A New Algorithm for Generalized Wavelet Transform

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Abstract. In order to enhance the speed of wavelet transform in signal processing, in this paper an accelerative computing theory is elaborated for generalized wavelet transform and the fast lifted wavelet transform from the perspective of the multi-resolution analysis theory. The capability of accelerative algorithm is proved in theory. Then the accelerative computing procedure for a series of bi-orthogonal Haar wavelet is demonstrated. Appling this idea to multi-resolution representation for medical image, the quality of image is retained and the running time is saved effectively.

Keywords: Accelerative algorithm, Multi-resolution, Embedded sub-analysis.

1 Introduction

A common application of the Discrete Wavelet Transform (DWT) is in image and signal processing[1-3]. Image compression is playing an important role in the modern life with a rapid increase in the amount of digital camera. But in many fields, the DWT is quite a costly operation. So some people proposed fast algorithm to speed it up[4-6].

On the other hand, the generalized fast wavelet transform is a linear time algorithm when the supports of filters are uniformly bounded[7]. But for general filters, the time complexity of wavelet transform algorithm may be nonlinear. So it is necessary to study the time complexity of wavelet transform and the accelerative algorithm when the supports of filters are not uniformly bounded.

In this paper the accelerative computing theory, algorithm for generalized fast wavelet transform is proposed. This accelerative algorithm can be applied into the lifted wavelet transform. The paper is organized as follows. In section 2, the accelerative algorithm for generalized wavelet transform is presented. The author intends to give analysis of time complexity for accelerative algorithm. Section 3 presents a series of bi-orthogonal Haar wavelet and accelerative computing procedure. Section 4 applies this idea to multi-resolution representation for medical image. Finally, section 5 contains the conclusion for this paper.

2 Accelerative Algorithm for Generalized Wavelet Transform

2.1 Accelerative Algorithm Theory

The generalized multi-resolution analysis theory is the foundation for fast wavelet transform. It is defined as follows[7].

L. Zhang, J. Kwok, and B.-L. Lu (Eds.): ISNN 2010, Part I, LNCS 6063, pp. 311–318, 2010.

Definition 1. A multi-resolution analysis M of L_2 is a sequence of closed subspaces $M = \{V_j \subset L_2 \mid j \in J \subset Z\}$, so that

1. $V_j \subset V_{j+1}$,
2. $\bigcup_{j \in J} V_j$ is dense in L_2,
3. For $\forall j \in J, V_j$ has a Riesz basis given by scaling functions $\{\varphi_{j,k} \mid k \in K(j)\}$.

Where $L_2 = L_2(X, \Sigma, \mu)$ is a general function space, with $X \subset R^n$ being the spatial domain, Σ is a σ-algebra, and μ is a non-atomic measure on Σ. We do not require the measure to be translation invariant, so weighted measures are allowed.

The $K(j)$ is regarded as general index set and $K(j) \subset K(j+1)$. Let M and M' are two multi-resolution analysis of L_2, where $M' = \{V_{j'} \subset L_2 \mid j' \in J' \subset Z\}$. For describing the procedure of accelerative algorithm theory, we give the following definition.

Definition 2. A multi-resolution analysis M' is called embedded sub-analysis of M if:

1. $M' \subset M$,
2. $\tilde{M}' \subset \tilde{M}$, where $\tilde{M}$ is a dual multi-resolution analysis of M,
3. For $\forall j' \in J'$, $V_{j'}$ has the same scaling functions base and wavelet functions base in M and M', either do $\tilde{V}_{j'}$.

This definition shows that for $\forall f \in L_2, f$ has the same projection coefficients in M and M'.

Assume $V_p \in M, V_q \in M$ and $p > q$.

For $\forall f \in L_2$, the wavelet transform from V_p to V_q is denoted as $Wf(M, p, q)$ in M.

If the V_p and V_q are belong to M', and M' is embedded sub-analysis of M. It is clear that the efficiency of $Wf(M', p, q)$ is higher than one of $Wf(M, p, q)$.

Because of $J \subset Z$, so the index set J is discrete.

Definition 3. For $\forall p, q \in J, p > q$, p and q are called partners in set J, if and only if for $\forall r \in J$, $r \geq p > q$ or $p > q \geq r$.

Specially, there is only one step in the $Wf(M', p, q)$ when the indices p and q are partners in set J'.

2.2 Accelerative Algorithm Procedure

In pseudo code the procedure of $Wf(M, p, q)$ is as follows[7].

```
for j = p-1  downto  q  step -1
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$$\lambda_j = \tilde{H}_j \lambda_{j+1}$$

$$\gamma_j = \tilde{G}_j \lambda_{j+1}$$

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next j
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When the index p and q are partners, the procedure of $Wf(M', p, q)$ is

$$\lambda_q = \tilde{H}\ \lambda_p,\ \gamma_q = \tilde{G}\ \lambda_p, \tag{1}$$

where $\tilde{H} = \tilde{H}_q \tilde{H}_{q+1} \cdots \tilde{H}_{p-1}$, $\tilde{G} = \tilde{G}_q \tilde{H}_{q+1} \cdots \tilde{H}_{p-1}$. It is the accelerative algorithm.

These two algorithms lead to the same results are followed:

$$\lambda_q = \tilde{H}_q \tilde{H}_{q+1} \cdots \tilde{H}_{p-1} \lambda_p, \gamma_q = \tilde{G}_q \tilde{H}_{q+1} \cdots \tilde{H}_{p-1} \lambda_p\ . \tag{2}$$

2.3 Analysis of Time Complexity for Accelerative Algorithm

Now we analyze the time complexity of these algorithms. The running time of the algorithm is decided by matrix multiplication, which contains scalar multiplication and addition. The number of scalar multiplication is denoted as $count(M, j)$ In the j-th iteration cycle. So we have:

$$\begin{aligned} count(M, j) &= r(\tilde{H}_j) \times c(\tilde{H}_j) \times c(\lambda_{j+1}) + r(\tilde{G}_j) \times c(\tilde{G}_j) \times c(\lambda_{j+1}) \\ &= r(\tilde{H}_j) \times c(\tilde{H}_j) + r(\tilde{G}_j) \times c(\tilde{G}_j) \end{aligned}, \tag{3}$$

where the $r(\cdot)$ and $c(\cdot)$ represent the dimensions of matrix.

Total number of scalar multiplication in $Wf(M, p, q)$ is:

$$count(M) = \sum_{j=p-1}^{q} [r(\tilde{H}_j) \times c(\tilde{H}_j) + r(\tilde{G}_j) \times c(\tilde{G}_j)]\ . \tag{4}$$

From the matrix multiplication and multi-resolution analysis definition, we have the following equations.

$$\forall j \in J, \quad c(\tilde{H}_j) = c(\tilde{G}_j) = r(\lambda_{j+1}),\ r(\tilde{H}_j) + r(\tilde{G}_j) = r(\lambda_{j+1})\ . \tag{5}$$

This results in

$$\begin{aligned} count(M) &= \sum_{j=p-1}^{q} [r(\tilde{H}_j) \times c(\tilde{H}_j) + r(\tilde{G}_j) \times c(\tilde{G}_j)] \\ &= \sum_{j=p-1}^{q} c(\tilde{H}_j)^2 \end{aligned}. \tag{6}$$

Similarly, the total number of scalar multiplication in $Wf(M', p, q)$ is:

$$count(M') = r(\tilde{H}) \times c(\tilde{H}) \times c(\lambda_p) + r(\tilde{G}) \times c(\tilde{G}) \times c(\lambda_p) = c(\tilde{H}_{p-1})^2\ . \tag{7}$$

So the efficiency of accelerative algorithm is defined as $1 - (c(\tilde{H}_{p-1})^2 \Big/ \sum_{j=p-1}^{q} c(\tilde{H}_j)^2)$. If $c(\tilde{H}_{k-1}) / c(\tilde{H}_k) = 1/2$ for $\forall k$, The accelerative efficiency are shown in Table 1.

In this case, we have $1-(c(\tilde{H}_{p-1})^2 \Big/ \sum_{j=p-1}^{q} c(\tilde{H}_j)^2) \to 1/3$, when $p - q \to \infty$.

Note that in the accelerative algorithm there is no restriction on the filters. If the support of wavelet filters is uniformly bounded, we only care for scaling coefficients. This conclusion is also correct.

Table 1. Accelerative efficiency with different $p-q$

$p-q$	2	3	4	5
Accelerative efficiency	20.00%	23.81%	24.71%	24.93%

3 Accelerative Algorithm for Bi-orthogonal Haar Wavelet

In [7] the generalized orthogonal Haar wavelet is presented. The index set $L(j,k)$ of this Haar wavelet contains either 2 or 3 elements. Now we generalize a series of Haar wavelet. In which the index set $L(j,k)$ contains t elements, instead of 2 or 3 elements. Then the accelerative computing procedure for Haar wavelet is demonstrated.

3.1 Generalized Bi-orthogonal Haar Wavelet

A set of measurable subsets $\{X_{j,k} \mid j \in J, k \in K(j)\}$ is called a nested set of partitioning[7] if it is a set of partitioning, and for $\forall j, \forall k$, there exists index set $L(j,k)$, so that

$$X_{j,k} = \bigcup_{l \in L(j,k)} X_{j+1,l} \quad . \tag{8}$$

The spaces V_j and $\tilde{V}_j$ are defined as[7]

$$V_j = closspan\{\varphi_{j,k} \mid k \in K(j)\} \subset L_2\,,\ \tilde{V}_j = closspan\{\tilde{\varphi}_{j,k} \mid k \in K(j)\} \subset L_2\,, \tag{9}$$

where $\varphi_{j,k} = \chi_{X_{j,k}}$, $\tilde{\varphi}_{j,k} = \chi_{X_{j,k}} / \mu(X_{j,k})$.

In the following part we assume for $\forall j, \forall k$, $|L(j,k)| = t \geq 2$, where t is a constant.

It is easy to proof the space V_j generate a multi-resolution analysis denoted as $M(t)$ of L_2, with the scaling function $\varphi_{j,k}$. The space $\tilde{V}_j$ generate a dual multi-resolution analysis with the dual scaling function $\tilde{\varphi}_{j,k}$.

Because there are t elements in index set $L(j,k)$, let

$$L(j,k) = \{m_1, m_2, m_3, \cdots, m_t\}\,, \tag{10}$$

where $m_1 = k$.

For every $j \in J$, wavelet ψ_{j,m_i} and dual wavelet $\tilde{\psi}_{j,m_i}$ are defined as

$$\psi_{j,m_i} = \tilde{\psi}_{j,m_i} = \frac{\sum_{s=1}^{i-1} \varphi_{j+1,m_s}}{2\sum_{s=1}^{i-1} \mu(X_{j+1,m_s})} - \frac{\varphi_{j+1,m_i}}{2\mu(X_{j+1,m_i})}, \tag{11}$$

where $i \in \{2,3,\cdots,t-1,t\}$.

From Eq.(9) we have

$$\tilde{V}_j = V_j,\ W_j = \tilde{W}_j, \tag{12}$$

where W_j is the orthogonal complement of V_j in V_{j+1}.

Now we get a series of multi-resolution analyses $M(t)$, $t = 2,3,\cdots$. In each of $M(t)$, the scaling functions are bi-orthogonal and orthogonal. Either do wavelets.

For every $M(t)$, because of $\tilde{\varphi}_{j,k} = \chi_{X_{j,k}} / \mu(X_{j,k})$ and $X_{j,k} = \bigcup_{l \in L(j,k)} X_{j+1,l}$, so the filters are below.

$$\forall l \in L(j,k), \quad \tilde{h}_{j,k,l} = \frac{\mu(X_{j+1,l})}{\mu(X_{j,k})}. \tag{13}$$

From Eq. (11), we have

$$\tilde{g}_{j,m_i,l} = \begin{cases} \dfrac{\mu(X_{j+1,l})}{2\sum_{s=1}^{i-1} \mu(X_{j+1,m_s})}, & l \in \{m_1, m_2, \cdots, m_{i-1}\} \\ 1/2, & l = m_i \end{cases}. \tag{14}$$

3.2 Accelerative Algorithm for Haar Wavelet

From the constructing of $M(t)$, we see that every multi-resolution analysis $M(t)$ comes from a nested set of partitioning $\{X_{j,k} \mid j \in J, k \in K(j)\}$. The nested set of partitioning which generate $M(t)$ is denoted as $\{X_{j,k}^{(t)} \mid j \in J, k \in K(j,t)\}$.

For fixed $t_0 \ge 2$, then there are $t_0 - 1$ Haar wavelet bases in every W_j of $M(t_0)$.

The procedure of $Wf(M(t_0), p, q)$ is given by

```
For  j = p-1  downto  q  step -1
```

$$\lambda_j = \tilde{H}_j \lambda_{j+1}$$

$$\gamma_j = \tilde{G}_j \lambda_{j+1}$$

```
next j
```

where the filters $\tilde{H}_j, \tilde{G}_j$ are described in Eq.(13) and (14).

Now for the fixed subspace V_p and V_q in $M(t_0)$, we will construct the embedded sub-analysis $M(t_0^{p-q})$ by the following procedure. So that the subspace V_p and V_q belong to $M(t_0^{p-q})$, where the indices p and q are partners.

The set $\{X_{j,k}^{(t_0)} \mid j \in J, k \in K(j,t_0)\}$ are denoted simply as $\{X_{j,k} \mid j \in J, k \in K(j)\}$, which generates the $M(t_0)$. For $\forall j, \forall k$, the $L(j,k)$ in Eq.(8) contain t_0 elements.

Now we define a new set as follows.

$$\Omega = \{X_{j,k} \mid j = p + (p-q)r, j \in J, r \in Z, k \in K(j)\} \ . \tag{15}$$

For $\forall j, \forall k$, there exists index set $L'(j,k)$, so that $X_{j,k} = \bigcup_{l \in L'(j,k)} X_{j+p-q,l}$, where the number of elements in $L'(j,k)$ is a fixed natural number t_0^{p-q}.

It is easy to validate the space V_j and $\tilde{V}_j$ satisfy three conditions of definition 1, where $j = p + (p-q)r, j \in J$. So a multi-resolution analysis and its dual multi-resolution analysis are generated by respectively space V_j and $\tilde{V}_j$, denoted as $M(t_0^{p-q})$ and $\tilde{M}(t_0^{p-q})$. Now we reach the following conclusion:

Theorem 1. For $\forall j = p + (p-q)r, j \in J$, the scaling functions which construct V_j in $M(t_0)$ are the same as in $M(t_0^{p-q})$, either do $\tilde{V}_j$.

This theorem shows that for $\forall f \in L_2$, f has the same projection coefficients in $M(t_0)$ and $M(t_0^{p-q})$.

So the accelerative procedure for $Wf(M(t_0), p, q)$ is $Wf(M(t_0^{p-q}), p, q)$ which is the wavelet transform from V_p to V_q in $M(t_0^{p-q})$. The procedure is

$$\lambda_q = \tilde{H}\ \lambda_p\ ,\ \gamma_q = \tilde{G}\ \lambda_p\ , \tag{16}$$

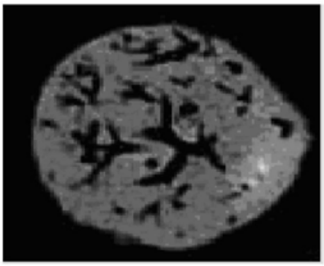
where the elements of filter $\tilde{H}$ are below

$$\forall l, k, \quad \tilde{h}_{k,l} = \mu(X_{p,l}) / \mu(X_{q,k}) \ . \tag{17}$$

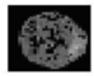
There are two options of filter $\tilde{G}$. First, dual wavelet bases in $\tilde{M}(t_0^{p-q})$ are the same as in $\tilde{M}(t_0)$, then filter $\tilde{G} = \tilde{G}_q \tilde{H}_{q+1} \cdots \tilde{H}_{p-1}$. Second, dual wavelet bases in $\tilde{M}(t_0^{p-q})$ are constructed with Eq.(11). So the elements of filter are described in Eq.(14) .

4 Experimental Results

In this session we apply the accelerative algorithm in multi-resolution representation for medicine computer tomography (CT) in order to validate its capability. First step is building the partitioning gridding for computer tomography data. Then construct bi-orthogonal Haar wavelet and accelerate the computing procedure. In this wavelet, we have $|L(j,k)|=2, \text{for } \forall j,k$. Fig.1 illustrates the original medicine CT and transformed image.



a. Original medicine CT



b. Transformed image with $p-q=4$

Fig. 1. Original medicine CT and transformed image

In this experiment the transformed images have the same quality whether the transform is accelerated. This results from theorem 1. But the accelerated transform has saved about 20 percent of time expenditure. This accelerative efficiency is less than theory one in table 1. In fact, besides the wavelet transform the experiment has I/O and other time expenditure.

5 Conclusion

In this paper, a new framework of generalized fast wavelet transform is presented, within which one can accelerate the computing of generalized fast wavelet transform. The analysis of time complexity shows that the running time is saved. This idea can be applicable to the fast lifted wavelet transform and dual lifting scheme as well. The accelerative algorithm is applied to multi-resolution representation for medical image and has a striking effect in running time.

Acknowledgement

This work is supported by NSF Grant #50608072, Natural Science Foundation Project of CQ CSTC 2009BB2375.

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