Chapter 5
Tight Closure in Positive Characteristic

In this chapter, p is a fixed prime number, and all rings are assumed to have characteristic p, unless explicitly mentioned otherwise. We review the notion of tight closure due to Hochster and Huneke (as a general reference, we will use [59]). The main protagonist in this elegant theory is the p-th power Frobenius map. We will focus on five key properties of tight closure, which will enable us to prove, virtually effortlessly, several beautiful theorems. Via these five properties, we can give a more axiomatic treatment, which lends itself nicely to generalization, and especially to a similar theory in characteristic zero (see Chapters 6 and 7).

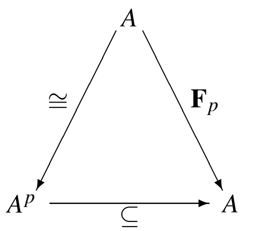
5.1 Frobenius

The major advantage of rings of positive characteristic is the presence of an algebraic endomorphism: the Frobenius. More precisely, let A be a ring of characteristic p, and let $\mathbf{F}_p$, or more accurately, $\mathbf{F}_{p,A}$, be the ring homomorphism $A \to A\colon a \mapsto a^p$, called the *Frobenius* on A. Recall that this is indeed a ring homomorphism, where the only thing to note is that the coefficients in the binomial expansion

$$\mathbf{F}_p(a+b) = \sum_{i=0}^{p} \binom{p}{i} a^i b^{p-i} = \mathbf{F}_p(a) + \mathbf{F}_p(b)$$

are divisible by p for all $0 < i < p$ whence zero in A, proving that $\mathbf{F}_p$ is additive.

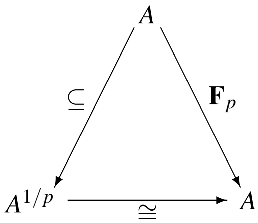
When A is reduced, $\mathbf{F}_p$ is injective whence yields an isomorphism with its image $A^p := \mathrm{Im}(\mathbf{F}_p)$ consisting of all p-th powers of elements in A (and not to be confused with the p-th Cartesian power of A). The inclusion $A^p \subseteq A$ is isomorphic with the Frobenius on A because we have a commutative diagram



(5.1)

H. Schoutens, *The Use of Ultraproducts in Commutative Algebra*,
Lecture Notes in Mathematics 1999, DOI 10.1007/978-3-642-13368-8_5,

When A is a domain, then we can also define the ring $A^{1/p}$ as the subring of the algebraic closure of the field of fractions of A consisting of all elements b satisfying $b^p \in A$. Hence $A \subseteq A^{1/p}$ is integral. Since, $\mathbf{F}_p(A^{1/p}) = A$ and $\mathbf{F}_p$ is injective, we get $A^{1/p} \cong A$. Moreover, we have a commutative diagram



(5.2)

showing that the Frobenius on A is also isomorphic to the inclusion $A \subseteq A^{1/p}$. It is sometimes easier to work with either of these inclusions rather than with the Frobenius itself, especially to avoid notational ambiguity between source and target of the Frobenius (instances where this approach would clarify the argument are the proofs of Theorem 5.1.2 and Corollary 5.1.3 below).

Often, the inclusion $A^p \subseteq A$ is even finite, and hence so is the Frobenius itself. One can show, using Noether normalization or Cohen's Structure Theorems that this is true when A is respectively a k-affine algebra or a complete Noetherian local ring with residue field k, and k is perfect, or more generally, $(k : k^p) < \infty$.

5.1.1 Frobenius Transforms

Given an ideal $I \subseteq A$, we will denote its extension under the Frobenius by $\mathbf{F}_p(I)A$, and call it the *Frobenius transform* of I. Note that $\mathbf{F}_p(I)A \subseteq I^p$, but the inclusion is in general strict. In fact, one easily verifies that

5.1.1 *If* $I = (x_1, \ldots, x_n)A$, *then* $\mathbf{F}_p(I)A = (x_1^p, \ldots, x_n^p)A$.

If we repeat this process, we get the *iterated Frobenius transforms* $\mathbf{F}_p^n(I)A$ of I, generated by the p^n-th powers of elements in I, and in fact, of generators of I. In tight closure theory, the simplified notation

$$I^{[p^n]} := \mathbf{F}_p^n(I)A$$

is normally used, but for reasons that will become apparent once we defined tight closure as a difference closure (see §6.1.1), we will use the 'heavier' notation. On the other hand, since we fix the characteristic, we may omit p from the notation and simply write $\mathbf{F} : A \to A$ for the Frobenius.

5.1.2 Kunz Theorem

The next result, due to Kunz, characterizes regular local rings in positive characteristic via the Frobenius. We will only prove the direction that we need.

Theorem 5.1.2 (Kunz). *Let R be a Noetherian local ring. If R is regular, then $\mathbf{F}_p$ is flat. Conversely, if R is reduced and $\mathbf{F}_p$ is flat, then R is regular.*

Proof. We only prove the direct implication; for the converse see [68, §42]. Let $\mathbf{x}$ be a system of parameters of R, whence an R-regular sequence. Since $\mathbf{F}(\mathbf{x})$ is also a system of parameters, it too is R-regular. Hence, R, viewed as an R-algebra via $\mathbf{F}$, is a balanced big Cohen-Macaulay algebra, whence is flat by Theorem 3.3.9. □

Corollary 5.1.3. *If R is a regular local ring, $I \subseteq R$ an ideal, and $a \in R$ an arbitrary element, then $a \in I$ if and only if $\mathbf{F}(a) \in \mathbf{F}(I)R$.*

Proof. One direction is of course trivial, so assume $\mathbf{F}(a) \in \mathbf{F}(I)R$. However, since $\mathbf{F}$ is flat by Theorem 5.1.2, the contraction of the extended ideal $\mathbf{F}(I)R$ along $\mathbf{F}$ is again I by Proposition 3.2.5, and a lies in this contraction (recall that $\mathbf{F}(I)R \cap R$ stands really for $\mathbf{F}^{-1}(\mathbf{F}(I)R)$.) □

5.2 Tight Closure

The definition of tight closure, although not complicated, is not that intuitive either. The idea is inspired by the ideal membership test of Corollary 5.1.3. Unfortunately, that test only works over regular local rings, so that it will be no surprise that whatever test we design, it will have to be more involved. Moreover, the proposed test will in fact fail in general, that is to say, the elements satisfying the test form an ideal which might be strictly bigger than the original ideal. But not too much bigger, so that we may view this bigger ideal as a closure of the original ideal, and as such, it is a 'tight' fit.

In the remainder of this section, A is a Noetherian ring, of characteristic p. A first obvious generalization of the ideal membership test from Corollary 5.1.3 is to allow iterates of the Frobenius: we could ask, given an ideal $I \subseteq A$, what are the elements x such that $\mathbf{F}^n(x) \in \mathbf{F}^n(I)A$ for some power n? They do form an ideal and the resulting closure operation is called the *Frobenius closure*. However, its properties are not sufficiently strong to derive all the results tight closure can.

The adjustment to make in the definition of Frobenius closure, although minor, might at first be a little surprising. To make the definition, we will call an element $a \in A$ a *multiplier*, if it is either a unit, or otherwise generates an ideal of positive height (necessarily one by Krull's Principal Ideal Theorem). Put differently, a is a multiplier if it does not belong to any minimal prime ideal of A. In particular, the product of two multipliers is again a multiplier. In a domain, a situation we can often reduce to, a multiplier is simply a non-zero element.

The name 'multiplier' comes from the fact that we will use such elements to multiply our test condition with. However, for this to make sense, we cannot just take one iterate of the Frobenius, we must take all of them, or at least all but finitely many. So we now define: an element $x \in A$ belongs to the *tight closure* $\mathrm{cl}_A(I)$ of an ideal $I \subseteq A$, if there exists a multiplier $c \in A$ and a positive integer N such that

$$c\mathbf{F}^n(x) \in \mathbf{F}^n(I)A \tag{5.3}$$

for all $n \geq N$. Note that the multiplier c and the bound N may depend on x and I, but not on n. We will write $\mathrm{cl}(I)$ for $\mathrm{cl}_A(I)$ if the ring A is clear from the context. In the literature, tight closure is invariably denoted I^*, but again for reasons that will become clear in the next chapter, our notation better suits our purposes. Let us verify some elementary properties of this closure operation:

5.2.1 *The tight closure of an ideal I in a Noetherian ring A is again an ideal, it contains I, and it is equal to its own tight closure. Moreover, we can find a multiplier c and a positive integer N which works simultaneous for all elements in* $\mathrm{cl}(I)$ *in criterion* (5.3)*.*

It is easy to verify that $\mathrm{cl}(I)$ is closed under multiples, and contains I. To show that it is closed under sums, whence an ideal, assume $x, x' \in A$ both lie in $\mathrm{cl}(I)$, witnessed by the equations (5.3) for some multipliers c and c', and some positive integers N and N' respectively. However, $cc'\mathbf{F}^n(x+x')$ then lies in $\mathbf{F}^n(I)A$ for all $n \geq \max\{N, N'\}$, showing that $x + x' \in \mathrm{cl}(I)$ since cc' is again a multiplier. Let $J := \mathrm{cl}(I)$ and choose generators $y_1, \dots, y_s$ of J. Let c_i and N_i be the corresponding multiplier and bound for y_i. It follows that $c := c_1c_2\cdots c_s$ is a multiplier such that (5.3) holds for all $n \geq N := \max\{N_1, \dots, N_s\}$ and all $x \in J$, since any such element is a linear combination of the y_i. In particular, $c\mathbf{F}^n(J)A \subseteq \mathbf{F}^n(I)A$ for all $n \geq N$. Hence if z lies in the tight closure of J, so that $d\mathbf{F}^n(z) \in \mathbf{F}^n(J)A$ for some multiplier d and for all $n \geq M$, then $cd\mathbf{F}^n(z) \in \mathbf{F}^n(I)A$ for all $n \geq \max\{M, N\}$, whence $z \in \mathrm{cl}(I) = J$. The last assertion now easily follows from the above analysis. In the sequel, we will therefore no longer make the bound N explicit and instead of "for all $n \geq N$" we will just write "for all $n \gg 0$".

Example 5.2.2. It is instructive to look at some examples. Let K be a field of characteristic $p > 3$, and let $A := K[\xi, \zeta, \eta]/(\xi^3 - \zeta^3 - \eta^3)K[\xi, \zeta, \eta]$ be the projective coordinate ring of the *cubic Fermat curve*. Let us show that ξ^2 is in the tight closure of $I := (\zeta, \eta)A$. For a fixed e, write $2p^e = 3h + r$ for some $h \in \mathbb{N}$ and some remainder $r \in \{1, 2\}$, and let c be the multiplier ξ^3. Hence

$$c\mathbf{F}^e(\xi^2) = \xi^{3(h+1)+r} = \xi^r(\zeta^3 + \eta^3)^{h+1}.$$

A quick calculation shows that any monomial in the expansion of $(\zeta^3+\eta^3)^{h+1}$ is a multiple of either $\mathbf{F}^e(\zeta)$ or $\mathbf{F}^e(\eta)$, showing that (5.3) holds for all e, and hence that $(\xi^2,\zeta,\eta)A \subseteq \mathrm{cl}(I)$.

It is often much harder to show that an element does not belong to the tight closure of an ideal. Shortly, we will see in Theorem 5.3.6 that any element outside the integral closure is also outside the tight closure. Since $(\xi^2,\zeta,\eta)A$ is integrally closed, we conclude that it is equal to $\mathrm{cl}(I)$.

Example 5.2.3. Let A be the coordinate ring of the hypersurface in $\mathbb{A}^3_K$ given by the equation $\xi^2-\zeta^3-\eta^7=0$. By a similar calculation as in the previous example, one can show that ξ lies in the tight closure of $(\zeta,\eta)A$.

A far more difficult result is to show that this is not true if we replace η^7 by η^5 in the above equation. In fact, in this new coordinate ring A', any ideal is tightly closed, that is to say, in the terminology from Definition 5.2.7 below, A' is F-regular, but this is a deep fact, following from it being log-terminal (see the discussion following Theorem 5.5.6).

It is sometimes cumbersome to work with multipliers in arbitrary rings, but in domains they are just non-zero elements. Fortunately, we can always reduce to the domain case when calculating tight closure:

Proposition 5.2.4. *Let A be a Noetherian ring, let $\mathfrak{p}_1,\dots,\mathfrak{p}_s$ be its minimal primes, and put $\bar{A}_i := A/\mathfrak{p}_i$. For all ideals $I\subseteq A$ we have*

$$\mathrm{cl}_A(I)=\bigcap_{i=1}^{s}\mathrm{cl}_{\bar{A}_i}(I\bar{A}_i)\cap A. \tag{5.4}$$

Proof. The same equations which exhibit x as en element of $\mathrm{cl}_A(I)$ also show that it is in $\mathrm{cl}_{\bar{A}_i}(I\bar{A}_i)$ since any multiplier in A remains, by virtue of its definition, a multiplier in $\bar{A}_i$ (moreover, the converse also holds: by prime avoidance, we can lift any multiplier in $\bar{A}_i$ to one in A). So one inclusion in (5.4) is clear.

Conversely, suppose x lies in the intersection on the right hand side of (5.4). Let $c_i\in A$ be a multiplier in A (so that its image is a multiplier in $\bar{A}_i$), such that

$$c_i\mathbf{F}^n_{\bar{A}_i}(x)\in\mathbf{F}^n_{\bar{A}_i}(I)\bar{A}_i$$

for all $n\gg 0$. This means that each $c_i\mathbf{F}^n_A(x)$ lies in $\mathbf{F}^n_A(I)A+\mathfrak{p}_i$ for $n\gg 0$. Choose for each i, an element $t_i\in A$ inside all minimal primes except $\mathfrak{p}_i$, and let $c:=c_1t_1+\dots+c_st_s$. A moment's reflection yields that c is again a multiplier. Moreover, since $t_i\mathfrak{p}_i\subseteq\mathfrak{n}$, where $\mathfrak{n}:=\mathrm{nil}(R)$ is the nilradical of A, we get

$$c\,\mathbf{F}^n_A(x)\in\mathbf{F}^n_A(I)A+\mathfrak{n}$$

for all $n \gg 0$. Choose m such that $\mathfrak{n}^{p^m}$ is zero, whence also the smaller ideal $\mathbf{F}_A(\mathfrak{n})$. Applying $\mathbf{F}_A^m$ to the previous equations, yields

$$\mathbf{F}_A^m(c)\mathbf{F}_A^{m+n}(x) \in \mathbf{F}_A^{m+n}(I)A$$

for all $n \gg 0$, which means that $x \in \mathrm{cl}_A(I)$ since $\mathbf{F}_A^m(c)$ is again a multiplier. □

We will encounter many operations similar to tight closure, and so we formally define:

Definition 5.2.5 (Closure Operation). A *closure operation* on a ring A is any order-preserving, increasing, idempotent endomorphism on the set of ideals of A ordered by inclusion.

For instance, taking the radical of an ideal is a closure operation, and so is *integral closure* discussed below. Tight closure too is a closure operation on A, since it clearly also preserves inclusion: if $I \subseteq I'$, then $\mathrm{cl}(I) \subseteq \mathrm{cl}(I')$. An ideal that is equal to its own tight closure is called *tightly closed*. Recall that the *colon ideal* $(I : J)$ is the ideal of all elements $a \in A$ such that $aJ \subseteq I$; here $I \subseteq A$ is an ideal, but $J \subseteq A$ can be any subset, which, however, most of the time is either a single element or an ideal. Almost immediately from the definitions, we get

5.2.6 *If I is tightly closed, then so is $(I : J)$ for any $J \subseteq A$.* □

One of the longest outstanding open problems in tight closure theory was its behavior under localization: do we always have

$$\mathrm{cl}_A(I)A_{\mathfrak{p}} \stackrel{?}{=} \mathrm{cl}_{A_{\mathfrak{p}}}(IA_{\mathfrak{p}}) \tag{5.5}$$

for every prime ideal $\mathfrak{p} \subseteq A$. Recently, Brenner and Monsky have announced (see [15]) a negative answer to this question. The full extent of this phenomenon is not yet understood, and so one has proposed the following two definitions (the above cited counterexample still does not contradict that both notions are the same).

Definition 5.2.7. A Noetherian ring A is called *weakly F-regular* if each of its ideals is tightly closed. If all localizations of A are weakly F-regular, then A is called *F-regular*.

5.3 Five Key Properties of Tight Closure

In this section we derive five key properties of tight closure, all of which admit fairly simple proofs. It is important to keep this in mind, since these five properties will already suffice to prove in the next section some deep theorems in commutative algebra. In fact, as we will see, any closure operation with these five properties on a class of Noetherian local rings would establish these deep theorems for that particular class (and there are still classes for which these statements remain conjectural). Moreover, the proofs of the five properties themselves rest on a few simple facts about the Frobenius, so that this will allow us to also carry over our arguments to characteristic zero in Chapters 6 and 7.

The first property, stated here only in its weak version, is merely an observation. Namely, any equation (5.3) in a ring A extends to a similar equation in any A-algebra B. In order for the latter to calculate tight closure, the multiplier $c \in A$ should remain a multiplier in B, and so we proved:

Theorem 5.3.1 (Weak Persistence). *Let $A \to B$ be a ring homomorphism, and let $I \subseteq A$ be an ideal. If $A \to B$ is injective and B is a domain, or more generally, if $A \to B$ preserves multipliers, then* $\mathrm{cl}_A(I) \subseteq \mathrm{cl}_B(IB)$. □

The remarkable fact is that this is also true if $A \to B$ is arbitrary and A is of finite type over an excellent Noetherian local ring (see [59, Theorem 2.3]). We will not need this stronger version, the proof of which requires another important ingredient of tight closure theory: the notion of a test element. A multiplier $c \in A$ is called a *test element* for A, if for every $a \in \mathrm{cl}(I)$, we have $c\mathbf{F}^n(a) \in \mathbf{F}^n(I)A$ for all n. The existence of test elements is not easy, and lies outside the scope of these notes, but once one has established their existence, many arguments become even more streamlined.

Theorem 5.3.2 (Regular Closure). *In a regular local ring, every ideal is tightly closed. In fact, a regular ring is F-regular.*

Proof. Let R be a regular local ring. Since any localization of R is again regular, the second assertion follows from the first. To prove the first, let I be an ideal and $x \in \mathrm{cl}(I)$. Towards a contradiction, assume $x \notin I$. In particular, we must have $(I : x) \subseteq \mathfrak{m}$. Choose a non-zero element c such that (5.3) holds for all $n \gg 0$. This means that c lies in the colon ideal $(\mathbf{F}^n(I)R : \mathbf{F}^n(x))$, for all $n \gg 0$. Since $\mathbf{F}$ is flat by Theorem 5.1.2, the colon ideal is equal to $\mathbf{F}^n(I : x)R$ by Theorem 3.3.14. Since $(I : x) \subseteq \mathfrak{m}$, we get $c \in \mathbf{F}^n(\mathfrak{m})R \subseteq \mathfrak{m}^{p^n}$. Since this holds for all $n \gg 0$, we get $c = 0$ by Theorem 2.4.14, clearly a contradiction. □

Theorem 5.3.3 (Colon Capturing). *Let R be a Noetherian local domain which is a homomorphic image of a Cohen-Macaulay local ring, and let $(x_1, \dots, x_d)$ be a system of parameters in R. Then for each i, the colon ideal $((x_1, \dots, x_i)R : x_{i+1})$ is contained in* $\mathrm{cl}((x_1, \dots, x_i)R)$.

Proof. Let S be a local Cohen-Macaulay ring such that $R = S/\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subseteq S$ of height h. By prime avoidance, we can lift the x_i to elements in S, again denoted for simplicity by x_i, and find elements $y_1, \dots, y_h \in \mathfrak{p}$ such that $(y_1, \dots, y_h, x_1, \dots, x_d)$ is a system of parameters in S, whence an S-regular sequence. Since $\mathfrak{p}$ contains the ideal $J := (y_1, \dots, y_h)S$ of the same height, it is a minimal prime of J. Let $J = \mathfrak{g}_1 \cap \dots \cap \mathfrak{g}_s$ be a minimal primary decomposition of J, with $\mathfrak{g}_1$ the $\mathfrak{p}$-primary component of J. In particular, some power of $\mathfrak{p}$ lies in $\mathfrak{g}_1$, and we may assume that this power is of the form p^m for some m. Choose c inside all

$\mathfrak{g}_i$ with $i > 1$, but outside $\mathfrak{p}$ (note that this is possible by prime avoidance). Putting everything together, we have

$$c\mathbf{F}^m(\mathfrak{p}) \subseteq c\,\mathfrak{p}^{p^m} \subseteq J. \tag{5.6}$$

Fix some i, let $I := (x_1, \ldots, x_i)S$ and assume $zx_{i+1} \in IR$, for some $z \in S$. Lifting this to S, we get $zx_{i+1} \in I + \mathfrak{p}$. Applying the n-th power of Frobenius to this for $n > m$, we get $\mathbf{F}^n(z)\mathbf{F}^n(x_{i+1}) \in \mathbf{F}^n(I)S + \mathbf{F}^n(\mathfrak{p})S$. By (5.6), this means that $c\mathbf{F}^n(z)\mathbf{F}^n(x_{i+1})$ lies in $\mathbf{F}^n(I)S + \mathbf{F}^{n-m}(J)S$. Since the $\mathbf{F}^{n-m}(y_j)$ together with the $\mathbf{F}^n(x_j)$ form again an S-regular sequence, we conclude that

$$c\mathbf{F}^n(z) \in \mathbf{F}^n(I)S + \mathbf{F}^{n-m}(J)S \subseteq \mathbf{F}^n(I)S + J$$

whence $c\mathbf{F}^n(z) \in \mathbf{F}^n(I)R$ for all $n > m$. By the choice of c, it is non-zero in R, so that the latter equations show that $z \in \mathrm{cl}(IR)$. □

The condition that R is a homomorphic image of a regular local ring is satisfied either if R is a local affine algebra, or, by Cohen's Structure Theorems, if R is complete. These are the two only cases in which we will apply the previous theorem. With a little effort, one can extend the proof without requiring R to be a domain (see for instance [59, Theorem 3.1]).

Theorem 5.3.4 (Finite Extensions). *If $A \to B$ is a finite, injective homomorphism of domains, and $I \subseteq A$ be an ideal, then* $\mathrm{cl}_B(IB) \cap A = \mathrm{cl}_A(I)$.

Proof. One direction is immediate by Theorem 5.3.1. For the converse, there exists an A-module homomorphism $\varphi\colon B \to A$ such that $c := \varphi(1) \neq 0$, by Lemma 5.3.5 below. Suppose $x \in \mathrm{cl}_B(IB) \cap A$, so that for some non-zero $d \in B$, we have $d\mathbf{F}^n(x) \in \mathbf{F}^n(I)B$ for $n \gg 0$. Since B is finite over A, some non-zero multiple of d lies in A, and hence without loss of generality, we may assume $d \in A$. Applying φ to these equations, we get

$$cd\,\mathbf{F}^n(x) \in \mathbf{F}^n(I)A$$

showing that $x \in \mathrm{cl}_A(I)$, since cd is a multiplier. □

Lemma 5.3.5. *If $A \subseteq B$ is a finite extension of domains, then there exists an A-linear map $\varphi\colon B \to A$ with $\varphi(1) \neq 0$.*

Proof. Suppose B is generated over A by the elements $b_1, \ldots, b_s$. Let K and L be the fields of fractions of A and B respectively. Since B is a domain, it lies inside the K-vector subspace $V \subseteq L$ generated by the b_i. Choose an isomorphism $\gamma\colon V \to K^t$ of K-vector spaces. After renumbering, we may assume that the first entry of $\gamma(1)$ is non-zero. Let $\pi\colon K^t \to K$ be the projection onto the first coordinate, and let $d \in A$ be the common denominator of the $\pi(\gamma(b_i))$ for $i = 1, \ldots, s$. Now define an A-linear homomorphism φ by the rule $\varphi(y) = d\pi(\gamma(y))$ for $y \in B$. Since y is an A-linear combination of the b_i and since $d\pi(\gamma(b_i)) \in A$, also $\varphi(y) \in A$. Moreover, by construction, $\varphi(1) \neq 0$. □

Note that a special case of Theorem 5.3.4 is the fact that tight closure measures the extent to which an extension of domains $A \subseteq B$ fails to be cyclically pure: $IB \cap A$ is contained in the tight closure of I, for any ideal $I \subseteq A$. In particular, in view of Theorem 5.3.2, this reproves the well-known fact that if $A \subseteq B$ is an extension of domains with A regular, then $A \subseteq B$ is cyclically pure. The next and last property involves another closure operation, integral closure. It will be discussed in more detail below (§5.4), and here we just state its relationship with tight closure:

Theorem 5.3.6 (Integral Closure). *For every ideal $I \subseteq A$, its tight closure is contained in its integral closure. In particular, radical ideals, and more generally integrally closed ideals, are tightly closed.*

Proof. The second assertion is an immediate consequence of the first. We verify condition (5.4.1.iv) below to show that if x belongs to the tight closure $\mathrm{cl}_A(I)$, then it also belongs to the integral closure $\bar{I}$. Let $A \to V$ be a homomorphism into a discrete valuation ring V, such that its kernel is a minimal prime of A. We need to show that $x \in IV$. However, this is clear since $x \in \mathrm{cl}_V(IV)$ by Theorem 5.3.1 (note that $A \to V$ preserves multipliers), and since $\mathrm{cl}_V(IV) = IV$, by Theorem 5.3.2 and the fact that V is regular. □

It is quite surprising that there is no proof, as far as I am aware of, that a prime ideal is tightly closed without reference to integral closure.

5.4 Integral Closure

The *integral closure* $\bar{I}$ of an ideal I is the collection of all elements $x \in A$ satisfying an integral equation of the form

$$x^d + a_1 x^{d-1} + \cdots + a_d = 0 \tag{5.7}$$

with $a_j \in I^j$ for all $j = 1, \ldots, d$. We say that I is *integrally closed* if $I = \bar{I}$. Since clearly $\bar{I} \subseteq \mathrm{rad}(I)$, radical ideals are integrally closed. It follows from either characterization (5.4.1.ii) or (5.4.1.iv) below that $\bar{I}$ is an ideal.

Theorem 5.4.1. *Let A be an arbitrary Noetherian ring (not necessarily of characteristic p). For an ideal $I \subseteq A$ and an element $x \in A$, the following are equivalent*

- 5.4.1.i. *x belongs to the integral closure, $\bar{I}$;*
- 5.4.1.ii. *there is a finitely generated A-module M with zero annihilator such that $xM \subseteq IM$;*
- 5.4.1.iii. *there is a multiplier $c \in A$ such that $cx^n \in I^n$ for infinitely many (respectively, for all sufficiently large) n;*
- 5.4.1.iv. *for every homomorphism $A \to V$ into a discrete valuation ring V with kernel equal to a minimal prime of A, we have $x \in IV$.*

Proof. We leave it to the reader to show that x lies in the integral closure of an ideal I if and only if it lies in the integral closure of each $I(A/\mathfrak{p})$, for $\mathfrak{p}$ a minimal prime of A. Hence we may moreover assume that A is a domain. Suppose x satisfies an integral equation (5.7), and let $J := x^{d-1}A + x^{d-2}I + \cdots + I^d$. An easy calculation shows that $xJ \subseteq IJ$, proving (5.4.1.i) ⇒ (5.4.1.ii). Moreover, by induction, $x^nJ \subseteq I^nJ$, and hence for any non-zero element $c \in J$, we get $cx^n \in I^n$, proving (5.4.1.iii). Note that in particular, $x^nI^d \subseteq I^n$ for all n. The implication (5.4.1.ii) ⇒ (5.4.1.i) is proven by a 'determinantal trick': apply [69, Theorem 2.1] to the multiplication with x on M. To prove (5.4.1.iii) ⇒ (5.4.1.iv), suppose there is some non-zero $c \in A$ such that $cx^n \in I^n$ for infinitely many n. Let $A \subseteq V$ be an injective homomorphism into a discrete valuation ring V, and let v be the valuation on V. Hence $v(c) + nv(x) \geq nv(I)$ for infinitely many n, where $v(I)$ is the minimum of all $v(a)$ with $a \in I$. It follows that $v(x) \geq v(I)$, and hence $x \in IV$.

Remains to prove (5.4.1.iv) ⇒ (5.4.1.i), so assume $x \in IV$ for every embedding $A \subseteq V$ into a discrete valuation ring V. Let $I = (a_1, \dots, a_n)A$, and consider the homomorphism $A[\xi] \to A_x$ given by $\xi_i \mapsto a_i/x$, where $\xi := (\xi_1, \dots, \xi_n)$. Let B be its image, so that $A \subseteq B \subseteq A_x$ (one calls B the *blowing-up* of $I + xA$ at x). Let $\mathfrak{m} := (\xi_1, \dots, \xi_n)A[\xi]$. I claim that $\mathfrak{m}B = B$. Assuming the claim, we can find $f \in \mathfrak{m}$ such that $f(\mathbf{a}/x) = 1$ in A_x, where $\mathbf{a} := (a_1, \dots, a_n)$. Write $f = f_1 + \cdots + f_d$ in its homogeneous parts f_j of degree j, so that

$$1 = x^{-1}f_1(\mathbf{a}) + \cdots + x^{-d}f_d(\mathbf{a}).$$

Multiplying with x^d, and observing that $f_j(\mathbf{a}) \in I^j$, we see that x satisfies an integral equation (5.7), and hence $x \in \bar{I}$.

To prove the claim ex absurdum, suppose $\mathfrak{m}B$ is not the unit ideal, whence is contained in a maximal ideal $\mathfrak{n}$ of B. Let $(x_1, \dots, x_n)$ be a generating tuple of $\mathfrak{n}$. Let R be the $B_\mathfrak{n}$-algebra generated by the fractions x_i/x_1 with $i = 1, \dots, n$ (the blowing-up of $B_\mathfrak{n}$ at $\mathfrak{n}$). Since $\mathfrak{n}R = x_1R$, there exists a height one prime ideal $\mathfrak{p}$ in R containing $\mathfrak{n}R$. Let V be the normalization of $R_\mathfrak{p}$. It follows that V is a discrete valuation ring (see [69, Theorem 11.2]) containing $B_\mathfrak{n}$ as a local subring. In particular, $A \subseteq V$, and $\mathfrak{m}V$ lies in the maximal ideal πV. Since $\xi_i \mapsto a_i/x$, we get $a_i \in x\pi V$ for all i, and hence $IV \subseteq x\pi V$, contradicting that $x \in IV$. □

From this we readily deduce:

Corollary 5.4.2. *A domain A is normal if and only if each principal ideal is integrally closed if and only if each principal ideal is tightly closed.* □

In one of our applications below (Theorem 5.5.1), we will make use of the following nice application of the chain rule:

Proposition 5.4.3. *Let K be a field of characteristic zero, and let R be either the power series ring $K[[\xi]]$, the ring of convergent power series $K\{\xi\}$ (assuming K is a normed field), or the localization of $K[\xi]$ at the ideal generated by the indeterminates $\xi := (\xi_1, \dots, \xi_n)$. If f is a non-unit, then it lies in the integral closure of its Jacobian ideal* $\operatorname{Jac}(f) := (\partial f/\partial \xi_1, \dots, \partial f/\partial \xi_n)R$.

Proof. Recall that $K\{\xi\}$ consists of all formal power series f such that $f(\mathbf{u})$ is a convergent series for all $\mathbf{u}$ in a small enough neighborhood of the origin. Put $J := \mathrm{Jac}(f)$. In view of (5.4.1.iv), we need to show that given an embedding $R \subseteq V$ into a discrete valuation ring V, we have $f \in JV$. Since completion is faithfully flat, we may replace V by its completion, and hence already assume V is complete. By Cohen's Structure Theorems, V is a power series ring $\kappa[[\zeta]]$ in a single variable over a field extension κ of K. Viewing the image of f in $\kappa[[\zeta]]$ as a power series in ζ, the multi-variate chain rule yields

$$\frac{df}{d\zeta} = \sum_{i=1}^{n} \frac{\partial f}{\partial \xi_i} \cdot \frac{d\xi_i}{d\zeta} \in JV.$$

However, since f has order $e \geq 1$ in V, its derivative $df/d\zeta$ has order $e-1$, and hence $f \in (df/d\zeta)V \subseteq JV$. Note that for this to be true, however, the characteristic needs to be zero. For instance, in characteristic p, the power series ξ^p would already be a counterexample to the proposition. □

Since the integral closure is contained in the radical closure, we get that some power of f lies in its Jacobian ideal $\mathrm{Jac}(f)$. A famous theorem due to Briançon-Skoda states that in fact already the n-th power lies in the Jacobian, where n is the number of variables. We will prove this via an elegant tight closure argument in Theorem 5.5.1 below.

5.5 Applications

We will now discuss three important applications of tight closure. Perhaps surprisingly, the original statements all were in characteristic zero (with some of them in their original form plainly false in positive characteristic), and their proofs required deep and involved arguments, some even based on transcendental/analytic methods. However, they each can be reformulated so that they also make sense in positive characteristic, and then can be established by surprisingly elegant tight closure arguments. As for the proofs of their characteristic zero counterparts, they must wait until we have developed the theory in characteristic zero in Chapters 6 and 7 (or one can use the 'classical' tight closure in characteristic zero discussed in §5.6).

5.5.1 The Briançon-Skoda Theorem

We already mentioned this famous result, proven first in [16].

Theorem 5.5.1 (Briançon-Skoda). *Let R be either the ring of formal power series $\mathbb{C}[[\xi]]$, or the ring of convergent power series $\mathbb{C}\{\xi\}$, or the localization of the* *polynomial ring* $\mathbb{C}[\xi]$ *at the ideal generated by* ξ, *where* $\xi := (\xi_1, \ldots, \xi_n)$ *are some indeterminates. If* f *is not a unit, then* $f^n \in \mathrm{Jac}(f) := (\partial f/\partial \xi_1, \ldots, \partial f/\partial \xi_n)R$.

This theorem will follow immediately from the characteristic zero analogue of the next result (with $l = 1$), in view of Proposition 5.4.3; we will do this in Theorem 6.2.5 below.

Theorem 5.5.2 (Briançon-Skoda—Tight Closure Version). *Let* A *be a Noetherian ring of characteristic* p, *and* $I \subseteq A$ *an ideal generated by* n *elements. Then we have for all* $l \geq 1$ *an inclusion*

$$\overline{I^{n+l-1}} \subseteq \mathrm{cl}(I^l).$$

In particular, if A *is a regular local ring, then the integral closure of* I^{n+l-1} *lies inside* I^l *for* $l \geq 1$.

Proof. For simplicity, I will only prove the case $l = 1$ (which gives the original Briançon-Skoda theorem). Assume z lies in the integral closure of I^n. By (5.4.1.iii), there exists a multiplier $c \in A$ such that $cz^k \in I^{kn}$ for all $k \gg 0$. Since $I := (f_1, \ldots, f_n)A$, we have an inclusion $I^{kn} \subseteq (f_1^k, \ldots, f_n^k)A$. Hence with k equal to p^m, we get $c\mathbf{F}^m(z) \in \mathbf{F}^m(I)A$ for all $m \gg 0$. In conclusion, $z \in \mathrm{cl}(I)$. The last assertion then follows from Theorem 5.3.2. □

5.5.2 *The Hochster-Roberts Theorem*

We will formulate the next result without defining in detail all the concepts involved, except when we get to its algebraic formulation. A *linear algebraic group* G is an affine subscheme of the general linear group $\mathrm{GL}(K,n)$ over an algebraically closed field K such that its K-rational points form a subgroup of the latter group. When G acts (as a group) on a closed subscheme $X \subseteq \mathbb{A}^n_K$ (more precisely, for each algebraically closed field L containing K, there is an action of the L-rational points of $G(L)$ on $X(L)$), we can define the *quotient space* X/G, consisting of all orbits under the action of G on X, as the affine space $\mathrm{Spec}(R^G)$, where R^G denotes the subring of G-invariant sections in $R := \Gamma(X, \mathscr{O}_X)$ (the action of G on X induces an action on the sections of X, and hence in particular on R). For this to work properly, we also need to impose a certain finiteness condition: G has to be *linearly reductive*. Although not usually its defining property, we will here take this to mean that there exists an R^G-linear map $R \to R^G$ which is the identity on R^G, called the *Reynolds operator* of the action. For instance, if $K = \mathbb{C}$, then an algebraic group is linearly reductive if and only if it is the complexification of a real Lie group, where the Reynolds operator is obtained by an integration process. This is the easiest to understand if G is finite, when the integration is just a finite sum

$$\rho\colon R \to R^G\colon a \mapsto \frac{1}{|G|}\sum_{\sigma\in G} a^\sigma,$$

where a^σ denotes the result of $\sigma \in G$ acting on $a \in R$. In fact, as indicated by the above formula, a finite group is linearly reductive over a field of positive characteristic, provided its cardinality is not divisible by the characteristic. If X is non-singular and G is linearly reductive, then we will call X/G a *quotient singularity*.[1] The celebrated Hochster-Roberts theorem now states:

Theorem 5.5.3. *Any quotient singularity is Cohen-Macaulay.*

To state a more general result, we need to take a closer look at the Reynolds operator. A ring homomorphism $A \to B$ is called *split*, if there exists an A-linear map $\sigma\colon B \to A$ which is the identity on A (note that σ need not be multiplicative, that is to say, is not a ring homomorphism, only a module homomorphism). We call σ the *splitting* of $A \to B$. Hence the Reynolds operator is a splitting of the inclusion $R^G \subseteq R$. The only property of split maps that will matter is the following:

5.5.4 *A split homomorphism $A \to B$ is cyclically pure.*

See the discussion at the beginning of §2.4.3 for the definition of cyclic purity. Let $a \in IB \cap A$ with $I = (f_1,\dots,f_s)A$ an ideal in A. Hence $a = f_1b_1 + \dots + f_sb_s$ for some $b_i \in B$. Applying the splitting σ, we get by A-linearity $a = f_1\sigma(b_1) + \dots + f_s\sigma(b_s) \in I$, proving that A is cyclically pure in B. □

We also need the following result on the preservation of cyclic purity under completions:

Lemma 5.5.5. *Let R and S be Noetherian local rings with respective completions $\widehat{R}$ and $\widehat{S}$. If $R \to S$ is cyclically pure, then so is its completion $\widehat{R} \to \widehat{S}$.*

Proof. The homomorphism $S \to \widehat{S}$ is faithfully flat, hence cyclically pure; thus the composition $R \to S \to \widehat{S}$ is cyclically pure. So from now on we may suppose that $S = \widehat{S}$. It suffices to show that $\widehat{R} \to S$ is injective, since the completion of $R/\mathfrak{a}$ is equal to $\widehat{R}/\mathfrak{a}\widehat{R}$, for any ideal $\mathfrak{a}$ in R. Let $a \in \widehat{R}$ be such that $a = 0$ in S, and for each i choose $a_i \in R$ such that $a \equiv a_i \mod \mathfrak{m}^i\widehat{R}$, where $\mathfrak{m}$ is the maximal ideal of R. Then a_i lies in $\mathfrak{m}^iS$, hence by cyclical purity, in $\mathfrak{m}^i$. Therefore $a \in \mathfrak{m}^i\widehat{R}$ for all i, showing that $a = 0$ in $\widehat{R}$ by Krull's Intersection Theorem (Theorem 2.4.14). □

We can now state a far more general result, of which Theorem 5.5.3 is just a special case.

Theorem 5.5.6. *If $R \to S$ is a cyclically pure homomorphism and if S is regular, then R is Cohen-Macaulay.*

[1] The reader should be aware that other authors might use the term more restrictively, only allowing X to be affine space $\mathbb{A}^n_K$, or G to be finite.

Proof. The problem is clearly local, and so we assume that $(R,\mathfrak{m})$ and $(S,\mathfrak{n})$ are local. By Lemma 5.5.5, we may further reduce to the case that R and S are both complete. We split the proof in two parts: we first show that R is F-regular (see Definition 5.2.7), and then show that any complete local F-regular domain is Cohen-Macaulay.

5.5.7 *A cyclically pure subring of a regular ring is F-regular.*

Indeed, since both cyclic purity and regularity are preserved under localization, we only need to show that every ideal in R is tightly closed. To this end, let $I \subseteq R$ and $x \in \mathrm{cl}(I)$. Hence x lies in the tight closure of IS by (weak) persistence (Theorem 5.3.1), and therefore in IS by Theorem 5.3.2. Hence by cyclic purity, $x \in I = IS \cap R$, proving that R is weakly F-regular. Note that we actually proved that a cyclically pure subring of a (weakly) F-regular ring is again (weakly) F-regular.

5.5.8 *A complete local F-regular domain is Cohen-Macaulay.*

Assume R is F-regular and let $(x_1,\dots,x_d)$ be a system of parameters in R. To show that x_{i+1} is $R/(x_1,\dots,x_i)R$-regular, assume $zx_{i+1} \in (x_1,\dots,x_i)R$. Colon Capturing (Theorem 5.3.3) yields that z lies in the tight closure of $(x_1,\dots,x_i)R$, whence in the ideal itself since R is F-regular. □

Remark 5.5.9. In fact, R is then also normal (this follows easily from 5.5.7 and Corollary 5.4.2). A far more difficult result is that R is then also *pseudo-rational* (a concept that lies beyond the scope of these notes; see for instance [59, 99] for a discussion of what follows). This was first proven by Boutot in [14] for $\mathbb{C}$-affine algebras by means of deep vanishing theorems. The positive characteristic case was proven by Smith in [108] by tight closure methods, where she also showed that pseudo-rationality is in fact equivalent with the weaker notion of F-rationality (a local ring is *F-rational* if some parameter ideal is tightly closed). I proved the general characteristic zero case in [99] by means of ultraproducts. In fact, being F-regular is equivalent under the $\mathbb{Q}$-Gorenstein assumption with having log-terminal singularities (see [38, 95]; for an example see Example 5.2.3). It should be noted that 'classical' tight closure theory in characteristic zero (see §5.6 below) is not sufficiently versatile to derive these results: so far, only our present ultraproduct method seems to work.

5.5.3 The Ein-Lazardsfeld-Smith Theorem

The next result, although elementary in its formulation, was only proven recently in [26] using quite complicated methods (which only work over $\mathbb{C}$), but then soon after in [55] by an elegant tight closure argument (see also [90]), which proves the result over any field K.

Theorem 5.5.10. *Let $V \subseteq K^2$ be a finite subset with ideal of definition $I := \mathfrak{I}(V)$. For each k, let $J_k(V)$ be the ideal of all polynomials f having multiplicity at least k at each point $x \in V$. Then $J_{2k}(V) \subseteq I^k$, for all k.*

To formulate the more general result of which this is just a corollary, we need to introduce symbolic powers. We first do this for a prime ideal $\mathfrak{p}$: its k-th *symbolic power* is the contracted ideal $\mathfrak{p}^{(k)} := \mathfrak{p}^k R_{\mathfrak{p}} \cap R$. In general, the inclusion $\mathfrak{p}^k \subseteq \mathfrak{p}^{(k)}$ may be strict, and, in fact, $\mathfrak{p}^{(k)}$ is the $\mathfrak{p}$-primary component of $\mathfrak{p}^k$. If $\mathfrak{a}$ is a radical ideal (we will not treat the more general case), then we define its k-th *symbolic power* $\mathfrak{a}^{(k)}$ as the intersection $\mathfrak{p}_1^{(k)} \cap \dots \cap \mathfrak{p}_s^{(k)}$, where the $\mathfrak{p}_i$ are all the minimal overprimes of $\mathfrak{a}$. The connection with Theorem 5.5.10 is given by:

5.5.11 *The k-th symbolic power of the ideal of definition $I := \mathfrak{I}(V)$ of a finite subset $V \subseteq K^2$ is equal to the ideal $J_k(V)$ of all polynomials that have multiplicity at least k at any point of V.*

Indeed, for $\mathbf{x} \in V$, let $\mathfrak{m} := \mathfrak{m}_{\mathbf{x}}$ be the corresponding maximal ideal. By definition, a polynomial f has multiplicity at least k at each $\mathbf{x} \in V$, if $f \in \mathfrak{m}^k A_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ containing I. The latter condition simply means that $f \in \mathfrak{m}^{(k)}$, so that the claim follows from the definition of symbolic power. □

Hence, in view of this, Theorem 5.5.10 is an immediate consequence of the following theorem (at least in positive characteristic; for the characteristic zero case, see Theorems 6.2.6 and 7.2.4 below):

Theorem 5.5.12. *Let A be a regular domain of characteristic p. Let $\mathfrak{a} \subseteq A$ be a radical ideal and let h be the maximal height of its minimal overprimes. Then we have an inclusion $\mathfrak{a}^{(hn)} \subseteq \mathfrak{a}^n$, for all n.*

Proof. We start with proving the following useful inclusion:

$$\mathfrak{a}^{(hp^e)} \subseteq \mathbf{F}^e(\mathfrak{a})A \tag{5.8}$$

for all e. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be the minimal prime ideals of $\mathfrak{a}$. We first prove (5.8) locally at one of these minimal primes $\mathfrak{p}$. Since $A_{\mathfrak{p}}$ is regular and $\mathfrak{a}A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$, we can find $f_i \in \mathfrak{a}$ such that $\mathfrak{a}A_{\mathfrak{p}} = (f_1, \dots, f_h)A_{\mathfrak{p}}$. By definition of symbolic powers, $\mathfrak{a}^{(hp^e)}A_{\mathfrak{p}} = \mathfrak{a}^{hp^e}A_{\mathfrak{p}}$. On the other hand, $\mathfrak{a}^{hp^e}A_{\mathfrak{p}}$ consists of monomials in the f_i of degree hp^e, and hence any such monomial lies in $\mathbf{F}^e(\mathfrak{a})A_{\mathfrak{p}}$. This establishes (5.8) locally at $\mathfrak{p}$. To prove this globally, take $z \in \mathfrak{a}^{(hp^e)}$. By what we just proved, there exists $s_i \notin \mathfrak{p}_i$ such that $s_i z \in \mathbf{F}^e(\mathfrak{a})A$ for each $i = 1, \dots, m$. For each i, choose an element t_i in all $\mathfrak{p}_j$ except $\mathfrak{p}_i$, and put $s := t_1 s_1 + \dots + s_m t_m$. It follows that s multiplies z inside $\mathbf{F}^e(\mathfrak{a})A$, whence a fortiori, so does $\mathbf{F}^e(s)$. Hence

$$z \in (\mathbf{F}^e(\mathfrak{a})A : \mathbf{F}^e(s)) = \mathbf{F}^e(\mathfrak{a} : s)A$$

where we used Theorem 3.3.14 and the fact that $\mathbf{F}$ is flat on A by Theorem 5.1.2. However, s does not lie in any of the $\mathfrak{p}_i$, whence $(\mathfrak{a} : s) = \mathfrak{a}$, proving (5.8).

To prove the theorem, let $f \in \mathfrak{a}^{(hn)}$, and fix some e. We may write $p^e = an + r$ for some $a, r \in \mathbb{N}$ with $0 \leq r < n$. Since the usual powers are contained in the symbolic powers, and since $r < n$, we have inclusions

$$\mathfrak{a}^{hn} f^a \subseteq \mathfrak{a}^{hr} f^a \subseteq \mathfrak{a}^{(han+hr)} = \mathfrak{a}^{(hp^e)} \subseteq \mathbf{F}^e(\mathfrak{a})A \tag{5.9}$$

where we used (5.8) for the last inclusion. Taking n-th powers in (5.9) shows that $\mathfrak{a}^{hn^2} f^{an}$ lies in the n-th power of $\mathbf{F}^e(\mathfrak{a})A$, and this in turn lies inside $\mathbf{F}^e(\mathfrak{a}^n)A$. Choose some non-zero c in $\mathfrak{a}^{hn^2}$. Since $p^e \geq an$, we get $c\mathbf{F}^e(f) \in \mathbf{F}^e(\mathfrak{a}^n)A$ for all e. In conclusion, f lies in $\mathrm{cl}(\mathfrak{a}^n)$ whence in $\mathfrak{a}^n$ by Theorem 5.3.2. □

One might be tempted to try to prove a more general form which does not assume A to be regular, replacing $\mathfrak{a}^n$ by its tight closure. However, we used the regularity assumption not only via Theorem 5.3.2 but also via Kunz's Theorem that the Frobenius is flat. Hence the above proof does not work in arbitrary rings.

5.6 Classical Tight Closure in Characteristic Zero

To prove the previous three theorems in a ring of equal characteristic zero, Hochster and Huneke also developed tight closure theory for such rings. One of the precursors to tight closure theory was the proof of the Intersection Theorem by Peskine and Szpiro in [75]. They used properties of the Frobenius together with a method to transfer results from characteristic p to characteristic zero, which was then generalized by Hochster in [43]. This same technique is also used to obtain a tight closure theory in equal characteristic zero, as we will discuss briefly in this section. However, using ultraproducts, we will bypass in Chapters 6 and 7 this rather heavy-duty machinery, to arrive much quicker at proofs in equal characteristic zero.

Let A be a Noetherian ring containing the rationals. The idea is to associate to A some rings in positive characteristic, its *reductions modulo p*, and calculate tight closure in the latter. More precisely, let $\mathfrak{a} \subseteq A$ be an ideal, and $z \in A$. We say that z lies in the *HH-tight closure* of $\mathfrak{a}$ (where "HH" stands for Hochster-Huneke), if there exists a $\mathbb{Z}$-affine subalgebra $R \subseteq A$ containing z, such that (the image of) z lies in the tight closure of $I(R/pR)$ for all primes numbers p, where $I := \mathfrak{a} \cap R$.

It is not too hard to show that this yields a closure operation on A (in the sense of Definition 5.2.5). Much harder is showing that it satisfies all the necessary properties from §5.3. For instance, to prove the analogue of Theorem 5.3.2, one needs some results on generic flatness, and some deep theorems on Artin Approximation (see for instance [59, Appendix 1] or [54]; for a brief discussion of Artin Approximation, see §7.1 below). In contrast, using ultraproducts, one can avoid all these complications in the affine case (Chapter 6), or get by with a more elementary version of Artin Approximation in the general case (Chapter 7).