Symmetry Matters for the Sizes of Extended Formulations

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Abstract. In 1991, Yannakakis [17] proved that no symmetric extended formulation for the matching polytope of the complete graph K_n with n nodes has a number of variables and constraints that is bounded subexponentially in n. Here, symmetric means that the formulation remains invariant under all permutations of the nodes of K_n. It was also conjectured in [17] that "asymmetry does not help much," but no corresponding result for general extended formulations has been found so far. In this paper we show that for the polytopes associated with the matchings in K_n with $\lfloor \log n \rfloor$ edges there are non-symmetric extended formulations of polynomial size, while nevertheless no symmetric extended formulation of polynomial size exists. We furthermore prove similar statements for the polytopes associated with cycles of length $\lfloor \log n \rfloor$. Thus, with respect to the question for smallest possible extended formulations, in general symmetry requirements may matter a lot.

1 Introduction

Linear Programming techniques have proven to be extremely fruitful for combinatorial optimization problems with respect to both structural analysis and the design of algorithms. In this context, the paradigm is to represent the problem by a polytope $P \subseteq \mathbb{R}^m$ whose vertices correspond to the feasible solutions of the problem in such a way that the objective function can be expressed by a linear functional $x \mapsto \langle c, x \rangle$ on $\mathbb{R}^m$ (with some $c \in \mathbb{R}^m$). If one succeeds in finding a description of P by means of linear constraints, then algorithms as well as structural results from Linear Programming can be exploited. In many cases, however, the polytope P has exponentially (in m) many facets, thus P can only be described by exponentially many inequalities. Also it may be that the inequalities needed to describe P are too complicated to be identified.

In some of these cases one may find an *extended formulation for P*, i.e., a (preferably small and simple) description by linear constraints of another polyhedron $Q \subseteq \mathbb{R}^d$ in some higher dimensional space that projects to P via some (simple) linear map $p : \mathbb{R}^d \to \mathbb{R}^m$ with $p(y) = Ty$ for all $y \in \mathbb{R}^d$ (and some matrix $T \in \mathbb{R}^{m \times d}$). Indeed, if $p^\star : \mathbb{R}^m \to \mathbb{R}^d$ with $p^\star(x) = T^{\mathrm{t}} x$ for all $x \in \mathbb{R}^m$ denotes the linear map that is adjoint to p (with respect to the standard bases), then we have $\max\{\langle c, x \rangle : x \in P\} = \max\{\langle p^\star(c), y \rangle : y \in Q\}$.

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As for an example, let us consider the spanning tree polytope $\mathrm{P}_{\mathrm{spt}}(n) = \mathrm{conv}\{\chi(T) \in \{0,1\}^{E_n} : T \subseteq E_n \text{ spanning tree of } K_n\}$, where $K_n = ([n], E_n)$ denotes the complete graph with node set $[n] = \{1, \ldots, n\}$ and edge set $E_n = \{\{v,w\} : v,w \in [n], v \neq w\}$, and $\chi(A) \in \{0,1\}^B$ is the *characteristic vector* of the subset $A \subseteq B$ of B, i.e., for all $b \in B$, we have $\chi(A)_b = 1$ if and only if $b \in A$. Thus, $\mathrm{P}_{\mathrm{spt}}(n)$ is the polytope associated with the bases of the graphical matroid of K_n, and hence (see [7]), it consists of all $x \in \mathbb{R}_+^{E_n}$ satisfying $x(E_n) = n-1$ and $x(E_n(S)) \leq |S| - 1$ for all $\subseteq [n]$ with $2 \leq |S| \leq n-1$, where $\mathbb{R}_+^E$ is the nonnegative orthant of $\mathbb{R}^E$, we denote by $E_n(S)$ the subset of all edges with both nodes in S, and $x(F) = \sum_{e \in F} x_e$ for $F \subseteq E_n$. This linear description of $\mathrm{P}_{\mathrm{spt}}(n)$ has an exponential (in n) number of constraints, and as all the inequalities define pairwise disjoint facets, none of them is redundant.

The following much smaller exended formulation for $\mathrm{P}_{\mathrm{spt}}(n)$ (with $O(n^3)$ variables and constraints) appears in [5] (and a similar one in [17], who attributes it to [13]). Let us introduce additional 0/1-variables $z_{e,v,u}$ for all $e \in E_n$, $v \in e$, and $u \in [n] \setminus e$. While each spanning tree $T \subseteq E_n$ is represented by its characteristic vector $x^{(T)} = \chi(T)$ in $\mathrm{P}_{\mathrm{spt}}(n)$, in the extended formulation it will be represented by the vector $y^{(T)} = (x^{(T)}, z^{(T)})$ with $z^{(T)}_{e,v,u} = 1$ (for $e \in E_n$, $v \in e$, $u \in [n] \setminus e$) if and only if $e \in T$ and u is contained in the component of v in $T \setminus e$. The polyhedron $Q_{\mathrm{spt}}(n) \subseteq \mathbb{R}^d$ defined by the nonnegativity constraints $x \geq \mathbf{0}$, $z \geq \mathbf{0}$, the equations $x(E_n) = n-1$, $x_{\{v,w\}} - z_{\{v,w\},v,u} - z_{\{v,w\},w,u} = 0$ for all pairwise distinct $v,w,u \in [n]$, as well as $x_{\{v,w\}} + \sum_{u \in [n] \setminus \{v,w\}} z_{\{v,u\},u,w} = 1$ for all distinct $v,w \in [n]$, satisfies $p(Q_{\mathrm{spt}}(n)) = \mathrm{P}_{\mathrm{spt}}(n)$, where $p : \mathbb{R}^d \to \mathbb{R}^E$ is the orthogonal projection onto the x-variables.

For many other polytopes (with exponentially many facets) associated with polynomial time solvable combinatorial optimization problems polynomially sized extended formulations can be constructed as well (see, e.g., the recent survey [5]). Probably the most prominent problem in this class for which, however, no such small formulation is known, is the matching problem. In fact, Yannakakis [17] proved that no *symmetric* polynomially sized extended formulation of the matching polytope exists.

Here, *symmetric* refers to the symmetric group $\mathfrak{S}(n)$ of all permutations $\pi : [n] \to [n]$ of the node set $[n]$ of K_n acting on E_n via $\pi.\{v,w\} = \{\pi(v), \pi(w)\}$ for all $\pi \in \mathfrak{S}(n)$ and $\{v,w\} \in E_n$. Clearly, this action of $\mathfrak{S}(n)$ on E_n induces an action on the set of all subsets of E_n. For instance, this yields an action on the spanning trees of K_n, and thus, on the vertices of $\mathrm{P}_{\mathrm{spt}}(n)$. The extended formulation of $\mathrm{P}_{\mathrm{spt}}(n)$ discussed above is *symmetric* in the sense that, for every $\pi \in \mathfrak{S}(n)$, replacing all indices associated with edges $e \in E_n$ and nodes $v \in [n]$ by $\pi.e$ and $\pi.v$, respectively, does not change the set of constraints in the formulation. Phrased informally, all subsets of nodes of K_n of equal cardinality play the same role in the formulation. For a general definition of symmetric extended formulations see Section 2.

In order to describe the main results of Yannakakis paper [17] and the contributions of the present paper, let us denote by $\mathcal{M}^{\ell}(n) = \{M \subseteq E_n : M \text{ matching in } K_n, |M| = \ell\}$ the set of all matchings of size ℓ (a matching being a subset of edges no two of which share a node), and by $\mathrm{P}^{\ell}_{\mathrm{match}}(n) = \mathrm{conv}\{\chi(M) \in \{0,1\}^{E_n} \,:\, M \in \mathcal{M}^{\ell}(n)\}$ the associated polytope. According to Edmonds [6] the *perfect matching polytope* $\mathrm{P}^{n/2}_{\mathrm{match}}(n)$ (for even n) is described by

$$\mathrm{P}^{n/2}_{\mathrm{match}}(n) = \{x \in \mathbb{R}^{E_n}_{+} \,:\, x(\delta(v)) = 1 \text{ for all } v \in [n], \\ x(E(S)) \leq (|S|-1)/2 \text{ for all } S \subseteq [n], 3 \leq |S| \text{ odd}\} \quad (1)$$

(with $\delta(v) = \{e \in E_n \,:\, v \in e\}$). Yannakakis [17, Thm.1 and its proof] shows that there is a constant $C > 0$ such that, for every extended formulation for $\mathrm{P}^{n/2}_{\mathrm{match}}(n)$ (with n even) that is symmetric in the sense above, the number of variables and constraints is at least $C \cdot \binom{n}{\lfloor n/4 \rfloor} = 2^{\Omega(n)}$. This in particular implies that there is no polynomial size symmetric extended formulation for the *matching polytope* of K_n (the convex hulls of characteristic vectors of *all* matchings in K_n), of which the *perfect* matching polytope is a face.

Yannakakis [17] also obtains a similar (maybe less surprising) result on traveling salesman polytopes. Denoting the set of all (simple) cycles of length ℓ in K_n by $\mathcal{C}^{\ell}(n) = \{C \subseteq E_n \,:\, C \text{ cycle in } K_n, |C| = \ell\}$, and the associated polytopes by $\mathrm{P}^{\ell}_{\mathrm{cycl}}(n) = \mathrm{conv}\{\chi(C) \in \{0,1\}^{E_n} \,:\, C \in \mathcal{C}^{\ell}(n)\}$, the *traveling salesman polytope* is $\mathrm{P}^{n}_{\mathrm{cycl}}(n)$. Identifying $\mathrm{P}^{n/2}_{\mathrm{match}}(n)$ (for even n) with a suitable face of $\mathrm{P}^{3n}_{\mathrm{cycl}}(3n)$, Yannakakis concludes that all symmetric extended formulations for $\mathrm{P}^{n}_{\mathrm{cycl}}(n)$ have size at least $2^{\Omega(n)}$ as well [17, Thm. 2 and its proof].

Yannakakis' results in a fascinating way illuminate the borders of our principal abilities to express combinatorial optimization problems like the matching or the traveling salesman problem by means of linear constraints. However, they only refer to linear descriptions that respect the inherent symmetries in the problems. In fact, the second open problem mentioned in the concluding section of [17] is described as follows: "We do not think that asymmetry helps much. Thus, prove that the matching and TSP polytopes cannot be expressed by polynomial size LP's without the asymmetry assumption."

The contribution of our paper is to show that, in contrast to the assumption expressed in the quotation above, asymmetry can help much, or, phrased differently, that symmetry requirements on extended formulations indeed can matter significantly with respect to the minimal sizes of extended formulations. Our main results are that both $\mathrm{P}^{\lfloor \log n \rfloor}_{\mathrm{match}}(n)$ and $\mathrm{P}^{\lfloor \log n \rfloor}_{\mathrm{cycl}}(n)$ do not admit symmetric extended formulations of polynomial size, while they have non-symmetric extended formulations of polynomial size (see Cor. 1 and 2 for matchings, as well as Cor. 3 and 4 for cycles). The corresponding theorems from which these corollaries are derived provide some more general and more precise results for $\mathrm{P}^{\ell}_{\mathrm{match}}(n)$ and $\mathrm{P}^{\ell}_{\mathrm{cycl}}(n)$. In order to establish the lower bounds for symmetric extensions, we generalize the techniques developed by Yannakakis [17]. The constructions of the compact non-symmetric extended formulations rely on small families of perfect hash functions [1,8,15].

The paper is organized as follows. In Section 2, we provide definitions of extensions, extended formulations, their sizes, the crucial notion of a section of an extension, and we give some auxilliary results. In Section 3, we present Yannakakis' method to derive lower bounds on the sizes of symmetric extended formulations for perfect matching polytopes in a general setting, which we then exploit in Section 4 in order to derive lower bounds on the sizes of symmetric extended formulations for the polytopes $\mathrm{P}^{\ell}_{\mathrm{match}}(n)$ associated with cardinality restricted matchings. In Section 5, we describe our non-symmetric extended formultions for these polytopes. Finally, in Section 6 we present the results on $\mathrm{P}^{\ell}_{\mathrm{cycl}}(n)$. Some remarks conclude the paper in Section 7.

2 Extended Formulations, Extensions, and Symmetry

An *extension* of a polytope $P \subseteq \mathbb{R}^m$ is a polyhedron $Q \subseteq \mathbb{R}^d$ together with a projection (i.e., a linear map) $p : \mathbb{R}^d \to \mathbb{R}^m$ with $p(Q) = P$; it is called a *subspace extension* if Q is the intersection of an affine subspace of $\mathbb{R}^d$ and the nonnegative orthant $\mathbb{R}^d_+$. For instance, the polyhedron $Q_{\mathrm{spt}}(n)$ defined in the Introduction is a subspace extension of the spanning tree polytope $\mathrm{P}_{\mathrm{spt}}(n)$. A (finite) system of linear equations and inequalities whose solutions are the points in an extension Q of P is an *extended formulation* for P. The *size* of an extension is the number of its facets plus the dimension of the space it lies in. The *size* of an extended formulation is its number of inequalities (including nonnegativity constraints, but not equations) plus its number of variables. Clearly, the size of an extended formulation is at least as large as the size of the extension it describes. Conversely, every extension is described by an extended formulation of at most its size.

Extensions or extended formulations of a family of polytopes $P \subseteq \mathbb{R}^m$ (for varying m) are *compact* if their sizes and the encoding lengths of the coefficients needed to describe them can be bounded by a polynomial in m and the maximal encoding length of all components of all vertices of P. Clearly, the extension $Q_{\mathrm{spt}}(n)$ of $\mathrm{P}_{\mathrm{spt}}(n)$ from the Introduction is compact.

In our context, *sections* $s : X \to Q$ play a crucial role, i.e., maps that assign to every vertex $x \in X$ of P some point $s(x) \in Q \cap p^{-1}(x)$ in the intersection of the polyhedron Q and the *fiber* $p^{-1}(x) = \{y \in \mathbb{R}^d \;:\; p(y) = x\}$ of x under the projection p. Such a section induces a bijection between X and its image $s(X) \subseteq Q$, whose inverse is given by p. In the spanning tree example from the Introduction, the assignment $\chi(T) \mapsto y^{(T)} = (x^{(T)}, z^{(T)})$ defined such a section. Note that, in general, sections will not be induced by linear maps. In fact, if a section is induced by a linear map $s : \mathbb{R}^m \to \mathbb{R}^d$, then the intersection of Q with the affine subspace of $\mathbb{R}^d$ generated by $s(X)$ is isomorphic to P, thus Q has at least as many facets as P.

For a family $\mathcal{F}$ of subsets of X, an extension $Q \subseteq \mathbb{R}^d$ is said to be *indexed by* $\mathcal{F}$ if there is a bijection between $\mathcal{F}$ and $[d]$ such that (identifying $\mathbb{R}^{\mathcal{F}}$ with $\mathbb{R}^d$ via this bijection) the map $\mathbf{1}_{\mathcal{F}} = (\mathbf{1}_F)_{F \in \mathcal{F}} : X \to \{0,1\}^{\mathcal{F}}$ whose component functions are the characteristic functions $\mathbf{1}_F : X \to \{0,1\}$ (with $\mathbf{1}_F(x) = 1$ if and only if $x \in F$), is a section for the extension, i.e., $\mathbf{1}_{\mathcal{F}}(X) \subseteq Q$ and $p(\mathbf{1}_{\mathcal{F}}(x)) = x$ hold for all $x \in X$. For instance, the extension $Q_{\mathrm{spt}}(n)$ of $\mathrm{P}_{\mathrm{spt}}(n)$ is indexed by the family $\{\mathcal{T}(e) : e \in E_n\} \cup \{\mathcal{T}(e,v,u) : e \in E_n, v \in e, u \in [n] \setminus e\}$, where $\mathcal{T}(e)$ contains all spanning trees using edge e, and $\mathcal{T}(e,v,u)$ consists of all spanning trees in $\mathcal{T}(e)$ for which u and v are in the same component of $T \setminus \{e\}$.

In order to define the notion of symmetry of an extension precisely, let the group $\mathfrak{S}(d)$ of all permutations of $[d] = \{1,\ldots,d\}$ act on $\mathbb{R}^d$ by coordinate permutations. Thus we have $(\sigma.y)_j = y_{\sigma^{-1}(j)}$ for all $y \in \mathbb{R}^d$, $\sigma \in \mathfrak{S}(d)$, and $j \in [d]$.

Let $P \subseteq \mathbb{R}^m$ be a polytope and G be a group acting on $\mathbb{R}^m$ with $\pi.P = P$ for all $\pi \in G$, i.e., the action of G on $\mathbb{R}^m$ induces an action of G on the set X of vertices of P. An extension $Q \subseteq \mathbb{R}^d$ of P with projection $p : \mathbb{R}^d \to \mathbb{R}^m$ is *symmetric* (with respect to the action of G), if for every $\pi \in G$ there is a permutation $\kappa_\pi \in \mathfrak{S}(d)$ with $\kappa_\pi.Q = Q$ and

$$p(\kappa_\pi.y) = \pi.p(y) \quad \text{for all } y \in \mathbb{R}^d. \tag{2}$$

The prime examples of symmetric extensions arise from extended formulations that "look symmetric". To be more precise, we define an extended formulation $A^{=}y = b^{=}$, $A^{\leq}y \leq b^{\leq}$ describing the polyhedron $Q = \{y \in \mathbb{R}^d : A^{=}y = b^{=}, A^{\leq}y \leq b^{\leq}\}$ extending $P \subseteq \mathbb{R}^m$ as above to be *symmetric* (with respect to the action of G on the set X of vertices of P), if for every $\pi \in G$ there is a permutation $\kappa_\pi \in \mathfrak{S}(d)$ satisfying (2) and there are two permutations $\varrho^{=}_\pi$ and $\varrho^{\leq}_\pi$ of the rows of $(A^{=}, b^{=})$ and $(A^{\leq}, b^{\leq})$, respectively, such that the corresponding simultaneous permutations of the columns and the rows of the matrices $(A^{=}, b^{=})$ and $(A^{\leq}, b^{\leq})$ leaves them unchanged. Clearly, in this situation the permutations κ_π satisfy $\kappa_\pi.Q = Q$, which implies the following.

Lemma 1. *Every symmetric extended formulation describes a symmetric extension.*

One example of a symmetric extended formulation is the extended formulation for the spanning tree polytope described in the Introduction (with respect to the group G of all permutations of the nodes of the complete graph).

For the proof of the central result on the non-existence of certain symmetric subspace extensions (Theorem 1), a weaker notion of symmetry will be sufficient. We call an extension as above *weakly symmetric* (with respect to the action of G) if there is a section $s : X \to Q$ for which the action of G on $s(X)$ induced by the bijection s works by permutation of variables, i.e., for every $\pi \in G$ there is a permutation $\kappa_\pi \in \mathfrak{S}(d)$ with $s(\pi.x) = \kappa_\pi.s(x)$ for all $x \in X$. The following statement (and its proof, for which we refer to [12]) generalizes the construction of sections for symmetric extensions of matching polytopes described in Yannakakis' paper [17, Claim 1 in the proof of Thm. 1].

Lemma 2. *Every symmetric extension is weakly symmetric.*

Finally, the following result (again, we refer to [12] for a proof) will turn out to be useful in order to derive lower bounds on the sizes of symmetric extensions for one polytope from bounds for another one.

Lemma 3. *Let $Q \subseteq \mathbb{R}^d$ be an extension of the polytope $P \subseteq \mathbb{R}^m$ with projection $p : \mathbb{R}^d \to \mathbb{R}^m$, and let the face P' of P be an extension of a polytope $R \subseteq \mathbb{R}^k$ with projection $q : \mathbb{R}^m \to \mathbb{R}^k$. Then the face $Q' = p^{-1}(P') \cap Q \subseteq \mathbb{R}^d$ of Q is an extension of R via the composed projection $q \circ p : \mathbb{R}^d \to \mathbb{R}^k$.*

If the extension Q of P is symmetric with respect to an action of a group G on $\mathbb{R}^m$ (with $\pi.P = P$ for all $\pi \in G$), and a group H acts on $\mathbb{R}^k$ such that, for every $\tau \in H$, we have $\tau.R = R$, and there is some $\pi_\tau \in G$ with $\pi_\tau.P' = P'$ and $q(\pi_\tau.x) = \tau.q(x)$ for all $x \in \mathbb{R}^m$, then the extension Q' of R is symmetric (with respect to the action of the group H).

3 Yannakakis' Method

Here, we provide an abstract view on the method used by Yannakakis [17] in order to bound from below the sizes of symmetric extensions for perfect matching polytopes, without referring to these concrete poytopes. That method is capable of establishing lower bounds on the number of variables of weakly symmetric subspace extensions of certain polytopes. By the following lemma, which is basically Step 1 in the proof of [17, Theorem 1], such bounds imply similar lower bounds on the dimension of the ambient space and the number of facets for general symmetric extensions (that are not necessarily subspace extensions).

Lemma 4. *If, for a polytope P, there is a symmetric extension in $\mathbb{R}^{\tilde{d}}$ with f facets, then P has also a symmetric subspace extension in $\mathbb{R}^d$ with $d \leq 2\tilde{d} + f$.*

The following simple lemma provides the strategy for Yannakakis' method, which we need to extend slightly by allowing restrictions to affine subspaces.

Lemma 5. *Let $Q \subseteq \mathbb{R}^d$ be a subspace extension of the polytope $P \subseteq \mathbb{R}^m$ with vertex set $X \subseteq \mathbb{R}^m$, and let $s : X \to Q$ be a section for the extension. If $S \subseteq \mathbb{R}^m$ is an affine subspace, and, for some $X^\star \subseteq X \cap S$, the coefficients $c_x \in \mathbb{R}$ ($x \in X^\star$) yield an affine combination of a nonnegative vector*

$$\sum_{x \in X^\star} c_x s(x) \geq \mathbf{0}_d \quad \textit{with} \quad \sum_{x \in X^\star} c_x = 1\,, \tag{3}$$

from the section images of the vertices in $X^\star$, then $\sum_{x \in X^\star} c_x x \in P \cap S$ holds.

Proof. Since Q is a subspace extension, we obtain $\sum_{x \in X^\star} c_x s(x) \in Q$ from $s(x) \in Q$ (for all $x \in X^\star$). Thus, if $p : \mathbb{R}^d \to \mathbb{R}^m$ is the projection of the extension, we derive

$$P \ni p(\sum_{x \in X^\star} c_x s(x)) = \sum_{x \in X^\star} c_x p(s(x)) = \sum_{x \in X^\star} c_x x\,. \tag{4}$$

As S is an affine subspace containing $X^\star$, we also have $\sum_{x \in X^\star} c_x x \in S$.

Due to Lemma 5 one can prove that subspace extensions of some polytope P with certain properties do not exist by finding, for such a hypothetical extension, a subset $X^\star$ of vertices of P and an affine subspace S containing $X^\star$, for which one can construct coefficients $c_x \in \mathbb{R}$ satisying (3) such that $\sum_{x\in X^\star} c_x x$ violates some inequality that is valid for $P \cap S$.

Actually, following Yannakakis, we will not apply Lemma 5 directly to a hypothetical small weakly symmetric subspace extension, but we will rather first construct another subspace extension from the one assumed to exist that is indexed by some convenient family $\mathcal{F}$. We say that an extension Q of a polytope P is *consistent* with a family $\mathcal{F}$ of subsets of the vertex set X of P if there is a section $s : X \to Q$ for the extension such that, for every component function s_j of s, there is a subfamily $\mathcal{F}_j$ of $\mathcal{F}$ such that s_j is constant on every set in $\mathcal{F}_j$, and the sets in $\mathcal{F}_j$ partition X. In this situation, we also call the section s *consistent* with $\mathcal{F}$. The proof of the following lemma can be found in [12].

Lemma 6. *If $P \subseteq \mathbb{R}^m$ is a polytope and $\mathcal{F}$ is a family of vertex sets of P for which there is some extension Q of P that is consistent with $\mathcal{F}$, then there is some extension Q' for P that is indexed by $\mathcal{F}$. If Q is a subspace extension, then Q' can be chosen to be a subspace extension as well.*

Lemmas 5 and 6 suggest the following strategy for proving that subspace extensions of some polytope P with certain properties (e.g., being weakly symmetric and using at most B variables) do not exist by (a) exhibiting a family $\mathcal{F}$ of subsets of the vertex set X of P with which such an extension would be consistent and (b) determining a subset $X^\star \subset X$ of vertices and an affine subspace S containing $X^\star$, for which one can construct coefficients $c_x \in \mathbb{R}$ satisying

$$\sum_{x\in X^\star} c_x \mathbf{1}_{\mathcal{F}}(x) \geq \mathbf{0}_{\mathcal{F}} \quad \text{with} \quad \sum_{x\in X^\star} c_x = 1\,, \tag{5}$$

such that $\sum_{x\in X^\star} c_x x$ violates some inequality that is valid for $P \cap S$.

Let us finally investigate more closely the sections that come with weakly symmetric extensions. In particular, we will discuss an approach to find suitable families $\mathcal{F}$ within the strategy mentioned above in the following setting. Let $Q \subseteq \mathbb{R}^d$ be a weakly symmetric extension of the polytope $P \subseteq \mathbb{R}^m$ (with respect to an action of the group G on the vertex set X of P) along with a section $s : X \to Q$ such that for every $\pi \in G$ there is a permutation $\kappa_\pi \in \mathfrak{S}(d)$ that satisfies $s(\pi.x) = \kappa_\pi.s(x)$ for all $x \in X$ (with $(\kappa_\pi.s(x))_j = s_{\kappa_\pi^{-1}(j)}(x)$).

In this setting, we can define an action of G on the set $\mathcal{S} = \{s_1, \ldots, s_d\}$ of the component functions of the section $s : X \to Q$ with $\pi.s_j = s_{\kappa^{-1}_{\pi^{-1}}(j)} \in \mathcal{S}$ for each $j \in [d]$. In order to see that this definition indeed is well-defined (note that $s_1, \ldots, s_d$ need not be pairwise distinct functions) and yields a group action, observe that, for each $j \in [d]$ and $\pi \in G$, we have

$$(\pi.s_j)(x) = s_{\kappa^{-1}_{\pi^{-1}}(j)}(x) = (\kappa_{\pi^{-1}}.s(x))_j = s_j(\pi^{-1}.x) \quad \text{for all } x \in X\,, \tag{6}$$

from which one deduces $1.s_j = s_j$ for the one-element 1 in G as well as $(\pi\pi').s_j = \pi.(\pi'.s_j)$ for all $\pi, \pi' \in G$. The *isotropy* group of $s_j \in \mathcal{S}$ under this action is $\text{iso}_G(s_j) = \{\pi \in G \,:\, \pi.s_j = s_j\}$. From (6) one sees that, for all $x \in X$ and

$\pi \in \mathrm{iso}_G(s_j)$, we have $s_j(x) = s_j(\pi^{-1}.x)$. Thus, s_j is constant on every orbit of the action of the subgroup $\mathrm{iso}_G(s_j)$ of G on X. We conclude the following.

Remark 1. In the setting described above, if $\mathcal{F}$ is a family of subsets of X such that, for each $j \in [d]$, there is a sub-family $\mathcal{F}_j$ partitioning X and consisting of vertex sets each of which is contained in an orbit under the action of $\mathrm{iso}_G(s_j)$ on X, then s is consistent with $\mathcal{F}$.

In general, it will be impossible to identify the isotropy groups $\mathrm{iso}_G(s_j)$ without more knowledge on the section s. However, for each isotropy group $\mathrm{iso}_G(s_j)$, one can at least bound its index $(G : \mathrm{iso}_G(s_j))$ in G.

Lemma 7. *In the setting described above, we have* $(G : \mathrm{iso}_G(s_j)) \le d$ *.*

Proof. This follows readily from the fact that the index $(G : \mathrm{iso}_G(s_j))$ of the isotropy group of the element $s_j \in \mathcal{S}$ under the action of G on $\mathcal{S}$ equals the cardinality of the orbit of s_j under that action, which due to $|\mathcal{S}| \le d$, clearly is bounded from above by d.

The bound provided in Lemma 7 can become useful, in case one is able to establish a statement like "if $\mathrm{iso}_G(s_j)$ has index less than τ in G then it contains a certain subgroup H_j". Choosing $\mathcal{F}_j$ as the family of orbits of X under the action of the subgroup H_j of G, then $\mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_d$ is a familiy as in Remark 1. If this family (or any refinement of it) can be used to perform Step (b) in the strategy outlined in the paragraph right after the statement of Lemma 6, then one can conclude the lower bound $d \ge \tau$ on the number of variables d in an extension as above.

4 Bounds on Symmetric Extensions of $\mathrm{P}^{\ell}_{\mathrm{match}}(n)$

In this section, we use Yannakakis' method described in Section 3 to prove the following result.

Theorem 1. *For every* $n \ge 3$ *and odd* ℓ *with* $\ell \le \frac{n}{2}$*, there exists no weakly symmetric subspace extension for* $\mathrm{P}^{\ell}_{\mathrm{match}}(n)$ *with at most* $\binom{n}{(\ell-1)/2}$ *variables (with respect to the group* $\mathfrak{S}(n)$ *acting via permuting the nodes of* K_n *as described in the Introduction).*

From Theorem 1, we can derive the following more general lower bounds. Since we need it in the proof of the next result, and also for later reference, we state a simple fact on binomial coefficients first.

Lemma 8. *For each constant* $b \in \mathbb{N}$ *there is some constant* $\beta > 0$ *with* $\binom{M-b}{N} \ge \beta\binom{M}{N}$ *for all large enough* $M \in \mathbb{N}$ *and* $N \le \frac{M}{2}$*.*

Theorem 2. *There is a constant* $C > 0$ *such that, for all* n *and* $1 \le \ell \le \frac{n}{2}$*, the size of every extension for* $\mathrm{P}^{\ell}_{\mathrm{match}}(n)$ *that is symmetric (with respect to the group* $\mathfrak{S}(n)$ *acting via permuting the nodes of* K_n *as described in the Introduction) is bounded from below by* $C \cdot \binom{n}{\lfloor(\ell-1)/2\rfloor}$*.*

Proof. For odd ℓ, this follows from Theorem 1 using Lemmas 1, 2, and 4. For even ℓ, the polytope $\mathrm{P}^{\ell-1}_{\mathrm{match}}(n-2)$ is (isomorphic to) a face of $\mathrm{P}^{\ell-1}_{\mathrm{match}}(n)$ defined by $x_e = 1$ for an arbitrary edge e of K_n. From this, as $\ell - 1$ is odd (and not larger than $(n-2)/2$) with $\lfloor(\ell-2)/2\rfloor = \lfloor(\ell-1)/2\rfloor$, and due to Lemma 8, the theorem follows by Lemma 3.

For even n and $\ell = n/2$, Theorem 2 provides a similar bound to Yannakakis result (see Step 2 in the proof of [17, Theorem 1]) that no weakly symmetric subspace extension of the perfect matching polytope of K_n has a number of variables that is bounded by $\binom{n}{k}$ for any $k < n/4$.

Theorem 2 in particular implies that the size of every symmetric extension for $\mathrm{P}^{\ell}_{\mathrm{match}}(n)$ with $\Omega(\log n) \le \ell \le n/2$ is bounded from below by $n^{\Omega(\log n)}$, which has the following consequence.

Corollary 1. *For $\Omega(\log n) \le \ell \le n/2$, there is no compact extended formulation for $\mathrm{P}^{\ell}_{\mathrm{match}}(n)$ that is symmetric (with respect to the group $G = \mathfrak{S}(n)$ acting via permuting the nodes of K_n as described in the Introduction).*

The rest of this section is devoted to indicate the proof of Theorem 1. Throughout, with $\ell = 2k+1$, we assume that $Q \subseteq \mathbb{R}^d$ with $d \le \binom{n}{k}$ is a weakly symmetric subspace extension of $\mathrm{P}^{2k+1}_{\mathrm{match}}(n)$ for $4k+2 \le n$. We will only consider the case $k \ge 1$, as for $\ell = 1$ the theorem trivially is true (note that we restrict to $n \ge 3$). Weak symmetry is meant with respect to the action of $G = \mathfrak{S}(n)$ on the set X of vertices of $\mathrm{P}^{2k+1}_{\mathrm{match}}(n)$ as described in the Introduction, and we assume $s : X \to Q$ to be a section as required in the definition of weak symmetry. Thus, we have $X = \{\chi(M) \in \{0,1\}^{E_n} : M \in \mathcal{M}^{2k+1}(n)\}$, where $\mathcal{M}^{2k+1}(n)$ is the set of all matchings $M \subseteq E_n$ with $|M| = 2k+1$ in the complete graph $K_n = (V, E)$ (with $V = [n]$), and $(\pi.\chi(M))_{\{v,w\}} = \chi(M)_{\{\pi^{-1}(v),\pi^{-1}(w)\}}$ holds for all $\pi \in \mathfrak{S}(n)$, $M \in \mathcal{M}^{2k+1}(n)$, and $\{v,w\} \in E$.

In order to identify suitable subgroups of the isotropy groups $\mathrm{iso}_{\mathfrak{S}(n)}(s_j)$ (see the remarks at the end of Section 3), we use the following result on subgroups of the symmetric group $\mathfrak{S}(n)$, where $\mathfrak{A}(n) \subseteq \mathfrak{S}(n)$ is the *alternating group* formed by all even permutations of $[n]$. This result is Claim 2 in the proof of Thm. 1 of Yannakakis paper [17]. Its proof relies on a theorem of Bochert's [3] stating that any subgroup of $\mathfrak{S}(m)$ that acts primitively on $[m]$ and does not contain $\mathfrak{A}(m)$ has index at least $\lfloor(m+1)/2\rfloor!$ in $\mathfrak{S}(m)$ (see [16, Thm. 14.2]).

Lemma 9. *For each subgroup U of $\mathfrak{S}(n)$ with $(\mathfrak{S}(n) : U) \le \binom{n}{k}$ for $k < \frac{n}{4}$, there is a $W \subseteq [n]$ with $|W| \le k$ and $H_j = \{\pi \in \mathfrak{A}(n) : \pi(v) = v \text{ for all } v \in W\} \subseteq U$.*

As we assumed $d \le \binom{n}{k}$ (with $k < \frac{n}{4}$ due to $4k+2 \le n$), Lemmas 7 and 9 imply $H_j \subseteq \mathrm{iso}_{\mathfrak{S}(n)}(s_j)$ for all $j \in [d]$. For each $j \in [d]$, two vertices $\chi(M)$ and $\chi(M')$ of $\mathrm{P}^{2k+1}_{\mathrm{match}}(n)$ (with $M, M' \in \mathcal{M}^{2k+1}(n)$) are in the same orbit under the action of the group H_j if and only if we have

$$M \cap E(V_j) = M' \cap E(V_j) \quad \text{and} \quad V_j \setminus M = V_j \setminus M' \,. \tag{7}$$

Indeed, it is clear that (7) holds if we have $\chi(M') = \pi.\chi(M)$ for some permutation $\pi \in H_j$. In turn, if (7) holds, then there clearly is some permutation $\pi \in \mathfrak{S}(n)$ with $\pi(v) = v$ for all $v \in V_j$ and $M' = \pi.M$. Due to

$|M| = 2k+1 > 2|V_j|$ there is some edge $\{u,w\} \in M$ with $u,w \notin V_j$. Denoting by $\tau \in \mathfrak{S}(n)$ the transposition of u and w, we thus also have $\pi\tau(v) = v$ for all $v \in V_j$ and $M' = \pi\tau.M$. As one of the permutations π and $\pi\tau$ is even, say π', we find $\pi' \in H_j$ and $M' = \pi'.M$, proving that M and M' are contained in the same orbit under the action of H_j.

As it will be convenient for Step (b) (referring to the strategy described after the statement of Lemma 6), we will use the following refinements of the partitionings of X into orbits of H_j (as mentioned at the end of Section 3). Clearly, for $j \in [d]$ and $M, M' \in \mathcal{M}^{2k+1}(n)$,

$$M \setminus E(V \setminus V_j) = M' \setminus E(V \setminus V_j) \tag{8}$$

implies (7). Thus, for each $j \in [d]$, the equivalence classes of the equivalence relation defined by (8) refine the partitioning of X into orbits under H_j, and we may use the collection of all these equivalence classes (for all $j \in [d]$) as the family $\mathcal{F}$ in Remark 1. With

$$\Lambda = \{(A,B) \,:\, A \subseteq E \text{ matching and there is some } j \in [d] \text{ with } A \subseteq E \setminus E(V \setminus V_j), B = V_j \setminus V(A)\}\,,$$

(with $V(A) = \bigcup_{a \in A} a$) we hence have $\mathcal{F} = \{F(A,B) \,:\, (A,B) \in \Lambda\}$, where

$$F(A,B) = \{\chi(M) \,:\, M \in \mathcal{M}^{2k+1}(n), A \subseteq M \subseteq E(V \setminus B)\}\,.$$

In order to construct a subset $X^\star \subseteq X$ which will be used to derive a contradiction as mentioned after Equation (5), we choose two arbitrary disjoint subsets $V_\star, V^\star \subset V$ of nodes with $|V_\star| = |V^\star| = 2k+1$, and define $\mathcal{M}^\star = \{M \in \mathcal{M}^{2k+1}(n) \,:\, M \subseteq E(V_\star \cup V^\star)\}$ as well as $X^\star = \{\chi(M) \,:\, M \in \mathcal{M}^\star\}$. Thus, $\mathcal{M}^\star$ is the set of perfect matchings on $K(V_\star \cup V^\star)$. Clearly, $X^\star$ is contained in the affine subspace S of $\mathbb{R}^E$ defined by $x_e = 0$ for all $e \in E \setminus E(V_\star \cup V^\star)$. In fact, $X^\star$ is the vertex set of the face $\mathrm{P}^{2k+1}_{\mathrm{match}}(n) \cap S$ of $\mathrm{P}^{2k+1}_{\mathrm{match}}(n)$, and for this face the inequality $x(V_\star : V^\star) \geq 1$ is valid (where $(V_\star : V^\star)$ is the set of all edges having one node in $V_\star$ and the other one in $V^\star$), since every matching $M \in \mathcal{M}^\star$ intersects $(V_\star : V^\star)$ in an odd number of edges. Therefore, in order to derive the desired contradiction, it suffices to find $c_x \in \mathbb{R}$ (for all $x \in X^\star$) with

$\sum_{x \in X^\star} c_x = 1$, $\sum_{x \in X^\star} c_x \cdot \mathbf{1}_{\mathcal{F}}(x) \geq \mathbf{0}_{\mathcal{F}}$, and $\sum_{x \in X^\star} c_x \sum_{e \in (V_\star : V^\star)} x_e = 0$. For the details on how this can be done we refer to [12].

5 A Non-symmetric Extension for $\mathrm{P}^{\ell}_{\mathrm{match}}(n)$

We shall establish the following result on the existence of extensions for cardinality restricted matching polytopes in this section.

Theorem 3. *For all n and ℓ, there are extensions for $\mathrm{P}^{\ell}_{\mathrm{match}}(n)$ whose sizes can be bounded by $2^{O(\ell)} n^2 \log n$ (and for which the encoding lengths of the coefficients needed to describe the extensions by linear systems can be bounded by a constant).*

In particular, Theorem 3 implies the following, although, according to Corollary 1, no compact symmetric extended formulations exist for $\mathrm{P}^{\ell}_{\mathrm{match}}(n)$ with $\ell = \Theta(\log n)$.

Corollary 2. *For all n and $\ell \leq O(\log n)$, there are compact extended formulations for $\mathrm{P}^{\ell}_{\mathrm{match}}(n)$.*

The proof of Theorem 3 relies on the following result on the existence of small families of *perfect-hash functions*, which is from [1, Sect. 4]. Its proof is based on results from [8,15].

Theorem 4 (Alon, Yuster, Zwick [1]). *There are maps $\phi_1, \ldots, \phi_{q(n,r)} : [n] \to [r]$ with $q(n,r) \leq 2^{O(r)} \log n$ such that, for every $W \subseteq [n]$ with $|W| = r$, there is some $i \in [q(n,r)]$ for which the map ϕ_i is bijective on W.*

Furthermore, we will use the following two auxilliary results that can be derived from general results on *polyhedral branching systems* [11, see Cor. 3 and Sect. 4.4]. The first one (Lemma 10) provides a construction of an extension of a polytope that is specified as the convex hull of some polytopes of which extensions are already available. In fact, in this section it will be needed only for the case that these extensions are the polytopes themselves (this is a special case of a result of Balas', see [2, Thm.2.1]). However, we will face the slightly more general situation in our treatment of cycle polytopes in Section 6.

Lemma 10. *If the polytopes $P_i \subseteq \mathbb{R}^m$ (for $i \in [q]$) have extensions Q_i of size s_i, respectively, then $P = \mathrm{conv}(P_1 \cup \cdots \cup P_q)$ has an extension of size $\sum_{i=1}^{q}(s_i+2)+1$.*

The second auxilliary result that we need deals with describing a 0/1-polytope that is obtained by splitting variables of a 0/1-polytope of which a linear description is already available.

Lemma 11. *Let $\mathcal{S}$ be a set of subsets of $[t]$, $P = \mathrm{conv}\{\chi(S) \in \{0,1\}^t : S \in \mathcal{S}\} \subseteq \mathbb{R}^t$, the corresponding 0/1-polytope, $J = J(1) \uplus \cdots \uplus J(t)$ a disjoint union of finite sets $J(i)$,*

$$\mathcal{S}^\star = \{S^\star \subseteq J : \text{There is some } S \in \mathcal{S} \text{ with } |S^\star \cap J(i)| = 1 \text{ for all } i \in S, |S^\star \cap J(i)| = 0 \text{ for all } i \notin S\}\,, \quad (9)$$

and $P^\star = \mathrm{conv}\{\chi(S^\star) \in \{0,1\}^J : S^\star \in \mathcal{S}^\star\}$. If $P = \{y \in [0,1]^t : Ay \leq b\}$ for some $A \in \mathbb{R}^{s \times t}$ and $b \in \mathbb{R}^s$, then

$$P^\star = \{x \in [0,1]^J : \sum_{i=1}^{t} A_{\star,i} \cdot \sum_{j \in J(i)} x_j \leq b_i \text{ for all } i \in [t]\}\,. \quad (10)$$

In order to prove Theorem 3, let $\phi_1, \ldots, \phi_q$ be maps as guaranteed to exist by Theorem 4 with $r = 2\ell$ and $q = q(n, 2\ell) \leq 2^{O(\ell)} \log n$, and denote $\mathcal{M}_i = \{M \in \mathcal{M}^\ell(n) : \phi_i \text{ is bijective on } V(M)\}$ for each $i \in [q]$. By Theorem 4, we have $\mathcal{M}^\ell(n) = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_q$. Consequently,

$$\mathrm{P}^{\ell}_{\mathrm{match}}(n) = \mathrm{conv}(P_1 \cup \cdots \cup P_q) \quad (11)$$

with $P_i = \text{conv}\{\chi(M) : M \in \mathcal{M}_i\}$ for all $i \in [q]$, where we have

$$P_i = \{x \in \mathbb{R}^E_+ : x_{E \setminus E_i} = \mathbf{0}, x(\delta(\phi_i^{-1}(s))) = 1 \text{ for all } s \in [2\ell], \\ x(E_i(\phi_i^{-1}(S))) \leq (|S| - 1)/2 \text{ for all } S \subseteq [2\ell], |S| \text{ odd}\},$$

where $E_i = E \setminus \bigcup_{j \in [2\ell]} E(\phi_i^{-1}(j))$. This follows by Lemma 11 from Edmonds' linear description (1) of the perfect matching polytope $\mathrm{P}^{\ell}_{\text{match}}(2\ell)$ of $K_{2\ell}$. As the sum of the number of variables and the number of inequalities in the description of P_i is at most $2^{O(\ell)} + n^2$ (the summand n^2 comes from the nonnegativity constraints on $x \in \mathbb{R}^E_+$ and the constant in $O(\ell)$ is independent of i), we obtain an extension of $\mathrm{P}^{\ell}_{\text{match}}(n)$ of size $2^{O(\ell)} n^2 \log n$ by Lemma 10. This proves Theorem 3.

6 Extensions for Cycle Polytopes

By a modification of Yannakakis' construction for the derivation of lower bounds on the sizes of symmetric extensions for traveling salesman polytopes from the corresponding lower bounds for matching polytopes [17, Thm. 2], we obtain lower bounds on the sizes of symmetric extensions for $\mathrm{P}^{\ell}_{\text{cycl}}(n)$. The lower bound $\ell \geq 42$ in the statement of the theorem (whose proof can be found in [12]) is convenient with respect to both formulating the bound and proving its validity.

Theorem 5. *There is a constant $C' > 0$ such that, for all n and $42 \leq \ell \leq n$, the size of every extension for $\mathrm{P}^{\ell}_{\text{cycl}}(n)$ that is symmetric (with respect to the group $\mathfrak{S}(n)$ acting via permuting the nodes of K_n as described in the Introduction) is bounded from below by $C' \cdot \binom{\lfloor \frac{n}{3} \rfloor}{\lfloor (\lfloor \frac{\ell}{6} \rfloor - 1)/2 \rfloor}$.*

Corollary 3. *For $\Omega(\log n) \leq \ell \leq n$, there is no compact extended formulation for $\mathrm{P}^{\ell}_{\text{cycl}}(n)$ that is symmetric (with respect to the group $\mathfrak{S}(n)$ acting via permuting the nodes of K_n as described in the Introduction).*

On the other hand, if we drop the symmetry requirement, we find extensions of the following size.

Theorem 6. *For all n and ℓ, there are extensions for $\mathrm{P}^{\ell}_{\text{cycl}}(n)$ whose sizes can be bounded by $2^{O(\ell)} n^3 \log n$ (and for which the encoding lengths of the coefficients needed to describe the extensions by linear systems can be bounded by a constant).*

Before we prove Theorem 6, we state a consequence that is similar to Corollary 1 for matching polytopes. It shows that, despite the non-existence of symmetric extensions for the polytopes associated with cycles of length $\Theta(\log n)$ (Corollary 3), there are non-symmetric compact extensions of these polytopes.

Corollary 4. *For all n and $\ell \leq O(\log n)$, there are compact extended formulations for $\mathrm{P}^{\ell}_{\text{cycl}}(n)$.*

The rest of the section is devoted to prove Theorem 6, i.e., to construct an extension of $\mathrm{P}^{\ell}_{\text{cycl}}(n)$ whose size is bounded by $2^{O(\ell)} n^3 \log n$. We proceed similarly to the proof of Theorem 3 (the construction of extensions for matching polytopes), this time starting with maps $\phi_1, \dots, \phi_q$ as guaranteed to exist by Theorem 4 with $r = \ell$ and $q = q(n, \ell) \leq 2^{O(\ell)} \log n$, and defining $\mathcal{C}_i = \{C \in \mathcal{C}^\ell(n) : \phi_i \text{ is bijective on } V(C)\}$ for each $i \in [q]$. Thus, we have $\mathcal{C}^\ell(n) = \mathcal{C}_1 \cup \dots \cup \mathcal{C}_q$, and hence, $\mathrm{P}^\ell_{\mathrm{cycl}}(n) = \mathrm{conv}(P_1 \cup \dots \cup P_q)$ with $P_i = \mathrm{conv}\{\chi(C) : C \in \mathcal{C}_i\}$ for all $i \in [q]$. Due to Lemma 10, it suffices to exhibit, for each $i \in [q]$, an extension of P_i of size bounded by $O(2^\ell \cdot n^3)$ (with the constant independent of i). Towards this end, let, for $i \in [q]$, $V_c = \phi_i^{-1}(c)$ for all $c \in [\ell]$, and define $P_i(v^\star) = \mathrm{conv}\{\chi(C) : C \in \mathcal{C}_i, v^\star \in V(C)\}$ for each $v^\star \in V_\ell$. Thus, we have $P_i = \mathrm{conv} \bigcup_{v^\star \in V_\ell} P_i(v^\star)$, and hence, due to Lemma 10, it suffices to construct extensions of the $P_i(v^\star)$, whose sizes are bounded by $O(2^\ell \cdot n^2)$.

In order to derive such extensions define, for each $i \in [q]$ and $v^\star \in V_\ell$, a directed acyclic graph D with nodes (A, v) for all $A \subseteq [\ell - 1]$ and $v \in \phi_i^{-1}(A)$, as well as two additional nodes s and t, and arcs $\big(s, (\{\phi_i(v)\}, v)\big)$ and $\big(([\ell-1], v), t\big)$ for all $v \in \phi_i^{-1}([\ell - 1])$, as well as $\big((A, v), (A \cup \{\phi_i(w)\}, w)\big)$ for all $A \subseteq [\ell - 1]$, $v \in \phi_i^{-1}(A)$, and $w \in \phi_i^{-1}([\ell - 1] \setminus A)$. This is basically the dynamic programming digraph (using an idea going back to [10]) from the color-coding method for finding paths of prescribed lengths described in [1]. Each s-t-path in D corresponds to a cycle in $\mathcal{C}_i$ that visits $v^\star$, and each such cycle, in turn, corresponds to two s-t-paths in D (one for each of the two directions of transversal).

Defining $Q_i(v^\star)$ as the convex hull of the characteristic vectors of all s-t-paths in D in the arc space of D, we find that $P_i(v^\star)$ is the image of $Q_i(v^\star))$ under the projection whose component function corresponding to the edge $\{v, w\}$ of K_n is given by the sum of all arc variables corresponding to arcs $((A, v), (A', w))$ (for $A, A' \subseteq [\ell - 1]$) if $v^\star \notin \{v, w\}$, and by the sum of the two arc variables corresponding to $(s, (\{\phi_i(w)\}, w))$ and $(([\ell - 1], w), t)$ in case of $v = v^\star$. Clearly, $Q_i(v^\star)$ can be described by nonnegativity constraints, flow conservation constraints for all nodes in D different from s and t, and by the equation stating that there must be exactly one flow-unit leaving s. As the number of arcs of D is in $O(2^\ell \cdot n^2)$, we thus have found an extension of $P_i(v^\star)$ of the desired size.

7 Conclusions

The results presented in this paper demonstrate that there are polytopes which have compact extended formulations though they do not admit symmetric ones. These polytopes are associated with matchings (or cycles) of some prescribed cardinalities (see [4] for a recent survey on general cardinality restricted combinatorial optimization problems). Similarly, for the permutahedron associated with $[n]$ there is a gap between the smallest sizes $\Theta(n \log n)$ of a non-symmetric extension [9] and $\Theta(n^2)$ of a symmetric extension [14].

Nevertheless, the question whether there are compact extended formulations for general matching polytopes (or for perfect matching polytopes), remains one of the most interesting open question here. In fact, it is even unknown whether there are (non-symmetric) extended formulations of these polytopes of size $2^{o(n)}$.

Actually, it seems that there are almost no lower bounds known on the sizes of (not necessarily symmetric) extensions, except for the one obtained by the observation that every extension Q of a polytope P with f faces has at least f faces itself, thus Q has at least $\log f$ facets (since a face is uniquely determined by the subset of facets it is contained in) [9]. It would be most interesting to obtain other lower bounds, including special ones for 0/1-polytopes.

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