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Hans Schoutens

The Use of Ultraproducts in Commutative Algebra

$$\mathbb{C} \cong \operatorname*{ulim}_{p \to \infty} \mathbb{F}_p^{alg}$$

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Hans Schoutens
City University of New York
CUNY Graduate Center
Department of Mathematics
365 Fifth Avenue
New York, NY, 10016
USA
hschoutens@citytech.cuny.edu

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To my mother, Jose Van Passel,
for giving me wisdom;
to my father, Louis Schoutens,
for giving me knowledge;
to my teacher, Pierre Gevers,
for giving me the love for mathematics;
to my mentor, Jan Denef,
for giving me inspiration;
and to my one and true love,
Parvaneh Pourshariati,
for giving me purpose.

Contents

Chapter 1
Introduction

Unbeknownst to the majority of algebraists, ultraproducts have been around in model-theory for more than half a century, since their first appearance in a paper by Łoś ([65]), although the construction goes even further back, to work of Skolem in 1938 on non-standard models of Peano arithmetic. Through Kochen's seminal paper [61] and his joint work [9] with Ax, ultraproducts also found their way into algebra. They did not leave a lasting impression on the algebraic community though, shunned perhaps because there were conceived as non-algebraic, belonging to the alien universe of set-theory and non-standard arithmetic, a universe in which most mathematicians did not, and still do not feel too comfortable.

The present book intends to debunk this common perception of ultraproducts: when applied to algebraic objects, their construction is quite natural, yet very powerful, and requires hardly any knowledge of model-theory. In particular, when applied to a collection of rings A_w, where w runs over some infinite index set W, the construction is entirely algebraic: the ultraproduct of the A_w is realized as a certain residue ring of the Cartesian product $A_\infty := \prod A_w$ modulo the so-called *null-ideal* (see below). Any ring arising in this way will be denoted $A_\natural$, and called an *ultra-ring*;[1] and the A_w are then called *approximations* of this ultra-ring. As this terminology suggests, we may think of ultraproducts as certain kinds of limits. This is the perspective of [102], which I will not discuss in these notes.

Whereas the classical Cartesian product performs a parallel computation, so to speak, within each A_w, the ultraproduct, on the other hand, computes things generically: elements in the ultraproduct $A_\natural$ satisfy certain algebraic relations if and only if their corresponding entries satisfy the same relations in the approximations A_w *with probability one*. To make this latter condition explicit, an ostensibly extrinsic component has to be introduced: we must impose some (degenerated) probability measure on the index set W of the family. The classical way is to choose a (non-principal) ultrafilter on W, and then say that an *event holds with probability one* (or, more informally, *almost always*) if the set of indices for which it holds belongs to the ultrafilter. Fortunately, the dependence on the choice of

[1] For the rather unorthodox notation, see below.

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ultrafilter/probability measure turns out to be, for all our intents and purposes, irrelevant, and so ultraproducts behave almost as if they were intrinsically defined.[2]

Once we have chosen a (non-principal) ultrafilter, we can define the ultraproduct $A_\natural$ as the residue ring of the Cartesian product $A_\infty := \prod_w A_w$ modulo the null-ideal of *almost zero elements*, that is to say, those elements in the product almost all of whose entries are zero. However, we can make this construction entirely algebraic, without having to rely on an ultrafilter/probability measure (although the latter perspective is more useful when we have to prove things about ultraproducts). Namely, A_∞ carries naturally the structure of a $\mathbb{Z}_\infty$-algebra, where $\mathbb{Z}_\infty$ is the corresponding Cartesian power of the ring of integers $\mathbb{Z}$. Given any minimal prime ideal $\mathfrak{P}$ in $\mathbb{Z}_\infty$, the base change $A_\natural := A_\infty/\mathfrak{P}A_\infty$ is an ultra-ring (with corresponding null-ideal $\mathfrak{P}A_\infty$). Moreover, all possible ultraproducts of the A_w arise in this way (see §2.5). Principal null-ideals, corresponding to principal ultrafilters, have one of the A_w as residue rings, and therefore are of little use. Hence from now on, when talking about ultra-rings, we always assume that the null-ideal is not principal—it follows that it is then infinitely generated—and this is equivalent with the ultrafilter containing all co-finite subsets, and also with $\mathfrak{P}$ containing the direct sum ideal $\bigoplus \mathbb{Z}$. Perhaps even more surprisingly familiar is the alternative definition given in §2.6 (communicated to me by Macintyre): an ultra-ring is simply a stalk at a point x of a sheaf of rings on a Boolean scheme, where a scheme is called *Boolean* if each residue field is isomorphic to $\mathbb{F}_2$ (and the null-ideal is non-principal if and only if the prime ideal of x is infinitely generated).

I already alluded to the main property of ultraproducts: they have the same (first-order) properties than almost all their approximations A_w; this is known to model-theorists as Łoś' Theorem. Although it may not always be easy to determine whether a property carries over, that is to say, is first-order, this is the case if it is expressible in arithmetic terms. *Arithmetical* here refers to algebraic formulas between ring elements, 'first-order objects,' but not between 'higher-order objects,' like ideals or modules. For instance, properties such as being a domain, reduced, normal, local, or Henselian, are easily seen to be preserved. Among those that do not carry over, is, unfortunately, the Noetherian property. Ultra-rings, therefore, are hardly ever Noetherian; the ultraproduct construction takes us outside our category! In particular, tools from commutative algebra seem no longer applicable. However, as we will show, there is still an awful lot, especially in the local case, that does carry through, with a few minor adaptations of the definitions. In fact, we will introduce two variant constructions that are designed to overcome altogether this obstacle. I have termed these *chromatic products*, for they, too, are denoted using musical notation: the *protoproduct* $A_\flat$, and the *cataproduct* $A_\sharp$. The latter is defined as soon all A_w are Noetherian local rings of bounded embedding dimension (that is to say, whose maximal ideal is gener-

[2] This does not mean that ultraproducts of the same rings, but with respect to different ultrafilters, are necessarily isomorphic.

ated by n elements, for some n independent from w). Its main advantage over the ultraproduct itself, of which it is a further residue ring, is that a cataproduct is always Noetherian and complete. To define protoproducts, we need some additional data on the approximations, namely, some uniform grading, analogous to polynomial degree. Although protoproducts do not need to be Noetherian, they often are. In case both are defined, we get a *chromatic scale of homomorphisms* $A_\flat \to A_\natural \to A_\sharp$.

However, as we shall see, it is in combination with certain flatness results that ultraproducts, and more generally chromatic products, acquire their real power. Already in their 1984 paper [86], Schmidt and van den Dries observed how a certain flatness property of ultraproducts, discovered five years prior to this by van den Dries in [25], translates into the existence of uniform bounds in polynomial rings (see our discussion in §4.2). This paper was soon followed by others exploiting this new method: [11,23,84]. The former two papers brought in a third theme that we will encounter in this book on occasion: Artin Approximation (see §7.1). So germane to almost every single application of ultraproducts is flatness, that I have devoted a separate chapter, Chapter 3, to it. It contains several flatness results, old and new,[3] that will be of use later in the book. Prior to this chapter, I introduce first our main protagonist, the ultra-ring, and prove some elementary facts. Noteworthy is a model-theoretic version of the Lefschetz Principle, Theorem 2.4.3, which will provide the basis of most transfer results from positive to zero characteristic: we may realize the field of complex numbers as an ultraproduct of fields of positive characteristic!

The subsequent chapters—except for Chapter 5, which is a brief survey on classical tight closure theory—then contain deeper results and properties of ultra-rings. Since an ultraproduct averages or captures the generic behavior of its approximations, it should not come as a surprise that as a tool, it is particularly well suited to derive uniformity results. This is done in Chapter 4, whose material is both thematically and chronologically the closest to its above mentioned paradigmatic forebear [86]. A second, more profound application of the method to commutative algebra is described in Chapters 6 and 7: we use ultraproducts to give an alternative treatment of tight closure theory in characteristic zero. Tight closure theory, introduced by Hochster and Huneke in an impressive array of beautiful articles—[47,48,50,53,51], to name only a few—is an extremely powerful tool, which relies heavily on the algebraicity of the Frobenius in positive characteristic, and as such is primarily a positive characteristic tool. Without going into details (these can be found in Chapter 5), one associates, using the p-th power Frobenius homomorphisms, to any ideal $\mathfrak{a}$ in a ring of characteristic $p > 0$, its *tight closure* $\mathfrak{a}^*$, an overideal contained in the integral closure of $\mathfrak{a}$, but often much closer or "tighter" to the original $\mathfrak{a}$. What really attracted people to the method was not only the

[3] Some of the well-known criteria are given here with a new proof; see, for instance, §3.3.6 on the Local Flatness Criterium.

apparent ease with which deep, known results could be reproved, but also its new, and sometimes unexpected applications, both in commutative algebra and algebraic geometry, derived almost all by means of fairly elementary arguments.

Although essentially a positive characteristic method, its authors also conceived of tight closure theory in characteristic zero in [54], by a generic reduction to positive characteristic. In fact, this reduction method, using Artin Approximation, as well as the method in positive characteristic itself were both inspired by the equally impressive work of Peskine and Szpiro [75] on Intersection Conjectures, and Hochster's own early work on big Cohen-Macaulay modules ([56]) and homological conjectures ([43,44]). However, to develop the method in characteristic zero some extremely deep results on Artin Approximation[4] were required, and the elegance of the positive characteristic method was entirely lost. No wonder! In characteristic zero, there is no Frobenius, nor any other algebraic endomorphism that could take over its role. To the rescue, however, come our ultraproducts. Keeping in mind that an ultraproduct is some kind of averaging process, it follows that the ultraproduct of rings of different positive characteristic is an ultra-ring of characteristic zero, for which reason we call it a *Lefschetz ring*. Furthermore, the ultraproduct of the corresponding Frobenius maps—one of the many advantages of ultraproducts, they can be taken of almost anything!—yields an *ultra-Frobenius* on this Lefschetz ring. Notwithstanding that it is no longer a power map, this ultra-Frobenius can easily fulfill the role played by the Frobenius in the positive characteristic theory. The key observation now is that many rings of characteristic zero—for instance, all Noetherian local rings, and all rings of finite type over a field—embed in a Lefschetz ring via a faithfully flat homomorphism. Flatness is essential here: it guarantees that the embedded ring preserves its ideal structure within the Lefschetz ring, which makes it possible to define the tight closure of its ideals inside that larger ring. In this manner, we can restore the elegant arguments from the positive characteristic theory, and prove the same results with the same elegant arguments as before. The present theory of characteristic zero tight closure is the easiest to develop for rings of finite type over an algebraically closed field, and this is explained in Chapter 6. The general local case is more complicated, and does require some further results on Artin Approximation, although far less deep than the ones Hochster and Huneke need for their theory. In fact, conversely, one can deduce certain Artin Approximation results from the fact that any Noetherian local ring has a faithfully flat Lefschetz extension (see in particular §7.1.4). Chapter 7 only develops the parts necessary to derive all the desired applications; for a more thorough treatment, one can consult [6].

In a parallel development, Hochster and Huneke's work on tight closure also led them to their discovery of canonically defined, big balanced Cohen-Macaulay algebras in positive characteristic: any system of parameters in a excellent local domain of positive characteristic becomes a regular sequence in the absolute integral closure of the ring. The same statement is plainly false in characteristic

[4] The controversy initially shrouding these results is a tale on its own.

zero, and the authors had to circumvent this obstruction again using complicated reduction techniques. Using ultraproducts, one constructs, quite canonically, big balanced Cohen-Macaulay algebras in characteristic zero simply by (faithfully flatly) embedding the ring inside a Lefschetz ring and then taking the ultraproduct of the absolute integral closures of the positive characteristic approximations of this Lefschetz ring. With aid of these new techniques, I was able to give new characterizations of rational and log-terminal singularities. Furthermore, exploiting the canonical properties of the ultra-Frobenius, I succeeded in settling some of the conjectures that hitherto had remained impervious to tight closure methods. All these results, unfortunately, fall outside the scope of this book, and the reader is referred to the articles [94, 95, 99], or to the survey paper [100].

The next two chapters, Chapter 8 on cataproducts, and Chapter 9 on protoproducts, develop the theory of the chromatic products mentioned already above. Most of the applications are on uniform bounds. For instance, we discuss some of the characterizations from [101] of several ring-theoretic properties of Noetherian local rings, such as being analytically unramified, Cohen-Macaulay, unmixed, etc., in terms of uniform behavior of two particular ring-invariants: order (with respect to the maximal ideal) and *degree*. This latter invariant measures to which extent an element is a parameter of the ring, and is a spin-off of our analysis of the dimension theory for ultra-rings (Krull dimension is one of the many invariants that are not preserved under ultraproducts, requiring a different approach via systems of parameters). Protoproducts, on the other hand, are designed to study rings with a generalized grading, called *proto-grading*, and most applications are again on uniform bounds in terms of these. This is in essence a formalization of the method coming out of the aforementioned [86].

In the last chapter, we discuss some open problems, commonly known as *homological conjectures*. Whereas these are now all settled in equal characteristic, either by the older methods, or by the recent tight closure methods, the case when the Noetherian local ring has different characteristic than its residue field, the *mixed characteristic case*, is for the most part still wide open (other than the recent breakthrough in dimension three by Heitman [40] and Hochster [46]). We will settle some of them, at least *asymptotically*, meaning, for large enough residual characteristic. This is still far from a complete solution, and our asymptotic results would only gain considerable interest if the actual conjectures turned out to be false. The method is inspired by Ax and Kochen's solution of a problem posed by Artin about C_2-fields, historically the first application of ultraproducts outside logic (see §10.1.2). Their main result, generalized latter by Eršhov ([29, 30]), is that an ultraproduct of mixed characteristic discrete valuation rings of different residual characteristics is isomorphic to an ultraproduct of equal characteristic discrete valuation rings. So, we can transfer results from equal characteristic, the known case, to results in mixed characteristic. However, the fact that properties only hold with probability one in an ultraproduct accounts for the asymptotic nature of our results. In §10.3, I propose a variant method, using cataproducts instead. Here the asymptotic nature can also be expressed in terms of the *ramification index*, that is to say, the order of the residual characteristic, rather than just the residual characteristic itself. Although this gives often more general results, in terms of more natural invariants, some of the homological problems still elude treatment. We conclude with a result, Theorem 10.3.7, showing how these asymptotic results could nonetheless lead to a positive solution of the corresponding full conjecture, provided we understand the growth rate of these uniform bounds better.

This book also includes two appendices, which contain some applications of the present theory, but also some material used at various points in the main text. Appendix A gives a new construction for the Henselization of a Noetherian local ring. The constructive nature of the process allows us then to define a proto-grading on this Henselization, called the *etale proto-grade*, and apply the theory from Chapter 9 to the ring of algebraic power series rings. Appendix B discusses Boolean rings and some of their generalizations (J-rings, n-Boolean and ω-Boolean rings, periodic rings). In particular, we prove, by means of ultraproducts, some representation theorems analogous to Stone's theorem for Boolean rings, which seem to have been unnoticed hitherto.

Notations and Conventions We follow the common convention to let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}$, $\mathbb{Q}_p$, $\mathbb{R}$, and $\mathbb{C}$ denote respectively, the natural numbers, the integers, the ring of p-adic integers, the field of rational, of p-adic, of real, and of complex numbers. The q-element field, for q a power of a prime number p, will be denoted $\mathbb{F}_q$; its algebraic closure is denoted $\mathbb{F}_p^{\text{alg}}$. The complement of a set $D \subset W$ is denoted $-D$, and more generally, the difference between two subsets $D, E \subseteq W$ is denoted $D - E$.

All rings are assumed to be commutative. More often than not, the image of an element $a \in A$ under a ring homomorphism $A \to B$ is still denoted a. In particular, IB denotes the ideal generated by the images of elements in the ideal $I \subseteq A$, and $J \cap A$ denotes the ideal of all elements in A whose image lies in the ideal $J \subseteq B$.

Chapter 2
Ultraproducts and Łoś' Theorem

In this chapter, W denotes an infinite set, always used as an index set, on which we fix a non-principal ultrafilter.[1] Given any collection of (first-order) structures indexed by W, we can define their ultraproduct. However, in this book, we will be mainly concerned with the construction of an ultraproduct of rings, an *ultra-ring* for short, which is then defined as a certain residue ring of their Cartesian product. From this point of view, the construction is purely algebraic, although it is originally a model-theoretic one (we only provide some supplementary background on the model-theoretic perspective). We review some basic properties (deeper theorems will be proved in the later chapters), the most important of which is Łoś' Theorem, relating properties of the approximations with their ultraproduct. When applied to algebraically closed fields, we arrive at a result that is pivotal in most of our applications: the Lefschetz Principle (Theorem 2.4.3), allowing us to transfer many properties between positive and zero characteristic.

2.1 Ultraproducts

We start with the classical definition of ultraproducts via ultrafilters; for different approaches, see §§2.5 and 2.6 below.

2.1.1 Ultrafilters

By a (*non-principal*) *ultrafilter* $\mathfrak{W}$ on W, we mean a collection of infinite subsets of W closed under finite intersection, with the property that for any subset $D \subseteq W$, either D or its complement $-D$ belongs to $\mathfrak{W}$. In particular, the empty set does not belong to $\mathfrak{W}$, and if $D \in \mathfrak{W}$ and E is an arbitrary set containing D, then also

[1] We will drop the adjective 'non-principal' since these are the only ultrafilters we are interested in; if we want to talk about principal ones, we just say *principal filter*; and if we want to talk about both, we say *maximal filter*.

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$E \in \mathfrak{W}$, for otherwise $-E \in \mathfrak{W}$, whence $\emptyset = D \cap -E \in \mathfrak{W}$, contradiction. Since every set in $\mathfrak{W}$ must be infinite, it follows that any co-finite set belongs to $\mathfrak{W}$. The existence of ultrafilters follows from the Axiom of Choice, and we make this set-theoretic assumption henceforth. It follows that for any infinite subset of W, we can find an ultrafilter containing this set.

More generally, a proper collection of subsets of W is called a *filter* if it is closed under intersection and supersets. Any ultrafilter is a filter which is maximal with respect to inclusion. If we drop the requirement that all sets in $\mathfrak{W}$ must infinite, then some singleton must belong to $\mathfrak{W}$; such a filter is called *principal*, and these are the only other maximal filters. A maximal filter is an ultrafilter if and only if it contains the *Frechet filter* consisting of all co-finite subsets (for all these properties, see for instance [81, §4] or [57, §6.4]).

In the remainder of these notes, unless stated otherwise, we fix an ultrafilter $\mathfrak{W}$ on W, and (almost always) omit reference to this fixed ultrafilter from our notation. No extra property of the ultrafilter is assumed, with the one exception described in Remark 8.1.5, which is nowhere used in the rest of our work anyway. Ultrafilters play the role of a decision procedure on the collection of subsets of W by declaring some subsets 'large' (those belonging to $\mathfrak{W}$) and declaring the remaining ones 'small'. More precisely, let o_w be elements indexed by $w \in W$, and let $\mathscr{P}$ be a property. We will use the expressions *almost all o_w satisfy property $\mathscr{P}$* or *o_w satisfies property $\mathscr{P}$ for almost all w* as an abbreviation of the statement that there exists a set D in the ultrafilter $\mathfrak{W}$, such that property $\mathscr{P}$ holds for the element o_w, whenever $w \in D$. Note that this is also equivalent with the statement that the set of all $w \in W$ for which o_w has property $\mathscr{P}$, lies in the ultrafilter (read: *is large*).

2.1.2 Ultraproducts

Let O_w be sets, for $w \in W$. We define an equivalence relation on the Cartesian product $O_\infty := \prod O_w$, by calling two sequences (a_w) and (b_w), for $w \in W$, equivalent, if a_w and b_w are equal for almost all w. In other words, if the set of indices $w \in W$ for which $a_w = b_w$ belongs to the ultrafilter. We will denote the equivalence class of a sequence (a_w) by

$$\operatorname*{ulim}_{w \to \infty} a_w, \quad \text{or} \quad \operatorname{ulim} a_w, \quad \text{or} \quad a_\natural.$$

The set of all equivalence classes on $\prod O_w$ is called the *ultraproduct* of the O_w and is denoted

$$\operatorname*{ulim}_{w \to \infty} O_w, \quad \text{or} \quad \operatorname{ulim} O_w, \quad \text{or} \quad O_\natural.$$

If all O_w are equal to the same set O, then we call their ultraproduct the *ultrapower* $O_\natural$ of O. There is a canonical map $O \to O_\natural$, sometimes called the *diagonal embedding*, sending an element o to the image of the constant sequence o in $O_\natural$. To see that it is an injection, assume o' has the same image as o in $O_\natural$. This means that for almost all w, and hence for at least one, the elements o and o' are equal.

Note that the element-wise and set-wise notations are reconciled by the fact that

$$\operatorname*{ulim}_{w\to\infty}\{o_w\} = \{\operatorname*{ulim}_{w\to\infty} o_w\}.$$

The more common notation for an ultraproduct one usually finds in the literature is O^*; in the past, I also have used O_∞, which in this book is reserved to denote Cartesian products. The reason for using the particular notation $O_\natural$ in these notes is because we will also introduce the remaining chromatic products $O_\flat$ and $O_\sharp$ (at least for certain local rings; see Chapters 9 and 8 respectively).

We will also often use the following terminology: if o is an element in an ultraproduct $O_\natural$, then any choice of elements $o_w \in O_w$ with ultraproduct equal to o will be called an *approximation* of o. Although an approximation is not uniquely determined by the element, any two agree almost everywhere. Below we will extend our usage of the term approximation to include other objects as well.

2.1.3 Properties of Ultraproducts

For the following properties, the easy proofs of which are left as an exercise, let O_w be sets with ultraproduct $O_\natural$.

2.1.1 *If Q_w is a subset of O_w for each w, then* $\operatorname{ulim} Q_w$ *is a subset of $O_\natural$.*

In fact, $\operatorname{ulim} Q_w$ consists of all elements of the form $\operatorname{ulim} o_w$, with almost all o_w in Q_w.

2.1.2 *If each O_w is the graph of a function $f_w\colon A_w \to B_w$, then $O_\natural$ is the graph of a function $A_\natural \to B_\natural$, where $A_\natural$ and $B_\natural$ are the respective ultraproducts of A_w and B_w. We will denote this function by*

$$\operatorname*{ulim}_{w\to\infty} f_w \quad \text{or} \quad f_\natural.$$

Moreover, we have an equality

$$\operatorname*{ulim}_{w\to\infty}(f_w(a_w)) = (\operatorname*{ulim}_{w\to\infty} f_w)(\operatorname*{ulim}_{w\to\infty} a_w), \tag{2.1}$$

for $a_w \in A_w$.

2.1.3 *If each O_w comes with an operation $*_w\colon O_w \times O_w \to O_w$, then*

$$*_\natural := \operatorname*{ulim}_{w\to\infty} *_w$$

is an operation on $O_\natural$. If all (or, almost all) O_w are groups with multiplication $*_w$ and unit element 1_w, then $O_\natural$ is a group with multiplication $*_\natural$ and unit element $1_\natural := \operatorname{ulim} 1_w$. If almost all O_w are Abelian groups, then so is $O_\natural$.

2.1.4 *If each O_w is a (commutative) ring under the addition $+_w$ and the multiplication $\cdot_w$, then $O_\natural$ is a (commutative) ring with addition $+_\natural$ and multiplication $\cdot_\natural$.*

In fact, in that case, $O_\natural$ is just the quotient of the product $O_\infty := \prod O_w$ modulo the *null-ideal*, the ideal consisting of all sequences (o_w) for which almost all o_w are zero (for more on this ideal, see §2.5 below). From now on, we will drop subscripts on the operations and denote the ring operations on the O_w and on $O_\natural$ simply by $+$ and $\cdot$.

2.1.5 *If almost all O_w are fields, then so is $O_\natural$. More generally, if almost each O_w is a domain with field of fractions K_w, then the ultraproduct $K_\natural$ of the K_w is the field of fractions of $O_\natural$.*

Just to give an example of how to work with ultraproducts, let me give the proof: if $a \in O_\natural$ is non-zero, with approximation a_w (recall that this means that $\operatorname{ulim} a_w = a$), then by the previous description of the ring structure on $O_\natural$, almost all a_w will be non-zero. Therefore, letting b_w be the inverse of a_w whenever this makes sense, and zero otherwise, one verifies that $\operatorname{ulim} b_w$ is the inverse of a. □

2.1.6 *If C_w are rings and O_w is an ideal in C_w, then $O_\natural$ is an ideal in $C_\natural := \operatorname{ulim} C_w$. In fact, $O_\natural$ is equal to the subset of all elements of the form $\operatorname{ulim} o_w$ with almost all $o_w \in O_w$. Moreover, the ultraproduct of the C_w/O_w is isomorphic to $C_\natural/O_\natural$. If almost every O_w is generated by e elements, then so is $O_\natural$.*

In other words, the ultraproduct of ideals $O_w \subseteq C_w$ is equal to the image of the ideal $\prod O_w$ in the product $C_\infty := \prod C_w$ under the canonical residue homomorphism $C_\infty \to C_\natural$. As for the last assertion, suppose $o_{1w}, \dots, o_{e(w),w}$ generate O_w, for each w, and let $o_{i\natural}$ be the ultraproduct of the $o_{i,w}$, where we put the latter equal to 0 if $i > e(w)$. The ideal generated by the $o_{i\natural}$ can be strictly contained in $O_\natural$ (an example is the ideal of infinitesimals, defined below in 2.4.13), but it is equal to it if almost all e_w are equal, say, to e. Indeed, any element $o_\natural$ of $O_\natural$ is an ultraproduct of elements $o_w \in O_w$, which therefore can be written as a linear combination $o_w = r_{1w}o_{1w} + \dots + r_{e,w}o_{e,w}$, for some $r_{i,w} \in C_w$. Let $r_{i\natural} \in C_\natural$ be the ultraproduct of the $r_{i,w}$, for $i = 1, \dots, e$. By Łoś' Theorem (see Theorem 2.3.2 below), we have $o_\natural = r_{1\natural}o_{1\natural} + \dots + r_{e\natural}o_{e\natural}$.

2.1.7 *If $f_w \colon A_w \to B_w$ are ring homomorphisms, then the ultraproduct $f_\natural$ is again a ring homomorphism. In particular, if σ_w is an endomorphism on A_w, then the ultraproduct $\sigma_\natural$ is a ring endomorphism on $A_\natural := \operatorname{ulim} A_w$.*

2.2 Model-theory in Rings

The previous examples are just instances of the general principle that 'algebraic structure' carries over to the ultraproduct. The precise formulation of this principle is called *Łoś' Theorem* (Łoś is pronounced 'wôsh') and requires some terminology from model-theory. However, for our purposes, a weak version of Łoś' Theorem (namely Theorem 2.3.1 below) suffices in almost all cases, and its proof is entirely algebraic. Nonetheless, for a better understanding, the reader is invited to indulge in some elementary model-theory, or rather, an ad hoc version for rings only (if this not satisfies him/her, (s)he should consult any textbook, such as [57, 67, 81]).

2.2.1 Formulae

By a *quantifier free formula without parameters* in the free variables $\xi = (\xi_1, \dots, \xi_n)$, we will mean an expression of the form

$$\varphi(\xi) := \bigvee_{j=1}^{m} f_{1j} = 0 \wedge \ldots \wedge f_{sj} = 0 \wedge g_{1j} \neq 0 \wedge \ldots \wedge g_{tj} \neq 0, \tag{2.2}$$

where each f_{ij} and g_{ij} is a polynomial with integer coefficients in the variables ξ, and where $\wedge$ and $\vee$ are the logical connectives *and* and *or*. If instead we allow the f_{ij} and g_{ij} to have coefficients in a ring R, then we call $\varphi(\xi)$ a *quantifier free formula with parameters in R*. We allow all possible degenerate cases as well: there might be no variables at all (so that the formula simply declares that certain elements in $\mathbb{Z}$ or in R are zero and others are non-zero) or there might be no equations or no negations or perhaps no conditions at all. Put succinctly, a quantifier free formula is a Boolean combination of polynomial equations using the connectives $\wedge$, $\vee$ and $\neg$ (negation), with the understanding that we use distributivity and De Morgan's Laws to rewrite this Boolean expression in the (disjunctive normal) form (2.2).

By a *formula without parameters* in the free variables ξ, we mean an expression of the form

$$\varphi(\xi) := (\mathrm{Q}_1\, \zeta_1) \cdots (\mathrm{Q}_p\, \zeta_p)\, \psi(\xi, \zeta),$$

where $\psi(\xi, \zeta)$ is a quantifier free formula without parameters in the free variables ξ and $\zeta = (\zeta_1, \dots, \zeta_p)$ and where Q_i is either the universal quantifier $\forall$ or the existential quantifier $\exists$. If instead $\psi(\xi, \zeta)$ has parameters from R, then we call $\varphi(\xi)$ a *formula with parameters* in R. A formula with no free variables is called a *sentence*.

2.2.2 Satisfaction

Let $\varphi(\xi)$ be a formula in the free variables $\xi = (\xi_1, \dots, \xi_n)$ with parameters from R (this includes the case that there are no parameters by taking $R = \mathbb{Z}$ and the case that there are no free variables by taking $n = 0$). Let A be an R-algebra and let

$\mathbf{a} = (a_1, \ldots, a_n)$ be a tuple with entries from A. We will give meaning to the expression $\mathbf{a}$ *satisfies* the formula $\varphi(\xi)$ in A (sometimes abbreviated to $\varphi(\mathbf{a})$ *holds in* A or *is true in* A) by induction on the number of quantifiers. Suppose first that $\varphi(\xi)$ is quantifier free, given by the Boolean expression (2.2). Then $\varphi(\mathbf{a})$ holds in A, if for some j_0, all $f_{ij_0}(\mathbf{a}) = 0$ and all $g_{ij_0}(\mathbf{a}) \neq 0$. For the general case, suppose $\varphi(\xi)$ is of the form $(\exists\zeta)\,\psi(\xi,\zeta)$ (respectively, $(\forall\zeta)\,\psi(\xi,\zeta)$), where the satisfaction relation is already defined for the formula $\psi(\xi,\zeta)$. Then $\varphi(\mathbf{a})$ holds in A, if there is some $b \in A$ such that $\psi(\mathbf{a},b)$ holds in A (respectively, if $\psi(\mathbf{a},b)$ holds in A, for all $b \in A$). The subset of A^n consisting of all tuples satisfying $\varphi(\xi)$ will be called the *subset defined* by φ, and will be denoted $\varphi(A)$. Any subset that arises in such way will be called a *definable subset* of A^n.

Note that if $n = 0$, then there is no mention of tuples in A. In other words, a sentence is either true or false in A. By convention, we set A^0 equal to the singleton $\{\emptyset\}$ (that is to say, A^0 consists of the empty tuple $\emptyset$). If φ is a sentence, then the set defined by it is either $\{\emptyset\}$ or $\emptyset$, according to whether φ is true or false in A.

2.2.3 Constructible Sets

There is a connection between definable subsets and Zariski-constructible subsets, where the relationship is the most transparent over algebraically closed fields, as we will explain below. In general, we can make the following observations.

Let R be a ring. Let $\varphi(\xi)$ be a quantifier free formula with parameters from R, given as in (2.2). Let $\Sigma_{\varphi(\xi)}$ denote the constructible subset of $\mathbb{A}^n_R = \operatorname{Spec}(R[\xi])$ consisting of all prime ideals $\mathfrak{p}$ which, for some j_0, contain all f_{ij_0} and do not contain any g_{ij_0}. In particular, if $n = 0$, so that $\mathbb{A}^0_R$ is by definition $\operatorname{Spec}(R)$, then the constructible subset Σ_φ associated to φ is a subset of $\operatorname{Spec}(R)$.

Let A be an R-algebra and assume moreover that A is a domain (we will never use constructible sets associated to formulae if A is not a domain). For an n-tuple $\mathbf{a}$ over A, let $\mathfrak{p}_\mathbf{a}$ be the (prime) ideal in $A[\xi]$ generated by the $\xi_i - a_i$, where $\xi = (\xi_1, \ldots, \xi_n)$. Since $A[\xi]/\mathfrak{p}_\mathbf{a} \cong A$, we call such a prime ideal an *A-rational point* of $A[\xi]$. It is not hard to see that this yields a bijection between n-tuples over A and A-rational points of $A[\xi]$, which we therefore will identify with one another. In this terminology, $\varphi(\mathbf{a})$ holds in A if and only if the corresponding A-rational point $\mathfrak{p}_\mathbf{a}$ lies in the constructible subset $\Sigma_{\varphi(\xi)}$ (strictly speaking, we should say that it lies in the base change $\Sigma_{\varphi(\xi)} \times_{\operatorname{Spec}(R)} \operatorname{Spec}(A)$, but for notational clarity, we will omit any reference to base changes). If we denote the collection of A-rational points of the constructible set $\Sigma_{\varphi(\xi)}$ by $\Sigma_{\varphi(\xi)}(A)$, then this latter set corresponds to the definable subset $\varphi(A)$ under the identification of A-rational points of $A[\xi]$ with n-tuples over A. If φ is a sentence, then Σ_φ is a constructible subset of $\operatorname{Spec}(R)$ and hence its base change to $\operatorname{Spec}(A)$ is a constructible subset of $\operatorname{Spec}(A)$. Since A is a domain, $\operatorname{Spec}(A)$ has a unique A-rational point (corresponding to the zero-ideal) and hence φ holds in A if and only if this point belongs to Σ_φ.

Conversely, if Σ is an R-constructible subset of $\mathbb{A}^n_R$, then we can associate to it a quantifier free formula $\varphi_\Sigma(\xi)$ with parameters from R as follows. However, here there is some ambiguity, as a constructible subset is more intrinsically defined than a formula. Suppose first that Σ is the Zariski closed subset $\mathrm{V}(I)$, where I is an ideal in $R[\xi]$. Choose a system of generators, so that $I = (f_1,\dots,f_s)R[\xi]$ and set $\varphi_\Sigma(\xi)$ equal to the quantifier free formula $f_1(\xi) = \dots = f_s(\xi) = 0$. Let A be an R-algebra without zero-divisors. It follows that an n-tuple $\mathbf{a}$ is an A-rational point of Σ if and only if $\mathbf{a}$ satisfies the formula φ_Σ. Therefore, if we make a different choice of generators $I = (f'_1,\dots,f'_s)R[\xi]$, although we get a different formula φ', it defines in any R-algebra A without zero-divisors the same definable subset, to wit, the collection of A-rational points of Σ. To associate a formula to an arbitrary constructible subset, we do this recursively by letting $\varphi_\Sigma \wedge \varphi_\Psi$, $\varphi_\Sigma \vee \varphi_\Psi$ and $\neg\varphi_\Sigma$ correspond to the constructible sets $\Sigma \cap \Psi$, $\Sigma \cup \Psi$ and $-\Sigma$ respectively.

We say that two formulae $\varphi(\xi)$ and $\psi(\xi)$ in the same free variables $\xi = (\xi_1,\dots,\xi_n)$ are *equivalent* over a ring A, if they hold on exactly the same tuples from A (that is to say, if they define the same subsets in A^n). In particular, if φ and ψ are sentences, then they are equivalent in A if they are simultaneously true or false in A. If $\varphi(\xi)$ and $\psi(\xi)$ are equivalent for all rings A in a certain class $\mathcal{K}$, then we say that $\varphi(\xi)$ and $\psi(\xi)$ are *equivalent modulo the class* $\mathcal{K}$. In particular, if Σ is a constructible subset in $\mathbb{A}^n_R$, then any two formulae associated to it are equivalent modulo the class of all R-algebras without zero-divisors. In this sense, there is a one-one correspondence between constructible subsets of $\mathbb{A}^n_R$ and quantifier free formulae with parameters from R up to equivalence.

2.2.4 Quantifier Elimination

For certain rings (or classes of rings), every formula is equivalent to a quantifier free formula; this phenomenon is known under the name *Quantifier Elimination*. We will only encounter it for the following class.

Theorem 2.2.1 (Quantifier Elimination for Algebraically Closed Fields). *If $\mathcal{K}$ is the class of all algebraically closed fields, then any formula without parameters is equivalent modulo $\mathcal{K}$ to a quantifier free formula without parameters.*

More generally, if F is a field and $\mathcal{K}(F)$ the class of all algebraically closed fields containing F, then any formula with parameters from F is equivalent modulo $\mathcal{K}(F)$ to a quantifier free formula with parameters from F.

Proof (Sketch of proof). These statements can be seen as translations in model-theoretic terms of Chevalley's Theorem which says that the projection of a constructible subset is again constructible. I will only explain this for the first assertion. As already observed, a quantifier free formula $\varphi(\xi)$ (without parameters) corresponds to a constructible set $\Sigma_{\varphi(\xi)}$ in $\mathbb{A}^n_\mathbb{Z}$ and the tuples in K^n satisfying $\varphi(\xi)$ are precisely the K-rational points $\Sigma_{\varphi(\xi)}(K)$ of $\Sigma_{\varphi(\xi)}$. The key observation is now the following. Let $\psi(\xi,\zeta)$ be a quantifier free formula and put

$\gamma(\xi) := (\exists \zeta)\, \psi(\xi,\zeta)$, where $\xi = (\xi_1,\dots,\xi_n)$ and $\zeta = (\zeta_1,\dots,\zeta_m)$. Let $\Psi := \psi(K)$ be the subset of K^{n+m} defined by $\psi(\xi,\zeta)$ and let $\Gamma := \gamma(K)$ be the subset of K^n defined by $\gamma(\xi)$. Therefore, if we identify K^{n+m} with the collection of K-rational points of $\mathbb{A}_K^{n+m}$, then

$$\Psi = \Sigma_{\psi(\xi,\zeta)}(K).$$

Moreover, if $p\colon \mathbb{A}_K^{n+m} \to \mathbb{A}_K^n$ is the projection onto the first n coordinates then $p(\Psi) = \Gamma$. By Chevalley's Theorem (see for instance [27, Corollary 14.7] or [39, II. Exercise 3.19]), $p(\Sigma_{\psi(\xi,\zeta)})$ (as a subset in $\mathbb{A}_{\mathbb{Z}}^n$) is again constructible, and therefore, by our previous discussion, of the form $\Sigma_{\chi(\xi)}$ for some quantifier free formula $\chi(\xi)$. Hence $\Gamma = \Sigma_{\chi(\xi)}(K)$, showing that $\gamma(\xi)$ is equivalent modulo K to $\chi(\xi)$. Since $\chi(\xi)$ does not depend on K, we have in fact an equivalence of formulae modulo the class $\mathscr{K}$. To get rid of an arbitrary chain of quantifiers, we use induction on the number of quantifiers, noting that the complement of a set defined by $(\forall\zeta)\,\psi(\xi,\zeta)$ is the set defined by $(\exists\zeta)\neg\psi(\xi,\zeta)$, where $\neg(\cdot)$ denotes negation.

For some alternative proofs, see [57, Corollary A.5.2] or [67, Theorem 1.6]. □

2.3 Łoś' Theorem

Thanks to Quantifier Elimination (Theorem 2.2.1), when dealing with algebraically closed fields, we may forget altogether about formulae and use constructible subsets instead. However, we will not always be able to work just in algebraically closed fields and so we need to formulate a general transfer principle for ultraproducts. For most of our purposes, the following version suffices:

Theorem 2.3.1 (Equational Łoś' Theorem). *Suppose each A_w is an R-algebra, and let $A_\natural$ denote their ultraproduct. Let ξ be an n-tuple of variables, let $f \in R[\xi]$, and let $\mathbf{a}_w$ be n-tuples in A_w with ultraproduct $\mathbf{a}_\natural$. Then $f(\mathbf{a}_\natural) = 0$ in $A_\natural$ if and only if $f(\mathbf{a}_w) = 0$ in A_w for almost all w.*

Moreover, instead of a single equation $f = 0$, we may take in the above statement any system of equations and negations of equations over R.

Proof. Let me only sketch a proof of the first assertion. Suppose $f(\mathbf{a}_\natural) = 0$. One checks (do this!), making repeatedly use of (2.1), that $f(\mathbf{a}_\natural)$ is equal to the ultraproduct of the $f(\mathbf{a}_w)$. Hence the former being zero simply means that almost all $f(\mathbf{a}_w)$ are zero. The converse is proven by simply reversing this argument. □

On occasion, we might also want to use the full version of Łoś' Theorem, which requires the notion of a formula as defined above. Recall that a sentence is a formula without free variables.

Theorem 2.3.2 (Łoś' Theorem). *Let R be a ring and let A_w be R-algebras. If φ is a sentence with parameters from R, then φ holds in almost all A_w if and only if φ holds in the ultraproduct $A_\natural$.*

More generally, let $\varphi(\xi_1,\dots,\xi_n)$ be a formula with parameters from R and let $\mathbf{a}_w$ be an n-tuple in A_w with ultraproduct $\mathbf{a}_\natural$. Then $\varphi(\mathbf{a}_w)$ holds in almost all A_w if and only if $\varphi(\mathbf{a}_\natural)$ holds in $A_\natural$.

The proof is tedious but not hard; one simply has to unwind the definition of formula (see [57, Theorem 9.5.1] for a more general treatment). Note that $A_\natural$ is naturally an R-algebra, so that it makes sense to assert that φ is true or false in $A_\natural$. Applying Łoś' Theorem to a quantifier free formula proves Theorem 2.3.1.

2.4 Ultra-rings

An *ultra-ring* is simply an ultraproduct of rings. Probably the first examples of ultra-rings appearing in the literature are the so-called *non-standard integers*, that is to say, the ultrapowers $\mathbb{Z}_\natural$ of $\mathbb{Z}$,[2] and the *hyper-reals*, that is to say, the ultrapower $\mathbb{R}_\natural$ of the reals, which figure prominently in *non-standard analysis* (see, for instance, [36, 80]). Ultra-rings will be our main protagonists, but for the moment we only establish some very basic facts about them.

2.4.1 Ultra-fields

Let K_w be a collection of fields and $K_\natural$ their ultraproduct, which is again a field by 2.1.5 (or by an application of Łoś' Theorem). Any field which arises in this way is called an *ultra-field*.[3] Since an ultraproduct is either finite or uncountable, $\mathbb{Q}$ is an example of a field which is not an ultra-field.

2.4.1 *If for each prime number p, only finitely many K_w have characteristic p, then $K_\natural$ has characteristic zero.*

Indeed, for every prime number p, the equation $p\xi - 1 = 0$ has a solution in all but finitely many of the K_w and hence it has a solution in $K_\natural$, by Theorem 2.3.1. We will call an ultra-field $K_\natural$ of characteristic zero which arises as an ultraproduct of fields of positive characteristic, a *Lefschetz field* (the name is inspired by Theorem 2.4.3 below); and more generally, an ultra-ring of characteristic zero given as the ultraproduct of rings of positive characteristic will be called a *Lefschetz ring* (see §7.2.1 for more).

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[2] Logicians study these under the guise of *models of Peano arithmetic*, where, instead of $\mathbb{Z}_\natural$, one traditionally looks at the sub-semi-ring $\mathbb{N}_\natural$, the ultrapower of $\mathbb{N}$ (see, for instance, [63]). Caveat: not all non-standard models are realizable as ultrapowers.

[3] In case the K_w are finite but of unbounded cardinality, their ultraproduct $K_\natural$ is also called a *pseudo-finite field*; in these notes, however, we prefer the usage of the prefix *ultra-*, and so we would call such fields instead *ultra-finite fields*.

2.4.2 *If almost all K_w are algebraically closed fields, then so is $K_\natural$.*

The quickest proof is by means of Łoś' Theorem, although one could also give an argument using just Theorem 2.3.1.

Proof. For each $n \geq 2$, consider the sentence σ_n given by

$$(\forall \zeta_0, \ldots, \zeta_n)\,(\exists \xi)\,\zeta_n = 0 \,\vee\, \zeta_n \xi^n + \cdots + \zeta_1 \xi + \zeta_0 = 0.$$

This sentence is true in any algebraically closed field, whence in almost all K_w, and therefore, by Łoś' Theorem, in $K_\natural$. However, a field in which every σ_n holds is algebraically closed. □

We have the following important corollary which can be thought of as a model-theoretic Lefschetz Principle (here $\mathbb{F}_p^{\mathrm{alg}}$ is the algebraic closure of the p-element field $\mathbb{F}_p$; and, more generally, F^{alg} denotes the algebraic closure of a field F).

Theorem 2.4.3 (Lefschetz Principle). *Let W be the set of prime numbers, endowed with some ultrafilter. The ultraproduct of the fields $\mathbb{F}_p^{alg}$ is isomorphic with the field $\mathbb{C}$ of complex numbers, that is to say, we have an isomorphism*

$$\mathbb{C} \cong \operatorname*{ulim}_{p\to\infty} \mathbb{F}_p^{alg}$$

Proof. Let $\mathbb{F}_\natural$ denote the ultraproduct of the fields $\mathbb{F}_p^{\mathrm{alg}}$. By 2.4.2, the field $\mathbb{F}_\natural$ is algebraically closed, and by 2.4.1, its characteristic is zero. Using elementary set theory, one calculates that the cardinality of $\mathbb{F}_\natural$ is equal to that of the continuum. The theorem now follows since any two algebraically closed fields of the same uncountable cardinality and the same characteristic are (non-canonically) isomorphic by Steinitz's Theorem (see [57] or Theorem 2.4.7 below). □

Remark 2.4.4. We can extend the above result as follows: any algebraically closed field K of characteristic zero and cardinality 2^κ, for some infinite cardinal κ, is a Lefschetz field. Indeed, for each p, choose an algebraically closed field K_p of characteristic p and cardinality κ. Since the ultraproduct of these fields is then an algebraically closed field of characteristic zero and cardinality 2^κ, it is isomorphic to K by Steinitz's Theorem (Theorem 2.4.7). Under the generalized Continuum Hypothesis, any uncountable cardinal is of the form 2^κ, and hence any uncountable algebraically closed field of characteristic zero is then a Lefschetz field. We will tacitly assume this, but the reader can check that nowhere this assumption is used in an essential way.

Remark 2.4.5. Theorem 2.4.3 is an embodiment of a well-known heuristic principle in algebraic geometry regarding transfer between positive and zero characteristic, which Weil [113] attributes to Lefschetz. Essentially metamathematical in nature, there have been some attempts to formulate this principle in a formal, model-theoretic language in [10, 28]; for a more general version than ours, see [32, Theorem 8.3]. In fact, Theorem 2.4.3 is a special instance of model-theoretic compactness applied to the theory of algebraically closed fields. For instance, the next result, due to Ax [8], is normally proven using compactness, but here is a proof using Theorem 2.4.3 instead:

2.4.6 *If a polynomial map $\mathbb{C}^n \to \mathbb{C}^n$ is injective, then it is surjective.*

Indeed, by the Pigeon Hole Principle, the result is true if we replace $\mathbb{C}$ by any finite field; since $\mathbb{F}_p^{\mathrm{alg}}$ is a union of finite fields, the assertion remains true over it; an application of Theorem 2.4.3 then finishes the proof. □

Theorem 2.4.7 (Steinitz's Theorem). *If K and L are algebraically closed fields of the same characteristic and the same uncountable cardinality, then they are isomorphic.*

Proof (Sketch of proof). Let k be the common prime field of K and L (that is to say, either $\mathbb{Q}$ in characteristic zero, or $\mathbb{F}_p$ in positive characteristic p). Let Γ and Δ be respective transcendence bases of K and L over k. Since K and L have the same uncountable cardinality, Γ and Δ have the same cardinality, and hence there exists a bijection $f\colon \Gamma \to \Delta$. This naturally extends to a field isomorphism $k(\Gamma) \to k(\Delta)$. Since K is the algebraic closure of $k(\Gamma)$, and similarly, L of $k(\Delta)$, this isomorphism then extends to an isomorphism $K \to L$. □

The previous results might lead the reader to think that the choice of ultrafilter never matters. As we shall see later, for most of our purposes this is indeed true, but there are many situations were the ultrafilter determines the ultraproduct. For instance, consider the ultraproduct of fields F_w, where F_w is either $\mathbb{F}_2$ or $\mathbb{F}_3$. Since almost all F_w are therefore equal to one, and only one, among these two fields, so will their ultraproduct be (to see the latter, note that there is a first-order sentence expressing that a field has exactly two elements, and now use the model-theoretic version of Łoś' Theorem, Theorem 2.3.2). More precisely, the ultraproduct is equal to $\mathbb{F}_2$ if and only if the set I_2 of indices w for which $F_w = \mathbb{F}_2$ belongs to the ultrafilter. If I_2 is infinite, then there exists always an ultrafilter containing it, and if I_2 is also co-finite, then there exists another one not containing I_2, so that in the former case, the ultraproduct is equal to $\mathbb{F}_2$, and in the latter case to $\mathbb{F}_3$. We will prove a theorem below, Theorem 2.5.4, which tells us exactly all possible ultraproducts a given collection of rings can produce (see also Theorem 2.6.4).

2.4.2 Ultra-rings

Let A_w be a collection of rings. Their ultraproduct $A_\natural$ will be called, as already mentioned, an *ultra-ring*.

2.4.8 *If each A_w is local with maximal ideal $\mathfrak{m}_w$ and residue field $k_w := A_w/\mathfrak{m}_w$, then $A_\natural$ is local with maximal ideal $\mathfrak{m}_\natural := \operatorname{ulim}\mathfrak{m}_w$ and residue field $k_\natural := \operatorname{ulim} k_w$.*

Indeed, a ring is local if and only if the sum of any two non-units is again a non-unit. This statement is clearly expressible by means of a sentence, so that by Łoś' Theorem (Theorem 2.3.2), $A_\natural$ is local. Again we can prove this also directly, or using the equational version, Theorem 2.3.1. The remaining assertions now follow easily from 2.1.6. In fact, the same argument shows that the converse is also true: if $A_\natural$ is local, then so are almost all A_w. □

Recall that the *embedding dimension* of a local ring is the minimal number of generators of its maximal ideal. The next result is therefore immediate from 2.1.6 and 2.4.8.

2.4.9 *If A_w are local rings of embedding dimension e, then so is $A_\natural$.* □

As being a domain is captured by the fact that the equation $\xi\zeta = 0$ has no solution by non-zero elements; and being reduced by the fact that the equation $\xi^2 = 0$ has no non-zero solutions, we immediately get from Łoś' Theorem:

2.4.10 *Almost all A_w are domains (respectively, reduced) if and only if $A_\natural$ is a domain (respectively, reduced).* □

In particular, using 2.1.6, we see that an ultraproduct of ideals is a prime (respectively, radical, maximal) ideal if and only if almost all ideals are prime (respectively, reduced, maximal).

2.4.11 *If I_w are ideals in the local rings $(A_w, \mathfrak{m}_w)$, such that in $(A_\natural, \mathfrak{m}_\natural)$, their ultraproduct $I_\natural$ is $\mathfrak{m}_\natural$-primary, then almost all I_w are $\mathfrak{m}_w$-primary.*

Recall that an ideal I in a local ring $(R, \mathfrak{m})$ is called $\mathfrak{m}$*-primary* if its radical is equal to $\mathfrak{m}$. So, $\mathfrak{m}_\natural^N \subseteq I_\natural$ for some N, and therefore, $\mathfrak{m}_w \subseteq I_w$ for almost all w, by Łoś' Theorem. □

Note that here the converse may fail to hold: not every ultraproduct of $\mathfrak{m}_w$-primary ideals need to be $\mathfrak{m}_\natural$-primary (see Proposition 2.4.17 for a partial converse). For instance, the ultraproduct of the $\mathfrak{m}^w$ is no longer $\mathfrak{m}R_\natural$-primary in the ultrapower $R_\natural$ (see 8.1.3). An ideal in an ultra-ring is called an *ultra-ideal*, if it is an ultraproduct of ideals.[4]

2.4.12 *Any finitely generated, or more generally, any finitely related ideal $\mathfrak{a}$ in an ultra-ring $A_\natural$ is an ultra-ideal, and $A_\natural/\mathfrak{a}$ is again an ultra-ring.*

Let $A_\natural$ be the ultraproduct of rings A_w. Recall that an ideal $\mathfrak{a}$ is called *finitely related*, if it is of the form $(I : J)$ with I and J finitely generated. Suppose $I = (f_1, \dots, f_n)A_\natural$ and $J = (g_1, \dots, g_m)A_\natural$. Choose $f_{iw}, g_{iw} \in A_w$ with ultraproduct equal to f_i and g_i respectively, and put

$$\mathfrak{a}_w := ((f_{1w}, \dots, f_{nw})A_w : (g_{1w}, \dots, g_{mw})A_w).$$

[4] In the literature, such ideals are often called *internal* ideals.

It is now an easy exercise on Łoś' Theorem, using 2.1.6, that $\mathfrak{a}$ is the ultraproduct of the $\mathfrak{a}_w$, and $A_\natural/\mathfrak{a}$ the ultraproduct of the $A_w/\mathfrak{a}_w$. □

Not every ideal in an ultra-ring is an ultra-ideal; for an example, see the discussion at the start of §4.2. Another counterexample is provided by the following ideal, which will play an important role in the study of local ultra-rings (see Proposition 2.4.19 for an example).

Definition 2.4.13 (Ideal of Infinitesimals). For an arbitrary local ring $(R,\mathfrak{m})$, define its *ideal of infinitesimals*, denoted $\mathfrak{I}_R$, as the intersection

$$\mathfrak{I}_R := \bigcap_{n\geq 0} \mathfrak{m}^n.$$

The $\mathfrak{m}$-adic topology on R is Hausdorff (=separated) if and only if $\mathfrak{I}_R = 0$. Therefore, we will refer to the residue ring $R/\mathfrak{I}_R$ as the *separated quotient* of R. In commutative algebra, the ideal of infinitesimals hardly ever appears simply because of:

Theorem 2.4.14 (Krull's Intersection Theorem). *If R is a Noetherian local ring, then $\mathfrak{I}_R = 0$.*

Proof. This is an immediate consequence of the Artin-Rees Lemma (for which see [69, Theorem 8.5] or [7, Proposition 10.9]), or of its weaker variant proven in Theorem 8.2.1 below. Namely, for $x \in \mathfrak{I}_R$, there exists, according to the latter theorem, some c such that $xR \cap \mathfrak{m}^c \subseteq x\mathfrak{m}$. Since $x \in \mathfrak{m}^c$ by assumption, we get $x \in x\mathfrak{m}$, that is to say, $x = ax$ with $a \in \mathfrak{m}$. Hence $(1-a)x = 0$. As $1-a$ is a unit in R, we get $x = 0$. □

It would be dishonest to claim that the above yields a non-standard proof of Krull's theorem via Theorem 8.2.1, as the latter proof uses the flatness of cataproducts (Theorem 8.1.15), which is obtained via Cohen's Structure Theorems, and therefore, ultimately relies on Krull's Intersection Theorem. The exact connection between both results is given by Theorem 8.2.3.

Corollary 2.4.15. *In a Noetherian local ring $(R,\mathfrak{m})$, every ideal is the intersection of $\mathfrak{m}$-primary ideals.*

Proof. For $I \subseteq R$ an ideal, an application of Theorem 2.4.14 to the ring R/I shows that I is the intersection of all $I + \mathfrak{m}^n$, and the latter are indeed $\mathfrak{m}$-primary. □

Most local ultra-rings have a non-zero ideal of infinitesimals.

2.4.16 *If R_w are local rings with non-nilpotent maximal ideal, then the ideal of infinitesimals of their ultraproduct $R_\natural$ is non-zero. In particular, $R_\natural$ is not Noetherian.*

Indeed, by assumption, we can find non-zero $a_w \in \mathfrak{m}^w$ (let us for the moment assume that the index set is equal to $\mathbb{N}$) for all w. Hence their ultraproduct $a_\natural$ is non-zero and lies inside $\mathfrak{I}_{R_\natural}$. □

As we shall see later, being Noetherian is not preserved under ultraproducts. However, under certain restrictive conditions, of which the field case (2.1.5) is a special instance, we do have preservation (this also gives a more quantitative version of 2.4.11):

Proposition 2.4.17. *An ultraproduct $A_\natural$ of rings A_w is Artinian of length l if and only if almost all A_w are Artinian of length l.*

Proof. By the Jordan-Holder Theorem, there exist elements $a_0 = 0, a_1, \dots, a_l = 1$ in $A_\natural$ such that

$$a_0 A_\natural \subsetneq (a_0, a_1)A_\natural \subsetneq (a_0, a_1, a_2)A_\natural \subsetneq \cdots \subsetneq (a_0, \dots, a_l)A_\natural = A_\natural$$

is a maximal chain of ideals. Choose, for each $i = 0, \dots, l$, elements $a_{iw} \in A_w$ whose ultraproduct is a_i. By Łoś' Theorem, for a fixed $i < l$, almost all inclusions

$$(a_{0w}, \dots, a_{iw})A_w \subseteq (a_{0w}, \dots, a_{i+1w})A_w \tag{2.3}$$

are strict. This shows that almost all A_w have length at least l. If almost all of them would have length bigger than l, then for at least one i, we can insert in almost all inclusions (2.3) an ideal I_w different from both ideals. By Łoś' Theorem, the ultraproduct $I_\natural$ of the I_w would then be strictly contained between $(a_0, \dots, a_i)A_\natural$ and $(a_0, \dots, a_{i+1})A_\natural$, implying that $A_\natural$ has length at least $l+1$, contradiction. □

Proposition 2.4.18. *An* ultra-Dedekind domain*, that is to say, an ultraproduct of Dedekind domains, is a Prüfer domain.*

Proof. Recall that a domain is *Prüfer* if any localization at a maximal ideal is a valuation ring. By [34, §1.4], this equivalent with the property that every finitely generated ideal is projective, and so we verify the latter. Let A_w be Dedekind domains, that is to say, one-dimensional normal domains, and let $A_\natural$ be their ultraproduct. Let $I_\natural$ be a finitely generated ideal. By 2.4.12, we can find non-zero ideals $I_w \subseteq A_w$ such that their ultraproduct equals $I_\natural$. Since each I_w is generated by at most two elements we can find a split exact sequence

$$0 \to J_w \to A_w^2 \to I_w \to 0$$

for some submodule $J_w \subseteq A_w^2$. Since ultraproducts commute with direct sums, we get an isomorphism $I_\natural \oplus J_\natural \cong A_\natural^2$, where $J_\natural$ is the ultraproduct of the J_w, showing that $I_\natural$ is projective. □

Proposition 2.4.19. *An* ultra-discrete valuation ring $V_\natural$*, that is to say, an ultraproduct of discrete valuation rings V_w, is a valuation domain. Its ideal of infinitesimals $\mathfrak{I}_{V_\natural}$ is an infinitely generated prime ideal, and the separated quotient $V_\natural/\mathfrak{I}_{V_\natural}$—in Chapter 8 we will call this the* cataproduct $V_\sharp$ *of the V_w—is again a discrete valuation ring.*

Proof. Recall that a *valuation ring* is a domain such that for all a in the field of fractions of V, at least one of a or $1/a$ belongs to V. By 2.1.5, the field of fractions $K_\natural$ of $V_\natural$ is the ultraproduct of the field of fractions K_w of the V_w. Let $a_w \in K_w$ be an approximation of $a \in K_\natural$. For almost each w, either a_w or $1/a_w$ belongs to V_w. Therefore, by Łoś' Theorem, either $a \in V_\natural$ or $1/a \in V_\natural$, proving the first claim. If $\mathfrak{I}_{V_\natural}$ is finitely generated, then it is principal, say, of the form $b_\natural V_\natural$, since $V_\natural$ is a valuation domain. Let $b_w \in V_w$ be an approximation of $b_\natural$, and let $c_w := b_w/\pi_w$, where π_w is a uniformizing parameter of V_w. Since, for each n, almost all b_w have order at least n, almost all c_w have order at least $n-1$. Hence their ultraproduct $c_\natural$ also belongs to $\mathfrak{I}_{V_\natural} = b_\natural V_\natural$. Let $\pi_\natural$ be the ultraproduct of the π_w, so that it generates the maximal ideal $\mathfrak{m}_\natural$ of $V_\natural$ by the proof of 2.4.9. By Łoś' Theorem, $b_\natural/\pi_\natural = c_\natural$, so that $b_\natural \in c_\natural \mathfrak{m}_\natural \subseteq b_\natural \mathfrak{m}_\natural$, contradiction. Finally, to show that $V_\sharp := V_\natural/\mathfrak{I}_{V_\natural}$ is a discrete valuation ring, and hence, in particular, $\mathfrak{I}_{V_\natural}$ is prime, observe that for any non-zero element a in $V_\sharp$, there is a largest n such that $a \in \mathfrak{m}_\natural^n = \pi_\natural^n V_\natural$. The assignment $a \mapsto n$ is now easily seen to be a discrete valuation. □

The previous proof in fact shows that an ultraproduct of valuation rings is again a valuation ring.

2.4.3 Ultrapowers

An important instance of an ultra-ring is the ultrapower $A_\natural$ of a ring A. It is easy to see that the diagonal embedding $A \to A_\natural$ is a ring homomorphism. We will see in the next chapter that this embedding is often flat (see Corollary 3.3.3 and Theorem 3.3.4). However, an easy application of Łoś' Theorem immediately yields that this map is at least cyclically pure. Recall that a homomorphism $A \to B$ is called *cyclically pure*, if $IB \cap A = I$ for all ideals $I \subseteq A$. Examples of cyclically pure homomorphisms are, as we shall see, faithfully flat (Proposition 3.2.5) and split maps (see 5.5.4). It follows from Proposition 2.4.17 that the ultrapower of an Artinian ring is again Artinian. However, by 2.4.16 and Theorem 2.4.14, these are the only rings whose ultrapower is Noetherian. The next result is immediate from 2.1.6 and its proof:

2.4.20 *If I is a finitely generated ideal in a ring A, then its ultrapower in the ultrapower $A_\natural$ of A is equal to $IA_\natural$. In particular, the ultrapower of A/I is $A_\natural/IA_\natural$.*

The following is a counterexample if I is not finitely generated: let A be the polynomial ring over a field in countably many variables ξ_i, and let I be the ideal generated by all these variables. The ultraproduct f of the polynomials $f_w := \xi_1 + \dots + \xi_w$ is an element in the ultrapower $I_\natural$ of I but does not belong to $IA_\natural$, for if it were, then f must be a sum of finitely many generators of I, say, $\xi_1, \dots, \xi_i$, and therefore by 2.1.6, so must almost all f_w be, a contradiction whenever $w > i$.

2.4.4 Ultra-exponentiation

Let $A_\natural$ be an ultra-ring, given as the ultraproduct of rings A_w. Let $\mathbb{N}_\natural$ be the ultrapower of the natural numbers, and let $\alpha \in \mathbb{N}_\natural$ with approximations α_w. The *ultra-exponentiation map* on A with exponent α is defined as follows. Given $x \in A$, let $x_w \in A_w$ be an approximation of x, that is to say, $\operatorname{ulim} x_w = x$, and set

$$x^\alpha := \operatorname{ulim} x_w^{\alpha_w}.$$

One easily verifies that this definition does not depend on the choice of approximation of x nor of α: if x'_w and α'_w are also respectively approximations of x and α, then almost all x_w and x'_w are the same, and so are almost all α_w and α'_w, whence almost all $x_w^{\alpha_w}$ are equal to $(x'_w)^{\alpha'_w}$, and, therefore, they have the same ultraproduct. By Łoś' Theorem, ultra-exponentiation satisfies the same rules as regular exponentiation:

$$(xy)^\alpha = x^\alpha \cdot y^\alpha \quad \text{and} \quad x^\alpha \cdot x^\beta = x^{\alpha+\beta} \quad \text{and} \quad (x^\alpha)^\beta = x^{\alpha\cdot\beta}$$

for all $x, y \in A_\natural$ and all $\alpha, \beta \in \mathbb{N}_\natural$.

If A is local and x a non-unit, then x^α is an infinitesimal for any α in $\mathbb{N}_\natural$ not in $\mathbb{N}$. In these notes, the most important instance will be the ultra-exponentiation map obtained as the ultraproduct of Frobenius maps. More precisely, let $A_\natural$ be a Lefschetz ring, say, realized as the ultraproduct of rings A_p of characteristic p (here we assumed for simplicity that the underlying index set is just the set of prime numbers, but this is not necessary). On each A_p, we have an action of the *Frobenius*, given as $\mathbf{F}_p(x) := x^p$ (for more, see §5.1).

Definition 2.4.21 (Ultra-Frobenius). The ultraproduct of these Frobenii yields an endomorphism $\mathbf{F}_\natural$ on $A_\natural$, called the *ultra-Frobenius*, given by $\mathbf{F}_\natural(x) := x^\pi$, where $\pi \in \mathbb{N}_\natural$ is the ultraproduct of all prime numbers. Since each Frobenius is an endomorphism, so is any ultra-Frobenius by 2.1.7. In particular, we have

$$(x+y)^\pi = x^\pi + y^\pi$$

for all $x, y \in A_\natural$.

2.5 Algebraic Definition of Ultra-rings

Let A_w, for $w \in W$, be rings with Cartesian product $A_\infty := \prod_w A_w$ and direct sum $A_{(\infty)} := \bigoplus A_w$. Note that $A_{(\infty)}$ is an ideal in A_∞. Call an element $a \in A_\infty$ a *strong idempotent* if each of its entries is either zero or one. In other words, an element in A_∞ is a strong idempotent if and only if it is the characteristic function 1_D of a subset $D \subseteq W$. For any ideal $\mathfrak{a} \subseteq A_\infty$, let $\mathfrak{a}^\circ$ be the ideal generated by all strong idempotents in $\mathfrak{a}$, and let $\mathfrak{W}_\mathfrak{a}$ be the collection of subsets $D \subseteq W$ such that $1-1_D \in \mathfrak{a}$. Using the identities $(1-1_D)(1-1_E) = 1-1_E$ for $D \subseteq E$ and $1-1_{D\cap E} = 1_E(1-1_D)+1-1_E$, one verifies that $\mathfrak{W}_\mathfrak{a}$ is a filter.

2.5.1 *Given an ideal $\mathfrak{a} \subseteq A_\infty$, the filter $\mathfrak{W}_\mathfrak{a}$ is maximal if and only if $\mathfrak{a}$ is a prime ideal; it is principal if and only if the ideal $\mathfrak{a}^\circ$ is principal, if and only if $\mathfrak{a}$ does not contain the ideal $A_{(\infty)}$.*

Indeed, given an idempotent e, its *complement* $1-e$ is again idempotent, and the product of both is zero, that is to say, they are orthogonal. It follows that any prime ideal contains exactly one among e and $1-e$. Hence, if $\mathfrak{a}$ is prime, then $\mathfrak{W}_\mathfrak{a}$ consists of those subsets $D \subseteq W$ such that $1_D \notin \mathfrak{a}$. Since $1-1_D$ is the characteristic function of the complement of D, it follows that either D or its complement belongs to $\mathfrak{W}_\mathfrak{a}$. Moreover, if $D \in \mathfrak{W}_\mathfrak{a}$ and $D \subseteq E$, then $1_D \cdot 1_E = 1_D$ does not belong to $\mathfrak{a}$, whence neither does 1_E, showing that $E \in \mathfrak{W}_\mathfrak{a}$. This proves that $\mathfrak{W}_\mathfrak{a}$ is a maximal filter. It is not hard to see that if $\mathfrak{a}^\circ$ is principal, then it must be generated by the characteristic function of the complement of a singleton, and hence $\mathfrak{W}_\mathfrak{a}$ must be principal (the other direction is immediate). The last equivalence is left as an exercise to the reader. □

We can now formulate the following entirely algebraic characterization of an ultra-ring.

2.5.2 *Let $\mathfrak{P}$ be a prime ideal of A_∞ containing the direct sum ideal $A_{(\infty)}$. The ultraproduct of the A_w with respect to the ultrafilter $\mathfrak{W}_\mathfrak{P}$ is equal to $A_\infty/\mathfrak{P}^\circ$, that is to say, $\mathfrak{P}^\circ$ is the null-ideal determined by $\mathfrak{W}_\mathfrak{P}$. Furthermore, any ultra-ring having the A_w as approximations is of the form $A_\infty/\mathfrak{P}^\circ$, for some prime ideal $\mathfrak{P}$ containing $A_{(\infty)}$.*

Let $\mathfrak{n}$ be the null-ideal determined by $\mathfrak{W}_\mathfrak{P}$, that is to say, the collection of sequences in A_∞ almost all of whose entries are zero. If $D \in \mathfrak{W}_\mathfrak{P}$, then almost all entries of $1-1_D$ are zero, and hence $1-1_D \in \mathfrak{n}$. Since this is a typical generator of $\mathfrak{P}^\circ$, we get $\mathfrak{P}^\circ \subseteq \mathfrak{n}$. Conversely, suppose $a = (a_w) \in \mathfrak{n}$. Hence $a_w = 0$ for all w belonging to some $D \in \mathfrak{W}_\mathfrak{P}$. Since $1-1_D \in \mathfrak{P}^\circ$ and $a = a(1-1_D)$, we get $a \in \mathfrak{P}^\circ$.
Conversely, if $\mathfrak{W}$ is an ultrafilter with corresponding null-ideal $\mathfrak{n} \subseteq A_\infty$, then one easily checks that any prime ideal $\mathfrak{P}$ containing $\mathfrak{n}$ satisfies $\mathfrak{n} = \mathfrak{P}^\circ$. □

In fact, if $\mathfrak{P} \subseteq \mathfrak{Q}$ are prime ideals, then $\mathfrak{P}^\circ = \mathfrak{Q}^\circ$, showing that already all minimal prime ideals of A_∞ determine all possible ultrafilters.

Corollary 2.5.3. *If all A_w are domains, then $A_\natural$ is the coordinate ring of an irreducible component of* $\operatorname{Spec}(A_\infty/A_{(\infty)})$. *More precisely, the residue rings $A_\infty/\mathfrak{G}$, for $\mathfrak{G} \subseteq A_\infty$ a minimal prime containing $A_{(\infty)}$, are precisely the ultraproducts $A_\natural$ having the domains A_w for approximations. Moreover, these irreducible components are then also the connected components of* $\operatorname{Spec}(A_\infty/A_{(\infty)})$, *that is to say, they are mutually disjoint.*

Proof. Since the ultraproduct $A_\natural$ determined by $\mathfrak{G}$ is equal to $A_\infty/\mathfrak{G}^\circ$ by 2.5.2, and a domain by 2.4.10, the ideal $\mathfrak{G}^\circ$ must be prime. By minimality, $\mathfrak{G}^\circ = \mathfrak{G}$.

To prove the last assertion, let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ be two distinct minimal prime ideals of A_∞ containing $A_{(\infty)}$. Suppose $\mathfrak{G}_1 + \mathfrak{G}_2$ is not the unit ideal. Hence there exists a maximal ideal $\mathfrak{M} \subseteq A_\infty$ such that $\mathfrak{G}_1, \mathfrak{G}_2 \subseteq \mathfrak{M}$, and hence

$$\mathfrak{G}_1 = \mathfrak{G}_1^\circ = \mathfrak{M}^\circ = \mathfrak{G}_2^\circ = \mathfrak{G}_2,$$

contradiction. Hence $\mathfrak{G}_1 + \mathfrak{G}_2 = 1$, showing that any two irreducible components of $\operatorname{Spec}(A_\infty/A_{(\infty)})$ are disjoint. □

Note that the connected components of $\operatorname{Spec}(A_\infty)$, apart from the $\operatorname{Spec}(A_\natural)$, are the $\operatorname{Spec}(A_w)$ corresponding to the principal (maximal) filters. In the following structure theorem, $\mathbb{Z}_\infty := \mathbb{Z}^W$ denotes the Cartesian power of $\mathbb{Z}$. Any Cartesian product $A_\infty := \prod A_w$ is naturally a $\mathbb{Z}_\infty$-algebra.

Theorem 2.5.4. *Any ultra-ring is a base change of a ring of non-standard integers $\mathbb{Z}_\natural$. More precisely, the ultra-rings with approximation A_w are precisely the rings of the form $A_\infty/\mathfrak{G}A_\infty$, where $\mathfrak{G}$ is a minimal prime of $\mathbb{Z}_\infty$ containing the direct sum ideal.*

Proof. If $\mathfrak{P}$ is a prime ideal in A_∞ containing the direct sum ideal $A_{(\infty)}$, then the generators of $\mathfrak{P}^\circ$ already live in $\mathbb{Z}_\infty$, and generate the null-ideal in $\mathbb{Z}_\infty$ corresponding to the ultrafilter $\mathfrak{W}_{\mathfrak{P}}$. By Corollary 2.5.3, the latter ideal therefore is a minimal prime ideal $\mathfrak{G} \subseteq \mathbb{Z}_\infty$ of $\mathbb{Z}_{(\infty)}$. Since $\mathfrak{G}A_\infty = \mathfrak{P}^\circ$, one direction is clear from 2.5.2. Conversely, again by Corollary 2.5.3, any minimal prime ideal $\mathfrak{G} \subseteq \mathbb{Z}_\infty$ is the null-ideal determined by the ultrafilter $\mathfrak{W}_{\mathfrak{G}}$, and one easily checks that the same is therefore true for its extension $\mathfrak{G}A_\infty$. □

2.6 Sheaf-theoretic Definition of Ultra-rings

We say that a topological Hausdorff space X admits a *Hausdorff compactification* $X^\vee$, if $X \subseteq X^\vee$ such that for every compact Hausdorff space Y and every continuous map $f\colon X \to Y$, there is a unique map $f^\vee\colon X^\vee \to Y$ extending f. Since this is a universal problem, a Hausdorff compactification is unique, if it exists.

Proposition 2.6.1. *Every infinite discrete space X has a Hausdorff compactification.*

Proof. Let $X^\vee$ be the *Stone-Čech compactification* of X consisting of all maximal filters on X. We identify the principal filters with their generators, so that X becomes a subset of $X^\vee$. For a subset $U \subseteq X$, let $\tau(U) \subseteq X^\vee$ consist of all maximal filters containing U. For any $U \subseteq X$, we have

$$X^\vee - \tau(U) = \tau(X - U), \tag{2.4}$$

by the ultrafilter condition. We define a topology on X by taking the $\tau(U)$, for $U \subseteq X$, as a basis of open subsets. This works, since the intersection of two basic opens $\tau(U_1)$ and $\tau(U_2)$ is the basic open $\tau(U_1 \cap U_2)$. Note that $U \subseteq \tau(U)$ with equality if and only if U is finite. In fact, by the definition of the embedding $X \subseteq X^\vee$, we have $\tau(U) \cap X = U$, and hence the topology induced on X is just the discrete topology. In particular, every non-empty open has a non-empty intersection with X, showing that X is a dense (open) subset of $X^\vee$.

To see that $X^\vee$ is Hausdorff, take two distinct points in $X^\vee$, that is to say, distinct maximal filters on X. In particular, there exists a subset $U \subseteq X$ belonging to one but not the other. Hence $\tau(U)$ and $\tau(X - U)$ are disjoint opens, each containing exactly one of these two points. To prove compactness, we need to verify that the finite intersection property holds, that is to say, that any collection of non-empty closed subsets which is closed under finite intersections, has non-empty intersection. By (2.4), any basic open subset is also closed, that is to say, is a *clopen*, and hence any closed subset is an intersection of basic opens. Without loss of generality, we may therefore assume that $\{\tau(U_i)\}_i$ is a collection of non-empty closed subsets which is closed under finite intersections, and we have to show that their intersection is also non-empty. Since $X \cap \tau(U_i) = U_i$, it follows that the $\{U_i\}_i$ are closed under finite intersections. Let $\mathfrak{Y}$ be the collection of all subsets $U \subseteq X$ such that some U_i is contained in U. One checks that $\mathfrak{Y}$ is a filter, whence is contained in some maximal filter $\mathfrak{W}$. By construction, $U_i \in \mathfrak{W}$, for all i, showing that $\mathfrak{W}$ lies in the intersection of all $\tau(U_i)$.

Finally, we verify the universal property. Let $f\colon X \to Y$ be an (automatically continuous) map with Y a compact Hausdorff space and fix a point in $X^\vee$, that is to say, a maximal filter $\mathfrak{W}$ on X. Let $F_{\mathfrak{W}}$ be the intersection of all closures $\operatorname{clos}(f(U))$, where U runs over all subsets in $\mathfrak{W}$. Since any finite intersection

$$\operatorname{clos}(f(U_1)) \cap \cdots \cap \operatorname{clos}(f(U_s)),$$

for $U_i \in \mathfrak{W}$ contains the (non-empty) image of $U_1 \cap \cdots \cap U_s \in \mathfrak{W}$ under f, and since Y is compact, $F_{\mathfrak{W}}$ is non-empty. Suppose y and y' are two distinct elements in $F_{\mathfrak{W}}$. Since Y is Hausdorff, we can find disjoint opens V and V' containing respectively y and y'. In particular, their pre-images $f^{-1}V$ and $f^{-1}V'$ are disjoint, and so one of them, say $f^{-1}V$ cannot belong to $\mathfrak{W}$. It follows that $X - f^{-1}V$ belongs to $\mathfrak{W}$ and hence $F_{\mathfrak{W}}$ is contained in the closure of $f(X - f^{-1}V) = f(X) - V$. Since V is an open containing $y \in F_{\mathfrak{W}}$, it must therefore have non-empty intersection with $f(X) - V$, contradiction. Hence $F_{\mathfrak{W}}$ is a singleton, and we now define $f^\vee(\mathfrak{W})$ to be the unique element belonging to $F_{\mathfrak{W}}$. Immediate from the definitions we get that $f^\vee(\mathfrak{W}) = f(\mathfrak{W})$ in case $\mathfrak{W} \in X$, that is to say, is principal. So remains to show that $f^\vee$ is continuous.

To this end, let $V \subseteq Y$ be open and $\mathfrak{W} \in X^\vee$ a point in $(f^\vee)^{-1}(V)$. We need to find an open containing $\mathfrak{W}$ and contained in $(f^\vee)^{-1}(V)$. By construction, the intersection of all $\operatorname{clos}(f(U))$ with $U \in \mathfrak{W}$ is contained in V. By compactness, already finitely many of the $\operatorname{clos}(f(U))$ have an intersection contained in V (since their complements together with V form an open cover of Y). Letting $U \in \mathfrak{W}$ be the intersection of these finitely many members of $\mathfrak{W}$, then, as above, $\operatorname{clos}(f(U)) \subseteq V$. To see that $\tau(U) \subseteq (f^\vee)^{-1}(V)$, take $\mathfrak{Y} \in \tau(U)$. So $U \in \mathfrak{Y}$ and hence, per construction, $f^\vee(\mathfrak{Y}) \in \operatorname{clos}(f(U)) \subseteq V$. □

If U is infinite, then intersecting each set in $\mathfrak{W} \in \tau(U)$ with U yields a maximal filter on U, so that we get an induced map $\tau(U) \to U^\vee$. It is not hard to show that this is in fact an homeomorphism. Since U is dense in $U^\vee$, we showed that the closure of U in $X^\vee$ is just $\tau(U) = U^\vee$. Let A_w be rings, indexed by $w \in X$. Define a sheaf of rings $\mathscr{A}$ on X by taking for stalk $\mathscr{A}_w := A_w$ in each point $w \in X$ (note that since X is discrete, this completely determines the sheaf $\mathscr{A}$). Let $i\colon X \to X^\vee$ be the above embedding and let $\mathscr{A}^\vee := i_*\mathscr{A}$ be the direct image sheaf of $\mathscr{A}$ under i. By general sheaf theory, this is a sheaf on $X^\vee$. For instance, on a basic open $\tau(U)$ the ring of sections of $\mathscr{A}^\vee$ is $\mathscr{A}(\tau(U) \cap X) = \mathscr{A}(U)$, and the latter is just the Cartesian product of all A_w for $w \in U$.

2.6.2 *The stalk of $\mathscr{A}^\vee$ in a boundary point $\mathfrak{W} \in X^\vee - X$ is isomorphic to the ultraproduct* $\operatorname{ulim} A_w$ *with respect to the ultrafilter $\mathfrak{W}$.*

Indeed, by definition of stalk, $\mathscr{A}^\vee_{\mathfrak{W}}$ is the direct limit of all $\mathscr{A}^\vee(V)$ where V runs over all open subsets of $X^\vee$ containing $\mathfrak{W}$. It suffices to take the direct limit over all basic opens $\tau(U)$ containing $\mathfrak{W}$, that is to say, for $U \in \mathfrak{W}$. Now, as we already observed above,

$$\mathscr{A}^\vee(\tau(U)) = \mathscr{A}(U) = \prod_{w \in U} A_w \cong A_\infty/(1 - 1_U)A_\infty.$$

Hence this direct limit is equal to the residue ring of A_∞ modulo the ideal generated by all $1 - 1_U$ for $U \in \mathfrak{W}$, that is to say, by 2.5.2, modulo the null-ideal corresponding to $\mathfrak{W}$. □

2.6.3 *Under the identification of $\mathscr{P}(X)$ with the Cartesian power $(\mathbb{F}_2)_\infty$ (see Example B.2.3), the assignment $\mathfrak{p} \mapsto \mathfrak{W}_\mathfrak{p}$ defined in §2.5 yields a homeomorphism between the affine scheme* $\operatorname{Spec}(\mathscr{P}(X))$ *and $X^\vee$. Infinitely generated prime ideals then correspond to ultrafilters.*

The only thing to observe is that the inverse image of the basic open $\tau(U)$ for $U \subseteq X$ is the basic open $D(1_{-U})$ in $\operatorname{Spec}(\mathscr{P}(X))$. Note that if we view $X^\vee$ as the set of maximal filters, and hence as a subset of $\mathscr{P}(X)$, then $\mathfrak{p}$ is sent to its complement under this homeomorphism. □

Let us call a scheme X *Boolean* if it admits an open covering by affine schemes of the form $\operatorname{Spec} B$ with B a Boolean ring (see Proposition B.1.5 for some basic properties of Boolean rings). Equivalently, any section ring is Boolean, and this is also equivalent by (B.1.5.vii) to all stalks having two elements. In particular, $\operatorname{Spec}(\mathscr{P}(W))$ is Boolean, for any set W. We call $x \in X$ a *finite point* if the prime ideal associated to x is finitely generated, whence principal by (B.1.5.iii); in the remaining case, we call x an *infinite point*. By (B.1.5.x), the infinite points form a closed subset with ideal of definition the ideal generated by all atoms. We call X *atomless* if every point is infinite, and by (B.1.5.x) this is equivalent with any section ring of an open subset being atomless. The dichotomy between finite and infinite points is robust by Corollary B.1.8, in the sense given in 2.6.5 below.

Theorem 2.6.4. *Let $\mathscr{A}$ be a sheaf of rings on a Boolean scheme X. If $x \in X$ is infinite, then the stalk $\mathscr{A}_x$ is an ultra-ring. If X is the affine scheme of a power set ring $\mathscr{P}(W)$, then $\mathscr{A}_x$ is the ultraproduct of the stalks $\mathscr{A}_y$ at finite points $y \in X$ with respect to the ultrafilter given as the image of x under the homeomorphism $X \cong W^\vee$ from 2.6.3.*

Proof. Let us first show this in the case X is $\operatorname{Spec}(\mathscr{P}(W))$. Infinite points correspond to ultrafilters by 2.6.3, and the result follows by 2.6.2. In the general case, since stalks are local objects, we may assume that X is an affine scheme with Boolean coordinate ring B. By the Stone Representation Theorem (see Theorem B.2.7 below for a proof), there exists a faithfully flat embedding $B \subseteq C := \mathscr{P}(W)$ for some W (we will actually show that one can take W equal to an ultrapower of $\mathbb{N}$). Since x is infinite, B must be infinite by (B.1.5.viii), whence so must W be. Let $Y := \operatorname{Spec}(C)$. We need:

2.6.5 *If $f\colon Y \to X$ is a dominant morphism of Boolean schemes and $x \in X$ is infinite, then there exists an infinite $y \in Y$ with $f(y) = x$.*

Indeed, we may reduce to the affine case, in which case we have an injective homomorphism $B \to C$ between Boolean rings and a non-principal maximal ideal $\mathfrak{p} \subseteq B$ corresponding to x. The fiber $f^{-1}(x)$ has coordinate ring $C/\mathfrak{p}C$. If $C/\mathfrak{p}C$ is infinite, then it contains a non-principal maximal ideal by (B.1.5.viii), and its pre-image in C must then also be non-principal, so that we are done in this case. So assume $C/\mathfrak{p}C$ is finite and any maximal ideal containing $\mathfrak{p}C$ is principal. Since $C/\mathfrak{p}C$ is finite, $\mathfrak{p}C$ is the intersection of the finitely many maximal ideals containing $\mathfrak{p}$ by (B.1.5.vi). Hence $\mathfrak{p}C$ is an intersection of principal ideals whence is principal by (B.1.5.iii). Since $B \to C$ is an embedding, $\mathfrak{p}$ must be principal by Corollary B.1.8, contradiction. □

So, returning to the case at hand, there exists an infinite $y \in Y$ such that $f(y) = x$. Let $f^{-1}\mathscr{A}$ be the inverse image of $\mathscr{A}$ under the morphism $f\colon Y \to X$. Since $(f^{-1}\mathscr{A})_y$ is isomorphic to $\mathscr{A}_x$ and the former is an ultra-ring by the above, so is therefore the latter. □

In particular, any stalk over an atomless Boolean scheme is an ultra-ring!

Chapter 3
Flatness

To effectively apply ultraproducts to commutative algebra, we will use, as our main tool, flatness. Since it is neither as intuitive nor as transparent as many other concepts from commutative algebra, we review quickly some basic facts, and then discuss some flatness criteria that will be used later on. Flatness is an extremely important and versatile property, which underlies many deeper results in commutative algebra and algebraic geometry. In fact, I dare say that many a theorem or conjecture in commutative algebra can be recast as a certain flatness result; an instance is Proposition 6.4.6. With David Mumford, the great geometer, we observe:

> "The concept of flatness is a riddle that comes out of algebra, but which technically is the answer to many prayers."
>
> [22, p. 214]

3.1 Flatness

Flatness is in essence a homological notion, so we start off with reviewing some homological algebra.

3.1.1 Complexes

Recall that by a *complex*, over some ring A, we mean a (possibly infinite) sequence of A-module homomorphisms $M_i \xrightarrow{d_i} M_{i-1}$, for $i \in \mathbb{Z}$, such that the composition of any two consecutive maps is zero. We often simply will say that

$$\dots \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} M_{i-2} \xrightarrow{d_{i-2}} \dots \qquad (M_\bullet)$$

is a complex. The d_i are called the *boundary maps* of the complex, and often are omitted from the notation. Of special interest are those complexes in which all

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modules from a certain point on, either on the left or on the right, are zero (which forces the corresponding maps to be zero as well). Such a complex will be called *bounded* from the left or right respectively. In that case, one often renumbers so that the first non-zero module is labeled with $i=0$. If $M_\bullet$ is bounded from the left, one also might reverse the numbering, indicate this notationally by writing $M^\bullet$, and refer to this situation as a *co-complex* (and more generally, add for emphasis the prefix 'co-' to any object associated to it).

3.1.2 Homology

Since the composition $d_{i+1} \circ d_i$ is zero, we have in particular an inclusion $\mathrm{Im}(d_{i+1}) \subseteq \mathrm{Ker}(d_i)$. To measure in how far this fails to be an equality, we define the *homology* $\mathrm{H}_\bullet(M_\bullet)$ of $M_\bullet$ as the collection of modules

$$\mathrm{H}_i(M_\bullet) := \mathrm{Ker}(d_i)/\mathrm{Im}(d_{i+1}).$$

If all homology modules are zero, $M_\bullet$ is called *exact*. More generally, we say that $M_\bullet$ is *exact at i* (or at M_i) if $\mathrm{H}_i(M_\bullet) = 0$. Note that $M_1 \xrightarrow{d_1} M_0 \to 0$ is exact (at zero) if and only if d_1 is surjective, and $0 \to M_0 \xrightarrow{d_0} M_{-1}$ is exact if and only if d_0 is injective. An exact complex is often also called an *exact sequence*. In particular, this terminology is compatible with the nomenclature for short exact sequences. If $M_\bullet$ is bounded from the right (indexed so that the last non-zero module is M_0), then the *cokernel* of $M_\bullet$ is the cokernel of $d_1\colon M_1 \to M_0$. Put differently, the cokernel is simply the zero-th homology module $\mathrm{H}_0(M_\bullet)$. We say that $M_\bullet$ is *acyclic*, if all $\mathrm{H}_i(M_\bullet) = 0$ for $i > 0$. In that case, the *augmented* complex obtained by adding the cokernel of $M_\bullet$ to the right is then an exact sequence.

We will use the following property of ultraproducts on occasion, and although its proof is straightforward, it is instructive for learning to work with ultraproducts:

Theorem 3.1.1. *[Ultraproduct Commutes with Homology] For each w, let $M_{\bullet w}$ be a complex over a ring A_w and let $M_{\bullet\natural}$ and $A_\natural$ be the respective ultraproducts. Then $M_{\bullet\natural}$ is a complex over $A_\natural$ and its i-th homology $\mathrm{H}_i(M_{\bullet\natural})$ is isomorphic to the ultraproduct of the i-th homologies $\mathrm{H}_i(M_{\bullet w})$.*

Proof. It suffices to prove this at a fixed spot i, and so we may assume that $M_{\bullet w}$ is the complex

$$F_w \xrightarrow{e_w} G_w \xrightarrow{d_w} H_w.$$

Taking ultraproducts, we get a diagram $M_{\bullet\natural}$ of homomorphism of $A_\natural$-modules (we leave it to the reader to verify that the ultraproduct construction extends to the category of modules):

$$F_\natural \xrightarrow{e_\natural} G_\natural \xrightarrow{d_\natural} H_\natural$$

and it is an easy exercise on Łoś' Theorem that $d_\natural \circ e_\natural = 0$ since all $d_w \circ e_w = 0$. In other words, $M_{\bullet\natural}$ is a complex. Let I_w and Z_w be respectively the image of e_w and the kernel of d_w, and let $I_\natural$ and $Z_\natural$ be their respective ultraproducts. The homology of $M_{\bullet w}$ is given by Z_w/I_w, and we have to show that the homology of $M_{\bullet\natural}$ is isomorphic to the ultraproduct of the Z_w/I_w. An element $x_\natural \in G_\natural$ with approximations $x_w \in G_w$ belongs to $Z_\natural$ (respectively, to $I_\natural$) if and only if almost all $d_w(x_w) = 0$ in H_w (respectively, there exist $y_w \in F_w$ such that $x_w = e_w(y_w)$ for almost all w) if and only if $d_\natural(x_\natural) = 0$, that is to say, $x_\natural$ lies in the kernel of $d_\natural$ (respectively, $x_\natural = e_\natural(y_\natural)$ where $y_\natural \in F_\natural$ is the ultraproduct of the y_w, that is to say, $x_\natural$ lies in the image of $e_\natural$). Since the ultraproduct of the Z_w/I_w is isomorphic to $Z_\natural/I_\natural$ by the module analogue of 2.1.6, the claim follows. □

3.1.3 Flatness

Let A be a ring and M an A-module. Recall that $\cdot \otimes_A M$, that is to say, tensoring with respect to M, is a right exact functor, meaning that given an exact sequence

$$0 \to N_2 \to N_1 \to N_0 \to 0 \tag{3.1}$$

we get an exact sequence

$$N_2 \otimes_A M \to N_1 \otimes_A M \to N_0 \otimes_A M \to 0. \tag{3.2}$$

See [7, Proposition 2.18], where one also can find a good introduction to tensor products. We now call a module M *flat* if any short exact sequence (3.1) remains exact after tensoring, that is to say, we may add an additional zero on the left of (3.2). Put differently, M is flat if and only if $N' \otimes_A M \to N \otimes_A M$ is injective whenever $N' \to N$ is an injective homomorphism of A-modules. By breaking down a long exact sequence into short exact sequences, we immediately get:

3.1.2 *An exact complex remains exact after tensoring with a flat module.*

Well-known examples of flat modules are free modules, and more generally projective modules. In particular, $A[\xi]$, being free over A, is flat as an A-module. The same is true for the power series ring $A[[\xi]]$. Any localization of A is flat, and more generally, any localization of a flat module is again flat. In fact, flatness is preserved under base change in the following sense:

3.1.3 *If M is a flat A-module, then M/IM is a flat A/I-module for each ideal $I \subseteq A$. More generally, if $A \to B$ is any homomorphism, then $M \otimes_A B$ is a flat B-module.*

Immediate from the definition and fact that tensoring with $M \otimes_A B$ over B is the same as tensoring with M over A. □

3.1.4 Tor modules

Let M be an A-module. A *projective resolution* of M is a complex $P_\bullet$, bounded from the right, in which all the modules P_i are projective, and such that the augmented complex

$$P_i \to P_{i-1} \to \cdots \to P_0 \to M \to 0$$

is exact. Put differently, a projective resolution of M is an acyclic complex $P_\bullet$ of projective modules whose cokernel is equal to M. Tensoring this augmented complex with a second A-module N, yields a (possibly non-exact) complex

$$P_i \otimes_A N \to P_{i-1} \otimes_A N \to \cdots \to P_0 \otimes_A N \to M \otimes_A N \to 0.$$

The homology of the non-augmented part $P_\bullet \otimes N$ (that is to say, without the final module $M \otimes N$), is denoted

$$\mathrm{Tor}_i^A(M,N) := \mathrm{H}_i(P_\bullet \otimes_A N).$$

As the notation indicates, this does not depend on the choice of projective resolution $P_\bullet$. Moreover, we have for each i an isomorphism $\mathrm{Tor}_i^A(M,N) \cong \mathrm{Tor}_i^A(N,M)$ ([27, Appendix 3] or [69, Appendix B]). Since tensoring is right exact, $\mathrm{Tor}_0^A(M,N) \cong M \otimes_A N$. The next result is a general fact of 'derived functors' (Tor is indeed the *derived functor* of the tensor product as discussed for instance in [69, Appendix B]).

3.1.4 *Given a short exact sequence of A-modules*

$$0 \to N' \to N \to N'' \to 0,$$

we get for every A-module M, a long exact sequence

$$\cdots \to \mathrm{Tor}_{i+1}^A(M,N'') \xrightarrow{\delta_{i+1}} \mathrm{Tor}_i^A(M,N') \to \\ \mathrm{Tor}_i^A(M,N) \to \mathrm{Tor}_i^A(M,N'') \xrightarrow{\delta_i} \mathrm{Tor}_{i-1}^A(M,N') \to \dots$$

where the δ_i are the so-called connecting homomorphisms, *and the remaining maps are induced by the original maps.*

3.1.5 Tor-criterion for Flatness

We can now formulate a homological criterion for flatness (see for instance [69, Theorem 7.8]; more flatness criteria will be discussed in §3.3 below).

Theorem 3.1.5. *For an A-module M, the following are equivalent*

3.1.5.i. *M is flat;*
3.1.5.ii. $\mathrm{Tor}_i^A(M,N) = 0$ *for all* $i > 0$ *and all A-modules N;*
3.1.5.iii. $\mathrm{Tor}_1^A(M,A/I) = 0$ *for all finitely generated ideals* $I \subseteq A$. □

For Noetherian rings we can even restrict the test in (3.1.5.iii) to prime ideals (but see also Theorem 3.3.18 below, which reduces the test to a single ideal):

Corollary 3.1.6. *Let A be a Noetherian ring and M an A-module. If* $\mathrm{Tor}_1^A(M,A/\mathfrak{p})$ *vanishes for all prime ideals* $\mathfrak{p} \subseteq A$*, then M is flat. More generally, if, for some* $i \geq 1$*, every* $\mathrm{Tor}_i^A(M,A/\mathfrak{p})$ *vanishes for* $\mathfrak{p}$ *running over the prime ideals in A, then* $\mathrm{Tor}_i^A(M,N)$ *vanishes for all (finitely generated) A-modules N.*

Proof. The first assertion follows from the last by (3.1.5.iii). The last assertion, for finitely generated modules, follows from the fact that every such module N admits a *prime filtration*, that is to say, a finite ascending chain of submodules

$$0 = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_e = N \tag{3.3}$$

such that each successive quotient N_j/N_{j-1} is isomorphic to the (cyclic) A-module $A/\mathfrak{p}_j$ for some prime ideal $\mathfrak{p}_j \subseteq A$, for $j = 1,\dots,e$ (see [69, Theorem 6.4]). By induction on j, one then derives from the long exact sequence (3.1.4) that $\mathrm{Tor}_i^A(M,N_j) = 0$, whence in particular $\mathrm{Tor}_i^A(M,N) = 0$. To prove the result for N arbitrary, one reduces to the case $i = 1$ by taking syzygies of M, and then applies Theorem 3.1.5. □

3.2 Faithful Flatness

We call an A-module M *non-degenerated*, if $\mathfrak{m}M \neq M$ for all (maximal) ideals $\mathfrak{m}$ of A. By Nakayama's Lemma, we immediately get:

3.2.1 *Any finitely generated module over a local ring is non-degenerated.* □

3.2.1 Faithfully Flat Homomorphisms

Of particular interest are the non-degenerated modules which are moreover flat, called *faithfully flat* modules. One has the following homological characterization of faithful flatness (see [69, Theorem 7.2] for a proof):

3.2.2 *For an A-module M to be faithfully flat, it is necessary and sufficient that an arbitrary complex* $N_\bullet$ *is exact if and only if* $N_\bullet \otimes_A M$ *is exact.* □

It is not hard to see that any free or projective module is faithfully flat. On the other hand, no proper localization of A is faithfully flat. The analogue of 3.1.3 holds: the base change of a faithfully flat module is again faithfully flat.

3.2.3 *If M is a faithfully flat A-module, then $M \otimes_A N$ is non-zero, for every non-zero A-module N.*

Indeed, let $N \neq 0$ and choose a non-zero element $n \in N$. Since $I := \mathrm{Ann}_A(n)$ is then a proper ideal, it is contained in some maximal ideal $\mathfrak{m} \subseteq A$. Note that $An \cong A/I$. Tensoring the induced inclusion $A/I \hookrightarrow N$ with M gives by assumption an injection $M/IM \hookrightarrow M \otimes_A N$. The first of these modules is non-zero, since $IM \subseteq \mathfrak{m}M \neq M$, whence so is the second, as we wanted to show. □

In most of our applications, the A-module has the additional structure of an A-algebra. In particular, we call a ring homomorphism $A \to B$ *(faithfully) flat* if B is (faithfully) flat as an A-module. Since by definition a local homomorphism of local rings $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a ring homomorphism with the additional property that $\mathfrak{m} \subseteq \mathfrak{n}$, we get immediately:

3.2.4 *Any local homomorphism which is flat, is faithfully flat.* □

Proposition 3.2.5. *A faithfully flat map is cyclically pure, whence, in particular, injective.*

Proof. We need to show that if $A \to B$ is faithfully flat, and $I \subseteq A$ an ideal, then $I = IB \cap A$. For I equal to the zero ideal, this just says that $A \to B$ is injective. Suppose this last statement is false, and let $a \in A$ be a non-zero element in the kernel of $A \to B$, that is to say, $a = 0$ in B. However, by 3.2.3, the module $aA \otimes_A B$ is non-zero, say, containing the non-zero element x. Hence x is of the form $ra \otimes b$ for some $r \in A$ and $b \in B$, and therefore equal to $r \otimes ab = r \otimes 0 = 0$, contradiction.

To prove the general case, note that B/IB is a flat A/I-module by 3.1.3. It is clearly also non-degenerated, so that applying our first argument to the natural homomorphism $A/I \to B/IB$ yields that it must be injective, which precisely means that $I = IB \cap A$. □

We can paraphrase the previous result as *faithful flatness preserves the ideal structure of a ring.* In particular, from its definition as the ascending chain condition on ideals, we get immediately the following Noetherianity criterion:

Corollary 3.2.6. *Let $A \to B$ be a faithfully flat, or more generally, a cyclically pure homomorphism. If B is Noetherian, then so is A.* □

A similar argument shows:

3.2.7 *If $R \to S$ is a faithfully flat, or more generally, a cyclically pure homomorphism of local rings, and if $I \subseteq R$ is minimally generated by e elements, then so is IS.*

Clearly, IS is generated by at most e elements. By way of contradiction, suppose it is generated by strictly fewer elements. By Nakayama's Lemma, we may choose these generators already in I. So there exists an ideal $J \subseteq I$, generated by less than e elements, such that $JS = IS$. However, by cyclic purity, we have $J = JS \cap R = IS \cap R = I$, contradicting that I requires at least e generators. □

If $A \to B$ is a flat or faithfully flat homomorphism, then we also will call the corresponding morphism $Y := \mathrm{Spec}(B) \to X := \mathrm{Spec}(A)$ *flat* or *faithfully flat* respectively.

Theorem 3.2.8. *A morphism $Y \to X$ of affine schemes is faithfully flat if and only if it is flat and surjective.*

Proof. Let $A \to B$ be the corresponding homomorphism. Assume $A \to B$ is faithfully flat, and let $\mathfrak{p} \subseteq A$ be a prime ideal. Surjectivity of the morphism amounts to showing that there is at least one prime ideal of B lying over $\mathfrak{p}$. The base change $A_\mathfrak{p} \to B_\mathfrak{p}$ is again faithfully flat, and hence in particular $\mathfrak{p}B_\mathfrak{p} \neq B_\mathfrak{p}$. In other words, the fiber ring $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ is non-empty, which is what we wanted to prove (indeed, take any maximal ideal $\mathfrak{n}$ of $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ and let $\mathfrak{q} := \mathfrak{n} \cap B$; then verify that $\mathfrak{q} \cap A = \mathfrak{p}$.)

Conversely, assume $Y \to X$ is flat and surjective, and let $\mathfrak{m}$ be a maximal ideal of A. Let $\mathfrak{q} \subseteq B$ be a prime ideal such that $\mathfrak{m} = \mathfrak{q} \cap A$. Hence $\mathfrak{m}B \subseteq \mathfrak{q} \neq B$, showing that B is non-degenerated over A. □

3.2.2 *Flatness and Regular Sequences*

A finite sequence $(x_1,\dots,x_n)$ in a ring A is called *pre-regular*, if each x_i is a non-zero divisor on $A/(x_1,\dots,x_{i-1})A$; if $(x_1,\dots,x_n)A$ is moreover a proper ideal, then we say that $(x_1,\dots,x_n)$ is a *regular sequence*. If (x_1) is a regular sequence, that is to say, a non-zero divisor and a non-unit, then we also express this by saying that x_1 is an *A-regular element*

Proposition 3.2.9. *If $A \to B$ is a flat homomorphism and* $\mathbf{x}$ *is an A-pre-regular sequence, then* $\mathbf{x}$ *is also B-pre-regular. If $A \to B$ is faithfully flat, and* $\mathbf{x}$ *is an A-regular sequence, then* $\mathbf{x}$ *is also B-regular.*

Proof. We induct on the length n of $\mathbf{x} := (x_1,\dots,x_n)$. Assume first $n = 1$. Multiplication by x_1, that is to say, the homomorphism $A \xrightarrow{x_1} A$, is injective, whence remains so after tensoring with B by 3.1.3. It is not hard to see that the resulting homomorphism is again multiplication $B \xrightarrow{x_1} B$, showing that x_1 is B-pre-regular. For $n > 1$, the base change $A/x_1A \to B/x_1B$ is flat, so that by induction $(x_2,\dots,x_n)$ is B/x_1B-pre-regular. Hence we are done, since x_1 is B-pre-regular by the previous argument. The last statement now follows from this, since then B is non-degenerated, and hence, in particular, $\mathbf{x}B \neq B$. □

Tor modules behave well under deformation by a regular sequence in the following sense.

Proposition 3.2.10. *Let* $\mathbf{x}$ *be a regular sequence in a ring A, and let M and N be two A-modules. If* $\mathbf{x}$ *is M-regular and* $\mathbf{x}N = 0$*, then we have for each i an isomorphism*

$$\operatorname{Tor}_i^A(M,N) \cong \operatorname{Tor}_i^{A/\mathbf{x}A}(M/\mathbf{x}M,N).$$

Proof. By induction on the length of the regular sequence, we may assume that we have a single A-regular and M-regular element x. Put $B := A/xA$. From the short exact sequence

$$0 \to A \xrightarrow{x} A \to B \to 0$$

we get after tensoring with M, a long exact sequence of Tor-modules as in 3.1.4. Since $\mathrm{Tor}_i^A(A,M)$ vanishes for all i, so must each $\mathrm{Tor}_i^A(M,B)$ in this long exact sequence for $i > 1$. Furthermore, the initial part of this long exact sequence is

$$0 \to \mathrm{Tor}_1^A(M,B) \to M \xrightarrow{x} M \to M/xM \to 0$$

proving that $\mathrm{Tor}_1^A(M,B)$ too vanishes as x is M-regular. Now, let $P_\bullet$ be a projective resolution of M. The homology of $\bar{P}_\bullet := P_\bullet \otimes_A B$ is by definition $\mathrm{Tor}_i^A(M,B)$, and since we showed that this is zero, $\bar{P}_\bullet$ is exact, whence a projective resolution of M/xM. Hence we can calculate $\mathrm{Tor}_i^B(M/xM,N)$ as the homology of $\bar{P}_\bullet \otimes_B N$ (note that by assumption, N is a B-module). However, the latter complex is equal to $P_\bullet \otimes_A N$ (which we can use to calculate $\mathrm{Tor}_i^A(M,N)$), and hence both complexes have the same homology, as we wanted to show. □

3.2.3 Scalar Extensions

Recall that a homomorphism $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ of local rings is called *unramified*, if $\mathfrak{m}S = \mathfrak{n}$, or equivalently, if the *closed fiber $S/\mathfrak{m}S$ is trivial.* A homomorphism which is at the same time faithfully flat and unramified is sometimes called *formally etale*, although some authors in addition require that the residue field extension be separable. To not cause any confusion, we will call such a homomorphism a *scalar extension* (see below for the terminology). By [37, 0_{III} 10.3.1], for any Noetherian local ring R and any extension l of its residue field, there exists a scalar extension of R with residue field l; we will reprove this in Theorem 3.2.13 below.

Proposition 3.2.11. *Consider the following commutative triangle of local homomorphisms between Noetherian local rings*

$$\begin{array}{ccccc} & & (R,\mathfrak{m}) & & \\ & {\scriptstyle f}\swarrow & & \searrow{\scriptstyle h} & \\ (S,\mathfrak{n}) & & \xrightarrow[g]{} & & (T,\mathfrak{p}) \end{array} \tag{3.4}$$

If any two are scalar extensions, then so is the third.

Proof. It is clear that the composition of two scalar extensions is again scalar. Assume g and h are scalar extensions. Then f is faithfully flat by an easy argument using 3.2.2, and $\mathfrak{m}T = \mathfrak{p} = \mathfrak{n}T$. Since g is faithfully flat, we get $\mathfrak{m}S = \mathfrak{m}T \cap S = \mathfrak{n}T \cap S = \mathfrak{n}$ by Proposition 3.2.5, showing that f is also a scalar extension. Finally, assume f and h are scalar extensions. Let

$$\dots R^{b_2} \to R^{b_1} \to R \to R/\mathfrak{m} \to 0 \tag{3.5}$$

be a free resolution of $R/\mathfrak{m}$. Since S is flat over R, tensoring yields a free resolution

$$\dots S^{b_2} \to S^{b_1} \to S \to S/\mathfrak{m}S \to 0. \tag{3.6}$$

By assumption $S/\mathfrak{m}S$ is the residue field l of S. Therefore, $\operatorname{Tor}_\bullet^S(T,l)$ can be calculated as the homology of the complex

$$\dots T^{b_2} \to T^{b_1} \to T \to T/\mathfrak{m}T \to 0 \tag{3.7}$$

obtained from (3.6) by the base change $S \to T$. However, (3.7) can also be obtained by tensoring (3.5) over R with T. Since T is flat over R, the sequence (3.7) is exact, whence, in particular, $\operatorname{Tor}_1^S(T,l) = 0$. By the Local Flatness Criterion (see Corollary 3.3.22 below), T is flat over S. Since $\mathfrak{n} = \mathfrak{m}S$ and $\mathfrak{p} = \mathfrak{m}T$, we get $\mathfrak{p} = \mathfrak{n}T$, showing that g, too, is a scalar extension. □

The following are examples of scalar extensions (for the notion of catapower, see Chapter 8; for the proof of (3.2.12.iii), see Corollary 3.3.3 and Theorem 8.1.15 below).

3.2.12 *Let R be a Noetherian local ring.*

- 3.2.12.i. *The natural map $R \to \widehat{R}$, given by completion, is a scalar extension.*
- 3.2.12.ii. *Any etale map is a scalar extension.*
- 3.2.12.iii. *The diagonal embeddings $R \to R_\natural$ and $R \to R_\sharp$ are scalar extensions, where $R_\natural$ and $R_\sharp$ are respectively an ultrapower and a catapower of R.* □

The next result, which extends Cohen's Structure Theorems, explains the terminology (for a version in mixed characteristic case, see [102]).

Theorem 3.2.13. *Let $(R,\mathfrak{m})$ be a Noetherian local ring of equal characteristic with residue field k. Every extension $k \subseteq l$ of fields can be lifted to a faithfully flat extension $R \to \widehat{R_l}$ inducing the given extension on the residue fields, such that $\widehat{R_l}$ is a complete local ring with maximal ideal $\mathfrak{m}\widehat{R_l}$ and residue field l. In other words, $R \to \widehat{R_l}$ is a scalar extension.*

In fact, $\widehat{R_l}$ is a solution to the following universal property: any complete Noetherian local R-algebra T with residue field l has a unique structure of a local $\widehat{R_l}$-algebra. In particular, $\widehat{R_l}$ is uniquely determined by R and l up to isomorphism, and is called the complete scalar extension *of R along l.*

Proof. By Cohen's Structure Theorems, the completion $\widehat{R}$ of R is isomorphic to $k[[\xi]]/\mathfrak{a}$ for some ideal $\mathfrak{a}$ and some tuple of indeterminates ξ. Put

$$\widehat{R_l} := l[[\xi]]/\mathfrak{a}l[[\xi]],$$

that is to say, $\widehat{R_l}$ is the $\mathfrak{n}$-adic completion of $\widehat{R} \otimes_k l$, where $\mathfrak{n} := \mathfrak{m}(\widehat{R} \otimes_k l)$. It is now easy to check that this ring has the desired properties.

To prove the universal property, let T be any complete Noetherian local R-algebra, given by the local homomorphism $R \to T$. By the universal property of completions, we have a unique extension $k[[\xi]]/\mathfrak{a} \cong \widehat{R} \to T$, and by the universal properties of tensor product and completion, this uniquely extends to a homomorphism $\widehat{R_l} = l[[\xi]]/\mathfrak{a}l[[\xi]] \to T$. □

Note that complete scalar extension is actually a functor, that is to say, any local homomorphism $R \to S$ of Noetherian local rings whose residue fields are subfields of l extends to a local homomorphism $\widehat{R_l} \to \widehat{S_l}$. In particular, complete scalar extension commutes with homomorphic images:

$$(R/\mathfrak{a})\widehat{_l} \cong \widehat{R_l}/\mathfrak{a}\widehat{R_l}, \tag{3.8}$$

for all ideals $\mathfrak{a} \subseteq R$. Scalar extensions preserve many good properties:

3.2.14 *If $R \to S$ is a scalar extension, then R and S have the same dimension, and one is regular (respectively, Cohen-Macaulay) if and only if the other is.*

Indeed, the equality of dimension follows from (3.16) (see our discussion below). Since both have also the same embedding dimension by 3.2.7, the claim about regularity follows. In general, let $\mathbf{x}$ be a system of parameters in R. It follows that $\mathbf{x} = (x_1, \dots, x_d)$ is also a system of parameters in S. So if R is Cohen-Macaulay, $\mathbf{x}$ is an R-regular whence S-regular sequence, by Proposition 3.2.9, proving that S is Cohen-Macaulay. The converse follows from the fact that each $R/(x_1, \dots, x_i)R$ is a subring of $S/(x_1, \dots, x_i)S$ by Proposition 3.2.5. □

3.3 Flatness Criteria

Because flatness will play such a crucial role in our later work, we want several ways of detecting it. In this section, we will see six such criteria.

3.3.1 Equational Criterion for Flatness

Our first criterion is very useful in applications (see for instance Theorem 4.4.3), and works without any hypothesis on the ring or module. To give a streamlined presentation, let us introduce the following terminology: given an A-module N, and tuples $\mathbf{b}_i$ in A^n, by an *N-linear combination* of the $\mathbf{b}_i$, we mean a tuple in N^n of the form $n_1\mathbf{b}_1 + \cdots + n_s\mathbf{b}_s$ where $n_i \in N$. Of course, if N has the structure of an A-algebra, this is just the usual terminology. Given a (finite) homogeneous linear system of equations

$$L_1(t) = \cdots = L_s(t) = 0 \tag{$\mathscr{L}$}$$

over A in the n variables t, we denote the A-submodule of N^n consisting of all solutions of $\mathscr{L}$ in N by $\mathrm{Sol}_N(\mathscr{L})$, and we let $f_{\mathscr{L}}\colon N^n \to N^s$ be the map given by substitution $\mathbf{x} \mapsto (L_1(\mathbf{x}), \dots, L_s(\mathbf{x}))$. In particular, we have an exact sequence

$$0 \to \mathrm{Sol}_N(\mathscr{L}) \to N^n \xrightarrow{f_{\mathscr{L}}} N^s. \qquad (\dagger_{\mathscr{L}/N})$$

Theorem 3.3.1. *A module M over a ring A is flat if and only if every solution in M of a homogeneous linear equation in finitely many variables over A is an M-linear combination of solutions in A. Moreover, instead of a single linear equation, we may take any finite, linear system of equations in the above criterion.*

Proof. We will only prove the first assertion, and leave the second for the reader. Let $L = 0$ be a homogeneous linear equation in n variables with coefficients in A. If M is flat, then the exact sequence $(\dagger_{L/A})$ remains exact after tensoring with M, that is to say,

$$0 \to \mathrm{Sol}_A(L) \otimes_A M \to M^n \xrightarrow{f_L} M,$$

and hence by comparison with $(\dagger_{L/M})$, we get

$$\mathrm{Sol}_M(L) = \mathrm{Sol}_A(L) \otimes_A M.$$

From this it follows easily that any tuple in $\mathrm{Sol}_M(L)$ is an M-linear combination of tuples in $\mathrm{Sol}_A(L)$, proving the direct implication.

Conversely, assume the condition on the solution sets of linear forms holds. To prove that M is flat, we will verify condition (3.1.5.iii) in Theorem 3.1.5. To this end, let $I := (a_1, \dots, a_k)A$ be a finitely generated ideal of A. Tensor the exact sequence $0 \to I \to A \to A/I \to 0$ with M to get by 3.1.4 an exact sequence

$$0 = \mathrm{Tor}_1^A(A, M) \to \mathrm{Tor}_1^A(A/I, M) \to I \otimes_A M \to M. \qquad (3.10)$$

Suppose y is an element in $I \otimes M$ that is mapped to zero in M. Writing $y = a_1 \otimes m_1 + \dots + a_k \otimes m_k$ for some $m_i \in M$, we get $a_1 m_1 + \dots + a_k m_k = 0$. Hence by assumption, there exist solutions $\mathbf{b}^{(1)}, \dots, \mathbf{b}^{(s)} \in A^k$ of the linear equation $a_1 t_1 + \dots + a_k t_k = 0$, such that

$$(m_1, \dots, m_k) = n_1 \mathbf{b}^{(1)} + \dots + n_s \mathbf{b}^{(s)}$$

for some $n_i \in M$. Letting $b_i^{(j)}$ be the i-th entry of $\mathbf{b}^{(j)}$, we see that

$$y = \sum_{i=1}^{k} a_i \otimes m_i = \sum_{i=1}^{k} \sum_{j=1}^{s} a_i \otimes n_j b_i^{(j)} = \sum_{j=1}^{s} \Big(\sum_{i=1}^{k} a_i b_i^{(j)}\Big) \otimes n_j = \sum_{j=1}^{s} 0 \otimes n_j = 0.$$

Hence $I \otimes_A M \to M$ is injective, so that $\mathrm{Tor}_1^A(A/I, M)$ must be zero by (3.10). Since this holds for all finitely generated ideals $I \subseteq A$, we proved that M is flat by (3.1.5.iii). □

It is instructive to view the previous result from the following perspective. To a homogeneous linear equation $L = 0$, we associated an exact sequence $(\dagger_{L/N})$. The image of f_L is of the form IN where I is the ideal generated by the coefficients of the linear form defining L. In case $N = B$ is an A-algebra, this leads to the following extended exact sequence

$$0 \to \mathrm{Sol}_B(L) \to B^n \xrightarrow{f_L} B \to B/IB \to 0. \qquad (\ddagger_{IB})$$

This justifies calling $\mathrm{Sol}_B(L)$ the *module of syzygies* of IB (one checks that it only depends on the ideal I). Therefore, we may paraphrase the equational flatness criterion for algebras as follows:

3.3.2 *A ring homomorphism $A \to B$ is flat if and only if taking syzygies commutes with extension in the sense that the module of syzygies of IB is the extension to B of the module of syzygies of an arbitrary ideal $I \subseteq A$.*

Here is one application of the equational flatness criterion.

Corollary 3.3.3. *The diagonal embedding of a Noetherian ring inside its ultrapower is faithfully flat.*

Proof. Let A be a ring and $A_\natural$ an ultrapower of A. Recall that $A \to A_\natural$ is given by sending an element $a \in A$ to the ultraproduct $\mathrm{ulim}_{w\to\infty} a$ of the constant sequence. If $\mathfrak{m} \subseteq A$ is a maximal ideal, then $\mathfrak{m}A_\natural$ is its ultraproduct by 2.4.20, whence again maximal by Łoś' Theorem (Theorem 2.3.1), showing that $A_\natural$ is non-degenerated. To show it is also flat, we use the equational criterion. Let $L = 0$ be a homogeneous linear equation with coefficients in A. Let $\mathbf{a} \in A_\natural^n$ be a solution of $L = 0$ in $A_\natural$. Write $\mathbf{a}$ as an ultraproduct of tuples $\mathbf{a}_w \in A^n$. By Łoś' Theorem, almost each $\mathbf{a}_w \in \mathrm{Sol}_A(L)$. Hence $\mathbf{a}$ lies in the ultrapower of $\mathrm{Sol}_A(L)$. By Noetherianity, $\mathrm{Sol}_A(L)$ is finitely generated, and hence, its ultrapower is simply the $A_\natural$-module generated by $\mathrm{Sol}_A(L)$, so that we are done by Theorem 3.3.1. □

3.3.2 Coherency Criterion

We can turn this into a criterion for coherency. Recall that a ring A is called *coherent*, if the solution set of any homogeneous linear equation over A is finitely generated. Clearly, Noetherian rings are coherent. We have:

Theorem 3.3.4. *A ring A is coherent if and only if the diagonal embedding into one of its ultrapowers is flat.*

Proof. The direct implication is proven by the same argument that proves Corollary 3.3.3, since we really only used that A is coherent in that argument. Conversely, suppose $A \to A_\natural$ is flat. Towards a contradiction, assume L is a linear form (in n indeterminates) over A whose solution set $\mathrm{Sol}_A(L)$ is infinitely generated. In particular, we can choose a sequence $\mathbf{a}_w$, for $w = 1, 2, \dots$, in $\mathrm{Sol}_A(L)$ which is contained in no finitely generated submodule of $\mathrm{Sol}_A(L)$. The ultraproduct $\mathbf{a}_\natural \in A_\natural^n$ of this sequence lies in $\mathrm{Sol}_{A_\natural}(L)$ by Łoś' Theorem. Hence, by Theorem 3.3.1, there exists a finitely generated submodule $H \subseteq \mathrm{Sol}_A(L)$ such that $\mathbf{a}_\natural \in H \cdot A_\natural$. Therefore, almost all $\mathbf{a}_j$ lie in H by Łoś' Theorem, contradiction. □

3.3.3 Quotient Criterion for Flatness

The next criterion is derived from our Tor-criterion (Theorem 3.1.5):

Theorem 3.3.5. *Let $A \to B$ be a flat homomorphism, and let $I \subseteq B$ be an ideal. The induced homomorphism $A \to B/I$ is flat if and only if $\mathfrak{a}B \cap I = \mathfrak{a}I$ for all finitely generated ideals $\mathfrak{a} \subseteq A$.*

Moreover, if A is Noetherian, we only need to check the above criterion for $\mathfrak{a}$ a prime ideal of A.

Proof. From the exact sequence $0 \to I \to B \to B/I \to 0$ we get after tensoring with $A/\mathfrak{a}$ an exact sequence

$$0 = \mathrm{Tor}_1^A(B, A/\mathfrak{a}) \to \mathrm{Tor}_1^A(B/I, A/\mathfrak{a}) \to I/\mathfrak{a}I \to B/\mathfrak{a}B$$

where we used the flatness of B for the vanishing of the first module. The kernel of $I/\mathfrak{a}I \to B/\mathfrak{a}B$ is easily seen to be $(\mathfrak{a}B \cap I)/\mathfrak{a}I$. Hence $\mathrm{Tor}_1^A(B/I, A/\mathfrak{a})$ vanishes if and only if $\mathfrak{a}B \cap I = \mathfrak{a}I$. This proves by Theorem 3.1.5 the stated equivalence in the first assertion; the second assertion follows by the same argument, this time using Corollary 3.1.6. □

To put this criterion to use, we need another definition (for another application, see Theorem 8.2.1 below). The (A-)*content* of a polynomial $f \in A[\xi]$ (or a power series $f \in A[[\xi]]$) is by definition the ideal generated by its coefficients.

Corollary 3.3.6. *Let A be a Noetherian ring, let ξ be a finite tuple of indeterminates, and let B denote either $A[\xi]$ or $A[[\xi]]$. If $f \in B$ has content one, then B/fB is flat over A.*

Proof. The natural map $A \to B$ is flat. To verify the second criterion in Theorem 3.3.5, let $\mathfrak{p} \subseteq A$ be a prime ideal. The forward inclusion $\mathfrak{p}fB \subseteq \mathfrak{p}B \cap fB$ is immediate. To prove the other, take $g \in \mathfrak{p}B \cap fB$. In particular, $g = fh$ for some $h \in B$. Since $\mathfrak{p} \subseteq A$ is a prime ideal, so is $\mathfrak{p}B$ (this is a property of polynomial or power series rings, not of flatness!). Since f has content one, $f \notin \mathfrak{p}B$ whence $h \in \mathfrak{p}B$. This yields $g \in \mathfrak{p}fB$, as we needed to prove. □

3.3.4 Cohen-Macaulay Criterion for Flatness

To formulate our next criterion, we need a definition.

Definition 3.3.7 (Big Cohen-Macaulay Modules). Let R be a Noetherian local ring, and let M be an arbitrary R-module. We call M a *big Cohen-Macaulay module*, if there exists a system of parameters in R which is M-regular. If moreover every system of parameters is M-regular, then we call M a *balanced big Cohen-Macaulay* module.

It has become tradition to add the somehow redundant adjective 'big' to emphasize that the module is not necessarily finitely generated. It is one of the greatest open problems in homological algebra to show that every Noetherian local ring has at least one big Cohen-Macaulay module, and, as we shall see, this is known to be the case for any Noetherian local ring containing a field (see §6.4 and §7.4).[1] A Cohen-Macaulay local ring is clearly a balanced big Cohen Macaulay module over itself, so the problem of the existence of these modules is only important for deriving results over Noetherian local rings with 'worse than Cohen-Macaulay' singularities.

Once one has a big Cohen-Macaulay module, one can always construct, using completion, a balanced big Cohen-Macaulay module from it (see for instance [17, Corollary 8.5.3]). Here is a criterion for a big Cohen-Macaulay module to be balanced taken from [6, Lemma 4.8] (recall that a regular sequence is called *permutable* if any permutation is again regular).

Proposition 3.3.8. *A big Cohen-Macaulay module M over a Noetherian local ring is balanced, if every M-regular sequence is permutable.* □

If R is a Cohen-Macaulay local ring, and M a flat R-module, then M is a balanced big Cohen-Macaulay module, since every system of parameters in R is R-regular, whence M-regular by Proposition 3.2.9. We have the following converse:

Theorem 3.3.9. *If M is a balanced big Cohen-Macaulay module over a regular local ring, then it is flat. More generally, over an arbitrary local Cohen-Macaulay ring, if M is a balanced big Cohen-Macaulay module of finite projective dimension, then it is flat.*

Proof. The first assertion is just a special case of the second since any module over a regular local ring has finite projective dimension. For simplicity, we will just prove the first, and leave the second as an exercise for the reader. So let M be a balanced big Cohen-Macaulay module over the d-dimensional regular local ring R. Since a finitely generated R-module N has finite projective dimension,

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[1] A related question is even open in these cases: does there exist a 'small' Cohen-Macaulay module, that is to say, a finitely generated one, if the ring is moreover complete? There are counterexamples to the existence of a small Cohen-Macaulay module if the ring is not complete.

all $\mathrm{Tor}_i^R(M,N) = 0$ for $i \gg 0$. Let e be maximal such that $\mathrm{Tor}_e^R(M,N) \neq 0$ for some finitely generated R-module N. If $e = 0$, then we are done by Theorem 3.1.5. So, by way of contradiction, assume $e \geq 1$. By Corollary 3.1.6, there exists a prime ideal $\mathfrak{p} \subseteq R$ such that $\mathrm{Tor}_e^R(M, R/\mathfrak{p}) \neq 0$. Let h be the height of $\mathfrak{p}$. Choose a system of parameters $(x_1, \dots, x_d)$ in R such that $\mathfrak{p}$ is a minimal prime of $I := (x_1, \dots, x_h)R$. Since (the image of) $\mathfrak{p}$ is then an associated prime of R/I, we get a short exact sequence

$$0 \to R/\mathfrak{p} \to R/I \to C \to 0$$

for some finitely generated R-module C. The relevant part of the long exact Tor sequence from 3.1.4, obtained by tensoring the above exact sequence with M, is

$$\mathrm{Tor}_{e+1}^R(M,C) \to \mathrm{Tor}_e^R(M,R/\mathfrak{p}) \to \mathrm{Tor}_e^R(M,R/I). \tag{3.12}$$

The first module in (3.12) is zero by the maximality of e. The last module is zero too since it is isomorphic to $\mathrm{Tor}_e^{R/I}(M/IM, R/I) = 0$ by Proposition 3.2.10 and the fact that $(x_1, \dots, x_d)$ is by assumption M-regular. Hence the middle module in (3.12) is also zero, contradiction. □

We derive the following criterion for Cohen-Macaulayness:

Corollary 3.3.10. *If X is an irreducible affine scheme of finite type over a field K, and $\phi\colon X \to \mathbb{A}_K^d$ is a Noether normalization, that is to say, a finite and surjective morphism, then X is Cohen-Macaulay if and only if ϕ is flat.*

Proof. Suppose $X = \mathrm{Spec}(B)$, so that ϕ corresponds to a finite and injective homomorphism $A \to B$, with $A := K[\xi_1, \dots, \xi_d]$ and B a d-dimensional affine domain. Let $\mathfrak{n}$ be a maximal ideal of B, and let $\mathfrak{m} := \mathfrak{n} \cap A$ be its contraction to A. Since flatness is a local property, it suffices to show that $A_\mathfrak{m} \to B_\mathfrak{n}$ is flat. Since $A/\mathfrak{m} \to B/\mathfrak{n}$ is finite and injective, and since the second ring is a field, so is the former by [69, §9 Lemma 1]. Hence $\mathfrak{m}$ is a maximal ideal of A, and $A_\mathfrak{m}$ is regular. Choose an ideal $I := (x_1, \dots, x_d)A$ whose image in $A_\mathfrak{m}$ is a parameter ideal. Since the natural homomorphism $A/I \to B/IB$ is finite, the latter ring is Artinian since the former is (note that $A/I = A_\mathfrak{m}/IA_\mathfrak{m}$). It follows that $IB_\mathfrak{n}$ is a parameter ideal in $B_\mathfrak{n}$.

Now, if B, whence also $B_\mathfrak{n}$ is Cohen-Macaulay, then $(x_1, \dots, x_d)$, being a system of parameters in $B_\mathfrak{n}$, is $B_\mathfrak{n}$-regular. This proves that $B_\mathfrak{n}$ is balanced big Cohen-Macaulay module over $A_\mathfrak{m}$, whence is flat by Theorem 3.3.9.

Conversely, assume $X \to \mathbb{A}_K^d$ is flat. Therefore, $A_\mathfrak{m} \to B_\mathfrak{n}$ is flat, and hence $(x_1, \dots, x_d)$ is a $B_\mathfrak{n}$-regular sequence by Proposition 3.2.9. Since we already showed that this sequence is a system of parameters, we see that $B_\mathfrak{n}$ is Cohen-Macaulay. Since this holds for all maximal prime ideals of B, we proved that B is Cohen-Macaulay. □

Remark 3.3.11. The above argument proves the following more general result in the local case: if $A \subseteq B$ is a finite and faithfully flat extension of local rings with A regular, then B is Cohen-Macaulay. For the converse, we can even formulate a stronger criterion; see Theorem 3.3.26 below.

We conclude with an application of the above Cohen-Macaulay criterion:

Corollary 3.3.12. *Any hypersurface in $\mathbb{A}_K^n$ is Cohen-Macaulay.*

Proof. Recall that a hypersurface Y is an affine closed subscheme of the form $\operatorname{Spec}(A/fA)$ with $A := K[\xi_1,\dots,\xi_n]$ and $f \in A$. Moreover, Y has dimension $n-1$, whence its Noether normalization is of the form $Y \to \mathbb{A}_K^{n-1}$. In fact, after a change of coordinates, we may assume that f is monic in ξ_n of degree d. It follows that A/fA is free over $A' := K[\xi_1,\dots,\xi_{n-1}]$ with basis $1,\xi_n,\dots,\xi_n^{d-1}$. Hence A/fA is flat over A', whence Cohen-Macaulay by Corollary 3.3.10. □

3.3.5 Colon Criterion for Flatness

Recall that $(I:a)$ denotes the *colon ideal* of all $x \in A$ such that $ax \in I$. Colon ideals are related to cyclic modules in the following way:

3.3.13 *For any ideal $I \subseteq A$ and any element $a \in A$, we have an isomorphism $a(A/I) \cong A/(I:a)$.*

Indeed, the homomorphism $A \to A/I\colon x \mapsto ax$ has image $a(A/I)$ whereas its kernel is $(I:a)$. We already saw that faithfully flat homomorphisms preserve the ideal structure of a ring. Using colon ideals, we can even give the following criterion:

Theorem 3.3.14. *A homomorphism $A \to B$ is flat if and only if*

$$(IB:a) = (I:a)B$$

for all elements $a \in A$ and all (finitely generated) ideals $I \subseteq A$.

Proof. Suppose $A \to B$ is flat. In view of 3.3.13, we have an exact sequence

$$0 \to A/(I:a) \to A/I \to A/(I+aA) \to 0 \tag{3.13}$$

which, when tensored with B gives the exact sequence

$$0 \to B/(I:a)B \to B/IB \xrightarrow{f} B/(IB+aB) \to 0.$$

However, the kernel of f is easily seen to be $a(B/IB)$, which is isomorphic to $B/(IB:a)$ by 3.3.13. Hence the inclusion $(I:a)B \subseteq (IB:a)$ must be an equality.

In view of Theorem 3.1.5, we need to show that $\operatorname{Tor}_1^A(B,A/J) = 0$ for every finitely generated ideal $J \subseteq A$ to prove the converse. We induct on the minimal number s of generators of J, where the case $s=0$ trivially holds. Write $J = I + aA$ with I an ideal generated by $s-1$ elements. Tensoring (3.13) with B, we get from 3.1.4 an exact sequence

$$0 = \operatorname{Tor}_1^A(B,A/I) \to \operatorname{Tor}_1^A(B,A/J) \xrightarrow{\delta} B/(I:a)B \to B/IB \xrightarrow{g} B/JB \to 0,$$

where the first module vanishes by induction. As above, the kernel of g is easily seen to be $B/(IB : a)$, so that our assumption on the colon ideals implies that δ is the zero map, whence $\operatorname{Tor}_1^A(B, A/J) = 0$ as we wanted to show. □

Here is a nice 'descent type' application of this criterion:

Corollary 3.3.15. *Let $A \to B \to C$ be homomorphisms whose composition is flat. If $B \to C$ is cyclically pure, then $A \to B$ is flat. In fact, it suffices that $B \to C$ is* cyclically pure with respect to ideals extended from A, *that is to say, that $JB = JC \cap B$ for all ideals $J \subseteq A$.*

Proof. Given an ideal $I \subseteq A$ and an element $a \in A$, we need to show in view of Theorem 3.3.14 that $(IB : a) = (I : a)B$. One inclusion is immediate, so take y in $(IB : a)$. By the same theorem, we have $(IC : a) = (I : a)C$, so that y lies in $(I : a)C \cap B$ whence in $(I : a)B$ by cyclical purity. □

The next criterion will be useful when dealing with non-Noetherian algebras in the next chapter. We call an ideal J in a ring B *finitely related*, if it is of the form $J = (I : b)$ with $I \subseteq B$ a finitely generated ideal and $b \in B$.

Theorem 3.3.16. *Let A be a Noetherian ring and B an arbitrary A-algebra. Suppose $\mathscr{P}$ is a collection of prime ideals in B such that every proper, finitely related ideal of B is contained in some prime ideal belonging to $\mathscr{P}$. If $A \to B_{\mathfrak{p}}$ is flat for every $\mathfrak{p} \in \mathscr{P}$, then $A \to B$ is flat.*

Proof. By Theorem 3.3.14, we need to show that $(IB : a) = (I : a)B$ for all $I \subseteq A$ and $a \in A$. Put $J := (I : a)$. Towards a contradiction, let x be an element in $(IB : a)$ but not in JB. Hence $(JB : x)$ is a proper, finitely related ideal, and hence contained in some $\mathfrak{p} \in \mathscr{P}$. However, $(IB_{\mathfrak{p}} : a) = JB_{\mathfrak{p}}$ by flatness and another application of Theorem 3.3.14, so that $x \in JB_{\mathfrak{p}}$, contradicting that $(JB : x) \subseteq \mathfrak{p}$. □

We can also derive a coherency criterion due to Chase ([21]):

Corollary 3.3.17. *A ring is coherent if and only if every finitely related ideal is finitely generated.*

Proof. The direct implication is a simple application of the coherency condition. For the converse, suppose every finitely related ideal is finitely generated. We will prove that $R \to R_\natural$ is flat, where $R_\natural$ is an ultrapower of R, from which it follows that R is coherent by Theorem 3.3.4. To prove flatness, we use the Colon Criterion, Theorem 3.3.14. To this end, let $I \subseteq R$ be finitely generated and let $a \in R$. We have to show that if b lies in $(IR_\natural : a)$ then it already lies in $(I : a)R_\natural$. Let b_w be an approximation of b. By Łoś' Theorem, almost each $b_w \in (I : a)$. By assumption, the colon ideal $(I : a)$ is finitely generated, say by $f_1, \dots, f_s$, and hence we can find c_{iw} such that $b_w = c_{1w} f_1 + \cdots + c_{sw} f_s$. Let $c_i \in R_\natural$ be the ultraproduct of the c_{iw}, for each $i = 1, \dots, s$. By Łoś' Theorem, $b = c_1 f_1 + \cdots + c_s f_s$, showing that it belongs to $(I : a)R_\natural$. □

3.3.6 Local Criterion for Flatness

For finitely generated modules, we have the following criterion:

Theorem 3.3.18 (Local Flatness Theorem–Finitely Generated Case). *Let R be a Noetherian local ring with residue field k. If M is a finitely generated R-module whose first Betti number vanishes, that is to say, if* $\operatorname{Tor}_1^R(M,k) = 0$, *then M is flat.*

Proof. Take a minimal free resolution

$$\cdots \to F_1 \to F_0 \to M \to 0$$

of M, that is to say, such that the kernel of each boundary map $d_i\colon F_i \to F_{i-1}$ lies inside $\mathfrak{m}F_i$. Therefore, since tensoring this complex with k yields the zero complex, the rank of F_i is equal to the *i-th Betti number* of M, that is to say, the vector space dimension of $\operatorname{Tor}_i^R(M,k)$. In particular, F_1 has rank zero, so that $M \cong F_0$ is free whence flat. □

There is a much stronger version of this result, where we may replace the condition that M is finitely generated over R by the condition that M is finitely generated over a Noetherian local R-algebra S (see for instance [69, Theorem 22.3] or [27, Theorem 6.8]). We will present here a new proof, for which we need to make some further definitions. The method is an extension of the work in [93], which primarily dealt with detecting finite projective dimension.

Let A be a (not necessarily Noetherian) ring, and let $\mathbf{mod}_A$ be the class of all finitely presented A-modules. We will call a subclass $\mathbf{N} \subseteq \mathbf{mod}_R$ a *deformation class* if it is closed under isomorphisms, direct summands, extensions, and deformations, that is to say, if it is closed under the following respective rules:[2]

3.3.18.i. if N belongs to $\mathbf{N}$ and $M \cong N$, then M belongs to $\mathbf{N}$;
3.3.18.ii. if $N \cong M \oplus M'$ belongs to $\mathbf{N}$, then so do M and M';
3.3.18.iii. if $0 \to K \to M \to N \to 0$ is an exact sequence in $\mathbf{mod}_R$ with $K, N \in \mathbf{N}$, then also $M \in \mathbf{N}$;
3.3.18.iv. if x is an M-regular element in the Jacobson radical of A and M/xM belongs to $\mathbf{N}$, then so does M.

Recall that the *Jacobson radical* of A is the intersection of all its maximal ideals; equivalently, it is the ideal of all x such that $1+ax$ is unit for all a. Condition 3.3.18.iv holds vacuously, if the Jacobson radical is equal to the *nilradical*, the ideal of all nilpotent elements. Clearly, $\mathbf{mod}_A$ itself is a deformation class. We leave it as an easy exercise to show that:

3.3.19 *Any intersection of deformation classes is again a deformation class. In particular, any class $\mathbf{K} \subseteq \mathbf{mod}_A$ sits inside a smallest deformation class, called the* deformation class of $\mathbf{K}$. □

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[2] A class satisfying the first three conditions is called a *net* in [93].

Let us call a subclass $\mathbf{K} \subseteq \mathbf{mod}_A$ *deformationally generating*, if its deformation class is equal to $\mathbf{mod}_A$, and *quasi-deformationally generating*, if its deformation class contains all cyclic modules of the form A/I with $I \subseteq A$ finitely generated. One easily shows, by induction on the number of generators, that if A is coherent, deformationally generating and quasi-deformationally generating are equivalent notions.

Proposition 3.3.20. *If R is a Noetherian local ring, then its residue field is deformationally generating.*

Proof. We need to show that any finitely generated module M belongs to the deformation class $\mathbf{N}$ generated by the residue field. Since any module generated by n elements is the extension of two modules generated by less than n elements, an induction on n using (3.3.18.iii) reduces to the case $n = 1$, that is to say, $M = R/\mathfrak{a}$. Suppose the assertion is false, and let $\mathfrak{a}$ be a maximal counterexample. If $\mathfrak{a}$ is not prime, then for $\mathfrak{p}$ a minimal prime ideal $\mathfrak{p}$ of $\mathfrak{a}$, we have an exact sequence

$$0 \to R/\mathfrak{p} \to R/\mathfrak{a} \to R/\mathfrak{a}' \to 0$$

for some $\mathfrak{a}' \subseteq R$ strictly containing $\mathfrak{a}$. The two outer modules belong to $\mathbf{N}$ by maximality, whence so does the inner one by (3.3.18.iii), contradiction. Hence $\mathfrak{a}$ is a prime ideal, which therefore must be different from the maximal ideal of R. Let x be an element in the maximal ideal not in $\mathfrak{a}$. By maximality $R/(\mathfrak{a} + xR)$ belongs to $\mathbf{N}$, whence so does $R/\mathfrak{a}$ by (3.3.18.iv), since x is $R/\mathfrak{a}$-regular, contradiction again. □

The main flatness criterion of this section is:

Theorem 3.3.21. *Let $A \to B$ be a homomorphism sending the Jacobson radical of A inside that of B, and let $\mathbf{K} \subseteq \mathbf{mod}_A$ be quasi-deformationally generating. A coherent B-module Q is flat over A if and only if $\mathrm{Tor}_1^A(Q, M) = 0$ for all $M \in \mathbf{K}$.*

Proof. One direction is immediate, so we only need to show the direct implication. Define a functor $\mathscr{F}$ on $\mathbf{mod}_R$, by $\mathscr{F}(M) := \mathrm{Tor}_1^A(Q, M)$. By Theorem 3.1.5, it suffices to show that $\mathscr{F}$ vanishes on each A/I with $I \subseteq A$ finitely generated. This will follow as soon as we can show that $\mathscr{F}(M) = 0$ for all M in the deformation class $\mathbf{N}$ of $\mathbf{K}$. By induction on the rules (3.3.18.i)–(3.3.18.iv), it will suffice to show that each new module M in $\mathbf{N}$ obtained from an application of one of these rules vanishes again on $\mathscr{F}$. The case of rule (3.3.18.i) is trivial; for (3.3.18.ii), we use that $\mathscr{F}$ is additive; and for (3.3.18.iii), we are done by the long exact sequence of Tor (3.1.4). So remains to verify the claim for rule (3.3.18.iv), that is to say, assume x is an M-regular element in the Jacobson radical of A such that $\mathscr{F}(M/xM) = 0$. Applying 3.1.4 to the exact sequence

$$0 \to M \xrightarrow{x} M \to M/xM \to 0$$

we get part of a long exact sequence

$$\mathscr{F}(M) \xrightarrow{x} \mathscr{F}(M) \to \mathscr{F}(M/xM) = 0. \tag{3.14}$$

Since M is finitely presented, we have an exact sequence

$$F \to A^m \to A^n \to M \to 0$$

with F some (possibly infinitely generated) free A-module. Tensoring with Q yields a complex

$$F \otimes_A Q \to Q^m \to Q^n \to M \otimes_A Q \to 0 \qquad (3.15)$$

whose first homology is by definition $\mathscr{F}(M)$. Since Q is a coherent module, so is any direct sum of Q by [35, Corollary 2.2.3], and hence the kernel of the morphism $Q^m \to Q^n$ in (3.15) is finitely generated by [35, Corollary 2.2.2]. Since $\mathscr{F}(M)$ is a quotient of this kernel, it, too, is finitely generated. By (3.14), we have an equality $\mathscr{F}(M) = x\mathscr{F}(M)$. By assumption, x belongs to the Jacobson radical of B, and hence, by Nakayama's Lemma, $\mathscr{F}(M) = 0$, as we needed to show. □

Combining Proposition 3.3.20 with Theorem 3.3.21 immediately gives the following well-known flatness criterion:

Corollary 3.3.22 (Local Flatness Criterion). *Let $R \to S$ be a local homomorphism of Noetherian local rings, and let k be the residue field of R. If M is a finitely generated S-module such that* $\operatorname{Tor}_1^R(M,k) = 0$*, then M is flat over R.* □

To extend this local flatness criterion to a larger class of rings, we make the following definition. Let us call a local ring R *ind-Noetherian*, if it is a direct limit of Noetherian local subrings R_i, indexed by a directed poset I, such that each $R_i \to R$ is a scalar extension (that is to say, faithfully flat and unramified; see §3.2.3). Clearly, any Noetherian local ring is ind-Noetherian (by taking $R_i = R$).

Lemma 3.3.23. *An ind-Noetherian local ring is coherent and has finite embedding dimension.*

Proof. Let $(R,\mathfrak{m})$ be ind-Noetherian. Since $\mathfrak{m}$ is in particular extended from a Noetherian local ring, it is finitely generated. We use Corollary 3.3.17 to prove coherency. To this end we must show that a finitely related ideal $(\mathfrak{a} : \mathfrak{b})$ is finitely generated. Since $\mathfrak{a}$ and $\mathfrak{b}$ are finitely generated, there exists a Noetherian local subring $S \subseteq R$ and ideals $I,J \subseteq S$ such that $S \to R$ is a scalar extension, and $\mathfrak{a} = IR$ and $\mathfrak{b} = JR$. Theorem 3.3.14 yields that $(I : J)R = (IR : JR) = (\mathfrak{a} : \mathfrak{b})$, whence in particular, is finitely generated. □

3.3.24 *If $R \to S$ is essentially of finite type and R is ind-Noetherian, then so is S.*

Indeed, S is isomorphic to the localization of $R[x]/(f_1,\dots,f_s)R[x]$ with respect to the ideal generated by the variables and by the maximal ideal of R. Hence, there is a directed subset $J \subseteq I$ such that $f_1,\dots,f_s$ are defined over each R_j with $j \in J$. It is now easy to see that the appropriate localization S_j of $R_j[x]/(f_1,\dots,f_s)R_j[x]$ forms a directed system with union equal to S, and each $S_j \to S$ is a scalar extension. □

Corollary 3.3.25. *Let $R \to S$ be a local homomorphism of ind-Noetherian rings. If Q is a finitely presented S-module such that $\mathrm{Tor}_1^R(Q,k) = 0$, where k is the residue field of R, then Q is flat over R. If Q is moreover Noetherian, then so is R.*

Proof. In view of Theorem 3.3.21, to prove the first assertion, we need to show that k is quasi-deformationally generating (note that S is coherent by Lemma 3.3.23, whence so is the finitely presented S-module Q). Let $\mathfrak{a} \subseteq R$ be a finitely generated ideal. Choose a Noetherian local subring T and an ideal $I \subseteq T$ such that $T \subseteq R$ is a scalar extension, and $IR = \mathfrak{a}$. By Proposition 3.3.20, the module T/I belongs to the deformation class of T-modules generated by the residue field l of T. Since each of the rules (3.3.18.i)–(3.3.18.iv) are preserved by faithfully flat extensions, $T/I \otimes_T R = R/\mathfrak{a}$ lies in the deformation class of $l \otimes_T R \cong k$, where the latter isomorphism follows from the unramifiedness of $T \to R$.

To prove that R is Noetherian, under the additional assumption that Q is Noetherian, let $\mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \dots$ be a chain of ideals in R. Choose i such that $\mathfrak{a}_i Q = \mathfrak{a}_j Q$ for all $j \geq i$. Hence $\mathfrak{a}_i/\mathfrak{a}_j \otimes Q = 0$, for $j \geq i$, and since Q is faithfully flat, as it is non-degenerated by 3.2.1, we get $\mathfrak{a}_i/\mathfrak{a}_j = 0$ by 3.2.3. □

3.3.7 Dimension Criterion for Flatness

If $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ is a local homomorphism of Noetherian local rings, then we have the following dimension inequality, with equality when $R \to S$ is flat (see [69, Theorem 15.1]):

$$\dim(S) \leq \dim(R) + \dim(S/\mathfrak{m}S). \tag{3.16}$$

Recall that we call $S/\mathfrak{m}S$ the *closed fiber* of $R \to S$: it defines the locus of all prime ideals in S which lie above $\mathfrak{m}$. Conversely, equality in (3.16) often implies flatness. We first discuss one well-known criterion, and then prove one new one.

Theorem 3.3.26. *Let $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a homomorphism of Noetherian local rings, with R regular and S Cohen-Macaulay. Then $R \to S$ is flat if and only if we have equality in* (3.16).

Proof. One direction holds always, as we discussed above. So assume we have equality in (3.16), that is to say, $e = d + h$ where d, h, and e, are the respective dimension of R, the closed fiber $S/\mathfrak{m}S$, and S. Let $(x_1,\dots,x_d)$ be a system of parameters of R. Since $S/\mathfrak{m}S$ has dimension $h = e - d$, there exist $x_{d+1},\dots,x_e$ in S such that their image in $S/\mathfrak{m}S$ is a system of parameters. Hence $(x_1,\dots,x_e)$ is a system of parameters in S, whence is an S-regular sequence. In particular, $(x_1,\dots,x_d)$ is S-regular, showing that S is a balanced big Cohen-Macaulay R-module, and therefore is flat by Theorem 3.3.9. □

For our last criterion, which generalizes a flatness criterion due to Kollár [62, Theorem 8], we impose some regularity condition on the closed fiber, weakening instead the conditions on the rings themselves.

Theorem 3.3.27. *Let $R \to S$ be a local homomorphism of Noetherian local rings. Assume R is either an excellent normal local domain with perfect residue field, or an analytically irreducible domain with algebraically closed residue field. If the closed fiber is regular, of dimension* $\dim(S) - \dim(R)$, *then $R \to S$ is faithfully flat.*

Proof. Let d and e be the respective dimensions of R and S. We will induct on the dimension $h := e - d$ of the closed fiber. If $h = 0$, then $R \to S$ is in fact unramified. It suffices to prove this case under the additional assumption that both R and S are complete. Indeed, if $R \to S$ is arbitrary, then $\widehat{R} \to \widehat{S}$ satisfies again the hypotheses of the theorem and therefore would be faithfully flat. Hence $R \to S$ is faithfully flat by Proposition 3.2.11.

So assume R and S are complete and let l be the residue field of S. Either assumption on R implies that $R\widehat{_l}$ is again a domain, of the same dimension as R (we leave this as an exercise to the reader; see [102, Corollary 3.10 and Proposition 3.11]). By the universal property of complete scalar extensions (Theorem 3.2.13—note that this result also holds in mixed characteristic, although we did not provide a proof in these notes; see [102, Corollary 3.3]), we get a local R-algebra homomorphism $R\widehat{_l} \to S$. By [69, Theorem 8.4], this homomorphism is surjective. It is also injective, since $R\widehat{_l}$ and S have the same dimension and $R\widehat{_l}$ is a domain. Hence $R\widehat{_l} \cong S$, so that $R \to S$ is a scalar extension, whence faithfully flat.

For the general case, $h > 0$, let $\tilde{R}$ be the localization of $R[\xi]$ at the ideal $\tilde{\mathfrak{m}}$ generated by $\mathfrak{m}$ and the variables $\xi := (\xi_1, \dots, \xi_h)$. By assumption, $\tilde{R}$ has the same dimension as S. Let $\mathbf{y}$ be an h-tuple whose image in the closed fiber is a regular system of parameters, that is to say, which generates $\mathfrak{n}(S/\mathfrak{m}S)$. Let $\tilde{R} \to S$ be the R-algebra homomorphism given by sending ξ to $\mathbf{y}$. Hence $\mathfrak{n} = \mathfrak{m}S + \mathbf{y}S = \tilde{\mathfrak{m}}S$, so that by the case $h = 0$, the homomorphism $\tilde{R} \to S$ is flat, whence so is $R \to S$. □

The requirement on R that we really need is that any complete scalar extension is again a domain, and for this, it suffices that the complete scalar extension over the algebraic closure of the residue field of R is a domain (see [102, Proposition 3.11]).

Chapter 4
Uniform Bounds

In this chapter, we will discuss our first application of ultraproducts: the existence of uniform bounds over polynomial rings. The method goes back to A. Robinson, but really gained momentum by the work of Schmidt and van den Dries in [86], where they brought in flatness as an essential tool. Most of our applications will be concerned with affine algebras over an ultra-field. For such an algebra, we construct its ultra-hull as a certain faithfully flat ultra-ring. As we will also use this construction in our alternative definition of tight closure in characteristic zero in Chapter 6, we study it in detail in §4.3. In particular, we study transfer between the affine algebra and its approximations. We conclude in §4.4 with some applications to uniform bounds, in the spirit of Schmidt and van den Dries.

4.1 Ultra-hulls

Let us fix an ultra-field K, realized as the ultraproduct of fields K_w for $w \in W$. For a concrete example, one may take $K := \mathbb{C}$ and $K_p := \mathbb{F}_p^{\mathrm{alg}}$ by Theorem 2.4.3 (with W the set of prime numbers). We make the construction of the ultra-hull in three stages.

4.1.1 Ultra-hull of a Polynomial Ring

In this section, we let $A := K[\xi]$, where $\xi := (\xi_1, \ldots, \xi_n)$ are indeterminates. We define the *ultra-hull* (called the *non-standard hull* in the earlier papers [88,89,92]) of A as the ultraproduct of the $A_w := K_w[\xi]$, and denote it $U(A)$. The inclusions $K_w \subseteq A_w$ induce an inclusion $K \subseteq U(A)$. Let ξ_i also denote the ultraproduct $\mathrm{ulim}_w\, \xi_i$ of the constant sequence ξ_i. By Łoś' Theorem, Theorem 2.3.1, the ξ_i are algebraically independent over K. Hence, we may view them as indeterminates over K in $U(A)$, thus yielding an embedding $A = K[\xi] \subseteq U(A)$. To see why this is called an ultra-hull, let us introduce the category of ultra-K-algebras: a K-algebra $B_\natural$ is

H. Schoutens, *The Use of Ultraproducts in Commutative Algebra*,
Lecture Notes in Mathematics 1999, DOI 10.1007/978-3-642-13368-8_4,

called an *ultra-K-algebra* if it is the ultraproduct of K_w-algebras B_w; a morphism of ultra-K-algebras $B_\natural \to C_\natural$ is any K-algebra homomorphism obtained as the ultraproduct of K_w-algebra homomorphisms $B_w \to C_w$. It follows that any ultra-K-algebra is a K-algebra. The ultra-hull $U(A)$ is clearly an ultra-K-algebra. We have:

4.1.1 *The ultra-hull $U(A)$ satisfies the following universal property: given an ultra-K-algebra $B_\natural$, and a K-algebra homomorphism $A \to B_\natural$, there exists a unique ultra-K-algebra homomorphism $U(A) \to B_\natural$ extending $A \to B$.*

Indeed, by assumption, $B_\natural$ is the ultraproduct of K_w-algebras B_w. Let $b_{i\natural}$ be the image of ξ_i under the homomorphism $A \to B_\natural$, and choose $b_{iw} \in B_w$ whose ultraproduct equals $b_{i\natural}$. Define K_w-algebra homomorphisms $A_w \to B_w$ by the rule $\xi_i \mapsto b_{iw}$. The ultraproduct of these homomorphisms is then the required ultra-K-algebra homomorphism $U(A) \to B_\natural$. Its uniqueness follows by an easy application of Łoś' Theorem. □

An intrinsic characterization of A as a subset of $U(A)$ is provided by the next result (in the terminology of Chapter 9, this exhibits A as a certain *protoproduct*):

4.1.2 *An ultraproduct $f_\natural \in U(A)$ belongs to A if and only if its approximations $f_w \in A_w$ have bounded degree, meaning that there is a d such that almost all f_w have degree at most d.*

Indeed, if $f \in A$ has degree d, then we can write it as $f = \sum_\nu u_\nu \xi^\nu$ for some $u_\nu \in K$, where ν runs over all n-tuples with $|\nu| \leq d$. Choose $u_{\nu w} \in K_w$ such that their ultraproduct is u_ν, and put

$$f_w := \sum_{|\nu|\leq d} u_{\nu w}\xi^\nu. \tag{4.1}$$

An easy calculation shows that the ultraproduct of the f_w is equal to f, viewed as an element in $U(A)$. Conversely, if almost each f_w has degree at most d, so that we can write it in the form (4.1), then

$$\operatorname*{ulim}_{w\to\infty} f_w = \sum_{|\nu|\leq d} (\operatorname*{ulim}_{w\to\infty} u_{\nu w})\xi^\nu$$

is a polynomial (of degree at most d). □

4.1.2 Ultra-hull of an Affine Algebra

More generally, let C be a *K-affine algebra*, that is to say, a finitely generated K-algebra, say of the form $C = A/I$ for some ideal $I \subseteq A$. We define the *ultra-hull* of C to be $U(A)/IU(A)$, and denote it $U(C)$. It is clear that the diagonal embedding $A \subseteq U(A)$ induces by base change a homomorphism $C \to U(C)$. Less obvious is that this is still an injective map, which we will prove in Corollary 4.2.3 below. To show that the construction of $U(C)$ does not depend on the choice of presentation $C = A/I$, we verify that $U(C)$ satisfies the same universal property 4.1.1 as $U(A)$: any K-algebra homomorphism $C \to B_\natural$ to some ultra-K-algebra $B_\natural$ extends uniquely to a homomorphism $U(C) \to B_\natural$ of ultra-K-algebras (recall that any solution to a universal property is necessarily unique). To see why the universal property holds, apply 4.1.1 to the composition $A \twoheadrightarrow A/I = C \to B_\natural$ to get a unique extension $U(A) \to B_\natural$. Since any element in I is sent to zero under the composition $A \to B_\natural$, this homomorphism factors through $U(A)/IU(A)$, yielding the required homomorphism $U(C) \to B_\natural$ of ultra-K-algebras. Uniqueness follows from the uniqueness of $U(A) \to B_\natural$.

Since $IU(A)$ is finitely generated, it is an ultra-ideal by 2.4.12, that is to say, an ultraproduct of ideals $I_w \subseteq A_w$, and the ultraproduct of the $C_w := A_w/I_w$ is equal to $U(C) = U(A)/IU(A)$ by 2.1.6. If $C = A'/I'$ is a different presentation of C as a K-algebra (with A' a polynomial ring in finitely many indeterminates), and $C'_w := A'_w/I'_w$ the corresponding K_w-algebras, then their ultraproduct $U(A')/I'U(A')$ is another way of defining the ultra-hull of C, whence it must be isomorphic to $U(C)$. Without loss of generality, we may assume $A \subseteq A'$ and hence $A_w \subseteq A'_w$. Since $U(A)/IU(A) \cong U(C) \cong U(A')/I'U(A')$, the homomorphisms $A_w \subseteq A'_w$ induce homomorphisms $C_w \to C'_w$, and by Łoś' Theorem, almost all are isomorphisms. This justifies the usage of calling the C_w *approximations* of C (in spite of the fact that they are not uniquely determined by C).

4.1.3 *The ultra-hull $U(\cdot)$ is a functor from the category of K-affine algebras to the category of ultra-K-algebras.*

The only thing which remains to be verified is that an arbitrary K-algebra homomorphism $C \to D$ of K-affine rings induces a homomorphism of ultra-K-algebras $U(C) \to U(D)$. However, this follows from the universal property applied to the composition $C \to D \to U(D)$, admitting a unique extension so that the following diagram is commutative

$$\begin{array}{ccc} C & \longrightarrow & D \\ \downarrow & & \downarrow \\ U(C) & \longrightarrow & U(D). \end{array} \tag{4.2}$$

□

4.1.3 Ultra-hull of a Local Affine Algebra

Recall that a K-affine local ring R is simply the localization $C_{\mathfrak{p}}$ of a K-affine algebra C at a prime ideal $\mathfrak{p}$. Let us call R *geometric*, if $\mathfrak{p}$ is a maximal ideal $\mathfrak{m}$ of C. A geometric K-affine local ring, in other words, is the local ring of a closed point on an affine scheme of finite type over K. Note that a local K-affine algebra is in general not finitely generated as a K-algebra; one usually says that R is *essentially of finite type* over K. The next result will enable us to define the ultra-hull of a geometric affine local ring; we shall discuss the general case in §4.3.2 below (see Remark 4.3.5):

4.1.4 *Let C be a K-affine algebra. If $\mathfrak{m}$ is a maximal ideal in C, then $\mathfrak{m}U(C)$ is a maximal ideal in $U(C)$, and $C/\mathfrak{m} \cong U(C)/\mathfrak{m}U(C)$.*

By our previous discussion, $U(L) := U(C)/\mathfrak{m}U(C)$ is the ultra-hull of the field $L := C/\mathfrak{m}$. By the Nullstellensatz, the extension $K \subseteq L$ is finite, and from this, it is easy to see that L is an ultra-field. By the universal property, L is equal to its own ultra-hull, and hence $\mathfrak{m}U(C)$ is a maximal ideal. □

We can now define the ultra-hull of a local K-affine algebra $R = C_{\mathfrak{m}}$ as the localization $U(R) := U(C)_{\mathfrak{m}U(C)}$. Note that $U(R)$ is again an ultra-ring: let C_w be approximations of C, and let $\mathfrak{m}_w \subseteq C_w$ be ideals whose ultraproduct is equal to $\mathfrak{m}U(C)$. Since the latter is maximal, so are almost all $\mathfrak{m}_w$. For those w, set $R_w := (C_w)_{\mathfrak{m}_w}$ (and arbitrary for the remaining w). One easily verifies that $U(R)$ is then isomorphic to the ultraproduct of the R_w, and for this reason we call the R_w again an *approximation* of R. We can formulate a similar universal property which is satisfied by $U(R)$, and then show that any local homomorphism $R \to S$ of local K-affine algebras induces a unique homomorphism $U(R) \to U(S)$. Moreover, any two approximations agree almost everywhere. In particular, for homomorphic images we have:

4.1.5 *If $I \subseteq C$ is an ideal in a K-affine (local) ring, then $U(C/I) = U(C)/IU(C)$.*

We extend our nomenclature also to elements and ideals: if $a \in C$ is an element or $I \subseteq C$ is an ideal, and $a_w \in C_w$ and $I_w \subseteq C_w$ are such that their ultraproduct equals $a \in U(C)$ and $IU(C)$ respectively, then we call the a_w and the I_w *approximations* of a and I respectively. In particular, by 4.1.4, the approximations of a maximal ideal are almost all maximal. The same holds true with 'prime' instead of 'maximal', but the proof is more involved, and we have to postpone it until Theorem 4.3.4 below.

4.2 The Schmidt-van den Dries Theorem

The ring $U(A)$ is highly non-Noetherian. In particular, although each $\mathfrak{m}U(A)$ is a maximal ideal for $\mathfrak{m}$ a maximal ideal of $A = K[\xi]$, these are not the only maximal ideals of $U(A)$. To see an example, choose, for each w, a polynomial $f_w \in A_w$ in ξ_1 of degree w with distinct roots in K_w (assuming K_w has at least size w), and let $f \in U(A)$ be their ultraproduct. Let $\mathfrak{a}$ be the ideal generated by all f/h where h runs over all elements in A such that $f \in hU(A)$. Since f has infinitely many roots,[1] $\mathfrak{a}$ is not the unit ideal, and hence is contained in some maximal ideal $\mathfrak{M}$ of $U(A)$. However, for the same reason, $\mathfrak{a}$ cannot be inside a maximal ideal of the form $\mathfrak{m}U(A)$ with $\mathfrak{m} \subseteq A$, showing that $\mathfrak{M}$ is not of the latter form. In fact, $\mathfrak{M}$ is not even an ultra-ideal.

Nonetheless, the maximal ideals that are extended from A 'cover' enough of $U(A)$ in order to apply Theorem 3.3.16. More precisely:

4.2.1 *If almost all K_w are algebraically closed, then any proper finitely related ideal of $U(A)$ is contained in some $\mathfrak{m}U(A)$ with $\mathfrak{m} \subseteq A$ a maximal ideal.*

Indeed, this is even true for any proper ultra-ideal $I \subseteq U(A)$. Namely, let I be the ultraproduct of ideals $I_w \subseteq A_w$. By Łoś' Theorem, almost each I_w is a proper ideal whence contained in some maximal ideal $\mathfrak{m}_w$. By the Nullstellensatz, we can write $\mathfrak{m}_w$ as $(\xi_1 - u_{1w}, \dots, \xi_n - u_{nw})A_w$ for some $u_{iw} \in K_w$. Let $u_i \in K$ be the ultraproduct of the u_{iw}, so that the ultraproduct of the $\mathfrak{m}_w$ is equal to $(\xi_1 - u_1, \dots, \xi_n - u_n)U(A)$, and by Łoś' Theorem it contains I. □

It is necessary that almost all K_w are algebraically closed. For instance, if all K_w are equal to $\mathbb{Q}$ (whence K is the ultrapower $\mathbb{Q}_\natural$), and we let $\mathfrak{m}_w$ be the ideal in $\mathbb{Q}[\xi]$ generated by the w-th cyclotomic polynomial, then the ultraproduct $\mathfrak{m}_\natural$ of the $\mathfrak{m}_w$ is principal but contains no non-zero element of $\mathbb{Q}_\natural[\xi]$.

Theorem 4.2.2. *For any K-affine algebra, the diagonal embedding $C \to U(C)$ is faithfully flat, whence in particular injective.*

Proof. If we have proven this result for the ultra-hull $U(A)$ of A, then it will follow from 3.1.3 for any $C \to U(C)$, since the latter is just a base change $C = A/I \to U(A)/IU(A) = U(C)$, where $C = A/I$ is some presentation of C. The non-degeneratedness of $U(A)$ is immediate from 4.1.4. So remains to show the flatness of $A \to U(A)$, and for this we may assume that K and all K_w are algebraically closed. Indeed, if K' is the ultraproduct of the algebraic closures of the K_w, then $A \to A' := K'[\xi]$ is flat by 3.1.3. By Łoś' Theorem, the canonical homomorphism $U(A) \to U(A')$ is cyclically pure with respect to ideals extended from A, where $U(A')$ is the ultra-K'-hull of A. Hence if we showed that $A' \to U(A')$ is flat, then so is $A \to U(A)$ by Corollary 3.3.15. Hence we may assume all K_w are algebraically closed. By Theorem 3.3.16 in conjunction with 4.2.1, we only need to show that $R := A_\mathfrak{m} \to U(R) = U(A)_{\mathfrak{m}U(A)}$ is flat for every maximal ideal $\mathfrak{m} \subseteq A$. After a translation, we may assume $\mathfrak{m} = (\xi_1, \dots, \xi_n)A$. By Łoś' Theorem, $(\xi_1, \dots, \xi_n)$ is an $U(A)$-regular sequence whence $U(R)$-regular. This proves that $U(R)$ is a big Cohen-Macaulay R-module. By Proposition 3.3.8 it is therefore a balanced big Cohen-Macaulay module, since any regular sequence in $U(R)$ is permutable by Łoś' Theorem, because this is so in the Noetherian local rings $(A_w)_{\mathfrak{m}_w}$ (see [69, Corollary to Theorem 16.3]). Hence $U(R)$ is flat over R by Theorem 3.3.9. □

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[1] Notwithstanding that f is only an *ultra-polynomial*, we may view it by 2.1.2 as a function on $K_\natural$, and a *root* of f then means an element $u \in K_\natural$ such that $f(u) = 0$ (which means that, for any approximation $u_w \in K_w$ of u, almost all $f_w(u_w) = 0$ are zero).

Immediately from this and the cyclic purity of faithfully flat homomorphisms (Proposition 3.2.5) we get:

Corollary 4.2.3. *The diagonal embedding $C \to U(C)$ is injective, and $IU(C) \cap C = I$ for any ideal $I \subseteq C$.* □

4.3 Transfer of Structure

We will use ultra-hulls in our definition of tight closure in characteristic zero (see §6), and to this end, we need to investigate more closely the relation between an affine algebra and its approximations. We start with the following far reaching generalization of 4.1.4.

4.3.1 Finite Extensions

Proposition 4.3.1. *If $C \to D$ is a finite homomorphism of K-affine algebras, then $U(D) \cong U(C) \otimes_C D$, and hence $U(C) \to U(D)$ is also finite.*

Proof. Since D is finite as a module over C, the tensor product $U(C) \otimes_C D$ is finite over $U(C)$, whence an ultra-K-algebra. By the universal property of the ultra-hull of D, we therefore have a unique homomorphism $U(D) \to U(C) \otimes_C D$ of ultra-K-algebras. On the other hand, by the universal property of tensor products, we have a unique homomorphism $U(C) \otimes_C D \to U(D)$. It is no hard to see that the latter is in fact a morphism of ultra-K-algebras. By uniqueness of both homomorphisms, they must therefore be each other's inverse. □

Corollary 4.3.2. *If C is an Artinian K-affine algebra, then $C \cong U(C)$.*

Proof. Since C is a direct product of local Artinian rings ([27, Corollary 9.1]), and since ultra-hulls are easily seen to commute with direct products, we may assume C is moreover local, with maximal ideal $\mathfrak{m}$, say. Let $L := C/\mathfrak{m}$, so that $L \cong U(L)$ by 4.1.4. Note that the vector space dimension of C over L is equal to the length of C. In any case, C is a finite L-module, so that by Proposition 4.3.1 we get $U(C) = U(L) \otimes_L C = C$. □

Corollary 4.3.3. *The dimension of a K-affine algebra is equal to the dimension of almost all of its approximations.*

Proof. Let C be an n-dimensional K-affine algebra, with approximations C_w. The assertion is trivial for $C = A$ a polynomial ring. By Noether normalization (see for instance [27, Theorem 13.3]), there exists a finite extension $A \subseteq C$. The induced homomorphism $U(A) \to U(C) \cong U(A) \otimes_A C$ is finite, by Proposition 4.3.1, and injective since $A \to U(A)$ is flat by Theorem 4.2.2. By Łoś' Theorem, almost all $A_w \to C_w$ are finite and injective. Hence almost all C_w have dimension n by [27, Proposition 9.2.]. □

4.3.2 Prime Ideals

We return to our discussion on the behavior of prime ideals under the ultra-hull, and we are ready to prove the promised generalization of 4.1.4 (this was originally proven in [86] by different means).

Theorem 4.3.4. *A K-affine algebra C is a domain if and only if $U(C)$ is, if and only if almost all of its approximations are. In particular, if $\mathfrak{p}$ is a prime ideal in an arbitrary K-affine algebra D, then $\mathfrak{p}U(D)$ is again a prime ideal, and so are almost all of its approximations $\mathfrak{p}_w$.*

Proof. By Łoś' Theorem, almost all C_w are domains if and only if $U(C)$ is a domain. If this holds, then C too is a domain since it is a subring of $U(C)$ by Corollary 4.2.3. Conversely, assume C is a domain, and let $A \subseteq C$ be a Noether normalization of C, that is to say a finite and injective extension. Let $A_w \subseteq C_w$ be the corresponding approximations implied by Proposition 4.3.1. Let $\mathfrak{p}_w$ be a prime ideal in C_w of maximal dimension, and let $\mathfrak{p}_\natural$ be their ultraproduct, a prime ideal in $U(C)$. An easy dimension argument shows that $\mathfrak{p}_w \cap A_w = (0)$ and hence by Łoś' Theorem, $\mathfrak{p}_\natural \cap U(A) = (0)$. Let $\mathfrak{p} := \mathfrak{p}_\natural \cap C$. Since $\mathfrak{p} \cap A$ is contained in $\mathfrak{p}_\natural \cap U(A)$, it is also zero. Hence $A \to C/\mathfrak{p}$ is again finite and injective. Since C is a domain, an easy dimension argument yields that $\mathfrak{p} = 0$. On the other hand, we have an isomorphism $U(C) = U(A) \otimes_A C$, so that by general properties of tensor products

$$U(C)/\mathfrak{p}_\natural = U(A)/(\mathfrak{p}_\natural \cap U(A)) \otimes_{A/(\mathfrak{p}_\natural \cap A)} C/(\mathfrak{p}_\natural \cap C) = U(A) \otimes_A C = U(C)$$

showing that $\mathfrak{p}_\natural$ is zero, whence so are almost all $\mathfrak{p}_w$. Hence almost all C_w are domains, and hence by Łoś' Theorem, so is $U(C)$.

The last assertion is immediate from the first applied to $C := D/\mathfrak{p}$. □

Remark 4.3.5. This allows us to define the ultra-hull of an arbitrary local K-affine algebra $C_\mathfrak{p}$ as the localization $U(C)_{\mathfrak{p}U(C)}$.

To show that a local affine algebra has the same dimension as almost all of its approximations, we introduce a new dimension notion. Let $(R, \mathfrak{m})$ be a local ring of finite embedding dimension (that is to say, with a finitely generated maximal ideal).

Definition 4.3.6 (Geometric Dimension). We define the *geometric dimension* of R, denoted $\operatorname{geodim}(R)$, as the least number of elements generating an $\mathfrak{m}$-primary ideal.

As the (Krull) dimension $\dim(R)$ equals the dimension of the topological space $V := \operatorname{Spec}(R)$, it is essentially a topological invariant. On the other hand, $\operatorname{geodim}(R)$ is the least number of equations defining the closed point x corresponding to the maximal ideal $\mathfrak{m}$, and hence is a geometric invariant. It is a well-known result from commutative algebra that for Noetherian local rings (Krull) dimension equals geometric dimension (see, for instance, [69, Theorem 13.4]).

The next result shows that this is no longer true if one drops the Noetherianity condition, since ultra-rings are in general infinite dimensional (for some calculations of their prime spectrum, see [72,73,74]).

Proposition 4.3.7. *If $(R,\mathfrak{m})$ is a d-dimensional local K-affine algebra, then $U(R)$ has geometric dimension d.*

Proof. We induct on the dimension d, where the case $d=0$ follows from Corollary 4.3.2. So assume $d > 0$, and let x be a parameter in R. Hence, R/xR has dimension $d-1$, so that by induction, $U(R/xR)$ has geometric dimension $d-1$. Since $U(R/xR) = U(R)/xU(R)$ by 4.1.5, we see that $U(R)$ has geometric dimension at most d. By way of contradiction, suppose its geometric dimension is at most $d-1$. In particular, there exists an $\mathfrak{m}U(R)$-primary ideal $\mathfrak{N}$ generated by $d-1$ elements. Put $\mathfrak{n} := \mathfrak{N} \cap R$, and let n be such that $\mathfrak{m}^n U(R) \subseteq \mathfrak{N}$. By faithful flatness, that is to say, by Corollary 4.2.3, we have an inclusion $\mathfrak{m}^n \subseteq \mathfrak{n}$, showing that $\mathfrak{n}$ is $\mathfrak{m}$-primary. Hence $R/\mathfrak{n} \cong U(R/\mathfrak{n}) = U(R)/\mathfrak{n}U(R)$ by Corollary 4.3.2. Hence $U(R)/\mathfrak{N}$ is a homomorphic image of $R/\mathfrak{n}$ whence equal to it by definition of $\mathfrak{n}$. In conclusion, $\mathfrak{N} = \mathfrak{n}U(R)$. Since R has geometric dimension d, the $\mathfrak{m}$-primary ideal $\mathfrak{n}$ requires at least d generators. Since $R \to U(R)$ is flat by Theorem 4.2.2, also $\mathfrak{n}U(R)$ requires at least d generators by 3.2.7, contradiction. □

We can now extend the result from Corollary 4.3.3 to the local case as well:

Corollary 4.3.8. *The dimension of a local K-affine algebra R is equal to the dimension of almost all of its approximations R_w. Moreover, if $\mathbf{x}$ is a sequence in R with approximations $\mathbf{x}_w$, then $\mathbf{x}$ is a system of parameters if and only if almost all $\mathbf{x}_w$ are.*

Proof. The second assertion follows immediately from the first and Łoś' Theorem. By Proposition 4.3.7, the geometric dimension of $U(R)$ is equal to $d := \dim(R)$. Let R_w be approximations of R, so that their ultraproduct equals $U(R)$. If I is an $\mathfrak{m}U(R)$-primary ideal generated by d elements, then its approximation I_w is an $\mathfrak{m}_w$-primary ideal generated by d elements for almost all w by 2.4.11. Hence almost all R_w have (geometric) dimension at most d.

Let $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_d = \mathfrak{m}$ be a chain of prime ideals in R of maximal length. By faithful flatness (in the form of Corollary 4.2.3), this chain remains strict when extended to $U(R)$, and by Theorem 4.3.4, it consists again of prime ideals. Hence if $\mathfrak{p}_{iw} \subseteq R_w$ are approximations of $\mathfrak{p}_i$, then by Łoś' Theorem, we get a strict chain of prime ideals $\mathfrak{p}_{0w} \subsetneq \cdots \subsetneq \mathfrak{p}_{dw} = \mathfrak{m}_w$ for almost all w, proving that almost all R_w have dimension at least d. □

Note that it is not true that if $\mathbf{x}_w$ are systems of parameters in the approximations, then their ultraproduct (which in general lies outside R) does not necessarily generate an $\mathfrak{m}U(R)$-primary ideal.

4.3.3 Singularities

Now that we know how dimension behaves under ultra-hulls, we can investigate singularities.

Theorem 4.3.9. *A local K-affine algebra is respectively regular or Cohen-Macaulay if and only if almost all its approximations are.*

Proof. Let R be a d-dimensional local K-affine algebra, and let R_w be its approximations. If R is regular, then its embedding dimension is d, whence so is the embedding dimension of $U(R)$, and by Łoś' Theorem, then so is the embedding dimension of R_w for almost each w, and conversely. Corollary 4.3.8 then proves the assertion for regularity. As for the Cohen-Macaulay condition, let $\mathbf{x}$ be a system of parameters with approximation $\mathbf{x}_w$. Hence almost each $\mathbf{x}_w$ is a system of parameters in R_w by Corollary 4.3.8. If R is Cohen-Macaulay, then $\mathbf{x}$ is an R-regular sequence, hence $U(R)$-regular by flatness (see Proposition 3.2.9), whence almost each $\mathbf{x}_w$ is R_w-regular by Łoś' Theorem, and almost all R_w are Cohen-Macaulay. The converse follows along the same lines. □

4.4 Uniform Bounds

In this last section, we discuss some applications of ultraproducts to the study of rings. The results as well as the proof method via ultraproducts are due to Schmidt and van den Dries from their seminal paper [86], and were further developed in [84, 88, 89, 98].

4.4.1 Linear Equations

The proof of the next result is very typical for an argument based on ultraproducts, and will be the template for all future proofs.

Theorem 4.4.1 (Schmidt-van den Dries). *For any pair of positive integers (d,n), there exists a uniform bound $b := b(d,n)$ with the following property. Let k be a field, and let $f_0,\dots,f_s \in k[\xi]$ be polynomials of degree at most d in at most n indeterminates ξ such that $f_0 \in (f_1,\dots,f_s)k[\xi]$. Then there exist $g_1,\dots g_s \in k[\xi]$ of degree at most b such that $f_0 = g_1f_1+\dots+g_sf_s$.*

Proof. By way of contradiction, suppose this result is false for some pair (d,n). This means that we can produce counterexamples requiring increasingly high degrees. Before we write these down, observe that the number s of polynomials in these counterexamples can be taken to be the same by Lemma 4.4.2 below (by adding zero polynomials if necessary). So, for each $w \in \mathbb{N}$, we can find counterexamples consisting of a field K_w, and polynomials $f_{0w},\dots,f_{sw} \in A_w := K_w[\xi]$ of degree at most d, such that f_{0w} can be written as an A_w-linear combination of the $f_{1w},\dots,f_{sw}$, but any such linear combination involves a polynomial of degree at least w. Let f_i be the ultraproduct of the f_{iw}. This is again a polynomial of degree d in A by 4.1.2. Moreover, by Łoś' Theorem, $f_0 \in (f_1,\dots,f_d)U(A)$. We use the flatness of $A \to U(A)$ via its corollary in 4.2.3, to conclude that $f_0 \in (f_1,\dots,f_s)U(A) \cap A = (f_1,\dots,f_s)A$. Hence we can find polynomials $g_i \in A$ such that

$$f_0 = g_1 f_1 + \dots + g_s f_s. \tag{4.3}$$

Let e be the maximum of the degrees of the g_i. By 4.1.2 again, we can choose approximations $g_{iw} \in A_w$ of each g_i, of degree at most e. By Łoś' Theorem, (4.3) yields for almost all w that $f_{0w} = \sum_i g_{iw} f_{iw}$, contradicting our assumption. □

Lemma 4.4.2. *Any ideal in A generated by polynomials of degree at most d requires at most* $b := \binom{d+n}{n}$ *generators.*

Proof. Note that b is equal to the number of monomials of degree at most d in n variables. Let $I := (f_1,\dots,f_s)A$ be an ideal in A with each f_i of degree at most d. Choose some (total) ordering $<$ on these monomials (e.g., the lexicographical ordering on the exponent vectors), and let $l(f)$ denote the largest monomial appearing in f with non-zero coefficient, for any $f \in A$ of degree at most d (where we put $l(0) := -\infty$). If $l(f_i) = l(f_j)$ for some non-zero f_i, f_j with $i < j$, then $l(uf_i - vf_j) < l(f_i)$ for some non-zero elements $u, v \in K$, and we may replace the generator f_j by the new generator $uf_i - vf_j$. Doing this recursively for all i, we arrive at a situation in which all non-zero f_i have different $l(f_i)$, and hence there can be at most b of these. □

We can reformulate the result in Theorem 4.4.1 to arrive at some further generalizations. The ideal membership condition in that theorem is really about solving an (inhomogeneous) linear equation in A: the equation $f_0 = f_1 t_1 + \dots + f_s t_s$, where the t_i are the unknowns of this equation (as opposed to ξ, which are indeterminates). One can then easily extend the previous argument to arbitrary systems of equations: there exists a uniform bound $b := b(d,n)$ such that for any field k, and for any linear system of equations $\lambda_1 = \dots = \lambda_s = 0$ with $\lambda_i \in k[\xi, t]$ of ξ-degree at most d and t-degree at most one, where ξ is an n-tuple of indeterminates and t is a finite tuple of variables, if the system admits a solution in $k[\xi]$, then it admits a solutions all of whose entries have degree at most b. In the homogeneous case we can say even more:

Theorem 4.4.3. *For any pair of positive integers (d,n), there exists a uniform bound $b := b(d,n)$ with the following property. Over a field k, any homogeneous linear system of equations with coefficients in $k[\xi_1,\dots,\xi_n]$ all of whose coefficients have degree at most d, admits a finite number of solutions of degree at most b such that any other solution is a linear combination of these finitely many solutions.*

Proof. The proof once more is by contradiction. Assume the statement is false for the pair (n,d). Hence we can find for each $w \in \mathbb{N}$, a field K_w, and a linear system of homogeneous equations

$$\lambda_{1w}(t) = \cdots = \lambda_{sw}(t) = 0 \qquad (\mathscr{L}_w)$$

in the variables $t = (t_1, \ldots, t_m)$ with coefficients in A_w, such that the module of solutions $\mathrm{Sol}_{A_w}(\mathscr{L}_w) \subseteq A_w^k$ requires at least one generator one of whose entries is a polynomial of degree at least w. Here, we may again take the number m of t-variables as well as the number s of equations to be the same in all counterexamples, by another use of Lemma 4.4.2. The ultraproduct of each λ_{iw} is, as before by 4.1.2, an element $\lambda_i \in A[t]$ which is a linear form in the t-variables (and has degree at most d in ξ). By the equational criterion for flatness, Theorem 3.3.1, the flatness of $A \to U(A)$, proven in Theorem 4.2.2, amounts to the existence of solutions $\mathbf{b}_1, \ldots, \mathbf{b}_l \in \mathrm{Sol}_A(\mathscr{L})$ such that any solution of the homogeneous linear system $\mathscr{L}$ of equations $\lambda_1 = \cdots = \lambda_s = 0$ in $U(A)$ lies in the $U(A)$-module generated by the $\mathbf{b}_i$. Let e be the maximum of the degrees occurring in the $\mathbf{b}_i$. In particular, we can find approximations $\mathbf{b}_{iw} \in A_w^m$ of $\mathbf{b}_i$ whose entries all have degree at most e. I claim that almost each $\mathrm{Sol}_{A_w}(\mathscr{L}_w)$ is equal to the submodule H_w generated by $\mathbf{b}_{1w}, \ldots, \mathbf{b}_{lw}$, which would then contradict our assumption.

To prove the claim, one inclusion is clear, so assume by way of contradiction that we can find for almost all w a solution $\mathbf{q}_w \in \mathrm{Sol}_{A_w}(\mathscr{L}_w)$ outside H_w. Let $\mathbf{q}_\natural \in U(A)^m$ be its ultraproduct (note that this time, we cannot guarantee that its entries lie in A since the degrees might be unbounded). By Łoś' Theorem, $\mathbf{q}_\natural \in \mathrm{Sol}_{U(A)}(\mathscr{L})$, whence can be written as an $U(A)$-linear combination of the $\mathbf{b}_i$. Writing this out and using Łoś' Theorem once more, we conclude that $\mathbf{q}_w$ lies in H_w for almost all w, contradiction. □

4.4.2 *Primality Testing*

The next result, with a slightly different proof from the original, is also due to Schmidt and van den Dries.

Theorem 4.4.4 (Schmidt-van den Dries). *For any pair of positive integers (d,n), there exists a uniform bound $b := b(d,n)$ with the following property. Let k be a field, and let $\mathfrak{p}$ be an ideal in $k[\xi_1, \ldots, \xi_n]$ generated by polynomials of degree at most d. Then $\mathfrak{p}$ is a prime ideal if and only if for any two polynomials f,g of degree at most b which do not belong to $\mathfrak{p}$, neither does their product.*

Proof. One direction in the criterion is obvious. Suppose the other is false for the pair (d,n), so that we can find for each $w \in \mathbb{N}$, a field K_w and a non-prime ideal $\mathfrak{a}_w \subseteq A_w$ generated by polynomials of degree at most d, such that any two polynomials of degree at most w not in $\mathfrak{a}_w$ have their product also outside $\mathfrak{a}_w$. Taking ultraproducts of the generators of the $\mathfrak{a}_w$ of degree at most d gives polynomials of degree at most d in A by 4.1.2, and by Łoś' Theorem if $\mathfrak{a} \subseteq A$ is the ideal they generate, then $\mathfrak{a}U(A)$ is the ultraproduct of the $\mathfrak{a}_w$. I claim that $\mathfrak{a}$ is a prime ideal. However, this implies that almost all $\mathfrak{a}_w$ must be prime ideals by Theorem 4.3.4, contradiction.

To verify the claim, let $f,g \notin \mathfrak{a}$. We want to show that $fg \notin \mathfrak{a}$. Let e be the maximum of the degrees of f and g. Choose approximations $f_w, g_w \in A_w$ of degree at most e, of f and g respectively. By Łoś' Theorem, $f_w, g_w \notin \mathfrak{a}_w$ for almost all w. For $w \geq e$, our assumption then implies that $f_w g_w \notin \mathfrak{a}_w$, whence by Łoś' Theorem, their ultraproduct $fg \notin \mathfrak{a}U(A)$. A fortiori, then neither does fg belong to $\mathfrak{a}$, as we wanted to show. □

The pattern by now must become clear: prove that a particular property of ideals is preserved under ultra-hulls, and use this to deduce uniform bounds. For instance, one can easily derive from Theorem 4.3.4 that:

Proposition 4.4.5. *The image of a radical ideal in the ultra-hull remains radical.* □

Since the radical of an ideal is the intersection of its minimal overprimes, we derive from this the following uniform bounds property:

Theorem 4.4.6. *For any pair of positive integers (d,n), there exists a uniform bound $b := b(d,n)$ with the following property. Let k be a field, and let I be an ideal in $k[\xi_1,\dots,\xi_n]$ generated by polynomials of degree at most d. Then its radical $J := \operatorname{rad}(I)$ is generated by polynomials of degree at most b. Moreover, $J^b \subseteq I$ and I has at most b distinct minimal overprimes, all of which are generated by polynomials of degree at most b.* □

Similarly, we can use Theorem 3.3.14, the Colon Criterion, to show that there exists a uniform bound $b := b(d,n)$ such that for any field k, any ideal $I \subseteq k[\xi]$ generated by polynomials of degree at most d, and any $a \in k[\xi]$ of degree at most d in the n indeterminates ξ, the ideal $(I : a)$ is generated by polynomials of degree at most b.

Realizing a finitely generated module as the cokernel of a matrix (acting on a free module) and using that ultraproducts commute with homology (Theorem 3.1.1), one can extend all the previous bounds to modules as well. This was the route taken in [88]. Without proof, I state one of the results of that paper proven by this technique.

Theorem 4.4.7 ([88, Theorem 4.5]). *For any triple of positive integers (d,n,i), there exists a uniform bound $b := b(d,n,i)$ with the following property. Let k be a field, let B be an affine algebra of the form $k[\xi_1,\dots,\xi_n]/I$ with I an ideal generated by polynomials of degree at most d, and let M and N be finitely generated B-modules realized as the cokernel of matrices of size at most d and with entries of degree at most d. If $M \otimes_A N$ has finite length, then the i-th Betti number, that is to say, the length of $\operatorname{Tor}_i^A(M,N)$, is bounded by b. Similarly, if $\operatorname{Hom}_A(M,N)$ has finite length, then the i-th Bass number, that is to say, the length of $\operatorname{Ext}_A^i(M,N)$, is at most b.* □

4.4.3 Comments

Our proof of the flatness of ultra-hulls (Theorem 4.2.2) is entirely different from the original proof of Schmidt and van den Dries, which uses an induction on the number of indeterminates based on classical arguments of Hermann from constructive commutative algebra. The present approach via big Cohen-Macaulay algebras has the advantage that one can extend this method to many other situations, like Theorem 7.1.6 below. Yet another approach, through a coherency result due to Vasconcelos, can be found in [5].

As already mentioned, some of the bounds proven here were already established by Hermann [42], based on work of Seidenberg [103, 104, 105] on constructions in algebra. Using Groebner bases, Buchberger obtained in [18] the same result, but by an explicit description of an algorithm (e.g., one that calculates the polynomials g_i in Theorem 4.4.1). This led to a direct implementation into various algebraic software programs, which was not practically feasible in the case of Hermann's explicit proof using elimination theory, in view of the exponential growth of degrees of polynomials involved in this elimination process. Model-theoretic proofs, such as the ones in this book, lack even more practical implementation, but they provide sometimes extra information. For instance, we show that there exist uniform bounds that are independent of the base field. With some additional work, it is sometimes possible to show that the bounds are recursive (see, for instance, [11]). But even these abstract methods can sometimes lead to explicit bounds, as is evident from Schmidt's work [84, 85].

Chapter 5
Tight Closure in Positive Characteristic

In this chapter, p is a fixed prime number, and all rings are assumed to have characteristic p, unless explicitly mentioned otherwise. We review the notion of tight closure due to Hochster and Huneke (as a general reference, we will use [59]). The main protagonist in this elegant theory is the p-th power Frobenius map. We will focus on five key properties of tight closure, which will enable us to prove, virtually effortlessly, several beautiful theorems. Via these five properties, we can give a more axiomatic treatment, which lends itself nicely to generalization, and especially to a similar theory in characteristic zero (see Chapters 6 and 7).

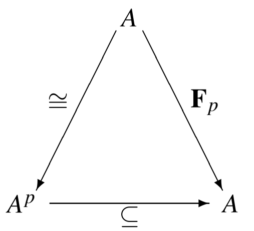
5.1 Frobenius

The major advantage of rings of positive characteristic is the presence of an algebraic endomorphism: the Frobenius. More precisely, let A be a ring of characteristic p, and let $\mathbf{F}_p$, or more accurately, $\mathbf{F}_{p,A}$, be the ring homomorphism $A \to A\colon a \mapsto a^p$, called the *Frobenius* on A. Recall that this is indeed a ring homomorphism, where the only thing to note is that the coefficients in the binomial expansion

$$\mathbf{F}_p(a+b) = \sum_{i=0}^{p} \binom{p}{i} a^i b^{p-i} = \mathbf{F}_p(a) + \mathbf{F}_p(b)$$

are divisible by p for all $0 < i < p$ whence zero in A, proving that $\mathbf{F}_p$ is additive.

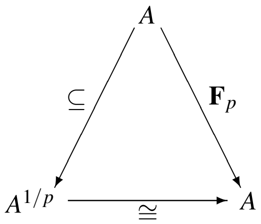
When A is reduced, $\mathbf{F}_p$ is injective whence yields an isomorphism with its image $A^p := \mathrm{Im}(\mathbf{F}_p)$ consisting of all p-th powers of elements in A (and not to be confused with the p-th Cartesian power of A). The inclusion $A^p \subseteq A$ is isomorphic with the Frobenius on A because we have a commutative diagram



(5.1)

H. Schoutens, *The Use of Ultraproducts in Commutative Algebra*,
Lecture Notes in Mathematics 1999, DOI 10.1007/978-3-642-13368-8_5,

When A is a domain, then we can also define the ring $A^{1/p}$ as the subring of the algebraic closure of the field of fractions of A consisting of all elements b satisfying $b^p \in A$. Hence $A \subseteq A^{1/p}$ is integral. Since, $\mathbf{F}_p(A^{1/p}) = A$ and $\mathbf{F}_p$ is injective, we get $A^{1/p} \cong A$. Moreover, we have a commutative diagram



(5.2)

showing that the Frobenius on A is also isomorphic to the inclusion $A \subseteq A^{1/p}$. It is sometimes easier to work with either of these inclusions rather than with the Frobenius itself, especially to avoid notational ambiguity between source and target of the Frobenius (instances where this approach would clarify the argument are the proofs of Theorem 5.1.2 and Corollary 5.1.3 below).

Often, the inclusion $A^p \subseteq A$ is even finite, and hence so is the Frobenius itself. One can show, using Noether normalization or Cohen's Structure Theorems that this is true when A is respectively a k-affine algebra or a complete Noetherian local ring with residue field k, and k is perfect, or more generally, $(k : k^p) < \infty$.

5.1.1 Frobenius Transforms

Given an ideal $I \subseteq A$, we will denote its extension under the Frobenius by $\mathbf{F}_p(I)A$, and call it the *Frobenius transform* of I. Note that $\mathbf{F}_p(I)A \subseteq I^p$, but the inclusion is in general strict. In fact, one easily verifies that

5.1.1 *If* $I = (x_1, \dots, x_n)A$, *then* $\mathbf{F}_p(I)A = (x_1^p, \dots, x_n^p)A$.

If we repeat this process, we get the *iterated Frobenius transforms* $\mathbf{F}_p^n(I)A$ of I, generated by the p^n-th powers of elements in I, and in fact, of generators of I. In tight closure theory, the simplified notation

$$I^{[p^n]} := \mathbf{F}_p^n(I)A$$

is normally used, but for reasons that will become apparent once we defined tight closure as a difference closure (see §6.1.1), we will use the 'heavier' notation. On the other hand, since we fix the characteristic, we may omit p from the notation and simply write $\mathbf{F} : A \to A$ for the Frobenius.

5.1.2 Kunz Theorem

The next result, due to Kunz, characterizes regular local rings in positive characteristic via the Frobenius. We will only prove the direction that we need.

Theorem 5.1.2 (Kunz). *Let R be a Noetherian local ring. If R is regular, then $\mathbf{F}_p$ is flat. Conversely, if R is reduced and $\mathbf{F}_p$ is flat, then R is regular.*

Proof. We only prove the direct implication; for the converse see [68, §42]. Let $\mathbf{x}$ be a system of parameters of R, whence an R-regular sequence. Since $\mathbf{F}(\mathbf{x})$ is also a system of parameters, it too is R-regular. Hence, R, viewed as an R-algebra via $\mathbf{F}$, is a balanced big Cohen-Macaulay algebra, whence is flat by Theorem 3.3.9. □

Corollary 5.1.3. *If R is a regular local ring, $I \subseteq R$ an ideal, and $a \in R$ an arbitrary element, then $a \in I$ if and only if $\mathbf{F}(a) \in \mathbf{F}(I)R$.*

Proof. One direction is of course trivial, so assume $\mathbf{F}(a) \in \mathbf{F}(I)R$. However, since $\mathbf{F}$ is flat by Theorem 5.1.2, the contraction of the extended ideal $\mathbf{F}(I)R$ along $\mathbf{F}$ is again I by Proposition 3.2.5, and a lies in this contraction (recall that $\mathbf{F}(I)R \cap R$ stands really for $\mathbf{F}^{-1}(\mathbf{F}(I)R)$.) □

5.2 Tight Closure

The definition of tight closure, although not complicated, is not that intuitive either. The idea is inspired by the ideal membership test of Corollary 5.1.3. Unfortunately, that test only works over regular local rings, so that it will be no surprise that whatever test we design, it will have to be more involved. Moreover, the proposed test will in fact fail in general, that is to say, the elements satisfying the test form an ideal which might be strictly bigger than the original ideal. But not too much bigger, so that we may view this bigger ideal as a closure of the original ideal, and as such, it is a 'tight' fit.

In the remainder of this section, A is a Noetherian ring, of characteristic p. A first obvious generalization of the ideal membership test from Corollary 5.1.3 is to allow iterates of the Frobenius: we could ask, given an ideal $I \subseteq A$, what are the elements x such that $\mathbf{F}^n(x) \in \mathbf{F}^n(I)A$ for some power n? They do form an ideal and the resulting closure operation is called the *Frobenius closure*. However, its properties are not sufficiently strong to derive all the results tight closure can.

The adjustment to make in the definition of Frobenius closure, although minor, might at first be a little surprising. To make the definition, we will call an element $a \in A$ a *multiplier*, if it is either a unit, or otherwise generates an ideal of positive height (necessarily one by Krull's Principal Ideal Theorem). Put differently, a is a multiplier if it does not belong to any minimal prime ideal of A. In particular, the product of two multipliers is again a multiplier. In a domain, a situation we can often reduce to, a multiplier is simply a non-zero element.

The name 'multiplier' comes from the fact that we will use such elements to multiply our test condition with. However, for this to make sense, we cannot just take one iterate of the Frobenius, we must take all of them, or at least all but finitely many. So we now define: an element $x \in A$ belongs to the *tight closure* $\mathrm{cl}_A(I)$ of an ideal $I \subseteq A$, if there exists a multiplier $c \in A$ and a positive integer N such that

$$c\mathbf{F}^n(x) \in \mathbf{F}^n(I)A \tag{5.3}$$

for all $n \geq N$. Note that the multiplier c and the bound N may depend on x and I, but not on n. We will write $\mathrm{cl}(I)$ for $\mathrm{cl}_A(I)$ if the ring A is clear from the context. In the literature, tight closure is invariably denoted I^*, but again for reasons that will become clear in the next chapter, our notation better suits our purposes. Let us verify some elementary properties of this closure operation:

5.2.1 *The tight closure of an ideal I in a Noetherian ring A is again an ideal, it contains I, and it is equal to its own tight closure. Moreover, we can find a multiplier c and a positive integer N which works simultaneous for all elements in* $\mathrm{cl}(I)$ *in criterion* (5.3).

It is easy to verify that $\mathrm{cl}(I)$ is closed under multiples, and contains I. To show that it is closed under sums, whence an ideal, assume $x,x' \in A$ both lie in $\mathrm{cl}(I)$, witnessed by the equations (5.3) for some multipliers c and c', and some positive integers N and N' respectively. However, $cc'\mathbf{F}^n(x+x')$ then lies in $\mathbf{F}^n(I)A$ for all $n \geq \max\{N,N'\}$, showing that $x+x' \in \mathrm{cl}(I)$ since cc' is again a multiplier. Let $J := \mathrm{cl}(I)$ and choose generators $y_1,\dots,y_s$ of J. Let c_i and N_i be the corresponding multiplier and bound for y_i. It follows that $c := c_1c_2\cdots c_s$ is a multiplier such that (5.3) holds for all $n \geq N := \max\{N_1,\dots,N_s\}$ and all $x \in J$, since any such element is a linear combination of the y_i. In particular, $c\mathbf{F}^n(J)A \subseteq \mathbf{F}^n(I)A$ for all $n \geq N$. Hence if z lies in the tight closure of J, so that $d\mathbf{F}^n(z) \in \mathbf{F}^n(J)A$ for some multiplier d and for all $n \geq M$, then $cd\mathbf{F}^n(z) \in \mathbf{F}^n(I)A$ for all $n \geq \max\{M,N\}$, whence $z \in \mathrm{cl}(I) = J$. The last assertion now easily follows from the above analysis. In the sequel, we will therefore no longer make the bound N explicit and instead of "for all $n \geq N$" we will just write "for all $n \gg 0$".

Example 5.2.2. It is instructive to look at some examples. Let K be a field of characteristic $p > 3$, and let $A := K[\xi,\zeta,\eta]/(\xi^3-\zeta^3-\eta^3)K[\xi,\zeta,\eta]$ be the projective coordinate ring of the *cubic Fermat curve*. Let us show that ξ^2 is in the tight closure of $I := (\zeta,\eta)A$. For a fixed e, write $2p^e = 3h+r$ for some $h \in \mathbb{N}$ and some remainder $r \in \{1,2\}$, and let c be the multiplier ξ^3. Hence

$$c\mathbf{F}^e(\xi^2) = \xi^{3(h+1)+r} = \xi^r(\zeta^3+\eta^3)^{h+1}.$$

A quick calculation shows that any monomial in the expansion of $(\zeta^3+\eta^3)^{h+1}$ is a multiple of either $\mathbf{F}^e(\zeta)$ or $\mathbf{F}^e(\eta)$, showing that (5.3) holds for all e, and hence that $(\xi^2,\zeta,\eta)A \subseteq \mathrm{cl}(I)$.

It is often much harder to show that an element does not belong to the tight closure of an ideal. Shortly, we will see in Theorem 5.3.6 that any element outside the integral closure is also outside the tight closure. Since $(\xi^2,\zeta,\eta)A$ is integrally closed, we conclude that it is equal to $\mathrm{cl}(I)$.

Example 5.2.3. Let A be the coordinate ring of the hypersurface in $\mathbb{A}^3_K$ given by the equation $\xi^2-\zeta^3-\eta^7=0$. By a similar calculation as in the previous example, one can show that ξ lies in the tight closure of $(\zeta,\eta)A$.

A far more difficult result is to show that this is not true if we replace η^7 by η^5 in the above equation. In fact, in this new coordinate ring A', any ideal is tightly closed, that is to say, in the terminology from Definition 5.2.7 below, A' is F-regular, but this is a deep fact, following from it being log-terminal (see the discussion following Theorem 5.5.6).

It is sometimes cumbersome to work with multipliers in arbitrary rings, but in domains they are just non-zero elements. Fortunately, we can always reduce to the domain case when calculating tight closure:

Proposition 5.2.4. *Let A be a Noetherian ring, let $\mathfrak{p}_1,\dots,\mathfrak{p}_s$ be its minimal primes, and put $\bar{A}_i := A/\mathfrak{p}_i$. For all ideals $I \subseteq A$ we have*

$$\mathrm{cl}_A(I) = \bigcap_{i=1}^{s} \mathrm{cl}_{\bar{A}_i}(I\bar{A}_i) \cap A. \tag{5.4}$$

Proof. The same equations which exhibit x as en element of $\mathrm{cl}_A(I)$ also show that it is in $\mathrm{cl}_{\bar{A}_i}(I\bar{A}_i)$ since any multiplier in A remains, by virtue of its definition, a multiplier in $\bar{A}_i$ (moreover, the converse also holds: by prime avoidance, we can lift any multiplier in $\bar{A}_i$ to one in A). So one inclusion in (5.4) is clear.

Conversely, suppose x lies in the intersection on the right hand side of (5.4). Let $c_i \in A$ be a multiplier in A (so that its image is a multiplier in $\bar{A}_i$), such that

$$c_i\mathbf{F}^n_{\bar{A}_i}(x) \in \mathbf{F}^n_{\bar{A}_i}(I)\bar{A}_i$$

for all $n \gg 0$. This means that each $c_i\mathbf{F}^n_A(x)$ lies in $\mathbf{F}^n_A(I)A+\mathfrak{p}_i$ for $n \gg 0$. Choose for each i, an element $t_i \in A$ inside all minimal primes except $\mathfrak{p}_i$, and let $c := c_1t_1+\cdots+c_st_s$. A moment's reflection yields that c is again a multiplier. Moreover, since $t_i\mathfrak{p}_i \subseteq \mathfrak{n}$, where $\mathfrak{n} := \mathrm{nil}(R)$ is the nilradical of A, we get

$$c\,\mathbf{F}^n_A(x) \in \mathbf{F}^n_A(I)A+\mathfrak{n}$$

for all $n \gg 0$. Choose m such that $\mathfrak{n}^{p^m}$ is zero, whence also the smaller ideal $\mathbf{F}_A(\mathfrak{n})$. Applying $\mathbf{F}_A^m$ to the previous equations, yields

$$\mathbf{F}_A^m(c)\mathbf{F}_A^{m+n}(x) \in \mathbf{F}_A^{m+n}(I)A$$

for all $n \gg 0$, which means that $x \in \mathrm{cl}_A(I)$ since $\mathbf{F}_A^m(c)$ is again a multiplier. □

We will encounter many operations similar to tight closure, and so we formally define:

Definition 5.2.5 (Closure Operation). A *closure operation* on a ring A is any order-preserving, increasing, idempotent endomorphism on the set of ideals of A ordered by inclusion.

For instance, taking the radical of an ideal is a closure operation, and so is *integral closure* discussed below. Tight closure too is a closure operation on A, since it clearly also preserves inclusion: if $I \subseteq I'$, then $\mathrm{cl}(I) \subseteq \mathrm{cl}(I')$. An ideal that is equal to its own tight closure is called *tightly closed*. Recall that the *colon ideal* $(I : J)$ is the ideal of all elements $a \in A$ such that $aJ \subseteq I$; here $I \subseteq A$ is an ideal, but $J \subseteq A$ can be any subset, which, however, most of the time is either a single element or an ideal. Almost immediately from the definitions, we get

5.2.6 *If I is tightly closed, then so is $(I : J)$ for any $J \subseteq A$.* □

One of the longest outstanding open problems in tight closure theory was its behavior under localization: do we always have

$$\mathrm{cl}_A(I)A_{\mathfrak{p}} \stackrel{?}{=} \mathrm{cl}_{A_{\mathfrak{p}}}(IA_{\mathfrak{p}}) \tag{5.5}$$

for every prime ideal $\mathfrak{p} \subseteq A$. Recently, Brenner and Monsky have announced (see [15]) a negative answer to this question. The full extent of this phenomenon is not yet understood, and so one has proposed the following two definitions (the above cited counterexample still does not contradict that both notions are the same).

Definition 5.2.7. A Noetherian ring A is called *weakly F-regular* if each of its ideals is tightly closed. If all localizations of A are weakly F-regular, then A is called *F-regular*.

5.3 Five Key Properties of Tight Closure

In this section we derive five key properties of tight closure, all of which admit fairly simple proofs. It is important to keep this in mind, since these five properties will already suffice to prove in the next section some deep theorems in commutative algebra. In fact, as we will see, any closure operation with these five properties on a class of Noetherian local rings would establish these deep theorems for that particular class (and there are still classes for which these statements remain conjectural). Moreover, the proofs of the five properties themselves rest on a few simple facts about the Frobenius, so that this will allow us to also carry over our arguments to characteristic zero in Chapters 6 and 7.

The first property, stated here only in its weak version, is merely an observation. Namely, any equation (5.3) in a ring A extends to a similar equation in any A-algebra B. In order for the latter to calculate tight closure, the multiplier $c \in A$ should remain a multiplier in B, and so we proved:

Theorem 5.3.1 (Weak Persistence). *Let $A \to B$ be a ring homomorphism, and let $I \subseteq A$ be an ideal. If $A \to B$ is injective and B is a domain, or more generally, if $A \to B$ preserves multipliers, then* $\mathrm{cl}_A(I) \subseteq \mathrm{cl}_B(IB)$. □

The remarkable fact is that this is also true if $A \to B$ is arbitrary and A is of finite type over an excellent Noetherian local ring (see [59, Theorem 2.3]). We will not need this stronger version, the proof of which requires another important ingredient of tight closure theory: the notion of a test element. A multiplier $c \in A$ is called a *test element* for A, if for every $a \in \mathrm{cl}(I)$, we have $c\mathbf{F}^n(a) \in \mathbf{F}^n(I)A$ for all n. The existence of test elements is not easy, and lies outside the scope of these notes, but once one has established their existence, many arguments become even more streamlined.

Theorem 5.3.2 (Regular Closure). *In a regular local ring, every ideal is tightly closed. In fact, a regular ring is F-regular.*

Proof. Let R be a regular local ring. Since any localization of R is again regular, the second assertion follows from the first. To prove the first, let I be an ideal and $x \in \mathrm{cl}(I)$. Towards a contradiction, assume $x \notin I$. In particular, we must have $(I : x) \subseteq \mathfrak{m}$. Choose a non-zero element c such that (5.3) holds for all $n \gg 0$. This means that c lies in the colon ideal $(\mathbf{F}^n(I)R : \mathbf{F}^n(x))$, for all $n \gg 0$. Since $\mathbf{F}$ is flat by Theorem 5.1.2, the colon ideal is equal to $\mathbf{F}^n(I : x)R$ by Theorem 3.3.14. Since $(I : x) \subseteq \mathfrak{m}$, we get $c \in \mathbf{F}^n(\mathfrak{m})R \subseteq \mathfrak{m}^{p^n}$. Since this holds for all $n \gg 0$, we get $c = 0$ by Theorem 2.4.14, clearly a contradiction. □

Theorem 5.3.3 (Colon Capturing). *Let R be a Noetherian local domain which is a homomorphic image of a Cohen-Macaulay local ring, and let $(x_1, \dots, x_d)$ be a system of parameters in R. Then for each i, the colon ideal $((x_1, \dots, x_i)R : x_{i+1})$ is contained in* $\mathrm{cl}((x_1, \dots, x_i)R)$.

Proof. Let S be a local Cohen-Macaulay ring such that $R = S/\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subseteq S$ of height h. By prime avoidance, we can lift the x_i to elements in S, again denoted for simplicity by x_i, and find elements $y_1, \dots, y_h \in \mathfrak{p}$ such that $(y_1, \dots, y_h, x_1, \dots, x_d)$ is a system of parameters in S, whence an S-regular sequence. Since $\mathfrak{p}$ contains the ideal $J := (y_1, \dots, y_h)S$ of the same height, it is a minimal prime of J. Let $J = \mathfrak{g}_1 \cap \dots \cap \mathfrak{g}_s$ be a minimal primary decomposition of J, with $\mathfrak{g}_1$ the $\mathfrak{p}$-primary component of J. In particular, some power of $\mathfrak{p}$ lies in $\mathfrak{g}_1$, and we may assume that this power is of the form p^m for some m. Choose c inside all

$\mathfrak{g}_i$ with $i > 1$, but outside $\mathfrak{p}$ (note that this is possible by prime avoidance). Putting everything together, we have

$$c\mathbf{F}^m(\mathfrak{p}) \subseteq c\,\mathfrak{p}^{p^m} \subseteq J. \tag{5.6}$$

Fix some i, let $I := (x_1, \ldots, x_i)S$ and assume $zx_{i+1} \in IR$, for some $z \in S$. Lifting this to S, we get $zx_{i+1} \in I + \mathfrak{p}$. Applying the n-th power of Frobenius to this for $n > m$, we get $\mathbf{F}^n(z)\mathbf{F}^n(x_{i+1}) \in \mathbf{F}^n(I)S + \mathbf{F}^n(\mathfrak{p})S$. By (5.6), this means that $c\mathbf{F}^n(z)\mathbf{F}^n(x_{i+1})$ lies in $\mathbf{F}^n(I)S + \mathbf{F}^{n-m}(J)S$. Since the $\mathbf{F}^{n-m}(y_j)$ together with the $\mathbf{F}^n(x_j)$ form again an S-regular sequence, we conclude that

$$c\mathbf{F}^n(z) \in \mathbf{F}^n(I)S + \mathbf{F}^{n-m}(J)S \subseteq \mathbf{F}^n(I)S + J$$

whence $c\mathbf{F}^n(z) \in \mathbf{F}^n(I)R$ for all $n > m$. By the choice of c, it is non-zero in R, so that the latter equations show that $z \in \mathrm{cl}(IR)$. □

The condition that R is a homomorphic image of a regular local ring is satisfied either if R is a local affine algebra, or, by Cohen's Structure Theorems, if R is complete. These are the two only cases in which we will apply the previous theorem. With a little effort, one can extend the proof without requiring R to be a domain (see for instance [59, Theorem 3.1]).

Theorem 5.3.4 (Finite Extensions). *If $A \to B$ is a finite, injective homomorphism of domains, and $I \subseteq A$ be an ideal, then* $\mathrm{cl}_B(IB) \cap A = \mathrm{cl}_A(I)$.

Proof. One direction is immediate by Theorem 5.3.1. For the converse, there exists an A-module homomorphism $\varphi\colon B \to A$ such that $c := \varphi(1) \neq 0$, by Lemma 5.3.5 below. Suppose $x \in \mathrm{cl}_B(IB) \cap A$, so that for some non-zero $d \in B$, we have $d\mathbf{F}^n(x) \in \mathbf{F}^n(I)B$ for $n \gg 0$. Since B is finite over A, some non-zero multiple of d lies in A, and hence without loss of generality, we may assume $d \in A$. Applying φ to these equations, we get

$$cd\,\mathbf{F}^n(x) \in \mathbf{F}^n(I)A$$

showing that $x \in \mathrm{cl}_A(I)$, since cd is a multiplier. □

Lemma 5.3.5. *If $A \subseteq B$ is a finite extension of domains, then there exists an A-linear map $\varphi\colon B \to A$ with $\varphi(1) \neq 0$.*

Proof. Suppose B is generated over A by the elements $b_1, \ldots, b_s$. Let K and L be the fields of fractions of A and B respectively. Since B is a domain, it lies inside the K-vector subspace $V \subseteq L$ generated by the b_i. Choose an isomorphism $\gamma\colon V \to K^t$ of K-vector spaces. After renumbering, we may assume that the first entry of $\gamma(1)$ is non-zero. Let $\pi\colon K^t \to K$ be the projection onto the first coordinate, and let $d \in A$ be the common denominator of the $\pi(\gamma(b_i))$ for $i = 1, \ldots, s$. Now define an A-linear homomorphism φ by the rule $\varphi(y) = d\pi(\gamma(y))$ for $y \in B$. Since y is an A-linear combination of the b_i and since $d\pi(\gamma(b_i)) \in A$, also $\varphi(y) \in A$. Moreover, by construction, $\varphi(1) \neq 0$. □

Note that a special case of Theorem 5.3.4 is the fact that tight closure measures the extent to which an extension of domains $A \subseteq B$ fails to be cyclically pure: $IB \cap A$ is contained in the tight closure of I, for any ideal $I \subseteq A$. In particular, in view of Theorem 5.3.2, this reproves the well-known fact that if $A \subseteq B$ is an extension of domains with A regular, then $A \subseteq B$ is cyclically pure. The next and last property involves another closure operation, integral closure. It will be discussed in more detail below (§5.4), and here we just state its relationship with tight closure:

Theorem 5.3.6 (Integral Closure). *For every ideal $I \subseteq A$, its tight closure is contained in its integral closure. In particular, radical ideals, and more generally integrally closed ideals, are tightly closed.*

Proof. The second assertion is an immediate consequence of the first. We verify condition (5.4.1.iv) below to show that if x belongs to the tight closure $\mathrm{cl}_A(I)$, then it also belongs to the integral closure $\bar{I}$. Let $A \to V$ be a homomorphism into a discrete valuation ring V, such that its kernel is a minimal prime of A. We need to show that $x \in IV$. However, this is clear since $x \in \mathrm{cl}_V(IV)$ by Theorem 5.3.1 (note that $A \to V$ preserves multipliers), and since $\mathrm{cl}_V(IV) = IV$, by Theorem 5.3.2 and the fact that V is regular. □

It is quite surprising that there is no proof, as far as I am aware of, that a prime ideal is tightly closed without reference to integral closure.

5.4 Integral Closure

The *integral closure* $\bar{I}$ of an ideal I is the collection of all elements $x \in A$ satisfying an integral equation of the form

$$x^d + a_1 x^{d-1} + \cdots + a_d = 0 \tag{5.7}$$

with $a_j \in I^j$ for all $j = 1, \ldots, d$. We say that I is *integrally closed* if $I = \bar{I}$. Since clearly $\bar{I} \subseteq \mathrm{rad}(I)$, radical ideals are integrally closed. It follows from either characterization (5.4.1.ii) or (5.4.1.iv) below that $\bar{I}$ is an ideal.

Theorem 5.4.1. *Let A be an arbitrary Noetherian ring (not necessarily of characteristic p). For an ideal $I \subseteq A$ and an element $x \in A$, the following are equivalent*

- 5.4.1.i. *x belongs to the integral closure, $\bar{I}$;*
- 5.4.1.ii. *there is a finitely generated A-module M with zero annihilator such that $xM \subseteq IM$;*
- 5.4.1.iii. *there is a multiplier $c \in A$ such that $cx^n \in I^n$ for infinitely many (respectively, for all sufficiently large) n;*
- 5.4.1.iv. *for every homomorphism $A \to V$ into a discrete valuation ring V with kernel equal to a minimal prime of A, we have $x \in IV$.*

Proof. We leave it to the reader to show that x lies in the integral closure of an ideal I if and only if it lies in the integral closure of each $I(A/\mathfrak{p})$, for $\mathfrak{p}$ a minimal prime of A. Hence we may moreover assume that A is a domain. Suppose x satisfies an integral equation (5.7), and let $J := x^{d-1}A + x^{d-2}I + \cdots + I^d$. An easy calculation shows that $xJ \subseteq IJ$, proving (5.4.1.i) $\Rightarrow$ (5.4.1.ii). Moreover, by induction, $x^nJ \subseteq I^nJ$, and hence for any non-zero element $c \in J$, we get $cx^n \in I^n$, proving (5.4.1.iii). Note that in particular, $x^nI^d \subseteq I^n$ for all n. The implication (5.4.1.ii) $\Rightarrow$ (5.4.1.i) is proven by a 'determinantal trick': apply [69, Theorem 2.1] to the multiplication with x on M. To prove (5.4.1.iii) $\Rightarrow$ (5.4.1.iv), suppose there is some non-zero $c \in A$ such that $cx^n \in I^n$ for infinitely many n. Let $A \subseteq V$ be an injective homomorphism into a discrete valuation ring V, and let v be the valuation on V. Hence $v(c) + nv(x) \geq nv(I)$ for infinitely many n, where $v(I)$ is the minimum of all $v(a)$ with $a \in I$. It follows that $v(x) \geq v(I)$, and hence $x \in IV$.

Remains to prove (5.4.1.iv) $\Rightarrow$ (5.4.1.i), so assume $x \in IV$ for every embedding $A \subseteq V$ into a discrete valuation ring V. Let $I = (a_1,\dots,a_n)A$, and consider the homomorphism $A[\xi] \to A_x$ given by $\xi_i \mapsto a_i/x$, where $\xi := (\xi_1,\dots,\xi_n)$. Let B be its image, so that $A \subseteq B \subseteq A_x$ (one calls B the *blowing-up* of $I + xA$ at x). Let $\mathfrak{m} := (\xi_1,\dots,\xi_n)A[\xi]$. I claim that $\mathfrak{m}B = B$. Assuming the claim, we can find $f \in \mathfrak{m}$ such that $f(\mathbf{a}/x) = 1$ in A_x, where $\mathbf{a} := (a_1,\dots,a_n)$. Write $f = f_1 + \cdots + f_d$ in its homogeneous parts f_j of degree j, so that

$$1 = x^{-1}f_1(\mathbf{a}) + \cdots + x^{-d}f_d(\mathbf{a}).$$

Multiplying with x^d, and observing that $f_j(\mathbf{a}) \in I^j$, we see that x satisfies an integral equation (5.7), and hence $x \in \bar{I}$.

To prove the claim ex absurdum, suppose $\mathfrak{m}B$ is not the unit ideal, whence is contained in a maximal ideal $\mathfrak{n}$ of B. Let $(x_1,\dots,x_n)$ be a generating tuple of $\mathfrak{n}$. Let R be the $B_\mathfrak{n}$-algebra generated by the fractions x_i/x_1 with $i = 1,\dots,n$ (the blowing-up of $B_\mathfrak{n}$ at $\mathfrak{n}$). Since $\mathfrak{n}R = x_1R$, there exists a height one prime ideal $\mathfrak{p}$ in R containing $\mathfrak{n}R$. Let V be the normalization of $R_\mathfrak{p}$. It follows that V is a discrete valuation ring (see [69, Theorem 11.2]) containing $B_\mathfrak{n}$ as a local subring. In particular, $A \subseteq V$, and $\mathfrak{m}V$ lies in the maximal ideal πV. Since $\xi_i \mapsto a_i/x$, we get $a_i \in x\pi V$ for all i, and hence $IV \subseteq x\pi V$, contradicting that $x \in IV$. □

From this we readily deduce:

Corollary 5.4.2. *A domain A is normal if and only if each principal ideal is integrally closed if and only if each principal ideal is tightly closed.* □

In one of our applications below (Theorem 5.5.1), we will make use of the following nice application of the chain rule:

Proposition 5.4.3. *Let K be a field of characteristic zero, and let R be either the power series ring $K[[\xi]]$, the ring of convergent power series $K\{\xi\}$ (assuming K is a normed field), or the localization of $K[\xi]$ at the ideal generated by the indeterminates $\xi := (\xi_1,\dots,\xi_n)$. If f is a non-unit, then it lies in the integral closure of its Jacobian ideal* $\operatorname{Jac}(f) := (\partial f/\partial\xi_1,\dots,\partial f/\partial\xi_n)R$.

Proof. Recall that $K\{\xi\}$ consists of all formal power series f such that $f(\mathbf{u})$ is a convergent series for all $\mathbf{u}$ in a small enough neighborhood of the origin. Put $J := \mathrm{Jac}(f)$. In view of (5.4.1.iv), we need to show that given an embedding $R \subseteq V$ into a discrete valuation ring V, we have $f \in JV$. Since completion is faithfully flat, we may replace V by its completion, and hence already assume V is complete. By Cohen's Structure Theorems, V is a power series ring $\kappa[[\zeta]]$ in a single variable over a field extension κ of K. Viewing the image of f in $\kappa[[\zeta]]$ as a power series in ζ, the multi-variate chain rule yields

$$\frac{df}{d\zeta} = \sum_{i=1}^{n} \frac{\partial f}{\partial \xi_i} \cdot \frac{d\xi_i}{d\zeta} \in JV.$$

However, since f has order $e \geq 1$ in V, its derivative $df/d\zeta$ has order $e-1$, and hence $f \in (df/d\zeta)V \subseteq JV$. Note that for this to be true, however, the characteristic needs to be zero. For instance, in characteristic p, the power series ξ^p would already be a counterexample to the proposition. □

Since the integral closure is contained in the radical closure, we get that some power of f lies in its Jacobian ideal $\mathrm{Jac}(f)$. A famous theorem due to Briançon-Skoda states that in fact already the n-th power lies in the Jacobian, where n is the number of variables. We will prove this via an elegant tight closure argument in Theorem 5.5.1 below.

5.5 Applications

We will now discuss three important applications of tight closure. Perhaps surprisingly, the original statements all were in characteristic zero (with some of them in their original form plainly false in positive characteristic), and their proofs required deep and involved arguments, some even based on transcendental/analytic methods. However, they each can be reformulated so that they also make sense in positive characteristic, and then can be established by surprisingly elegant tight closure arguments. As for the proofs of their characteristic zero counterparts, they must wait until we have developed the theory in characteristic zero in Chapters 6 and 7 (or one can use the 'classical' tight closure in characteristic zero discussed in §5.6).

5.5.1 The Briançon-Skoda Theorem

We already mentioned this famous result, proven first in [16].

Theorem 5.5.1 (Briançon-Skoda). *Let R be either the ring of formal power series $\mathbb{C}[[\xi]]$, or the ring of convergent power series $\mathbb{C}\{\xi\}$, or the localization of the* *polynomial ring* $\mathbb{C}[\xi]$ *at the ideal generated by* ξ, *where* $\xi := (\xi_1, \ldots, \xi_n)$ *are some indeterminates. If* f *is not a unit, then* $f^n \in \operatorname{Jac}(f) := (\partial f/\partial \xi_1, \ldots, \partial f/\partial \xi_n)R$.

This theorem will follow immediately from the characteristic zero analogue of the next result (with $l = 1$), in view of Proposition 5.4.3; we will do this in Theorem 6.2.5 below.

Theorem 5.5.2 (Briançon-Skoda—Tight Closure Version). *Let* A *be a Noetherian ring of characteristic* p, *and* $I \subseteq A$ *an ideal generated by* n *elements. Then we have for all* $l \geq 1$ *an inclusion*

$$\overline{I^{n+l-1}} \subseteq \operatorname{cl}(I^l).$$

In particular, if A *is a regular local ring, then the integral closure of* I^{n+l-1} *lies inside* I^l *for* $l \geq 1$.

Proof. For simplicity, I will only prove the case $l = 1$ (which gives the original Briançon-Skoda theorem). Assume z lies in the integral closure of I^n. By (5.4.1.iii), there exists a multiplier $c \in A$ such that $cz^k \in I^{kn}$ for all $k \gg 0$. Since $I := (f_1, \ldots, f_n)A$, we have an inclusion $I^{kn} \subseteq (f_1^k, \ldots, f_n^k)A$. Hence with k equal to p^m, we get $c\mathbf{F}^m(z) \in \mathbf{F}^m(I)A$ for all $m \gg 0$. In conclusion, $z \in \operatorname{cl}(I)$. The last assertion then follows from Theorem 5.3.2. □

5.5.2 The Hochster-Roberts Theorem

We will formulate the next result without defining in detail all the concepts involved, except when we get to its algebraic formulation. A *linear algebraic group* G is an affine subscheme of the general linear group $\operatorname{GL}(K, n)$ over an algebraically closed field K such that its K-rational points form a subgroup of the latter group. When G acts (as a group) on a closed subscheme $X \subseteq \mathbb{A}^n_K$ (more precisely, for each algebraically closed field L containing K, there is an action of the L-rational points of $G(L)$ on $X(L)$), we can define the *quotient space* X/G, consisting of all orbits under the action of G on X, as the affine space $\operatorname{Spec}(R^G)$, where R^G denotes the subring of G-invariant sections in $R := \Gamma(X, \mathcal{O}_X)$ (the action of G on X induces an action on the sections of X, and hence in particular on R). For this to work properly, we also need to impose a certain finiteness condition: G has to be *linearly reductive*. Although not usually its defining property, we will here take this to mean that there exists an R^G-linear map $R \to R^G$ which is the identity on R^G, called the *Reynolds operator* of the action. For instance, if $K = \mathbb{C}$, then an algebraic group is linearly reductive if and only if it is the complexification of a real Lie group, where the Reynolds operator is obtained by an integration process. This is the easiest to understand if G is finite, when the integration is just a finite sum

$$\rho: R \to R^G: a \mapsto \frac{1}{|G|}\sum_{\sigma\in G} a^\sigma,$$

where a^σ denotes the result of $\sigma \in G$ acting on $a \in R$. In fact, as indicated by the above formula, a finite group is linearly reductive over a field of positive characteristic, provided its cardinality is not divisible by the characteristic. If X is non-singular and G is linearly reductive, then we will call X/G a *quotient singularity*.[1] The celebrated Hochster-Roberts theorem now states:

Theorem 5.5.3. *Any quotient singularity is Cohen-Macaulay.*

To state a more general result, we need to take a closer look at the Reynolds operator. A ring homomorphism $A \to B$ is called *split*, if there exists an A-linear map $\sigma: B \to A$ which is the identity on A (note that σ need not be multiplicative, that is to say, is not a ring homomorphism, only a module homomorphism). We call σ the *splitting* of $A \to B$. Hence the Reynolds operator is a splitting of the inclusion $R^G \subseteq R$. The only property of split maps that will matter is the following:

5.5.4 *A split homomorphism $A \to B$ is cyclically pure.*

See the discussion at the beginning of §2.4.3 for the definition of cyclic purity. Let $a \in IB \cap A$ with $I = (f_1, \dots, f_s)A$ an ideal in A. Hence $a = f_1b_1 + \cdots + f_sb_s$ for some $b_i \in B$. Applying the splitting σ, we get by A-linearity $a = f_1\sigma(b_1) + \cdots + f_s\sigma(b_s) \in I$, proving that A is cyclically pure in B. □

We also need the following result on the preservation of cyclic purity under completions:

Lemma 5.5.5. *Let R and S be Noetherian local rings with respective completions $\widehat{R}$ and $\widehat{S}$. If $R \to S$ is cyclically pure, then so is its completion $\widehat{R} \to \widehat{S}$.*

Proof. The homomorphism $S \to \widehat{S}$ is faithfully flat, hence cyclically pure; thus the composition $R \to S \to \widehat{S}$ is cyclically pure. So from now on we may suppose that $S = \widehat{S}$. It suffices to show that $\widehat{R} \to S$ is injective, since the completion of $R/\mathfrak{a}$ is equal to $\widehat{R}/\mathfrak{a}\widehat{R}$, for any ideal $\mathfrak{a}$ in R. Let $a \in \widehat{R}$ be such that $a = 0$ in S, and for each i choose $a_i \in R$ such that $a \equiv a_i \bmod \mathfrak{m}^i\widehat{R}$, where $\mathfrak{m}$ is the maximal ideal of R. Then a_i lies in $\mathfrak{m}^iS$, hence by cyclical purity, in $\mathfrak{m}^i$. Therefore $a \in \mathfrak{m}^i\widehat{R}$ for all i, showing that $a = 0$ in $\widehat{R}$ by Krull's Intersection Theorem (Theorem 2.4.14). □

We can now state a far more general result, of which Theorem 5.5.3 is just a special case.

Theorem 5.5.6. *If $R \to S$ is a cyclically pure homomorphism and if S is regular, then R is Cohen-Macaulay.*

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[1] The reader should be aware that other authors might use the term more restrictively, only allowing X to be affine space $\mathbb{A}^n_K$, or G to be finite.

Proof. The problem is clearly local, and so we assume that $(R,\mathfrak{m})$ and $(S,\mathfrak{n})$ are local. By Lemma 5.5.5, we may further reduce to the case that R and S are both complete. We split the proof in two parts: we first show that R is F-regular (see Definition 5.2.7), and then show that any complete local F-regular domain is Cohen-Macaulay.

5.5.7 *A cyclically pure subring of a regular ring is F-regular.*

Indeed, since both cyclic purity and regularity are preserved under localization, we only need to show that every ideal in R is tightly closed. To this end, let $I \subseteq R$ and $x \in \operatorname{cl}(I)$. Hence x lies in the tight closure of IS by (weak) persistence (Theorem 5.3.1), and therefore in IS by Theorem 5.3.2. Hence by cyclic purity, $x \in I = IS \cap R$, proving that R is weakly F-regular. Note that we actually proved that a cyclically pure subring of a (weakly) F-regular ring is again (weakly) F-regular.

5.5.8 *A complete local F-regular domain is Cohen-Macaulay.*

Assume R is F-regular and let $(x_1,\dots,x_d)$ be a system of parameters in R. To show that x_{i+1} is $R/(x_1,\dots,x_i)R$-regular, assume $zx_{i+1} \in (x_1,\dots,x_i)R$. Colon Capturing (Theorem 5.3.3) yields that z lies in the tight closure of $(x_1,\dots,x_i)R$, whence in the ideal itself since R is F-regular. □

Remark 5.5.9. In fact, R is then also normal (this follows easily from 5.5.7 and Corollary 5.4.2). A far more difficult result is that R is then also *pseudo-rational* (a concept that lies beyond the scope of these notes; see for instance [59, 99] for a discussion of what follows). This was first proven by Boutot in [14] for $\mathbb{C}$-affine algebras by means of deep vanishing theorems. The positive characteristic case was proven by Smith in [108] by tight closure methods, where she also showed that pseudo-rationality is in fact equivalent with the weaker notion of F-rationality (a local ring is *F-rational* if some parameter ideal is tightly closed). I proved the general characteristic zero case in [99] by means of ultraproducts. In fact, being F-regular is equivalent under the $\mathbb{Q}$-Gorenstein assumption with having log-terminal singularities (see [38, 95]; for an example see Example 5.2.3). It should be noted that 'classical' tight closure theory in characteristic zero (see §5.6 below) is not sufficiently versatile to derive these results: so far, only our present ultraproduct method seems to work.

5.5.3 The Ein-Lazardsfeld-Smith Theorem

The next result, although elementary in its formulation, was only proven recently in [26] using quite complicated methods (which only work over $\mathbb{C}$), but then soon after in [55] by an elegant tight closure argument (see also [90]), which proves the result over any field K.

Theorem 5.5.10. *Let $V \subseteq K^2$ be a finite subset with ideal of definition $I := \mathfrak{I}(V)$. For each k, let $J_k(V)$ be the ideal of all polynomials f having multiplicity at least k at each point $x \in V$. Then $J_{2k}(V) \subseteq I^k$, for all k.*

To formulate the more general result of which this is just a corollary, we need to introduce symbolic powers. We first do this for a prime ideal $\mathfrak{p}$: its k-th *symbolic power* is the contracted ideal $\mathfrak{p}^{(k)} := \mathfrak{p}^k R_\mathfrak{p} \cap R$. In general, the inclusion $\mathfrak{p}^k \subseteq \mathfrak{p}^{(k)}$ may be strict, and, in fact, $\mathfrak{p}^{(k)}$ is the $\mathfrak{p}$-primary component of $\mathfrak{p}^k$. If $\mathfrak{a}$ is a radical ideal (we will not treat the more general case), then we define its k-th *symbolic power* $\mathfrak{a}^{(k)}$ as the intersection $\mathfrak{p}_1^{(k)} \cap \cdots \cap \mathfrak{p}_s^{(k)}$, where the $\mathfrak{p}_i$ are all the minimal overprimes of $\mathfrak{a}$. The connection with Theorem 5.5.10 is given by:

5.5.11 *The k-th symbolic power of the ideal of definition $I := \mathfrak{I}(V)$ of a finite subset $V \subseteq K^2$ is equal to the ideal $J_k(V)$ of all polynomials that have multiplicity at least k at any point of V.*

Indeed, for $\mathbf{x} \in V$, let $\mathfrak{m} := \mathfrak{m}_\mathbf{x}$ be the corresponding maximal ideal. By definition, a polynomial f has multiplicity at least k at each $\mathbf{x} \in V$, if $f \in \mathfrak{m}^k A_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ containing I. The latter condition simply means that $f \in \mathfrak{m}^{(k)}$, so that the claim follows from the definition of symbolic power. □

Hence, in view of this, Theorem 5.5.10 is an immediate consequence of the following theorem (at least in positive characteristic; for the characteristic zero case, see Theorems 6.2.6 and 7.2.4 below):

Theorem 5.5.12. *Let A be a regular domain of characteristic p. Let $\mathfrak{a} \subseteq A$ be a radical ideal and let h be the maximal height of its minimal overprimes. Then we have an inclusion $\mathfrak{a}^{(hn)} \subseteq \mathfrak{a}^n$, for all n.*

Proof. We start with proving the following useful inclusion:

$$\mathfrak{a}^{(hp^e)} \subseteq \mathbf{F}^e(\mathfrak{a})A \tag{5.8}$$

for all e. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be the minimal prime ideals of $\mathfrak{a}$. We first prove (5.8) locally at one of these minimal primes $\mathfrak{p}$. Since $A_\mathfrak{p}$ is regular and $\mathfrak{a}A_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$, we can find $f_i \in \mathfrak{a}$ such that $\mathfrak{a}A_\mathfrak{p} = (f_1, \dots, f_h)A_\mathfrak{p}$. By definition of symbolic powers, $\mathfrak{a}^{(hp^e)}A_\mathfrak{p} = \mathfrak{a}^{hp^e}A_\mathfrak{p}$. On the other hand, $\mathfrak{a}^{hp^e}A_\mathfrak{p}$ consists of monomials in the f_i of degree hp^e, and hence any such monomial lies in $\mathbf{F}^e(\mathfrak{a})A_\mathfrak{p}$. This establishes (5.8) locally at $\mathfrak{p}$. To prove this globally, take $z \in \mathfrak{a}^{(hp^e)}$. By what we just proved, there exists $s_i \notin \mathfrak{p}_i$ such that $s_i z \in \mathbf{F}^e(\mathfrak{a})A$ for each $i = 1, \dots, m$. For each i, choose an element t_i in all $\mathfrak{p}_j$ except $\mathfrak{p}_i$, and put $s := t_1 s_1 + \cdots + s_m t_m$. It follows that s multiplies z inside $\mathbf{F}^e(\mathfrak{a})A$, whence a fortiori, so does $\mathbf{F}^e(s)$. Hence

$$z \in (\mathbf{F}^e(\mathfrak{a})A : \mathbf{F}^e(s)) = \mathbf{F}^e(\mathfrak{a} : s)A$$

where we used Theorem 3.3.14 and the fact that $\mathbf{F}$ is flat on A by Theorem 5.1.2. However, s does not lie in any of the $\mathfrak{p}_i$, whence $(\mathfrak{a} : s) = \mathfrak{a}$, proving (5.8).

To prove the theorem, let $f \in \mathfrak{a}^{(hn)}$, and fix some e. We may write $p^e = an + r$ for some $a, r \in \mathbb{N}$ with $0 \le r < n$. Since the usual powers are contained in the symbolic powers, and since $r < n$, we have inclusions

$$\mathfrak{a}^{hn} f^a \subseteq \mathfrak{a}^{hr} f^a \subseteq \mathfrak{a}^{(han+hr)} = \mathfrak{a}^{(hp^e)} \subseteq \mathbf{F}^e(\mathfrak{a})A \tag{5.9}$$

where we used (5.8) for the last inclusion. Taking n-th powers in (5.9) shows that $\mathfrak{a}^{hn^2} f^{an}$ lies in the n-th power of $\mathbf{F}^e(\mathfrak{a})A$, and this in turn lies inside $\mathbf{F}^e(\mathfrak{a}^n)A$. Choose some non-zero c in $\mathfrak{a}^{hn^2}$. Since $p^e \ge an$, we get $c\mathbf{F}^e(f) \in \mathbf{F}^e(\mathfrak{a}^n)A$ for all e. In conclusion, f lies in $\mathrm{cl}(\mathfrak{a}^n)$ whence in $\mathfrak{a}^n$ by Theorem 5.3.2. □

One might be tempted to try to prove a more general form which does not assume A to be regular, replacing $\mathfrak{a}^n$ by its tight closure. However, we used the regularity assumption not only via Theorem 5.3.2 but also via Kunz's Theorem that the Frobenius is flat. Hence the above proof does not work in arbitrary rings.

5.6 Classical Tight Closure in Characteristic Zero

To prove the previous three theorems in a ring of equal characteristic zero, Hochster and Huneke also developed tight closure theory for such rings. One of the precursors to tight closure theory was the proof of the Intersection Theorem by Peskine and Szpiro in [75]. They used properties of the Frobenius together with a method to transfer results from characteristic p to characteristic zero, which was then generalized by Hochster in [43]. This same technique is also used to obtain a tight closure theory in equal characteristic zero, as we will discuss briefly in this section. However, using ultraproducts, we will bypass in Chapters 6 and 7 this rather heavy-duty machinery, to arrive much quicker at proofs in equal characteristic zero.

Let A be a Noetherian ring containing the rationals. The idea is to associate to A some rings in positive characteristic, its *reductions modulo p*, and calculate tight closure in the latter. More precisely, let $\mathfrak{a} \subseteq A$ be an ideal, and $z \in A$. We say that z lies in the *HH-tight closure* of $\mathfrak{a}$ (where "HH" stands for Hochster-Huneke), if there exists a $\mathbb{Z}$-affine subalgebra $R \subseteq A$ containing z, such that (the image of) z lies in the tight closure of $I(R/pR)$ for all primes numbers p, where $I := \mathfrak{a} \cap R$.

It is not too hard to show that this yields a closure operation on A (in the sense of Definition 5.2.5). Much harder is showing that it satisfies all the necessary properties from §5.3. For instance, to prove the analogue of Theorem 5.3.2, one needs some results on generic flatness, and some deep theorems on Artin Approximation (see for instance [59, Appendix 1] or [54]; for a brief discussion of Artin Approximation, see §7.1 below). In contrast, using ultraproducts, one can avoid all these complications in the affine case (Chapter 6), or get by with a more elementary version of Artin Approximation in the general case (Chapter 7).

Chapter 6
Tight Closure in Characteristic Zero. Affine Case

We will develop a tight closure theory in characteristic zero which is different from the Hochster-Huneke approach discussed briefly in §5.6. In this chapter we treat the affine case, that is to say, we develop the theory for algebras of finite type over an uncountable algebraically closed field K of characteristic zero; the general local case will be discussed in Chapter 7. Recall that under the Continuum Hypothesis, any uncountable algebraically closed field K of characteristic zero is a *Lefschetz field*, that is to say an ultraproduct of fields of positive characteristic, by Theorem 2.4.3 and Remark 2.4.4. In particular, without any set-theoretic assumption, $\mathbb{C}$, the field of complex numbers, is a Lefschetz field. The idea now is to use the ultra-Frobenius, that is to say, the ultraproduct of the Frobenii (see Definition 2.4.21), in the same manner in the definition of tight closure as in positive characteristic. However, the ultra-Frobenius does not act on the affine algebra but rather on its ultra-hull, so that we have to introduce a more general setup. It is instructive to do this first in an axiomatic manner (§6.1) and then specialize to the situation at hand (§6.2). We briefly discuss a variant construction in §6.3, and conclude in §6.4 with another example how ultraproducts can be used to transfer constructions from positive to zero characteristic, to wit, the balanced big Cohen-Macaulay algebras of Hochster and Huneke.

6.1 Difference Hulls

A ring C together with an endomorphism σ on C is called a *difference ring*, and for emphasis, we denote this as a pair (C,σ). If (C,σ) and $(C'\sigma')$ are difference rings, and $\varphi\colon C \to C'$ a ring homomorphism, then we call φ a *morphism of difference rings* if it commutes with the endomorphisms, that is to say, if $\varphi(\sigma(a)) = \sigma'(\varphi(a))$ for all $a \in C$. The example par excellence of a difference ring is any ring of positive characteristic endowed with his Frobenius. We will now reformulate tight closure from this perspective, but anticipating already the fact that the ultra-Frobenius acts only on a certain overring of the affine algebra, to wit, its ultra-hull defined in §4.1. Since we also want the theory to be compatible with ring homomorphisms ('Persistence'), we need to work categorically. Let $\mathfrak{C}$ be a

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category of Noetherian rings closed under homomorphic images (at this point we do not need to make any characteristic assumption). Often, the category will also be closed under localization, and we will tacitly assume this as well. In summary, $\mathfrak{C}$ is a collection of Noetherian rings so that for any A in $\mathfrak{C}$ any localization $S^{-1}A$ and any residue ring A/I belongs again to $\mathfrak{C}$ (and the canonical maps $A \to S^{-1}A$ and $A \to A/I$ are morphisms in $\mathfrak{C}$).

Definition 6.1.1 (Difference hull). A *difference hull* on $\mathfrak{C}$ is a functor $D(\cdot)$ from $\mathfrak{C}$ to the category of difference rings, and a natural transformation η from the identity functor to $D(\cdot)$ with the following three properties:

6.1.1.i. each $\eta_A : A \to D(A)$ is faithfully flat;
6.1.1.ii. the endomorphism σ_A of $D(A)$ preserves $D(A)$-regular sequences;
6.1.1.iii. for any ideal $I \subseteq A$, we have $\sigma_A(I) \subseteq I^2 D(A)$.

Spelling out this functoriality, we have, therefore, for each A in $\mathfrak{C}$, a difference ring $D(A)$ with endomorphism σ_A and a faithfully flat ring homomorphism $\eta_A : A \to D(A)$, and for each morphism $A \to B$ in $\mathfrak{C}$, an induced morphism of difference rings $D(A) \to D(B)$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \eta_A\ } & D(A) \\ \downarrow & & \downarrow \\ B & \xrightarrow[\ \eta_B\ ]{} & D(B) \end{array} \tag{6.1}$$

commutes. Since η_A is in particular injective (Proposition 3.2.5), we will henceforth view A as a subring of $D(A)$ and omit η_A from our notation.

6.1.1 *Difference Closure*

Given a difference hull $D(\cdot)$ on some category $\mathfrak{C}$, we define the *difference closure* $\mathrm{cl}^D(I)$ of an ideal $I \subseteq A$ of a member A of $\mathfrak{C}$ as follows: an element $z \in A$ belongs to $\mathrm{cl}^D(I)$ if there exists a multiplier $c \in A$ and a number $N \in \mathbb{N}$ such that

$$c\sigma^n(z) \in \sigma^n(I)D(A) \tag{6.2}$$

for all $n \geq N$. Here, $\sigma^n(I)D(A)$ denotes the ideal in $D(A)$ generated by all $\sigma^n(y)$ with $y \in I$, where σ is the endomorphism of the difference ring $D(A)$. It is crucial here that the multiplier c already belongs to A, although the membership relations in (6.2) are inside the bigger ring $D(A)$. We leave it as an exercise to show that the difference closure is indeed a closure operation in the sense of Definition 5.2.5. An ideal that is equal to its difference closure will be called *difference closed*.

Example 6.1.2 (Frobenius hull). It is clear that our definition is inspired by the membership test (5.3) for tight closure, and indeed, this is just a special case. Namely, for a fixed prime number p, let $\mathfrak{C}_p$ be the category of all Noetherian rings of characteristic p and let $D(\cdot)$ be the functor assigning to a ring A the difference ring $(A, \mathbf{F}_A)$. It is easy to see that this makes $D(\cdot)$ a difference hull in the above sense, and the difference closure with respect to this hull is just the tight closure of the ideal; we will refer to this construction as the *Frobenius hull*.

In the next section, we will view tight closure in characteristic zero as a difference closure too. For the remainder of this section, we fix a category $\mathfrak{C}$ endowed with a difference hull $D(\cdot)$, and study the corresponding difference closure on the members of $\mathfrak{C}$. For a given member A of $\mathfrak{C}$, we let σ_A, or just σ, be the endomorphism of $D(\cdot)$. In fact, we are mostly interested in the restriction of σ to A, and we also denote this homomorphism by σ (of course, this restriction is no longer an endomorphism).

6.1.2 Five Key Properties of Difference Closure

To derive the necessary properties of this closure operation, namely the analogues of the five key properties of §5.3, we again depart from a flatness result, the analogue of Kunz's Theorem (Theorem 5.1.2).

Proposition 6.1.3. *If A is a regular local ring in $\mathfrak{C}$, then $\sigma\colon A \to D(A)$ is faithfully flat.*

Proof. By Theorem 3.3.9, it suffices to show that $D(A)$ is a balanced big Cohen-Macaulay algebra via σ. To this end, let $(x_1, \dots, x_d)$ be an A-regular sequence. Since $A \subseteq D(A)$ is by assumption faithfully flat, $(x_1, \dots, x_d)$ is $D(A)$-regular by Proposition 3.2.9. By condition (6.1.1.ii), the sequence $(\sigma(x_1), \dots, \sigma(x_d))$ is also $D(A)$-regular, as we wanted to show. □

Corollary 6.1.4. *Any ideal of a regular ring in $\mathfrak{C}$ is difference closed.*

Proof. Suppose first that $(R, \mathfrak{m})$ is a regular local ring in $\mathfrak{C}$, and z lies in the difference closure of an ideal $I \subseteq R$. Hence, with c and N as in (6.2), the multiplier c lies in $(\sigma^n(I)D(R) : \sigma^n(z))$ for $n \geq N$, and hence by flatness (Proposition 6.1.3) and the Colon Criterion (Theorem 3.3.14), it lies in $\sigma^n(I : z)D(R)$. If z does not belong to I, then $(I : z) \subseteq \mathfrak{m}$, and hence c belongs to $\sigma^n(\mathfrak{m})D(R)$ which in turn lies inside $\mathfrak{m}^{2^n}D(R)$ by condition (6.1.1.iii). By faithful flatness, c therefore lies in $\mathfrak{m}^{2^n}$, for every $n \geq N$, contradicting, in view of Krull's Intersection Theorem 2.4.14, that it is a multiplier.

For the general case, assume z lies in the difference closure of an ideal I in a regular ring A in $\mathfrak{C}$. By weak persistence (see 6.1.6 below) and the local case,

$z \in IA_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m}$ of A. It follows that $(I : z)$ cannot be a proper ideal, whence $z \in I$. □

Remark 6.1.5. Let us call a difference hull *simple* if instead of condition (6.1.1.iii) we have the stronger condition that $\sigma(I)$ is contained in all powers of $ID(A)$, for $I \subseteq A$. In that case, we can define a variant of the difference closure, called *simple difference closure*, by requiring condition (6.2) to hold only for $n = 1$, that is to say, a single test suffices. Inspecting the above proof, one sees that for a simple difference hull, any ideal I in a regular ring is equal to its simple difference closure. We leave it to the reader to show that simple difference closure satisfies all the properties below of its non-simple counterpart.

Weak persistence holds for the same reasons as it does for tight closure, so for the record we state:

6.1.6 *If $A \to B$ is an injective morphism in $\mathfrak{C}$ with A and B domains, then* $\mathrm{cl}^D(I) \subseteq \mathrm{cl}^D(IB)$.

For the next result, we need the following strengthening of condition (6.1.1.ii):

Lemma 6.1.7. *Let $D(R)$ be a difference hull of a local ring R in $\mathfrak{C}$ with endomorphism σ. If $(x_1,\dots,x_h)$ is an R-regular sequence, then $(\sigma^{e_1}(x_1),\dots,\sigma^{e_h}(x_h))$ is a $D(R)$-regular sequence, for any $e_1,\dots,e_h \geq 0$.*

Proof. Let e be maximum of the e_i. Since the sequence $(x_1,\dots,x_h)$ is R-regular, it is $D(R)$-regular by (6.1.1.i). By repeated use of (6.1.1.ii), the sequence

$$(\sigma^e(x_1),\dots,\sigma^e(x_h))$$

is a permutable $D(R)$-regular sequence. By condition (6.1.1.iii), we can find $z_i \in D(R)$ such that $\sigma^e(x_i) = z_i\sigma^{e_i}(x_i)$. The assertion now follows from Lemma 6.1.8 below. □

Lemma 6.1.8. *In any ring A, if $(a_1b_1,\dots,a_hb_h)$ is a permutable A-regular sequence, then so is $(a_1,\dots,a_h)$.*

Proof. Since there is nothing to prove if all b_i are units, we may induct on the number of b_i which are not a unit. Therefore, it suffices to prove that if $(a_1b_1,\dots,a_hb_h)$ is A-regular, then so is

$$(a_1b_1,\dots,a_{N-1}b_{N-1},a_N,a_{N+1}b_{N+1},\dots,a_hb_h). \tag{6.3}$$

To this end, we have to show that the i-th element in the sequence (6.3) is not a zero-divisor modulo the first $i-1$ elements. This is part of the hypothesis for $i < N$ and immediate for $i = N$, for any factor of a non zero-divisor is also a non zero-divisor. So we may assume $N < i$. Suppose

$$za_ib_i \in (a_1b_1,\dots,a_{N-1}b_{N-1},a_N,a_{N+1}b_{N+1},\dots,a_{i-1}b_{i-1})A.$$

Multiplying with b_N, we obtain that $zb_Na_ib_i$ lies in $(a_1b_1,\dots,a_{i-1}b_{i-1})A$. Since a_ib_i is not a zero-divisor modulo this latter ideal, zb_N lies already in this ideal. Therefore, if we put

$$I := (a_1b_1,\dots,a_{N-1}b_{N-1},a_{N+1}b_{N+1},\dots,a_{i-1}b_{i-1})A,$$

then for some $r \in A$, we have $zb_N + ra_Nb_N \in I$. Since the original sequence is permutable, a_Nb_N is not a zero-divisor modulo I. Therefore, neither is b_N, as it is a factor. It follows that $z + ra_N \in I$, whence $z \in I + a_NA$, as required. □

Proposition 6.1.9 (Colon Capturing). *Let R be a Noetherian local domain which is a homomorphic image of a Cohen-Macaulay local ring in $\mathfrak{C}$, and let $(x_1,\dots,x_d)$ be a system of parameters in R. Then for each i, the colon ideal $((x_1,\dots,x_i)R : x_{i+1})$ is contained in $\mathrm{cl}^D((x_1,\dots,x_i)R)$.*

Proof. Let S be a local Cohen-Macaulay ring in $\mathfrak{C}$ such that $R = S/\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subseteq S$, and assume the x_i already belong to S. As in the proof of Theorem 5.3.3, we can find an S-regular sequence $(y_1,\dots,y_h,x_1,\dots,x_d)$ with $y_1,\dots,y_h \in \mathfrak{p}$, an element $c \notin \mathfrak{p}$, and a number $m \in \mathbb{N}$ such that

$$c\mathfrak{p}^{2^m} \subseteq J := (y_1,\dots,y_h)S. \tag{6.4}$$

Let τ denote the endomorphism of $D(S)$. By assumption, the canonical epimorphism $S \to R$ induces a morphism of difference rings $D(S) \to D(R)$. In particular, $\mathfrak{p}D(R) = 0$.

Fix for some i, let $I := (x_1,\dots,x_i)S$ and assume $zx_{i+1} \in IR$ for some $z \in S$. Hence $zx_{i+1} \in I + \mathfrak{p}$. Applying τ^n to this for $n > m$, we get $\tau^n(z)\tau^n(x_{i+1}) \in \tau^n(I)D(S) + \tau^n(\mathfrak{p})D(S)$. By (6.4) and (6.1.1.iii), this means that

$$c\tau^n(z)\tau^n(x_{i+1}) \in \tau^n(I)D(S) + \tau^{n-m}(J)D(S).$$

Since the $\tau^{n-m}(y_j)$ together with the $\tau^n(x_j)$ form a $D(S)$-regular sequence by Lemma 6.1.7, we conclude that

$$c\tau^n(z) \in \tau^n(I)D(S) + \tau^{n-m}(J)D(S) \subseteq \tau^n(I)D(S) + JD(S).$$

Therefore, under the induced morphism $D(S) \to D(R)$, we get

$$c\sigma^n(z) \in \sigma^n(I)D(R)$$

for all $n > m$, showing that $z \in \mathrm{cl}^D(IR)$. □

As in positive characteristic, a slight modification of the proof allows us to omit the domain condition. To prove the remaining two properties (the analogues of Theorems 5.3.4 and 5.3.6 respectively), some additional assumptions are needed. To compare with integral closure, we have to make a rather technical assumption on the underlying category $\mathfrak{C}$. We say that $\mathfrak{C}$ has the *Néron* property if for any homomorphism $A \to V$ with A in $\mathfrak{C}$ and V a discrete valuation ring (not necessarily belonging to $\mathfrak{C}$), there exists a faithfully flat extension $V \to W$ and a morphism $A \to R$ in $\mathfrak{C}$ with $R \in \mathfrak{C}$ a regular local ring such that the following diagram commutes

$$\begin{array}{ccc} A & \longrightarrow & V \\ \downarrow & & \downarrow \\ R & \longrightarrow & W. \end{array} \tag{6.5}$$

Clearly the Frobenius hull in prime characteristic trivially satisfies this property since we then may take $R = V = W$.

Proposition 6.1.10. *If $\mathfrak{C}$ is a difference hull satisfying the Néron property, then the difference closure of any ideal is contained in its integral closure.*

Proof. Let $I \subseteq A$ be an ideal of a ring A in $\mathfrak{C}$, and let $z \in A$ be in the difference closure of I. In order to show that z lies in the integral closure of I, we use criterion (5.4.1.iv). To this end, let $A \to V$ be a homomorphism into a discrete valuation ring V whose kernel is a minimal prime of A. We need to show that $z \in IV$. Since $\mathfrak{C}$ has the Néron property, we can find a faithfully flat extension $V \to W$ and a morphism $A \to R$ in $\mathfrak{C}$ with R a regular local ring, yielding a commutative diagram (6.5). By assumption, there exists a multiplier $c \in A$ and a number N such that (6.2) holds in $D(A)$. Since c does not lie in the kernel of $A \to V$, its image in R must, a fortiori, be non-zero. Hence the same ideal membership relations viewed in $D(R)$ show that z lies in the difference closure of IR. By Corollary 6.1.4, this implies that z already lies in IR whence in IW. By faithful flatness and Proposition 3.2.5, we get $z \in IV$, as we wanted to show. □

Let us say that the difference hull $D(\cdot)$ *commutes with finite homomorphisms* if for each finite homomorphism $A \to B$ in $\mathfrak{C}$, the canonical homomorphism $D(A) \otimes_A B \to D(B)$ is an isomorphism of $D(A)$-algebras. Once more, this property holds trivially for the Frobenius hull.

Proposition 6.1.11. *If $D(\cdot)$ commutes with finite homomorphisms, and if $A \subseteq B$ is a finite extension of domains, then $\mathrm{cl}^D(I) = \mathrm{cl}^D(IB) \cap A$ for any ideal $I \subseteq A$.*

Proof. As in the proof of Theorem 5.3.4, we have an A-linear map $\varphi\colon B \to A$ with $\varphi(1) \neq 0$. By base change, this yields a $D(A)$-linear map $D(A) \otimes_A B \to D(A)$, whence a $D(A)$-linear map $D(B) \to D(A)$. The remainder of the argument is now as in the proof of Theorem 5.3.4, and is left to the reader. □

6.2 Tight Closure

Our axiomatic treatment in terms of difference closure now only requires us to identify the appropriate difference hull. For the remainder of this chapter, K denotes a fixed algebraically closed Lefschetz field, and $\mathfrak{C}_K$ is the category of affine K-algebras (that is to say, the algebras essentially of finite type over K). By definition, we can realize K as an ultraproduct of fields K_p of characteristic p, where for simplicity we index these fields by their characteristic although this is not necessary. We remind the reader that $K = \mathbb{C}$ is an example of a Lefschetz field (Theorem 2.4.3). As difference hull, we now take the ultra-hull as defined in §4.1, viewing it as a difference ring by means of its ultra-Frobenius (see Definition 2.4.21).

Theorem 6.2.1. *The category $\mathfrak{C}_K$ has the Néron property, and the ultra-hull constitutes a simple difference hull which commutes with finite homomorphisms.*

Proof. We defer the proof of the Néron property to Proposition 6.2.2 below. The ultra-hull is functorial by 4.1.3. Property (6.1.1.i) holds by Theorem 4.2.2, and the two remaining properties (6.1.1.ii) and (6.1.1.iii) hold trivially. By Łoś' Theorem, the ultra-hull is a simple difference hull as defined in Remark 6.1.5; and it commutes with finite homomorphisms by Proposition 4.3.1. □

Proposition 6.2.2. *The category $\mathfrak{C}_K$ has the Néron property.*

Proof. Assume $A \to V$ is a homomorphism from a K-affine ring A into a discrete valuation ring V. Replacing A by its image in V, we may view A as a subring of V. By Cohen's Structure Theorems, the completion of V is isomorphic to $L[[t]]$ for some field L extending K and for t a single indeterminate. Let $\bar{L}$ be the algebraic closure of L and put $W := \bar{L}[[t]]$. By base change, the natural homomorphism $V \to W$ is faithfully flat. The image of A in W has the same (uncountable) cardinality as K, whence is already contained in a subring of the form $k[[t]]$ with k an algebraically closed subfield of $\bar{L}$ of the same cardinality as K. By Theorem 2.4.7, we have an isomorphism $k \cong K$, and so we may assume that the composition $A \to W$ factors through $K[[t]]$. Let B' be the A-subalgebra of W generated by t, and let B be its localization at $tW \cap B'$, so that B is a local V_0-affine algebra, where V_0 is the localization of $K[t]$ at the ideal generated by t. By Néron p-desingularization (see for instance [3, §4]), the embedding $B \subseteq K[[t]]$ factors through a regular local V_0-algebra R. Since R is then also a K-affine local algebra, it satisfies all the required properties. □

The difference closure obtained from this choice of difference hull on $\mathfrak{C}_K$ will simply be called again *tight closure* (in the paper [92] it was called *non-standard tight closure*). For ease of reference, we repeat its definition here: an element z in a K-affine ring A belongs to the tight closure of an ideal $I \subseteq A$ if there exists a multiplier $c \in A$ such that

$$c\mathbf{F}^n_\natural(z) \in \mathbf{F}^n_\natural(I)U(A) \tag{6.6}$$

for all $n \gg 0$. We will denote the tight closure of I by $\mathrm{cl}_A(I)$ or simply $\mathrm{cl}(I)$, and we adopt the corresponding terminology from positive characteristic. Immediately from Theorem 6.2.1, the fact that a local K-affine algebra is a residue ring of a regular ring (namely, of a localization of polynomial ring), and the results in the previous section we get:

Theorem 6.2.3. *Tight closure on K-affine rings satisfies the five key properties:*

6.2.3.i. *if $A \to B$ is an extension of K-affine domains, or more generally, a homomorphism of K-affine rings preserving multipliers, then* $\mathrm{cl}_A(I) \subseteq \mathrm{cl}_B(IB)$ *for every ideal $I \subseteq A$;*
6.2.3.ii. *if A is a K-affine regular ring, then any ideal in A is tightly closed, and in fact, A is F-regular;*
6.2.3.iii. *if R is a local K-affine algebra and $(x_1,\dots,x_d)$ a system of parameters in R, then $((x_1,\dots,x_i)R : x_{i+1}) \subseteq \mathrm{cl}((x_1,\dots,x_i)R)$ for all i;*
6.2.3.iv. *the tight closure of an ideal is contained in its integral closure;*
6.2.3.v. *if $A \subseteq B$ is a finite extension of K-affine domains, then* $\mathrm{cl}_A(I) = \mathrm{cl}_B(IB) \cap A$.

□

Of all five properties, only (6.2.3.iv) relies on a deeper theorem, to wit Néron p-desingularization (which, nonetheless, is a much weaker form of Artin Approximation than needed for the HH-tight closure as discussed in §5.6). Is there a more elementary argument, at least for proving that tight closure is inside the radical of an ideal? On the other hand, property (6.2.3.v) is not such a very impressive fact in characteristic zero since any finite extension of a Noetherian normal domain containing the rationals is split (see also the discussion following Theorem 6.4.1 below).

Since the ultra-hull is a simple difference hull, we can also define *simple tight closure* by requiring that (6.6) only holds for $n = 1$ (this was termed *non-standard closure* in [92]); it is a closure operation satisfying the five key properties of Theorem 6.2.3. As already remarked, these five properties form the foundation for deriving several deep theorems, as we now will show.

Theorem 6.2.4 (Hochster-Roberts—Affine Case). *If $R \to S$ is a cyclically pure homomorphism of local K-affine algebras and if S is regular, then R is Cohen-Macaulay.*

The argument is exactly as in positive characteristic: one shows first that R is weakly F-regular, and then that any weakly F-regular ring is Cohen-Macaulay because we have Colon Capturing (in fact, one can prove an analogue of this result in any difference hull). Note that by our discussion in §5.5.2, we have now completed the proof of Theorem 5.5.3 (to prove the result, we may always extend the base field to a Lefschetz field). The next result, however, cannot be proven—it seems—within the more general framework of difference hulls, although its tight closure proof is still elementary.

Theorem 6.2.5 (Briançon-Skoda—Affine Case). *Let A be a K-affine ring, and let $I \subseteq A$ be an ideal generated by n elements. If I has positive height, then we have for all $l \geq 1$ an inclusion*

$$\overline{I^{n+l-1}} \subseteq \mathrm{cl}(I^l).$$

In particular, if A is a K-affine regular local ring, then the integral closure of I^{n+l-1} lies inside I^l for all $l \geq 1$.

Proof. Again we only proof the case $l = 1$. Let z be in the integral closure of I^n, and let A_p, z_p and I_p be approximations of A, z and I respectively. The integral equation (similar to (5.7)), say, of degree d, witnessing that z lies in the integral closure of I^n, shows by Łoś' Theorem that almost each z_p satisfies a similar integral equation of degree d, and hence, in particular, z_p belongs to the integral closure of I_p^n. By the argument in the proof of Theorem 5.4.1, for those p we have

$$I_p^{dn} z_p^k \in I_p^{kn}$$

for all k. As in the proof of Theorem 5.5.2, this implies that $I_p^{dn}\mathbf{F}_p^e(z_p)$ is contained in $\mathbf{F}_p^e(I_p)A_p$ for all e. Taking ultraproducts then yields

$$I^{dn}\mathbf{F}_\natural^e(z) \subseteq \mathbf{F}_\natural^e(I)U(A).$$

Since I has positive height, we can find by prime avoidance a multiplier $c \in I^{dn}$. In particular, $c\mathbf{F}_\natural^e(z) \in \mathbf{F}_\natural^e(I)U(A)$ for all e, whence $z \in \mathrm{cl}(I)$, as we wanted to show. The last assertion then follows from Theorem 6.2.3. □

We would of course prefer a version in which no assumption on I needs to be made. This indeed exists, but requires an intermediary closure operation, *ultra-closure* (see §6.3). The argument is almost identical to the above; see [92, Theorem 9.2]. Using the previous result, we have now proven the polynomial case in the Briançon-Skoda theorem (Theorem 5.5.1). The last of our applications, the Ein-Lazardsfeld-Smith Theorem, can neither be carried out in the purely axiomatic setting of difference closure, but relies on some additional properties of the ultra-hull.

Theorem 6.2.6. *Let A be a K-affine regular domain, and let $\mathfrak{a} \subseteq A$ be a radical ideal, given as the intersection of finitely many prime ideals of height at most h. Then for all n, we have an inclusion $\mathfrak{a}^{(hn)} \subseteq \mathfrak{a}^n$.*

Proof. Let $z \in \mathfrak{a}^{(hn)}$, and let A_p, z_p and $\mathfrak{a}_p$ be approximations of A, z and $\mathfrak{a}$ respectively. By Theorem 4.3.9, almost all A_p are regular, and by Corollary 4.3.3 and Theorem 4.3.4, almost each $\mathfrak{a}_p$ is the intersection of finitely many prime ideals of height at most h. As in the proof of Theorem 5.5.12, for those p we therefore have $\mathfrak{a}_p^{hn^2}\mathbf{F}_p^e(z_p) \subseteq \mathbf{F}_p^e(\mathfrak{a}_p^n)A_p$ for all e. Taking ultraproducts then yields $\mathfrak{a}^{hn^2}\mathbf{F}_\natural^e(z) \subseteq \mathbf{F}_\natural^e(\mathfrak{a}^n)U(A)$, showing that z lies in $\mathrm{cl}(\mathfrak{a}^n)$ whence in $\mathfrak{a}^n$ by Theorem 6.2.3. □

6.3 Ultra-closure

In the two last proofs, we derived some membership relations in the approximations of an affine algebra and then took ultraproducts to get the same relations in its ultra-hull. However, each time the relations in the approximations already established tight closure membership in those rings. This suggests the following definition. Let A be a K-affine algebra, $I \subseteq A$ an ideal and $z \in A$. We say that z lies in the *ultra-closure* ultra-cl(I) of I (called the *generic tight closure* in [92, 95]), if z_p lies in the tight closure of I_p for almost all p, where A_p, z_p and I_p are approximations of A, z and I respectively. Put differently

$$\text{ultra-cl}(I) = (\operatorname*{ulim}_{p\to\infty} \mathrm{cl}_{A_p}(I_p)) \cap A,$$

where we view the ultraproduct of the tight closures as an ideal in $U(A)$.

With little effort one shows:

Proposition 6.3.1. *Ultra-closure is a closure operation satisfying the five key properties listed in Theorem 6.2.3.*

To relate ultra-closure with tight closure, some additional knowledge of the theory of test elements (see the discussion following Theorem 5.3.1) is needed. Since we did not discuss these in detail, I quote the following result without proof.

Proposition 6.3.2 ([92, Proposition 8.4]). *Given a K-affine algebra A, there exists a multiplier $c \in A$ with approximation $c_p \in A_p$ such that c_p is a test element in A_p for almost all p.* □

Theorem 6.3.3. *The ultra-closure of an ideal is contained in its tight closure (and also in its simple tight closure).*

Proof. Let $z \in$ ultra-cl(I), with I an ideal in a K-affine algebra A. Let A_p, z_p and I_p be approximations of A, z and I respectively. By definition, almost each z_p lies in the tight closure of I_p. Let c be a multiplier as in Proposition 6.3.2, with approximations c_p. For almost all p for which c_p is a test element, we get $c_p\mathbf{F}_p^e(z_p) \in \mathbf{F}_p^e(I_p)A_p$ for all $e \geq 0$. Taking ultraproducts then yields $c\mathbf{F}_\natural^e(z) \in \mathbf{F}_\natural^e(I)U(A)$ for all e, showing that z lies in the (simple) tight closure of I. □

Without proof, we state the following comparison between our theory and the classical theory due to Hochster and Huneke (see §5.6); for a proof see [92, Theorem 10.4].

Proposition 6.3.4. *The HH-tight closure of an ideal is contained in its ultra-closure, whence in its tight closure.* □

6.4 Big Cohen-Macaulay Algebras

Although the material in this section is strictly speaking not part of tight closure theory, the development of the latter was germane to the discovery by Hochster and Huneke of Theorem 6.4.1 below.

6.4.1 Big Cohen-Macaulay Algebras in Prime Characteristic

Recall that the *absolute integral closure* A^+ of a domain A with field of fractions F, is the integral closure of A inside an algebraic closure of F. Since algebraic closure is unique up to isomorphism, so is absolute integral closure. Nonetheless it is not functorial, and we only have the following quasi-functorial property: given a homomorphism $A \to B$ of domains, there exists a (not necessarily unique) homomorphism $A^+ \to B^+$ making the diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A^+ & \longrightarrow & B^+ \end{array} \qquad (6.7)$$

commute.

Theorem 6.4.1 ([49]). *For every excellent local domain R in characteristic p, the absolute integral closure R^+ is a balanced big Cohen-Macaulay algebra.*

The condition that a Noetherian local ring is excellent is for instance satisfied when R is either K-affine or complete (see [69, §32]). The proof of the above result is beyond the scope of these notes (see for instance [59, Chapters 7& 8]) although we will present a 'dishonest' proof shortly. It is quite a remarkable fact that the same result is completely false in characteristic zero: in fact any extension of a normal domain is split, and hence provides a counterexample as soon as R is not Cohen-Macaulay. One can use the absolute integral closure to define a closure operation in an excellent local domain R of prime characteristic as follows. For an ideal I, let the *plus closure* of I be the ideal $I^+ := IR^+ \cap R$. One can show that I^+ is a closure operation in the sense of Definition 5.2.5, satisfying the five key properties listed in Theorem 6.2.3. Moreover, unlike tight closure, it is not hard to show that it commutes with localization.

Proposition 6.4.2. *In an excellent local domain R of prime characteristic, the plus closure of an ideal $I \subseteq R$ is contained in its tight closure.*

Proof. Let $z \in I^+$. By definition, there exists a finite extension $R \subseteq S \subseteq R^+$ such that $z \in IS$ (note that R^+ is the direct limit of all finite extensions of R by local domains). Hence $z \in \mathrm{cl}(I)$ by Theorem 5.3.4. □

It was conjectured that plus closure always equals tight closure. In view of [15], this now seems unlikely, since plus closure is easily seen to commute with localization, whereas tight closure apparently does not (see our discussion of (5.5)). Nonetheless, Smith has verified a special case of the conjecture for an important class of ideals:

Theorem 6.4.3 ([107]). *Any ideal generated by part of a system of parameters in an excellent local domain of prime characteristic has the same plus closure as tight closure.*

Remark 6.4.4. The main ingredient in the proof of Proposition 6.4.2 is the following fact, which is immediate from Lemma 5.3.5: the dual of R^+ as an R-module is non-zero, that is to say, there exists a non-zero R-module morphism $R^+ \to R$. Hochster ([45, Theorem 10.5]) has proven this to be true for any big Cohen-Macaulay algebra over a complete local ring of positive characteristic. Using this fact in the same way as in the proof of Theorem 5.3.4, he shows that if B is a balanced big Cohen-Macaulay algebra over a Noetherian local ring of positive characteristic, then $IB \cap R$ is contained in the tight closure of an ideal $I \subseteq R$. In fact, conversely, any element in the tight closure of I lies in IB, for some balanced big Cohen-Macaulay R-algebra B ([45, Theorem 11.1]).

Proof of Theorem 6.4.1 (Affine or Complete Case) Assuming Theorem 6.4.3

The proof we will present here is dishonest in the sense that Smith made heavily use of Theorem 6.4.1 to derive her result. However, here is how the converse direction goes. Let $(x_1,\dots,x_d)$ be a system of parameters in a local domain R of characteristic p which is either affine or complete, and suppose $zx_{i+1} \in IR^+$ for some $z \in R^+$ and $I := (x_1,\dots,x_i)R$. Hence there already exists a finite extension $R \subseteq S \subseteq R^+$ containing z such that $zx_{i+1} \in IS$. Since $R \subseteq S$ is finite, $(x_1,\dots,x_d)$ is also a system of parameters in S. In either case, Colon Capturing applies (see the remark following Theorem 5.3.3) and we get $z \in \operatorname{cl}(IS)$. By Theorem 6.4.3, this implies that z lies in the plus closure of IS, whence in IS^+. However, it is not hard to see that $R^+ = S^+$, proving that $(x_1,\dots,x_d)$ is an R^+-regular sequence. □

6.4.5 *If R is an excellent regular local ring of prime characteristic, then R^+ is faithfully flat over R.*

This follows immediately from Theorem 6.4.1 and the Cohen-Macaulay criterion for flatness (Theorem 3.3.9). Interestingly, it also provides an alternative strategy to prove Theorem 6.4.1:

Proposition 6.4.6. *Let k be a field of positive characteristic. Suppose we can show that any k-affine (respectively, complete) regular local ring has a faithfully flat absolute integral closure, then the absolute integral closure of any k-affine (respectively, complete Noetherian) local domain is a balanced big Cohen-Macaulay algebra.*

Proof. I will only treat the affine case and leave the complete case as an exercise. Let R be a local k-affine domain, and let $\mathbf{x}$ be a system of parameters in R.

By Noether normalization with parameters ([27, Theorem 13.3]), we can find a k-affine regular local subring $S \subseteq R$ containing x, such that $S \subseteq R$ is finite and $\mathbf{x}S$ is the maximal ideal of S. By assumption, S^+ is faithfully flat over S, and hence $(x_1, \dots, x_d)$ is an S^+-regular sequence by Proposition 3.2.9. Finiteness yields $S^+ = R^+$, and so we are done. □

6.4.2 Big Cohen-Macaulay Algebras in Characteristic Zero

As already mentioned, if R is a K-affine local domain of characteristic zero, then R^+ will in general not be a big Cohen-Macaulay algebra. However, we can still associate to any such R (in a quasi-functorial way) a canonically defined balanced big Cohen-Macaulay algebra as follows. Let R_p be an approximation of R. By Theorem 4.3.4, almost all R_p are domains. Let $B(R)$ be the ultraproduct of the R_p^+. To show that this is independent from the choice of approximation, we will give an alternative, more intrinsic description of $B(R)$. Let $\mathbb{N}_\natural$ be the ultrapower of the set of natural numbers, and let t be an indeterminate. For an element $f \in U(R[t])$, define its *ultra-degree* $\alpha \in \mathbb{N}_\natural$ (with respect to t) to be the ultraproduct of the t-degrees α_p of the f_p, where f_p is an approximation of f. Call an element $f \in U(R[t])$ *ultra-monic* if there exists $\alpha \in \mathbb{N}_\natural$ such that $f - t^\alpha$ has ultra-degree strictly less than α (see §2.4.4 for ultra-exponentiation). By a *root* of $g \in U(R[t])$ in a Lefschetz field L containing K we mean an element $a \in L$ such that $g \in (t-a)U(R_L[t])$, where $R_L := R \otimes_K L$ and its ultra-hull is taken in the category $\mathfrak{C}_L$. One now easily shows that there exists an algebraically closed Lefschetz field L containing K such that $B(R)$ is isomorphic to the ring of all $a \in L$ that are a root of some ultra-monic element in $U(R_L[t])$. Moreover, this ring is independent from the choice of L.

By Łoś' Theorem, there is a canonical homomorphism $R \to B(R)$.

Theorem 6.4.7. *If R is a K-affine local domain, then $B(R)$ is a balanced big Cohen-Macaulay algebra over R.*

Proof. Since almost each approximation R_p is a K_p-affine (whence excellent) local domain, R_p^+ is a balanced big Cohen-Macaulay R_p-algebra by Theorem 6.4.1. Let $\mathbf{x}$ be a system of parameters of R, with approximation $\mathbf{x}_p$. By Corollary 4.3.8, almost each $\mathbf{x}_p$ is a system of parameters in R_p, whence an R_p^+-regular sequence. By Łoś' Theorem, $\mathbf{x}$ is therefore $B(R)$-regular, as we wanted to show. □

Hochster and Huneke ([52]) arrive differently at balanced big Cohen-Macaulay algebras in characteristic zero, via their lifting method discussed in §5.6. However, their construction, apart from being rather involved, is far less canonical. In contrast, although it appears that $B(R)$ depends on R, we have in fact:

6.4.8 *For each d, there exists a ring B_d such that for any K-affine local domain R, we have $B(R) \cong B_d$ if and only if R has dimension d. In other words, B_d is a balanced big Cohen-Macaulay algebra for R if and only if R has dimension d.*

Indeed, by Noether normalization, R is finite over the localization of $K[\xi]$ at the ideal generated by the indeterminates $\xi := (\xi_1,\dots,\xi_d)$. By Łoś' Theorem, the approximation R_p is finite over the corresponding localization of $K_p[\xi]$. If S_p is the absolute integral closure of this localization, then $S_p = R_p^+$. Hence the ultraproduct of the S_p only depends on d and is isomorphic to $B(R)$. □

In analogy with plus closure, we define the *B-closure* $\mathrm{cl}^B(I)$ of an ideal I in a K-affine local domain R as the ideal $IB(R) \cap R$. As in positive characteristic, it is a closure operation satisfying the five key properties of Theorem 6.2.3. Using Proposition 6.4.2 and Łoś' Theorem, together with Theorem 6.3.3 we get:

6.4.9 *For any ideal I in a K-affine local domain R, we have inclusions* $\mathrm{cl}^B(I) \subseteq \text{ultra-cl}(I) \subseteq \mathrm{cl}(I)$. □

Like tight closure theory, the existence of balanced big Cohen-Macaulay algebras does have many important applications. To illustrate this, we give an alternative proof of the Hochster-Roberts Theorem, as well as a proof of the Monomial Conjecture (as far as I am aware of, no tight closure argument proves the latter). We will treat only the affine characteristic zero case here, but the same argument applies in positive characteristic, and, once we have developed the theory in Chapter 7, for arbitrary equicharacteristic Noetherian local rings.

Alternative Proof of Theorem 6.2.4

Let $R \to S$ be a cyclically pure homomorphism of K-affine local domains with S regular, and let $\mathbf{x} := (x_1,\dots,x_d)$ be a system of parameters in R. To show that this is an R-regular sequence, assume $zx_{i+1} \in I := (x_1,\dots,x_i)R$. Since $\mathbf{x}$ is $B(R)$-regular by Theorem 6.4.7, we get $z \in IB(R)$. By quasi-functoriality (after applying Łoś' Theorem to (6.7)) we get a homomorphism $B(R) \to B(S)$ making the diagram

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ B(R) & \longrightarrow & B(S) \end{array} \qquad (6.8)$$

commute. In particular, $z \in IB(S)$. Since S is regular, $S \to B(S)$ is flat by the Cohen-Macaulay criterion for flatness (Theorem 3.3.9) and Theorem 6.4.7. Hence z belongs to IS by Proposition 3.2.5 whence to I by cyclical purity. □

As promised, we conclude with an application of the existence of big Cohen-Macaulay algebras to one of the homological conjectures (for further discussion, especially the still open mixed characteristic case, see Chapter 10). Let us call a tuple $(x_1,\dots,x_d)$ in a ring R *monomial*, if for all k, we have

$$(x_1\cdots x_d)^{k-1} \notin (x_1^k,\dots,x_d^k)R. \tag{6.9}$$

We say that the *Monomial Conjecture* holds for a Noetherian local ring R, if R satisfies the hypothesis in the next result:

Theorem 6.4.10 (Monomial Conjecture). *If R is a local K-affine algebra, then any system of parameters is monomial.*

Proof. Let $(x_1,\dots,x_d)$ be a system of parameters, let x be the product of the x_i, and suppose $x^{k-1} \in I_k := (x_1^k,\dots,x_d^k)R$ for some k. Let $\mathfrak{p}$ be a d-dimensional prime ideal. Since $(x_1,\dots,x_d)$ is then also a system of parameters in $R/\mathfrak{p}$, and $x^{k-1} \in I_k(R/\mathfrak{p})$, we may after replacing R by $R/\mathfrak{p}$ assume that R is a domain. Hence $(x_1,\dots,x_d)$ is $B(R)$-regular by Theorem 6.4.7. However, it is easy to see that for a regular sequence we can never have $x^{k-1} \in I_kB(R)$, that is to say, regular sequences are always monomial. □

Remark 6.4.11. By an argument on local cohomology, one can show that given any system of parameters $(x_1,\dots,x_d)$ in a Noetherian local ring R, there exists some t such that $(x_1^t,\dots,x_d^t)$ is monomial [17, Remark 9.2.4(b)]. Hence the real issue as far as the Monomial Conjecture is concerned is the fact that one can always take $t = 1$.

Chapter 7
Tight Closure in Characteristic Zero. Local Case

The goal of this chapter is to extend the tight closure theory from the previous chapter to include all Noetherian rings containing a field. However, the theory becomes more involved, especially if one wants to maintain full functoriality. We opt in these notes to forego this cumbersome route (directing the interested reader to the joint paper [6] with Aschenbrenner), and only develop the theory minimally as to still obtain the desired applications. In particular, we will only focus on the local case.

From our axiomatic point of view, we need to define a difference hull on the category of Noetherian local rings containing $\mathbb{Q}$. The main obstacle is how to define an ultra-hull-like object, on which we then have automatically an action of the ultra-Frobenius. By Cohen's Structure Theorems, the problem can be reduced to constructing a difference hull for the power series ring $R := K[[\xi]]$ in a finite number of indeterminates ξ over an algebraically closed Lefschetz field K. A candidate presents itself naturally: let $U(R)$ be the ultraproduct of the $K_p[[\xi]]$, where the K_p are algebraically closed fields of characteristic p whose ultraproduct is K. However, unlike in the polynomial case, there is no obvious homomorphism from R to $U(R)$, and in fact, the very existence of such a homomorphism implies already some form of Artin Approximation. It turns out, however, that we can embed R in an ultrapower of $U(R)$, and this is all we need, since the latter is still a Lefschetz ring. So we start with a discussion of this construction, and its underlying tool, Artin Approximation.

7.1 Artin Approximation

7.1.1 Constructing Algebra Homomorphisms

In this section, we study the following problem: Given two A-algebras S and T, when is there an A-algebra homomorphism $S \to T$? We will only provide a solution to the weaker version in which we are allowed to replace T by one of its ultrapowers. Since we want to apply this problem when T is equal to $U(R)$, we will merely have replaced one type of ultraproduct with another.

H. Schoutens, *The Use of Ultraproducts in Commutative Algebra*, 97
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Theorem 7.1.1. *For a Noetherian ring A, and A-algebras S and T, the following are equivalent:*

7.1.1.i. *every system of polynomial equations with coefficients from A which is solvable in S, is solvable in T;*

7.1.1.ii. *for each finitely generated A-subalgebra C of S, there exists an A-algebra homomorphism $\varphi_C\colon C \to T$;*

7.1.1.iii. *there exists an A-algebra homomorphism $\eta\colon S \to T_\natural$, where $T_\natural$ is some ultrapower of T.*

Proof. Suppose that (7.1.1.i) holds, and let $C \subseteq S$ be an A-affine subalgebra. Hence C is isomorphic to $A[\xi]/I$ with ξ a finite tuple of indeterminates and I some ideal in $A[\xi]$. Let $\mathbf{x}$ be the image of ξ in S, so that $\mathbf{x}$ is a solution of the system of equations $f_1 = \cdots = f_s = 0$, where $I = (f_1,\dots,f_s)A[\xi]$. By assumption, there exists therefore a solution $\mathbf{y}$ of this system of equations in T. Hence the A-algebra homomorphism $A[\xi] \to T$ given by sending ξ to $\mathbf{y}$ factors through an A-algebra homomorphism $\varphi_C\colon C \to T$, proving implication (7.1.1.i) $\Rightarrow$ (7.1.1.ii).

Assume next that (7.1.1.ii) holds. Let W be the collection of all A-affine subalgebras of S (there is nothing to show if S itself is A-affine, so we may assume W is in particular infinite). For each finite subset $\Sigma \subseteq S$ let $\langle\Sigma\rangle$ be the subset of W consisting of all A-affine subalgebras $C \subseteq S$ containing Σ. Any finite intersection of sets of the form $\langle\Sigma\rangle$ is again of that form. Hence we can find an ultrafilter on W containing each $\langle\Sigma\rangle$, where Σ runs over all finite subsets of S. Let $T_\natural$ be the ultrapower of T with respect to this ultrafilter. For each A-affine subalgebra $C \subseteq S$, let $\widetilde{\varphi}_C\colon S \to T$ be the map which coincides with φ_C on C and which is identically zero outside C. (This is of course no longer a homomorphism.) Define $\eta\colon S \to T_\natural$ to be the restriction to S of the ultraproduct of the $\widetilde{\varphi}_C$. In other words,

$$\eta(x) := \operatorname*{ulim}_{C\to\infty} \widetilde{\varphi}_C(x)$$

for any $x \in S$. Remains to verify that η is an A-algebra homomorphism. For $x,y \in S$, we have for each $C \in \langle\{x,y\}\rangle$ that

$$\widetilde{\varphi}_C(x+y) = \varphi_C(x+y) = \varphi_C(x) + \varphi_C(y) = \widetilde{\varphi}_C(x) + \widetilde{\varphi}_C(y),$$

since $\widetilde{\varphi_C}$ and φ_C agree on elements in C. Since this holds for almost all C, Łoś' Theorem yields $\eta(x+y) = \eta(x) + \eta(y)$. By a similar argument, one also shows that $\eta(xy) = \eta(x)\eta(y)$ and $\eta(ax) = a\eta(x)$ for $a \in A$, proving that η is an A-algebra homomorphism.

Finally, suppose that $\eta\colon S \to T_\natural$ is an A-algebra homomorphism, for some ultrapower $T_\natural$ of T. Let $f_1 = \cdots = f_s = 0$ be a system of polynomial equations with coefficients in A, and let $\mathbf{x}$ be a solution in S. Since η is an A-algebra homomorphism, $\eta(\mathbf{x})$ is a solution in $T_\natural$. Hence by Łoś' Theorem, this system must have a solution in T, proving (7.1.1.iii) $\Rightarrow$ (7.1.1.i). ☐

Remark 7.1.2. One can also get, with almost the same argument, embeddings rather than just homomorphisms: the following are equivalent for algebras S and T over a Noetherian ring A:

7.1.2.i. *every finite system of polynomial equations and negations of equations with coefficients from A which is solvable in S, is solvable in T;*
7.1.2.ii. *given an A-affine subalgebra $C \subseteq S$ and finitely many non-zero elements $c_1, \dots, c_n$ of C there exists an A-algebra homomorphism $C \to T$ sending each c_i to a non-zero element of T;*
7.1.2.iii. *there exists an embedding $S \to T_\natural$ of A-algebras into an ultrapower $T_\natural$ of T.*

We will use this criterion below to construct Lefschetz hulls; for another application, see Theorem B.2.7 below.

7.1.2 Artin Approximation

We already got acquainted with Artin Approximation in our discussion of HH-tight closure, or in the guise of Néron p-desingularization as used in Proposition 6.2.2. The time has come, therefore, to present a more detailed discussion. Let $(R, \mathfrak{m})$ be a Noetherian local ring. We say that R satisfies the *Artin Approximation property* if any system of polynomial equations $f_1 = \cdots = f_s = 0$ (in finitely many indeterminates) with coefficients in R which is solvable in the completion $\widehat{R}$ is already solvable in R. By some easy manipulations, we can formulate some stronger versions that are often useful in applications. Given finitely many congruence relations $f_i \equiv 0 \mod \mathfrak{m}^{c_i}\widehat{R}$ with $f_i \in R[\xi]$, one can turn these in to a system of equations, such that the congruences are solvable in $\widehat{R}$ or R if and only if the equations are. More precisely, let $\mathfrak{m} = (x_1, \dots, x_e)R$. For each i and each e-tuple $\mathbf{j}$ of non-negative integers whose entries sum up to c_i, let $\zeta_{i,\mathbf{j}}$ be a new indeterminate, and consider the polynomial

$$g_i := f_i(\xi) - \sum_{|\mathbf{j}|=c_i} \zeta_{i,\mathbf{j}} \mathbf{x}^{\mathbf{j}}$$

with coefficients in R. By assumption, the system of equations $g_1 = \cdots = g_s = 0$ has a solution in $\widehat{R}$, whence, by Artin Approximation, in R, which in turn means that the congruences $f_i \equiv 0 \mod \mathfrak{m}^{c_i}$ have a common solution in R.

A similar trick can be applied to a system of equations and negations of equations: if we have a solution of $f_1 = \cdots = f_s = 0$ and $g \neq 0$ in $\widehat{R}$, then by the Krull's Intersection Theorem (Theorem 2.4.14), there is some c such that this solution is also a solution to the negated congruence $g \not\equiv 0 \mod \mathfrak{m}^c\widehat{R}$. Since $R/\mathfrak{m}^cR \cong \widehat{R}/\mathfrak{m}^c\widehat{R}$, we can find some $r \in R$ not in $\mathfrak{m}^c$, such that the given solution satisfies $g - r \equiv 0 \mod \mathfrak{m}^c\widehat{R}$. By the previous case, we can find a solution in R of $f_1 = \cdots = f_s = 0$

and the congruence $g \equiv r \neq 0 \mod \mathfrak{m}^c$, which therefore in particular is a solution to $g \neq 0$. In conclusion, Artin Approximation is equivalent with either of the following two apparently stronger conditions:

7.1.2.a. *any system of polynomial equations and negations of equations over R which is solvable in $\widehat{R}$ is already solvable in R;*

7.1.2.b. *given some c and a system of equations over R with a solution $\widehat{\mathbf{x}}$ in $\widehat{R}$, we can find a solution $\mathbf{x}$ in R such that $\mathbf{x} \equiv \widehat{\mathbf{x}} \mod \mathfrak{m}^c\widehat{R}$, that is to say, a solution in $\widehat{R}$ can be 'approximated' arbitrarily close by solutions in R.*

The last condition also explains the name of this property; the first condition can be paraphrased for model-theorists simply as *R is existentially closed in $\widehat{R}$*. Immediately from Theorem 7.1.1, or rather by the embedding version of Remark 7.1.2, we get:

7.1.3 *A Noetherian local ring R has the Artin Approximation property if and only if its completion embeds in some ultrapower of R.* □

Not any Noetherian local ring can have the Artin Approximation property:

Proposition 7.1.4. *A Noetherian local ring $(R,\mathfrak{m})$ with the Artin Approximation property is Henselian.*

Proof. Recall that this means that R satisfies Hensel's Lemma: any simple root $\bar{a}$ in $R/\mathfrak{m}$ of a monic polynomial $f \in R[t]$ lifts to a root in the ring itself (see Appendix A). By Hensels' Lemma (Theorem A.1.1), we can find such a root in $\widehat{R}$, and therefore by Artin Approximation (in the guise of (7.1.2.b) above), we then also must have a root in R itself. □

Artin conjectured in [3] that the converse also holds if R is moreover excellent (it can be shown that any ring having the Artin Approximation property must be excellent). Although one has now arrived at a positive solution by means of very deep tools ([76, 109, 111]), the ride has been quite bumpy, with many false proofs appearing in print during the intermediate decades. Luckily, we only need this in the following special case due to Artin himself, admitting a fairly simple proof (which nonetheless is beyond the scope of these notes).

Theorem 7.1.5 ([3, Theorem 1.10]). *The Henselization $k[\xi]^\sim$, with k a field and ξ a finite tuple of indeterminates, admits the Artin Approximation property.* □

The Henselization $k[\xi]^\sim$ of $k[\xi]$ is the 'smallest' Henselian local ring contained in $k[[\xi]]$, and in this case, is equal to the ring of algebraic power series; see A.3.4.

7.1.3 Embedding Power Series Rings

From now on, unless stated otherwise, K denotes an arbitrary ultra-field, given as the ultraproduct of fields K_m (for simplicity we assume $m \in \mathbb{N}$). We fix a tuple of indeterminates $\xi := (\xi_1, \dots, \xi_n)$, define $A := K[\xi]$ and $R := K[[\xi]]$, and let

$\mathfrak{m} := (\xi_1, \dots, \xi_n)\mathbb{Z}[\xi]$. Similarly, for each m, we let $A_m := K_m[\xi]$ and $R_m := K_m[[\xi]]$, and in accordance with our notation from §4.1, we denote their respective ultraproducts by $U(A)$ and $U(R)$. By Łoś' Theorem, we get a homomorphism $U(A) \to U(R)$ so that $U(R)$ is in particular an A-algebra, but unlike the affine case, it is no longer clear how to make $U(R)$ into an R-algebra. Note that $U(R)$ is only quasi-complete (see the proof of Theorem 8.1.4), so that limits are not unique. In particular, although the truncations $f_n \in A$ of a power series $f \in R$ form a Cauchy sequence in $U(R)$, there is no obvious choice for their limit.

Theorem 7.1.6. *There exists an ultrapower $L(R)$ of $U(R)$ and a faithfully flat A-algebra homomorphism $\eta_R\colon R \to L(R)$.*

Proof. We start with proving the existence of an A-algebra homomorphism η_R from R to some ultrapower of $U(R)$. To this end, we need to show in view of Theorem 7.1.1 that any polynomial system of equations $\mathscr{L}$ over A which is solvable in R, is also solvable in $U(R)$. By Theorem 7.1.5, the system has a solution $\mathbf{y}$ in $A^\sim$. Since the complete local rings R_m are Henselian (Theorem A.1.1), so is $U(R)$ by Łoś' Theorem. By the universal property of Henselization, Theorem A.2.3, the canonical homomorphism $A \to U(R)$ extends to a (unique) A-algebra homomorphism $A^\sim \to U(R)$. Hence the image in $U(R)$ of $\mathbf{y}$ is a solution of $\mathscr{L}$ in $U(R)$, as we wanted to show.

Let $L(R)$ be the ultrapower of $U(R)$ given by Theorem 7.1.1 with corresponding A-algebra homomorphism $\eta\colon R \to L(R)$. Since $\eta(\xi_i) = \xi_i$, for all i, the maximal ideal of $L(R)$ is generated by the ξ, and so η is local. By the Cohen-Macaulay criterion for flatness (Theorem 3.3.9), it suffices to show that $L(R)$ is a balanced big Cohen-Macaulay algebra. Since ξ is an R_m-regular sequence, so is its ultraproduct $\eta(\xi) = \xi$ in $L(R)$. This proves that $L(R)$ is a big Cohen-Macaulay algebra, and we can now use Proposition 3.3.8 and Łoś' Theorem, to conclude that it is balanced, and hence that $\eta\colon R \to L(R)$ is faithfully flat. □

Being an ultrapower of an ultraproduct, $L(R) = U(R)_\natural$, itself is an ultra-ring. More precisely:

7.1.7 *There exists an index set W and an $\mathbb{N}$-valued function assigning to each $w \in W$ an index $m(w)$, such that*

$$L(R) = \operatorname*{ulim}_{w \to \infty} R_{m(w)}.$$

7.1.4 Strong Artin Approximation

We say that a local ring $(S, \mathfrak{n})$ has the *strong Artin Approximation property* if the following holds: given a system $\mathscr{L}$ of polynomial equations $f_1 = \dots = f_s = 0$ with coefficients in S, if $\mathscr{L}$ has an approximate solution in S modulo $\mathfrak{n}^m$ for all m, then $\mathscr{L}$ has a (true) solution in S. Here by an *approximate solution* of $\mathscr{L}$ modulo an ideal $\mathfrak{a} \subseteq S$, we mean a tuple $\mathbf{x}$ in S such that the congruences $f_1(\mathbf{x}) \equiv \cdots \equiv f_s(\mathbf{x}) \equiv 0 \mod \mathfrak{a}$ hold, that is to say, a solution of $\mathscr{L}$ in $S/\mathfrak{a}$.

We start with the following observation regarding the connection between R and its Lefschetz hull $L(R)$ (this will be explored in more detail in §8.1 where we will call the separated quotient the *cataproduct* of the R_m).

Proposition 7.1.8. *The separated quotient $U(R)/\mathfrak{I}_{U(R)}$ of $U(R)$ is isomorphic to $R = K[[\xi]]$; similarly, the separated quotient $L(R)/\mathfrak{I}_{L(R)}$ is isomorphic to $K_\natural[[\xi]]$, where $K_\natural$ is some ultrapower of K.*

Proof. The proof of both statements is similar, and so we will only prove the first (for the second, we let $K_\natural$ be the ultraproduct of the $K_{m(w)}$ given by 7.1.7). We start by defining a homomorphism $U(R) \to R$ as follows. Given $f \in U(R)$, choose approximations $f_m \in R_m$ and expand each as a power series

$$f_m = \sum_{\nu \in \mathbb{N}^n} a_{\nu,m} \xi^\nu$$

for some $a_{\nu,m} \in K_m$. Let $a_\nu \in K$ be the ultraproduct of the $a_{\nu,m}$ and define

$$\tilde{f} := \sum_{\nu \in \mathbb{N}^n} a_\nu \xi^\nu \in R.$$

One checks that the map $f \mapsto \tilde{f}$ is well-defined (that is to say, independent of the choice of approximation), and is a ring homomorphism. It is not hard to see that it is moreover surjective. So remains to show that its kernel equals the ideal of infinitesimals $\mathfrak{I}_{U(R)}$. Suppose $\tilde{f} = 0$, whence all $a_\nu = 0$. For fixed d, almost all $a_{\nu,m} = 0$ whenever $|\nu| < d$. Hence $f_m \in \mathfrak{m}^d R_m$ for almost all m, and therefore $f \in \mathfrak{m}^d U(R)$ by Łoś' Theorem. Since this holds for all d, we see that $f \in \mathfrak{I}_{U(R)}$. Conversely, any infinitesimal is easily seen to lie in the kernel by simply reversing this argument. □

In [11]—a paper the methods of which were germane for the development of the present theory—the following ultraproduct argument was used to derive a strong Artin Approximation result.

Theorem 7.1.9 (Becker-Denef-van den Dries-Lipshitz). *The ring $R := K[[\xi]]$, for K an arbitrary algebraically closed ultra-field and ξ a finite tuple of indeterminates, has the strong Artin Approximation.*

Proof. Let $\mathscr{L}$ be a system of equations over R, and for each m, let $\mathbf{x}_m$ be an approximate solution of $\mathscr{L}$ modulo $\mathfrak{m}^m R$. Let $R_\natural$ be some ultrapower of R, and let $\mathbf{x}$ be the ultraproduct of the $\mathbf{x}_m$. By Łoś' Theorem, $\mathbf{x}$ is an approximate solution of $\mathscr{L}$ modulo any $\mathfrak{m}^m R_\natural$, whence modulo $\mathfrak{I}_{R_\natural}$, the ideal of infinitesimals of $R_\natural$. By Proposition 7.1.8, the separated quotient $R_\natural/\mathfrak{I}_{R_\natural}$ is isomorphic to $K_\natural[[\xi]]$, where $K_\natural$ is the ultrapower of K. The image of $\mathbf{x}$ in $K_\natural[[\xi]]$ is therefore a solution of the system $\mathscr{L}$. Let $k \subseteq K$ be a countable algebraically closed subfield such that $\mathscr{L}$ is already defined over k, and let $L \subseteq K_\natural$ be the algebraic closure of the field generated over k by all the coefficients of the entries in the image of $\mathbf{x}$ in $K_\natural[[\xi]]$. Since L has the same cardinality as K, they are isomorphic as fields by Theorem 2.4.7, and in fact, by a simple modification of its proof, these fields are isomorphic over their common countable subfield k. In particular, the image of $\mathbf{x}$ under the induced $k[[\xi]]$-algebra isomorphism of $L[[\xi]]$ with $K[[\xi]]$, gives the desired solution of $\mathscr{L}$ in $R = K[[\xi]]$. □

Any version in which the same conclusion as in the strong Artin Approximation property can be reached just from the solvability modulo a single power $\mathfrak{n}^N$ of the maximal ideal $\mathfrak{n}$, where N only depends on (some numerical invariants of) the system of equations, is called the *uniform strong Artin Approximation* property. In [11], the uniform strong Artin Approximation for certain Henselizations was derived from the Artin Approximation property of those rings via ultraproducts. To get a uniform version in more general situations, additional restrictions have to be imposed on the equations (see [3, Theorem 6.1] or [11, Theorem 3.2]) and substantially more work is required [23, 24]. We will here only present a more restrictive version in which the equations have polynomial coefficients as well.

Theorem 7.1.10 (Uniform Strong Artin Approximation). *For any pair of positive integers (d,n), there exists a bound $b := b(d,n)$ with the following property. Let k be a field, put $A := k[\xi]$ with ξ an n-tuple of variables, and let $\mathfrak{m}$ be the ideal generated by these variables. Let $\mathscr{L}$ be a polynomial system of equations with coefficients from A, in at most n indeterminates t, such that each polynomial in $\mathscr{L}$ has total degree (with respect to both ξ and t) at most d. If $\mathscr{L}$ admits an approximate solution in A modulo $\mathfrak{m}^bA$, then it admits a true solution in $k[[\xi]]$.*

Proof. Towards a contradiction, assume such a bound does not exist for the pair (d,n), so that for each $m \in \mathbb{N}$ we can find a counterexample consisting of a field K_m, and of polynomials f_{im} for $i = 1,\dots,s$ over this field of total degree at most d in the indeterminates ξ and t, such that viewed as a system $\mathscr{L}_m$ of equations in the unknowns t, it has an approximate solution $\mathbf{x}_m$ in $A_m := K_m[\xi]$ modulo $\mathfrak{m}^m A_m$ but no actual solution in $R_m := K_m[[\xi]]$. Note that by Lemma 4.4.2 we may assume that the number of equations s is independent from m. Let K and $U(R)$ be the ultraproduct of the K_m and R_m respectively, and let f_i and $\mathbf{x}$ be the ultraproduct of the f_{im} and $\mathbf{x}_m$ respectively. By 4.1.2, the f_i are polynomials over K, and by Łoś' Theorem, $f_i(\mathbf{x}) \equiv 0 \mod \mathfrak{I}_{U(R)}$. By Proposition 7.1.8, we have an epimorphism $U(R) \to R$. In particular, the image of $\mathbf{x}$ in R is a solution of the system $\mathscr{L}$ given by $f_1 = \cdots = f_s = 0$.

Since we have an A-algebra homomorphism $R \to L(R)$ by Theorem 7.1.6, the image of $\mathbf{x}$ in $L(R)$ remains a solution of the system $\mathscr{L}$, and hence by Łoś' Theorem, we can find for almost each w, a solution of $\mathscr{L}_{m(w)}$ in $R_{m(w)}$, contradicting our assumption on the systems $\mathscr{L}_m$. □

Note that the above proof only uses the existence of a homomorphism from R to some ultrapower of $U(R)$, showing that mere existence is already a highly nontrivial result, and hence it should not come as a surprise that we needed at least some form of Artin Approximation to prove the latter. Of course, by combining this with Theorem 7.1.5, we may even conclude that $\mathscr{L}$ has a solution in $A^{\sim}$, thus recovering the original result [3, Theorem 6.1] (see also [11, Theorem 3.2]). If instead we use the filtered version of Theorem 7.1.6, to be discussed briefly after Proposition 7.3.2 below, we get filtered versions of this uniform strong Artin Approximation property, as explained in [6]. Here is an example of such a result, generalizing [11, Theorem 4.3] (which only treats the case $s = 1$):

Theorem 7.1.11. *For any pair of positive integers (d,n), there exists a bound $b := b(d,n)$ with the following property. Let k be a field and let $\mathscr{L}$ be a polynomial system of equations in at most n indeterminates t with coefficients in $A := k[\xi]$ with ξ an n-tuple of variables, such that the total degree (with respect to ξ and t) is at most d. If $\mathscr{L}$ has an approximate solution $(x_1,\dots,x_n)$ in A modulo $\mathfrak{m}^b A$ with $x_1,\dots,x_l$ depending only on $\xi_1,\dots,\xi_s$, for some l,s, then there exists a solution $(y_1,\dots,y_n)$ in $k[[\xi]]$ with $y_1,\dots,y_l$ depending only on $\xi_1,\dots,\xi_s$.*

Proof (Sketch). As always, we start with assuming towards a contradiction that there exist counterexamples $\mathscr{L}_m$ over $A_m := K_m[[\xi]]$ of degree at most d with an approximate solution modulo $\mathfrak{m}^m A_m$ whose first l entries belong to $A'_m := K_m[\xi_1,\dots,\xi_s]$, but having no solution in $R_m := K_m[[\xi]]$ whose first l entries belong to $R'_m := K_m[[\xi_1,\dots,\xi_s]]$. By Proposition 7.3.2 below, one gets a commutative diagram of corresponding Lefschetz hulls

$$\begin{array}{ccc} R' & \longrightarrow & R \\ \downarrow & & \downarrow \\ L(R') & \longrightarrow & L(R) \end{array} \tag{7.1}$$

where $R := K[[\xi]]$ and $R' = K[[\xi_1,\dots,\xi_s]]$, and where K is the ultraproduct of the K_m. Using the existence of these embeddings in the same way as in the proof of Theorem 7.1.10, one obtains the desired contradiction. □

We conclude with the non-linear analogue of Theorem 4.4.3. We cannot simply expect the same conclusion as in the linear case to hold: there is no uniform bound on the degree of polynomial solutions in terms of the degrees of the system of equations (a counterexample is discussed in [89, Theorem 9.1]). However, we can recover bounds when we allow for power series solutions. Of course degree makes no sense in this context, and so we introduce the following substitute. By §A.2, a power series y lies in the Henselization $A^{\sim}$ if there exists an N-tuple $\mathbf{y}$ in R with first coordinate equal to y, and a *Hensel system* $\mathscr{H}$, consisting of N polynomials $f_1,\dots,f_N \in A[t]$ in the N unknowns t such that the Jacobian matrix $\operatorname{Jac}(\mathscr{H})$ evaluated at $\mathbf{x}$ is invertible in R. We say that y has *etale proto-grade* as most d (see §9 for the nomenclature), if we can find such a Hensel system of size $N \leq d$ with all f_i of total degree at most d in ξ and t (see §A.3 for more details).

Theorem 7.1.12. *For any pair of positive integers (d,n), there exists a uniform bound $b := b(d,n)$ with the following property. Let k be a field and put $A := k[\xi]$ with ξ an n-tuple of variables. Let $\mathscr{L}$ be a system of polynomial equations in $A[t]$ in at most n indeterminates t, such that each polynomial in $\mathscr{L}$ has total degree (with respect to ξ and t) at most d. If $\mathscr{L}$ is solvable in $k[[\xi]]$, then it has a solution in $A^{\sim}$ of etale complexity at most b.*

Proof. Suppose no such bound on the etale complexity exists for the pair (d,n), yielding for each m a counterexample consisting of a field K_m, and a system of polynomial equations $\mathscr{L}_m$ over K_m of total degree at most d with a solution $\mathbf{y}_m$ in the power series ring R_m, such that, however, any solution in $A_m^{\sim}$ has etale complexity at least m (notation as before). Let $\mathscr{L}$ be the ultraproduct of the $\mathscr{L}_m$, a system of polynomial equations over K by 4.1.2 (and an application of Lemma 4.4.2), and let $\mathbf{y}$ be the ultraproduct of the $\mathbf{y}_m$, a solution of $\mathscr{L}$ in $U(R)$ by Łoś' Theorem. By Proposition 7.1.8, under the canonical epimorphism $U(R) \to R$, we get a solution of $\mathscr{L}$ in R, whence in $A^{\sim}$ by Theorem 7.1.5. Let $\mathscr{H}$ be a Hensel system for this solution $\mathbf{x}$ viewed as a tuple in $A^{\sim}$ (note that one can always combine Hensel systems for each entry of a tuple to a Hensel system for the whole tuple; see the discussion preceding A.2.2), and let d be its total degree. Since the ultraproduct $H_\natural$ of the $A_m^{\sim}$ is a Henselian local ring containing A by Łoś' Theorem, the universal property of Henselizations (Theorem A.2.3) yields an A-algebra homomorphism $A^{\sim} \to H_\natural$. Viewing therefore $\mathbf{x}$ as a solution of $\mathscr{L}$ in $H_\natural$, we can find approximations $\mathbf{x}_m$ in $A_m^{\sim}$ which are solutions of $\mathscr{L}_m$ for almost each m. If we let $\mathscr{H}_m$ be an approximation of $\mathscr{H}$, then by Łoś' Theorem, for almost all m, it is a Hensel system for $\mathbf{x}_m$ of degree at most d, thus contradicting our assumption. □

7.2 Tight Closure

For the remainder of this chapter, we specify the previous theory to the case that K is an algebraically closed Lefschetz field, given as the ultraproduct of the algebraically closed fields K_p of characteristic p.

7.2.1 Lefschetz Hulls

In particular, $L(R)$ is a Lefschetz ring, given as the ultraproduct of the power series rings $R_{p(w)} := K_{p(w)}[[\xi]]$, where $p(w)$ is equal to the underlying characteristic.

The ultraproduct $\mathbf{F}_\natural$ of the $\mathbf{F}_{p(w)}$ acts on $L(R)$, making it a difference ring. This immediately extends to homomorphic images:

Corollary 7.2.1. *The assignment $R/I \mapsto L(R/I) := L(R)/IL(R)$ constitutes a difference hull on the category of all homomorphic images of R.* □

Note that any complete Noetherian local ring with residue field K and embedding dimension at most n is a homomorphic image of R by Cohen's Structure Theorems. However, a local homomorphism between two such rings is not necessarily an epimorphism, so that the previous statement is much weaker than obtaining a difference hull on the category of complete Noetherian local rings with residue field K. We will address this issue further in §7.3 below. For now, we will extend our construction to include any Noetherian local ring S of equal characteristic zero. Our definition though will depend on some choices. We start by taking K sufficiently large so that it contains the residue field k of S as a subfield. Let $\widehat{S_K}$ be the complete scalar extension of S along K as given by Theorem 3.2.13. By Cohen's Structure Theorems, we may write $\widehat{S_K}$ as $R/\mathfrak{a}$ for some ideal $\mathfrak{a} \subseteq R = K[[\xi]]$ (assuming that the number n of indeterminates ξ is at least the embedding dimension of S). We now define $L(S) := L(\widehat{S_K}) = L(R)/\mathfrak{a}L(R)$. Since $S \to \widehat{S_K}$ is faithfully flat by Theorem 3.2.13, this assignment is a difference hull on the category of all homomorphic images of S by Corollary 7.2.1, called a *Lefschetz hull* of S (for another type of Lefschetz hull, see §7.4.1 below).

7.2.2 *Tight Closure*

The *tight closure* of an ideal $I \subseteq S$ is by definition the difference closure of I with respect to a (choice of) Lefschetz hull, and is again denoted $\mathrm{cl}_S(I)$ or simply $\mathrm{cl}(I)$ (although technically speaking, we should also include the Lefschetz hull in the notation). In other words, $z \in \mathrm{cl}(I)$ if and only if there exists a multiplier $c \in S$ such that

$$c\mathbf{F}_\natural^e(z) \in \mathbf{F}_\natural^e(I)L(S) \tag{7.2}$$

for all $e \gg 0$ (again we suppress the embedding $\eta_S\colon S \to L(S)$ in our notation).

By our axiomatic treatment of difference closure, we therefore immediately obtain the five key properties of Theorem 6.2.3 for the category of all residue rings of S. However, this category is a severely limited category, and the only two properties that do not rely on any functoriality with respect to general homomorphisms are:

7.2.2 *Any regular local ring of equal characteristic zero is F-regular, and any complete local domain S (or more generally, any equidimensional homomorphic image of a Cohen-Macaulay local ring) of equal characteristic zero admits Colon Capturing: for any system of parameters $(x_1,\dots,x_d)$ in S, we have $((x_1,\dots,x_i)S : x_{i+1}) \subseteq \mathrm{cl}((x_1,\dots,x_i)S)$ for all i.*

Inspecting the proofs of Theorems 6.2.5 and 6.2.6, we see that these carry over immediately to the present case, and hence we can now state:

Theorem 7.2.3 (Briançon-Skoda—local case). *Let S be a Noetherian local ring of equal characteristic zero, and let $I \subseteq S$ be an ideal generated by n elements. If I has positive height, then we have for all $l \geq 1$ an inclusion*

$$\overline{I^{n+l-1}} \subseteq \mathrm{cl}(I^l).$$

In particular, if S is moreover regular, then the integral closure of I^{n+l-1} lies inside I^l for all $l \geq 1$. □

In particular, we also proved the original version of the Briançon-Skoda theorem (Theorem 5.5.1).

Theorem 7.2.4. *Let S be a regular local ring of equal characteristic zero, and let $\mathfrak{a} \subseteq S$ be the intersection of finitely many prime ideals of height at most h. Then for all n, we have an inclusion $\mathfrak{a}^{(hn)} \subseteq \mathfrak{a}^n$.* □

7.3 Functoriality

Unfortunately, the last of our three applications, the Hochster-Roberts Theorem, requires functoriality beyond the one provided by Corollary 7.2.1. To this end, we briefly discuss how to extend some form of functoriality to the category of all Noetherian local rings of equal characteristic zero. As we will see shortly, functoriality requires a 'filtered' version of Theorem 7.1.1. To show that this version holds for power series rings over K, we require the following more sophisticated Artin Approximation result due to Rotthaus (its proof is still relatively simple in comparison with those of the general Artin Conjecture needed in the Hochster-Huneke version). As before, $R := K[[\xi]]$, and ζ is another finite tuple of indeterminates.

Theorem 7.3.1 ([82]). *The Henselization $R[\zeta]^{\sim}$ of the localization of $R[\zeta]$ at the maximal ideal generated by all the indeterminates admits the Artin Approximation property.*

We extend the terminology used in §4.1: given an ultra-ring $C_\natural$, realized as the ultraproduct of rings C_w, then by an *ultra-$C_\natural$-algebra*, we mean an ultraproduct $D_\natural$ of C_w-algebras D_w. If almost each C_w is local and D_w is a local C_w-algebra (meaning that the canonical homomorphism $C_w \to D_w$ is a local homomorphism), then we call $D_\natural$ an *ultra-local $C_\natural$-algebra*. Similarly, a *morphism of ultra-(local) $C_\natural$-algebras* is by definition an ultraproduct of (local) C_w-algebra homomorphisms.

For our purposes, we only will need the following quasi-functorial version of the Lefschetz hull.

Proposition 7.3.2. *Let S be a Noetherian local ring of equal characteristic zero with a given choice of Lefschetz hull $\eta_S\colon S \to \mathsf{L}(S)$. For every Noetherian local S-algebra T whose residue field embeds in K, there exists a choice of Lefschetz hull $\eta_T : T \to \mathsf{L}(T)$ on T, having in addition the structure of an ultra-local $\mathsf{L}(S)$-algebra.*

Proof. By taking an isomorphic copy of the S-algebra T, we may assume that the induced homomorphism on the residue fields is an inclusion of subfields of K. In that case, one easily checks that the complete scalar extension $\widehat{S_K} \to \widehat{T_K}$ of the canonical homomorphism $S \to T$ is in fact a K-algebra homomorphism. Taking n sufficiently large, $\widehat{S_K}$ and $\widehat{T_K}$ are homomorphic images of $R = K[[\xi]]$, and the K-algebra homomorphism $\widehat{S_K} \to \widehat{T_K}$ lifts to a K-algebra endomorphism φ of R. So without loss of generality, we may assume $S = T = R$. Let $\mathbf{x} := (x_1,\dots,x_n)$ be the image of ξ under φ, so that in particular, each x_i is a power series without constant term. Note that the K-algebra local homomorphism φ is completely determined by this tuple, namely $\varphi(f) = f(\mathbf{x})$ for any $f \in R$. Let $R' := R[[\zeta]]$, where ζ is another n-tuple of indeterminates, and put $R'_p := R_p[[\zeta]]$. Note that φ is isomorphic to the composition

$$R \subseteq R' \twoheadrightarrow R'/J \cong K[[\zeta]] \cong R,$$

where the first map is just inclusion, and where J is the ideal generated by all $\xi_i - x_i$. Since Lefschetz hulls commute with homomorphic images, we reduced the problem to finding a Lefschetz hull $\eta_{R'}\colon R' \to \mathsf{L}(R')$, together with a morphism $\mathsf{L}(R) \to \mathsf{L}(R')$ of ultra-local K-algebras extending the inclusion $R \subseteq R'$.

By Theorem 7.1.6, there exists some ultrapower of $\mathsf{U}(R)$ which is faithfully flat over R. Since we will have to further modify this ultrapower, we denote it by $Z_\natural$. Recall that it is in fact an ultraproduct of the R_p by 7.1.7. Let $Z'_\natural$ denote the corresponding ultraproduct of the $R'_{p(w)}$. In particular, we get a morphism $Z_\natural \to Z'_\natural$ of ultra-local K-algebras. Moreover, $Z'_\natural$ is an R-algebra via the composition $R \to Z_\natural \to Z'_\natural$, whence also an $R[\zeta]$-algebra, since in $Z'_\natural$, the indeterminates ζ remain algebraically independent over R. We will obtain $\mathsf{L}(R')$ as a (further) ultrapower of $Z'_\natural$ from an application of Theorem 7.1.1, which at the same time then also provides the desired R-algebra homomorphism $R' \to \mathsf{L}(R')$. So, given a polynomial system of equations $\mathscr{L}$ with coefficients in R having a solution in R', we need to find a solution in $Z'_\natural$. By Theorem 7.3.1, we can find a solution in $R[\zeta]^\sim$, since R' is the completion of the latter ring. By the universal property of Henselizations (see Theorem A.2.3), we get a local $R[\zeta]$-algebra homomorphism $R[\zeta]^\sim \to Z'_\natural$, and hence via this homomorphism, we get a solution for $\mathscr{L}$ in $Z'_\natural$, as we wanted to show. Let $R' \to \mathsf{L}(R')$ be the homomorphism given by Theorem 7.1.1, which is then faithfully flat by (the proof of) Theorem 7.1.6. Let $\mathsf{L}(R)$ be the corresponding ultrapower of $Z_\natural$, so that $R \to \mathsf{L}(R)$ too is faithfully flat. Moreover, the homomorphism $Z_\natural \to Z'_\natural$ then yields, after taking ultrapowers, a morphism of ultra-local K-algebras $\mathsf{L}(R) \to \mathsf{L}(R')$. We leave it to the reader to verify that it extends the inclusion $R \subseteq R'$, and admits all the desired properties. □

In [6], a much stronger form of functoriality is obtained, by making the ad hoc argument in the previous proof more canonical. In particular, we construct $\eta_R \colon R \to L(R)$ in such way that it maps each of the subrings $K[[\xi_1,\dots,\xi_i]]$ to the corresponding subring of $L(R)$ of all elements depending only on the indeterminates $\xi_1,\dots,\xi_i$, that is to say, the ultraproduct of the $K_{p(w)}[[\xi_1,\dots,\xi_i]]$ (our treatment of the inclusion $R \subseteq R'$ in the previous proof is a special instance of this). However, this is not a trivial matter, and caution has to be exercised as to how much we can preserve. For instance, in [6, §4.33], we show that 'unnested' subrings cannot be preserved, that is to say, there cannot exist an η_R which maps any subring $K[[\xi_{i_1},\dots,\xi_{i_s}]]$ into the corresponding subring of all elements depending only on the indeterminates $\xi_{i_1},\dots,\xi_{i_s}$ (the concrete counterexample requires $n=6$, and it would be of interest to get already a counterexample for $n=2$).

Proposition 7.3.2 is sufficiently strong to get the following form of weak persistence: if $S \to T$ is a local homomorphism of Noetherian local domains of equal characteristic zero, then we can define tight closure operations $\mathrm{cl}_S(\cdot)$ and $\mathrm{cl}_T(\cdot)$ on S and T respectively, such that $\mathrm{cl}_S(I) \subseteq \mathrm{cl}_T(IT)$ for all $I \subseteq S$ (see the argument in the next proof).

Theorem 7.3.3 (Hochster-Roberts). *If $S \to T$ is a cyclically pure homomorphism of Noetherian local rings of equal characteristic, and if T is regular, then S is Cohen-Macaulay.*

Proof. We already dealt with the positive characteristic case, so assume the characteristic is zero. Since the completion of a cyclically pure homomorphism is again cyclically pure by Lemma 5.5.5, we may assume S and T are complete, and by Proposition 7.3.2, we may assume that $L(T)$ is an ultra-$L(S)$-algebra (by taking K sufficiently large). Let $(x_1,\dots,x_d)$ be a system of parameters in S, and assume $zx_{i+1} \in I := (x_1,\dots,x_i)S$. By Colon Capturing (7.2.2), we get $z \in \mathrm{cl}(I)$, so that (7.2) holds for all $e \gg 0$. However, we may now view these relations also in $L(T)$ via the S-algebra homomorphism $L(S) \to L(T)$, showing that $z \in \mathrm{cl}(IT)$. By 7.2.2 therefore, $z \in IT$ whence by cyclic purity, $z \in I$, as we wanted to show. □

We can now also tie up another loose end, the last of our five key properties, namely the connection with integral closure (recall that (6.2.3.v) is not really an issue in characteristic zero; see the discussion after Theorem 6.4.1):

Theorem 7.3.4. *The tight closure of an ideal lies inside its integral closure.*

Proof. Let $I \subseteq S$ be an ideal in a Noetherian local ring $(S,\mathfrak{n})$ of equal characteristic zero, and let $z \in \mathrm{cl}(I)$. Since the integral closure of I is equal to the intersection of the integral closures of all $I+\mathfrak{n}^k$, we may reduce to the case that I is $\mathfrak{n}$-primary. In view of (5.4.1.iv), we need to show that $z \in IV$, for every homomorphism $S \to V$ into a discrete valuation ring V with kernel a minimal prime ideal of S. There is nothing to show if $\mathfrak{n}V = V$ whence $IV = V$, so that we may assume $S \to V$ is local. Moreover, by a similar cardinality argument as in Proposition 6.2.2, we may replace V by a sub-discrete valuation ring whose residue field embeds in K.

By Proposition 7.3.2, there exists a Lefschetz hull $L(V)$ on V which is an ultra-local $L(S)$-algebra. In particular, z lies in the tight closure of IV with respect to this choice of Lefschetz hull, and so we are done by an application of 7.2.2 to the regular ring V. □

7.4 Big Cohen-Macaulay Algebras

As in the affine case, we can also associate to each Noetherian local domain of equal characteristic zero a balanced big Cohen-Macaulay algebra. However, to avoid some complications caused by the fact that the completion of a domain need not be a domain, I will only discuss this in case S is a complete Noetherian local domain with residue field K (for the general case, see [6, §7]). But even in this case, the Lefschetz hull defined above does not have the desired properties: we do not know whether the approximations of S are again domains. So we discuss first a different construction of a Lefschetz hull.

7.4.1 *Relative Hulls*

Fix some Noetherian local ring $(S,\mathfrak{n})$ with residue field k contained in K, and let $L(S)$ be a Lefschetz hull for S with approximations S_w. We want to construct a Lefschetz hull on the category of S-affine algebras, extending the Lefschetz hull defined on §7.2.1. Let us first consider the polynomial ring $B := S[\zeta]$ in finitely many indeterminates ζ. Let $L_S(B)$ be defined as the ultraproduct of the $B_w := S_w[\zeta]$, so that $L_S(B)$ is an ultra-$L(S)$-algebra. The homomorphism $S \to L_S(B)$ extends naturally to a homomorphism $B \to L_S(B)$, since the ζ remain algebraically independent over $L(S)$. We call $L_S(B)$ the *relative Lefschetz hull of B* (with respect to the Lefschetz hull $S \to L(S)$). Similarly, if $C = B/I$ is an arbitrary S-affine algebra, then we define $L_S(C)$ as the residue ring $L_S(B)/IL_S(B)$, and we call this the *relative Lefschetz hull* of C (with respect to the choice of Lefschetz hull $L(S)$). By base change the homomorphism $B \to L_S(B)$ induces a homomorphism $C \to L_S(C)$. Moreover, $L_S(C)$ is an ultra-$L(S)$-algebra, since I is finitely generated. More precisely, $IL_S(B)$ is the ultraproduct of ideals $I_w \subseteq B_w$, and $L_S(C)$ is equal to the ultraproduct of the B_w/I_w, called therefore *relative approximations* of C.

This new hull agrees with the old one on the base ring: $L(S) = L_S(S)$. It is instructive to calculate $L_S(B)/\mathfrak{n}L_S(B) = L_S(B/\mathfrak{n}B) = L_S(k[\zeta])$, where k is the residue field of S. Since $\mathfrak{n}\widehat{S_K}$ is the maximal ideal in $\widehat{S_K}$, we get $L(S)/\mathfrak{n}L(S) = L(k) = L(K)$, and this field is just an ultrapower of $U(K)$. Note that, by construction, K is equal to its own ultra-hull $U(K)$. Hence the relative approximations of $L_S(B/\mathfrak{n}B)$ are equal to $B_w/\mathfrak{n}_w B_w = K_{p(w)}[\zeta]$, showing that $L_S(B/\mathfrak{n}B)$ is the ultrapower of $U(K[\zeta])$. Next, suppose T is a local S-affine algebra, say of the form $B_\mathfrak{p}/IB_\mathfrak{p}$, with $\mathfrak{p} \subseteq B$ a prime ideal containing I. Moreover, since we assume that $S \to T$ is local,

$\mathfrak{n}B \subseteq \mathfrak{p}$. In order to define the relative Lefschetz hull $L_S(T)$ of T as the localization of $L_S(B/IB)$ with respect to $\mathfrak{p}L_S(B/IB)$, we need:

7.4.1 *If $\mathfrak{p}$ is a prime ideal in B containing $\mathfrak{n}B$, then $\mathfrak{p}L_S(B)$ is prime.*

We need to show that $L_S(B)/\mathfrak{p}L_S(B) = L_S(B/\mathfrak{p})$ is a domain. Since $B/\mathfrak{p}$ is a homomorphic image of $B/\mathfrak{n}B$, it suffices to show that $\mathfrak{p}$ extends to a prime ideal in $L_S(B/\mathfrak{n}B)$. By Theorem 4.3.4, the extension of $\mathfrak{p}$ to $U(K[\zeta])$ remains prime. Since $L_S(B/\mathfrak{n}B)$ is an ultrapower of $U(K[\zeta])$, the extension of $\mathfrak{p}$ to the former is again prime by Łoś' Theorem. □

To prove that these are well-defined objects, that is to say, independent of the choice of presentation $C = B/I$ (or its localization), one easily proves a similar universal property as for ultra-hull:

7.4.2 *Any S-algebra homomorphism $C \to D_\natural$ with $D_\natural$ an ultra-$L(S)$-algebra, extends uniquely to a morphism $L_S(C) \to D_\natural$ of ultra-$L(S)$-algebras. Similarly, any local $L(S)$-algebra homomorphism $T \to D_\natural$ with $D_\natural$ an ultra-local $L(S)$-algebra, extends uniquely to a morphism $L_S(T) \to D_\natural$ of ultra-local $L(S)$-algebras.* □

Proposition 7.4.3. *On the category of S-affine algebras, $L_S(\cdot)$ is a difference hull.*

Proof. Let T be a local S-affine algebra (we leave the global case as an exercise for the reader). Clearly, the ultra-Frobenius $\mathbf{F}_\natural$ acts on each $L_S(T)$, making the latter into a difference ring. So remains to show that the canonical map $T \to L_S(T)$ is faithfully flat. By Cohen's Structure Theorems, $\widehat{S_K}$ is a homomorphic image of $R := K[[\xi]]$. A moment's reflection shows that $L_S(T) = L_R(\widehat{T_K})$, so that by an application of Theorem 3.2.13, we may reduce to the case that $S = R$. By another application of Cohen's structure theorem, T is a homomorphic image of a localization of $R[\zeta]$, and hence without loss of generality, we may assume that T is moreover regular. Flatness of $T \to L_R(T)$ then follows from the Cohen-Macaulay criterion of flatness in the same way as in the proof of Theorem 4.2.2. □

7.4.2 Big Cohen-Macaulay Algebras

For the remainder of this section, S is a complete Noetherian local domain with residue field K. By Cohen's Structure Theorems, we can find a finite extension $R \subseteq S$ (for an appropriate choice of n and $R := K[[\xi]]$ as before). The Lefschetz hull we will use for S to construct a balanced big Cohen-Macaulay algebra is the relative hull $L_R(S)$ (with respect to a fixed Lefschetz hull for R). Let S_w be the relative approximations of S with respect to this choice of Lefschetz hull, that is to say, S_w are the complete local $K_{p(w)}$-algebras whose ultraproduct is $L_R(S)$. By the above discussion, $L_R(S)$ is a domain, whence so are almost all S_w. Let $B(S)$

be the ultraproduct of the S_w^+, so that $B(S)$ is in particular an ultra-$L_R(S)$-algebra whence an S-algebra. It is now straightforward to prove:

Theorem 7.4.4. *For each complete Noetherian local domain S with residue field K, the S-algebra $B(S)$ is a balanced big Cohen-Macaulay algebra.* □

Theorem 7.4.5 (Monomial Conjecture). *The Monomial Conjecture holds for any Noetherian local ring S of equal characteristic, that is to say, any system of parameters is monomial.*

Proof. I will only explain the equal characteristic zero case; the positive characteristic case is analogous, using instead Theorem 6.4.1. Towards a contradiction, suppose $(x_1,\dots,x_d)$ is a counterexample, that is to say, a system of parameters which fails (6.9) for some k. After taking a complete scalar extension (which preserves the system of parameters), we may assume that S is complete with residue field K. After killing a prime ideal of maximal dimension (which again preserves the system of parameters), we then may assume moreover that S is a domain. The counterexample then also holds in $B(S)$, contradicting that $(x_1,\dots,x_d)$ is a $B(S)$-regular sequence by Theorem 7.4.4. □

As before, we can also define the *B-closure* of an ideal $I \subseteq S$ by the rule $\mathrm{cl}^B(I) := IB(S) \cap S$ and prove that it satisfies the five key properties.

Chapter 8
Cataproducts

One of the main obstacles in the study of ultra-rings is the absence of the Noetherian property, forcing us to modify several definitions from Commutative Algebra. This route is further pursued in [101]. However, there is another way to circumvent these problems: the cataproduct $A_\sharp$, the first of our chromatic products. We will mainly treat the local case, which turns out to yield always a Noetherian local ring. The idea is simply to take the separated quotient of the ultraproduct with respect to the maximal adic topology. The saturatedness property of ultraproducts—well-known to model-theorists—implies that the cataproduct is in fact a complete local ring. Obviously, we do no longer have the full transfer strength of Łoś' Theorem, although we shall show that many algebraic properties still persist, under some mild conditions. We conclude with some applications to uniform bounds. Whereas the various bounds in Chapter 4 were expressed in terms of polynomial degree, we will introduce a different notion of degree here,[1] in terms of which we will give the bounds. Conversely, we can characterize many local properties through the existence of such bounds.

8.1 Cataproducts

Recall from 2.4.9 that the ultraproduct of local rings of bounded embedding dimension is again a local ring of finite embedding dimension. In this chapter, we will be mainly concerned with the following subclass.

Definition 8.1.1 (Ultra-Noetherian Ring). We call a local ring $R_\natural$ *ultra-Noetherian* if it is the ultraproduct of Noetherian local rings of bounded embedding dimension, that is to say, of Noetherian local rings R_w such that the embedding dimension of R_w is at most e, for some e independent of w.

The Noetherian local rings R_w will be called *approximations* of $R_\natural$ (note the more liberal use of this term than in the previous chapters, which,

[1] In spite of the nomenclature, and unlike *proto-grade*, to be introduced in the next chapter, this new degree is not a generalization of polynomial degree.

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however, should not cause any confusion). It is important to keep in mind that approximations are not uniquely determined by $R_\natural$. A good example of this phenomenon is exhibited by Corollary 10.1.2 below.

Recall from Definition 4.3.6 that for a local ring $(S,\mathfrak{n})$ of finite embedding dimension, its geometric dimension $\operatorname{geodim}(S)$ is the least number d of elements $x_1,\dots,x_d \in \mathfrak{n}$ such that $S/(x_1,\dots,x_d)S$ is Artinian, that is to say, such that $(x_1,\dots,x_d)S$ is $\mathfrak{n}$-primary. Any tuple $(x_1,\dots,x_d)$ with this property is then called a *system of parameters* of R.[2] Any element of R which belongs to some system of parameters will be called a *parameter*. We immediately get:

8.1.2 *The geometric dimension of a local ring is at most its embedding dimension, whence in particular is finite for any ultra-Noetherian local ring. The geometric dimension of an ultra-Noetherian local ring is larger than or equal to the (geometric) dimension of its Noetherian approximations.*

We only need to verify the second assertion. Let $\mathbf{x}$ be a system of parameters, say of length d, in an ultra-Noetherian local ring $R_\natural$, and choose d-tuples $\mathbf{x}_w$ in the approximations R_w whose ultraproduct is equal to $\mathbf{x}$. By assumption, $R_\natural/\mathbf{x}R_\natural$ is Artinian, say of length l. By 2.1.6, the $R_w/\mathbf{x}_wR_w$ are approximations of $R_\natural/\mathbf{x}R_\natural$, and hence almost all are Artinian by Proposition 2.4.17. Hence almost all R_w have (geometric) dimension at most d. □

To see that this latter inequality can be strict, let $R_n := K[[\xi]]/\xi^nK[\xi]$ with ξ a single indeterminate over the field K; the ultraproduct $R_\natural$ of these Artinian rings has geometric dimension at least one since ξ is a parameter (and, in fact, $\operatorname{geodim}(R_\natural) = 1$; see 8.1.3 below). To study this phenomenon as well as further properties of ultra-Noetherian local rings, we first introduce a new kind of product:

8.1.1 Cataproducts

In 2.4.16 we saw that most ultra-Noetherian rings are not Noetherian (in model-theoretic terms this means that the class of Noetherian local rings of fixed embedding dimension is not first-order definable). However, there is a Noetherian local ring closely associated to any ultra-Noetherian local ring. Fix an ultra-Noetherian local ring

$$R_\natural := \operatorname*{ulim}_{w\to\infty} R_w,$$

and define the *cataproduct* of the R_w as the separated quotient of $R_\natural$, that is to say,

$$R_\sharp := R_\natural/\mathfrak{I}_{R_\natural},$$

[2] In [91, 97, 101] such a tuple was called *generic*.

where $\mathfrak{I}_{R_\natural}$ is the ideal of infinitesimals of $R_\natural$. If all R_w are equal to a fixed Noetherian local ring $(R,\mathfrak{m})$, then we call $R_\sharp$ the *catapower* of R. In this case, the natural, diagonal embedding $R \to R_\natural$ induces a natural homomorphism $R \to R_\sharp$. Since $\mathfrak{m}R_\natural$ is the maximal ideal of $R_\natural$, likewise, $\mathfrak{m}R_\sharp$ is the maximal ideal of $R_\sharp$. The relationship between the rings R_w and their cataproduct $R_\sharp$ is much less strong than in the ultraproduct case, as the following example illustrates.

8.1.3 *The catapower of a Noetherian local ring* $(R,\mathfrak{m})$ *is isomorphic to the cataproduct of the Artinian local rings* $R/\mathfrak{m}^n$.

Indeed, if $R_\natural$ and $S_\natural$ denote the ultrapower of R and the ultraproduct of the $R/\mathfrak{m}^n$ respectively, then, by 2.1.7, we get a surjective homomorphism $R_\natural \to S_\natural$. However, any element in the kernel of this homomorphism is an infinitesimal, so that the induced homomorphism $R_\sharp \to S_\sharp$ is an isomorphism. □

Nonetheless, as before, we will still refer to the R_w as *approximations* of $R_\sharp$, and given an element $x \in R_\sharp$, we call any choice of elements $x_w \in R_w$ whose ultraproduct is a lifting of x to $R_\natural$, an *approximation* of x.

Theorem 8.1.4. *The cataproduct of Noetherian local rings of bounded embedding dimension is complete and Noetherian.*

Proof. In almost all our applications,[3] the ultrafilter lives on a countable index set W, but nowhere did we exclude larger cardinalities. For simplicity, however, I will assume countability, and treat the general case in a separate remark below. Hence, we may assume $W = \mathbb{N}$. Let $(R_\natural,\mathfrak{m})$ be the ultraproduct of Noetherian local rings R_w of embedding dimension at most e. It follows that $R_\natural$ too has embedding dimension at most e. Let us first show that $R_\natural$ is quasi-complete (note that it is not Hausdorff in general, because $\mathfrak{I}_{R_\natural} \neq 0$). To this end, we have to show that any Cauchy sequence $\mathbf{a}$ has a limit in $R_\natural$. Without loss of generality, we may assume that $\mathbf{a}(n) \equiv \mathbf{a}(n+1) \mod \mathfrak{m}^n R_\natural$. Choose approximations $\mathbf{a}_w(n) \in R_w$ such that

$$\mathbf{a}(n) = \operatorname*{ulim}_{w\to\infty} \mathbf{a}_w(n)$$

for each $n \in \mathbb{N}$. By Łoś' Theorem, we have for a fixed n that

$$\mathbf{a}_w(n) \equiv \mathbf{a}_w(n+1) \mod \mathfrak{m}_w^n \tag{8.1}$$

for almost all w, say, for all w in D_n. I claim that we can modify the $\mathbf{a}_w(n)$ in such way that (8.1) holds for all n and all w. More precisely, for each n there exists an approximation $\tilde{\mathbf{a}}_w(n)$ of $\mathbf{a}(n)$, such that

$$\tilde{\mathbf{a}}_w(n) \equiv \tilde{\mathbf{a}}_w(n+1) \mod \mathfrak{m}_w^n \tag{8.2}$$

for all n and w. We will construct the $\tilde{\mathbf{a}}_w(n)$ recursively from the $\mathbf{a}_w(n)$. When $n = 0$, no modification is required (since by assumption $\mathfrak{m}_w^0 = R_w$), and hence we

[3] A notable exception is the construction of a Lefschetz hull given in Theorem 7.1.6.

set $\tilde{\mathbf{a}}_w(0) := \mathbf{a}_w(0)$ and $\tilde{\mathbf{a}}_w(1) := \mathbf{a}_w(1)$. So assume we have defined already the $\tilde{\mathbf{a}}_w(j)$ for $j \leq n$ such that (8.2) holds for all w. Now, for those w for which (8.1) fails for some $j \leq n$, that is to say, for $w \notin (D_0 \cup \cdots \cup D_n)$, let $\tilde{\mathbf{a}}_w(n+1)$ be equal to $\tilde{\mathbf{a}}_w(n)$; for the remaining w, that is to say, for almost all w, we make no changes: $\tilde{\mathbf{a}}_w(n+1) := \mathbf{a}_w(n+1)$. It is now easily seen that (8.2) holds for all w, and $\tilde{\mathbf{a}}_w(n)$ is another approximation of $\mathbf{a}(n)$, for all n, thus establishing our claim.

So we may assume (8.1) holds for all n and w. Define $b := \operatorname{ulim} \mathbf{a}_w(w)$. Since $\mathbf{a}_w(w) \equiv \mathbf{a}_w(n) \mod \mathfrak{m}_w^n$ for all $w \geq n$, Łoś' Theorem yields $b \equiv \mathbf{a}(n) \mod \mathfrak{m}^n R_\natural$, showing that b is a limit of $\mathbf{a}$.

Since the cataproduct $R_\sharp$ of the R_w is a homomorphic image of $R_\natural$, it is again quasi-complete. By construction, $R_\sharp$ is Hausdorff and therefore even complete, that is to say, every Cauchy sequence has a unique limit. Since $R_\sharp$ has finite embedding dimension, it is therefore Noetherian by [69, Theorem 29.4]. □

Remark 8.1.5. In order for the above argument to work for arbitrary index sets W, we need to make one additional assumption on the ultrafilter $\mathfrak{W}$: it needs to be *countably incomplete*, meaning that there exists a function $f\colon W \to \mathbb{N}$ such that for each n, almost all $f(w)$ are greater than or equal to n. Of course, if $W = \mathbb{N}$ such a function exists, namely the identity will already work. Countably incomplete ultrafilters exist on any infinite set. In fact, it is a strong set-theoretic condition to assume that not every ultrafilter is countably incomplete! Now, the only place where we need this assumption is to build the limit element b. This time we should take it to be the ultraproduct of the $\mathbf{a}_w(f(w))$. The reader can verify that this one modification makes the proof work for any index set.

Example 8.1.6. Whereas ultraproducts are impossible to compute, due to their non-constructive nature, cataproducts are much more accessible. Proposition 7.1.8 is just one instance, and below, we will present more evidence to corroborate this claim. Here is another, easy example: let K_w be fields with ultraproduct (whence cataproduct) equal to $K_\natural$. If R_w is the localization of $K_w[\xi]$ at the ideal generated by the variables $\xi = (\xi_1, \ldots, \xi_n)$, then their cataproduct $R_\sharp$ is equal to $K_\natural[[\xi]]$. Indeed, using 8.1.2, it is not hard to see that $R_\sharp$ has dimension n and that the variables generate its maximal ideal. Hence $R_\sharp$ is regular, and since it is complete by the previous theorem, and has residue field $K_\natural$, the claim follows from Cohen's Structure Theorems for complete regular local rings.

Proposition 8.1.7. *Let $R_\natural$ be an ultra-Noetherian local ring and let $R_\sharp$ be the corresponding cataproduct, that is to say, its separated quotient. For any ideal $I \subseteq R_\natural$, its $\mathfrak{m}$-adic closure is equal to $I + \mathfrak{I}_{R_\natural}$. In particular, the separated quotient of $R_\natural/I$ is $R_\sharp/IR_\sharp$.*

Proof. It suffices to show the first assertion. Clearly, $I + \mathfrak{I}_{R_\natural}$ is contained in the $\mathfrak{m}$-adic closure of I. To prove the other inclusion, assume a lies in the $\mathfrak{m}$-adic closure of I. Hence its image in $R_\sharp$ lies in the $\mathfrak{m}$-adic closure of $IR_\sharp$, and this is just $IR_\sharp$ by Theorem 2.4.14, since $R_\sharp$ is Noetherian by Theorem 8.1.4. Therefore, a lies in $IR_\sharp \cap R_\natural = I + \mathfrak{I}_{R_\natural}$. □

In particular, if R_w are approximations of $R_\natural$, and $I_w \subseteq R_w$ are ideals with ultraproduct $I \subseteq R_\natural$, then the cataproduct of the R_w/I_w is equal to $R_\sharp/IR_\sharp$.

8.1.2 *Dimension Theory for Cataproducts*

From a model-theoretic point of view, Łoś' Theorem determines which properties are preserved in ultraproducts: the first-order ones. Since cataproducts are residue rings, they, therefore, inherit any positive first-order property from their approximations. However, we do not want to derive properties of the cataproduct via a syntactical analysis, but instead use an algebraic approach. The first issue to address is the way dimension behaves under cataproducts. We already mentioned that the geometric dimension of an ultra-Noetherian ring can exceed that of its components (see 8.1.2 and its discussion). The same phenomenon occurs for cataproducts because we have:

8.1.8 *For an ultra-Noetherian local ring $(R_\natural, \mathfrak{m})$ its geometric dimension is equal to the dimension of its separated quotient $R_\sharp$, that is to say, ultraproduct and cataproduct have the same geometric dimension.*

Let $\mathbf{x} := (x_1, \dots, x_d)$ be a system of parameters in $R_\natural$ (recall that this means that $(x_1, \dots, x_d)R_\natural$ is an $\mathfrak{m}$-primary ideal, with d the geometric dimension of $R_\natural$). So $S_\natural := R_\natural/\mathbf{x}R_\natural$ is an Artinian local ring, whence, by Krull's Intersection Theorem (Theorem 2.4.14), must be equal to its separated quotient $S_\sharp$. Proposition 8.1.7 yields $S_\sharp = R_\sharp/\mathbf{x}R_\sharp$, showing that $R_\sharp$ has geometric dimension at most d. Since $R_\sharp$ is Noetherian by Theorem 8.1.4, it has therefore dimension at most d. Moreover, we may reverse the argument, for if $S_\sharp$ is Artinian, then necessarily $S_\natural = S_\sharp$. □

To investigate when the dimension of a cataproduct is equal to the dimension of almost all of its approximations, we need to introduce a new invariant.

Definition 8.1.9 (Parameter Degree). Given a local ring $(R, \mathfrak{m})$ of finite embedding dimension, its *parameter degree*, denoted $\mathrm{pardeg}(R)$, is by definition the least possible length of the residue rings $R/\mathbf{x}R$, where $\mathbf{x}$ runs over all systems of parameters.

Note that by definition of geometric dimension, the parameter degree of R is always finite. Closely related to this invariant, is the *degree*[4] $\deg_R(x)$ of an element $x \in R$, defined as follows: if x is a unit, then we set $\deg_R(x)$ equal to zero, and if x is not a parameter, then we set $\deg_R(x)$ equal to ∞; in the remaining case, we let $\deg_R(x)$ be the parameter degree of R/xR. We leave it as an exercise to show that:

[4] Hopefully, this will not cause confusion with the notion *degree of a polynomial*, as the present notion is always used in a local context whereas the latter is only used in an affine context.

8.1.10 *Let R be a d-dimensional Noetherian local ring, or more generally, a local ring of geometric dimension d, and let $x \in R$. Then the degree of x is equal to the minimal length of any residue ring of the form $R/(xR+I)$, where I runs over all ideals generated by $d-1$ non-units.*

In [101, Proposition 2.2 and Theorem 3.4], we prove the following generalization of Theorem 8.1.4:

8.1.11 *The completion of a local ring R of finite embedding dimension is Noetherian, and has dimension equal to the geometric dimension of R; moreover, both rings have the same Hilbert polynomial.* □

We define the *multiplicity* of R to be the leading coefficient of its Hilbert polynomial times $d!$ (this coincides with the classical definition in the Noetherian case). The multiplicity of R is always at most its parameter degree, and provided R is Noetherian with infinite residue field, both are equal if and only if R is Cohen-Macaulay (see [96, Lemma 3.3] for the Noetherian case, and [101, Lemma 6.10] for a generalization).

Theorem 8.1.12. *Let R_w be d-dimensional Noetherian local rings of embedding dimension at most e. Their cataproduct $R_\natural$ has dimension d if and only if almost all R_w have bounded parameter degree (that is to say, $\operatorname{pardeg}(R_w) \leq r$, for some r and for almost all w).*

Proof. Assume that almost all R_w have parameter degree at most r, so that their exists a d-tuple $\mathbf{x}_w$ in R_w such that $S_w := R_w/\mathbf{x}_w R_w$ has length at most r. Hence the cataproduct $S_\natural$ has length at most r by Proposition 2.4.17. By Proposition 8.1.7, the cataproduct $S_\natural$ is isomorphic to $R_\natural/\mathbf{x}R_\natural$, where $\mathbf{x}$ is the ultraproduct of the $\mathbf{x}_w$. Hence $R_\natural$, being Noetherian by Theorem 8.1.4, has dimension at most d, whence equal to d by 8.1.2 and 8.1.8.

Conversely, suppose $R_\natural$ has dimension d. Let $\mathbf{x}$ be a system of parameters of $R_\natural$ with approximation $\mathbf{x}_w$, and let r be the length of $R_\natural/\mathbf{x}R_\natural$. By Proposition 2.4.17, almost all $R_w/\mathbf{x}_w R_w$ have length at most r. It follows that almost each $\mathbf{x}_w$ is a system of parameters, and hence that R_w has parameter degree at most r. □

8.1.3 Catapowers

Let us apply the previous results to catapowers. In the next result, the first statement is immediate from Theorem 8.1.4 and Proposition 8.1.7; the second follows immediately from Theorem 8.1.12.

Corollary 8.1.13. *Let R be a Noetherian local ring with catapower $R_\natural$. For any ideal $I \subseteq R$, the catapower of R/I is $R_\natural/IR_\natural$. Moreover, R and $R_\natural$ have the same dimension.* □

Corollary 8.1.14. *The catapower of a regular local ring is again regular (of the same dimension).*

Proof. Let $(R, \mathfrak{m})$ be a d-dimensional regular local ring. If $d = 0$, then R is a field, and $R_\sharp$ is equal to the ultrapower $R_\natural$ whence a field. So we may assume $d > 0$. Let x be a minimal generator of $\mathfrak{m}$. Hence R/xR is regular of dimension $d-1$, so that by induction, its catapower is also regular of dimension $d-1$. But this catapower is just $R_\sharp/xR_\sharp$ by Corollary 8.1.13. It follows that $\mathfrak{m}R_\sharp$ is generated by at most d elements. Since $R_\sharp$ has dimension d by Corollary 8.1.13, it is regular. □

To further explore the connection between a ring and its catapower, we require a flatness result.

Theorem 8.1.15. *For each Noetherian local ring R, the induced homomorphism $R \to R_\sharp$ into its catapower $R_\sharp$ is faithfully flat.*

Proof. Since $R \to R_\sharp$ is local, we only need to verify flatness. Moreover, since $R_\sharp$ is complete by Theorem 8.1.4, we get $(\widehat{R})_\sharp = R_\sharp$ by a double application of 8.1.3, whence an induced homomorphism $\widehat{R} \to R_\sharp$. As $R \to \widehat{R}$ is flat, we only need to show that $\widehat{R} \to R_\sharp$ is flat, and hence we may already assume that R is complete.

Suppose first that R is moreover regular. By Corollary 8.1.14, so is then $R_\sharp$. In particular, the generators of $\mathfrak{m}$ form an $R_\sharp$-regular sequence, so that $R_\sharp$ is flat over R by Theorem 3.3.9. For R arbitrary, $R = S/I$ for some regular local ring S and some ideal $I \subseteq S$ by Cohen's Structure Theorems. By our previous argument the ultrapower $S_\sharp$ of S is flat, whence so is $R = S/I \to S_\sharp/IS_\sharp = R_\sharp$ by 3.1.3 (where we used Corollary 8.1.13 for the last equality). □

The reader who is willing to use some heavier commutative algebra can prove the following stronger fact:

Corollary 8.1.16. *If R is an excellent local ring, then the diagonal embedding $R \to R_\sharp$ is regular.*

Proof. For the notion of excellence and regular maps, see [69, §32]. By Theorem 8.1.15, the map $R \to R_\sharp$ is flat. It is also unramified (see §3.2.3), since $\mathfrak{m}R_\sharp$ is the maximal ideal of $R_\sharp$. If R is a field k, then $R_\sharp$ is just its ultrapower $k_\natural$. Using Maclane's Criterion for Separability, it is not hard to show that the extension $k \to k_\natural$ is separable. This shows in view of Corollary 8.1.13 that for R arbitrary, $R \to R_\sharp$ induces a separable extension of residue fields. Hence $R \to R_\sharp$ is formally etale by [69, Theorem 28.10], whence regular by [2]. □

We can now generalize the fact that catapowers preserve regularity (Corollary 8.1.14) to:

Corollary 8.1.17. *If R is an excellent local ring, then R is regular, normal, reduced or Cohen-Macaulay, if and only if $R_\sharp$ is.*

Proof. Immediate from Corollary 8.1.16 and the fact that regular maps preserve these properties in either direction (see [69, Theorem 32.2]). □

Corollary 8.1.18. *If R is a complete Noetherian local domain, then so is its catapower $R_\sharp$.*

Proof. Let S be the normalization of R (that is to say, the integral closure of R inside its field of fractions). By [69, §33], the extension $R \subseteq S$ is finite, and S is also a complete Noetherian local ring. I claim that the induced homomorphism of catapowers $R_\sharp \to S_\sharp$ is again finite and injective. Since $S_\sharp$ is normal by Corollary 8.1.17, it is a domain, whence so is its subring $R_\sharp$.

So remains to prove the claim. By the weak Artin-Rees Lemma applied to the finite R-module S (see Remark 8.2.2 below), we can find for each m a uniform bound $e(m)$ such that $\mathfrak{m}^{e(m)}S \cap R \subseteq \mathfrak{m}^m$. Let $\mathfrak{n}$ be the maximal ideal of S. Since $S/\mathfrak{m}S$ is finite over $R/\mathfrak{m}$ by base change, it is Artinian, and hence $\mathfrak{n}^l \subseteq \mathfrak{m}S$ for some l. Together with the weak Artin-Rees bound, this yields

$$\mathfrak{n}^{le(m)} \cap R \subseteq \mathfrak{m}^m \tag{8.3}$$

for all m.

Let $S_\natural$ be the ultrapower of S, so that $S_\natural$ is a finite $R_\natural$-module. The inclusion $\mathfrak{I}_{R_\natural} \subseteq \mathfrak{I}_{S_\natural} \cap R_\natural$ is clear, and we need to prove the converse, for then $R_\sharp \to S_\sharp$ will be injective. So let $z \in R_\natural$ be such that it is an infinitesimal in $S_\natural$, and let $z_w \in R$ be approximations of z. Fix some m. Since $z \in \mathfrak{n}^{le(m)}S_\natural$, by Łoś' Theorem $z_w \in \mathfrak{n}^{le(m)}$ for almost all w, whence $z_w \in \mathfrak{m}^m$ by (8.3). By another application of Łoś' Theorem, we get $z \in \mathfrak{m}^m R_\natural$, and since this holds for all m, we get $z \in \mathfrak{I}_{R_\natural}$, as we wanted to show. □

We conclude with a generalization of Proposition 7.1.8:

Theorem 8.1.19. *Let R be a Noetherian local ring of equal characteristic, with residue field k, and let $R_\sharp$ and $k_\sharp$ be their respective catapowers. Then $R_\sharp$ is isomorphic to the complete scalar extension $\widehat{R_{k_\sharp}}$ over $k_\sharp$.*

Proof. Since a ring and its completion have the same complete scalar extensions, we may assume R is complete. By Cohen's Structure Theorems, R is a homomorphic image of a power series ring $k[[\xi]]$, with ξ an n-tuple of indeterminates. Since complete scalar extensions (by (3.8)) as well as catapowers (Corollary 8.1.13) commute with homomorphic images, we may assume $R = k[[\xi]]$. So remains to show that $R_\sharp \cong k_\sharp[[\xi]]$. However, this is clear by Cohen's structure theorem, since $R_\sharp$ is regular by Corollary 8.1.14, with residue field $k_\sharp$, and having dimension n by Corollary 8.1.13. □

8.1.4 Cataproducts in the Non-local Case

Although below, we will only be interested in cataproducts of Noetherian local rings of bounded embedding dimension, precisely because we can now apply our tools from commutative algebra to them, it might be of interest to define cataproducts in general. For this, we must rely on the alternative description of ultraproducts from §2.5. Given a collection of rings A_w, with Cartesian product $A_\infty := \prod A_w$, choose a maximal ideal $\mathfrak{M}$ in A_∞ containing the direct sum ideal $A_{(\infty)} := \bigoplus A_w$. We define the ($\mathfrak{M}$-) *cataproduct* of the A_w as the $\mathfrak{M}$-adic separated quotient of A_∞, that is to say, the ring $A_\sharp := A_\infty/\mathfrak{M}^\infty$, where $\mathfrak{M}^\infty$ is the intersection of all powers of $\mathfrak{M}$. Note that $\mathfrak{M}^\circ \subseteq \mathfrak{M}^\infty$, showing that $A_\sharp$ is a residue ring of $A_\natural = A_\infty/\mathfrak{M}^\circ$. Theorem 8.1.4 is the essential ingredient to prove that both definitions agree in the local case. To prove the analogue of Theorem 8.1.4 in this more general setup, we make the following definition: a maximal ideal $\mathfrak{M}$ of A_∞ is called *algebraic* if it contains a product $\prod \mathfrak{m}_w$ of maximal ideals $\mathfrak{m}_w \subseteq A_w$ (whence in particular contains the direct sum ideal $A_{(\infty)}$); the corresponding cataproduct is then also called *algebraic*.

Theorem 8.1.20. *Any algebraic cataproduct is a complete local ring. More precisely, if $\mathfrak{M}$ is an algebraic maximal ideal of the product $A_\infty := \prod A_w$, then the corresponding $\mathfrak{M}$-cataproduct $A_\sharp$ is a complete local ring with maximal ideal $\mathfrak{M}A_\sharp$.*

Proof. Let $\mathfrak{m}_w \subseteq A_w$ be maximal ideals whose product $\mathfrak{m} := \prod \mathfrak{m}_w$ is contained in $\mathfrak{M}$. Let us first show that

$$\mathfrak{M} = \mathfrak{m} + \mathfrak{M}^\circ. \tag{8.4}$$

Indeed, in the ultraproduct $A_\natural := A_\infty/\mathfrak{M}^\circ$ (see 2.5.2) the extended ideal $\mathfrak{m}A_\natural$ is equal to the ultraproduct of the $\mathfrak{m}_w$, whence by Łoś' Theorem is maximal. Since it is contained in the maximal ideal $\mathfrak{M}A_\natural$, both ideals must be the same, proving (8.4). Since $\mathfrak{M}^\circ$ is idempotent (as it is generated by idempotents), we immediately get from this that

$$\mathfrak{M}^n = \mathfrak{m}^n + \mathfrak{M}^\circ,$$

for all n. In particular, the $\mathfrak{M}A_\natural$-adic topology is the same as the $\mathfrak{m}A_\natural$-adic one, and we have

$$A_\sharp = A_\infty/\mathfrak{M}^\infty = A_\natural/\mathfrak{m}^\infty A_\natural.$$

To prove that $A_\sharp$ is complete, it suffices therefore to show that $A_\natural$ is $\mathfrak{m}$-adically quasi-complete. A minor modification of the argument in Theorem 8.1.4 easily accomplishes this (nowhere did we explicitly use that the R_w were local, of bounded embedding dimension). It follows from [69, Theorem 8.14] that $A_\sharp$ is local with maximal ideal $\mathfrak{m}A_\sharp = \mathfrak{M}A_\sharp$. □

If all A_w are local, then any maximal ideal $\mathfrak{M} \subseteq A_\infty$ is algebraic, since $A_\natural = A_\infty/\mathfrak{M}^\circ$ is local, with maximal ideal $\mathfrak{m}A_\natural$ by Łoś' Theorem. Hence, $\mathfrak{M}A_\natural$, being also a maximal ideal, must be equal to $\mathfrak{m}A_\natural$, and hence $\mathfrak{m} \subseteq \mathfrak{M}$, proving that the latter is algebraic. To construct a non-algebraic maximal ideal, take any ultraring admitting a maximal ideal which is not an ultra-ideal; its pre-image in the Cartesian product is then non-algebraic by the previous argument. Although one could replace the maximal ideal $\mathfrak{M}$ in the above construction by an arbitrary prime ideal containing $A_{(\infty)}$, I do not know what this more general notion of cataproduct would entail. In any case, it is not hard to see that an algebraic prime ideal is always maximal.

Corollary 8.1.21. *If there is a uniform bound on the number of generators of the maximal ideals of all A_w, then any algebraic cataproduct is Noetherian.*

Proof. With notation as in the previous proof, $\mathfrak{m}A_\natural$ is finitely generated by Łoś' Theorem, whence so is $\mathfrak{m}A_\sharp$. The result now follows from [69, Theorem 29.4], since $A_\sharp$ is complete by Theorem 8.1.20. □

The corollary applies in particular to the approximations A_w of an affine algebra A over a Lefschetz field (see Chapter 4), for if A is generated, as an algebra, by at most n elements, then so is almost each A_w, and, by the Nullstellensatz, each maximal ideal is then generated by at most n elements.

8.2 Uniform Behavior

In Chapter 4 we amply illustrated how ultraproducts can be used to prove several uniformity results. This section contains more results derived by this technique.

8.2.1 Weak Artin-Rees

The Artin-Rees Lemma is an important tool in commutative algebra, especially when using 'topological' arguments. There is a weaker form of Artin-Rees, which is often really the only property one uses and which we can now prove easily by non-standard methods.

Theorem 8.2.1. *Let $(R,\mathfrak{m})$ be a Noetherian local ring, and let $\mathfrak{a} \subseteq R$ be an ideal. For each l, there exists a uniform bound $e := e(\mathfrak{a}, l)$ such that*

$$\mathfrak{a} \cap \mathfrak{m}^e \subseteq \mathfrak{m}^l \mathfrak{a}.$$

Proof. Suppose not, so that for some l, none of the intersections $\mathfrak{a} \cap \mathfrak{m}^n$ is contained in $\mathfrak{m}^l\mathfrak{a}$. Hence we can find elements $a_n \in \mathfrak{a} \cap \mathfrak{m}^n$ outside $\mathfrak{m}^l\mathfrak{a}$. Let $R_\natural$ and $R_\sharp$ be the respective ultrapower and catapower of R. The diagonal embeddings $R \to R_\natural$ and $R \to R_\sharp$ are both flat by Corollary 3.3.3 and Theorem 8.1.15 respectively. Since $R_\sharp = R_\natural/\mathfrak{I}_{R_\natural}$, the quotient criterion, Theorem 3.3.5, yields $\mathfrak{a}R_\natural \cap \mathfrak{I}_{R_\natural} = \mathfrak{a}\mathfrak{I}_{R_\natural}$. Let a be the ultraproduct of the a_n, so that by Łoś' Theorem, $a \in \mathfrak{a}R_\natural \cap \mathfrak{I}_{R_\natural} = \mathfrak{a}\mathfrak{I}_{R_\natural}$. The latter ideal is in particular contained in $\mathfrak{a}\mathfrak{m}^l R_\natural$, and hence by Łoś' Theorem once more, $a_n \in \mathfrak{m}^l\mathfrak{a}$ for almost all n, contradiction. □

Remark 8.2.2. Using the fact that M/N admits a finite prime filtration (3.3), for $N \subseteq M$ finitely generated modules over R, we get a module version of the above result: *there exists for each l, a uniform bound $e := e(N,M,l)$ such that*

$$N \cap \mathfrak{m}^e M \subseteq \mathfrak{m}^l N.$$

We may turn the previous result into a Noetherianity criterion:

Theorem 8.2.3. *For a coherent local ring $(R,\mathfrak{m})$ of finite embedding dimension the following are equivalent:*

8.2.3.i. *R is Noetherian;*
8.2.3.ii. *every finitely generated ideal is $\mathfrak{m}$-adically closed;*
8.2.3.iii. *for every finitely generated ideal $\mathfrak{a}$ and every l, there exists some $e := e(\mathfrak{a};l)$ such that $\mathfrak{a}\cap\mathfrak{m}^e \subseteq \mathfrak{a}\mathfrak{m}^l$;*
8.2.3.iii' *for every finitely generated ideal $\mathfrak{a}$, the $\mathfrak{m}$-adic topology on $\mathfrak{a}$ coincides with its induced topology as a subspace $\mathfrak{a}\subseteq R$.*

Proof. The equivalence of (8.2.3.iii) and (8.2.3.iii') is immediate from the definitions. The implication (8.2.3.i) $\Rightarrow$ (8.2.3.ii) is Krull's Intersection Theorem (Theorem 2.4.14), and (8.2.3.i) $\Rightarrow$ (8.2.3.iii) is just Theorem 8.2.1.

We will simultaneously prove both converses. Observe first that in either case, R is separated. Indeed, this is clear in case (8.2.3.ii) since the closure of the zero ideal is $\mathfrak{I}_R$. In case (8.2.3.iii), let $x\in\mathfrak{I}_R$ and choose e such that $xR\cap\mathfrak{m}^e\subseteq x\mathfrak{m}$. Since x lies in the former ideal, it belongs to the latter, and hence $x = ax$ for some $a\in\mathfrak{m}$. Since R is local, $1-a$ is a unit, showing that $x = 0$ and hence that $\mathfrak{I}_R = 0$. Let $R_\sharp$ be the catapower of R, that is to say, $R_\sharp = R_\natural/\mathfrak{I}_{R_\natural}$ (which is then also the completion of $R_\natural$), where $R_\natural$ is the ultrapower of R. By 8.1.11, regardless whether R is Noetherian, $R_\sharp$ is. Hence, if we can show that $R\to R_\sharp$ is faithfully flat, the assertion follows from Corollary 3.2.6.

By Theorem 3.3.4, the diagonal embedding $R\to R_\natural$ is flat. The quotient criterium (Theorem 3.3.5) then yields the desired flatness of $R\to R_\sharp$, provided we can show that $\mathfrak{a}R_\natural\cap\mathfrak{I}_{R_\natural} = \mathfrak{a}\mathfrak{I}_{R_\natural}$, for every finitely generated ideal $\mathfrak{a} := (f_1,\dots,f_s)R$. One direction in this inclusion is clear, so assume $x\in\mathfrak{a}R_\natural\cap\mathfrak{I}_{R_\natural}$. Let $x_w\in R$ be approximations of x. There is nothing to show if $x = 0$, so that without loss of generality, we may assume all x_w are non-zero. By separatedness, there is some $n(w)$ such that x_w belongs to $\mathfrak{m}^{n(w)}\mathfrak{a}$ but not to $\mathfrak{m}^{n(w)+1}\mathfrak{a}$. Let $y_{iw}\in\mathfrak{m}^{n(w)}$ such that

$$x_w = y_{1w}f_1+\cdots+y_{sw}f_s,$$

and let y_i be the ultraproduct of the y_{iw}. By Łoś' Theorem, $x = y_1f_1+\cdots+y_sf_s$, so we need to show that $y_i\in\mathfrak{I}_{R_\natural}$. Assume this fails, say, for $i=1$, so that $y_1\notin\mathfrak{m}^lR_\natural$ for some l. I claim that, in either case, almost all x_w belong to $\mathfrak{m}^l\mathfrak{a}$. In particular, $l\leq n(w)$, for almost all w, whence almost all y_{1w} belong to $\mathfrak{m}^l$. This in turn implies by Łoś' Theorem that $y_1\in\mathfrak{m}^lR_\natural$, contradiction.

So remains to prove the claim. Assuming (8.2.3.iii), there exists e such that $\mathfrak{a}\cap\mathfrak{m}^e\subseteq\mathfrak{m}^l\mathfrak{a}$. Since $x\in\mathfrak{a}R_\natural\cap\mathfrak{m}^eR_\natural$, Łoś' Theorem yields that almost all x_w belong to $\mathfrak{a}\cap\mathfrak{m}^e$ whence to $\mathfrak{m}^l\mathfrak{a}$. So assume (8.2.3.ii) holds. Let $S_\natural$ and $S_\sharp$ be respectively the ultrapower and catapower of $S := \widehat{R}$. For the same reason as above, S is Noetherian, so that the natural map $S\to S_\sharp$ is faithfully flat by Theorem 8.1.15. By Corollary 3.3.3, so is the map $S\to S_\natural$. Hence by Theorem 3.3.5, we have $\mathfrak{a}S_\natural\cap\mathfrak{I}_{S_\natural} = \mathfrak{a}\mathfrak{I}_{S_\natural}$. In particular, $x\in\mathfrak{a}\mathfrak{I}_{S_\natural}$. By Łoś' Theorem, almost each x_w

lies in $\mathfrak{a}\mathfrak{m}^l S$. Since $\mathfrak{a}\mathfrak{m}^l$ is finitely generated, it is adically closed, and hence $\mathfrak{a}\mathfrak{m}^l S \cap R = \mathfrak{a}\mathfrak{m}^l$ by Proposition 8.1.7 (or, rather its analogue for arbitrary local rings of finite embedding dimension which is proven in the same manner using 8.1.11 instead). Therefore, almost each x_w lies in $\mathfrak{a}\mathfrak{m}^l$, as we needed to show. □

8.2.2 *Uniform Arithmetic in a Complete Noetherian Local Ring*

In what follows, our invariants are allowed to take values in $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$. An *n-ary numerical function* is by definition a map $F\colon \overline{\mathbb{N}}^n \to \overline{\mathbb{N}}$ such that $F(\mathbf{x}) = \infty$ if and only if $\mathbf{x}$ does not belong to $\mathbb{N}^n$. The restriction of a numerical function $\overline{\mathbb{N}}^n \to \overline{\mathbb{N}}$ to $\mathbb{N}^n$ is $\mathbb{N}$-valued, and conversely, any $\mathbb{N}$-valued function has a unique extension to a numerical function. By the *order*, $\operatorname{ord}_R(x)$, of an element x in a local ring $(R,\mathfrak{m})$, we mean the supremum of all m such that $x \in \mathfrak{m}^m$ (so that in particular $\operatorname{ord}_R(x) = \infty$ if and only if $x \in \mathfrak{I}_R$).

Theorem 8.2.4. *A complete Noetherian local ring R is a domain if and only if there exists a binary numerical function F such that*

$$\operatorname{ord}_R(xy) \le F(\operatorname{ord}_R(x), \operatorname{ord}_R(y)) \tag{8.5}$$

for all $x,y \in R$.

Proof. Assume first that (8.5) holds for some F. If x and y are non-zero, then their order is finite by Theorem 2.4.14, whence $F(\operatorname{ord}(x), \operatorname{ord}(y))$ is finite per definition. In particular, xy must be non-zero, showing that R is a domain.

Conversely, assume towards a contradiction that no such function F can be defined on a pair $(a,b) \in \mathbb{N}^2$. This implies that there exist for each n, elements x_n and y_n in R of order at most a and b respectively, but such that their product $x_n y_n$ has order at least n. Let $R_\natural$ and $R_\sharp$ be the ultrapower and catapower of R respectively, and let x and y be the ultraproducts of x_n and y_n respectively. It follows from Łoś' Theorem that $\operatorname{ord}_{R_\natural}(x) \le a$ and $\operatorname{ord}_{R_\natural}(y) \le b$, and hence in particular, x and y are non-zero in $R_\sharp$. By Corollary 8.1.18, the catapower $R_\sharp$ is again a domain. In particular, xy is a non-zero element in $R_\sharp$, and hence has finite order, say, c, by Theorem 2.4.14. However, then also $\operatorname{ord}_{R_\natural}(xy) = c$ whence $\operatorname{ord}_R(x_n y_n) = c$ for almost all n by Łoś' Theorem, contradiction. □

Remark 8.2.5. It is not hard to show that an arbitrary Noetherian local ring R admits a numerical function with the above property if and only if its completion does. It follows that R is analytically irreducible (meaning that its completion is a domain) if and only if a numerical function as above exists. Theorem 8.2.4 was first proven by Rees [78] via a valuation argument. By [112, Theorem 3.4] and [58, Proposition 2.2], we may take F linear, or rather, of the form $F(a,b) := c\max\{a,b\}$, for some $c \in \mathbb{N}$ (one usually expresses this by saying that R has *c-bounded multiplication*).

Theorem 8.2.6. *A d-dimensional Noetherian local ring $(R,\mathfrak{m})$ is Cohen-Macaulay if and only if there exists a binary numerical function G such that*

$$\operatorname{ord}_{R/I}(xy) \leq G(\deg_{R/I}(x), \operatorname{ord}_{R/I}(y)) \tag{8.6}$$

for all $x, y \in R$ and all ideals $I \subseteq R$ generated by part of a system of parameters of length $d-1$.

Proof. Suppose a function G satisfying (8.6) exists, and let $(z_1,\dots,z_d)$ be a system of parameters in R. Fix some i and let $y \in (J : z_{i+1})$ with $J := (z_1,\dots,z_i)R$. We need to show that $y \in J$. For each m, let $I_m := J + (z_{i+2}^m,\dots,z_d^m)R$, and put $x := z_{i+1}$. Since $xy \in J \subseteq I_m$, the left hand side in (8.6) for $I = I_m$ is infinite, whence so must the right hand side be. However, x is a parameter in R/I_m, and therefore has finite degree. Hence, the second argument of G must be infinite, that is to say, $\operatorname{ord}_{R/I_m}(y) = \infty$. In other words, $y \in I_m$, and since this holds for all m, we get $y \in J$ by Theorem 2.4.14, as we wanted to show.

Conversely, towards a contradiction, suppose R is Cohen-Macaulay but no such function G can be defined on the pair $(a,b) \in \mathbb{N}^2$. This means that there exist elements $x_n, y_n \in R$ and a $d-1$-tuple $\mathbf{z}_n$ which is part of a system of parameters in R, such that $\deg_{S_n}(x_n) \leq a$ and $\operatorname{ord}_{S_n}(y_n) \leq b$, but $x_n y_n$ has order at least n in $S_n := R/\mathbf{z}_n R$. Let $R_\natural$ and $R_\sharp$ be the respective ultrapower and catapower of R. Since R is Cohen-Macaulay, so is $R_\sharp$ by Corollary 8.1.17. Let x, y and $\mathbf{z}$ be the ultraproduct of the x_n, y_n and $\mathbf{z}_n$ respectively. By Proposition 8.1.7, the cataproduct of the S_n is equal to $S_\sharp := R_\sharp/\mathbf{z}R_\sharp$. Since each S_n has dimension one, and parameter degree at most a by assumption on x_n, the dimension of $S_\sharp$ is again one by Theorem 8.1.12. Since $R_\sharp$ has dimension d by 8.1.8, the $d-1$-tuple $\mathbf{z}$ is part of a system of parameters in $R_\sharp$, whence is $R_\sharp$-regular. This in turn implies that $S_\sharp = R_\sharp/\mathbf{z}R_\sharp$ is Cohen-Macaulay. Moreover, by Łoś' Theorem, y has order b in $R_\natural/\mathbf{z}R_\natural$ whence also in $S_\sharp$, and x has degree a in $S_\sharp$. In particular, x is a parameter in $S_\sharp$ whence $S_\sharp$-regular. On the other hand, by Łoś' Theorem, xy is an infinitesimal in $R_\natural/\mathbf{z}R_\natural$, whence zero in $S_\sharp$. Since x is $S_\sharp$-regular, y is zero in $S_\sharp$, contradicting that its order in that ring is b. □

In [101, §12], similar characterizations are obtained by this method for other ring-theoretic properties, such as (quasi-)unmixed, analytically unramified, normal, ... We leave it to the reader to verify that a Noetherian local ring is regular if and only if order and degree coincide (the proof is straightforward and makes no use of ultraproducts).

Chapter 9
Protoproducts

In Chapter 4, we used ultraproducts to derive uniform bounds for various algebraic operations, where the bounds are given in terms of the degrees of the polynomials involved. This was done by constructing a faithfully flat embedding of the polynomial ring A into an ultraproduct $U(A)$ of polynomial rings, called its *ultra-hull*. Moreover, A is characterized as the subring of $U(A)$ of all elements of finite degree. In this chapter, we want to put these uniformity results in a more general context, by replacing the degree on A by what we will call a *proto-grading*. However, as the notion of ultra-hull is no longer available, we must replace the latter by the ultrapower $A_\natural$. Moreover, there now may be elements of finite proto-grade in $A_\natural$ outside A, leading to the notion of the *protopower* $A_\flat$ of A, sitting in between A and its ultrapower, and these embeddings may or may not be (faithfully) flat. The existence of uniform bounds in terms of the proto-grading follow from good properties of this protopower. This can be extended to several rings simultaneously by using protoproducts instead.

9.1 Protoproducts

Whereas as cataproducts are homomorphic images of ultraproducts, protoproducts will be subrings. To define them, we need to formalize the notion of the degree of a polynomial.

9.1.1 Proto-graded Rings

By a *pre-proto-grading* $\Gamma_\bullet(A)$ on a ring A, we mean an ascending chain of subsets

$$\Gamma_0(A) \subseteq \Gamma_1(A) \subseteq \Gamma_2(A) \subseteq \dots$$

and a unary function $F\colon \mathbb{N} \to \mathbb{N}$ such that the sum and the product of any two elements in $\Gamma_n(A)$ lies in $\Gamma_{F(n)}(A)$, and such that for any unit $u \in A$, if u lies in

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$\Gamma_n(A)$, then its inverse lies in $\Gamma_{F(n)}(A)$. The following terminology will prove to be quite convenient when discussing proto-gradings: we will say that an element $x \in A$ *has proto-grade at most* n, if it lies in $\Gamma_n(A)$. The minimal value n such that x has proto-grade at most n will occasionally be called the proto-grade of x, allowing the case that x lies in no $\Gamma_n(A)$, in which case it is said to have *infinite proto-grade*. Therefore, albeit less accurate, we may say that a proto-graded ring is one whose arithmetic (addition, product, and inverse) is uniformly bounded with respect to its proto-grade. If we want to emphasize the function F, we call $\Gamma_\bullet(A)$ a pre-proto-F-grading. We call $\Gamma_\bullet(A)$ a *proto-F-grading*, or simply a *proto-grading*, if moreover the union of all $\Gamma_n(A)$ is equal to A, that is to say, if there are no elements of infinite proto-grade. It is easy to show that:

9.1.1 *If $\Gamma_\bullet(A)$ is a pre-proto-F-grading on A, then the collection of all elements of finite proto-grade forms a subring A' of A, called the* proto-graded subring *associated to the pre-proto-grading, and $\Gamma_n(A') := \Gamma_n(A)$ defines a proto-F-grading on this subring.*

A *proto-graded* ring (A,Γ) is a ring endowed with a proto-grading $\Gamma_\bullet(A)$. Two proto-gradings $\Gamma_\bullet(A)$ and $\Theta_\bullet(A)$ are *equivalent* if there exists a unary function G such that $\Gamma_n(A) \subseteq \Theta_{G(n)}(A)$ and $\Theta_n(A) \subseteq \Gamma_{G(n)}(A)$ for all n. For all intent and purposes, as we shall see, we may replace any proto-grading by an equivalent one. For instance, since any proto-grading is equivalent with a proto-grading $\Gamma_\bullet(A)$ such that $0, \pm 1 \in \Gamma_0(A)$, there is no harm in assuming this already from the start.

The *trivial proto-grade* is given by letting all $\Gamma_n(A)$ be equal to A. The following standard structures all are instances of a proto-grade, the second one lending its name to the concept:

9.1.1.i. Any polynomial ring $A := Z[\xi]$ over an arbitrary ring Z is proto-graded by letting $\Gamma_n(A)$ consist of all elements of degree at most n. We refer to this as the *affine proto-grading* on A. Note that in particular $Z = \Gamma_0(A)$, or put differently, all coefficients have proto-grade zero.

9.1.1.ii. Any $\mathbb{N}$-graded ring $A = \oplus_n A_n$ is proto-graded by letting $\Gamma_n(A)$ be equal to the direct sum $A_0 \oplus A_1 \oplus \cdots \oplus A_n$.

9.1.1.iii. For a given subring A of $\mathbb{R}$, let $\Gamma_n(A)$ be the set of all elements $a \in A$ of absolute value at most n. In order for this to be a proto-grading, we need to exclude the possibility that there exist units in A converging to zero. For instance, if A is not dense at zero (that is to say, if there exists $\varepsilon > 0$ such that $A \cap [-\varepsilon, \varepsilon] = \{0\}$), then the absolute value yields a proto-grade on A.

We can easily extend the first example as follows:

9.1.2 *Let (A,Γ) be a proto-graded ring, an assume A is either reduced or Noetherian. Let ξ be a (finite) tuple of indeterminates, and put $B := A[\xi]$. Let $\Gamma_n(B)$ be the set of all polynomials of degree at most n each coefficient of which has proto-grade at most n. Then $\Gamma_\bullet(B)$ is a proto-grading on B, called the* extended degree proto-grade.

To prove this, it suffices, by induction, to treat the case that ξ is a single variable. Note that, in general, if A is F-proto-graded, then it is also F'-proto-graded, for any F' such that $F(n) \leq F'(n)$. Hence we may assume without loss of generality that $2n \leq F(n)$ and F is monotone. Let $G(n) := F^n(n)$, where F^n denotes the n-fold composition of F, and let $f, g \in B$ of proto-grade at most n. Clearly, $f+g$ has proto-grade at most $F(n)$. As for the product, note that fg has degree at most $2n$, and any coefficient in fg is a sum of $n+1$ products $a_i b_j$, with a_i and b_j respective coefficients of f and g. Hence each term in this sum has proto-grade at most $F(n)$, and the total sum therefore has proto-grade at most $G(n)$. If A is reduced, then the only units in B are constants, and the assertion is proven. If A is Noetherian, then $\mathfrak{n} := \mathrm{nil}(A)$ has finite nilpotency, that is to say, $\mathfrak{n}^N = 0$ for some N. If $f = a_0 + a_1\xi + \cdots + a_d\xi^d$ is a unit, then so is a_0 and all a_i for $i > 0$ belong to $\mathfrak{n}$. Multiplying f with the inverse of a_0, which increases the proto-grade up to at most $F(n)$, we may assume $a_0 = 1$. Hence $(f-1)^N = 0$. Expanding this expression shows that the inverse of f is of the form $f^{N-1} - Nf^{N-2} + \cdots \pm 1$. Each term has proto-graded at most $G(n)$ by our previous argument, so their sum has proto-grade at most $G^N(n)$. In conclusion, B is proto-graded with respect to the function $n \mapsto F^{nN}(n)$. □

In particular, the affine proto-grading on $A[\xi]$ is the extended degree proto-grading where A is given the trivial proto-grading. By (9.1.1.iii), we may view the ring of integers $\mathbb{Z}$ as a proto-graded ring. The degree proto-grading, as defined in 9.1.2, on $\mathbb{Z}[\xi]$ extending this absolute value proto-grading will be called the *Kronecker proto-grading* on $\mathbb{Z}[\xi]$, and will be studied further in §9.3.

9.1.2 The Category of Proto-graded Rings

A *morphism of proto-graded rings* $(A, \Gamma) \to (B, \Theta)$ is a ring homomorphism $A \to B$ for which there exists a unary function $G\colon \mathbb{N} \to \mathbb{N}$ such that $\Gamma_n(A) \subseteq \Theta_{G(n)}(B)$, for all n. In particular, if Γ and Γ' are equivalent proto-gradings on A, then the identity on A induces an isomorphism of proto-graded rings. These definitions give rise to the category of proto-graded rings.

Let (A, Γ) be a proto-F-graded ring, and let $\varphi\colon A \to B$ be a ring homomorphism. We define the *push-forward* of Γ by the rule $\Gamma_n(B) := \varphi(\Gamma_n(A))$ for all n. In general, $\Gamma_\bullet(B)$ is only a pre-proto-F-grading, since elements outside the image of the homomorphism have infinite proto-grade. In particular, the push-forward is a proto-grading if φ is surjective, that is to say, if B is of the form A/I for some ideal $I \subseteq A$. We call this proto-grading the *residual proto-grading* on A/I. In regard to localizations, we can show:

9.1.3 *Let (A, Γ) be a proto-graded rings and let $S \subseteq A$ be a multiplicative subset. There exists a natural proto-grading on the localization $S^{-1}A$, such that the natural map $A \to S^{-1}A$ is a morphism of proto-gradings.*

Indeed, suppose Γ is a proto-F-grading. On $B := S^{-1}A$, define a proto-grading by the rule that x/s has proto-grade at most n, for $x \in A$ and $s \in S$, if both x and s have proto-grade at most n. It is now easy to verify that this yields a proto-G-grading on B, where $G = F \circ F$. □

In view of 9.1.2, we can now extend a given proto-grading on a ring A to any A-affine algebra B. Namely, write B as $A[\xi]/I$ and give B the residual proto-grading of the extended degree proto-grading on $A[\xi]$ (and similarly, for local A-affine algebras, using 9.1.3). We leave it to the reader to verify that any two presentations of B as an A-affine algebra yield equivalent proto-gradings. In case the base ring Z is trivially proto-graded, then we refer to the thus obtained proto-grading on a (local) Z-affine algebra B as the *Z-affine proto-grading*, or simply, the *affine proto-grading* on B.

9.1.3 Protopowers

Let (A,Γ) be a pre-proto-graded ring, and let $A_\natural$ be some ultrapower of A. We define a pre-proto-grading on $A_\natural$ by letting $\Gamma_n(A_\natural)$ be the ultrapower of $\Gamma_n(A)$ (viewed as a subset of $A_\natural$). The *protopower* $A_\flat$ of A is defined as the proto-graded subring associated to this pre-proto-grading, that is to say,

$$A_\flat := \bigcup_n \Gamma_n(A_\natural).$$

By 9.1.1, the protopower is again a ring, and Γ induces a proto-grading on $A_\flat$. The following characterization of $A_\flat$ easily follows from Łoś' Theorem:

9.1.4 *An element in the ultrapower $A_\natural$ lies in the protopower $A_\flat$ if and only if for some n, almost all of its approximations have proto-grade at most n.*

We may express this more simply by saying that $x \in A_\natural$ belongs to $A_\flat$ if and only if some (equivalently, any) approximation of x has *uniformly bounded proto-grade*. In case A is proto-graded, the case we will be in almost all the time, we have a natural diagonal embedding $A \subseteq A_\flat$. Thus, we have completed the *chromatic scale* of a proto-graded Noetherian local ring A: there exist natural local A-algebra homomorphisms

$$A \hookrightarrow A_\flat \hookrightarrow A_\natural \twoheadrightarrow A_\sharp. \tag{9.1}$$

9.1.4 Protoproducts

To define protoproducts, we need to make an assumption on the sequence of (pre-)proto-graded rings A_w. We say that the A_w are *uniformly proto-graded* if there exists a unary function F, such that the (pre-)proto-grading on A_w is equivalent with a (pre-)proto-F-grading $\Gamma_\bullet(A_w)$ for (almost) all w. If this is the case, let $A_\natural$ be the ultraproduct of the A_w, and for each n, let $\Gamma_n(A_\natural)$ be the subset of $A_\natural$ given as the ultraproduct of the $\Gamma_n(A_w)$. By Łoś' Theorem, $\Gamma_\bullet(A_\natural)$ is a pre-proto-F-grading on $A_\natural$. The associated proto-graded subring $A_\flat$ is called the *protoproduct* of the A_w. One checks that this definition does not depend on the choice of the unary function F, or the particular equivalent proto-F-grading. Of course, a protopower is just a special instance of a protoproduct where all the rings are equal to a single proto-graded ring. The protoproduct of trivially proto-graded rings is just their ultraproduct, so that protoproducts generalize the notion of ultraproduct.

Lemma 9.1.5. *The protoproduct of uniformly proto-graded local rings is a local ring.*

Proof. Let $(R_m, \mathfrak{m}_w)$ be proto-F-graded local rings, and let $(R_\natural, \mathfrak{m}_\natural)$ be their ultraproduct. I claim that $\mathfrak{m}_\natural \cap R_\flat$ is the unique maximal ideal of the protoproduct $R_\flat$. To this end, we have to show that if $x \in R_\flat$ does not belong to $\mathfrak{m}_\natural$, then it is invertible in $R_\flat$. Let x_w be an approximation of x. In particular, almost all x_w are units, and have proto-grade at most n, for some n independent of w. Hence their respective inverses y_w have proto-grade at most $F(n)$. The ultraproduct y of the y_w lies therefore also in $R_\flat$. By Łoś' Theorem, $xy = 1$ holds in $R_\natural$, whence also in the subring $R_\flat$, as we wanted to show. □

9.1.6 *Let A_w be uniformly proto-graded rings with protoproduct $A_\flat$ and ultraproduct $A_\natural$, and let $I_w \subseteq A_w$ be ideals with ultraproduct $I_\natural \subseteq A_\natural$. The protoproduct $B_\flat$ of the $B_w := A_w/I_w$ viewed in their residual proto-grading is equal to $A_\flat/(I_\natural \cap A_\flat)$.*

Since the ultraproduct of the B_w is equal to $A_\natural/I_\natural$ by 2.1.6, an element $x \in A_\natural$ viewed as an element of $A_\natural/I_\natural$ has an approximation x_w of bounded proto-grade in B_w if and only if almost all x_w have proto-grade at most n, for some n. This in turn is equivalent with $x \in A_\flat$. Hence $B_\flat$ is equal to the image of $A_\flat$ in $A_\natural/I_\natural$, and this is just $A_\flat/(I_\natural \cap A_\flat)$. □

Applied to a single proto-graded ring A and a single finitely generated ideal $I \subseteq A$ (so that its ultrapower is just $IA_\natural$ by 2.4.20), we proved:

9.1.7 *Let A be a proto-graded ring with protopower $A_\flat$ and ultrapower $A_\natural$, and let $I \subseteq A$ be a finitely generated ideal. The protopower of A/I is equal to $A_\flat/(IA_\natural \cap A_\flat)$.* □

Protoproducts commute with the formation of a polynomial ring in the following sense:

Proposition 9.1.8. *Let A_w be uniformly proto-graded rings, which we assume to be either reduced or Noetherian, let ξ be a finite tuple of indeterminates, and view each $B_w := A_w[\xi]$ with its extended degree proto-grading. If $A_\flat$ and $B_\flat$ are the respective protoproducts of A_w and B_w, then $B_\flat = A_\flat[\xi]$.*

Proof. Note that it follows from the proof of 9.1.2 (which requires for the ring to be either reduced or Noetherian) that all B_w are also uniformly proto-graded, so that it makes sense to talk about their protoproduct. Let $A_\natural \subseteq B_\natural$ be the ultraproducts of the A_w and B_w respectively. By definition of the extended degree proto-grading, an element f in the ultraproduct $B_\natural$ of the B_w has an approximation f_w of bounded proto-grade if and only if, for some n, almost all f_w have degree at most n with coefficients of proto-grade at most n in A_w. Hence such an f belongs to $A_\natural[\xi]$ by an argument similar to the one used for 4.1.2. Moreover, by Łoś' Theorem, the coefficients of f are then all in $\Gamma_n(A_\flat)$, showing that $f \in A_\flat[\xi]$. Reversing the argument yields the converse inclusion. □

For instance, the protopower of $\mathbb{Z}[\xi]$ with respect to its Kronecker proto-grading is $\mathbb{Z}[\xi]$ itself, whereas with respect to its affine proto-grading we get $\mathbb{Z}_\natural[\xi]$, where $\mathbb{Z}_\natural$ is the ultrapower of $\mathbb{Z}$. It is instructive to revisit our construction of an ultra-hull in this new formalism: let K be the ultraproduct of fields K_w, each of which we view with its trivial proto-grading. In particular, the protoproduct of the K_w is just K, and hence by the above result, the protoproduct of the $A_w := K_w[\xi]$ in their affine proto-grading, is $A := K[\xi]$, whereas the ultraproduct of the A_w is the ultra-hull $U(A)$ of A.

9.1.5 Algebraic Protoproducts

As an illustration of our definitions, let me present the Lefschetz Principle (Theorem 2.4.3) in a new light, to wit, as an *ultraprotoproduct*. Given an extension of fields $K \subseteq L$, we say that an element $x \in L$ has *K-algebraic proto-grade* at most n, if it satisfies some (non-zero) polynomial of degree n with coefficients in K. From the fact that $x+y$, xy, and $1/x$ all belong to $K(x,y)$, it follows that this constitutes a proto-grade on L. The corresponding proto-graded subring is simply $L \cap K^{\text{alg}}$. If K is the prime field of L, then we simply refer to this proto-grade as the *algebraic proto-grade* on L.

9.1.9 *The algebraic protoproduct of all finite fields of characteristic p is* $\mathbb{F}_p^{alg}$.

Indeed, the ultrapower of all finite fields of characteristic p contains each of these finite fields, whence their union $\mathbb{F}_p^{\text{alg}}$, and the result now follows from the definition and the previous observation regarding the proto-graded subring. Hence we may paraphrase the Lefschetz Principle as:

9.1.10 *The field of complex numbers is an ultraproduct of the algebraic protoproduct of all finite fields.*

9.2 Uniform Bounds

In Chapter 4, the main tool for deriving uniform bounds was the faithful flatness of the ultra-hull. In the more general setup of proto-gradings, this is no longer a property holding automatically, but rather an hypothesis, and so we have to investigate when it is satisfied. Ideally, we should derive uniform bounds for a class of uniformly proto-graded rings (with the bounds only depending on some numerical invariants of the ring and the data), and for this we need good properties of protoproducts. An example of this method will be discussed in §10.2 in the next chapter. In this chapter, however, we content ourselves with bounds that work for a single proto-graded ring, for which it suffices to work with protopowers.

In what follows, A is a proto-graded ring, with proto-grading $\Gamma_\bullet(A)$, protopower $A_\flat$, and ultrapower $A_\natural$. We need to study the properties of the inclusions $A \subseteq A_\flat \subseteq A_\natural$. Since $A \to A_\natural$ is cyclically pure (see the discussion in §2.4.3), so is $A \subseteq A_\flat$. We may ask under which conditions will $A \subseteq A_\flat$ be faithfully flat, and we will see some examples below. However, keeping the example of an ultra-hull in the above discussion in mind, the more important question is the nature of the embedding $A_\flat \subseteq A_\natural$ (recall that this is the analogue of the embedding $A \subseteq U(A)$ of the ultra-hull). Unlike the ultra-hull case, this embedding may fail to be faithfully flat, an a priori obstruction for deriving uniform bounds à la Chapter 4, and so we make the following definition.

Definition 9.2.1. A proto-grading on a ring A is called respectively *flat*, *non-degenerated*, *faithfully flat*, or *cyclically pure*, if the natural embedding $A_\flat \subseteq A_\natural$ has the corresponding property.

To better formulate the next results, we introduce the following terminology. Let A be a proto-graded ring, with $A_\flat$ and $A_\natural$ its respective protopower and ultrapower. An ideal $I \subseteq A$ is said to have *proto-grade at most n*, if it can be generated by n elements of proto-grade at most n (note the bound on the number of generators!). In particular, for any $n \geq 1$, an element has proto-grade at most n if and only if the ideal it generates has proto-grade at most n. The usefulness of this concept is exhibited by the following result:

9.2.2 *Let $I_w \subseteq A$ be ideals of proto-grade at most n, for some n independent from w. Then there exists a finitely generated ideal $I \subseteq A_\flat$ such that $IA_\natural$ is equal to the ultraproduct $I_\natural$ of the I_w.*

Indeed, if $f_{1w}, \dots, f_{nw}$ are generators of I_w of proto-grade at most n, and if $f_1, \dots, f_n \in A_\natural$ are their respective ultraproducts, belonging therefore to the subring $A_\flat$, then, in view of 2.1.6, we may take $I := (f_1, \dots, f_n)A_\flat$. □

A note of caution: the ideal I is not necessarily equal to $I_\natural \cap A_\flat$, nor even uniquely determined by the I_w. We will return to this issue in Definition 9.4.1 below.

9.2.1 *Noetherian Proto-gradings*

Trivial proto-gradings are automatically faithfully flat, for then protopower and ultrapower agree. However, to derive meaningful bounds, some finiteness assumptions are required, the most natural of which is that the protopower should also be Noetherian. Let us therefore call a proto-grading *Noetherian* if its protopower $A_\flat$ is a Noetherian ring. If the proto-grading on A is Noetherian, then A itself must also be Noetherian by Corollary 3.2.6, since $A \to A_\flat$ is cyclically pure. The trivial proto-grading shows that the converse fails in general. So of real interest to us will be those proto-gradings which are at the same time Noetherian and faithfully flat. The example par excellence, of course, is the ultra-hull as discussed above. For technical purposes, we also need the following definition: a proto-grading on A is *coherent* if $A_\flat$ is a coherent ring (see §3.3.2).

Noetherianity of the proto-grading is characterized by the existence of certain uniform bounds as the following generalization of a theorem due to Seidenberg in [103] shows.

Theorem 9.2.3. *For a proto-graded ring A, the following are equivalent:*

9.2.3.i. *the proto-grading is Noetherian;*
9.2.3.ii. *there exists for each function $\lambda\colon \mathbb{N} \to \mathbb{N}$, a bound $n := n(\lambda)$ with the property that given a sequence of elements f_i of proto-grade at most $\lambda(i)$ for all $i \in \mathbb{N}$, then for some $i \leq n$, we can write $f_i = q_0 f_0 + \dots + q_{i-1} f_{i-1}$ with all q_j of proto-grade at most n.*

Proof. By way of contradiction, assume that $A_\flat$ is Noetherian, but that for some λ, no such bound exists. Therefore, we can find for each w, a counterexample consisting of the following data: elements $f_{iw} \in A$ of proto-grade at most $\lambda(i)$, for $i \leq w$, such that no f_{iw} can be written as a linear combination of the $f_{0w}, \dots, f_{i-1w}$ with coefficients of proto-grade at most w. For $i > w$, set f_{iw} equal to zero, and, for each i, let f_i be the ultraproduct of the f_{iw}, so that by construction, $f_i \in A_\flat$. Since $A_\flat$ is Noetherian, the ideal generated by all the f_i is equal to $(f_0, \dots, f_{m-1})A_\flat$ for some m. In particular, there exist $q_i \in A_\flat$ such that $f_m = q_0 f_0 + \dots + q_{m-1} f_{m-1}$. Choose $n \geq m$ such that all q_i for $i < m$ have proto-grade at most n. Hence, by Łoś' Theorem, f_{mw} is a linear combination of $f_{0w}, \dots, f_{m-1,w}$ with coefficients of proto-grade at most n, for almost all w, contradicting our assumption for $w > n$.

Conversely, assume that (9.2.3.ii) holds but that there exists an infinite strictly ascending chain of ideals $\mathfrak{a}_0 \subsetneq \mathfrak{a}_1 \subsetneq \mathfrak{a}_2 \subsetneq \dots$ in $A_\flat$. Choose for each i, an element f_i in $\mathfrak{a}_i$ but not in $\mathfrak{a}_{i-1}$. Let I be the ideal in $A_\flat$ generated by these f_i. For each i, choose $\lambda(i)$ so that f_i has proto-grade at most $\lambda(i)$. Let f_{iw} be an approximation of f_i of proto-grade at most $\lambda(i)$. By assumption, there is a bound $n := n(\lambda)$ such that for some $i \leq n$ and some q_{jw} of proto-grade at most n, we have

$$f_{iw} = q_{0w} f_{0w} + \dots + q_{i-1,w} f_{i-1,w}.$$

Let $q_j \in A_\flat$ be the ultraproduct of the q_{jw}. Since there are only finitely many possibilities for $i \leq n$, there is one such which holds for almost all w. For this i, we have therefore by Łoś' Theorem that

$$f_i = q_0 f_0 + \dots + q_{i-1} f_{i-1} \tag{9.2}$$

in $A_\natural$. Since all elements in (9.2) belong to the subring $A_\flat$, this equation itself holds in this subring, showing that $f_i \in \mathfrak{a}_{i-1}$, contradiction. □

Applying this to a constant function yields:

Corollary 9.2.4. *If A has a Noetherian proto-grading, then for any n there exists $n' \geq n$ with the property that any ideal generated by elements of proto-grade at most n is generated already by n' of these generators, that is to say, is an ideal of proto-grade at most n'.* □

We already mentioned that the affine proto-grading on a polynomial ring $K[\xi]$ over a field K is Noetherian, and in this case, one can give a more explicit bound in the previous Corollary 9.2.4: namely we may take n' equal to the number of monomials of degree at most n in the ξ (Lemma 4.4.2). Nonetheless, I am not aware of such an explicit characterization for other functions λ in Theorem 9.2.3, that is to say, for the result that: *for each field K, any function λ admits a uniform bound n, such that if f_i are polynomials in ξ over K of degree at most $\lambda(i)$, then for some $i \leq n$, the polynomial f_i is a linear combination of the previous f_j with coefficients themselves polynomials of degree at most n.* In §A.3, we will show that the ring of algebraic power series $K[[\xi]]^{\mathrm{alg}}$ over a field K admits a faithfully flat, Noetherian proto-grading, called the *etale proto-grading*. In particular, we can apply the previous bounds to this situation, but even the bound given by Corollary 9.2.4 seems no longer to admit a straightforward argument; see Theorem A.3.5.

Corollary 9.2.5. *Let A have a Noetherian proto-grading and let ξ be a finite tuple of indeterminates. For each function $\lambda : \mathbb{N} \to \mathbb{N}$, there exists a uniform bound $n := n(\lambda)$ with the property that given a sequence of polynomials f_i, for $i \in \mathbb{N}$, of degree at most $\lambda(i)$ all of whose coefficients have proto-grade at most $\lambda(i)$ as well, then for some $i \leq n$, we can write $f_i = q_0 f_0 + \cdots + q_{i-1} f_{i-1}$ with all q_j polynomials of degree at most n having coefficients of proto-grade at most n.*

Proof. Let $A_\flat$ be the protopower of A, which by assumption is Noetherian. Since A itself is in particular Noetherian, the extended degree proto-grading on $B := A[\xi]$ is well-defined by 9.1.2. By Proposition 9.1.8, the protopower of B is $A_\flat[\xi]$, again a Noetherian ring. Therefore the extended degree proto-grading is Noetherian and the bound now follows from Theorem 9.2.3. □

Another example of a Noetherian proto-grading to which we may apply the previous corollary is the Kronecker proto-grading on $\mathbb{Z}$ (since the protopower is trivial), yielding:

Corollary 9.2.6. *Given a tuple of indeterminates ξ, for each function $\lambda : \mathbb{N} \to \mathbb{N}$, there exists a uniform bound $n := n(\lambda)$ with the property that if f_i are polynomials of degree at most $\lambda(i)$ with integer coefficients of absolute value at most $\lambda(i)$, for all $i \in \mathbb{N}$, then for some $i \leq n$, we can write $f_i = q_0 f_0 + \cdots + q_{i-1} f_{i-1}$ with all q_j polynomials of degree at most n having integer coefficients of absolute value at most n.* □

9.2.2 Non-degenerated Proto-gradings

By Lemma 9.1.5, any proto-grading on a local ring is non-degenerated. We can characterize non-degenerated proto-gradings by a uniformity result:

Theorem 9.2.7. *A proto-grading on a ring A is non-degenerated if and only if for each pair (n,s) there exists a uniform bound $b := b(n,s)$ with the property that if $(f_1,\dots,f_s)A$ is the unit ideal, with each f_i of proto-grade at most n, then there exist g_i of proto-grade at most b such that $1 = f_1g_1 + \cdots + f_sg_s$.*

If the proto-grading is non-degenerated and Noetherian, then b can be taken to be independent from s.

Proof. The last assertion follows from the first and Corollary 9.2.4. Suppose first that proto-grading is non-degenerated, so that the natural embedding $A_\flat \to A_\natural$ is non-degenerated, but towards a contradiction, suppose no bound exists for the pair (n,s). Hence for each w, we have a counterexample consisting of s elements f_{iw} of proto-grade at most n generating the unit ideal, but any linear combination of the f_{iw} equal to 1 requires at least one of the coefficients to have proto-grade at least w. Let f_i be the ultraproduct of the f_{iw}. By construction, $f_i \in A_\flat$, and by Łoś' Theorem, $(f_1,\dots,f_s)A_\natural = A_\natural$. Since $A_\flat \to A_\natural$ is non-degenerated, this implies that $(f_1,\dots,f_s)A_\flat = A_\flat$. Choose $g_i \in A_\flat$ such that $f_1g_1 + \cdots + f_sg_s = 1$, and let m be large enough so that all g_i have of proto-grade at most m. For each i, choose an approximation g_{iw} of g_i of proto-grade at most m. Since by Łoś' Theorem,

$$f_{1w}g_{1w} + \cdots + f_{sw}g_{sw} = 1 \tag{9.3}$$

for almost all w, we get the desired contradiction for any of those $w > m$.

Conversely, suppose a uniform bound b as above exists for all pairs (n,s), and let I be an ideal in $A_\flat$ such that $IA_\natural = A_\natural$. We want to show that $I = A_\flat$. By assumption, there exist $f_1,\dots,f_s \in I$ and $h_1,\dots,h_s \in A_\natural$ such that $f_1h_1 + \cdots + f_sh_s = 1$. Choose n large enough so that all f_i have proto-grade at most n. Let f_{iw} be an approximation of f_i of proto-grade at most n. By Łoś' Theorem, $(f_{1w},\dots,f_{sw})A = A$ for almost all w, and hence by assumption, we can find g_{iw} of proto-grade at most b satisfying (9.3), for some b independent of w. By construction, the ultraproduct g_i of g_{iw} belongs to $A_\flat$, and by Łoś' Theorem, $f_1g_1 + \cdots + f_sg_s = 1$ in $A_\natural$, whence already in the subring $A_\flat$, as we wanted to show. □

For affine proto-gradings (recall that these are essentially given by the degree), being non-degenerated is already a very restrictive assumption as the next result shows (recall that a ring A is called *von Neumann regular* if for each non-zero x there exists a non-zero $y \in A$ such that $xy^2 = y$, and that this is equivalent with A being *absolutely flat*, meaning that any A-module is flat):

Theorem 9.2.8. *For a reduced ring A, the following are equivalent:*

9.2.8.i. *A is von Neumann regular;*
9.2.8.ii. *the A-affine proto-grading on $A[\xi]$ is non-degenerated, for all finite tuples ξ of indeterminates;*
9.2.8.iii. *every A-affine proto-grading is non-degenerated.*

Proof. The equivalence of (9.2.8.i) and (9.2.8.ii) is an immediate consequence of Theorem 9.2.7 and the characterization of von Neumann regularity given in the proof of [83, Proposition 5]. The equivalence of (9.2.8.ii) and (9.2.8.iii) follows by base change. □

In view of Proposition 9.1.8, condition (9.2.8.ii) is equivalent with the canonical map $A_\natural[\xi] \to B_\natural$ being non-degenerated, where $B_\natural$ is the ultrapower of $A[\xi]$. For a non-reduced example where this property holds, let A be an Artinian local ring: the embedding $A_\natural[\xi] \to B_\natural$ is then in fact faithfully flat by [98, Theorem 1.2] (note that $A_\natural$ is again an Artinian local ring). In our current terminology, any affine proto-grading over an Artinian local ring is Noetherian and faithfully flat.

A reduced Noetherian ring is von Neumann regular if and only if it is a direct sum of fields. In particular, the affine proto-grading on $A := \mathbb{Z}[\xi]$ is degenerated. This is exemplified by the following ideal: let $\omega := \operatorname{ulim}_{n\to\infty} n \in \mathbb{N}_\natural$, and let $I := (1-2\xi, 2^\omega)A_\flat$ (recall that $A_\flat = \mathbb{Z}_\natural[\xi]$ by Proposition 9.1.8). Since each $(1-2\xi, 2^n)A$ is the unit ideal, so is $IA_\natural$ by Łoś' Theorem. However, in order to write 1 as linear combination of $1-2\xi$ and 2^n, we require polynomials of degree at least n, namely,

$$(1-2\xi)(\sum_{i<n}(2\xi)^i) + 2^n(\xi^n) = 1,$$

and so I is proper ideal in $A_\flat$.

If we replace the condition of being non-degenerated in Theorem 9.2.7 by the stronger assumption that the proto-grading is cyclically pure, then virtually an identical proof yields:

Theorem 9.2.9. *A proto-grading on a ring A is cyclically pure if and only if for each pair (n,s), there exists a uniform bound $b := b(n,s)$ such that if $f_0,\dots,f_s$ are elements in A of proto-grade at most n, with f_0 in the ideal generated by the remaining f_i, then $f_0 = f_1g_1 + \dots + f_sg_s$ for some g_i of proto-grade at most b.*

Moreover, the bound m can be chosen independent from s if the proto-grading is cyclically pure and Noetherian. □

9.2.3 Flat Proto-gradings

The following theorem generalizes the results in §4.4.1.

Theorem 9.2.10. *For a proto-graded ring A, consider the following conditions:*

9.2.10.i. *for each n there exists a uniform bound n' such that if $I, J \subseteq A$ are ideals of proto-grade at most n, then their colon ideal $(I : J)$ is generated*

a. *by elements of proto-grade at most* n';
b. *by* n' *elements of proto-grade at most* n', *that is to say,* $(I:J)$ *has proto-grade at most* n';

9.2.10.ii. *for each triple* (n,s,m), *there exists a bound* $n'' := n''(n,s,m)$ *such that if* $\mathscr{L}$ *is a homogeneous linear system of* s *equations in* m *variables with coefficients of proto-grade at most* n, *then the* A*-module of solutions of* $\mathscr{L}$ *is generated*

a. *by solutions with entries of proto-grade at most* n'';
b. *by* n'' *solutions with entries of proto-grade at most* n'';

9.2.10.iii. *for each* n, *there exists a bound* n''' *with the property that if* I *is an ideal of proto-grade at most* n, *then its module of syzygies is generated*

a. *by syzygies with entries of proto-grade at most* n''';
b. *by* n''' *syzygies with entries of proto-grade at most* n'''.

If the proto-grading is flat, then (9.2.10.ia), (9.2.10.iia) *and* (9.2.10.iiia) *hold. If the proto-grading is moreover Noetherian, or more generally, coherent, then the proto-grading is flat if and only if* (9.2.10.iib) *holds if and only if* (9.2.10.iiib) *holds. If on the other hand, the proto-grading is cyclically pure, then* (9.2.10.ib) *holds if and only if the proto-grading is faithfully flat and coherent.*

Proof. Note that in view of ($\ddagger_{IB}$), conditions (9.2.10.iiia) and (9.2.10.iiib) are a special instance of (9.2.10.iia) and (9.2.10.iib) respectively, with $s = 1$ and $m = n$. We start by proving that flatness implies (9.2.10.ia). By way of contradiction, assume that for some n, no bound as asserted exists. Hence for each w, we can construct a counterexample consisting of two ideals I_w and J_w of proto-grade at most n, such that $(I_w : J_w)$ cannot be generated by elements of proto-grade at most w. In particular, there exists $f_w \in (I_w : J_w)$ not belonging to the ideal generated by all elements in $(I_w : J_w)$ of proto-grade at most w. Let $A_\flat$ and $A_\natural$ be the respective protopower and ultrapower of A. By 9.2.2, we can find finitely generated ideals $I, J \subseteq A_\flat$ (in fact, of proto-grade at most n), such that $IA_\natural$ and $JA_\natural$ are the respective ultraproducts of the I_w and J_w. By Łoś' Theorem, the ultraproduct $f \in A_\natural$ of the f_w belongs to $(IA_\natural : JA_\natural)$. By assumption, $A_\flat \to A_\natural$ is flat, so that $f \in (I:J)A_\natural$ by Theorem 3.3.14. Let $g_1, \dots, g_s \in (I:J)$ be such that f is a linear combination in $A_\natural$ of the g_i, and choose $N \geq n$ large enough so that all g_i have proto-grade at most N. Let g_{iw} be an approximation of g_i. Hence by Łoś' Theorem, almost each g_{iw} has proto-grade at most N and belongs to $(I_w : J_w)$. Moreover, almost each f_w is a linear combination of the g_{iw}, contradicting our assumption whenever $w > N$. If $A_\flat$ is coherent, then by Corollary 3.3.17, we may choose the g_i so that they generate $(I:J)$. In that case, any element in $(I_w : J_w)$ is a linear combination of the approximations g_{iw}, that is to say, has proto-grade at most N for almost all w, from which we can now derive (9.2.10.ib) by a similar ad absurdum argument. That flatness implies (9.2.10.iia) and, under the additional coherency assumption, (9.2.10.iib) are proven in the same way, using instead Theorem 3.3.1.

To prove that (9.2.10.iiib) yields flatness, we will verify the equational criterion for flatness as stated in Theorem 3.3.1. To this end, let $\mathbf{x}$ be a solution in $A_\natural$ of a linear equation $a_1t_1 + \cdots + a_st_s = 0$ with $a_i \in A_\flat$. Choose $n \geq s$ sufficiently large such that each a_i has proto-grade at most n, and choose approximations a_{iw} of each a_i of proto-grade at most n, and an approximation $\mathbf{x}_w$ of $\mathbf{x}$. By Łoś' Theorem, $\mathbf{x}_w$ is a solution of $a_{1w}t_1 + \cdots + a_{sw}t_s = 0$, and hence in view of ($\ddagger_{IB}$) and (9.2.10.iiib), there exists some n''' such that $\mathbf{x}_w$ is a linear combination of n''' solutions all of whose entries have proto-grade at most n'''. Hence the ultraproduct of these n''' solutions are solutions of $a_1t_1 + \cdots + a_st_s = 0$ in $A_\flat$, and $\mathbf{x}$ is an $A_\natural$-linear combination of these solutions, as we wanted to show.

So remains to show is that if (9.2.10.ib) holds and the proto-grading is cyclically pure, then it is also flat and coherent. To show that $A_\flat \to A_\natural$ is flat, we verify the colon criterion (Theorem 3.3.14). Let $I := (h_1, \dots, h_s)A_\flat$ be a finitely generated ideal, and let $a \in A_\flat$. We need to show that $(IA_\natural : a) = (I : a)A_\natural$. Choose $n \geq s$ so that both I and a have proto-grade at most n. Let $I_w := (h_{1w}, \dots, h_{sw})A$ and a_w be approximations of proto-grade at most n of I and a respectively. By (9.2.10.ib), almost all $(I_w : a_w)$ have proto-grade at most n', say, generated by the n' elements f_{iw} of proto-grade at most n'. By Theorem 9.2.9, there exists a bound n'' only depending on n' whence on n, and elements g_{ijw} of proto-grade at most n'' such that

$$a_w f_{iw} = g_{i1w}h_{1w} + \cdots + g_{isw}h_{sw}$$

for all $i = 1, \dots, n'$ and all w. Taking the respective ultraproducts of the f_{iw} and g_{ijw} yield elements f_i and g_{ij} in $A_\flat$. Moreover, by Łoś' Theorem, we have, for all i, an identity $af_i = g_{i1}h_1 + \cdots + g_{is}h_s$ in $A_\natural$, whence in the subring $A_\flat$. This shows that $f_i \in (I : a)$. On the other hand, an easy argument on Łoś' Theorem shows that $(IA_\natural : a) = (f_1, \dots, f_{n'})A_\natural$, from which it follows that $(IA_\natural : a) = (I : a)A_\natural$, as we wanted to show. This also shows that $(I : a)$ is finitely generated, from which it follows that $A_\flat$ is coherent by Corollary 3.3.17. □

Combining the previous results then easily yields:

Theorem 9.2.11. *Let A be a ring with a Noetherian, faithfully flat proto-grading. There exists, for each n, a uniform bound n', such that if $I = (f_1, \dots, f_s)A$ is an ideal of proto-grade at most n, then the module of syzygies of I is generated by n' syzygies with entries of proto-grade at most n'. Moreover, if $f \in I$ and has proto-grade at most n, then there exist g_i of proto-grade at most n' such that $f = g_1f_1 + \cdots + f_sg_s$.* □

We conclude this section with a uniform version of Krull's Intersection Theorem (Theorem 2.4.14).

Theorem 9.2.12. *Let $(R, \mathfrak{m})$ be a local ring with a faithfully flat, Noetherian proto-grading. For each n, there exists a uniform bound $e := e(n)$ with the property that for any $f_0, \dots, f_s$ of proto-grade at most n, if f_0 lies in $(f_1, \dots, f_s)R + \mathfrak{m}^e$ then f_0 lies already in $(f_1, \dots, f_s)R$.*

Proof. Suppose that for some n, no such bound exists, so that for each w, we can find a counterexample consisting of an ideal I_w generated by elements of proto-grade at most n and an element f_w of proto-grade at most n, so that f_w lies in $I_w + \mathfrak{m}^w$ but not in I_w. By Corollary 9.2.4, the I_w are generated by at most n' elements of proto-grade at most n', for some n' only depending on n. By 9.2.2, there exists $I \subseteq R_\flat$ such that $IR_\natural$ is the ultraproduct of the I_w. By Łoś' Theorem, the ultraproduct f of the f_w does not belong to $IR_\natural$, whence a fortiori $f \notin I$. On the other hand, by Łoś' Theorem, $f \in IR_\natural + \mathfrak{m}^N R_\natural$ for every N. By assumption $R_\flat \to R_\natural$ is faithfully flat, so that f belongs to $I + \mathfrak{m}^N R_\flat$, for all N. Since $R_\flat$ is Noetherian, Krull's Intersection Theorem (Theorem 2.4.14) yields $f \in I$, contradiction. □

9.3 Proto-gradings Over the Integers

In this section, we will discuss briefly the existence of some bounds over the integers originally due to Seidenberg (for instance, the bound proven in Corollary 9.3.1 below is shown to be actually doubly exponential in [103]), with improved bounds given by Aschenbrenner (the same bound is proven to be polynomial in [4]). More precisely, let $A := \mathbb{Z}[\xi]$, viewed in its Kronecker proto-grading. Since $A = A_\flat \subseteq A_\natural$ is faithfully flat by Corollary 3.3.3, the Kronecker proto-grading is faithfully flat, and therefore an application of Theorem 9.2.11 yields:

Corollary 9.3.1. *There exists for each pair (m,n) a uniform bound $b := b(m,n)$, such that if $I = (f_1, \dots, f_s)A$, with $A = \mathbb{Z}[\xi_1, \dots, \xi_m]$, is an ideal of Kronecker proto-grade at most n (that is to say, generated by n polynomials of degree at most n with coefficients of absolute value at most n), then the module of syzygies of I is generated by b syzygies with entries of Kronecker proto-grade at most b. Moreover, if f has Kronecker proto-grade at most n and belongs to I, then there exist g_i of Kronecker proto-grade at most b such that $f = g_1 f_1 + \dots + f_s g_s$.* □

We already argued that the above is false if we take the degree proto-grading on $\mathbb{Z}[\xi]$, since this proto-grading fails to be non-degenerated. However, Aschenbrenner observed that (9.2.10.iib) holds in this case, proving that the degree proto-grading is flat. I will give here an independent, direct proof of flatness, and hence via (9.2.10.iib), recover Aschenbrenner's result. We prove this in greater generality, as we will need this later to prove Theorem 10.2.2.

Theorem 9.3.2. *Let Z_w be principal ideal domains with ultraproduct equal to $Z_\natural$. If $A_\flat$ and $A_\natural$ are respectively the protoproduct and ultraproduct of the $A_w := Z_w[\xi]$, then the canonical map $A_\flat \to A_\natural$ is flat.*

Proof. By Proposition 9.1.8, the protoproduct $A_\flat$ is equal to $Z_\natural[\xi]$, so that we have to show that $Z_\natural[\xi] \to A_\natural$ is flat. We will do this by means of the Tor criterion (Theorem 3.1.5), that is to say, by showing that

$$T := \operatorname{Tor}_1^{A_\flat}(A_\natural, A_\flat / I)$$

vanishes for every finitely generated ideal $I \subseteq A_\flat$. Towards a contradiction, suppose that τ is a non-zero element in T. Let Q_w be the field of fractions of Z_w, and let $Q_\natural$ be their ultraproduct, so that by Łoś' Theorem, it is the field of fractions of $Z_\natural$. Viewing each $B_w := Q_w[\xi]$ in its affine proto-grading, their protoproduct $B_\flat$ is equal to $Q_\natural[\xi] = A_\flat \otimes_{Z_\natural} Q_\natural$. Moreover, since any polynomial in B_w is of the form af with $a \in Q_w$ and $f \in A_w$, the ultraproduct $B_\natural$ of the B_w is equal to $A_\natural \otimes_{Z_\natural} Q_\natural$. Since $B_\flat \to B_\natural$ is faithfully flat by Theorem 4.2.2, and since this is just the base change of $A_\flat \to A_\natural$ over $Q_\natural$ by our previous calculations, we get $T \otimes_{Z_\natural} Q_\natural = 0$. Therefore, there exists some non-zero $a \in Z_\natural$ such that $a\tau = 0$ in T. Since all Z_w are Dedekind domains, $Z_\natural$ is a Prüfer domain by Proposition 2.4.18. By [83, Proposition 3] or [34, Corollary 7.3.4], the polynomial ring $A_\flat = Z_\natural[\xi]$ is therefore coherent. In particular, I has a finitely generated module of syzygies, and hence there exists an exact sequence

$$A_\flat^m \xrightarrow{d_2} A_\flat^n \xrightarrow{d_1} A_\flat \to A_\flat/I \to 0.$$

By definition of Tor-modules (see §3.1.4), we can calculate T as the homology of the tensored complex

$$A_\natural^m \xrightarrow{d_2} A_\natural^n \xrightarrow{d_1} A_\natural.$$

In particular, τ is the image of a tuple $\mathbf{x} \in A_\natural^n$ such that $d_1(\mathbf{x}) = 0$. Moreover, $\mathbf{x}$ does not belong to $\mathrm{Im}(d_2)$, but $a\mathbf{x}$ does. Let $a_w \in Z_w$, $\mathbf{x}_w \in A_w^n$ and d_{iw} be approximations of a, $\mathbf{x}$ and d_i respectively, yielding for almost all w a complex

$$A_w^m \xrightarrow{d_{1w}} A_w^n \xrightarrow{d_{2w}} A_w.$$

By Theorem 3.1.1, almost each $\mathbf{x}_w$ lies in the kernel of d_{1w} and not in the image of d_{2w}, whereas $a_w\mathbf{x}_w$ does lie in that image. Since each Z_w is a unique factorization domain, we can find tuples $\mathbf{y}_w$ and prime elements $p_w \in Z_w$, such that almost each $\mathbf{y}_w$ lies in the kernel of d_{1w} but not in the image of d_{2w}, yet $p_w\mathbf{y}_w$ does. Let $\pi \in Z_\natural$ and $\mathbf{y} \in A_\natural^n$ be the respective ultraproducts of the p_w and the $\mathbf{y}_w$. Since $\mathbf{y}$ lies in the kernel of d_1 but outside the image of d_2 by Łoś' Theorem, its image in T is a non-zero element, annihilated by π.

On the other hand, since each Z_w has dimension one, Z_w/p_wZ_w is a field, whence so is their ultraproduct $Z_\natural/\pi Z_\natural$. Therefore, the base change $A_\flat/\pi A_\flat \to A_\natural/\pi A_\natural$ is faithfully flat by Theorem 4.2.2. Since π is $A_\natural$-regular by Łoś' Theorem, we get a short exact sequence

$$0 \to A_\natural \xrightarrow{\pi} A_\natural \to A_\natural/\pi A_\natural \to 0 \tag{9.4}$$

and a degenerated spectral sequence

$$\begin{aligned}\mathrm{Tor}_{i-1}^{A_\flat/\pi A_\flat}(A_\natural/\pi A_\natural, A_\flat/(I:\pi)) \to \mathrm{Tor}_i^{A_\flat}(A_\natural/\pi A_\natural, A_\flat/I)\\ \to \mathrm{Tor}_i^{A_\flat/\pi A_\flat}(A_\natural/\pi A_\natural, A_\flat/(I+\pi A_\flat))\end{aligned}$$

For $i = 2$, the two outer modules are zero by the flatness of $A_\flat/\pi A_\flat \to A_\natural/\pi A_\natural$, whence so is the inner module. Therefore, the relevant part of the long exact Tor sequence (3.1.4) associated to (9.4) becomes

$$0 = \mathrm{Tor}_2^{A_\flat}(A_\natural/\pi A_\natural, A_\flat/I) \to T \xrightarrow{\pi} T$$

showing that π is T-regular, contradicting the fact that $\pi\mathbf{y} = 0$ in T. □

The present proof seems to require that Z is a unique factorization domain, but perhaps this can be circumvented by using [98, Theorem 2] instead of Theorem 4.2.2, so that we may derive Corollary 9.3.3 for any Dedekind domain, or even for any Prüfer domain, thus recovering the result in [5, Theorem A]. In view of Theorem 9.2.10, the previous result applied in the ultrapower case, yields:

Corollary 9.3.3. *Let Z be a principal ideal domain and ξ a finite tuple of indeterminates. For each n, there exists a uniform bound n' such that*

9.3.3.i. *the module of syzygies of any ideal in $Z[\xi]$ generated by polynomials of degree at most n, is generated by n' many tuples all of whose entries have degree at most n';*

9.3.3.ii. *the $Z[\xi]$-module of solutions of any homogeneous linear system over $Z[\xi]$ of at most n equations in at most n variables with coefficients of proto-grade at most n, is generated by n' solutions with entries of proto-grade at most n'.*

□

Similarly, Theorem 9.3.2 applied with all Z_w equal to $\mathbb{Z}$ together with Theorem 9.2.10 yields:

Corollary 9.3.4. *In $\mathbb{Z}[\xi]$, the module of syzygies of a tuple of polynomials of degree at most n is generated by n' tuples whose entries have degree at most n', for some n' only depending on n and the number of indeterminates.* □

9.4 Prime Bounded Proto-gradings

Let A be a proto-graded ring, with protopower $A_\flat$ and ultrapower $A_\natural$. Since the proto-grading may fail to be cyclically pure, not every ideal of $A_\flat$ is the contraction of an ideal of $A_\natural$. Among the contracted ideals in $A_\flat$, the following class is particularly nice:

Definition 9.4.1. An ideal $\mathfrak{a} \subseteq A_\flat$ is called *finitary* if it is of the form $IA_\natural \cap A_\flat$ for some finitely generated ideal $I \subseteq A_\flat$.

Note that a finitary ideal need not be finitely generated. If the proto-grading is cyclically pure and Noetherian, then any ideal in the protopower is finitary.

Any finitely generated ideal $I \subseteq A_\flat$ admits an *approximation* $I_w \subseteq A$, that is to say, ideals whose ultraproduct is equal to $IA_\natural$. We can extend this construction to finitary ideals of the form $\mathfrak{a} := IA_\natural \cap A_\flat$ with $I \subseteq A_\flat$ finitely generated, by defining an approximation I_w of $\mathfrak{a}$ to be any approximation of I. This makes sense since $\mathfrak{a}A_\natural = IA_\natural$. Note that for some n, almost all I_w have proto-grade at most n. This construction also admits the following converse:

Definition 9.4.2. Given an n and a collection of ideals $I_w \subseteq A$ of proto-grade at most n, we call $I_\flat := I_\natural \cap A_\flat$ their *protoproduct*, where $I_\natural$ is the ultraproduct of the I_w.

Note that $I_\flat$ is finitary by 9.2.2, with approximation I_w, and that $I_\flat A_\natural = I_\natural$. Unlike the ideal given by 9.2.2, the protoproduct is uniquely determined by the I_w. The next definition generalizes the uniform primality results obtained previously in the case of a polynomial ring over a field (see §4.4.2).

Definition 9.4.3. We call the proto-grading on A *prime bounded* if the extension of any finitary prime ideal of $A_\flat$ remains prime in $A_\natural$.

An easy example of a prime bounded proto-grading is the Kronecker proto-grading on $\mathbb{Z}[\xi]$: since the protopower is then just $\mathbb{Z}[\xi]$ (by Proposition 9.1.8), the result follows from the general fact that any prime ideal in a ring remains prime in its ultrapower. In view of Theorem 4.3.4, the degree proto-grading on a polynomial ring over a field is prime bounded. For an example where it is necessary that $\mathfrak{p}$ be finitary, see the example following Corollary 9.4.9. The property of being prime bounded is again characterized by a certain uniformity result:

Theorem 9.4.4 (Uniform Primality). *For a proto-graded ring A, the following are equivalent:*

9.4.4.i. *The proto-grading is prime bounded;*
9.4.4.ii. *for each n, there exists a uniform bound n' with the following property: given an ideal I of proto-grade at most n, the ideal is prime if and only if for any two elements f and g of proto-grade at most n', if both do not belong to I, then neither does their product.*

Proof. Suppose the proto-grading is prime bounded, but no bound as in (9.4.4.ii) exists. Hence for some n, we can find non-prime ideals $I_w \subseteq A$ of proto-grade at most n, having the property that if a product of two elements of proto-grade at most w belongs to I_w, then already one of them belongs to I_w. Let $\mathfrak{a} := I_\flat$ be the protoproduct of the I_w, that is to say, let $I_\natural$ be the ultraproduct of the I_w, and put $\mathfrak{a} := I_\natural \cap A_\flat$. I claim that $\mathfrak{a}$ is prime. Indeed, suppose we have elements $f, g \in A_\flat$ such that $fg \in \mathfrak{a}$. Choose n' large enough so that f and g have both proto-grade at most n'. Let f_w, g_w be respective approximations of f and g of proto-grade at most n'. Since $fg \in I_\natural$, almost each $f_w g_w$ lies in I_w. For those w which are also bigger than n', we then have by assumption that one of the two, say f_w, belongs to I_w. It follows that f lies in $I_\natural$, whence in $\mathfrak{a}$, proving the claim. By definition of prime boundedness, $\mathfrak{a}A_\natural = I_\natural$ is then also a prime ideal. However, since the latter ideal is the ultraproduct of the I_w, almost all of these ideals must be prime ideals by Łoś' Theorem, contradiction.

Conversely, suppose a bound to test primality as asserted in (9.4.4.ii) exists and let $\mathfrak{p}$ be a finitary prime ideal in $A_\flat$. We want to show that $\mathfrak{p}A_\natural$ is also prime. Let $\mathfrak{p}_w \subseteq A$ be an approximation of $\mathfrak{p}$. If almost no $\mathfrak{p}_w$ is prime, then, since almost all have uniformly bounded proto-grade, there exists by (9.4.4.ii), some n' and elements $f_w, g_w \in A$ of proto-grade at most n' not belonging to $\mathfrak{p}_w$ but whose product does. If f, g are their respective ultraproducts, then f and g already lie in $A_\flat$. Moreover, by Łoś' Theorem, f and g do not belong to $\mathfrak{p}A_\natural$ but their product does. Since $\mathfrak{p}$ is finitary, it is equal to $\mathfrak{p}A_\natural \cap A_\flat$, and hence $fg \in \mathfrak{p}$. Therefore, at least one among f or g belongs to the prime ideal $\mathfrak{p}$, contradiction. Hence almost all $\mathfrak{p}_w$ must be prime ideals, whence so is their ultraproduct $\mathfrak{p}A_\natural$, as we wanted to show. □

Theorem 9.4.5. *Let A be a ring with a faithfully flat, prime bounded, Noetherian proto-grading, then there exists for each n a uniform bound $e := e(n)$, such that for any choice of elements $a_1, \ldots, a_s$ of proto-grade at most n, the ideal $I := (a_1, \ldots, a_s)A$ has at most e minimal prime ideals, each of proto-grade at most e, its radical* $\operatorname{rad} I$ *has proto-grade at most e, and* $(\operatorname{rad} I)^e$ *lies inside I.*

Proof. We will prove all properties simultaneously. Assume no bound exists for some n, so that we can construct, after an application of Corollary 9.2.4, for each w, a counterexample I_w of proto-grade at most n with radical J_w, so that, respectively, J_w cannot be realized as the intersection of w prime ideals of proto-grade at most w, or has proto-grade at least w, or $(J_w)^w$ is not contained in I_w. Let $\mathfrak{a}$ be the protoproduct of the I_w, and let $\mathfrak{b}$ be its radical. Since the protopower $A_\flat$ is by assumption Noetherian, we can find some e such that $\mathfrak{b}^e \subseteq \mathfrak{a}$. From the inclusions $\mathfrak{b}^e A_\natural \subseteq \mathfrak{a}A_\natural \subseteq \mathfrak{b}A_\natural$, we conclude that both $\mathfrak{a}A_\natural$ and $\mathfrak{b}A_\natural$ have the same radical. Let $\mathfrak{p}_i$ be the (finitely many) minimal prime ideals of $\mathfrak{a}$. Since $\mathfrak{b} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s$ and $A_\flat \to A_\natural$ is flat by assumption, [69, Theorem 7.4] yields

$$\mathfrak{b}A_\natural = \mathfrak{p}_1 A_\natural \cap \cdots \cap \mathfrak{p}_s A_\natural. \tag{9.5}$$

Furthermore, each $\mathfrak{p}_i$ is finitary (since by faithful flatness it is equal to $\mathfrak{p}_i A_\natural \cap A_\flat$), and hence by prime boundedness, $\mathfrak{p}_i A_\natural$ is again a prime ideal. In particular, (9.5) shows that the extended ideal $\mathfrak{b}A_\natural$ is also radical.

Let $\mathfrak{b}_w$ and $\mathfrak{p}_{iw}$ be approximations of $\mathfrak{b}$ and $\mathfrak{p}_i$ respectively. By Łoś' Theorem, $\mathfrak{b}_w = \mathfrak{p}_{1w} \cap \cdots \cap \mathfrak{p}_{sw}$, almost all $\mathfrak{p}_{iw}$ are prime, and almost all $\mathfrak{b}_w$ are radical. Moreover, by Łoś' Theorem, $\mathfrak{b}_w^e \subseteq I_w \subseteq \mathfrak{b}_w$, for almost all w. This shows that almost each $\mathfrak{b}_w$ is equal to the radical J_w of I_w and the $\mathfrak{p}_{iw}$ are minimal prime ideals of I_w, contradicting either of our assumptions. □

The next result provides a larger class of prime bounded proto-gradings, to which we therefore may apply the previous results.

Theorem 9.4.6. *Let Z_w be domains of dimension at most d, and let $Z_\natural$ be their ultraproduct. Let ξ be a finite tuple of indeterminates and let $A_\natural$ be the ultraproduct of the $A_w := Z_w[\xi]$. Let $I \subseteq Z_\natural[\xi]$ be a finitely generated ideal. If $\mathfrak{p} := IA_\natural \cap Z_\natural[\xi]$ is prime, then so is $\mathfrak{p}A_\natural$.*

In particular, the affine proto-grade on a polynomial ring over a finite dimensional domain Z is prime bounded.

Proof. Since the A_w are uniformly proto-graded, their protoproduct $A_\flat$ is equal to $Z_\natural[\xi]$ by Proposition 9.1.8. In particular, the last assertion follows from the first, by taking $Z_w = Z$ for all w. To prove the first, we induct on d, where the case $d = 0$ is trivial. Let $\mathfrak{p}$ be as in the statement, that is to say, a finitary prime ideal of $A_\flat$, and let $\mathfrak{q} := \mathfrak{p} \cap Z_\natural$. Choose an approximation $\mathfrak{p}_w \subseteq A_w$ of $\mathfrak{p}$, and let $\mathfrak{q}_w := \mathfrak{p}_w \cap Z_w$. By Łoś' Theorem, the ultraproduct of the $\mathfrak{q}_w$ is equal to

$$\mathfrak{p}A_\natural \cap Z_\natural = (\mathfrak{p}A_\natural \cap A_\flat) \cap Z_\natural = \mathfrak{p} \cap Z_\natural = \mathfrak{q}.$$

Assume first that $\mathfrak{q}$ is zero, whence, by Łoś' Theorem, so are almost all $\mathfrak{q}_w$. Let $Q_\natural$ be the field of fractions of $Z_\natural$. In other words, $Q_\natural$ is the ultraproduct of the Q_w, where Q_w is the field of fractions of Z_w. Since $\mathfrak{p} \cap Z_\natural = (0)$, the extended ideal $\mathfrak{p}Q_\natural[\xi]$ is also prime. Let $B_\natural$ be the ultrapower of the polynomial ring $Q[\xi]$. By Theorem 4.3.4, the extension $\mathfrak{p}B_\natural$ remains prime. Hence we are done in this case if we can show that

$$\mathfrak{p}A_\natural = \mathfrak{p}B_\natural \cap A_\natural.$$

To this end, let f be in the right hand side and let $f_w \in A_w$ be an approximation of f. It follows that almost each f_w lies in $\mathfrak{p}_w Q_w[\xi]$. Hence, for some non-zero $s_w \in Z_w$, we have $s_w f_w \in \mathfrak{p}_w$ for almost all w. Since s_w cannot belong to $\mathfrak{p}_w$, as almost all $\mathfrak{q}_w$ are zero, we must have $f_w \in \mathfrak{p}_w$, and therefore $f \in \mathfrak{p}A_\natural$, as we wanted to show.

In the remaining case, almost all $\bar{Z}_w := Z_w/\mathfrak{q}_w$ have dimension strictly less than d. Since $A_\natural/\mathfrak{q}A_\natural$ is then the ultraproduct of the polynomial rings $\bar{Z}_w[\xi]$, our induction hypothesis yields that $\mathfrak{p}(A_\natural/\mathfrak{q}A_\natural)$ is prime, whence so is $\mathfrak{p}A_\natural$ as we wanted to show. □

It is worthwhile to formulate the application of Theorem 9.4.4 to the second part of Theorem 9.4.6 separately:

Theorem 9.4.7. *For every finite-dimensional domain A, for every finite tuple of indeterminates ξ, and for every positive integer n, there exists a uniform bound n' with the following property: given an ideal $\mathfrak{p} \subseteq A[\xi]$ generated by n polynomials of degree at most n, if for any two polynomials of degree at most n' outside $\mathfrak{p}$, neither does their product belong to $\mathfrak{p}$, then $\mathfrak{p}$ is prime.* □

Lemma 9.4.8. *Let A be a faithfully flat, Noetherian proto-graded ring with protopower $A_\flat$. Any associated prime ideal of $A_\flat$ is the extension of an associated prime ideal of A, whence its extension to the ultrapower $A_\natural$ remains prime.*

Proof. Let $\mathfrak{p}$ be an associated prime of $A_\flat$, say, $\mathfrak{p} = \mathrm{Ann}_{A_\flat}(a)$. Let $a_w \in A$ be an approximation of a, so that almost all a_w have proto-grade at most n, for some n. By Łoś' Theorem, almost all $\mathrm{Ann}_A(a_w)$ are non-zero. Since A is Noetherian, it has only finitely many associated primes. Since any minimal prime of a non-zero $\mathrm{Ann}_A(a_w)$ is an associated prime of A, we conclude that there exists an associated prime $\mathfrak{q}$ of A which is minimal over almost all $\mathrm{Ann}_A(a_w)$. Remains to show that $\mathfrak{p} = \mathfrak{q}A_\flat$.

By (9.2.10.ib), there is a uniform bound n' only depending on n', so that each $(\mathrm{Ann}_A(a_w) : \mathfrak{q})$ has proto-grade at most n'. Choose an element s_w of this colon ideal, of proto-grade at most n' and outside $\mathrm{Ann}_A(a_w)$. Hence, if s is the ultraproduct of the s_w, then $s \in A_\flat$, and by Łoś' Theorem, $sa \neq 0$, so that $s \notin \mathfrak{p}$. For $y \in \mathfrak{q}$, we have $s_w a_w y = 0$, so that by Łoś' Theorem, $say = 0$, that is to say, $sy \in \mathfrak{p}$. Since $s \notin \mathfrak{p}$, we get $y \in \mathfrak{p}$, showing that $\mathfrak{q}A_\flat \subseteq \mathfrak{p}$. To prove the other inclusion, let $x \in \mathfrak{p}$ and choose $x_w \in A$ so that their ultraproduct is x. Since $ax = 0$, Łoś' Theorem yields $a_w x_w = 0$ for almost all w. In particular, $x_w \in \mathrm{Ann}_A(a_w) \subseteq \mathfrak{q}$, so that by another application of Łoś' Theorem, $x \in \mathfrak{q}A_\natural$. Finally, by faithful flatness of $A_\flat \to A_\natural$, we get $x \in \mathfrak{q}A_\flat$. The final assertion is now immediate from Łoś' Theorem, since $\mathfrak{q}A_\natural$ is the ultrapower of $\mathfrak{q}$, whence prime.

Since the maximal ideal of the protopower of a Noetherian local ring is finitary and extends to the maximal ideal of the ultrapower, by Lemma 9.1.5, we get immediately from Lemma 9.4.8:

Corollary 9.4.9. *Any faithfully flat, Noetherian proto-grade on a one-dimensional Noetherian local ring is prime bounded.* □

Here is an example of a prime bounded proto-grading which is degenerated: the affine proto-grading on $\mathbb{Z}[\xi]$ with ξ a single variable. Namely, in $\mathbb{Z}_\natural[\xi]$, let $\mathfrak{p}$ be the ideal generated by $1-2\xi$ and the intersection of all powers $2^n\mathbb{Z}_\natural[\xi]$. One checks that $\mathfrak{p}$ is a prime ideal, but its extension to the ultrapower $A_\natural$ of $\mathbb{Z}[\xi]$ is the unit ideal. In particular, this implies that $\mathfrak{p}$ cannot be a finitary ideal.

We finish this section with a uniform elimination result. To this end, we must first prove some form of transfer result:

Proposition 9.4.10. *Let A have a Noetherian, faithfully flat, prime bounded proto-grading and let $\mathfrak{a} \subseteq A_\flat$ be an ideal in the protopower, with approximation $\mathfrak{a}_w \subseteq A$.*

9.4.10.i. $\mathfrak{a}$ *is prime (radical) if and only if almost all* $\mathfrak{a}_w$ *are prime (radical).*
9.4.10.ii. $\mathfrak{a}$ *has height h if and only if almost all* $\mathfrak{a}_w$ *have height h.*

Proof. Note that since the proto-grading is Noetherian and faithfully flat, $\mathfrak{a}$ is finitary. If $\mathfrak{a}$ is prime, then so is $\mathfrak{a}A_\natural$ by prime boundedness, and hence by Łoś' Theorem, so are almost all $\mathfrak{a}_w$. Conversely, if almost all $\mathfrak{a}_w$ are prime, then so is their ultraproduct $\mathfrak{a}A_\natural$, whence so is $\mathfrak{a} = \mathfrak{a}A_\natural \cap A_\flat$.

If $\mathfrak{a}$ is radical, then it can be written as $\mathfrak{a} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s$ with all $\mathfrak{p}_i$ prime ideals in $A_\flat$. Choose an approximation $\mathfrak{p}_{iw}$ of the $\mathfrak{p}_i$. Since

$$\mathfrak{a}A_\natural = \mathfrak{p}_1 A_\natural \cap \cdots \cap \mathfrak{p}_s A_\natural$$

by flatness and [69, Theorem 7.4], Łoś' Theorem yields $\mathfrak{a}_w = \mathfrak{p}_{1w} \cap \cdots \cap \mathfrak{p}_{sw}$, for almost all w. By the previous case, almost all $\mathfrak{p}_{iw}$ are prime, showing that almost all $\mathfrak{a}_w$ are radical. Conversely, suppose almost all $\mathfrak{a}_w$ are radical and $a^n \in \mathfrak{a}$. Choose $a_w \in R$ whose ultraproduct is a. By Łoś' Theorem, $\mathfrak{a}_w$ contains a_w^n, whence a_w, for almost all w. Hence a lies in $\mathfrak{a}A_\natural$, and hence by faithful flatness, in $\mathfrak{a}$, showing that $\mathfrak{a}$ is radical.

To prove (9.4.10.ii), assume first that $\mathfrak{a}$ is prime, and hence, by the previous argument, so are then almost all $\mathfrak{a}_w$. We induct on the height h of $\mathfrak{a}$. If $h = 0$, then $\mathfrak{a}$ is a minimal prime of $A_\flat$ and hence the extension of a minimal prime of A by Lemma 9.4.8. It follows that almost all $\mathfrak{a}_w$ are equal to this minimal prime, and the assertion is clear in this case. So assume $h > 0$ and choose a height $h-1$ prime ideal $\mathfrak{p}$ inside $\mathfrak{a}$. If $\mathfrak{p}_w$ is an approximation of $\mathfrak{p}$, then by induction almost all are height $h-1$ prime ideals. By Łoś' Theorem, $\mathfrak{p}_w \subsetneq \mathfrak{a}_w$, so that almost all $\mathfrak{a}_w$ have height at least h. Choose some a in $\mathfrak{a}$ but not in $\mathfrak{p}$ and let $a_w \in A$ be an approximation. In particular, $\mathfrak{a}$ is a minimal prime of $\mathfrak{p} + aA_\flat$. Let $\mathfrak{g}_w$ be a minimal prime ideal of $\mathfrak{p}_w + a_wA$ contained in $\mathfrak{a}_w$ (note that by Łoś' Theorem, a_w lies in $\mathfrak{a}_w$, for almost all w). By the Krull's Principal Ideal Theorem, almost all $\mathfrak{g}_w$ have height h. Choose n sufficiently large so that $\mathfrak{a}$ and a both have proto-grade at most n, whence so do almost all $\mathfrak{a}_w$ and a_w. By Theorem 9.4.5, therefore, almost all $\mathfrak{g}_w$ have proto-grade at most n', for some n' depending only on n. Let $\mathfrak{g}$ be their protoproduct, that is to say, the ideal $\mathfrak{g}_\natural \cap A_\flat$, where $\mathfrak{g}_\natural$ is the ultraproduct of the $\mathfrak{g}_w$. By (9.4.10.i), the ideal $\mathfrak{g}$ is prime. By Łoś' Theorem and faithful flatness, $\mathfrak{p} + aA_\flat \subseteq \mathfrak{g} \subseteq \mathfrak{a}$, so that $\mathfrak{g}$ and $\mathfrak{a}$, both being minimal prime ideals of $\mathfrak{p} + aA_\natural$, must be equal. Hence also almost all $\mathfrak{g}_w = \mathfrak{a}_w$ are equal, whence have height h. This proves (9.4.10.ii) for $\mathfrak{a}$ a prime ideal.

Assume finally that $\mathfrak{a}$ is arbitrary, of height h, and let $\mathfrak{p}$ be a minimal prime of $\mathfrak{a}$, with approximation $\mathfrak{p}_w$. By Łoś' Theorem and what we already established, $\mathfrak{p}_w$ is a height h prime ideal containing $\mathfrak{a}_w$, for almost all w. It follows that almost each $\mathfrak{a}_w$ has height at most h. If almost all $\mathfrak{a}_w$ would have height less than h, then we can choose for each w, a minimal prime $\mathfrak{g}_w$ of $\mathfrak{a}_w$ of that height and of proto-grade at most n, for some n independent from w, by Theorem 9.4.5, so that by the same argument as before, the protoproduct of these $\mathfrak{g}_w$ would be a prime ideal of height less than h containing $\mathfrak{a}$, contradiction. □

Proposition 9.4.11. *Let A have a Noetherian, flat proto-grading. For each n, there exists n' such that if I and J are ideals of proto-grade at most n, then $I \cap J$ has proto-grade at most n'.*

Proof. Suppose the assertion is false for some n, so that we can construct counterexamples consisting, for each w, of two ideals $I_w, J_w \subseteq A$ of proto-grade at most n such that $I_w \cap J_w$ has proto-grade at least w. Let $I_\flat, J_\flat \subseteq A_\flat$ and $I_\natural, J_\natural \subseteq A_\natural$ be the respective protoproducts and ultraproducts of $I_w, J_w \subseteq A$. Since $A_\flat$ is by assumption Noetherian, there exist finitely many $f_i \in A_\flat$ such that $I_\flat \cap J_\flat = (f_1, \dots, f_s)A_\flat$. Let n be the maximum of the proto-grades of the f_i and choose, for each i, an approximation $f_{iw} \in A$ of f_i of proto-grade at most n. Since $A_\flat \to A_\natural$ is flat, [69, Theorem 7.4] yields

$$(f_1,\dots,f_s)A_\natural = I_\flat A_\natural \cap J_\flat A_\natural = I_\natural \cap J_\natural$$

and hence by Łoś' Theorem, $(f_{1w},\dots,f_{sw})A_w = I_w \cap J_w$, contradicting our assumptions for $w > n, s$. □

Theorem 9.4.12 (Uniform Elimination). *Let A have a Noetherian, faithfully flat, prime bounded proto-grading. Let ξ be a finite tuple of indeterminates and view $A[\xi]$ in its extended degree proto-grading. For each n, there exists a uniform bound n' so that any radical ideal $\mathfrak{P} \subseteq A[\xi]$ of proto-grade at most n contracts to an ideal $\mathfrak{P} \cap A$ of proto-grade at most n'.*

Proof. By induction on the number of indeterminates, we only need to treat the case that ξ is a single variable. Since any radical ideal is the intersection of prime ideals, and their number is bounded by Theorem 9.4.5, we may by Proposition 9.4.11 reduce to the case that $\mathfrak{P}$ is a prime ideal. Suppose that for some n, no bound as claimed exists. Hence we can find prime ideals $\mathfrak{P}_w \subseteq A[\xi]$ of proto-grade at most n such that $\mathfrak{g}_w := \mathfrak{P}_w \cap A$ has proto-grade at least w. Let $B_\natural$ denote the ultrapower of $A[\xi]$. Recall that the protopower of $A[\xi]$ is equal to $A_\flat[\xi]$ by Proposition 9.1.8. Let $\mathfrak{P}_\natural$ and $\mathfrak{P} := \mathfrak{P}_\natural \cap A_\flat[\xi]$ be the respective ultraproduct and protoproduct of the $\mathfrak{P}_w$ and put $\mathfrak{p} := \mathfrak{P} \cap A_\flat$. Whence both $\mathfrak{P}$ and $\mathfrak{p}$ are prime ideals. Choose an approximation $\mathfrak{p}_w \subseteq A$ of $\mathfrak{p}$. It follows from Proposition 9.4 and Łoś' Theorem that almost each $\mathfrak{p}_w$ is a prime ideal contained in $\mathfrak{g}_w$, of proto-grade at most n, for some n independent of w.

If $\mathfrak{g}_\natural$ is the ultraproduct of the $\mathfrak{g}_w$, then by Łoś' Theorem,

$$\mathfrak{g}_\natural = \mathfrak{P} B_\natural \cap A_\natural. \tag{9.6}$$

Let h be the height of $\mathfrak{P}$. By [27, Exercise 10.2], the contraction $\mathfrak{p}$ has height h if and only if $\mathfrak{P} = \mathfrak{p} A_\flat[\xi]$. Assuming that this is the case, we have by (9.6) that $\mathfrak{g}_\natural = \mathfrak{p} B_\natural \cap A_\natural$. However, by Łoś' Theorem, this means that $\mathfrak{g}_w = \mathfrak{p}_w A[\xi] \cap A = \mathfrak{p}_w$, contradicting that the $\mathfrak{g}_w$ have unbounded proto-grade.

The remaining possibility is for $\mathfrak{p}$ to have height $h-1$. By Proposition 9.4, then so have almost all $\mathfrak{p}_w$. We already observed that $\mathfrak{p}_w \neq \mathfrak{g}_w$, and so $\mathfrak{g}_w$ has height at least h. On the other hand, almost all $\mathfrak{P}_w$ have height h by Proposition 9.4, so that by another application of [27, Exercise 10.2], we have $\mathfrak{P}_w = \mathfrak{g}_w A[\xi]$. It follows that almost all $\mathfrak{g}_w$ then have proto-grade at most n, contradiction. □

Chapter 10
Asymptotic Homological Conjectures in Mixed Characteristic

In this final chapter, we discuss some of the homological conjectures. Although now theorems in equal characteristic, many remain conjectures in mixed characteristic. Whereas there may be no consensus as to which conjectures count as 'homological', an extensive list of them together with their interconnections, can be found in Hochster's authoritative treatise [43].

We already encountered one of these conjectures—and proved it in equal characteristic; see Theorems 6.4.10 and 7.4.5—, when discussing big Cohen-Macaulay algebras: the Monomial Conjecture. In fact, Hochster has established most of the homological conjectures in equal characteristic by means of the existence of big Cohen-Macaulay modules. Hence probably the 'mother' of all homological conjectures in mixed characteristic is the very existence of a (balanced) big Cohen-Macaulay module (or, preferably, algebra); the best result to date is the existence of these up to dimension three (see [46], based upon the positive solution of the Direct Summand Conjecture in mixed characteristic in dimension three due to Heitmann [40]).

The ultraproduct method is a priori—but see §10.3.3—insufficiently powerful to derive the full versions of these conjectures from their equal characteristic counterparts. As the idea is to transfer the proven theorems in equal characteristic to the mixed characteristic case via ultraproducts, but as properties only hold almost everywhere on the approximations, we will only be able to deduce 'asymptotic' versions. This roughly amounts to the conjecture holding for a particular ring of mixed characteristic provided its residue characteristic is sufficiently large with respect to some other invariants associated to the particular problem. The first successful implementation of this strategy goes back to the work of Ax-Kochen ([9]), in which they solve a conjecture of Artin over the p-adics (see Theorem 10.1.3 below). Using the same method, I derived asymptotic versions of various homological conjectures in mixed characteristic in [91, 97], where the lower bounds on the residue characteristic are in terms of the degrees of the polynomials defining the data. In the terminology of these notes, this is in essence a protoproduct method, and will be discussed in §10.2. However, as in [101], using cataproducts instead, lower bounds in terms of much more natural invariants (dimension, multiplicity, etc.) can be derived, and this will be discussed in §10.3. We conclude with a discussion how our asymptotic bounds can even

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lead to a solution of the full conjectures. Since most of the arguments are outside the scope of these notes, we will most of the time only discuss the method, and leave the details of the proofs to the cited sources.

10.1 The Ax-Kochen-Eršhov Principle

10.1.1 Ax-Kochen-Eršhov Principle

One normally states this model-theoretic principle in terms of valued fields, but for our purposes, it is more natural to phrase it as a certain Lefschetz Principle for discrete valuation rings, formulated as an isomorphism of certain ultra-discrete valuation rings (recall that the latter are simply ultraproducts of discrete valuation rings; see Proposition 2.4.19). In this formalism, the principle states:

Theorem 10.1.1 ([9,29,30]). *If V and V' are two Henselian ultra-discrete valuation rings of the same uncountable cardinality with isomorphic residue fields of characteristic zero, then $V \cong V'$.*

We will use this principle in the following form. For each p, let V_p be a complete discrete valuation ring of mixed characteristic, with residue field k_p of characteristic p. Let ξ be a single indeterminate, and put

$$V_p^{\mathrm{eq}} := k_p[[\xi]] \tag{10.1}$$

We have:

Corollary 10.1.2. *The ultraproduct of all V_p is isomorphic to that of all V_p^{eq}.*

Proof. As stated, one might need to assume the Continuum Hypothesis, but this can be avoided by taking an ultraproduct with respect to a larger underlying set than just the prime numbers. All we need is that the ultraproduct $V_\natural$ of the V_p has the same cardinality as the ultraproduct $W_\natural$ of the V_p^{eq}, and so we will for sake of simplicity just assume this. Since the residue field of both $V_\natural$ and $W_\natural$ is the field of characteristic zero $k_\natural$, given as the ultraproduct of the k_p, the desired isomorphism now follows immediately from Proposition 2.4.19 and Theorem 10.1.1. □

10.1.2 Artin's Problem

A field K is called C_2 if for every homogeneous polynomial $f(\xi) \in K[\xi]$ of degree d in more than d^2 variables ξ, there exists a non-trivial solution in K. Lang proved in [64] that the field of fractions of $\mathbb{F}_p[[\xi]]$ is C_2, and Artin conjectured that the field of p-adics $\mathbb{Q}_p$ too is C_2. However, some counterexamples to the latter conjecture were found, and the optimal result is now:

Theorem 10.1.3. *For each d, there is a bound d' so that if p is a prime number bigger than d', then any homogeneous equation of degree d in more than d^2 variables has a non-trivial solution in $\mathbb{Q}_p$.*

Proof. The existence of a non-trivial solution of $f = 0$ in $\mathbb{Q}_p$ yields after clearing denominators a non-trivial solution in ring of p-adic integers, $\mathbb{Z}_p$. By Corollary 10.1.2, the ultraproduct of all $\mathbb{Z}_p$ is equal to the ultraproduct of the $\mathbb{F}_p[[\xi]]$. Since the assertion can be formulated by a first-order sentence (depending on d), which holds for all $\mathbb{F}_p[[\xi]]$, it holds in their ultraproduct, whence in almost all $\mathbb{Z}_p$, by a double application of Łoś' Theorem (here, one needs the full, model-theoretic version, Theorem 2.3.2). This shows that the exceptional set of prime numbers for this fixed d must lie outside any ultrafilter. Since any infinite set belongs to at least one ultrafilter, this exceptional set of prime numbers must be finite. □

10.2 Asymptotic Homological Conjectures via Protoproducts

The extent to which Artin's question has been answered is indicative of what follows: the truth of a certain property can only be established for sufficiently large p, depending on the complexity of the data. This is best described using the formalism of proto-gradings from Chapter 9.

10.2.1 Affine Proto-grade

We will work inside the class $\mathfrak{A}_{\text{DVR}}$ of local affine algebras over a complete discrete valuation ring. More precisely, a local ring $(R,\mathfrak{m})$ belongs to $\mathfrak{A}_{\text{DVR}}$, if its a local V-affine algebra with V a complete discrete valuation ring. Recall that this means that R is a localization of a finitely generated V-algebra with respect to a prime ideal containing the maximal ideal of V. We will view R with its V-affine proto-grading. For instance, if R is the localization of $V[\xi]$ at the maximal ideal generated by the uniformizing parameter of V and the indeterminates ξ, then $\Gamma_n(R)$ consists of all fractions f/g with $f,g \in V[\xi]$ of degree at most n and $g(0)$ a unit in V.

Although any sequence of rings in $\mathfrak{A}_{\text{DVR}}$ is uniformly proto-graded, and hence their protoproduct is well-defined, we cannot expect in general for it to capture much of the information stored in the sequence. For instance, let S be the localization of $V[\xi,\zeta]$ at the maximal ideal generated by the indeterminates and the uniformizing parameter of the discrete valuation ring V, let $f_n := \xi^n - \zeta^{n-1}$, and put $R_n := S/f_nS$. By 9.1.6, the protoproduct $R_\flat$ of the R_n is isomorphic to $S_\flat/fS_\natural \cap S_\flat$, where $S_\flat$ is the protopower of S and f the ultraproduct of the f_n. Since no multiple of f has finite degree, whence finite proto-grade, $fS_\natural \cap S_\flat = 0$, showing that $S_\flat = R_\flat$. Of course, what goes wrong in this example is that the f_n have unbounded proto-grade.

As we will shall see shortly, we can avoid this phenomenon by introducing the following terminology. For an arbitrary member R of $\mathfrak{A}_{\mathrm{DVR}}$, say of the form $(V[\xi]/I)_{\mathfrak{p}}$, we say that R itself has *affine proto-grade* at most n, if the number of indeterminates ξ is at most n, and both I and $\mathfrak{p}$ have proto-grade at most n (recall that the latter means that they are generated by at most n elements of degree at most n).[1] There is some ambiguity here in our definition of the affine proto-grade of a ring, because different affine presentations might yield different values. However, since we are only interested in uniform behavior, this will not matter. Depending on the situation, we will also make explicit what it means for some additional data to have bounded affine proto-grade. For instance, a homomorphism $\varphi\colon R \to S$ of rings in $\mathfrak{A}_{\mathrm{DVR}}$ has *proto-grade at most n*, if both R and S are of the form $(V[\xi]/I)_{\mathfrak{p}}$ and $(W[\zeta]/J)_{\mathfrak{q}}$, with V and W complete discrete valuation rings, with ξ and ζ tuples of indeterminates of size at most n, and with I, J, $\mathfrak{p}$ and $\mathfrak{q}$ ideals (with the latter two prime) of proto-grade at most n, such that $R \to S$ is induced by a homomorphism $V[\xi] \to W[\zeta]$ sending V inside W, and each ξ to a polynomial of degree at most n. Put differently, there is a homomorphism $V \to W$ which induces a homomorphism $R' := R \otimes_V W \to S$ making S into a local affine R'-algebra of proto-grade at most n. In particular, if $R_w \to S_w$ are morphisms in $\mathfrak{A}_{\mathrm{DVR}}$ of proto-grade at most n, then under the induced map among the ultraproducts $R_\natural \to S_\natural$, the image of the protoproduct $R_\flat$ of the R_w lies inside the protoproduct $S_\flat$ of the S_w. In other words, we showed:

10.2.1 *Given morphisms $R_w \to S_w$ in $\mathfrak{A}_{DVR}$ of proto-grade at most n, they induce a homomorphism $R_\flat \to S_\flat$ between the protoproducts.* □

10.2.2 Approximations and Transfer

The method to derive asymptotic properties is a mixture of the methods from Chapter 4, using ultra-hulls, and Chapter 9, using protoproducts. Crucial to either method in deriving bounds was a certain flatness result, which in the present context becomes:

Theorem 10.2.2. *If R_w are members of $\mathfrak{A}_{DVR}$ of affine proto-grade at most n, for some n, then their protoproduct $R_\flat$ is a local $V_\natural$-affine algebra, with $V_\natural$ an ultra-discrete valuation ring, and the canonical map $R_\flat \to R_\natural$ to the ultraproduct is faithfully flat.*

Proof. Since $R_\flat \to R_\natural$ is by construction local, we only need to show its flatness. Since this is a local question, I claim that we may reduce to the following case: for each w, let V_w be a complete discrete valuation ring, and let $A_w := V_w[\xi]$, viewed with its affine proto-grading, then the natural homomorphism $A_\flat \to A_\natural$ is flat. Indeed, assuming this flatness result, let $I_w \subseteq \mathfrak{p}_w$ have affine proto-grade at most n, so that $R_w = (A_w/I_w)_{\mathfrak{p}_w}$. By 9.2.2, there exist ideals $I \subseteq \mathfrak{p} \subseteq A_\flat$ whose extension

[1] In the articles [91, 97] the affine proto-grade of R was called its *(degree) complexity*.

to $A_\natural$ are the ultraproducts of respectively the I_w and $\mathfrak{p}_w$. Since each $\mathfrak{p}_w$ contains the uniformizing parameter π_w of V_w, an application of Theorem 4.3.4 over the fields $V_w/\pi_w V_w$ yields that $\mathfrak{p}$ is prime. By base change, $(A_\flat)_\mathfrak{p} \to (A_\natural)_{\mathfrak{p}A_\natural}$ is then faithfully flat. By 9.1.6, the protoproduct of the A_w/I_w is equal to $A_\flat/(IA_\natural \cap A_\flat)$. Since

$$I(A_\natural)_{\mathfrak{p}A_\natural} \cap (A_\flat)_\mathfrak{p} = I(A_\flat)_\mathfrak{p}$$

by faithful flatness, we showed that $R_\flat = (A_\flat/I)_\mathfrak{p}$. Since $R_\natural = (A_\natural/IA_\natural)_{\mathfrak{p}A_\natural}$, flatness follows by base change. Note that $A_\flat = V_\natural[\xi]$ by Proposition 9.1.8, where $V_\natural$ is the ultraproduct of the V_w, proving that $R_\flat$ is a local $V_\natural$-affine algebra.

So remains to show that $A_\flat \to A_\natural$ is flat, and this follows from Theorem 9.3.2. □

As we have observed before, the converse process of constructing an ultraproduct is taking approximations. This also applies here. Let as above V_w be complete discrete valuation rings with ultraproduct $V_\natural$. Given a local $V_\natural$-affine algebra $R := (V_\natural[\xi]/I)_\mathfrak{p}$, with ξ a finite tuple of indeterminates, I a finitely generated ideal, and $\mathfrak{p}$ a prime ideal lying above the maximal ideal of $V_\natural$ and containing I, we define the approximations of R as follows. Put $A_w := V_w[\xi]$, and let $I_w \subseteq \mathfrak{p}_w \subseteq A_w$ be respective approximations of I and $\mathfrak{p}$. By the argument in the above proof, almost each $\mathfrak{p}_w$ is prime. Moreover, if R has affine proto-grade at most n, then almost all I_w and $\mathfrak{p}_w$ have proto-grade at most n. Hence almost all $R_w := (V_w[\xi]/I_w)_{\mathfrak{p}_w}$ are well-defined members of $\mathfrak{A}_{\mathrm{DVR}}$ and have proto-grade at most n. It is easy to check that their protoproduct $R_\flat$ is equal to R. We will therefore refer to the R_w as *approximations* of R. These approximations, however, depend on the choice of components V_w of $V_\natural$, a fact that has to be borne in mind.

We can also look at this construction from an ultra-hull perspective as follows. In this point of view, the ultraproduct $R_\natural$ of the R_w functions as an ultra-hull of R, called the *ultra-$V_\natural$-hull* of R. By Theorem 10.2.2, this ultra-hull is faithfully flat. Note that $V_\natural$ is no longer Noetherian, which will account for some of the difficulties below in developing the theory, but it is still a valuation ring by Proposition 2.4.19, of embedding dimension one since its maximal ideal is generated by the ultraproduct π of the uniformizing parameters π_w. In particular, $V_\natural/\pi V_\natural$ is a field whence Noetherian, and hence any ideal in the above protoproduct $R_\flat$ containing π is finitely generated. This applies in particular to the maximal ideal of the protoproduct, showing that the protoproduct has finite embedding dimension.

10.2.3 *Let $R_\flat$ and $R_\natural$ respectively be the protoproduct and ultraproduct of local affine V_w-algebras R_w of affine proto-grade at most n, and let π be the ultraproduct of the uniformizing parameters of the V_w. Then $R_\natural/\pi R_\natural$ is the ultra-hull, in the sense of §4.1, of $R_\flat/\pi R_\flat$.*

Indeed, we established in the proof of Theorem 10.2.2 that $R_\flat$ is a local $V_\natural$-affine algebra of the form $(V_\natural[\xi]/I)_\mathfrak{p}$, where $V_\natural$ is the ultraproduct of the V_w and $\mathfrak{p}$ a prime ideal containing π. Hence $R_\flat/\pi R_\flat$ is a local $k_\natural$-affine algebra where

$k_\natural := V_\natural/\pi V_\natural$ is the ultraproduct of the residue fields k_w of the V_w. From the construction of I and $\mathfrak{p}$, it follows that the $R_w/\pi_w R_w$ are approximations of $R_\flat/\pi R_\flat$, and since their ultraproduct is equal to $R_\natural/\pi R_\natural$, the latter is the ultra-hull of $R_\flat/\pi R_\flat$. □

10.2.3 *Equal and Mixed Characteristic Approximations*

Let us specialize to the case we will encounter shortly. For each prime number p, let V_p be a complete discrete valuation ring of mixed characteristic with residue characteristic p, and let $V_\natural$ be their ultraproduct. Let R_p be a local V_p-affine algebra of affine proto-grade at most n, and let $R_\flat$ be their protoproduct. As we just proved, $R_\flat$ is a local $V_\natural$-affine algebra with approximations R_p and ultra-$V_\natural$-hull $R_\natural$. By Corollary 10.1.2, we may realize $V_\natural$ also as the ultraproduct of the complete discrete valuation rings V_p^{eq} of equal characteristic p (see (10.1)). Hence from this point of view, $R_\flat$ has approximations defined over the various V_p^{eq}, which we therefore denote by R_p^{eq}, and call *equal characteristic approximations* of the R_p (note that they have also bounded affine proto-grade). From this point of view, we then also may refer to the original R_p as *mixed characteristic approximations* of $R_\flat$. The ultraproduct of the R_p^{eq} will be denoted $R_\natural^{\mathrm{eq}}$. To distinguish it from $R_\natural$, we will call $R_\natural^{\mathrm{eq}}$ the *equal characteristic ultra-hull* of $R_\flat$, and $R_\natural$ its *mixed characteristic ultra-hull*. Similarly, if $x \in R_\flat$, then an *equal characteristic approximation* of x means an approximation of x viewed as an element in $R_\natural^{\mathrm{eq}}$, that is to say, elements $x_p \in R_p^{\mathrm{eq}}$ with ultraproduct equal to x. Since by construction $R_\flat$ is also the protoproduct of the R_p^{eq}, the canonical embedding $R_\flat \to R_\natural^{\mathrm{eq}}$ is again faithfully flat by Theorem 10.2.2. Of course, we may also reverse the process, going from equal to mixed characteristic instead.

The fact that both ultra-hulls are faithfully flat over the (common) protoproduct will guarantee a fair amount of transfer between the R_p and their equal characteristic approximations. The following result is but an example of this.

Theorem 10.2.4. *For some n and for each prime number p, let R_p be a local V_p-affine algebra of affine proto-grade at most n over a mixed characteristic complete discrete valuation ring V_p of residue characteristic p, and let R_p^{eq} be an equal characteristic approximation of the R_p. Almost all R_p are regular if and only if almost all R_p^{eq} are regular.*

Proof. Because of symmetry, it suffices to show only one direction, and so we may assume that almost all R_p are regular. Since by assumption each R_p has embedding dimension less than or equal to its affine proto-grade n, there are only finitely many possibilities for its dimension, and hence almost all R_p will have the same dimension, say, d. Let $\mathbf{x}_p$ be a regular system of parameters of R_p. Note that in particular each $\mathbf{x}_p$ is an R_p-regular sequence (of length d) minimally generating the maximal ideal. By construction, we may choose its entries to be of proto-grade (=degree) at most n, for all p. Hence their ultraproduct $\mathbf{x} := (x_1, \dots, x_d)$

has entries in the protoproduct $R_\flat$. By faithful flatness, moreover, $\mathbf{x}R_\flat = \mathbf{x}R_\natural \cap R_\flat$ is the maximal ideal of $R_\flat$. Let R_p^{eq} be equal characteristic approximations of the R_p, let $R_\natural$ and $R_\natural^{\mathrm{eq}}$ be the respective ultraproducts of R_p and R_p^{eq}, and let $\mathbf{x}_p^{\mathrm{eq}} := (x_{1p}^{\mathrm{eq}}, \dots, x_{dp}^{\mathrm{eq}})$ be equal characteristic approximations of $\mathbf{x}$, that is to say, tuples in R_p^{eq} whose ultraproduct is equal to $\mathbf{x}$, where we view the latter as a tuple over $R_\natural^{\mathrm{eq}}$ via the canonical embedding $R_\flat \to R_\natural^{\mathrm{eq}}$. By Łoś' Theorem, almost each $\mathbf{x}_p^{\mathrm{eq}}$ generates the maximal ideal of R_p^{eq}. So remains to show that almost each $\mathbf{x}_p^{\mathrm{eq}}$ is a regular sequence, whence a regular system of parameters. Fix some $i \le d$, let $I_p^{\mathrm{eq}} := (x_{1p}^{\mathrm{eq}}, \dots, x_{i-1,p}^{\mathrm{eq}})R_p^{\mathrm{eq}}$, and, towards a contradiction, assume $z_p^{\mathrm{eq}} x_{ip}^{\mathrm{eq}} \in I_p^{\mathrm{eq}}$, for some z_p^{eq} not in I_p^{eq}. Let $I := (x_1, \dots, x_{i-1})R_\flat$. Since $\mathbf{x}$ is an $R_\natural$-regular sequence by Łoś' Theorem, $(IR_\natural : x_i) = IR_\natural$. Since $R_\flat \to R_\natural$ is faithfully flat by Theorem 10.2.2, we also have $(IR_\natural : x_i) = (I : x_i)R_\natural$ by Theorem 3.3.14. Faithful flatness then yields $(I : x_i) = I$. On the other hand, by Łoś' Theorem, the ultraproduct $z \in R_\natural^{\mathrm{eq}}$ of the z_p^{eq} belongs to $(IR_\natural^{\mathrm{eq}} : x_i)$ but not to $IR_\natural^{\mathrm{eq}}$. Since also $R_\flat \to R_\natural^{\mathrm{eq}}$ is faithfully flat, another application of Theorem 3.3.14 yields $(IR_\natural^{\mathrm{eq}} : x_i) = (I : x_i)R_\natural^{\mathrm{eq}}$, which is then equal to $IR_\natural^{\mathrm{eq}}$ by what we just proved. Hence, z lies in $IR_\natural^{\mathrm{eq}}$, contradiction. □

To formulate analogous transfer results for arbitrary rings, we have to also face the complications encountered in Chapter 8, where the ultraproduct (and hence the protoproduct and the equal characteristic approximations) may have larger geometric dimension than the components. To control this bad behavior, one has to also bound the parameter degree. In the present setup, this is in fact easier, as the below domain case (Proposition 10.2.6) shows; we refer to [97, §6] for a discussion of the general case. A second complication arises from the fact that protoproducts are hardly ever Noetherian. The same obstacle for ultraproducts was overcome by passing to the cataproduct, that is to say, by taking the separated quotient, and so we will follow the same strategy. This leads to the following notion:

10.2.4 Cataprotoproducts

Let R_w be V_w-algebras in $\mathfrak{A}_{\mathrm{DVR}}$ of proto-grade at most n and let $R_\flat$ be their protoproduct. We define their *cataprotoproduct* as the separated quotient $R_{\flat\sharp} := R_\flat/\mathfrak{I}_{R_\flat}$.

10.2.5 *The cataprotoproduct $R_{\flat\sharp}$ of V_w-algebras of bounded proto-grade R_w is a Noetherian local ring, equal to $R_\flat/\mathfrak{I}_{V_\natural}R_\flat$, where $V_\natural$ is the ultraproduct of the V_w. Moreover, if $R_\flat$ is a domain, then $R_{\flat\sharp}$ has the same dimension as almost all R_w.*

Indeed, by Proposition 2.4.19, the cataproduct $V_\sharp := V_\natural/\mathfrak{I}_{V_\natural}$ of the discrete valuation rings V_w is a discrete valuation ring, whence in particular Noetherian. Since $R_\flat/\mathfrak{I}_{V_\natural}R_\flat$ is finitely generated over $V_\sharp$, it too is Noetherian. Hence, $\mathfrak{I}_{V_\natural}R_\flat = \mathfrak{I}_{R_\flat}$ by Theorem 2.4.14 and $R_{\flat\sharp}$ is Noetherian. To calculate its dimension in case $R_\flat$ is a domain, note that π is $R_\natural$-regular since $R_\flat \to R_\natural$ is flat by Theorem 10.2.2, where π is the ultraproduct of the uniformizing parameters π_w of the V_w. By Łoś' Theorem, almost all π_w are therefore R_w-regular. Let us show that π is also $R_{\flat\sharp}$-regular: if $x \in R_\flat$ is such that $\pi x \in \mathfrak{I}_{R_\flat} = \mathfrak{I}_{V_\natural} R_\flat$, then for each n, almost all $\pi_w x_w \in \pi_w^n R_w$, where $x_w \in R_w$ are approximations of x of bounded proto-grade. Since almost each π_w is R_w-regular, $x_w \in \pi_w^{n-1} R_w$, for every n, and hence, by Łoś' Theorem, $x \in \mathfrak{I}_{R_\natural}$. Since $x \in R_\flat$ and $R_\flat \to R_\natural$ is flat, we get $x \in \mathfrak{I}_{R_\flat}$, showing that π is $R_{\flat\sharp}$-regular.

Suppose almost all R_w have dimension d, whence almost all $R_w/\pi_w R_w$ have dimension $d-1$. By 10.2.3, the $R_w/\pi_w R_w$ are approximations of $R_\flat/\pi R_\flat \cong R_{\flat\sharp}/\pi R_{\flat\sharp}$, and therefore the latter also has dimension $d-1$ by Corollary 4.3.3. Since π is $R_{\flat\sharp}$-regular, $R_{\flat\sharp}$ has therefore also dimension d. □

Proposition 10.2.6. *For some n and for each prime number p, let R_p be a local V_p-affine algebra of affine proto-grade at most n over a mixed characteristic complete discrete valuation ring V_p of residue characteristic p, and let R_p^{eq} be an equal characteristic approximation of the R_p. Then almost all R_p are domains if and only if almost all R_p^{eq} are. Moreover, if this is the case, then almost all R_p and R_p^{eq} have the same dimension.*

Proof. The second assertion follows from the first and 10.2.5 since both approximations have the same protoproduct whence cataprotoproduct. So, it suffices to show that the protoproduct $R_\flat$ is a domain if and only if its approximations (of either type) are. Since $R_\flat \subseteq R_\natural$, one direction in the equivalence is immediate. So assume $R_\flat$ is a domain, and we need to show that then so is $R_\natural$. Let $V_\natural$ be the ultraproduct of the V_p, let $\pi \in V_\natural$ be a generator of its maximal ideal with approximations $\pi_p \in V_p$, and let $Q_\natural$ be the field of fractions of $V_\natural$. If $\pi = 0$ in $R_\flat$, then $R_\flat$ is in fact a local affine algebra over the residue field $V_\natural/\pi V_\natural$, and hence $R_\natural$ is a domain by Theorem 4.3.4. Moreover, the R_p are then the approximations of $R_\flat$, whence almost all have dimension d by Corollary 4.3.3. So we may assume $\pi \neq 0$ in $R_\flat$. Since $R_\flat$ is a domain, π is then $R_\flat$-regular. I claim that $R_\flat$ is therefore torsion-free over $V_\natural$. Indeed, suppose $x \in R_\flat$ is annihilated by some non-unit $a \in V_\natural$. Let $a_p \in V_p$ and $x_p \in R_p$ be approximations of a and x respectively. By Łoś' Theorem, almost each a_p is a non-unit and annihilates x_p. Hence, $\pi_p x_p = 0$, which in turn by Łoś' Theorem implies that $\pi x = 0$. Since π is $R_\flat$-regular, we get $x = 0$, as claimed.

In particular, $R_\flat \otimes_{V_\natural} Q_\natural$ is again a domain. By Theorem 4.3.4 once more, $R_\natural \otimes_{V_\natural} Q_\natural$ being the ultra-hull of $R_\flat \otimes_{V_\natural} Q_\natural$, is a domain too. On the other hand, since $R_\flat \to R_\natural$ is faithfully flat, $R_\natural$ is torsion-free over $V_\natural$ by Theorem 3.3.14. Hence the natural map $R_\natural \to R_\natural \otimes_{V_\natural} Q_\natural$ is injective, showing that $R_\natural$ is a domain. □

As a further illustration of the connection between mixed and equal characteristic approximations, we prove:

10.2.7 *With notation as before, the cataproduct of the R_p is isomorphic to the cataproduct of the equal characteristic approximations R_p^{eq}. In fact, their cataproduct is equal to the completion of their protoproduct.*

It suffices to prove the second assertion, so let $R_\flat$, $R_{\flat\sharp}$, $R_\natural$ and $R_\sharp$ be the equal characteristic chromatic products of the R_p. The map $R_\flat \to R_\natural$ induces by base change a map $R_{\flat\sharp} \to R_\sharp$, leading to a chromatic square

$$\begin{array}{ccc} R_\flat & \longrightarrow & R_\natural \\ \big\downarrow & & \big\downarrow \\ R_{\flat\sharp} & \longrightarrow & R_\sharp. \end{array} \tag{10.2}$$

At the cost of adding some extra variables representing the generators of its maximal ideal—which at most doubles the affine proto-grade—we may write each R_p as the localization of $V_p[\xi_1,\dots,\xi_s]/I_p$ with respect to the prime ideal generated by π_p and the first i variables $\xi_1,\dots,\xi_i$, for some ideal I_p of affine proto-grade at most n. Replacing V_p then by the discrete valuation ring given as the completion of $V_p[\xi_{i+1},\dots,\xi_s]$ with respect to the ideal generated by π_p, we may assume that $i = s$. By a similar calculation as in Example 8.1.6, the cataproduct $R_\sharp$ is equal to $V_\sharp[[\xi]]/IV_\sharp[[\xi]]$, where $I \subseteq V_\natural[\xi]$ is the protoproduct of the I_p. On the other hand, $R_\flat$ is the localization of $V_\natural[\xi]/I$ at the ideal generated by π and the variables, and hence the cataprotoproduct $R_{\flat\sharp}$ is the corresponding localization of $V_\sharp[\xi]/IV_\sharp[\xi]$, showing that the completion of the latter is equal to $R_\sharp$. Since $R_\flat$ and $R_{\flat\sharp}$ have the same completion, the result follows. □

10.2.5 *Asymptotic Direct Summand Conjecture*

Let $\mathscr{P}$ be a property of Noetherian local rings and some additional finite amount of data (to be made precise in each case). In the terminology of affine proto-grade (see §10.2.1), we can now define what it means for a property to hold asymptotically.

Definition 10.2.8. We will say that property $\mathscr{P}$ *holds asymptotically in mixed characteristic*, if for each n, there exists n' only depending on n, such that a Noetherian local ring of mixed characteristic R in $\mathfrak{A}_{\mathrm{DVR}}$ satisfies $\mathscr{P}$ provided its residue characteristic p is at least n', where R and the additional data have affine proto-grade at most n.

We will illustrate this terminology by means of the Direct Summand Conjecture, which states that given a finite extension of local rings $R \to S$, if R is regular, then R is a direct summand of the R-module S (the reader should convince him/herself that this is weaker than saying that $R \to S$ is split). The Direct Summand Conjecture is related to another of the homological conjectures, to wit the Monomial Conjecture, which we already encountered before (see Theorems 6.4.10 and 7.4.5, and also 10.2.13 below):

Theorem 10.2.9 (Direct Summand Conjecture). *If S is a Noetherian local ring for which the Monomial Conjecture holds, then for any finite extension $R \subseteq S$ with R regular, R is a direct summand of S.*

In particular, the Direct Summand Conjecture holds for any Noetherian local ring of equal characteristic.

Proof. The second assertion follows from the first, in view of Theorem 7.4.5. To prove the first, one shows that R is a direct summand of S if and only if some regular system of parameters of R is monomial when viewed as a tuple in S; see [17, Lemma 9.2.2]. □

In mixed characteristic, the Direct Summand Conjecture is still wide open, but we can now show:

Theorem 10.2.10. *The Direct Summand Conjecture holds asymptotically in mixed characteristic.*

Proof. Let us be more precise as to the exact statement. There exists for each n a uniform bound n' with the following property. Let V be a complete discrete valuation ring of mixed characteristic, let R and S be local V-affine algebras of affine proto-grade at most n, such that $R \subseteq S$ is a finite extension and R is regular. Now, if the residue characteristic of R is at least n', then R is a direct summand of S. Here we must view the affine proto-grade of S via its presentation as a finite R-module. In order to prove this, we suppose by way of contradiction that no such bound exists for n. Hence we can find for each prime number p a counterexample consisting of a complete discrete valuation ring V_p of residue characteristic p, and local V_p-affine algebras $R_p \subseteq S_p$ of affine proto-grade at most n with R_p regular and S_p finitely generated as an R_p-module, such that R_p is not a direct summand of S_p. Let $R_\flat \subseteq S_\flat$ be the respective protoproducts. It is not hard to show that this is again a finite extension. Let $V_\natural$ be the ultraproduct of the V_p, so that by the above discussion $R_\flat$ and $S_\flat$ are local $V_\natural$-affine algebras. Let R_p^{eq} and S_p^{eq} be the equal characteristic approximations of the R_p and S_p respectively, and let $R_\natural^{\mathrm{eq}}$ and $S_\natural^{\mathrm{eq}}$ be their respective ultraproducts. By Theorem 10.2.4, almost all R_p^{eq} are regular. Moreover, it is not hard to show that $R_p^{\mathrm{eq}} \to S_p^{\mathrm{eq}}$ is a finite extension for almost all p. By Theorem 10.2.9, almost each R_p^{eq} is a direct summand of S_p^{eq}, and by Łoś' Theorem, this in turn implies that $R_\natural^{\mathrm{eq}}$ is a direct summand of $S_\natural^{\mathrm{eq}}$. By faithful flatness (Theorem 10.2.2), this then yields that $R_\flat$ is a direct summand of $S_\flat$, whence $R_\natural$ is a direct summand of $S_\natural$, and by Łoś' Theorem, we finally arrive at the contradiction that almost each R_p is a direct summand of S_p. □

10.2.6 *The Asymptotic Weak Monomial Conjecture and Big Cohen-Macaulay Algebras*

To prove an asymptotic version of the Monomial Conjecture, we introduce the following terminology. Let $(R,\mathfrak{m})$ be a local ring of finite embedding dimension. Recall that the *Monomial Conjecture* holds in R if every system of parameters is monomial, that is to say, satisfies (6.9) (see the discussion preceding Theorem 6.4.10). By a *strong system of parameters* $\mathbf{x}$ in R, we mean a system of parameters of R which is also part of a minimal system of generators of $\mathfrak{m}$. In other words, if R has geometric dimension d and embedding dimension e, then a d-tuple $\mathbf{x}$ is a strong system of parameters if and only if $R/\mathbf{x}R$ is an Artinian local ring of embedding dimension $e-d$. We say that the *weak Monomial Conjecture* holds in R if some strong system of parameters is monomial. Even this weak version is not known to hold in general for Noetherian local rings of mixed characteristic (the monomial systems of parameters given by Remark 6.4.11 never generate the maximal ideal when $t>1$).

Before we prove an asymptotic version of this weaker conjecture, we must introduce big Cohen-Macaulay algebras in the present setup. We call an R-algebra B a *big Cohen-Macaulay algebra* if some system of parameters is a B-regular sequence, and a *balanced big Cohen-Macaulay algebra* if any system of parameters is B-regular. The Monomial Conjecture holds in any local ring admitting a balanced big Cohen-Macaulay algebra by the argument in proof of Theorem 6.4.10. In the course of our proof, we also will encounter an 'ultra' version of this conjecture: we say that a tuple $(x_1,\dots,x_d)$ in an ultra-Noetherian local ring R is *ultra-monomial*, if

$$(x_1\cdots x_d)^{\alpha-1}\notin(x_1^{\alpha},\dots,x_d^{\alpha})R$$

for every positive ultra-integer $\alpha\in\mathbb{N}_\natural$ (see §2.4.4 for the definition of ultra-exponentiation). We say that the *weak ultra-Monomial Conjecture* holds in R, if R admits a strong system of parameters $\mathbf{x}$ which is ultra-monomial; and the *ultra-Monomial Conjecture* holds in R, if any system of parameters is ultra-monomial.

Proposition 10.2.11. *For some n and for each prime number p, let R_p be a local V_p-affine domain of affine proto-grade at most n over a mixed characteristic complete discrete valuation ring V_p of residue characteristic p. Then the protoproduct $R_\flat$ of the R_p admits a balanced big Cohen-Macaulay algebra $B(R_\flat)$. In particular, the Monomial Conjecture holds in $R_\flat$.*

Furthermore, $B(R_\natural):=B(R_\flat)\otimes_{R_\flat}R_\natural$ is a big Cohen-Macaulay algebra over the ultraproduct $R_\natural$ of the R_p, and the weak ultra-Monomial Conjecture holds in $R_\natural$.

Proof. Let R_p^{eq} be equal characteristic approximations of the R_p. By Proposition 10.2.6, almost all R_p^{eq} are domains, and hence almost each $(R_p^{\mathrm{eq}})^+$ is a balanced big Cohen-Macaulay algebra by Theorem 6.4.1. Let $B(R_\flat)$ be the ultraproduct of the $(R_p^{\mathrm{eq}})^+$. Let d be the geometric dimension of $R_\flat$, and let $\mathbf{x}$ be a system of parameters in $R_\flat$ with equal characteristic approximation $\mathbf{x}_p^{\mathrm{eq}}$ (so that each $\mathbf{x}_p^{\mathrm{eq}}$ is a d-tuple in R_p^{eq}). By Proposition 10.2.6 (and using the same argument as in the proof of Corollary 4.3.8), almost all R_p^{eq} as well as $R_\natural$ have geometric dimension d. In particular, $\mathbf{x}$ is also a system of parameters in $R_\natural$, which is strong if and only if the original system is. Furthermore, by 2.4.11, almost each $\mathbf{x}_p^{\mathrm{eq}}$ is a system of parameters in R_p^{eq}, whence $(R_p^{\mathrm{eq}})^+$-regular. By Łoś' Theorem, $\mathbf{x}$ is therefore $B(R_\flat)$-regular, as we wanted to show. The second assertion follows since any (permutable) regular sequence is monomial.

Since $R_\flat \to R_\natural$ is flat by Theorem 10.2.2, so is the base change $B(R_\flat) \to B(R_\natural)$. Since $\mathbf{x}$ is $B(R_\flat)$-regular, it is also $B(R_\natural)$-regular by Proposition 3.2.9. This proves that $B(R_\natural)$ is a big Cohen-Macaulay algebra over $R_\natural$. It is not clear whether this is in fact a *balanced* big Cohen-Macaulay algebra (as we only proved regularity for systems of parameters coming from $R_\flat$), but we will construct such an algebra for a large class of ultra-rings, which includes the present case, in 10.3.2 below. Contrary to our present construction, the latter will no longer contain $R_\natural$ as a subring.

Finally, assume $\mathbf{x}$ is strong. I claim that, since $\mathbf{x} = (x_1,\dots,x_d)$ is $B(R_\flat)$-regular by our previous discussion, so is any 'ultra-exponential power' of $\mathbf{x}$, that is to say, any tuple of the form $\mathbf{x}^\beta := (x_1^{\beta_1},\dots,x_d^{\beta_d})$ for $\beta := (\beta_1,\dots,\beta_d) \in \mathbb{N}_\natural^d$. Indeed, letting $\beta_p \in \mathbb{N}^d$ be an approximation of β, then $(\mathbf{x}_p^{\mathrm{eq}})^{\beta_p}$ is $(R_p^{\mathrm{eq}})^+$-regular, for all those p for which $\mathbf{x}_p^{\mathrm{eq}}$ is $(R_p^{\mathrm{eq}})^+$-regular. The claim then follows from Łoś' Theorem. Moreover, any permutation of these ultra-exponential powers is then also $B(R_\natural)$-regular, since any permutation of $\mathbf{x}$ is also a strong system of parameters. By flatness, all ultra-exponential powers $\mathbf{x}^\beta$ are permutable $B(R_\natural)$-regular sequences. It is not hard to see, that $\mathbf{x}$ must therefore be ultra-monomial in $B(R_\natural)$, whence in $R_\natural$. □

Remark 10.2.12. Since any local $V_\natural$-affine algebra is a protoproduct, any such algebra therefore admits a balanced big Cohen-Macaulay algebra.

Theorem 10.2.13. *The weak Monomial Conjecture holds asymptotically for domains of mixed characteristic.*

Proof. Suppose not, so that for some n, we can find for each p, a mixed characteristic complete discrete valuation ring V_p of residue characteristic p, and a local V_p-affine domain R_p of affine proto-grade at most n, such that any strong system of parameters fails to be monomial. Let $R_\flat$ and $R_\natural$ be the respective protoproduct and ultraproduct of the R_p. Let $\mathbf{x}$ be a strong system of parameters in $R_\flat$ with approximation $\mathbf{x}_p$. By Proposition 10.2.6, almost all R_p and $R_\flat$ have the same geometric dimension as well as the same embedding dimension. Therefore, almost each $\mathbf{x}_p$ is a strong system of parameters in R_p. Hence, by assumption, almost each $\mathbf{x}_p$ fails to satisfy (6.9) for at least one exponent, say $k = \alpha_p$. Let $\alpha \in \mathbb{N}_\natural$ be the ultraproduct of the α_p, so that by Łoś' Theorem, we have

$$(x_1 \cdots x_d)^{\alpha-1} \in (x_1^\alpha,\dots,x_d^\alpha)R_\natural,$$

contradicting the last statement in Proposition 10.2.11. □

In the papers [91, 97] many more homological conjectures are proven to hold asymptotically, including the Hochster-Roberts Conjecture, and we conclude with a brief discussion of the latter. To this end, we need to study the singularity theory of ultra-rings. This is done in greater generality in [91, 97, 101], but here we only discuss what we need to prove an asymptotic version of the conjecture.

10.2.7 Pseudo-singularities

Let $(R,\mathfrak{m})$ be a local ring. For our purposes, we define the *(naive) depth* of R as the maximal length of an R-regular sequence—in a more general setup this is the wrong definition,[2] but it is fine if R is an ultra-ring, the only case we are interested in. We call R *pseudo-regular* if it has the same depth as embedding dimension, and *pseudo-Cohen-Macaulay* if it has the same depth as geometric dimension. Equivalently, R is respectively pseudo-regular or pseudo-Cohen-Macaulay, if there exists an R-regular sequence which generates the maximal ideal, respectively, which is a system of parameters. In particular, a pseudo-regular local ring is pseudo-Cohen-Macaulay. Note that in the Noetherian case, these two conditions simply become regular and Cohen-Macaulay, respectively. It is an easy consequence of Łoś' Theorem that an ultraproduct of regular local rings of the same dimension is pseudo-regular. As for an ultraproduct $R_\natural$ of Cohen-Macaulay local rings R_w, it is necessary that both their dimension and their multiplicity be bounded, say by d and l respectively, for $R_\natural$ to be pseudo-Cohen-Macaulay. Indeed, in that case, we can choose a system of parameters $\mathbf{x}_w$ of length d in each R_w such that $R_w/\mathbf{x}_wR_w$ has length at most l, and hence, so does their ultraproduct $R_\natural/\mathbf{x}_\natural R_\natural$ by Proposition 2.4.17, where $\mathbf{x}_\natural$ is the ultraproduct of the $\mathbf{x}_w$. Hence $R_\natural$ has geometric dimension d by 8.1.2, and since $\mathbf{x}_\natural$ is $R_\natural$-regular by Łoś' Theorem, $R_\natural$ is pseudo-Cohen-Macaulay.

Example 10.2.14. A note of caution: a pseudo-regular local ring need not be a domain. Let (V,π) be a discrete valuation ring with ultrapower $V_\natural$. Let $\mathfrak{J} \subseteq V_\natural$ be the ideal generated by all $\pi^{\omega-i}$ for $i = 0,1,\dots$, where

$$\pi^{\omega-i} := \operatorname*{ulim}_{n\to\infty} \pi^{n-i}.$$

Although $\mathfrak{J}$ is contained in $\mathfrak{I}_{V_\natural}$, it is not equal to it: the ultraproduct

$$z := \operatorname*{ulim}_{n\to\infty} \pi^{\mathrm{int}(n/2)}$$

[2] The correct definition of the depth of R is $n-h$, where h is the largest value i for which the i-th Koszul homology $H_i(\mathbf{x};R)$ is non-zero, and where $\mathbf{x}$ is an n-tuple generating the maximal ideal.

does not belong to $\mathfrak{I}$ (where int(r) denotes the integer part of a real number r). Nonetheless $z^2 \in \mathfrak{I}$, showing that $R := V_\natural/\mathfrak{I}$ is not a domain, and, in fact, not even reduced. However, π is R-regular, for if $\pi x \in \mathfrak{I}$ for some $x \in V_\natural$, then $\pi x = \pi^{\omega - i}u$ for some unit $u \in V_\natural$, and hence $x \in \pi^{\omega - i - 1}V_\natural \subseteq \mathfrak{I}$. Since π also generates the maximal ideal of R, the latter is pseudo-regular. Since $\mathfrak{I}$ is not an ultra-ideal, R is not an ultra-ring (and neither is it a local affine $V_\natural$-algebra, since $\mathfrak{I}$ is infinitely generated).

As before, let R_p be a local V_p-affine algebra of affine proto-grade at most n over a mixed characteristic complete discrete valuation ring V_p of residue characteristic p, and let $R_\flat$ and $R_\natural$ be their respective protoproduct and ultraproduct. Let $V_\natural$ be the ultraproduct of the V_p, so that $R_\flat$ is a local $V_\natural$-affine algebra, and let π be a generator of the maximal ideal of $V_\natural$. We have the following two transfer results.

Proposition 10.2.15. *Almost all R_p are regular if and only if $R_\flat$ is a pseudo-regular domain.*

Proof. Since regular local rings are always domains, and since their protoproduct lies inside their ultraproduct, the former is a domain, because the latter is by Łoś' Theorem. This proves the necessity of the domain condition. In particular, almost all R_p have dimension $d \leq n$, equal to the geometric dimension of $R_\flat$ by Proposition 10.2.6. If almost each R_p is regular, then there exists an R_p-regular sequence $\mathbf{x}_p$ of length d generating the maximal ideal. Moreover, the entries of this regular sequence, by definition of proto-grade, may be chosen to have proto-grade at most n. Hence the ultraproduct $\mathbf{x}_\natural$ of the $\mathbf{x}_p$ lies already in $R_\flat$, and generates the latter's maximal ideal by the flatness of $R_\flat \to R_\natural$ (see the proof of Theorem 10.2.4). Moreover, by Łoś' Theorem, $\mathbf{x}_\natural$ is an $R_\natural$-regular, whence $R_\flat$-regular sequence. Conversely, if $\mathbf{x}$ is a regular sequence of length d generating the maximal ideal of $R_\flat$, then it is $R_\natural$-regular by flatness, and hence almost all its approximations are regular sequences generating the maximal ideal, that is to say, almost all R_p are regular. □

Together with our previous discussion, this shows that all chromatic powers of a regular local ring in $\mathfrak{A}_{\mathrm{DVR}}$ are pseudo-regular.

Proposition 10.2.16. *If $R_\flat$ is a domain, then almost all R_p are Cohen-Macaulay if and only if $R_\flat$ is pseudo-Cohen-Macaulay.*

Proof. As in the previous proof, the domain condition allows us to assume that $R_\flat$ and almost all R_p have geometric dimension d. If $R_\flat$ is pseudo-Cohen-Macaulay, that is to say, has depth d, then so does $R_\natural$, by flatness, whence so do almost all R_p by Łoś' Theorem, proving that they are Cohen-Macaulay. To complete the proof, we now simply reverse this argument. □

We also need a flatness result:

Theorem 10.2.17. *If $R_\flat$ is pseudo-regular, then the natural map $R_\flat \to \mathcal{B}(R_\flat)$ is faithfully flat.*

Proof. We will use the equational criterion for flatness, Theorem 3.3.1. Let L be a linear form in n indeterminates with coefficients in $R_\flat$ and let $\mathbf{b}$ be a solution in $B := B(R_\flat)$ of $L = 0$. Let R_p^{eq}, L_p^{eq} and $\mathbf{b}_p^{\mathrm{eq}}$ be equal characteristic approximations of $R_\flat$, L and $\mathbf{b}$ respectively. By Łoś' Theorem, $\mathbf{b}_p^{\mathrm{eq}}$ is a solution in $(R_p^{\mathrm{eq}})^+$ of the linear equation $L_p^{\mathrm{eq}} = 0$. By Corollary 9.3.3, we can find tuples $\mathbf{a}_{1p}^{\mathrm{eq}}, \dots, \mathbf{a}_{sp}^{\mathrm{eq}}$ over R_p^{eq} generating the module of solutions of $L_p^{\mathrm{eq}} = 0$, all of proto-grade at most c, for some c independent from p and s. Let $\mathbf{a}_1, \dots, \mathbf{a}_s$ be the respective ultraproducts, which are therefore defined over $R_\flat$. By Łoś' Theorem, $L(\mathbf{a}_i) = 0$, for each i. On the other hand, almost all R_p^{eq} are regular, by (the equal characteristic variant of) Proposition 10.2.15. Therefore, $R_p^{\mathrm{eq}} \to (R_p^{\mathrm{eq}})^+$ is flat by Theorem 3.3.9. Hence we can write $\mathbf{b}_p^{\mathrm{eq}}$ as a linear combination over $(R_p^{\mathrm{eq}})^+$ of the $\mathbf{a}_{ip}^{\mathrm{eq}}$. By Łoś' Theorem, $\mathbf{b}$ is then a B-linear combination of the solutions $\mathbf{a}_i$, showing that $R_\flat \to B$ is flat whence faithfully flat. □

Proposition 10.2.18. *Let $R \to S$ be a local homomorphism of local $V_\natural$-affine domains, and put $\bar{R} := R/\pi R$ and $\bar{S} := S/\pi S$. If $\bar{R} \to \bar{S}$ is cyclically pure and S is pseudo-regular, then R is pseudo-Cohen-Macaulay.*

Proof. If $\pi = 0$ in R, then R and S are equal characteristic Noetherian local rings, a case already dealt with (Theorem 7.3.3). In the remaining case, π is R-regular, whence part of a system of parameters of R by the argument in the proof of 10.2.5 (note, contrary to the Noetherian case, not every regular element is a parameter!). So we may choose a system of parameters $(x_1 := \pi, x_2, \dots, x_d)$ of R. By Remark 10.2.12, this system of parameters is $B(R)$-regular, and we want to show that it is in fact R-regular. So suppose $rx_i \in J := (x_1, \dots, x_{i-1})R$ for some $i \geq 1$ and some $r \in R$, and we need to show $r \in J$. The case $i = 1$ is immediate, since R is a domain, and so we may assume $\pi \in J$. Since $\mathbf{x}$ is $B(R)$-regular, r belongs to $JB(R)$ whence to $JB(S)$, since we have a commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ B(R) & \longrightarrow & B(S) \end{array} \tag{10.3}$$

by the same argument proving (6.8). By Theorem 10.2.17, the map $S \to B(S)$ is flat, whence cyclically pure, and so $r \in JB(S) \cap S = JS$. Finally, since $\bar{R} \to \bar{S}$ is cyclically pure, $r \in J\bar{S} \cap \bar{R} = J\bar{R}$, whence $r \in J$, as we wanted to show. □

Theorem 10.2.19. *For each n, there exists a uniform bound n' such that if $R \to S$ is a cyclically pure homomorphism of affine proto-grade at most n of mixed characteristic local rings in $\mathfrak{A}_{DVR}$, and if S is regular, then R is Cohen-Macaulay provided its residue characteristic is at least n'.*

Proof. Suppose the assertion is false for n, so that we can find, for each p, a mixed characteristic, complete discrete valuation ring V_p of residue characteristic p, and a cyclically pure homomorphism $R_p \to S_p$ of mixed characteristic local V_p-domains of affine proto-grade at most n (that is to say, the rings have proto-grade at most n and the morphism between them is given by elements of proto-grade at most n), with S_p regular but R_p not Cohen-Macaulay. Let $R_\flat$ and $S_\flat$ be the respective protoproducts of the R_p and S_p. By our assumption on the proto-grade, we have, by 10.2.1, an induced homomorphism $R_\flat \to S_\flat$. By Proposition 10.2.15, the protoproduct $S_\flat$ is pseudo-regular. I claim that $R_\flat/\pi R_\flat \to S_\flat/\pi S_\flat$ is cyclically pure, where π is the ultraproduct of uniformizing parameters of the V_p. Assuming the claim, $R_\flat$ is then pseudo-Cohen-Macaulay by Proposition 10.2.18, whence almost all R_p are Cohen-Macaulay by Proposition 10.2.16, contradiction.

So, remains to prove the claim, and to this end, it suffices to show that if $I \subseteq R_\flat$ is an ideal containing π, then $IS_\flat \cap R_\flat = I$. Let $r \in IS_\flat \cap R_\flat$, and choose approximations $I_p \subseteq R_p$ and $r_p \in R_p$ of I and r respectively. By Łoś' Theorem, r_p lies in $I_pS_p \cap R_p$ and the latter is equal to I_p by assumption. Hence $r \in IR_\natural$, where $R_\natural$ is the ultraproduct of the R_p, and hence, by faithful flatness of $R_\flat \to R_\natural$ (Theorem 10.2.2), we get $r \in I$, as we wanted to show. □

For a variant of this result, which holds only for R of dimension three, but without any restriction on its proto-grade, see Theorem 10.3.5 below.

10.3 Asymptotic Homological Conjectures via Cataproducts

We now discuss a second method for obtaining asymptotic properties in mixed characteristic, via cataproducts. Moreover, this method, when applicable, will give sharper results, where the residue characteristic has to be only large with respect to some more natural invariants than the affine proto-grade, and where in fact we no longer need to assume that the ring is affine. Moreover, there is a second version, where this time not the residue characteristic, but the ramification index (see below) has to be sufficiently large. Unfortunately, asymptotic versions of some of the homological conjectures, like the Direct Summand and the Monomial Conjecture that were treated by the previous method, elude at present treatment by the cataproduct method.

10.3.1 Ramification

Let us first discuss how to go from mixed to equal characteristic by means of cataproducts. One way, of course, is already quite familiar to us: the cataproduct of local rings of different residue characteristic has (residue) characteristic zero. However, there is a second way. Given a local ring $(R,\mathfrak{m})$ of residue characteristic p, we call the $\mathfrak{m}$-adic order of p its *ramification index*, that is to say, the ramification index of R is the largest n such that $p \in \mathfrak{m}^n$. If the ramification index is one, we call R *unramified*, and if the ramification index is infinite (that is to say, p is an infinitesimal, including the case that $p = 0$, the equal characteristic case), we say that it is *infinitely ramified*. Since a Noetherian local ring does not have non-zero infinitesimals, being infinitely ramified is the same as having equal characteristic, but not so for arbitrary local rings, and here lies the clue to obtain equal characteristic cataproducts:

10.3.1 *Let R_w be mixed characteristic local rings of bounded embedding dimension, with residue characteristic p. If the R_w have unbounded ramification index (that is to say, if, for all n, almost all R_w have ramification index at least n), then the cataproduct $R_\sharp$ has equal characteristic p.*

Indeed, the ultraproduct is infinitely ramified by Łoś' Theorem, whence the cataproduct has equal characteristic p, as it is Noetherian by Theorem 8.1.4. Balanced big Cohen-Macaulay algebras are available in this setup too:

10.3.2 *If an ultra-Noetherian local ring R has either equal characteristic or is infinitely ramified, then it admits a balanced big Cohen-Macaulay algebra $B(R)$.*

Indeed, under either assumption, the separated quotient $R_\sharp$ is an equal characteristic Noetherian local ring by Theorem 8.1.4. If $\mathfrak{p}$ is a maximal dimensional prime ideal in $R_\sharp$, then any system of parameters in R remains one in $R_\sharp$ whence in $R_\sharp/\mathfrak{p}$, and therefore is $B(R_\sharp/\mathfrak{p})$-regular by Theorem 7.4.4. Hence $B(R) := B(R_\sharp/\mathfrak{p})$ yields the desired balanced big Cohen-Macaulay algebra. □

10.3.2 Asymptotic Improved New Intersection Conjecture

The last of the homological conjectures that we will discuss is an 'intersection' conjecture. The original conjecture, called the *Intersection Conjecture* was proven by Peskine and Szpiro in [75], using properties of the Frobenius in positive characteristic, and lifting the result to characteristic zero by means of Artin Approximation (virtually the same lifting technique as for HH-tight closure discussed in §5.6). Hochster and others (see, for instance, [31, 43, 44, 56]) formulated and subsequently proved generalizations of this result in equal characteristic, called 'new' and 'improved' intersection theorems. In fact, the New Intersection Theorem (whence also the original one) was established in mixed characteristic as well by Roberts in [79]. However, the most general of them all, the so-called Improved New Intersection Conjecture is only known to hold in equal characteristic. It is concerned with the length of a finite free complex with finite homology. Its asymptotic version reads:

Theorem 10.3.3 (Asymptotic Improved New Intersection Theorem). *For each triple of non-negative integers (m,r,l), there exists a uniform bound $e(m,r,l)$ with the* *following property. Let R be a Noetherian local ring of mixed characteristic and let $F_\bullet$ be a finite complex of finitely generated free R-modules. Assume R has embedding dimension at most m and each module in $F_\bullet$ has rank at most r.*

If each $H_i(F_\bullet)$, for $i > 0$, has length at most l and if $H_0(F_\bullet)$ has a minimal generator generating a submodule of length at most l, then the dimension of R is less than or equal to the length of the complex $F_\bullet$, provided R has either residue characteristic or ramification index at least $e(m,r,l)$.

Proof. We will give the proof modulo one result, Theorem 10.3.4 below. Since the dimension of R is at most m, there is nothing to show for complexes of length m or higher. Suppose the result is false for some triple (m,r,l), so that we can find for each w a counterexample consisting of a d_w-dimensional mixed characteristic Noetherian local ring R_w of embedding dimension at most m such that each R_w has either residue characteristic or ramification index at least w, and a complex $F_{w\bullet}$ of length $s_w \leq m$ consisting of finitely generated free R_w-modules of rank at most r such that all its higher homology has length at most l and such that its cokernel admits a minimal generator μ_w generating a submodule of length at most l, but such that $s_w < d_w$. Let $R_\natural$ and $R_\sharp$ be the respective ultraproduct and cataproduct of the R_w, and let μ, s and d be the ultraproduct of the μ_w, s_w and d_w respectively. In particular, $s < d \leq m$ and almost all s_w and d_w are equal to s and d respectively. By 8.1.2, the geometric dimension of $R_\natural$ is at least d. Let $F_\bullet$ be the ultraproduct of the complexes $F_{w\bullet}$. Since the ranks are at most r, each module in $F_\bullet$ will be a free $R_\natural$-module of rank at most r. Since ultraproducts commute with homology (Theorem 3.1.1) and preserve uniformly bounded length by the module version of Proposition 2.4.17, the higher homology $H_i(F_\bullet)$ has finite length (at most l) and so has the $R_\natural$-submodule of $H_0(F_\bullet)$ generated by μ. In particular, $F_\bullet$ is acyclic when localized at a non-maximal prime ideal, so that s is at least the geometric dimension of $R_\natural$ by Theorem 10.3.4 below, and hence $s \geq d$, contradiction. □

For the homological terminology used in the next result, see §3.1.2.

Theorem 10.3.4. *Let $(R,\mathfrak{m})$ be an ultra-Noetherian local ring, and assume R has either equal characteristic or is infinitely ramified. Let $F_\bullet$ be a finite complex of finitely generated free R-modules, and let M be its cokernel. If $F_\bullet$ is acyclic when localized at any prime ideal of R different from $\mathfrak{m}$, and if there exists a non-zero minimal generator of M whose annihilator is $\mathfrak{m}$-primary, then the geometric dimension of R is less than or equal to the length of $F_\bullet$.*

Proof. The proof is really just a modification of the classical proof (see [101, Corollary 10.9] for details). As with most homological conjectures, they become easy to prove if the ring is moreover Cohen-Macaulay, and in this particular instance, this is because of the Buchsbaum-Eisenbud Acyclicity Criterion ([17, Theorem 9.1.6]). It was Hochster's ingenious observation that instead of the ring being Cohen-Macaulay, it suffices for the proofs to go through that there exists a balanced big Cohen-Macaulay module. In the present situation, this is indeed the case due to 10.3.2. □

We conclude with a variant of the asymptotic Hochster-Roberts Theorem (see Theorem 10.2.19). Since big Cohen-Macaulay algebras are known to exist in dimension three, this theorem is therefore valid when $R \subseteq S$ both have dimension at most three. If we allow arbitrary dimension for S, then we can get the following asymptotic result. To compare this with Theorem 10.2.19, note that we only require for S to have bounded affine proto-grade as an R-algebra.

Theorem 10.3.5. *For each triple (p,d,l) with p a prime number, there exists a uniform bound $r := r(p,d,l)$ with the following property. Let R be a mixed characteristic three-dimensional Noetherian local ring with parameter degree at most l and residual characteristic p. If there exists a cyclically pure regular local R-algebra S having proto-grade at most d, and if the ramification index of R is at least r, then R is Cohen-Macaulay.*

Proof. By way of contradiction, suppose the assertion is false for some triple (p,d,l). Hence, there exists for each w, a mixed characteristic three-dimensional Noetherian local ring $(R_w, \mathfrak{m}_w)$ with parameter degree at most l and ramification index at least w, and a regular, local R_w-algebra S_w of proto-grade at most d, such that $R_w \to S_w$ is cyclically pure but R_w is not Cohen-Macaulay. Let I_w be a parameter ideal in R_w such that R_w/I_w has length at most l. By [46], each R_w admits a balanced big Cohen-Macaulay algebra B_w. In view of Lemma 10.3.6 below, we will have reached our desired contradiction once we show that

$$I_w R_w = I_w B_w \cap R_w, \tag{10.4}$$

for almost all w, as R_w is then Cohen-Macaulay for those w.

To prove (10.4) by way of contradiction, suppose x_w is in $I_w B_w \cap R_w$ but not in I_w. Let $(R_\natural, \mathfrak{m})$, x, and I be the respective ultraproducts of $(R_w, \mathfrak{m}_w)$, x_w, and I_w. By [17, Corollary 8.5.3], the $\mathfrak{m}$-adic completion B of the ultraproduct $B_\natural$ of the B_w is a balanced big Cohen-Macaulay over the cataproduct $R_\sharp$ of the R_w. By Łoś' Theorem, x lies in $IB_\natural$ but not in I. Moreover, $R_\sharp = R_\natural/\mathfrak{I}_{R_\natural}$ has equal characteristic p by 10.3.2. It follows from Remark 6.4.4 that $IB \cap R_\sharp$ lies in the tight closure of $IR_\sharp$. By 10.2.5, the cataprotoproduct $S_{\flat\sharp}$ is a regular local ring. I claim that $R_\sharp \to S_{\flat\sharp}$ is cyclically pure. Assuming this claim, $R_\sharp$ is then F-regular by 5.5.7, and hence $IB \cap R_\sharp = IR_\sharp$. Since $x \in IB \cap R_\sharp = IR_\sharp$ and since $\mathfrak{I}_{R_\natural} \subseteq I$, we get $x \in I$, whence by Łoś' Theorem, almost each x_w lies in I_w, contradiction.

So remains to prove the claim, and since any ideal is the intersection of $\mathfrak{m}$-primary ideals by the Krull's Intersection Theorem (Theorem 2.4.14), it suffices to show that $\mathfrak{a}S_{\flat\sharp} \cap R_\sharp = \mathfrak{a}R_\sharp$ for any $\mathfrak{m}$-primary ideal $\mathfrak{a} \subseteq R_\natural$. Let $y \in R_\natural$ be such that its image in $R_\sharp$ lies in $\mathfrak{a}S_{\flat\sharp}$. Since $\mathfrak{I}_{R_\natural} \subseteq \mathfrak{a}$ and since $S_{\flat\sharp} = S_\flat/\mathfrak{I}_{V_\natural}S_\flat$ by 10.2.5, we get $y \in \mathfrak{a}S_\flat$, where $S_\flat$ is the protoproduct of the S_w. In particular, $y \in \mathfrak{a}S_\natural$. Taking approximations $y_w \in R_w$ of y, Łoś' Theorem yields that almost all y_w lie in $\mathfrak{a}_w S_w$, whence by cyclical purity in $\mathfrak{a}_w$. Łoś' Theorem in turn then yields $y \in \mathfrak{a}R_\natural$, as we wanted to show.

In the previous proof, we used the following Cohen-Macaulay characterization. It should be noted that if in this criterion we require that it holds for all balanced big Cohen-Macaulay algebras, we get, at least in positive characteristic, a characterization of F-rationality, whence of pseudo-rationality by Remarks 5.5.9 and 6.4.4.

Lemma 10.3.6. *A Noetherian local ring R is Cohen-Macaulay if and only if there exists a system of parameters $\mathbf{x}$ in R and a balanced big Cohen-Macaulay algebra B over R such that $\mathbf{x}R = \mathbf{x}B \cap R$.*

Proof. If R is Cohen-Macaulay, then we may take $B = R$. To prove the converse, let us first show by downward induction on $i \leq d$ that $I_i = I_iB \cap R$, where $I_i := (x_1,\dots,x_i)R$ and $\mathbf{x} = (x_1,\dots,x_d)$. The case $i = d$ is just our assumption, so that we may take $i < d$ and assume we already established that $I_{j+1} = I_{j+1}B \cap R$. Let z be an element of $J_i := I_iB \cap R$. By our induction hypothesis, $z \in I_{i+1}$, say $z = y + ax_{i+1}$ with $y \in I_i$ and $a \in R$. Since $ax_{i+1} = z - y \in I_iB$ and $\mathbf{x}$ is a B-regular sequence, a lies in I_iB whence in J_i. In conclusion, we showed that $J_i = I_i + x_{i+1}J_i$, so that by Nakayama's Lemma, $J_i = I_i$, as claimed.

To complete the proof, we must show that $\mathbf{x}$ is R-regular. To this end, suppose $zx_{i+1} \in I_i$ for some $z \in R$. Since $\mathbf{x}$ is B-regular, z lies in I_iB whence in I_i, by what we just proved. □

10.3.3 Towards a Proof of the Improved New Intersection Theorem

Although our methods can in principle only prove asymptotic versions, a better understanding of the uniform bounds can in certain cases lead to a complete solution of the conjecture. To formulate such a result, let us say that a numerical function f *grows sub-linearly* if there exists some $0 \leq \alpha < 1$ such that $f(n)/n^\alpha$ remains bounded when n goes to infinity.

Theorem 10.3.7. *Suppose that for each pair (m,r) the numerical function $f_{m,r}(l) := e(m,r,l)$ grows sub-linearly, where e is the bound given by Theorem 10.3.3, then the Improved New Intersection Theorem holds.*

Proof. Let $\mathscr{I}_{m,r,l}$ be the collection of counterexamples with invariants (m,r,l), that is to say, all mixed characteristic Noetherian local rings R of embedding dimension at most m, admitting a finite free complex $F_\bullet$ of rank at most r such that each $H_i(F_\bullet)$, for $i > 0$, has length at most l and $H_0(F_\bullet)$ has a minimal generator generating a submodule of length at most l, but such that the length of $F_\bullet$ is strictly less than the dimension of R. We have to show that $\mathscr{I}_{m,r,l}$ is empty for all (m,r,l), so by way of contradiction, assume it is not for the triple (m,r,l). For each n, let $f(n)$ be the supremum of the ramification indexes of counterexamples in $\mathscr{I}_{m,r,n}$ (and equal to 0 if there is no counterexample). By Theorem 10.3.3, this supremum is always finite. By assumption, f grows sub-linearly, so that for some positive real numbers c and $\alpha < 1$, we have $f(n) \leq cn^{\alpha}$, for all n. In particular, for n larger than the $(1-\alpha)$-th root of $\frac{cl^{\alpha}}{f(l)}$, we have

$$f(ln) < nf(l). \tag{10.5}$$

Let $(R, \mathfrak{m})$ be a counterexample in $\mathscr{I}_{m,r,l}$ of ramification index $f(l)$, witnessed by the finite free complex $F_\bullet$ of length strictly less than the dimension of R. Since the completion of R will be again a counterexample in $\mathscr{I}_{m,r,l}$ of the same ramification index, we may assume R is complete, whence by Cohen's Structure Theorems of the form $R = V[[\xi]]/I$ for some complete discrete valuation ring V, some tuple of indeterminates ξ, and some ideal $I \subseteq V[[\xi]]$. Let $n \gg 0$ so that (10.5) holds, and let $W := V[t]/(t^n - \pi)V[t]$, where π is a uniformizing parameter of V. Let $S := W[[\xi]]/IW[[\xi]]$, so that $R \to S$ is faithfully flat and S has the same dimension and embedding dimension as R. By construction, the ramification index of S is equal to $nf(l)$. By faithful flatness, $F_\bullet \otimes_R S$ is a finite free complex of length strictly less than the dimension of S, with homology equal to $\mathrm{H}_\bullet(F_\bullet) \otimes_R S$. I claim that if H is an R-module of length a, then $H \otimes_R S$ has length na. Assuming this claim, it follows that S belongs to $\mathscr{I}_{m,r,nl}$, and hence its ramification index is by definition at most $f(ln)$, contradicting (10.5).

The claim is easily reduced by induction to the case $a = 1$, that is to say, when H is equal to the residue field $R/\mathfrak{m} = V/\pi V = k$. In that case, $H \otimes_R S = S/\mathfrak{m}S = W/\pi W$, and this is isomorphic to $k[t]/t^n k[t]$, a module of length n. □

Appendix A
Henselizations

In this appendix, I have gathered some facts about Henselizations that can be found scattered in the literature (some sources dealing more extensively with Henselizations are [70,71,77,106]). Hensel observed that solving an equation over the p-adics can be reduced to finding a root in the residue field, provided this root is simple. This property, now known as Hensel's Lemma—and a ring satisfying it, is called *Henselian*—, extends easily to any complete local ring; see Theorem A.1.1. Although any Noetherian local ring admits a uniquely defined, smallest complete overring, its completion—which inherits many of the good properties of the original ring, and in particular is Henselian—, the process introduces transcendental elements. The Henselization of a local ring is much closer to it than its completion, since it is a direct limit of finite etale extensions. As Eisenbud remarks

> "... [i]t can thus be used to give the same microscopic view of a variety as the completion, but without passing out of the category of algebraic varieties."
>
> [27, p. 186]

The main objective of this appendix is to give a direct construction of the Henselization which, to my knowledge, never appeared in print.[1]

A.1 Hensel's Lemma

A very important algebraic tool in studying local properties of a variety, or equivalently, properties of Noetherian local rings, is the completion $\widehat{R}$ of a Noetherian local ring R. It is again a Noetherian local ring, which inherits many of the properties of the original ring, and in fact, there is natural homomorphism $R \to \widehat{R}$, which is flat and unramified (recall that the latter means that the maximal ideal of R extends to the maximal ideal of its completion $\widehat{R}$). Whereas there is no

[1] Jan Denef, who was my promotor at the time, suggested the construction to me in 1981, which I then subsequently worked out and wrote up as part of my license thesis [87].

known classification of arbitrary Noetherian local rings, we do have many structure theorems, due mostly to Cohen, for complete Noetherian local rings. In particular, the equal characteristic complete regular local rings are completely classified by their residue field k and their dimension d: any such ring is isomorphic to the power series ring $k[[\xi_1,\dots,\xi_d]]$. Also extremely useful is the fact that we have an analogue of Noether normalization for complete Noetherian local domains: any such ring admits a regular subring over which it is finite. Another nice property of complete local rings is the following formal version of Newton's method for finding approximate roots.

Theorem A.1.1 (Hensel's Lemma). *Let $(R,\mathfrak{m})$ be a complete local ring with residue field k. Let $f \in R[t]$ be a monic polynomial in the single variable t, and let $\bar{f} \in k[t]$ denote its reduction modulo $\mathfrak{m}R[t]$. For every simple root $u \in k$ of $\bar{f} = 0$, we can find $a \in R$ such that $f(a) = 0$ and u is the image of a in k.*

Proof. Let $a_1 \in R$ be any lifting of u. Since $\bar{f}(u) = 0$, we get $f(a_1) \equiv 0 \mod \mathfrak{m}$. We will define elements $a_n \in R$ recursively such that $f(a_n) \equiv 0 \mod \mathfrak{m}^n$ and $a_n \equiv a_{n-1} \mod \mathfrak{m}^{n-1}$ for all $n > 1$. Suppose we already defined $a_1,\dots,a_n$ satisfying the above conditions. Consider the Taylor expansion

$$f(a_n + t) = f(a_n) + f'(a_n)t + g_n(t)t^2 \tag{A.1}$$

where $g_n \in R[t]$ is some polynomial. Since the image of a_n in k is equal to u, and since $\bar{f}'(u) \neq 0$ by assumption, $f'(a_n)$ does not lie in $\mathfrak{m}$ whence is a unit, say, with inverse u_n. Define $a_{n+1} := a_n - u_n f(a_n)$. Substituting $t = -u_n f(a_n)$ in (A.1), we get

$$f(a_{n+1}) \in (u_n f(a_n))^2 R \subseteq \mathfrak{m}^{2n},$$

as required.

To finish the proof, note that the sequence a_n is by construction Cauchy, and hence by assumption admits a limit $a \in R$ (whose residue is necessarily again equal to u). By continuity, $f(a)$ is equal to the limit of the $f(a_n)$ whence is zero. □

There are sharper versions of this result, where the root in the residue field need not be simple (see [27, Theorem 7.3]), or even involving systems of equations (see [13, §4.6]; but see also the next section).

A local ring satisfying the hypothesis of the above theorem is normally called a *Henselian* ring, although we will deviate from that practice in the next section. For some equivalent definitions, we refer once more to the literature [70, 71, 77, 106]. From a model-theoretic point of view, it is more convenient to work with Henselian local rings than with complete ones, since they form a first-order definable class (as is clear from the defining condition).

As with completion, there exists a 'smallest' Henselian overring. More precisely, for each Noetherian local ring R, there exists a Noetherian local R-algebra $R^\sim$, its *Henselization*, satisfying the following universal property: any local homomorphism $R \to H$ with H a Henselian local ring, factors uniquely through an R-algebra homomorphism $R^\sim \to H$. Below, we will show the existence of such a

Henselization by giving a concrete construction of $R^{\sim}$. Note that Theorem A.1.1 and the universal property imply that $R^{\sim}$ is a subring of $\widehat{R}$, and in particular, the latter is the completion of the former.

A.2 Construction of the Henselization

Let $(R,\mathfrak{m})$ be a Noetherian local ring. By a *Hensel system* over R of size N, we mean a pair $(\mathscr{H},\mathbf{a})$ consisting of a system $\mathscr{H}(t)$ of N polynomial equations $h_1,\dots,h_N \in R[t]$ in the N unknowns $t := (t_1,\dots,t_N)$, and an approximate solution $\mathbf{a}$ modulo $\mathfrak{m}$ in R (meaning that $h_i(\mathbf{a}) \equiv 0 \mod \mathfrak{m}$ for all i), such that associated Jacobian matrix

$$\operatorname{Jac}(\mathscr{H}) := \begin{pmatrix} \partial h_1/\partial t_1 & \partial h_1/\partial t_2 & \dots & \partial h_1/\partial t_N \\ \partial h_2/\partial t_1 & \partial h_2/\partial t_2 & \dots & \partial h_2/\partial t_N \\ \vdots & \vdots & \ddots & \vdots \\ \partial h_N/\partial t_1 & \partial h_N/\partial t_2 & \dots & \partial h_N/\partial t_N \end{pmatrix} \tag{A.2}$$

evaluated at $\mathbf{a}$ is invertible over R, that is to say, the *Jacobian determinant* $\det(\operatorname{Jac}(\mathscr{H}))$ evaluated at $\mathbf{a}$ is a unit in R. We express the latter condition also by saying that $\mathbf{a}$ is a *non-singular* approximate solution. An N-tuple $\mathbf{s}$ in some local R-algebra S is called a *solution* of the Hensel system $(\mathscr{H},\mathbf{a})$, if it is a solution of the system $\mathscr{H}$ and $\mathbf{s} \equiv \mathbf{a} \mod \mathfrak{m}S$. Note that $(\mathscr{H},\mathbf{s})$ is then a Hensel system over S, and therefore, we sometimes call $\mathscr{H}$ a Hensel system, without mentioning the (approximate) non-singular solution. A Hensel system of size $N = 1$ is just a Hensel equation together with a solution in the residue field, as in the statement of Hensel's lemma. In fact, R satisfies Hensel's lemma if and only if any Hensel system over R has a solution in R. The proof of this equivalence is not that easy (one can give for instance a proof using standard etale extensions as in [70]).

Instead, we alter out definition by calling a local ring R *Henselian*, if any Hensel system (of any size) over R has a solution in R. In conclusion, being Henselian in the new sense implies that in the old sense, and the converse also holds, but is harder to prove. An easy modification of the proof of Theorem A.1.1, left to the reader, shows that complete local rings are Henselian in this new sense. In fact, using multivariate Taylor expansion, we obtain the following stronger version.

A.2.1 *Any Hensel system $(\mathscr{H},\mathbf{a})$ over R admits a unique solution in the completion $\widehat{R}$.* □

We call an element $r \in \widehat{R}$ a *Hensel element* (over R) if there exists a Hensel system $(\mathscr{H},\mathbf{a})$ over R such that r is the first entry of the unique solution of this system in $\widehat{R}$. We will express this by saying that *$\mathscr{H}$ is a Hensel system for r*. Note that if $\mathbf{r} = (r_1,\dots,r_N)$ is a solution of a Hensel system $\mathscr{H}$ over R, then any r_i is a Hensel element. This is true by definition for r_1. For $i > 1$, let $\mathscr{H}'$ be obtained by interchanging the unknowns t_1 and t_i, as well as, h_1 with h_i. It follows that $\mathscr{H}'$ is a Hensel system for $(r_i,r_2,\dots,r_{i-1},r_1,r_{i+1},\dots,r_N)$, showing that r_i is a Hensel element.

Let $R^\sim$ be the subset of $\widehat{R}$ of all Hensel elements. For given Hensel elements r and r', we construct from their associated Hensel systems $(\mathscr{H}(t),\mathbf{a})$ and $(\mathscr{H}'(t'),\mathbf{a}')$ of size N and N' respectively, a new Hensel system for $r+r'$ as follows: let $N'' := N+N'+1$, let t'' be the N''-tuple of unknowns (u,t,t'), with u a single variable, and consider the system $\mathscr{H}''$ of N'' equations in t'' given by the equation $u = t_1 + t'_1$, and the systems $\mathscr{H}(t)$ and $\mathscr{H}'(t')$. One checks that $(\mathscr{H}'', a_1 + a'_1, \mathbf{a}, \mathbf{a}')$ is a Hensel system—since its Jacobian determinant is the product of the Jacobian determinants of $\mathscr{H}$ and $\mathscr{H}'$—whose unique solution in $\widehat{R}$ has first entry equal to $r+r'$, showing that the latter is again a Hensel element. The same argument can be used to prove that the product of Hensel elements is again a Hensel element. With little effort one actually shows:

A.2.2 *The collection of all Hensel elements is a local ring $R^\sim$ with maximal ideal $\mathfrak{m}R^\sim$. Moreover, $R^\sim$ is Henselian, with completion equal to $\widehat{R}$.*

Indeed, let $\mathfrak{m}^\sim := \mathfrak{m}\widehat{R} \cap R^\sim$. To show that $R^\sim$ is local with maximal ideal $\mathfrak{m}^\sim$, it suffices to show that any element $r \in R^\sim$ not in $\mathfrak{m}^\sim$ is a unit in $R^\sim$. Since r does not belong to $\mathfrak{m}\widehat{R}$, it has an inverse in $\widehat{R}$. Using an auxiliary variable u and the equation $t_1 u = 1$, it is now not hard to show that $1/r$ is an Hensel element. In particular, R, $R^\sim$ and $\widehat{R}$ all have the same residue field k. To prove that $R^\sim$ is Henselian, we must verify the multivariate Hensel lemma, that is to say, let $(\tilde{\mathscr{H}}(t),\mathbf{a})$ be a Hensel system over $R^\sim$. Since $R^\sim$ and R have the same residue field, we may choose $\mathbf{a}$ in R. By A.2.1, there exists a unique solution $\mathbf{r}$ over $\widehat{R}$ of this Hensel system. Remains to show that $\mathbf{r}$ has its entries already in $R^\sim$, and to this end, it suffices by the above discussion to construct a Hensel system over R of which $\mathbf{r}$ is part of a solution.

Let $\mathbf{s} = (s_1,\dots,s_d)$ be the tuple of coefficients in $R^\sim$ of the equations $\tilde{\mathscr{H}}$ (listed in a fixed order), and let $\mathscr{H}(t,u)$ be obtained from $\tilde{\mathscr{H}}$ by replacing each of these coefficients by a new variable u_i, so that $\tilde{\mathscr{H}}(t) = \mathscr{H}(t,\mathbf{s})$. For each s_i, choose $b_i \in R$ such that $s_i \equiv b_i \mod \mathfrak{m}\widehat{R}$. Let $(\mathscr{H}_i(u_i,t_i),(b_i,\mathbf{c}_i))$ be a Hensel system for each Hensel element s_i, with t_i a finite tuple of auxiliary unknowns and $\mathbf{c}_i$ a tuple of the corresponding length in R, for $i = 1,\dots,d$. One easily checks that the system $\mathscr{G}$ in the unknowns $t,u_1,t_1,\dots,u_d,t_d$ at the tuple $\mathbf{c} := (\mathbf{a},b_1,\mathbf{c}_1,\dots,b_d,\mathbf{c}_d)$ given by $\mathscr{H}$ and all $\mathscr{H}_i$ is a Hensel system, since the Jacobian determinant of $(\mathscr{G},\mathbf{c})$ is the product of the Jacobian determinants of $(\mathscr{H},\mathbf{a})$ and the $(\mathscr{H}_i,(b_i,\mathbf{c}_i))$. By A.2.1, the unique solution of this Hensel system in $\widehat{R}$ must be of the form $(\mathbf{r},s_1,\mathbf{r}_1,\dots,s_d,\mathbf{r}_d)$, for some $\mathbf{r}_i$ in $\widehat{R}$, showing that $\mathbf{r} \in R^\sim$. □

It is unfortunately less easy to prove that $R^\sim$ is also Noetherian, and we postpone the discussion until after we proved our main result:

Theorem A.2.3. *The ring $R^\sim$ satisfies the universal property of Henselization: any Henselian local R-algebra S admits a unique structure of $R^\sim$-algebra.*

Proof. We need to show that there exists a (unique) R-algebra homomorphism $R^\sim \to S$. Given $r \in R^\sim$, let $(\mathscr{H},\mathbf{a})$ be a Hensel system admitting a solution with first entry r. Since $\mathbf{a}$ is an approximate solution of $\mathscr{H}$ in R, it remains so in S. By (the revised) definition of Henselian, the approximate solution $\mathbf{a}$ lifts uniquely to a solution $\mathbf{s}$ in S. We define the image of r in S now as the first entry of this solution $\mathbf{s}$. Uniqueness guarantees firstly that this is an R-algebra homomorphism, an secondly that it is unique. □

Returning to the issue of Noetherianity, we will use the local flatness criterion discussed §3.3.6. We start with the flatness of the Henselization:

Proposition A.2.4. *For any ideal $I \subseteq R$, the Henselization of R/I is isomorphic to $R^\sim/IR^\sim$. Moreover, $R \to R^\sim$ is faithfully flat, whence a scalar extension, and $R^\sim$ is ind-Noetherian.*

Proof. Let $S := R/I$. It is not hard to show that any homomorphic image of a Henselian local ring is again Henselian. Hence $R^\sim/IR^\sim$ is Henselian, and the universal property of Henselizations then yields a unique homomorphism $S^\sim \to R^\sim/IR^\sim$. The composition of this homomorphism with $R^\sim/IR^\sim \to \widehat{R}/I\widehat{R}$ is injective, since the latter is the completion of S. Hence $S^\sim \to R^\sim/IR^\sim$ must also be injective. To prove surjectivity, let $r \in R^\sim$ and let $\mathscr{H}$ be a Hensel system for r. The reduction modulo I of this Hensel system therefore has a unique solution in $S^\sim$, and by uniqueness, the first entry of this solution must map to the image of r in $R^\sim/IR^\sim$. This proves the first assertion, and in particular that $I\widehat{R} \cap R^\sim = IR^\sim$, for any ideal $I \subseteq R$. The second assertion then follows from the flatness of $R \to \widehat{R}$ and Corollary 3.3.15. Since $R \to R^\sim$ is unramified by A.2.2, it is therefore a scalar extension (see §3.2.3).

So remains to show that $R^\sim$ is ind-Noetherian (defined after Corollary 3.3.22). Let $\mathbf{x}$ be a finite tuple in $R^\sim$. As already remarked before, we can find a Hensel system $\mathscr{H}(t)$ over R such that $\mathbf{x}$ is part of its unique solution. Hence, if $S_{\mathbf{x}}$ is the localization of $R[t]/(\mathscr{H})$ with respect to the ideal generated by $\mathfrak{m}$, then $\mathbf{x}$ is already a tuple in $S_{\mathbf{x}}$. It follows from the construction of $R^\sim$, that $S_{\mathbf{x}}^\sim = R^\sim$. In particular, $S_{\mathbf{x}} \to R^\sim$ is a scalar extension by what we just proved, and $R^\sim$ is the direct limit of the $S_{\mathbf{x}}$. □

Theorem A.2.5. *The Henselization of a Noetherian local ring is again Noetherian.*

Proof. It suffices to show that $R^\sim \to \widehat{R}$ is faithfully flat, since $\widehat{R}$ is Noetherian. To obtain flatness, it suffices in view of Corollary 3.3.25 and Proposition A.2.4 to show that $\mathrm{Tor}_1^{R^\sim}(\widehat{R}, k) = 0$, where k is the residue field of R. To this end, let

$$R^m \to R^n \to R \to k \to 0 \tag{A.3}$$

be an exact sequence. By Proposition A.2.4, tensoring with $R^\sim$ yields an exact sequence

$$(R^\sim)^m \to (R^\sim)^n \to R^\sim \to k \to 0.$$

By definition, $\mathrm{Tor}_1^{R^\sim}(\widehat{R}, k)$ is the homology of the complex obtained from tensoring this exact sequence with $\widehat{R}$, that is to say, of the complex

$$(\widehat{R})^m \to (\widehat{R})^n \to \widehat{R} \to k \to 0.$$

However, this latter complex is actually exact since it is obtained from tensoring (A.3) with the flat extension $\widehat{R}$, showing that $\operatorname{Tor}_1^{R^\sim}(\widehat{R},k) = 0$. □

A.3 Etale Proto-grade

We conclude with a proto-graded version of the previous construction by constructing a proto-grading on the Henselization of a proto-graded Noetherian local ring $(R,\mathfrak{m})$, and giving conditions under which this proto-grading is Noetherian and faithfully flat. Define a proto-grading on $R^\sim$ by the condition that a Hensel element $y \in R^\sim$ has proto-grade at most n if it admits a Hensel system $(\mathscr{H},\mathbf{u})$ of length $N \leq n$, in which all polynomials have degree at most n, and all coefficients as well as all entries of $\mathbf{u}$ have proto-grade at most n.

A.3.1 *This yields a proto-grading on $R^\sim$, called the* etale proto-grading *on $R^\sim$, extending the proto-grading on R. Moreover, $R \to R^\sim$ is a morphism of proto-graded rings.*

The fact that this is a proto-grading follows from the proof that $R^\sim$ is a ring, since we explicitly constructed Hensel systems for sums, products, and inverses (of units). Since $t-a$ is a Hensel system of $a \in R$, its etale proto-grade is equal to its proto-grade in R, and the last assertion is now immediate. □

The following result enables us to calculate protopowers:

Proposition A.3.2. *If R is a proto-graded Noetherian local ring and $R^\sim$ is viewed with its etale proto-grading extending the proto-grading on R, then we have an isomorphism*

$$(R^\sim)_\flat \cong (R_\flat)^\sim.$$

Proof. A special instance of the above isomorphism is the fact that if R is Henselian, then so is $R_\flat$. We prove this first, and so, let $(\mathscr{H},\mathbf{u})$ be a Hensel system over $R_\flat$ of proto-grade at most n, say. Choose approximations $\mathscr{H}_w$ and $\mathbf{u}_w$ over R of proto-grade at most n, with respective ultraproduct $\mathscr{H}$ and $\mathbf{u}$. By Łoś' Theorem, almost all $(\mathscr{H}_w,\mathbf{u}_w)$ are Hensel systems. Since R is Henselian by assumption, these Hensel systems have a (unique) solution $\mathbf{x}_w$. By definition, the $\mathbf{x}_w$ have etale proto-grade at most n, and hence their ultraproduct $\mathbf{x}$ lies in $R_\flat$. By Łoś' Theorem, $\mathbf{x}$ is then a solution of the Hensel system $(\mathscr{H},\mathbf{u})$.

Let R now be an arbitrary proto-graded Noetherian local ring. The embedding $R \to R^\sim$ induces an embedding $R_\flat \to (R^\sim)_\flat$. By our previous argument, $(R^\sim)_\flat$ is Henselian, whence by the universal property of a Henselization, we have a unique $R_\flat$-algebra embedding $(R_\flat)^\sim \to (R^\sim)_\flat$. To see that this is surjective, let x be an element in $(R^\sim)_\flat$, say of etale proto-grade at most n. Choose an approximation $x_w \in R^\sim$ of proto-grade at most n. Hence, almost each x_w is the first entry of the unique solution $\mathbf{x}_w$ of a Hensel system $(\mathscr{H}_w, \mathbf{u}_w)$ over R of proto-grade at most n. Since the ultraproduct $(\mathscr{H}, \mathbf{u})$ of the $(\mathscr{H}_w, \mathbf{u}_w)$ is a Hensel system of proto-grade at most n, whence defined over $R_\flat$, the ultraproduct $\mathbf{x}$ of the $\mathbf{x}_w$ is a solution of etale proto-grade at most n, belonging therefore to $(R_\flat)^\sim$. Since x is its first entry, $x \in (R_\flat)^\sim$, as we wanted to show. □

Theorem A.3.3. *If R is a local ring with a Noetherian proto-grading, then the etale proto-grading on $R^\sim$ is also Noetherian. If R is moreover regular and the proto-grading on R is faithfully flat, then the etale proto-grading on $R^\sim$ is also faithfully flat.*

Proof. The first assertion follows from Proposition A.3.2 and Theorem A.2.5. To prove the second assertion, assuming that $(R, \mathfrak{m})$ is moreover regular, we first show that $R_\flat$ is also regular, by induction on the dimension d of R. Since the proto-grade is faithfully flat, $(R/I)_\flat = R_\flat/IR_\flat$ for all ideals $I \subseteq R$ by 9.1.7. Applied to $I = x_1R$, where $\mathfrak{m} = (x_1, \dots, x_d)R$, we have by induction that $(R/x_1R)_\flat = R_\flat/x_1R_\flat$ is regular, whence so is $R_\flat$, since x_1 is $R_\flat$-regular (as $R \to R_\flat$ is flat). Since $R_\flat \to (R_\flat)^\sim$ is a scalar extension by Proposition A.2.4, also $(R_\flat)^\sim$ is regular by 3.2.14. Hence, in view of Proposition A.3.2, we proved that $(R^\sim)_\flat$ is regular. Since $(x_1, \dots, x_d)$ is an $(R^\sim)_\natural$-regular sequence by the flatness of $R^\sim \to (R^\sim)_\natural$ (using Theorem A.2.5 and Corollary 3.3.3), the Cohen-Macaulay criterion for flatness (Theorem 3.3.9) together with Proposition 3.3.8, yields the desired flatness of $(R^\sim)_\flat \to (R^\sim)_\natural$. □

Let k be a field and ξ a finite tuple of indeterminates. For simplicity, we denote the Henselization of the localization of $k[\xi]$ with respect to the variables also by $k[\xi]^\sim$. A power series $f \in k[[\xi]]$ is called *algebraic* if it is a root of a non-zero polynomial in one variable with coefficients in $k[\xi]$. We denote the subring of algebraic power series by $k[[\xi]]^{\mathrm{alg}}$. The following result is well-known (see, for instance, [3, 77]).

A.3.4 *For any field k, the ring $k[[\xi]]^{alg}$ is equal to the Henselization $k[\xi]^\sim$ of $k[\xi]_\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal generated by the indeterminates.*

In particular, viewing $k[\xi]$ with its affine proto-grade given by degree (see (9.1.1.i)), we get an etale proto-grade on $k[[\xi]]^{\mathrm{alg}}$: an algebraic power series f has proto-grade at most n, if there exists a Hensel system in $k[\xi, t]$ for f of size at most n, such that the total degree of each polynomial in the system is at most n. Theorem 9.2.11, in conjunction with Theorem A.3.3 and Corollary 9.2.4, applied to this etale proto-grade on the ring of algebraic power series, immediately yields:

Theorem A.3.5. *For each pair (n, m) there exists a uniform bound $n' := n'(n, m)$ with the property that if k is an arbitrary field, $R := k[[\xi]]^{alg}$ the ring of algebraic power series with ξ an m-tuple of indeterminates, and $I := (f_1, \dots, f_s)R$ an ideal generated by elements f_i of etale proto-grade at most n, then I is generated by at most n' of the f_i, and its module of syzygies is generated by n' syzygies with entries of proto-grade at most n'. Moreover, if $f \in I$ has etale proto-grade at most n, then there exist algebraic power series g_i of etale proto-grade at most n' such that $f = g_1f_1 + \dots + f_sg_s$.* □

In fact, we can now even give a non-linear version (as already mentioned, ideal membership amounts to solving a linear equation), which also extends Theorem 7.1.10 to arbitrary coefficients over the algebraic power series ring (and a similar argument using Theorem 7.1.6 would then also yield a uniform analogue of the next result).

Theorem A.3.6. *For each pair (n,m) there exists a uniform bound $n' := n'(n,m)$ with the property that if k is an arbitrary field, $R := k[[\xi]]^{alg}$ the ring of algebraic power series with ξ an m-tuple of indeterminates, and if $f_1 = \cdots = f_s = 0$, with $f_i \in R[t]$, is a polynomial system of $s \leq n$ equations in at most n unknowns t, of (extended) etale proto-grade at most n and admitting a solution in formal power series, then it admits an algebraic solution of etale proto-grade at most n'.*

Proof. Suppose no such bound holds for the pair (m,n), yielding for each w, a counterexample over a ring of algebraic power series $R_w := K_w[[\xi]]^{\mathrm{alg}}$, consisting of an n-tuple of equations $\mathbf{f}_w$ in $R_w[t]$ of extended etale proto-grade at most n, and a solution $\mathbf{y}_w$ in $K_w[[\xi]]$ of this system of equations (viewed as a system of equations in the unknowns t), but no such solution in algebraic power series of etale proto-grade at most w exists. Let K be the ultraproduct of the K_w, and let $\mathbf{y}_\natural$, $\mathbf{f}_\natural$, and $S_\natural$ be the respective ultraproducts of the $\mathbf{y}_w$, $\mathbf{f}_w$ and $K_w[[\xi]]$. By definition of the extended etale proto-grade (see 9.1.2), the $\mathbf{f}_w$ are polynomials of degree at most n with coefficients of etale proto-grade at most n, and hence their ultraproduct $\mathbf{f}_\natural$ lies in $K[[\xi]]^{\mathrm{alg}}[t]$. Moreover, by Łoś' Theorem, $\mathbf{y}_\natural$ is a solution of this system of equations in $S_\natural$, whence in $S_\sharp \cong K[[\xi]]$, by Proposition 7.1.8. Hence, by Theorem 7.1.5, we can find a solution $\mathbf{z}$ of this system of equations in $K[[\xi]]^{\mathrm{alg}}$. Let N be its etale proto-grade. By Proposition A.3.2, the ring $K[[\xi]]^{\mathrm{alg}}$ is just the protoproduct of the R_w, and hence by Łoś' Theorem, the approximations of $\mathbf{z}$ yield a counterexample for any w bigger than N. □

Appendix B
Boolean Rings

We mentioned Boolean rings in our sheaf-theoretic construction of an ultraproduct, Theorem 2.6.4. In this appendix, we generalize the notion of a Boolean ring to an n-Boolean ring, replacing the condition that all elements are idempotent by the condition that they are all n-potent (see below for definitions). The Stone Representation Theorem gives a description of Boolean rings in terms of power set rings. We give a proof, using the embedding theorems into ultrapowers from §7.1.3, in the more general context of n-Boolean rings, recovering in particular Henkin's version [41] of the Stone Representation Theorem [110]. In §B.3, we prove a similar result for an ω-Boolean ring, that is to say, a ring in which all elements are potent (but possibly of unbounded potency).[1] Not surprisingly, to control this unboundedness, we have to use protoproducts instead of ultraproducts. The last section then deals with a generalization studied already by Chacron, Bell, et al. ([19, 20, 12]), to wit, periodic rings.

B.1 n-Boolean Rings

A ring is called *torsion* if it has positive characteristic, that is to say, if $d = 0$ in R for some $0 \neq d \in \mathbb{N}$ (not necessarily prime). Any such ring admits a decomposition as a finite direct sum of rings of prime power characteristic, its so-called *primary components*:

B.1.1 *If R has torsion, say, of characteristic $p_1^{e_1} \cdots p_s^{e_s}$, with p_i distinct primes and $e_1 \geq 1$, then $R \cong R_1 \oplus \cdots \oplus R_s$, with the characteristic of R_i equal to $p_i^{e_i}$. If R is reduced, then all $e_i = 1$.*

Indeed, for each i, let d_i be the product of all $q_j := p_j^{e_j}$ except for $i = j$ and put $R_i := d_i R$. Since $R_i \cong R/\operatorname{Ann}_R(d_i)$, we may view it as a ring of characteristic q_i.

[1] This type of rings already occurs in [33], without any special name; elsewhere they are referred to as *J-rings*, after Jacobson [60], who proved that they are always commutative. In [1], the name n-Boolean refers to a different generalization of Boolean rings.

Since the d_i are relatively prime, there exist $r_i \in \mathbb{N}$ such that $1 = r_1d_1 + \cdots + r_sd_s$. For any $x \in R$, let $x_i := r_id_ix$. It follows that $x_i \in R_i$, and $x = x_1 + \cdots + x_n$. Since $x_ix_j = 0$ for $i \neq j$, the result follows readily. □

Recall that an element x in a ring R is called *nilpotent* if $x^n = 0$ for some $n \in \mathbb{N}$. By the *reduction*, R_{red}, of a ring we mean the residue ring $R/\mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of R, that is to say, the ideal of all nilpotent elements. Let $n \geq 2$. We say that an element x in a (commutative) ring R is *n-potent*, if $x^n = x$. The least such $n \geq 2$ is called the *potency* of x. Instead of 2-potent, one usually says *idempotent*. We will call an element *potent* if it is n-potent for some n, that is to say, if it has finite potency. We define the *potency* of an ideal as the maximal potency of its members. In order to discuss potencies, the following terminology will be helpful. We say that *m pre-divides n* if $m-1$ divides $n-1$, for $m,n \in \mathbb{N}$. We may also express this by saying that m is a *pre-divisor* of n, or that n is a *pre-multiple* of m. Similarly, we define the *least common pre-multiple* and the *greatest common pre-divisor* of m and n in the obvious way. Note that 2 pre-divides any number, whereas 1 only pre-divides itself.

Lemma B.1.2. *Let R be a ring, $m,n \geq 2$ integers, and $x \in R$.*

- B.1.2.i. *If x is m-potent, then it is n-potent for any pre-multiple n of m.*
- B.1.2.ii. *If x is n-potent, and m pre-divides n then $x^{\frac{n-1}{m-1}}$ is m-potent.*
- B.1.2.iii. *If x is both m-potent and n-potent, then it is also d-potent, where d is the greatest common pre-divisor of m and n.*

Proof. Let $(m-1)d = n-1$, whence $n = m + (d-1)(m-1)$. Multiplying $x^m = x$ with x^{m-1} yields $x^{2m-1} = x^m = x$. Continuing in this way, we get $x^n = x$. To prove (B.1.2.ii), observe that

$$(x^d)^m = x^{n-1+d} = x^n \cdot x^{d-1} = x \cdot x^{d-1} = x^d.$$

To prove (B.1.2.iii), suppose $m \leq n$ and we induct on n. If $m = n$, we are done, so we may assume $m < n$. Multiplying $x = x^m$ with x^{n-m} we get $x^{n-m+1} = x^n = x$. Since the greatest common pre-divisor of $(n-m+1)$ and m is also d, we are done by induction. □

By an *n-Boolean* ring B, we mean a ring in which every element is n-potent. Hence a 2-Boolean ring is just a Boolean ring, and any Boolean ring is n-Boolean for any n. If p is prime, then $\mathbb{Z}/p\mathbb{Z}$ is p-Boolean, since it is invariant under the Frobenius. Let us call a ring B *properly n-Boolean* if it is n-Boolean, but not m-Boolean for any $m < n$, that is to say, if B contains an element of potency n. It follows from Galois theory that any finite field $\mathbb{F}_q$ is properly q-Boolean. Immediately from Lemma B.1.2, we get:

Corollary B.1.3. *Let B be a ring, and $m,n \geq 2$.*

- B.1.3.i. *If B is m-Boolean, then it is n-Boolean for any pre-multiple n of m.*
- B.1.3.ii. *If B is both m-Boolean and n-Boolean, then it is also d-Boolean, where d is the greatest common pre-divisor of m and n.* □

Note that, in general, the characteristic of an n-Boolean ring can be larger than n. For instance, $\mathbb{Z}/6\mathbb{Z}$ is a 3-Boolean ring, and more generally, $\mathbb{Z}/2p\mathbb{Z}$ is p-Boolean, for every odd prime number p. We also observe that $\mathbb{Z}/15\mathbb{Z}$ and $\mathbb{Z}/30\mathbb{Z}$ are 5-Boolean but not 3-Boolean. To determine the possible characteristics, let $\alpha(n)$ be the greatest common divisor of all $k^n - k$, for $k \in \mathbb{N}$. For instance, $\alpha(2) = 2$, $\alpha(3) = 6$, $\alpha(4) = 2$, and $\alpha(5) = 30$. Note that by Fermat's Little Theorem, $\alpha(p)$ is divisible by p if p is prime, but from the previous examples, it is clear that in general it will be bigger. We will compute it in Corollary B.1.9 below, but for now we observe:

Lemma B.1.4. *If B is an n-Boolean ring, then its characteristic is a divisor of $\alpha(n)$. Moreover, $\alpha(n)$ is always square-free.*

Proof. Let d be the characteristic of B. For each k, since $k^n - k$ is zero in B, it must be divisible by d. To see the last assertion, we check that $\alpha(n)$ has p-adic order at most one, that is to say, is not divisible p^2, for any prime p. However, this is immediate since $\alpha(n)$ is a divisor of $p^n - p$, which has p-adic order one. □

In a Boolean ring B, one defines a partial order as follows: $a \leq b$ if $ab = a$, for $a, b \in B$. An element which is minimal among the non-zero elements of B is called an *atom*. Note that any multiple of an atom a is either zero or a itself, so that a is an atom if and only if aB has cardinality two. We call a Boolean ring *atomless*, if it has no atoms. In case B is a power set ring $\mathscr{P}(W)$ (see Example B.2.3), the order is given by inclusion, and so, atoms are precisely singletons. Moreover, the set of finite subsets is an ideal in B (by (B.1.5.x) below) whose residue ring is an example of an atomless Boolean ring.

Unlike Boolean rings, we cannot define a partial order on an n-Boolean ring B, for $n > 2$. Instead, we look at ideals with respect to inclusion. We call an ideal *atomic*, if it is a minimal non-zero ideal. The sum of all atomic ideals is called the *ideal of finite elements*. We call an element $x \in B$ an *atom* if it is idempotent and generates an atomic ideal. Note that this definition agrees with the older one in Boolean rings. In the next result, we gathered some basic facts about n-Boolean rings (recall that a prime ideal is called an *associated prime* if it is of the form $\mathrm{Ann}(x)$).

Proposition B.1.5. *Let B be an n-Boolean ring, $x, y \in B$ elements, $I \subseteq B$ an ideal, and $\mathfrak{p} \in \operatorname{Spec} B$ a prime ideal.*

- B.1.5.i. *$x^iB = xB$ for all $i > 0$, and any ideal is idempotent, that is to say, $I = I^2$;*
- B.1.5.ii. *x is idempotent if and only if it is of the form y^{n-1};*
- B.1.5.iii. *$xB \cap yB = xyB$ and $xB + yB = (x^{n-1} + y^{n-1} - x^{n-1}y^{n-1})B$. In particular, every finitely generated ideal is principal;*
- B.1.5.iv. *the annihilator of xB is equal to $(1 - x^{n-1})B$. It is a prime ideal if and only if xB is atomic;*
- B.1.5.v. *x is a unit in B if and only if $x^{n-1} = 1$ if and only if x is not a zero-divisor;*

B.1.5.vi. *any residue ring of B is n-Boolean. In particular, I is radical, whence equal to an intersection of prime ideals;*

B.1.5.vii. $\mathfrak{p}$ *is maximal with residue field a finite field of size at most n, isomorphic to $B_{\mathfrak{p}}$. In particular, B and any of its subrings have (Krull) dimension zero (one says that B is* hereditarily zero-dimensional*);*

B.1.5.viii. *if I is atomic, then it contains at most n elements, any non-zero element is a generator, and I contains a unique atom. Moreover, I with its induced addition and multiplication is isomorphic to a finite field $\mathbb{F}_q$, for some prime power $q = p^m$ pre-dividing n (with p dividing $\alpha(n)$), and q is the potency of I;*

B.1.5.ix. *any two distinct atoms are orthogonal, that is to say, their product is equal to zero;*

B.1.5.x. $\mathfrak{p}$ *is associated if and only if it is finitely generated, whence principal, if and only if it does not contain all finite elements. The ideal of finite elements is generated by all atoms and defines the closed subset of* Spec B *consisting of all non-principal prime ideals;*

B.1.5.xi. *there exists a non-principal prime ideal if and only if B is infinite. In particular, B is Noetherian if and only if B is finite if and only if* 1 *is a finite element;*

B.1.5.xii. *any B-module is flat, that is to say, B is absolutely flat, whence von Neumann regular.*

Proof. The inclusion $x^iB \subseteq xB$ is immediate. To prove the other inclusion, choose k such that $n^k \geq i$, and observe that $x = x^{n^k - i} \cdot x^i \in x^iB$, proving the first part of (B.1.5.i). Similarly, $I^2 \subseteq I$ and, for the converse, if $x \in I$ then $x \in x^2B \subseteq I^2$ by the first assertion. If x is idempotent, then $x = x^{n-1}$, and the converse of (B.1.5.ii) follows immediately from (B.1.2.ii) with $m = 2$. To prove (B.1.5.iii), let $z \in xB \cap yB$. If we write $z = ax = by$, then $z = ax = ax^n = byx^{n-1} \in xyB$. The second equality follows from the identity

$$(x^{n-1} + y^{n-1} - x^{n-1}y^{n-1})x = x + xy^{n-1} - xy^{n-1} = x$$

and the analogous identity for y. One direction in the first assertion of (B.1.5.iv) is clear since $x(1 - x^{n-1}) = x - x = 0$, so assume $ax = 0$. Hence $a = a(1 - x^{n-1}) \in (1 - x^{n-1})B$. This also proves (B.1.5.v), for if x is a unit, then $0 = \mathrm{Ann}(x) = (1 - x^{n-1})B$. To prove the direct inclusion in the second assertion in (B.1.5.iv), assume $\mathrm{Ann}(x)$ is prime. Given a non-zero ideal aB contained in xB, we need to show that $x \in aB$. By (B.1.5.iii), we have $axB = aB \cap xB = aB$, showing that $a \notin \mathrm{Ann}(x)$. Since $a(1 - a^{n-1}) = 0$ and $\mathrm{Ann}(x)$ is prime, $1 - a^{n-1} \in \mathrm{Ann}(x)$, showing that $x = xa^{n-1} \in aB$. Conversely, suppose xB is atomic, and we need to show that if a and b do not belong to $\mathrm{Ann}(x)$ then neither does their product. From $ax \neq 0$ and $bx \neq 0$ and the fact that xB is atomic, we get $axB = xB = bxB$ and hence $abxB = xB \neq 0$, so that, a fortiori, $abx \neq 0$.

The first assertion in (B.1.5.vi) is clear. Applying the observation that an n-Boolean ring is reduced to B/I, shows that I is radical. Since $\bar{B} := B/\mathfrak{p}$ is n-Boolean, any element in $\bar{B}$ satisfies the equation $\xi^n - \xi = 0$. However, in a domain, this equation can have at most n solutions, showing that $\bar{B}$ has cardinality at most n. As any finite domain is a field, $\mathfrak{p}$ is maximal. If $x \in \mathfrak{p}B_{\mathfrak{p}}$, then $1 - x^{n-1}$ is a unit in $B_{\mathfrak{p}}$ killing x by (B.1.5.iv), showing that $\mathfrak{p}B_{\mathfrak{p}} = 0$ and hence $B_{\mathfrak{p}} \cong B/\mathfrak{p}$, thus completing the proof of (B.1.5.vii). To prove (B.1.5.viii), suppose I is atomic. Since I must be principal, it is isomorphic to $F := B/\operatorname{Ann}(I)$. Since $\operatorname{Ann}(I)$ is maximal by (B.1.5.iv) and (B.1.5.vii), the cardinality of I is at most n by (B.1.5.vii) and F is a field. If x is a non-zero element in I, then the inclusion $xB \subseteq I$ must be an equality, proving the second assertion in (B.1.5.viii). In particular, $y := x^{n-1}$ is a non-zero idempotent in I by (B.1.5.ii), whence an atom. To show that y is the unique atom in I, suppose z is another non-zero idempotent in I. By what we just proved, $yB = zB$, so that $y = az$ and $z = by$, for some $a, b \in B$. Multiplying the first equality with z, we get $zy = az^2 = az = y$, and similarly, multiplying the second with y yields $zy = by^2 = by = z$, and hence $z = y$. To prove the last assertion in (B.1.5.viii), let q be the cardinality of the field F (recall that $q = p^m$ for some m, with p the characteristic of F). Let $I^\times := I - \{0\}$ and let $x \in I^\times$. We argued that $y := x^{n-1}$ is the unique atom of I. Multiplying with x, we get $xy = x^n = x$. It is now easy to check that $I^\times$ is a multiplicative group with unit element y and that the isomorphism $I = yB \cong F$ sending $x = xy$ to the residue of x in F yields a group isomorphism between $I^\times$ and the multiplicative group of F. Since the latter is cyclic, so is the former. In particular, there exists $x \in I^\times$ such that the powers of x generate $I^\times$. Since x has therefore potency q, so does I. If q does not pre-divide n, then x has potency at most $d < q$ by Lemma B.1.2, with d the greatest common pre-divisor of q and n, contradiction. Finally, since the characteristic of B divides $\alpha(n)$ by Lemma B.1.4, so must p, concluding the proof of (B.1.5.viii). To prove (B.1.5.ix), let x and y be distinct atoms. By (B.1.5.viii), they generate different atomic ideals $xB \neq yB$, and hence their intersection, equal to xyB by (B.1.5.iii), must be a proper subideal of either atomic ideal, whence equal to zero, showing that $xy = 0$.

The first equivalence in (B.1.5.x) is immediate by (B.1.5.iii) and (B.1.5.iv). To prove the second, let zB be an arbitrary atomic ideal. If $\mathfrak{p}$ is not an associated prime, then $z\mathfrak{p}$ cannot be zero, lest $\mathfrak{p}$ is contained in $\operatorname{Ann}(z)$ whence by maximality, equal to it. Let $a \in \mathfrak{p}$ be such that $az \neq 0$. Since zB is atomic, $zB = azB$ and hence belongs to $\mathfrak{p}$. Conversely, if every atomic ideal is contained in $\mathfrak{p}$, then $\mathfrak{p}$ cannot be associated, since otherwise by the above, $\mathfrak{p} = \operatorname{Ann}(x) = (1 - x^{n-1})B$ with xB atomic, and so $1 = x + (1 - x^{n-1}) \in \mathfrak{p}$, contradiction. The last assertion in (B.1.5.x) is then clear from the above and (B.1.5.viii). By (B.1.5.x), in order to prove (B.1.5.xi), it suffices to show that 1 is a finite element if and only if B is finite, if and only if every ideal is principal. If $1 = a_1 + \cdots + a_s$ is a sum of atomic elements a_i, then $B = a_1B + \cdots + a_sB$. Since each a_iB is finite by (B.1.5.viii), so is B. Assume next that B is finite, then any ideal is finitely generated, whence principal by (B.1.5.iii). Finally, if every maximal ideal is principal, then the ideal of finite elements must be the unit ideal by (B.1.5.x). This cycle of implications concludes the proof of (B.1.5.xi). Finally, (B.1.5.xii) follows from (B.1.5.vii) since each localization at a prime ideal is a field. Alternatively, by (B.1.5.iii), any finitely generated ideal $I \subseteq B$ is principal, say, of the form xB, and $B \cong xB \oplus B/xB$ by (B.1.5.iv). In particular, B/I, being a direct summand of B, is projective, whence flat, and hence $\mathrm{Tor}_1^B(B/I, \cdot)$ is identical zero. Absolute flatness now follows from Theorem 3.1.5. □

Remark B.1.6. Gilmer [33] showed that condition (B.1.5.xii) together with the first assertion of (B.1.5.vii) implies in turn that B is n-Boolean.

Corollary B.1.7. *Any embedding of n-Boolean rings is faithfully flat.*

Proof. Let $B \to C$ be an injective homomorphism of n-Boolean rings. The flatness of this homomorphism follows from (B.1.5.xii), so remains to show that C is non-degenerated. Suppose not, so that there exists a proper ideal $I \subseteq B$ such that $IC = C$. It follows that there must already exist a finitely generated ideal I with this property. Since I is principal by (B.1.5.iii), its generator a must be a unit in C. In particular, $a^{n-1} = 1$ in C by (B.1.5.v). Since $B \to C$ is injective, already $a^{n-1} = 1$ in B, showing that a is a unit in B. □

Corollary B.1.8. *Let $B \to C$ be an injective homomorphism of n-Boolean rings. Then an ideal $I \subseteq B$ is principal if and only if its image IC in C is.*

Proof. For the non-trivial direction, assume $I = yC$ for some $y \in C$. Hence there exist $a_1, \dots, a_n \in I$ such that y is a linear combination of these elements, that is to say, belongs to $(a_1, \dots, a_n)C$. By (B.1.5.iii), this ideal is generated by a single element a belonging to I. Hence $IC = aC$ and therefore by faithful flatness (Corollary B.1.7), we get $I = IC \cap B = aC \cap B = aB$. □

Corollary B.1.9. *For each n, the number $\alpha(n)$ is equal to the product of all prime numbers p pre-dividing n.*

Proof. If p pre-divides n, then we can write $n - 1 = (p-1)d$ for some d. Hence $k^n \equiv (k^{p-1})^d \cdot k \equiv k \mod p$ for each k, by Fermat's Little Theorem, showing that p divides each $k^n - k$, whence $\alpha(n)$.

Conversely, if p divides $\alpha(n)$, then by construction, $\mathbb{Z}/p\mathbb{Z}$ is n-Boolean. Let d be the greatest common pre-divisor of p and n. Hence, $\mathbb{Z}/p\mathbb{Z}$ is d-Boolean, by Corollary B.1.3. Since $\mathbb{Z}/p\mathbb{Z}$ is properly p-Boolean, we must have $p = d$, as we wanted to show. Since $\alpha(n)$ is square-free by Lemma B.1.4, we are done. □

Immediately from B.1.1 and the fact that an n-Boolean ring is reduced, we get:

Corollary B.1.10. *Any n-Boolean ring is a finite direct sum of n-Boolean rings of prime characteristic.* □

Proposition B.1.11. *Let n be even. Any n-Boolean ring has characteristic* 2. *In particular, if n is not a power of two, then there are no properly n-Boolean rings.*

Proof. By Corollary B.1.9, since $n - 1$ is odd, the only prime number p pre-dividing n is $p = 2$. □

For each n, let $\mathscr{B}_n$ be the collection of all finite fields whose cardinality pre-divides n. Note that any field in $\mathscr{B}_n$ is n-Boolean by Corollary B.1.3.

Theorem B.1.12. *For each n, a finite n-Boolean ring B is a direct sum of fields belonging to $\mathscr{B}_n$.*

Proof. Let $a_1,\dots,a_s$ be the atoms of B. By (B.1.5.xi), any element is a linear combination of the a_i, and by (B.1.5.ix), any two are orthogonal. In other words, B, as a ring, is isomorphic to the direct product $a_1B \oplus \cdots \oplus a_sB$, and by (B.1.5.viii), each direct summand is a field belonging to $\mathscr{B}_n$. □

Remark B.1.13. The number of atoms, whence the number of direct summands, is equal to the length $\ell(B)$ of the Artinian ring B.

Corollary B.1.14. *A finite p-Boolean ring B of characteristic p, for p a prime number, is isomorphic as a ring to $\mathbb{F}_p^s$, where $s = \ell(B)$. More generally, a finite p^m-Boolean ring of characteristic p is isomorphic to a finite direct sum of fields of characteristic p.*

Proof. The second assertion is immediate from Theorem B.1.12. For the first, observe that the only field in $\mathscr{B}_p$ of characteristic p is $\mathbb{F}_p$, since p is the maximal cardinality by (B.1.5.viii), so that the result follows from Remark B.1.13. □

Together with Corollary B.1.10, this reproves the main theorem in [66].

B.2 Stone Representation Theorem

Given a fixed ring U, we say that a ring R is *U-like*, if every finitely generated subring of R embeds, as a ring, in a finite direct product U^s. This definition applies to the current situation as follows. Define the *universal n-Boolean* ring $\mathbb{B}_n$ as the direct sum of all fields in $\mathscr{B}_n$. Immediately from Theorem B.1.12 and Corollary B.1.7, we get:

B.2.1 *If B is a finite n-Boolean ring, then there exists a faithfully flat embedding $B \to \mathbb{B}_n^s$, where $s = \ell(B)$.* □

This embedding is in general not unique, since finite fields which are not prime fields have non-trivial isomorphisms. However, if p is prime, then the isomorphism between a finite p-Boolean ring B of characteristic p and $\mathbb{F}_p^l$, with $l = \ell(B)$, is canonical up to a permutation of the factors, since the atoms of B are unique and $\mathbb{F}_p$ has no non-trivial automorphisms.

Corollary B.2.2. *A ring is n-Boolean if and only if it is $\mathbb{B}_n$-like.*

Proof. Suppose B is n-Boolean. Let V be a finitely generated subring of B. Since V is in particular Noetherian, it is finite by (B.1.5.xi), and hence, by B.2.1, embeds in some direct product $\mathbb{B}_n^s$, showing that B is $\mathbb{B}_n$-like. Conversely, suppose B is $\mathbb{B}_n$-like, and take some $x \in B$. Let V be the subring of B generated by x. By assumption, it is a subring of $\mathbb{B}_n^s$ for some s, whence is n-Boolean. In particular, x is n-potent, showing that B is n-Boolean. □

Table B.1 Cardinalities of n-Boolean fields

n	$q = p^m$ pre-divisor of n
3	$2,3$
5	$2,3,5$
7	$2,3,4,7$
9	$2,3,5,9$
11	$2,3,11$
13	$2,3,4,5,7,13$
15	$2,3,8$
17	$2,3,5,9,17$
19	$2,3,4,7,19$
21	$2,3,5,11$
23	$2,3,23$
25	$2,3,4,5,7,9,13,25$
27	$2,3,27$

Table B.1 calculates the cardinalities of the finite fields in $\mathscr{B}_n$ for some odd values of n. Note that if n is odd, then 2 and 3 are always present in this list, as they pre-divide any odd number. The case $n = 13$ shows nonetheless that even if we assume that 2 and 3 are invertible, there can still be characteristics other than n, even if n is prime. Comparing $n = 5$ with $n = 25$ shows that more characteristics can appear when we take powers. In comparison, the list for 125 is $q = 2,3,5,32,125$.

Given a ring C, recall that we previously denoted an infinite Cartesian product of C by C_∞. Since we will work over various index sets, we amend this notation by as follows: the Cartesian power over the index set X will be denoted $C_{\infty(X)}$. Note that $C_{\infty(X)}$ can be identified with the ring of all maps $f\colon X \to C$, with addition and multiplication given component-wise.

Example B.2.3. We already remarked that for a set X, the power set ring $\mathscr{P}(X)$, with addition given by the symmetric difference and multiplication by intersection is a Boolean ring. We may view it as a Cartesian power $(\mathbb{F}_2)_{\infty(X)}$, by letting C be the two-element field $\mathbb{F}_2$, identifying a subset with its characteristic function.

If C is U-like, then so is any Cartesian power of C. Note that an ultrapower of a Cartesian power is no longer a Cartesian power, but we do have:

Lemma B.2.4. *Let U be a finite ring. For each set X, the ultrapower of the Cartesian power $U_{\infty(X)}$ embeds into the Cartesian power $U_{\infty(X_\natural)}$, where $X_\natural$ is the corresponding ultrapower of X.*

Proof. Note that since U is finite, it is equal to its own ultrapower. Let $C_\natural$ be the ultrapower of $C := U_{\infty(X)}$. Viewing C as the collection of maps $X \to U$, given $f \in C_\natural$, choose maps $f_w\colon X \to U$ with ultraproduct equal to f. Define $f_\natural\colon X_\natural \to U$ as follows. For $x \in X_\natural$, choose approximations $x_w \in X$ of x, and let $f_\natural(x)$ be the ultraproduct (in U) of the elements $f_w(x_w)$. Put differently, $f_\natural(x)$ is the unique value in U equal to almost all $f_w(x_w)$. The assignment $f \mapsto f_\natural$ yields a map $C_\natural \to$

$U_{\infty(X_\natural)}$. Since addition and multiplication are defined componentwise, one easily checks that $C_\natural \to U_{\infty(X_\natural)}$ is a homomorphism. Suppose $f \in C_\natural$ is not identical zero, whence neither are almost all f_w. In particular, for almost each w, there exists $x_w \in X$ such that $f_w(x_w) \neq 0$. It follows that $f_\natural(x) \neq 0$, where x is the ultraproduct of the x_w, proving that $f_\natural$ is non-zero, and hence $C_\natural \to U_{\infty(X_\natural)}$ is injective, as we wanted to show. □

Theorem B.2.5 (Stone Representation). *Let U be a finite ring. A ring is U-like if and only if it is a subring of a Cartesian power of U. More precisely, B is U-like if and only if it admits an embedding into $U_{\infty(\mathbb{N}_\natural)}$, for some ultrapower $\mathbb{N}_\natural$ of $\mathbb{N}$.*

Proof. One direction is immediate since a subring of a U-like ring is U-like. For the opposite direction, assume B is U-like. We will show, using Remark 7.1.2, that there exists an embedding of B into some ultrapower $C_\natural$ of $C := U_{\infty(\mathbb{N})}$. Since $C_\natural$ is a subring of $U_{\infty(\mathbb{N}_\natural)}$ by Lemma B.2.4, this completes the proof. We want to verify the validity of (7.1.2.ii). Let V be a finitely generated subring of B. Hence V embeds in some finite direct product U^s, whence into C, showing that (7.1.2.ii) holds, whence also (7.1.2.iii). □

The finiteness of U is important here, and we will see in the next section how in certain instances, we can circumvent this restriction.

Theorem B.2.6. *Let $n \geq 2$. A ring is n-Boolean if and only if it is a subring of a Cartesian power of the universal n-Boolean ring $\mathbb{B}_n$. More precisely, B is n-Boolean if and only if it admits a faithfully flat embedding into $\mathbb{B}_{n\infty(\mathbb{N}_\natural)}$, for some ultrapower $\mathbb{N}_\natural$ of $\mathbb{N}$.*

If $q = p^m$ is a prime power, then a ring of characteristic p is q-Boolean if and only if it is a subring of a Cartesian power of $\mathbb{F}_q$.

Proof. The first assertion is immediate by Theorem B.2.5 with $U = \mathbb{B}_n$, and Corollary B.2.2. Faithful flatness follows from Corollary B.1.7. The last assertion follows from the fact that the only q-Boolean rings of characteristic p are the subfields of $\mathbb{F}_q$ by finite field theory (such a field must be a subfield of the field of invariants of the q-Frobenius map acting on $\mathbb{F}_p^{\text{alg}}$, and this field of invariants is precisely $\mathbb{F}_q$). □

The special case when $n = 2$, yields a version of the Stone Representation Theorem for Boolean rings a la Henkin [41], since $\mathbb{B}_2 = \mathbb{F}_2$.

Theorem B.2.7 (Stone Representation). *For each Boolean ring B, there exists an ultrapower $\mathbb{N}_\natural$ of $\mathbb{N}$ and a faithfully flat embedding $B \to \mathscr{P}(\mathbb{N}_\natural)$.* □

B.3 ω-Boolean Rings

We say that a ring B is *ω-Boolean* if each element is potent (with possibly different potency). In particular, n-Boolean rings are ω-Boolean. For an example of an ω-Boolean ring that for no n is n-Boolean, take the algebraic closure $\mathbb{F}_p^{\mathrm{alg}}$ of $\mathbb{F}_p$: every element is p^m-potent for some m, but m can be arbitrarily large. Unlike n-Boolean rings, ω-Boolean rings are no longer closed under taking ultraproducts. For instance, by Theorem 2.4.3, the ultraproduct of the ω-Boolean fields $\mathbb{F}_p^{\mathrm{alg}}$ is equal to $\mathbb{C}$, which clearly fails to be ω-Boolean. In fact, already any infinite Cartesian product of ω-Boolean rings is no longer ω-Boolean.

Lemma B.3.1. *For an ω-Boolean ring B, the following properties hold:*

B.3.1.i. *B is torsion;*
B.3.1.ii. *if B is finitely generated, then it is n-Boolean, for some n. In fact, a finite ring is ω-Boolean if and only if it is reduced, if and only if it is n-Boolean, for some n;*
B.3.1.iii. *any ideal in B is radical, any prime ideal is maximal, each localization at a prime ideal is a field, and B is hereditarily zero-dimensional;*
B.3.1.iv. *B is von Neumann regular, and any injective homomorphism between ω-Boolean rings is faithfully flat;*

Proof. Since $2^n - 2$ is zero in B, for some $n > 1$, we get (B.3.1.i). Let d be the characteristic of B. Since each generator has finite potency, it is integral over $\mathbb{Z}/d\mathbb{Z}$, and hence B is finite as a $\mathbb{Z}/d\mathbb{Z}$-module, whence finite. If n is the least common pre-multiple of all m, where m runs over all the finitely many potencies of elements in B, then B is n-Boolean by (B.1.2.i), proving the first assertion in (B.3.1.ii). We already observed that an ω-Boolean ring must be reduced, so suppose B is reduced and finite. In particular, it is Artinian, and hence a direct sum of local Artinian rings. Since the latter are reduced, they must be (finite) fields, whence, in particular, ω-Boolean. The first assertion in (B.3.1.iii) is clear, as $\bar{B} := B/I$ is ω-Boolean, whence reduced, for any ideal $I \subseteq B$. If I is prime, so that $\bar{B}$ is a domain, it must have prime characteristic, say p. Since any element is potent whence algebraic over $\mathbb{F}_p$, it is contained in $\mathbb{F}_p^{\mathrm{alg}}$. The result now follows since any subring of $\mathbb{F}_p^{\mathrm{alg}}$ is a field. By the same argument as in (B.1.5.vii), any localization of B at a prime ideal is then also a field, showing that B is absolutely flat, proving the first half of (B.3.1.iv). Suppose C is a degenerated ω-Boolean B-algebra. Hence it must already be degenerated over a finitely generated subalgebra $V \subseteq B$. Therefore, V cannot be a subring of C by Corollary B.1.7, whence a fortiori, neither can B, proving (B.3.1.iv). □

By B.1.1, a ring is ω-Boolean if and only if its primary components are. In analogy with our previous nomenclature, we say that a ring is *finite-like* (also called *locally finite*), if every finitely generated subring is finite, or, equivalently, if it is a direct limit of finite rings. By a *subquotient* of a ring A, we mean any ring of the form C/I with $C \subseteq A$ a subring and $I \subseteq C$ an ideal.

Corollary B.3.2. *For a ring B, the following are equivalent:*

B.3.2.i. *B is ω-Boolean;*
B.3.2.ii. *B is reduced and finite-like;*

B.3.2.iii. *any subquotient of B is reduced;*
B.3.2.iv. *any ideal in any subring of B is idempotent.*

Proof. Let us first show that in arbitrary ring R, every ideal is idempotent if and only if every ideal is radical. For the direct implication, let I be an ideal and $x^2 \in I$. Since $xR = x^2R$ is idempotent, we get $x \in I$, proving that I is radical. For the converse, let I be any ideal and $x \in I$. Hence $x^2 \in I^2$, and since I^2 is radical, we get $x \in I^2$, showing that $I \subseteq I^2$. This then also proves that (B.3.2.iii) and (B.3.2.iv) are equivalent. Applying either of the last two conditions to the prime subring of B, that is to say, the subring generated by 1, it is clear that B must be torsion and reduced in any of the four cases. Taking primary components, we may therefore assume that R has characteristic p. The equivalence of (B.3.2.i) and (B.3.2.ii) is immediate from (B.3.1.ii). Implication (B.3.2.i) $\Rightarrow$ (B.3.2.iii) is immediate from (B.3.1.iii). To prove the converse, let $x \in B$ and let V be the subring generated by x, that is to say, the image of $\mathbb{F}_p[\xi]$ in B under the map sending ξ to x. Since $\mathbb{F}_p[\xi]$ does not satisfy (B.3.2.iii), the kernel of this map must be non-zero. This implies that x is integral over $\mathbb{F}_p$. Therefore, V is finite, whence ω-Boolean by (B.3.1.ii), showing that x is potent. □

As already observed, it suffices to study ω-Boolean rings of prime characteristic. In that case, we can be more explicit:

Lemma B.3.3. *Any element in an ω-Boolean ring B of characteristic p is p^m-potent for some m. More generally, a potent element in a reduced ring R of characteristic p is p^m-potent, for some m.*

Proof. We only need to show the second assertion since B is reduced. Let V be the $\mathbb{F}_p$-subalgebra of R generated by x. Since x is potent, V is integral over $\mathbb{F}_p$, whence finite. Since it is reduced, it is isomorphic to a direct sum of fields, necessarily of characteristic p. Hence, for $q = p^m$ sufficiently large, all these fields are q-Boolean, whence so is their direct sum V, and hence x is q-potent. □

Example B.3.4. Without the assumption that R is reduced, the second assertion is false. For instance, in the ring $R := \mathbb{F}_3[\xi]/\xi^2\mathbb{F}_3[\xi]$, the element $\xi + 1$ is 4-potent, but for any power q of 3, we have $(\xi + 1)^q = \xi^q + 1 = 1$.

Corollary B.3.5. *A ring of characteristic p is ω-Boolean if and only if it is $\mathbb{F}_p^{alg}$-like.*

Proof. One direction is clear, since $\mathbb{F}_p^{\text{alg}}$ is ω-Boolean. Conversely, if B is ω-Boolean, then any finitely generated subring is q-potent for some power q of p by (B.3.1.ii) and Lemma B.3.3, and hence, by B.2.1, embeds in a finite direct product $(\mathbb{F}_p^{\text{alg}})^s$. □

Lemma B.3.3 suggests the following proto-grading. On a reduced ring R of characteristic p, we define a pre-proto-grading, called the *potency proto-grade*, by the condition that x has proto-grade at most n, if it is p^s-potent for some $s \leq n$.

Note that by (B.1.2.i), an element has potency proto-grade at most n if it is $p^{n!}$-potent, since p^s pre-divides $p^{n!}$ for any $s \leq n$. To verify that this constitutes a pre-proto-grading, observe that if x is p^m-potent and y is p^n-potent, then $x+y$ and xy are $q := p^{mn}$-potent. Indeed, q is a pre-multiple of both p^m and p^n, and hence x and y are both q-potent by (B.1.2.i). Therefore, $x+y$ and xy have proto-grade at most mn since $(x+y)^q = x^q + y^q = x+y$ and $(xy)^q = x^q y^q = xy$. The subring of all elements of finite potency proto-grade is called the *potency subring* of R, and is denoted $\omega(R)$. It follows immediately from Lemma B.3.3 that $\omega(R)$ is the largest ω-Boolean subring of R. Using B.1.1 and the fact that the sum of mutually orthogonal potent elements is again potent, we may generalize all this to any reduced torsion ring R:

B.3.6 *In a reduced torsion ring, the potent elements form a subring* $\omega(R)$. □

Since rings of characteristic p are uniformly proto-graded with respect to potency, we can define their protoproduct, and we have:

B.3.7 *The protoproduct* $R_\flat$ *of rings* R_w *of characteristic* p *is equal to the potency subring* $\omega(R_\natural)$ *of their ultraproduct* $R_\natural$. □

For instance, the protopower of $\mathbb{F}_p^{\mathrm{alg}}$ is $\mathbb{F}_p^{\mathrm{alg}}$ itself, since the collection of elements of proto-grade at most n in $\mathbb{F}_p^{\mathrm{alg}}$ is $\mathbb{F}_{p^n}$. Using this observation, one easily shows:

B.3.8 *For each set* X*, the potency subring of the Cartesian power* $(\mathbb{F}_p^{alg})_{\infty(X)}$ *is the direct limit of the Cartesian powers* $(\mathbb{F}_{p^n})_{\infty(X)}$*, for* $n \to \infty$. □

Theorem B.3.9. *A ring* B *of characteristic* p *is* ω*-Boolean if and only if there is a faithfully flat map from* B *into the direct limit of the* $(\mathbb{F}_{p^n})_{\infty(\mathbb{N}_\natural)}$*, where* $\mathbb{N}_\natural$ *is some ultrapower of* $\mathbb{N}$.

Proof. One direction is again clear. We can imitate the proof of Theorem B.2.5, to obtain an embedding of B into an ultrapower $C_\natural$ of $C := (\mathbb{F}_p^{\mathrm{alg}})_{\infty(\mathbb{N})}$, since any finitely generated subring is p^n-Boolean, for some n, by (B.3.1.ii). However, since a homomorphism sends potent elements to potent elements, the image of the embedding $B \to C_\natural$ must lie in $\omega(C_\natural)$, that is to say, B embeds into the protopower $C_\flat$. So, in view of B.3.8 and the fact that potent elements are sent to potent elements, it remains to show that $C_\flat$ embeds into the Cartesian power $(\mathbb{F}_p^{\mathrm{alg}})_{\infty(\mathbb{N}_\natural)}$. Let $f \in C_\flat$, having proto-grade at most N. Hence there exist approximations $f_w \in (\mathbb{F}_p^{\mathrm{alg}})_{\infty(\mathbb{N})}$ of f of proto-grade at most N, that is to say, with $q := p^N$, almost all $f_w \in (\mathbb{F}_q)_{\infty(\mathbb{N})}$. By Lemma B.2.4, we can then view f as an element in $(\mathbb{F}_q)_{\infty(\mathbb{N}_\natural)}$, and this is clearly a subring of $(\mathbb{F}_p^{\mathrm{alg}})_{\infty(\mathbb{N}_\natural)}$, as we wanted to show. □

B.4 Periodic Rings

It follows from Example B.3.4 that in an arbitrary (non-reduced) ring, the set of potent elements is in general not closed under addition. To circumvent this problem, we generalize the notion of potency: an element x in a ring R is called *periodic*, if there exist $0 < m < n$ such that $x^n = x^m$, that is to say, if the multiplicative set of all powers of x is finite. Potent and nilpotent elements are periodic, and these are essentially the source of all periodic elements, at least in torsion rings:

Lemma B.4.1. *In a torsion ring R, an element is periodic if and only if it is a sum of a nilpotent and a potent element if and only if its image in R_{red} is potent.*

Proof. Let R be a torsion ring. Since the sum of potent, nilpotent or periodic orthogonal elements is again of the same respective type, we may reduce to the primary case by B.1.1, and hence assume that R has characteristic p^m, with p prime. Let x be a periodic element in R. Assume first that R is reduced (whence of characteristic p). We want to show that x is potent. Let V be the subalgebra generated by x. Since x satisfies some equation $\xi^{i+j} - \xi^i$ for $i, j > 0$, it is integral over $\mathbb{F}_p$. Therefore, V is finite, and hence n-Boolean by (B.3.1.ii), and, in fact, q-Boolean for some power q of p by Lemma B.3.3, concluding the proof in the reduced case. Assume next that R is arbitrary, and let $R_{\text{red}} := R/\mathfrak{n}$ be the reduction of R, where $\mathfrak{n}$ is the nilradical of R. Since the image of x is periodic in R_{red}, whence q-potent for some power $q := p^e$, by the reduced case, we have $x - x^q = a$ for some $a \in \mathfrak{n}$. So remains to show that x^q is potent. It is well-known that the p-adic order of the binomial $\binom{q}{r}$ is equal to e minus the p-adic order of r. Hence, if the p-adic order of r is at most $e - m$, then $\binom{q}{r}$ is zero in R. In other words, in R, we have an identity

$$(\xi + \zeta)^q = \xi^q + \zeta^{p^{e-m+1}} f(\xi, \zeta) \tag{B.1}$$

for some polynomial $f \in \mathbb{Z}[\xi, \zeta]$. Increasing q if necessary (by taking some premultiple, see (B.1.2.i)), we may assume $a^{p^{e-m+1}} = 0$, and hence

$$x^q = (x^q + a)^q = (x^q)^q + a^{p^{e-m+1}} f(x, a) = x^{q^2},$$

by (B.1), showing that x^q is potent.

For the converse, we are left with showing that if $x = y + a$ is the sum of a potent y and a nilpotent a, then it is periodic. Applying the direct application to the potent whence periodic element y, we can write it as $y = y^q + b$ such that y^q is q-potent and b is nilpotent. Taking q sufficiently large, we may assume that the p^{e-m+1}-th powers of a, $a + b$, and b are all zero. Hence, by (B.1) applied to $x = y^q + a + b$, we get $x^q = y^{q^2}$. Since $y^q = y^{q^2}$ by assumption, we get $x^q = y^q$ and taking q-th powers gives $x^{q^2} = y^{q^2} = y^q = x^q$, showing that x is periodic. □

We generalize the definition of $\omega(R)$ to an arbitrary torsion ring R as the collection of all periodic elements; this agrees with our previous definition for reduced rings by Lemma B.4.1. Using the criterion from Lemma B.4.1 in conjunction with B.3.6, we get:

Corollary B.4.2. *In a torsion ring R, the periodic elements form a subring $\omega(R)$.* □

The example $\mathbb{Q}[[\xi]]/\xi^2\mathbb{Q}[[\xi]]$ shows that the previous results are false in non-torsion rings: the sum of the potent element 1 and the nilpotent element ξ is not periodic. Following Chacron [19], we call a ring R *periodic* if all of its elements are periodic. Clearly, ω-Boolean rings are periodic. Since $2^n - 2^m = 0$ in R, for some $m \neq n$, a periodic ring must have torsion. In view of the primary decomposition given by B.1.1, it suffices therefore to study periodic rings of prime power characteristic. The following is the Stone Representation Theorem for (commutative) periodic rings (some equivalencies were already proven in [33]).

Theorem B.4.3. *For a ring R of prime power characteristic $q = p^m$, the following are equivalent:*

B.4.3.i. *R is periodic;*
B.4.3.ii. *the reduction R_{red} of R is ω-Boolean;*
B.4.3.iii. *every element of R is the sum of a potent and a nilpotent element;*
B.4.3.iv. *R is finite-like;*
B.4.3.v. *R is integral over $\mathbb{Z}/q\mathbb{Z}$;*
B.4.3.vi. *R is hereditarily zero-dimensional;*
B.4.3.vii. *the reduction R_{red} of R embeds into the direct limit of all $(\mathbb{F}_{p^n})_{\infty(\mathbb{N}_\natural)}$, where $\mathbb{N}_\natural$ is some ultrapower of $\mathbb{N}$.*

Proof. The equivalence of (B.4.3.i) and (B.4.3.iii) is given by Lemma B.4.1. The equivalence of (B.4.3.ii) and (B.4.3.vii) is given by Theorem B.3.9. Since periodic elements are integral, (B.4.3.i) implies (B.4.3.v). The equivalence of (B.4.3.v) and (B.4.3.iv), is clear since any finitely generated integral subring is finite over $\mathbb{Z}/q\mathbb{Z}$, whence finite. Since finite rings are zero-dimensional, this then also shows (B.4.3.v) $\Rightarrow$ (B.4.3.vi). Assume next (B.4.3.vi), and let B be a finitely generated subring of R. Since B is Noetherian and by assumption zero-dimensional, it is Artinian, and hence of finite length over $\mathbb{Z}/q\mathbb{Z}$. Therefore, B itself is finite, proving (B.4.3.iv). If R is finite-like, then so is its reduction R_{red}, and hence the latter is ω-Boolean by Corollary B.3.2, proving (B.4.3.iv) $\Rightarrow$ (B.4.3.ii). Finally, the implication (B.4.3.ii) $\Rightarrow$ (B.4.3.iii) is given by Lemma B.4.1. □

Remark B.4.4. Without requiring q to be a prime power, all conditions, except the last one, (B.4.3.vii), are still equivalent by B.1.1. It follows from (B.4.3.v) that in an arbitrary ring R of characteristic d, the subring $\omega(R)$ of periodic elements is equal to the integral closure of $\mathbb{Z}/d\mathbb{Z}$ in R.

References

1. Anderson, D.: Generalizations of Boolean rings, Boolean-like rings and von Neumann regular rings. Comment. Math. Univ. St. Paul. **35**(1), 69–76 (1986) 179
2. André, M.: Localisation de la lissité formelle. Manuscripta Math. **13**, 297–307 (1974) 119
3. Artin, M.: Algebraic approximation of structures over complete local rings. Inst. Hautes Études Sci. Publ. Math. **36**, 23–58 (1969) 87, 100, 103, 104, 177
4. Aschenbrenner, M.: Ideal membership in polynomial rings over the integers. J. Amer. Math. Soc. **17**(2), 407–441 (2004) 140
5. Aschenbrenner, M.: Bounds and definability in polynomial rings. Quart. J. Math. **56**(3), 263–300 (2005) 63, 142
6. Aschenbrenner, M., Schoutens, H.: Lefschetz extensions, tight closure and big Cohen-Macaulay algebras. Israel J. Math. **161**, 221–310 (2007) 4, 42, 97, 104, 109, 110
7. Atiyah, M., Macdonald, G.: Introduction to commutative algebra. Addison-Wesley Publishing Co., Reading, Mass. (1969) 19, 31
8. Ax, J.: The elementary theory of finite fields. Ann. of Math. **88**, 239–271 (1968) 17
9. Ax, J., Kochen, S.: Diophantine problems over local fields I, II. Amer. J. Math. **87**, 605–630, 631–648 (1965) 1, 149, 150
10. Barwise, J., Eklof, P.: Lefschetz's principle. J. Algebra **13**, 554–570 (1969) 16
11. Becker, J., Denef, J., van den Dries, L., Lipshitz, L.: Ultraproducts and approximation in local rings I. Invent. Math. **51**, 189–203 (1979) 3, 63, 102, 103, 104
12. Bell, H.: On commutativity of periodic rings and near-rings. Acta Math. Acad. Sci. Hungar. **36**(3-4), 293–302 (1981) (1980) 179
13. Bourbaki, N.: Commutative Algebra, vol. 1-7. Springer-Verlag, New York (1983) 172
14. Boutot, J.F.: Singularités rationelles et quotients par les groupes réductifs. Invent. Math. **88**, 65–68 (1987) 78
15. Brenner, H., Monsky, P.: Tight closure does not commute with localization (2007). ArXiv:0710.2913 70, 92
16. Briançon, J., Skoda, H.: Sur la clôture intégrale d'un idéal de germes de fonctions holomorphes en un point de C^n. C. R. Acad. Sci. Paris **278**, 949–951 (1974) 75
17. Bruns, W., Herzog, J.: Cohen-Macaulay Rings. Cambridge University Press, Cambridge (1993) 42, 95, 158, 166, 167
18. Buchberger, B.: An algorithmic criterion for the solvability of algebraic systems of equations. Aequationes Math. **4**, 374–383 (1970) 63
19. Chacron, M.: Certains anneaux périodiques. Bull. Soc. Math. Belg. **20**, 66–78 (1968) 179, 192
20. Chacron, M.: On quasi periodic rings. J. Algebra **12**, 49–60 (1969) 179
21. Chase, S.: Direct products of modules. Trans. Amer. Math. Soc. **97**, 457–473 (1960) 45
22. D. Mumford, D.: The red book of varieties and schemes, *Lect. Notes in Math.*, vol. 1358, expanded edn. Springer-Verlag, Berlin (1999). Includes the Michigan lectures (1974) on curves and their Jacobians; with contributions by Enrico Arbarello 29

23. Denef, J., Lipshitz, L.: Ultraproducts and approximation in local rings II. Math. Ann. **253**, 1–28 (1980) 3, 103
24. Denef, J., Schoutens, H.: On the decidability of the existential theory of $\mathbb{F}_p[[t]]$. In: Valuation theory and its applications, Vol. II (Saskatoon, 1999), *Fields Inst. Commun.*, vol. 33, pp. 43–60. Amer. Math. Soc. (2003) 103
25. van den Dries, L.: Algorithms and bounds for polynomial rings. In: Logic Colloquium, pp. 147–157 (1979) 3
26. Ein, L., Lazarsfeld, R., Smith, K.: Uniform bounds and symbolic powers on smooth varieties. Invent. Math. **144**, 241–252 (2001) 78
27. Eisenbud, D.: Commutative Algebra with a View toward Algebraic Geometry, *Graduate Texts in Mathematics*, vol. 150. Springer-Verlag, New York (1995) 14, 32, 46, 56, 93, 148, 171, 172
28. Eklof, P.: Lefschetz's principle and local functors. Proc. Amer. Math. Soc. **37**, 333–339 (1973) 16
29. Eršhov, Y.: On the elementary theory of maximal normed fields I. Algebra i Logica **4**, 31–69 (1965) 5, 150
30. Eršhov, Y.: On the elementary theory of maximal normed fields II. Algebra i Logica **5**, 8–40 (1966) 5, 150
31. Evans, E., Griffith, P.: The syzygy problem. Ann. of Math. **114**, 323–333 (1981) 165
32. Fried, M., Jarden, M.: Field Arithmetic. University of Michigan Press (1986) 16
33. Gilmer, R.: Commutative rings with periodic multiplicative semigroup. Comm. Algebra **21**(11), 4025–4028 (1993) 179, 184, 192
34. Glaz, S.: Commutative coherent rings, *Lect. Notes in Math.*, vol. 1371. Springer-Verlag, Berlin (1989) 20, 141
35. Glaz, S.: Commutative coherent rings: Historical perspective and current developments. Nieuw Arch. Wisk. **10**, 37–56 (1992) 48
36. Goldblatt, R.: Lectures on the hyperreals, *Graduate Texts in Mathematics*, vol. 188. Springer-Verlag, New York (1998). An introduction to nonstandard analysis 15
37. Grothendieck, A., Dieudonné, J.: Elements de Géométrie Algébrique I-IV, *Inst. Hautes Études Sci. Publ. Math.*, vol. 4, 8, 11, 17, 20, 24, 28. Presses universitaires de France, Paris (1960-1965) 36
38. Hara, N.: A characterization of rational singularities in terms of injectivity of Frobenius maps. Amer. J. Math. **120**, 981–996 (1998) 78
39. Hartshorne, R.: Algebraic Geometry. Springer-Verlag, New York (1977) 14
40. Heitmann, R.: The direct summand conjecture in dimension three. Ann. of Math. **156**, 695–712 (2002) 5, 149
41. Henkin, L.: Some interconnections between modern algebra and mathematical logic. Trans. Amer. Math. Soc. **74**, 410–427 (1953) 179, 187
42. Hermann, G.: Die Frage der endlich vielen Schritte in der Theorie der Polynomideale. Math. Ann. **95**, 736–788 (1926) 63
43. Hochster, M.: Topics in the Homological Theory of Modules over Commutative Rings, *CBMS Regional Conf. Ser. in Math*, vol. 24. Amer. Math. Soc., Providence, RI (1975) 4, 80, 149, 165
44. Hochster, M.: Canonical elements in local cohomology modules and the direct summand conjecture. J. Algebra **84**, 503–553 (1983) 4, 165
45. Hochster, M.: Solid closure. In: Commutative algebra: syzygies, multiplicities, and birational algebra, *Contemp. Math.*, vol. 159, pp. 103–172. Amer. Math. Soc., Providence (1994) 92
46. Hochster, M.: Big Cohen-Macaulay algebras in dimension three via Heitmann's theorem. J. Algebra **254**, 395–408 (2002) 5, 149, 167
47. Hochster, M., Huneke, C.: Tightly closed ideals. Bull. Amer. Math. Soc. **18**(1), 45–48 (1988) 3
48. Hochster, M., Huneke, C.: Tight closure, invariant theory, and the Briançon-Skoda theorem. J. Amer. Math. Soc. **3**, 31–116 (1990) 3

49. Hochster, M., Huneke, C.: Infinite integral extensions and big Cohen-Macaulay algebras. Ann. of Math. **135**, 53–89 (1992) 91
50. Hochster, M., Huneke, C.: F-regularity, test elements, and smooth base change. Trans. Amer. Math. Soc. **346**, 1–62 (1994) 3
51. Hochster, M., Huneke, C.: Tight closure of parameter ideals and splitting in module-finite extensions. J. Alg. Geom. **3** (1994) 3
52. Hochster, M., Huneke, C.: Applications of the existence of big Cohen-Macaulay algebras. Adv. in Math. **113**, 45–117 (1995) 93
53. Hochster, M., Huneke, C.: Tight closure. In: Commutative Algebra, vol. 15, pp. 305–338 (1997) 3
54. Hochster, M., Huneke, C.: Tight closure in equal characteristic zero (2000). Preprint on http://www.math.lsa.umich.edu/˜hochster/tcz.ps.Z 4, 80
55. Hochster, M., Huneke, C.: Comparison of symbolic and ordinary powers of ideals. Invent. Math. **147**, 349–369 (2002) 78
56. Hochster, M., Roberts, J.: Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay. Adv. in Math. **13**, 115–175 (1974) 4, 165
57. Hodges, W.: Model Theory. Cambridge University Press, Cambridge (1993) 8, 11, 14, 15, 16
58. Hübl, R., Swanson, I.: Discrete valuations centered on local domains. J. Pure Appl. Algebra **161**(1-2), 145–166 (2001) 124
59. Huneke, C.: Tight Closure and its Applications, *CBMS Regional Conf. Ser. in Math*, vol. 88. Amer. Math. Soc. (1996) 65, 71, 72, 78, 80, 91
60. Jacobson, N.: Structure theory for algebraic algebras of bounded degree. Ann. of Math. (2) **46**, 695–707 (1945) 179
61. Kochen, S.: Ultraproducts in the theory of models. Ann. of Math. **74**, 221–261 (1961) 1
62. Kollár, J.: Flatness criteria. J. Algebra **175**, 715–727 (1995) 49
63. Kossak, R., Schmerl, J.: The structure of models of Peano arithmetic, *Oxford Logic Guides*, vol. 50. The Clarendon Press Oxford University Press, Oxford (2006) 15
64. Lang, S.: On quasi algebraic closure. Ann. of Math. **55**, 373–390 (1952) 150
65. Łoś, J.: Quelques remarques, théorèmes et problèmes sur les classes définissables d'algèbres. In: Mathematical interpretation of formal systems, pp. 98–113. North-Holland Publishing Co., Amsterdam (1955) 1
66. Luh, J.: On the structure of J-rings. Amer. Math. Monthly **74**, 164–166 (1967) 185
67. Marker, D.: Introduction to the model theory of fields. In: Model Theory of Fields, *Lect. Notes in Logic*, vol. 5, pp. 1–37. Springer-Verlag, Berlin (1996) 11, 14
68. Matsumura, H.: Commutative Algebra. W.A. Benjamin (1970) 67
69. Matsumura, H.: Commutative Ring Theory. Cambridge University Press, Cambridge (1986) 19, 32, 33, 43, 46, 49, 50, 55, 57, 74, 91, 116, 119, 120, 121, 122, 144, 147, 148
70. Milne, J.: Etale Cohomology. 33. Princeton Math. (1980) 171, 172, 173
71. Nagata, M.: Local Rings. Interscience Tracts in Pure and Applied Mathematics. Interscience Publishers (1962) 171, 172
72. Olberding, B., Saydam, S.: Ultraproducts of commutative rings. In: Commutative ring theory and applications (Fez, 2001), *Lecture Notes in Pure and Appl. Math.*, vol. 231, pp. 369–386. Dekker, New York (2003) 58
73. Olberding, B., Saydam, S., Shapiro, J.: Completions, valuations and ultrapowers of Noetherian domains. J. Pure Appl. Algebra **197**(1-3), 213–237 (2005) 58
74. Olberding, B., Shapiro, J.: Prime ideals in ultraproducts of commutative rings. J. Algebra **285**(2), 768–794 (2005) 58
75. Peskine, C., Szpiro, L.: Dimension projective finie et cohomologie etale. Inst. Hautes Études Sci. Publ. Math. **42**, 47–119 (1972) 4, 80, 165
76. Popescu, D.: General Néron desingularization and approximation. Nagoya Math. J. **104**, 85–115 (1986) 100
77. Raynaud, M.: Anneaux Locaux Henséliens, *Lect. Notes in Math.*, vol. 169. Springer-Verlag (1970) 171, 172, 177

78. Rees, D.: A note on analytically unramified local rings. J. London Math. Soc. **36**, 24–28 (1961) 124
79. Roberts, P.: Le théorème d'intersections. C. R. Acad. Sci. Paris **304**, 177–180 (1987) 165
80. Robinson, A.: Non-standard analysis. North-Holland Publishing Co., Amsterdam (1966) 15
81. Rothmaler, P.: Introduction to model theory, *Algebra, Logic and Applications*, vol. 15. Gordon and Breach Science Publishers, Amsterdam (2000) 8, 11
82. Rotthaus, C.: On the approximation property of excellent rings. Invent. Math. **88**, 39–63 (1987) 107
83. Sabbagh, G.: Coherence of polynomial rings and bounds in polynomial ideals. J. Algebra **31**, 499–507 (1974) 137, 141
84. Schmidt, K.: Bounds and definability over fields. J. Reine Angew. Math. **377**, 18–39 (1987) 3, 59, 63
85. Schmidt, K.: Polynomial bounds in polynomial rings over fields. J. Algebra **125**(1), 164–180 (1989) 63
86. Schmidt, K., van den Dries, L.: Bounds in the theory of polynomial rings over fields. A non-standard approach. Invent. Math. **76**, 77–91 (1984) 3, 5, 51, 57, 59
87. Schoutens, H.: Artin approximation. Master's thesis, KULeuven, Leuven (1981) 171
88. Schoutens, H.: Bounds in cohomology. Israel J. Math. **116**, 125–169 (2000) 51, 59, 62
89. Schoutens, H.: Uniform bounds in algebraic geometry and commutative algebra. In: Connections between model theory and algebraic and analytic geometry, *Quad. Mat.*, vol. 6, pp. 43–93. Dept. Math., Seconda Univ. Napoli, Caserta (2000) 51, 59, 104
90. Schoutens, H.: Lefschetz principle applied to symbolic powers. J. Algebra Appl. **2**, 177–187 (2003) 78
91. Schoutens, H.: Mixed characteristic homological theorems in low degrees. C. R. Acad. Sci. Paris **336**, 463–466 (2003) 114, 149, 152, 161
92. Schoutens, H.: Non-standard tight closure for affine $\mathbb{C}$-algebras. Manuscripta Math. **111**, 379–412 (2003) 51, 87, 88, 89, 90
93. Schoutens, H.: Projective dimension and the singular locus. Comm. Algebra **31**, 217–239 (2003) 46
94. Schoutens, H.: Canonical big Cohen-Macaulay algebras and rational singularities. Illinois J. Math. **48**, 131–150 (2004) 5
95. Schoutens, H.: Log-terminal singularities and vanishing theorems via non-standard tight closure. J. Alg. Geom. **14**, 357–390 (2005) 5, 78, 90
96. Schoutens, H.: Absolute bounds on the number of generators of Cohen-Macaulay ideals of height at most two. Bull. Soc. Math. Belg. **13**, 719–732 (2006) 118
97. Schoutens, H.: Asymptotic homological conjectures in mixed characteristic. Pacific J. Math. **230**, 427–468 (2007) 114, 149, 152, 155, 161
98. Schoutens, H.: Bounds in polynomial rings over Artinian local rings. Monatsh. Math. **150**, 249–261 (2007) 59, 137, 142
99. Schoutens, H.: Pure subrings of regular rings are pseudo-rational. Trans. Amer. Math. Soc. **360**, 609–627 (2008) 5, 78
100. Schoutens, H.: Characteristic p methods in characteristic zero via ultraproducts. In: B.O. M. Fontana S. Kabbaj, I. Swanson (eds.) Recent Developments in Commutative Algebra. Springer-Verlag (2010) 5
101. Schoutens, H.: Dimension theory for local rings of finite embedding dimension (2010). ArXiv:0809.5267v1 5, 113, 114, 118, 125, 149, 161, 166
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103. Seidenberg, A.: Constructions in algebra. Trans. Amer. Math. Soc. **197**, 273–313 (1974) 63, 134, 140
104. Seidenberg, A.: Construction of the integral closure of a finite integral domain. Proc. Amer. Math. Soc. **52**, 368–372 (1975) 63
105. Seidenberg, A.: Constructions in a polynomial ring over the ring of integers. Amer. J. Math. **100**, 685–703 (1978) 63

106. Serre, J.: Corps Locaux. Hermann, Paris (1968) 171, 172
107. Smith, K.: Tight closure of parameter ideals. Invent. Math. **115**, 41–60 (1994) 92
108. Smith, K.: F-rational rings have rational singularities. Amer. J. Math. **119**, 159–180 (1997) 78
109. Spivakovsky, M.: A new proof of D. Popescu's theorem on smoothing of ring homomorphisms. J. Amer. Math. Soc. **12**, 381–444 (1999) 100
110. Stone, M.: The theory of representation for Boolean algebras. Transactions of the American Mathematical Society **40**(1), 37–111 (1936) 179
111. Swan, R.: Néron-Popescu desingularization (Spring 1995). Expanded notes from a Univ. of Chicago series of lectures 100
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Addresses:

Professor J.-M. Morel, CMLA,
École Normale Supérieure de Cachan,
61 Avenue du Président Wilson,
94235 Cachan Cedex, France
E-mail: Jean-Michel.Morel@cmla.ens-cachan.fr

Professor F. Takens, Mathematisch Instituut,
Rijksuniversiteit Groningen, Postbus 800,
9700 AV Groningen, The Netherlands
E-mail: F.Takens@rug.nl

Professor B. Teissier,
Institut Mathématique de Jussieu,
UMR 7586 du CNRS,
Équipe "Géométrie et Dynamique",
175 rue du Chevaleret,
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