Range Parameter Induced Bifurcation in a Single Neuron Model with Delay-Dependent Parameters

Min Xiao[1,2] and Jinde Cao[2]

[1] School of Mathematics and Information Technology,
Nanjing Xiaozhuang University, Nanjing 210017, China
[2] Department of Mathematics, Southeast University, Nanjing 210096, China
candymanxm2003@yahoo.com.cn

Abstract. This paper deals with the single neuron model involving delay-dependent parameters proposed by Xu et al. [Phys. Lett. A, 354, 126-136, 2006]. The dynamics of this model are still largely undetermined, and in this paper, we perform some bifurcation analysis to the model. Unlike the article [Phys. Lett. A, 354, 126-136, 2006], where the delay is used as the bifurcation parameter, here we will use range parameter as bifurcation parameter. Based on the linear stability approach and bifurcation theory, sufficient conditions for the bifurcated periodic solution are derived, and critical values of Hopf bifurcation are assessed. The amplitude of oscillations always increases as the range parameter increases; the robustness of period against change in the range parameter occurs.

Keywords: Bifurcation, Oscillations, Delay-dependent parameters.

1 Introduction

For single neuron dynamics, Gopalsamy and Leung [1] proposed the following model of differential equation:

$$\dot{x}(t) = -x(t) + a\tanh[x(t)] - ab\tanh[x(t-\tau)]. \tag{1}$$

Here, $x(t)$ denotes the neuron response, and a and b are the range of the continuous variable $x(t)$ and measure of the inhibitory influence from the past history, respectively. Recently, Pakdaman and Malta [2], Ruan et al. [3] and Liao et al. [4] studied the stability, bifurcation and chaos of (1). But, the aforementioned studies on (1) suppose that the parameters in this model are constant independent of time delay. However, memory performance of the biological neuron usually depends on time history, and its memory intensity is usually lower and lower as time is gradually far away from the current time. It is natural to conceive that these neural networks may involve some delay-dependent parameters. Therefore, Xu et al. [5,6] considered (1) with parameter b depending on time delay τ described by

$$\dot{x}(t) = -\mu x(t) + a\tanh[x(t)] - ab(\tau)\tanh[x(t-\tau)], \tag{2}$$

L. Zhang, J. Kwok, and B.-L. Lu (Eds.): ISNN 2010, Part I, LNCS 6063, pp. 9–16, 2010.

where $\mu > 0, a > 0, \tau \geq 0$ is the time delay and $b(\tau) > 0$, which is called memory function, is a strictly decreasing function of τ. The presence of such dependence often greatly complicates the task of an analytical study of such model. Most existing methods for studying bifurcation fail when applied to such a class of delay models. Compared with the intensive studies on the neural networks with delay-independent parameters, little progress has been achieved for the systems that have delay-dependent parameters. Although a detailed analysis on the stability switches, Hopf bifurcation and chaos of (2) with delay-dependent parameters is given in [5,6], the dynamics analysis of (2) is far from complete. The purpose of this paper is to perform a thorough bifurcation analysis on (2). Unlike in Xu et al. [5,6], where the delay τ is used as the bifurcation parameter, here we will use the range parameter a as bifurcation parameter. Based on the linear stability approach and bifurcation theory, critical values of Hopf bifurcation are assessed, and sufficient conditions for the bifurcated periodic solution are derived. Moreover, the amplitude of oscillations always increases as the range parameter increases; the robustness of period against change in the range parameter occurs.

2 Bifurcation from Range Parameter a

The linearization of (2) at $x = 0$ is

$$\dot{x}(t) = (-\mu + a)x(t) - ab(\tau)x(t - \tau), \tag{3}$$

whose characteristic equation is

$$\lambda = -\mu + a - ab(\tau)e^{-\lambda\tau}. \tag{4}$$

In what follows, we regard range parameter a as the bifurcation parameter to investigate the distribution of the roots to (4).

Lemma 1. *For each fixed $\tau > 0$, if $0 < a \leq \mu/(1 + b(\tau))$, then all the roots of (4) have negative real parts.*

Proof. When $a = 0, \lambda = -\mu < 0$. For $a > 0$, clear $\lambda = 0$ is not a root of (4) since $a(1 - b(\tau)) < a(1 + b(\tau)) \leq \mu$. Let $i\omega(\omega > 0)$ be a root of (4), it is straightforward to obtain that

$$ab(\tau)\cos(\omega\tau) = a - \mu, \qquad ab(\tau)\sin(\omega\tau) = \omega, \tag{5}$$

yielding $\omega^2 = [ab(\tau)]^2 - (a - \mu)^2$. If $a \leq \mu/[1 + b(\tau)]$ holds, we have $ab(\tau) \leq |\mu - a|$. Thus, (4) has no imaginary root. In other words, (4) has no root appearing on the imaginary axis for $a \in (0, \mu/(1 + b(\tau))]$. Recalling that the root of (4) with $a = 0$ has negative real part, the conclusion follows.

Lemma 2. *For each fixed $\tau > 0$, there exists a sequence of a, denoted as $a_j, j = 1, 2, \ldots,$ such that (4) has a pair of purely imaginary roots $\pm i\omega_j$ when $a = a_j$; where*

$$a_j = \frac{\omega_j}{b(\tau)\sin(\omega_j\tau)}, \tag{6}$$

ω_j *is the root of the equation*

$$\omega\cot(\omega\tau)+\mu=\frac{\omega}{b(\tau)\sin(\omega\tau)}. \tag{7}$$

Proof. If $\lambda = \mathrm{i}\omega$ is a pure imaginary solution of (4), it must satisfy (5). Then, (7) can be directly from (5). Solutions of this equation are the horizontal coordinates of the intersecting points between the curve $y = \omega\cot(\omega\tau)$ and $y = \omega/[b(\tau)\sin(\omega\tau)] - \mu$. There are infinite number of intersecting points for these two curves that are graphically illustrated in Fig. 1. Denote ω_j as the solution of (7), and define a_j as in (6) , then (ω_j, a_j) is a solution of (5). Clearly, (4) has a pair of purely imaginary roots $\pm\mathrm{i}\omega_j$ when $a = a_j$. This completes the proof.

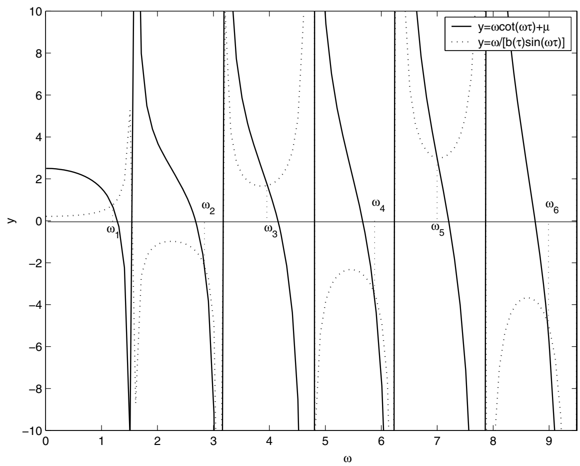

Fig. 1. Illustration for intersecting points between curve $y = \omega\cot(\omega\tau) + \mu$ and $y = \omega/[b(\tau)\sin(\omega\tau)]$

The last condition for the occurrence of a Hopf bifurcation at $a = a_j$ is

$$\frac{\mathrm{d}}{\mathrm{d}a}[\mathrm{Re}\lambda]_{a=a_j} \neq 0. \tag{8}$$

Substituting $\lambda(a)$ into (4) and differentiating the resulting equation with respect to a, we get

$$\frac{\mathrm{d}\lambda}{\mathrm{d}a}=\frac{\lambda+\mu}{a[1+\tau(\lambda+\mu-a)]},$$

and hence,

$$\left.\frac{\mathrm{d}\lambda}{\mathrm{d}a}\right|_{a=a_j}=\frac{\mu[1+\tau(\mu-a_j)]+\tau\omega_j^2+(1-\tau a_j)\omega_j\mathrm{i}}{a_j[(1+\tau(\mu-a_j))^2+\tau^2\omega_j^2]}.$$

Thus, we have

$$\frac{\mathrm{d}}{\mathrm{d}a}[\mathrm{Re}\lambda]_{a=a_j} = \frac{\mu[1+\tau(\mu-a_j)]+\tau\omega_j^2}{a_j[(1+\tau(\mu-a_j))^2+\tau^2\omega_j^2]}.$$

Theorem 1. *Suppose that* $\frac{d}{da}[Re\lambda]_{a=a_j} \neq 0$. *Then, for the system (2), there exists a Hopf bifurcation emerging from its trivial equilibrium* $x = 0$, *when the range parameter, a, passes through the critical value,* $a = a_j, j = 1, 2, \ldots$, *where* a_j *is defined by (6)-(7).*

Proof. The transversality condition (8) for Hopf bifurcation is satisfied. Applying Lemma 2 and Hopf bifurcation theorems for functional differential equations in [7], we obtain that Hopf bifurcation occurs at $a = a_j$ for the system (2). This completes the proof.

Without loss of generality, we only consider the intersecting points with positive horizontal coordinates $\omega_j, j = 1, 2, \ldots$, in Fig.1. It is clear that $\omega_1 < \omega_2 < \omega_3 < \cdots$, and $\omega_j \to \infty$ monotonically when $j \to \infty$. For these ω_j, form (6) and Fig.1, we obtain the following ordering

$$\cdots < a_6 < a_4 < a_2 < 0 < a_1 < a_3 < a_5 < \cdots.$$

Thus, there exists a minimum positive number a_1 such that (4) has a pair of purely imaginary roots $\pm \mathrm{i}\omega_1$ at $a = a_1$. From Lemmas 1 and 2, we easily obtain the following results about the stability of the trivial equilibrium $x = 0$ of system (2).

Theorem 2. *Suppose that* $\frac{d}{da}[Re\lambda]_{a=a_1} > 0$. *Then, for each fixed* $\tau > 0$, *the trivial equilibrium* $x = 0$ *of system (2) is asymptotically stable when* $a \in (0, a_1)$, *and unstable when* $a > a_1$.

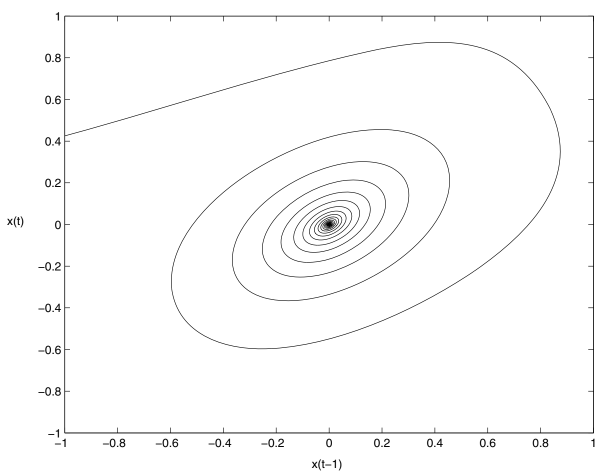

Fig. 2. Waveform plot and phase portrait of system (2) with $a = 0.68$

Proof. It is well known that the solution is locally asymptotically stable if all the roots of the characteristic equation have negative real parts and unstable if at least one root has positive real part. Therefore, conclusions is straightforward from Lemmas 1 and 2. This completes the proof.

3 Numerical Simulations

To verify the results obtained in the previous section, some examples are given as following. For comparison, the similar model (2), used in [5], is discussed.

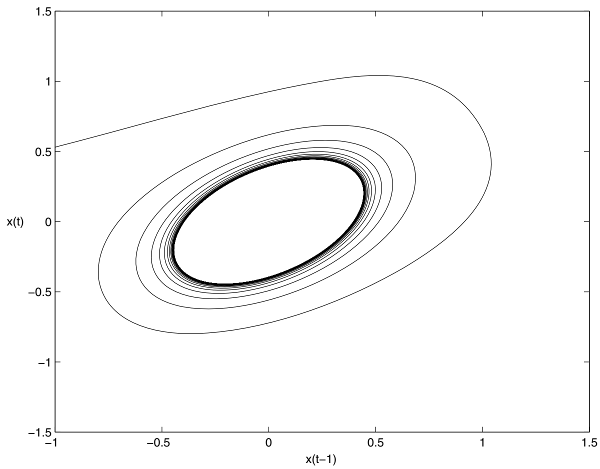

Fig. 3. Phase portrait of system (2) with $a = 0.77$

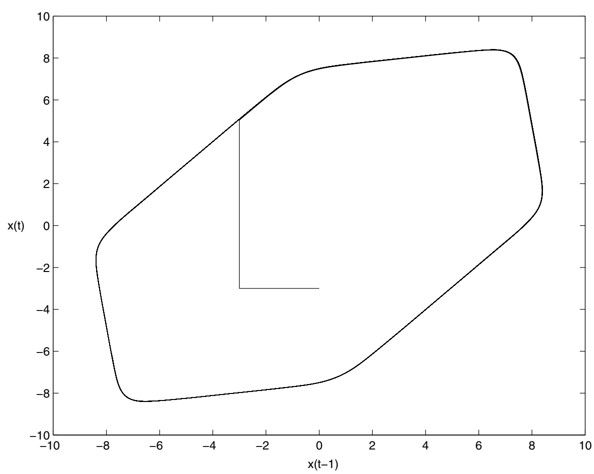

Fig. 4. Phase portrait of system (2) with $a = 5$

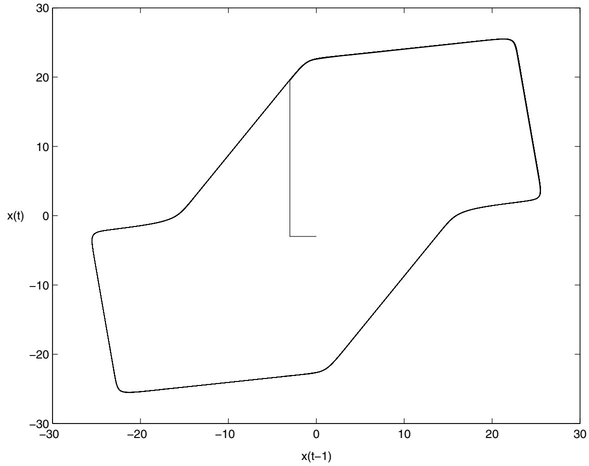

Fig. 5. Phase portrait of system (2) with $a = 15$

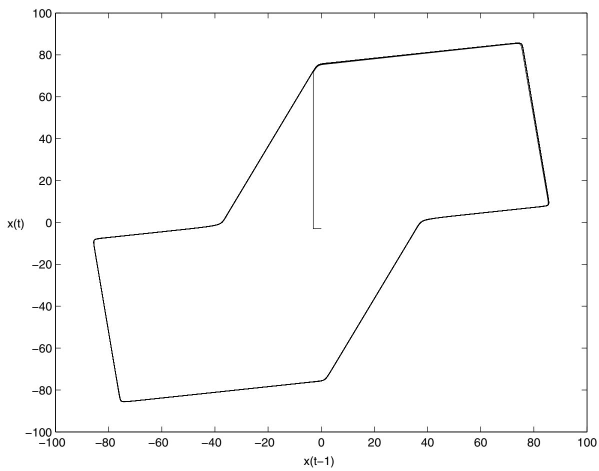

Fig. 6. Phase portrait of system (2) with $a = 50$

Table 1. Period and amplitude of the oscillations at different values of a

Range parameter a	0.77	5	15	50
Period	5.8	6.1	6.38	6.8
Amplitude	0.9	16.4	50	166

In our simulations of (2) with memory function $b(\tau) = be^{-\alpha\tau}(b > 0, \alpha > 0)$, $\mu = 2, \tau = 2, b = 3$ and $\alpha = 0.12$. We can apply (6)-(7) in Lemma 2 to obtain $a_1 = 0.7351$. From Theorem 2, we know that the trivial equilibrium $x = 0$ of system (2) is asymptotically stable when $a \in (0, 0.7351)$. This is illustrated by the numerical simulation shown in Fig. 2 in which $a = 0.68$. Further, from Theorem 1, when a is increased to the critical value 0.7351, the trivial equilibrium $x = 0$ losses its stability and Hopf bifurcation occurs. The bifurcation is supercritical and the bifurcating periodic solution is asymptotically stable (see Figs. 3-6).

Table 1 shows the effect of the range parameter a on the oscillation period and amplitude. The amplitudes increase clearly with the range parameter a, which means that the amplitudes of the oscillations can be controlled by regulating range parameter a. The amplitude is always sensitive to the change of the range parameter a. In addition to amplitude, the period of oscillation remains around 6.5 when the range parameter a varies. The change of period is not sensitive to a. The robustness of period against change in the range paramete occurs. Figs. 3-6 show the sustained oscillations generated by (2) with different a.

4 Concluding Remarks

This paper deals with the dynamics of a neuron model with delay-dependent parameters, which is far from complete. Unlike in Xu et al. [5], where the delay is used as the bifurcation parameter, here we choose range parameter as bifurcation parameter. A series of critical range parameters are determined and a simple stable criterion is given according to the range parameter. Through the analysis for the bifurcation, it is shown that the trivial equilibrium may lose stability via a Hopf bifurcation. The amplitudes of oscillations always increase clearly as the range parameter increases. In addition, the robustness of period against change in range parameter occurs.

The range parameter play an important role in dynamical behaviors of neural network model (2) with delayed dependent parameters. We can control the dynamical behaviors of the model (2) by modulating the range parameter. The method proposed in this paper is important for understanding the regulatory mechanisms of neural network. Moreover, the method provides a control mechanism to ensure a transition from an equilibrium to a periodic oscillation with a desired and robust amplitude and period.

Acknowledgement. This work was jointly supported by the National Natural Science Foundation of China under Grant 60874088, the 333 Project of Jiangsu Province of China, and the Specialized Research Fund for the Doctoral Program of Higher Education under Grant 20070286003. This work was also jointly sponsored by the Qing Lan Project of Jiangsu Province, the China Postdoctoral Science Foundation funded project under Grant 20090461056, the Jiangsu Planned Projects for Postdoctoral Research Funds under Grant 0901025C, the Jiangsu Ordinary University Natural Science Research Project under Grant 09KJD110007 and the Fundamental Discipline Construction Foundation of Nanjing Xiaozhuang University.

References

1. Gopalsamy, K., Leung, I.: Convergence under Dynamical Thresholds with Delays. IEEE Trans. Neural Netw. 8, 341–348 (1997)
2. Pakdaman, K., Malta, C.P.: A Note on Convergence under Dynamical Thresholds with Delays. IEEE Trans. Neural Netw. 9, 231–233 (1998)
3. Ruan, J., Li, L., Lin, W.: Dynamics of Some Neural Network Models with Delay. Phys. Rev. E 63, 051906 (2001)
4. Liao, X.F., Wong, K.W., Leung, C.S., Wu, Z.F.: Hopf Bifurcation and Chaos in A Single Delayed Neuron Equation with Non-Monotonic Activation Function. Chaos, Solitons and Fractals 12, 1535–1547 (2001)
5. Xu, X., Hua, H.Y., Wang, H.L.: Stability Switches, Hopf Bifurcation and Chaos of A Neuron Model with Delay-Dependent Parameters. Phys. Lett. A 354, 126–136 (2006)
6. Xu, X., Liang, Y.C.: Stability and Bifurcation of A Neuron Model with Delay-Dependent Parameters. In: Wang, J., Liao, X.-F., Yi, Z. (eds.) ISNN 2005. LNCS, vol. 3496, pp. 334–339. Springer, Heidelberg (2005)
7. Hale, J.: Theory of Functional Differential Equations. Springer, New York (1977)