac{dx}{dt} = \frac{\omega}{k} \tag{3.3}$$

Because of the constant light speed in the homogeneous medium, this equation signifies that the light wave can be represented either by the angular frequency ω in the time domain or equivalently by the wavenumber k in the space. The light speed in Eq. (3.3) can also be expressed as $c = \nu\lambda$ with $\nu = 1/T$ as the oscillation frequency of the light wave. In the vacuum, the light speed is measured as $c = 2.99792458 \times 10^8$ m/s. In the dielectric or non-conducting medium, like water for instance, the light speed is less than that in the vacuum. The ratio between them is denoted by n and called the refractive index of the respective dielectric medium. For the case that the light propagates from one medium (n_1) into another (n_2), the light speed then changes from c_1 to c_2 according to the following relation

$$\frac{c_2}{c_1} = \frac{n_1}{n_2} \tag{3.4}$$

Since the light frequency does not change while the light is refracted at the interface between two mediums, Eq. (3.4) can further be written with respect to $c = \nu\lambda$ as

$$\frac{c_2}{c_1} = \frac{n_1}{n_2} = \frac{\lambda_2}{\lambda_1} \tag{3.5}$$

This means that the wavelength of the light in a medium is reciprocally proportional to the medium refractive index n.

The refractive index n is an optical and physical parameter of a dielectric medium. It is not only a function of the medium and the medium temperature, but also a function of the light wavelength (color). The latter phenomenon is known as the dispersion or the chromatic aberration. In LDA measurements, the optical dispersion is generally neglected without causing any significant measurement errors (Chap. 13).

The transmission of light from one medium into another is also related to the change in the direction of light propagation. This is described by the law of refraction (also known as Snell's law) according to Fig. 3.1:

$$n_1 \sin\varepsilon_1 = n_2 \sin\varepsilon_2 \tag{3.6}$$

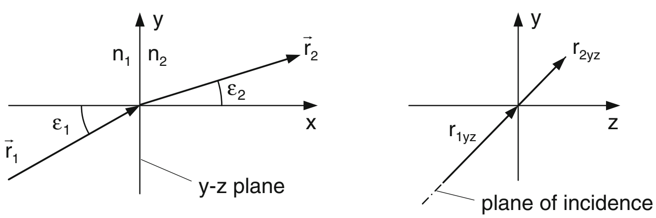

Fig. 3.1 Light ray refraction and the plane of incidence

In this equation, ε_1 and ε_2 represent the angles of incidence and refraction (or transmission), respectively. They are measured from the normal of the medium interface. Corresponding mediums are denoted as the incident and the refractive mediums.

The law of refraction given above can be generalized. According to Fig. 3.1 the x-axis agrees to the normal of the medium interface. The incident ray and the normal of the medium interface determine a plane that is known as the plane of incidence. The law of refraction also specifies that the refracted ray must also lie in the plane of incidence. For the purpose of performing extended calculations, the propagation directions of both the incident and the refracted rays are denoted by unit vectors $\vec{r}_1$ and $\vec{r}_2$, respectively. Their projections onto the medium interface ($y - z$ plane) are also shown in Fig. 3.1. Corresponding vector components are given by $r_{1\text{yz}}$ and $r_{2\text{yz}}$ in the $y - z$ plane. Clearly, these two vector components are calculated by $\sin\varepsilon_1$ and $\sin\varepsilon_2$, respectively. Equation (3.6) is then expressed as

$$r_{2\text{yz}} = \frac{n_1}{n_2} r_{1\text{yz}} \tag{3.7}$$

Because two sub-vectors $\vec{r}_{1\text{yz}}$ and $\vec{r}_{2\text{yz}}$ are parallel vectors in the $y - z$ plane, following expressions can be immediately obtained

$$r_{2\text{y}} = \frac{n_1}{n_2} r_{1\text{y}} \tag{3.8}$$

$$r_{2\text{z}} = \frac{n_1}{n_2} r_{1\text{z}} \tag{3.9}$$

These expressions represent the law of refraction in the form of using vectors i.e. vector components. For later convenience of completing some special calculations, Eq. (3.6) is further considered in the form

$$1 - \frac{n_1^2}{n_2^2} = \frac{\sin^2\varepsilon_1 - \sin^2\varepsilon_2}{\sin^2\varepsilon_1} = \frac{\cos^2\varepsilon_2 - \cos^2\varepsilon_1}{\sin^2\varepsilon_1} \tag{3.10}$$

With respect to $r_{1\text{x}} = \cos\varepsilon_1$ and $r_{2\text{x}} = \cos\varepsilon_2$ it follows

$$1 - \frac{n_1^2}{n_2^2} = \frac{r_{2\text{x}}^2 - r_{1\text{x}}^2}{\sin^2\varepsilon_1} = \frac{r_{1\text{x}}^2}{\sin^2\varepsilon_1}\left(\frac{r_{2\text{x}}^2}{r_{1\text{x}}^2} - 1\right) = \frac{1}{\tan^2\varepsilon_1}\left(\frac{r_{2\text{x}}^2}{r_{1\text{x}}^2} - 1\right) \tag{3.11}$$

from which one obtains

$$\frac{r_{2\text{x}}^2}{r_{1\text{x}}^2} - 1 = \left(1 - \frac{n_1^2}{n_2^2}\right)\tan^2\varepsilon_1 \tag{3.12}$$

The law of refraction is thus interpreted in the form of unit vector components $r_{1\text{x}}$ and $r_{2\text{x}}$. This equation will be applied in Chap. 14 to simplify the characterization of astigmatism which, as an optical aberration, is related to the refraction of a focused laser beam.

3.2 The Doppler Effect

The Doppler effect in optics is associated with the light propagation and accounts for the frequency shift when the light source is moving or light is reflected off a moving surface. Because there is no absolute motion according to the special theory of relativity, the Doppler effect must be described by the same mathematical formula, whether it is arising from the moving light source or the moving observer. This statement also relies on the physical reality that the light emitted from a moving light source is independent of the light source motion.

In Fig. 3.2, an optical interaction system with a moving light source and a fixed receiver has been shown. The initial distance between the light source and the receiver is given by s. The time used to transform the light through this distance is t, so that there is $s = ct$. For simplicity it is assumed that the light source emits the monochromatic light of wavelength λ_0. In the first instance, the light source is assumed to be seated in the space. The number of waves on the path s is then given by $s/\lambda_0 = ct/\lambda_0$. In the second instance, the light source is assumed to move at a velocity equal to $\vec{u}_s$. Because the light velocity is independent of the motion of the light source, the time used for wave transmission through the path s is still equal to t. Within this time, the light source itself moves from the plane a to the plane b. The waves which were initially distributed on the path s are now squeezed into the path $ct - (\vec{u}_s \cdot \vec{l})t$. Because it deals with the same number of waves there is

$$\frac{ct}{\lambda_0} = \frac{ct - \left(\vec{u}_s \cdot \vec{l}\right) t}{\lambda_1} \tag{3.13}$$

The wavelength of the light wave perceived at the receiver is then obtained as

$$\lambda_1 = \left(1 - \frac{\vec{u}_s \cdot \vec{l}}{c}\right) \lambda_0 \tag{3.14}$$

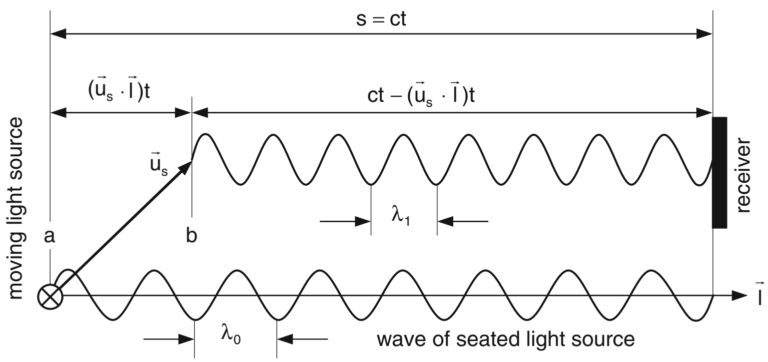

Fig. 3.2 Doppler effect in an optical system with moving light source

With respect to the constant light speed given by $\lambda_1 \nu_1 = \lambda_0 \nu_0 = c$, the frequency of the light wave is calculated as

$$\nu_1 = \frac{\nu_0}{1 - \vec{u}_\mathrm{s} \cdot \vec{l}/c} \tag{3.15}$$

This frequency of the light wave perceived at the receiver is shifted against the frequency ν_0 of the light wave that is emitted by the light source. The associated phenomenon is called the Doppler effect. It depends on the relative motion between the light source and the receiver.

Because of $\vec{u}_\mathrm{s} \cdot \vec{l}/c << 1$ the above equation is simplified to

$$\nu_1 = \nu_0 \left(1 + \frac{\vec{u}_\mathrm{s} \cdot \vec{l}}{c}\right) \tag{3.16}$$

Equation (3.15) i.e. (3.16) is derived by assuming the moving light source and the fixed receiver. According to the special theory of relativity, such a system is totally equivalent to the system with a fixed light source and a moving receiver, when the moving velocity of the receiver is set by $\vec{u}_\mathrm{r} = -\vec{u}_\mathrm{s}$. Because of this total equivalence the frequency of the light wave received by the moving receiver is obtained directly from Eq. (3.16) as

$$\nu_1 = \nu_0 \left(1 - \frac{\vec{u}_\mathrm{r} \cdot \vec{l}}{c}\right) \tag{3.17}$$

In relying on LDA principles, a scattering system is considered to consist of a fixed light source, a moving object (i.e. a small particle) and a fixed observer to receive the light scattered by the moving particle. The corresponding optical arrangement has been illustrated in Fig. 3.3. The particle moves at a velocity equal to $\vec{u}_\mathrm{p}$. The light frequency from the light source is ν_0. The frequency which is observed by the moving particle is ν_1 and can be calculated by Eq. (3.17) using substitutions $\vec{u}_\mathrm{p} = \vec{u}_\mathrm{r}$ and $\vec{l}_1 = \vec{l}$. The particle in its moving system scatters the incident light at the same frequency ν_1. In a certain spatial direction ($\vec{l}_2$), the scattered light is then received as the light of another frequency ν_2 by the fixed receiver because of the Doppler effect.

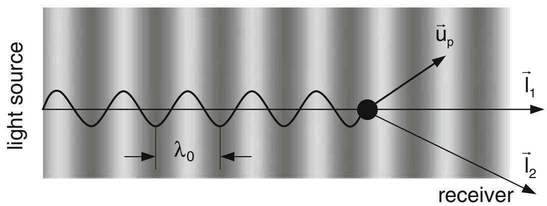

Fig. 3.3 Optical interaction system with a light source, a moving object and a receiver to explain the Doppler effect in LDA measurements

The relationship between the frequencies ν_1 and ν_2 is obtained from Eq. (3.16) by simply changing the indices

$$\nu_2 = \nu_1 \left(1 + \frac{\vec{u}_\mathrm{p} \cdot \vec{l}_2}{c}\right) \tag{3.18}$$

Combining Eqs. (3.17) and (3.18) yields

$$\nu_2 = \nu_0 \left(1 - \frac{\vec{u}_\mathrm{p} \cdot \vec{l}_1}{c}\right)\left(1 + \frac{\vec{u}_\mathrm{p} \cdot \vec{l}_2}{c}\right) \tag{3.19}$$

This equation is found as the basic theory of the laser Doppler anemometry. The shifted frequency according to Eq. (3.19) is then a function of the particle velocity which is considered to be equal to the flow velocity. Because the flow velocity is always negligible against the light speed as given by $\vec{u}_\mathrm{p} \cdot \vec{l}_1/c << 1$ and $\vec{u}_\mathrm{p} \cdot \vec{l}_2/c << 1$, Eq. (3.19) is further simplified to

$$\nu_2 \approx \nu_0 \left(1 - \frac{\vec{u}_\mathrm{p} \cdot \vec{l}_1}{c} + \frac{\vec{u}_\mathrm{p} \cdot \vec{l}_2}{c}\right) \tag{3.20}$$

The shifted frequency ν_2 is in the order of ν_0 and therefore still too high to be measured by conventional devices that are found in usual laboratories. In order to make use of the Doppler effect for flow measurements, the dual beam configuration has been confirmed to be highly effective. In fact, the configuration of using two laser beams has been widest applied and comes to be the standard in LDA measurements. The physical background of this configuration is given in Sect. 3.4. The key technique in it is the superposition of two light waves with different frequencies, as described in the next section.

3.3 Superposition of Two Plane Light Waves

The dual beam configuration of LDA optics (see Sect. 3.4) relies upon the superposition of two light waves that are differently shifted by the Doppler effect. As known in general, light is the electromagnetic oscillation in the form of waves. The superposition of two light waves of different frequencies leads to the so-called optical interference. For simplicity only the plane waves are considered, which propagate in the x-direction. According to Fig. 3.4a and b two harmonic waves are assumed to have different amplitudes and frequencies, as given by

$$E_\mathrm{a} = E_\mathrm{a0} \cos\left(\omega_\mathrm{a} t - k_\mathrm{a} x\right) \tag{3.21}$$

$$E_\mathrm{b} = E_\mathrm{b0} \cos\left(\omega_\mathrm{b} t - k_\mathrm{b} x\right) \tag{3.22}$$

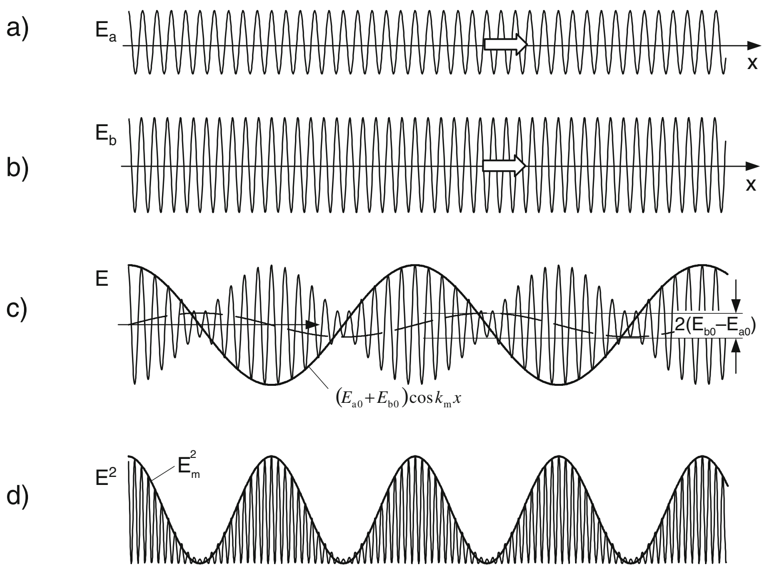

Fig. 3.4 Superposition of two light waves

Different amplitudes of two waves have been assumed, because in LDA measurements with the dual beam configuration the intensities of two chains of laser lights scattered by the particle are always different. This is true even if two laser beams are initially of equal light intensity.

The superposition of these two waves is simply given as

$$E = E_{a0} \cos(\omega_a t - k_a x) + E_{b0} \cos(\omega_b t - k_b x) \tag{3.23}$$

The spatial distribution of this superimposed wave can be obtained at a given time, as shown in Fig. 3.4c for the case $E_{b0} = 1.5 E_{a0}$ and $\omega_b = 1.1\omega_a$ i.e. $k_b = 1.1 k_a$. Obviously the superimposed wave possesses both a high frequency and a low modulation frequency. For calculating these two frequencies Eq. (3.23) is rearranged as

$$E = E_{a0} [\cos(\omega_a t - k_a x) + \cos(\omega_b t - k_b x)] + (E_{b0} - E_{a0}) \cos(\omega_b t - k_b x) \tag{3.24}$$

By applying the trigonometric identity

$$\cos\alpha + \cos\beta = 2\cos\frac{1}{2}(\alpha + \beta)\cos\frac{1}{2}(\alpha - \beta) \tag{3.25}$$

to the first term on the r.h.s. of Eq. (3.24), then the following equation is obtained:

$$E = 2E_{a0}\cos\left(\frac{\omega_a+\omega_b}{2}t - \frac{k_a+k_b}{2}x\right)\cos\left(\frac{\omega_a-\omega_b}{2}t - \frac{k_a-k_b}{2}x\right) \\ + (E_{b0}-E_{a0})\cos(\omega_b t - k_b x) \tag{3.26}$$

For further calculations the following abbreviations are applied

$$\overline{\omega} = \frac{1}{2}(\omega_a+\omega_b), \quad \omega_m = \frac{1}{2}(\omega_a-\omega_b) \tag{3.27}$$

and

$$\overline{k} = \frac{1}{2}(k_a+k_b), \quad k_m = \frac{1}{2}(k_a-k_b) \tag{3.28}$$

Herein ω_m and k_m represent the modulation frequency and the modulation wavenumber, respectively.

To the cosine function $\cos(\omega_b t - k_b x)$ in Eq. (3.26), $\omega_b = \overline{\omega} - \omega_m$ and $k_b = \overline{k} - k_m$ are substituted so that

$$\cos(\omega_b t - k_b x) = \cos\left(\overline{\omega}t - \overline{k}x\right)\cos(\omega_m t - k_m x) + \sin\left(\overline{\omega}t - \overline{k}x\right)\sin(\omega_m t - k_m x) \tag{3.29}$$

Eq. (3.26) is then converted to

$$E = (E_{a0}+E_{b0})\cos\left(\overline{\omega}t - \overline{k}x\right)\cdot\cos(\omega_m t - k_m x) \\ + (E_{b0}-E_{a0})\sin\left(\overline{\omega}t - \overline{k}x\right)\cdot\sin(\omega_m t - k_m x) \tag{3.30}$$

At the time $t = 0$, the superimposed wave represents a spatial wave distribution given by

$$E = (E_{a0}+E_{b0})\cos\left(\overline{k}x\right)\cos(k_m x) + (E_{b0}-E_{a0})\sin\left(\overline{k}x\right)\sin(k_m x) \tag{3.31}$$

In accordance with Fig. 3.4c, the first term on the r.h.s. of this equation represents the main form of the superimposed wave, whose maximum amplitude is given by $E_{a0}+E_{b0}$. Correspondingly the second term depicts an auxiliary wave with a maximum amplitude equal to $E_{b0}-E_{a0}$. It usually represents a negligible value and only disappears, if two plane waves of equal amplitude are superimposed.

The main form of the superimposed wave comprises the high angular frequency equal to $\overline{\omega}$ (i.e. $\overline{k}$ in the spatial wave distribution) and the low modulation frequency equal to ω_m (k_m). The amplitude of the high frequency oscillation is given by the modulated wave

$$E_m = (E_{a0}+E_{b0})\cos(\omega_m t - k_m x) \tag{3.32}$$

It indeed represents the envelope of the high frequency wave, as shown in Fig. 3.4c for $t=0$. Such a modulated wave form can be imagined to be obtainable by transmitting the wave given in Eq. (3.30) through a low pass filter.

The intensity of a light wave that is sensed by the human eyes or the photonic detectors such as photomultiplier tubes is given by the flux density that is proportional to the wave amplitude squared. From the superposition of two waves, as given by Eq. (3.30) and illustrated in Fig. 3.4c, the amplitude of the main wave oscillation has been confirmed to be the modulated wave and given in Eq. (3.32). Because of this the time and spatial distribution of intensities of the superimposed wave can thus be expressed by

$$E_{\mathrm{m}}^2 = (E_{\mathrm{a0}} + E_{\mathrm{b0}})^2 \cos^2 (\omega_{\mathrm{m}} t - k_{\mathrm{m}} x) \tag{3.33}$$

or equivalently as

$$E_{\mathrm{m}}^2 = \frac{1}{2} (E_{\mathrm{a0}} + E_{\mathrm{b0}})^2 [1 + \cos 2 (\omega_{\mathrm{m}} t - k_{\mathrm{m}} x)] \tag{3.34}$$

The flux density that is proportional to E_{m}^2 oscillates with an angular frequency of $2\omega_{\mathrm{m}} = \omega_{\mathrm{a}} - \omega_{\mathrm{b}}$ which is known as the beat frequency. The corresponding spatial distribution of such an oscillation is shown in Fig. 3.4d for the considered example. It is evident that even in the applied example with large amplitude difference ($E_{\mathrm{b0}} = 1.5 E_{\mathrm{a0}}$) the superimposed wave can be well approximated by its main part which is specified by Eq. (3.32) for the amplitude and by Eq. (3.34) for the light intensity.

At this moment it should be mentioned that in LDA measurement techniques the comparable beat frequency of the flux density oscillation is considered and measured to calculate the flow velocities. Because this frequency is many orders smaller than the light frequency (of about $6 \cdot 10^{14}$ Hz), it can be accurately measured by means of usual measurement devices.

3.4 LDA Principle

After the Doppler effect and the superposition of two light waves have been treated in foregoing sections, the optical configuration of an LDA-system and its functionality will be demonstrated. A standard one-component LDA system consists of two laser beams. For simplicity two laser beams (A and B) of equal frequencies (ν_0) are considered to intersect at an angle 2α (Fig. 3.5). The cross area of these two laser beams in the flow is called the measurement volume. A particle that is suspended in the flow is assumed to pass through the measurement volume and so scatters the lights of two laser beams simultaneously. Because of the different spatial layout of two laser beams, the moving particle of velocity $\vec{u}_{\mathrm{p}}$ perceives the different light frequencies resulted from different Doppler effects. A detector is spatially located along $\vec{l}_2$ for receiving the light that is scattered from the measurement volume. The considered system including the laser light source, a moving particle and a detector is completely comparable with the optical interaction system that has already been presented in Fig. 3.3 for explaining the Doppler effect. Hence according to

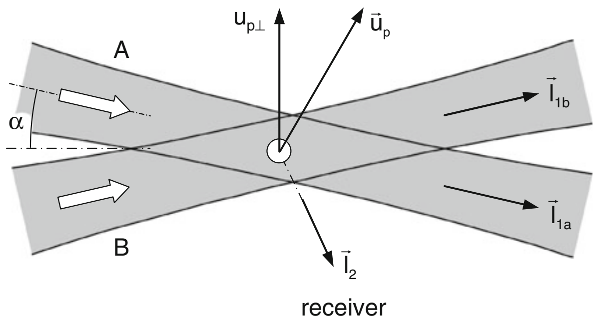

Fig. 3.5 Particle motion through the measurement volume

Eq. (3.20) the frequencies of two light waves received by the detector along $\vec{l}_2$ are given by

$$\nu_{2a} \approx \nu_0 \left(1 - \frac{\vec{u}_p \cdot \vec{l}_{1a}}{c} + \frac{\vec{u}_p \cdot \vec{l}_2}{c}\right) \tag{3.35}$$

and

$$\nu_{2b} \approx \nu_0 \left(1 - \frac{\vec{u}_p \cdot \vec{l}_{1b}}{c} + \frac{\vec{u}_p \cdot \vec{l}_2}{c}\right) \tag{3.36}$$

respectively.

While being received by the photodetector, the two light waves of frequencies ν_{2a} and ν_{2b} get to be superimposed. In accordance with Eq. (3.34), the flux density of the resultant light wave exhibits a low frequency that is known as the beat frequency and equal to twice the modulation frequency ($2\omega_m = \omega_a - \omega_b$). It indeed deals with a low frequency that can be easily measured by means of conventional measurement devices. In the terminology of LDA measurement techniques, this low frequency is called the Doppler frequency. It is calculated from Eqs. (3.35) and (3.36) by

$$\nu_D = |\nu_{2a} - \nu_{2b}| = \frac{\nu_0}{c}\left|\vec{u}_p \cdot \left(\vec{l}_{1b} - \vec{l}_{1a}\right)\right| \tag{3.37}$$

Because of $c/\nu_0 = \lambda_0$ and $\left|\vec{u}_p \cdot \left(\vec{l}_{1b} - \vec{l}_{1a}\right)\right| = 2u_{p\perp} \sin\alpha$ with $u_{p\perp}$ as the component of the particle velocity perpendicular to the bisector of the two laser beams, the above equation becomes

$$\nu_D = 2\frac{u_{p\perp}}{\lambda_0} \sin\alpha \tag{3.38}$$

The Doppler frequency is directly proportional to the velocity component $u_{p\perp}$ of the particle motion, however, independent of the particle motion direction. In assuming the particle velocity to be equal to the velocity of the fluid flow, the corresponding flow velocity component can be obtained by measuring the Doppler frequency. It yields then from Eq. (3.38)

$$u_{\perp} = \frac{\lambda_0}{2\sin\alpha}\nu_D \tag{3.39}$$

The factor that is multiplied to the Doppler frequency is a physical and geometrical constant. This circumstance implies that the LDA method for flow measurements is a method without system calibration. In addition, as seen from Eq. (3.38), the Doppler frequency detected in the scattered light is independent of the spatial position of the detector. This property enables the detector to be freely positioned for measurements. By using the separate detector, however, it is always time-consuming to align the detector optics to be focused onto the measurement volume. For this reason, most LDA systems are configured as the backscattering system in which the detector unit is integrated into the transmitting unit, see Chap. 4.

Because the Doppler frequency is always positive, independent of the particle flow direction, the velocity component $u_{\perp}$ that is calculated from Eq. (3.39) only corresponds to its absolute value. It is yet impossible to determine the sign of this velocity component i.e. the flow direction. A method to remove the ambiguity of the flow direction is the use of a Bragg cell to shift the frequency in one of two laser beams. This method has come to be a standard in most LDA optics.

Detailed descriptions of an LDA system including photodetectors (photomultiplier) and the use of Bragg cells will be given in Sect. 3.6.

3.5 Fringe Model on the Light Interference

The Doppler frequency in the light scattered by the particle while passing through the measurement volume, as given in Eq. (3.38), can also be calculated by accounting for the interference of laser lights in the measurement volume. The associated calculation method is known as the fringe model. As a matter of fact, the fringe model has most often been applied to explain the principle of LDA method in flow measurements because of its illustrative capability for easy understanding.

Two plane light waves of equal amplitudes (E_0) and frequencies (ω) are postulated to propagate in the directions of $\vec{k}_a$ and $\vec{k}_b$, respectively, and intersect at an angle 2α, as shown in Fig. 3.6. Because of equal angular frequencies and thus equal wavelengths of two plane waves, equal wavenumbers are given as $k = \left|\vec{k}_a\right| = \left|\vec{k}_b\right| = 2\pi/\lambda_0$. For the convenience of analysis, the two-dimensional $z - x$ coordinate system is applied with z as the optical axis which coincides with

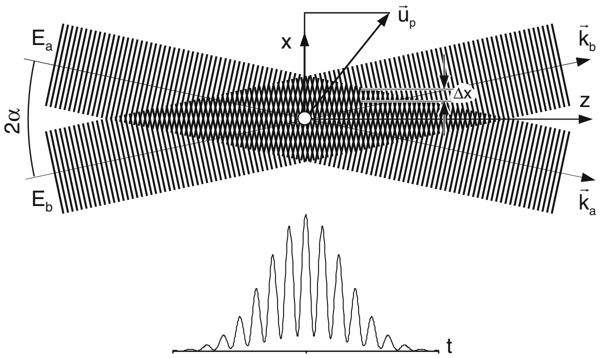

Fig. 3.6 Fringe model of LDA principle and the burst signal

the bisector of two wave vectors $\vec{k}_a$ and $\vec{k}_b$. In the field of the plane wave E_a for instance, the electromagnetic wave at $\vec{r} = (z, x)$ is given by

$$E_a = E_0 \cos\left(\omega t - \vec{k}_a \cdot \vec{r}\right) \tag{3.40}$$

Let us remember, the case of the vector $\vec{r}$ coinciding with the wave vector $\vec{k}$ has already been treated and given in Eq. (3.21).

With respect to the wavenumber $\vec{k}_a = (k\cos\alpha, -k\sin\alpha)$ in the $z - x$ plane Eq. (3.40) becomes

$$E_a = E_0 \cos\left[\omega t - k\left(z\cos\alpha - x\sin\alpha\right)\right] \tag{3.41}$$

Similarly, the electromagnetic wave at the same point, however, in the field of the plane wave E_b can be obtained. The wavenumber in this case is given as $\vec{k}_b = (k\cos\alpha,\ k\sin\alpha)$, so that

$$E_b = E_0 \cos\left[\omega t - k\left(z\cos\alpha + x\sin\alpha\right)\right] \tag{3.42}$$

The superposition of these two plane waves at the point $\vec{r} = (z, x)$ in the $z - x$ plane is obtained by again applying the trigonometric identity according to Eq. (3.25)

$$E = E_a + E_b = 2E_0 \cos\left(kx\sin\alpha\right) \cdot \cos\left(\omega t - kz\cos\alpha\right) \tag{3.43}$$

Obviously the resultant wave at the given point $\vec{r} = (z, x)$ shows the high angular frequency equal to ω. The amplitude of this wave oscillation is $2E_0\cos\left(kx\sin\alpha\right)$ that is a constant and geometrically only a function of the coordinate x. The light intensity that is proportional to the square of the wave amplitude is calculated by

$$E_m^2 = 4E_0^2\cos^2\left(kx\sin\alpha\right) = 2E_0^2\left[1 + \cos 2\left(kx\sin\alpha\right)\right] \tag{3.44}$$

With respect to $k = 2\pi/\lambda_0$ this equation is again written as

$$E_{\mathrm{m}}^2 = 2E_0^2\left[1 + \cos 2\pi\left(\frac{2\sin\alpha}{\lambda_0}x\right)\right] = 2E_0^2\left[1 + \cos\left(2\pi\frac{x}{\Delta x}\right)\right] \tag{3.45}$$

In the direction perpendicular to the optical axis i.e. parallel to the x-axis, the light intensity alternates with a distance equal to

$$\Delta x = \frac{\lambda_0}{2\sin\alpha} \tag{3.46}$$

This distance is known as the fringe spacing in the measurement volume. It is resulting from the interference of two light waves that takes place by overlapping them. To show the dimension of the fringe spacing in the measurement volume, two laser beams of equal wavelength $\lambda_0 = 514.5$ nm are considered to intersect at an half intersection angle $\alpha = 3°$. From the above equation the fringe spacing is calculated as $\Delta x = 5\mu$m. Obviously it deals with a quite small value. By use of a magnified image, the fringe pattern i.e. the light intensity distribution in the measurement volume could be visualized for instance by Miles and Witze (1994, 1996).

In comparing with Eq. (3.39), the velocity component of a particle passing through the measurement volume is calculated by

$$u_\perp = \Delta x \cdot \nu_{\mathrm{D}} \tag{3.47}$$

This equation points out that the Doppler frequency can be considered to be the alternating frequency in the intensity of light that is scattered out by a particle passing through the measurement volume. The corresponding light signal has also been shown in Fig. 3.6. In the terminology of LDA measurements, such a signal is called the Doppler burst. Because the laser beams used in LDA measurements have a Gaussian distribution in the intensity (see Sect. 3.7), the Doppler burst shows its maximum amplitude, as the particle is found in the centre of the measurement volume.

The fringe model presented above is a very useful tool to understand the LDA measurement principle. It also represents a very convenient means to make further studies of diverse optical phenomena influencing the measurement accuracy. To be mentioned here are for instance the change in the fringe spacing at flow measurements in the circular pipes, the fringe distortion caused by either improper optical layout or astigmatism arising from laser beam refractions, and the signal properties in measurements of particle size using Phase Doppler Anemometry (PDA). All of these points excluding that in PDA measurements can be found in the corresponding chapters of this book.

3.6 Frequency Shift Method to Resolve the Flow Direction

The LDA method is based on the evaluation of the burst signals that are generated by particles passing through the measurement volume. An ambiguity to the flow direction of the particle, however, exists because a positive and a negative velocity of the same magnitude will cause the same Doppler frequency. The burst signals thus only involve the magnitude but not the signs of respective velocities. In order to resolve the flow direction from each Doppler burst, the technique of using Bragg cells to slightly shift the frequency in one or both of two laser beams in each laser beam pair has become a standard. The physical principle of the Bragg cell can be found for instance in Albrecht et al. (2003). The purpose of shifting the light frequency is to create the moving fringes in the measurement volume in a predefined direction. The principle of using the frequency shift method to resolve the flow direction is explained here.

According to Fig. 3.7 the light wave frequency of the laser beam A is assumed to be shifted up by ν_{sh}, so that $\nu_{a0} = \nu_0 + \nu_{sh}$. Two laser beams thus show different frequencies. Correspondingly, the light frequencies perceived at the detector in the direction $\vec{l}_2$ can be directly obtained from Eqs. (3.35) and (3.36) as

$$\nu_{2a} = (\nu_0 + \nu_{sh}) \left(1 - \frac{\vec{u} \cdot \vec{l}_{1a}}{c} + \frac{\vec{u} \cdot \vec{l}_2}{c} \right) \tag{3.48}$$

and

$$\nu_{2b} = \nu_0 \left(1 - \frac{\vec{u} \cdot \vec{l}_{1b}}{c} + \frac{\vec{u} \cdot \vec{l}_2}{c} \right) \tag{3.49}$$

respectively. Herein both $\vec{l}_1$ and $\vec{l}_2$ are unit vectors.

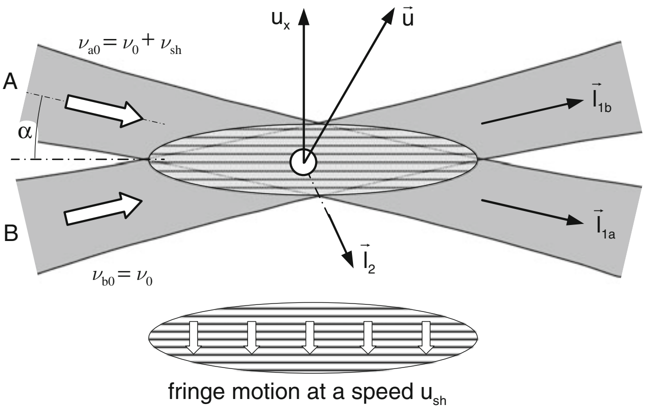

Fig. 3.7 Frequency shift at the laser beam A and the resultant fringe motion in the measurement volume with a shift speed u_{sh}

In the practical application, the shift frequency ν_{sh} is selected in the order of megahertz. It is much lower against the light frequency, however, sufficiently high in comparison to the Doppler frequency caused by the maximum flow velocity. Indeed, the value of the shift frequency should ensure that the difference $\nu_{2a} - \nu_{2b}$ is constantly positive. Under this condition the effective frequency of light signals measured by the receiving unit like the photomultiplier (PM) is simply

$$\nu_{PM} = \nu_{2a} - \nu_{2b} = \nu_{sh} + \frac{\nu_0}{c}\vec{u}\cdot\left(\vec{l}_{1b} - \vec{l}_{1a}\right) + \frac{\nu_{sh}}{c}\vec{u}\cdot\left(\vec{l}_2 - \vec{l}_{1a}\right) \tag{3.50}$$

The third term on the r.h.s. of above equation is negligible against the second term because of $\nu_{sh} << \nu_0$.

With respect to the definition of the positive sign of the velocity component u_x according to Fig. 3.7, the vector difference $\vec{l}_{1b} - \vec{l}_{1a}$ coincides with the positive x-axis. Because of $\vec{u}\cdot\left(\vec{l}_{1b} - \vec{l}_{1a}\right) = 2u_x \sin\alpha$ Eq. (3.50) is then simplified to

$$\nu_{PM} = \nu_{sh} + 2\nu_0\frac{u_x}{c}\sin\alpha \tag{3.51}$$

The second term on the r.h.s. of this equation represents the Doppler frequency caused by the flow velocity. With respect to $c = \lambda_0\nu_0$ and in using the fringe model according to Eq. (3.46), Eq. (3.51) is then rewritten as

$$\nu_{PM} = \nu_{sh} + \nu_D = \nu_{sh} + \frac{u_x}{\Delta x} \tag{3.52}$$

The velocity component u_x is then resolved as

$$u_x = \Delta x\,(\nu_{PM} - \nu_{sh}) \tag{3.53}$$

Thus from direct comparison between the frequency ν_{PM} which is detected at the photomultiplier and the preset shift frequency ν_{sh} both the value and the sign of the velocity component u_x can be exactly determined. According to Eq. (3.53) there is $u_x > 0$ from $\nu_{PM} > \nu_{sh}$. In particular, the zero velocity of the flow i.e. a particle seated in the measurement volume can also be measured. In this case, there is $\nu_{PM} - \nu_{sh} = 0$.

The technique of using the shift frequency to detect the flow direction is to generate the moving fringe pattern in the measurement volume. According to Eq. (3.52) the detected frequency is simply given as the superposition of the Doppler frequency and the shift frequency. By rewriting Eq. (3.52) to be $\nu_{PM} = (\nu_{sh}\Delta x + u_x)/\Delta x$ it is evident that the detected frequency results from the superposition of the particle velocity and a velocity equal to $\nu_{sh}\Delta x$. This implies that fringes in the measurement volume move in the negative x-direction at the constant speed equal to $u_{sh} = -\nu_{sh}\Delta x$ which is called fringe shift speed. The postulated fringe motion in the measurement volume can be demonstrated in the following way.

3.6.1 Fringe Shift Speed

According to Fig. 3.7 the angular frequency of two laser beams are assumed to be $\omega_a = 2\pi\nu_a$ and $\omega_b = 2\pi\nu_b$, respectively. The corresponding wave equations can be obtained from Eqs. (3.41) and (3.42). For simplicity, the fringe distribution along the x-axis will be considered. This means that $z = 0$ has to be applied, so that

$$E_a = E_0 \cos(\omega_a t + k_a x \sin\alpha) \tag{3.54}$$

$$E_b = E_0 \cos(\omega_b t - k_b x \sin\alpha) \tag{3.55}$$

The superposition of these two plane waves is obtained by again applying the trigonometric identity according to Eq. (3.25)

$$E = E_a + E_b = 2E_0 \cos\left(\omega_m t + \overline{k} x \sin\alpha\right) \cdot \cos(\overline{\omega} t + k_m x \sin\alpha) \tag{3.56}$$

with $\overline{\omega} = \frac{1}{2}(\omega_a + \omega_b)$, $\omega_m = \frac{1}{2}(\omega_a - \omega_b)$, $\overline{k} = \frac{1}{2}(k_a + k_b)$ and $k_m = \frac{1}{2}(k_a - k_b)$.

The resultant light wave at the given x shows the high angular frequency equal to $\overline{\omega}$. The amplitude of this wave oscillation is again the modulated wave $2E_0 \cos\left(\omega_m t + \overline{k} x \sin\alpha\right)$ with the low frequency ω_m. The light intensity that is proportional to the square of the wave amplitude is calculated by

$$E_m^2 = 4E_0^2 \cos^2(kx \sin\alpha) = 2E_0^2\left[1 + \cos 2\left(\omega_m t + \overline{k} x \sin\alpha\right)\right] \tag{3.57}$$

In the current calculation with respect to the shift frequency, there are $\omega_a = \omega_0 + \omega_{sh}$ and $\omega_b = \omega_0$, so that $\omega_m = \omega_{sh}/2 = \pi\nu_{sh}$. Correspondingly there is $\overline{k} = \frac{1}{2}(k_a + k_b) \approx 2\pi/\lambda_0$ because of $\nu_{sh} << \nu_0$. With such simplifications as well as with respect to $\Delta x = \lambda_0/(2\sin\alpha)$ from Eq. (3.46) one obtains from Eq. (3.57)

$$E_m^2 = 2E_0^2\left[1 + \cos 2\pi\left(\nu_{sh} t + \frac{x}{\Delta x}\right)\right] \tag{3.58}$$

It represents an apparent harmonic wave that would move in the negative x-direction. In reality, it only signifies the unsteadiness of the fringe pattern as if this is rolling at a constant speed. It does not indicate any energy transport. For $\nu_{sh} = 0$ one obtains Eq. (3.45).

The speed with which the fringe pattern rolls is determined from the condition

$$\nu_{sh} t + \frac{x}{\Delta x} = \text{const} \tag{3.59}$$

to

$$u_{sh} = \frac{dx}{dt} = -\nu_{sh}\Delta x \tag{3.60}$$

This speed is denoted as the fringe shift speed and has been postulated before based on Eq. (3.52).

The shift frequency, created by means of Bragg cells, is for instance 40 MHz in some LDA applications. With respect to a fringe spacing $\Delta x = 5\mu$m, see Sect. 3.5, the shift speed of fringes in the measurement volume is calculated to $u_{sh} = -200$ m/s. Because this speed is sufficiently higher than all possible fluctuation velocities in most flows, the sign of each measured velocity component u_x in a turbulent flow can be determined without ambiguity.

Some LDA users would sometimes like to check the accuracy of the shift frequency preset in the LDA system. A simple and accurate method for performing this task is presented in Chap. 18.

3.7 Gaussian Beam Properties

3.7.1 Geometrical Specifications of the Gaussian Beam

The laser beams that are applied in LDA techniques are generally single-mode laser beams. The intensity distribution in the cross section of such a laser beam can be approximated by the Gaussian distribution as given by

$$I(r) = I_0 \mathrm{e}^{-2(r/w)^2} \tag{3.61}$$

The light intensity on the beam axis is denoted by I_0. The thickness of a Gaussian beam is confirmed to be equal to $2w$. At $r = w$ the light intensity falls down to a level of about $\mathrm{e}^{-2} = 13.5\%$ of the light intensity on the beam axis ($r = 0$), as shown in Fig. 3.8.

The light intensity represents the time rate of flow of radiant energy i.e. the radiant flux density. The total power involved in a light beam is obtained by integrating the light intensity distribution across the light beam section, yielding

$$P = 2\pi I_0 \int_0^\infty \mathrm{e}^{-2(r/w)^2} r\mathrm{d}r = \frac{1}{2}\pi w^2 I_0 \tag{3.62}$$

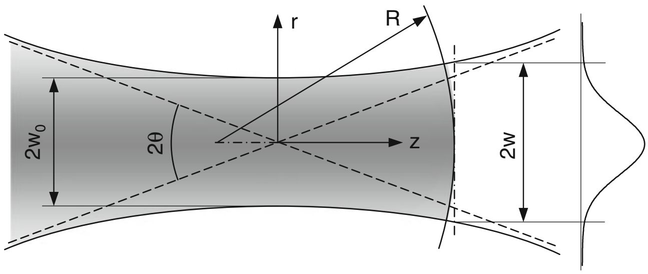

Fig. 3.8 Geometrical and optical specifications of the Gaussian beam

Light beams with the Gaussian distribution always comprise a well-defined beam waist. This property indicates that the Gaussian beam is a focused beam. In fact, such a light beam is geometrically completely describable by merely using the beam waist diameter. According to Fig. 3.8 the geometrical feature of a Gaussian beam mainly includes the curvature radius R of the wave front and the beam thickness $2w$ representing the beam divergence. The wave front is considered as a surface on which the phase is constant. From the wave optics the curvature radius of the front surface of a Gaussian beam and the beam thickness at the distance z from the beam waist are given by

$$R = z\left[1+\left(\frac{\pi w_0^2}{\lambda z}\right)^2\right] \tag{3.63}$$

and

$$w = w_0\sqrt{1+\left(\frac{\lambda z}{\pi w_0^2}\right)^2} \tag{3.64}$$

respectively.

For large values of the distance z the thickness of the light beam linearly increases with the distance.

Obviously the beam thickness at the beam waist, given by $2w_0$, is the most essential parameter that determines all geometrical features of a Gaussian beam. According to Eq. (3.63) the curvature radii of the beam front, both at the beam waist ($z = 0$) and at the large distance ($z = \infty$), are infinite. The beam can thus be considered as the plane wave beam. The position that the Gaussian beam possesses the smallest curvature radius ($R_{\min}$) is obtained from Eq. (3.63) under the condition

$$\frac{\mathrm{d}R}{\mathrm{d}z} = 0 \tag{3.65}$$

to

$$z_{\mathrm{R}} = \frac{\pi \cdot w_0^2}{\lambda} \tag{3.66}$$

This distance from the beam waist is called the Rayleigh length. It is again a function of the beam waist thickness. The smallest curvature radius in a Gaussian beam is then obtained from Eq. (3.63)

$$R_{\min} = 2z_{\mathrm{R}} = \frac{2\pi \cdot w_0^2}{\lambda} \tag{3.67}$$

This smallest curvature radius in the wave front is relevant when evaluating the uniformity of the fringe pattern in the LDA measurement volume. In the case that the measurement volume does not coincide with the waists of two laser beams, non-uniform fringe in the measurement volume will be created. The largest fringe distortion occurs when the measurement volume is formed by laser beam crossing on the Rayleigh length of both laser beams. More about this property and the influence of the fringe distortion on the measurement accuracy will be described in Chap. 16.

In using the Rayleigh length as a characteristic parameter, both the curvature radius of the front surface and the thickness of a Gaussian beam at a distance z from the beam waist are expressed by

$$R = z\left[1 + \left(\frac{z_R}{z}\right)^2\right] \tag{3.68}$$

and

$$w = w_0\sqrt{1 + \left(\frac{z}{z_R}\right)^2} \tag{3.69}$$

respectively.

Especially at the Rayleigh length there is

$$w_R = \sqrt{2}w_0 \tag{3.70}$$

At great distance it yields from Eq. (3.68) with $z >> z_R$

$$R = z \tag{3.71}$$

This last equation signifies that the front surface of a Gaussian beam is a circular surface which has its centre at the beam waist.

The light intensity in the Gaussian beam at the Rayleigh length can be calculated from Eq. (3.62). Because the total power in the beam remains constant, the light intensity in the center of the beam, if compared to the center light intensity at the beam waist, is given by

$$\frac{I_{0R}}{I_{0w}} = \frac{w_0^2}{w_R^2} = \frac{1}{2} \tag{3.72}$$

The divergence of a Gaussian beam can be expressed by the corresponding divergence angle 2θ under the condition $z \to \infty$. This can be obtained by accounting for $\tan\theta = \mathrm{d}w/\mathrm{d}z$ and carrying out the corresponding calculation from Eq. (3.69). Because it usually deals with a very small angle, the approximation $\tan\theta \approx \theta$ can be applied. The half divergence angle of a Gaussian beam is then obtained as

$$\theta = \frac{w_0}{z_R} \tag{3.73}$$

With regard to the Rayleigh length given in Eq. (3.66) this divergence angle of a Gaussian beam is again expressed as

$$\theta = \frac{\lambda}{\pi w_0} \tag{3.74}$$

Obviously laser beams that have large diameters at the beam waist show the negligible divergence angle. For instance for a laser beam of $\lambda = 500$ nm and $w_0 = 1$ mm, the divergence angle is only about 0.009°.

3.7.2 Transmission Performance of the Gaussian Beam

The transmission performance of a Gaussian beam through a lens of the focal length f is considered here according to Fig. 3.9. The Gaussian beam considered is assumed to have a beam waist thickness $2w_0'$. The corresponding Rayleigh length is given by z_R'. Based on lens optics the following geometrical relationships concerning the Gaussian beam prior to and after the lens are available

$$w_0 = \frac{f}{\sqrt{(s'-f)^2 + z_R'^2}} w_0' \tag{3.75}$$

$$s = f + \frac{f^2\left(s'-f\right)}{(s'-f)^2 + z_R'^2} \tag{3.76}$$

In most cases and for simplicity, a Gaussian beam after passing through a lens can be still considered as a Gaussian beam. All geometrical specifications of a Gaussian beam, as presented in Sect. 3.7.1, remain unchanged.

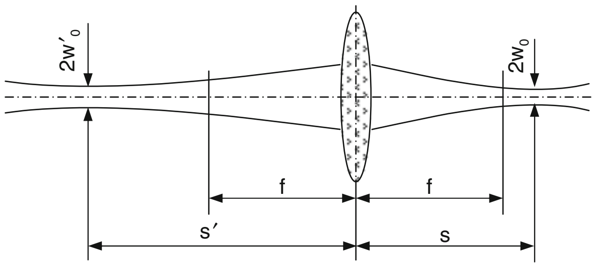

Fig. 3.9 Transmission performance of the Gaussian beam

3.8 Measurement Volume Size

On the optical side of LDA techniques, the measurement volume behaves as the key element in system operations. Both the measurement volume size and the specified optical performance determine the quality of flow measurements. In general, the measurement volume should always be created by arranging the intersection of two laser beams on their waists respectively. On one side, this requirement facilitates the high light intensity in the measurement volume, as this is necessary for detecting small particles passing through the measurement volume. On the other side, the plane wave front at the laser beam waist enables one to create uniform fringes in the measurement volume and hence to enhance the reliability and accuracy of measurements. Otherwise fringe distortion in the measurement volume will occur and lead to measurement errors (see Chap. 16).

For this reason and in general, the measurement volume is created at the waists of two laser beams. The form of the measurement volume can be approximated to be an ellipsoid, as illustrated in Fig. 3.10. The thickness i.e. the diameter of the measurement volume is given by the laser beam thickness at the beam waist

$$d_{\mathrm{mv}} = \frac{2w_0}{\cos\alpha} \tag{3.77}$$

with α as the half intersection angle between two laser beams.

The thickness of the measurement volume is proportional to the laser beam thickness. It depends therefore on the optical arrangement of laser beams regarding the focal length of the used optical lens because of Eq. (3.75). More about this dependence will be presented in Chap. 4 on the optical configuration of a concrete LDA system. In general, the thickness of the measurement volume is in the order of about 0.05–0.1 mm.

With regard to the fringe spacing in the measurement volume, as given in Eq. (3.46), the number of fringes in the measurement volume is then calculated as

$$N = \frac{d_{\mathrm{mv}}}{\Delta x} = 4\frac{w_0}{\lambda_0}\tan\alpha \tag{3.78}$$

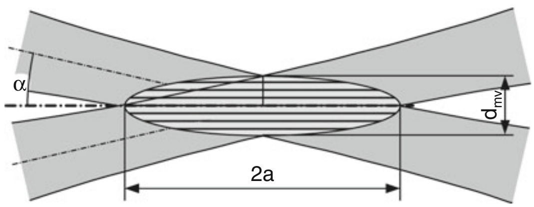

Fig. 3.10 Specification of the measurement volume size

The length of the measurement volume depends, for the same reason, both on the laser beam waist thickness and the crossing angle between two laser beams. According to Fig. 3.10 it is calculated as

$$2a = \frac{d_{\mathrm{mv}}}{\tan\alpha} = \frac{2w_0}{\sin\alpha} \tag{3.79}$$

In comparison with the thickness of the measurement volume, the measurement volume has usually a finite length of about 0.5–3 mm, again depending on the optical arrangement of laser beams.

The geometrical dimension of the measurement volume is defined here independent of the particle size. In reality, the calculated measurement volume size is applicable to small particles of diameters that are comparable to or smaller than the fringe spacing. For large particles that could still scatter the laser light and hence be detected even if the particle center is outside of the measurement volume, the effective detection volume is larger than the independent geometrical measurement volume. Such a dependence of the detection volume size on the particle size is especially crucial in the particle size and mass flux measurements (Zhang et al. 1998, Zhang and Ziada 2000) by means of the Phase Doppler Anemometry (PDA), which is an extended method of LDA, see Albrecht et al. (2003).