Chapter 8
Strongly Nonlinear Vibrations

Abstract. This chapter presents analytical successive approximations algorithms for different oscillators with strongly nonlinear characteristics. In general terms, such algorithms approximate temporal mode shapes of vibrations by polynomials and other simple functions of the triangular sine wave. In order to develope the algorithms, the triangular wave is introduced into dynamical systems as a new temporal argument. The corresponding manipulations with dynamical systems are described in the first three sections. Then the description focuses on the algorithm implementations for different essentially unharmonic cases including oscillators whose characteristics may approach nonsmooth or even discontinuous limits.

8.1 Periodic Solutions for First Order Dynamical Systems

Let us consider a dynamical system described by first-order differential equation with respect to the vector-function $x(t) \in R^n$,

$$\dot{x} = f(x) \tag{8.1}$$

where $f(x)$ is a continuous vector-function, and the over dot indicates time derivative.

We consider the class of periodic motions of the period $T = 4a$. Note that, in the autonomous case, the period is a priori unknown. Periodic solutions usually require specific, a priory unknown, initial conditions. Practically, however, such kind of the specific initial conditions are determined in a backward way, after some periodic family of solutions is obtained under the assumption of periodicity. In our case, the assumption of periodicity is imposed automatically by the form of representation for periodic solutions

$$x = X(\tau) + Y(\tau)e \tag{8.2}$$

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where $\tau = \tau(t/a)$ and $e = e(t/a)$ are the standard triangular sine and rectangular cosine, respectively, and $X(\tau)$ and $Y(\tau)$ are unknown components of the solution.

Substituting (8.2) in (8.1), gives

$$(Y' - aR_f) + (X' - aI_f)e \quad + Ye' = 0$$

where

$$R_f = R_f(X,Y) = \frac{1}{2}[f(X+Y) + f(X-Y)]$$
$$I_f = I_f(X,Y) = \frac{1}{2}[f(X+Y) - f(X-Y)]$$

Eliminating the periodic singular term $e' = de(t/a)/d(t/a)$ by means of the boundary condition for $Y(\tau)$, gives the non-linear boundary value problem on the standard interval, $-1 \leq \tau \leq 1$,

$$\begin{aligned} Y' &= aR_f(X,Y) \\ X' &= aI_f(X,Y) \\ Y|_{\tau=\pm 1} &= 0 \end{aligned} \tag{8.3}$$

Note that, the entire interval $-1 \leq \tau \leq 1$ is completely covered by a half of the period, $-a \leq t \leq a$, however, representation (8.2) unfolds the corresponding fragment on the entire time interval $-\infty < t < \infty$.

8.2 Second Order Dynamical Systems

Consider now the differential equation of motion in the standard Newtonian form

$$\ddot{x} + f(x, \dot{x}, t) = 0 \tag{8.4}$$

where $x(t) \in R^n$ is a positional vector-function, and the vector-function f is assumed to be sufficiently smooth and periodic with respect to the explicit time t with the period $T = 4a$; the autonomous case, when the period is a priory unknown, is under consideration as well.

Substituting (8.2) into (8.4), using the differential and algebraic properties of substitution (8.2), and imposing the boundary (smoothness) conditions, gives

$$(X'' + a^2 R_f) + (Y'' + a^2 I_f)e = 0$$

where

$$R_f = \frac{1}{2}\left[f\left(X+Y,\ \frac{X'+Y'}{a},\ a\tau\right) + f\left(X-Y,\ -\frac{X'-Y'}{a},\ 2a-a\tau\right)\right] \tag{8.5}$$

$$I_f = \frac{1}{2}\left[f\left(X+Y,\ \frac{X'+Y'}{a},\ a\tau\right) - f\left(X-Y,\ -\frac{X'-Y'}{a},\ 2a-a\tau\right)\right] \tag{8.6}$$

This leads to the boundary value problem

$$X'' + a^2 R_f(X, Y, X', Y', \tau) = 0, \quad X'|_{\tau=\pm1} = 0 \tag{8.7}$$

$$Y'' + a^2 I_f(X, Y, X', Y', \tau) = 0, \quad Y\,|_{\tau=\pm1} = 0 \tag{8.8}$$

Let us discuss the form of equations (8.7) and (8.8).

Firstly, despite of the obvious formal complication, equations (8.7) and (8.8) possess certain symmetries dictated by substitution (8.2). For instance, introducing the new unknown variables,

$$\begin{aligned} U(\tau) &= X(\tau) + Y(\tau) \\ V(\tau) &= X(\tau) - Y(\tau) \end{aligned}$$

brings the boundary value problem, (8.7) and (8.8), to the form, in which the differential equations are decoupled at cost of coupling the boundary conditions though,

$$\begin{aligned} U'' + a^2 f\left(U,\ U'/a,\ 2a - a\tau\right) &= 0 \\ V'' + a^2 f\left(V,\ \ -V'/a, 2a - \ a\tau\right) &= 0 \\ U' + V'|_{\tau=\pm1} &= 0 \\ U - V|_{\tau=\pm1} &= 0 \end{aligned} \tag{8.9}$$

In case when analytical methods are applied, the differential equations (8.9) for $U(\tau)$ and $V(\tau)$ can be usually solved in a similar way. Nevertheless, the previous boundary value problem, (8.7) and (8.8), may appear to have some advantages for analyses. For instance, the problem may admit families of solutions with either $Y(\tau) \equiv 0$ or $X(\tau) \equiv 0$.

Secondly, due to substitution (8.2), the major qualitative property of solutions, such as periodicity, is a priory captured by the new argument, τ. As a result, the following simplified system can be employed as a generating model for analytical algorithms of successive approximations

$$\begin{aligned} X'' &= 0 \\ Y'' &= 0 \end{aligned} \tag{8.10}$$

Indeed, substituting obvious solutions of equations (8.10) in (8.2), gives a family of nonsmooth *periodic* motions with respect to the original time parameter, t,

$$x(t) = X(0) + X'(0)\tau(t/a) + (Y(0) + Y'(0)\tau(t/a))e(t/a) \tag{8.11}$$

Thirdly, from the physical standpoint, linear equations (8.10) describe a strongly nonlinear (nonsmooth) generating model. In particular, if $Y(0) = 0$ and $Y'(0) = 0$, then vector-function (8.11) describes vibrations of basic vibroimpact models.

The analytical algorithms developed in few sections below are based on the idea of approximation of smooth vibrating systems by the basic vibroimpact models. In other words, the triangular sine wave is assumed to be a dominant component of temporal mode shapes of vibrations. Such an idea indeed follows the analogy with the quasi harmonic approaches. In particular, the harmonic balance method approximates vibrating systems by effective harmonic oscillators regardless types or magnitudes of the system nonlinearities. This is justified by the fact that Fourier coefficients usually decay in a fast enough rate so that, for instance, second term can be considered as a small correction to the first term. The corresponding "small parameter" is therefore hidden in the iterative procedure itself rather than explicitly present in the differential equations of motion.

Finally, equations (8.10) make sense due to the temporal substitution $t \longrightarrow \tau(t/a)$. In terms of the original variables, the corresponding equation, $\ddot{x} = 0$, contains too little information about the original system (8.4) and captures no global properties of the dynamics.

8.3 Periodic Solutions of Conservative Systems

8.3.1 The Vibroimpact Approximation

Let us consider the case of n-degrees-of-freedom conservative system

$$\ddot{x} + f(x) = 0 \tag{8.12}$$

where $f(x)$ is an odd analytical vector-function of the positional vector-column $x(t) \in R^n$.

A one-parameter family of periodic solutions will be built such that $X(-\tau) \equiv -X(\tau)$. Since equation (8.12) admits the group of time translations then another arbitrary parameter can be always added to the time variable. Taking into account the symmetry of system (8.12), enables one of considering the particular case of substitution (8.2)

$$x(t) = X(\tau(t/a)), \quad Y \equiv 0 \tag{8.13}$$

Based on the conditions assumed, the boundary value problem (8.7) and (8.8) is reduced to the following one

$$\begin{aligned} X'' + a^2 f(X) &= 0 \\ X'|_{\tau=1} &= 0 \end{aligned} \tag{8.14}$$

We seek solutions of the boundary value problem (8.14) in the form of series of successive approximations

$$X = X^0(\tau) + X^1(\tau) + X^2(\tau) + ... \tag{8.15}$$

$$a^2 = h_0(1 + \gamma_1 + \gamma_2 + ...) \tag{8.16}$$

In order to organize the corresponding iterative procedure, it is assumed that

$$\begin{gathered} O(\|X^i\|) \gg O(\|X^{i+1}\|) \\ O(\gamma_{i+1}) \gg O(\gamma_{i+2}) \\ (i = 0, 1, 2, ...) \end{gathered} \tag{8.17}$$

where the norm of vector-functions is defined by $\|X\| = \max_{\tau} \|X\|_{R^n}$.

Based on assumptions (8.17), series (8.15) and (8.16) generate the sequences of equations and boundary conditions as, respectively,

$$X^{0\prime\prime} = 0 \tag{8.18}$$

$$X^{1\prime\prime} = -h_0 f(X^0) \tag{8.19}$$

$$X^{2\prime\prime} = -h_0[\gamma_1 f(X^0) + f'_x(X^0)X^1] \tag{8.20}$$

$$\cdots$$

and

$$(X^{0\prime} + X^{1\prime})|_{\tau=1} = 0 \tag{8.21}$$

$$X^{2\prime}|_{\tau=1} = 0 \tag{8.22}$$

$$\cdots$$

Note that condition (8.21) includes first two approximations as the only way to proceed with a non-zero generating solution. In particular, the generating solution is found from equation (8.18) in the form

$$X^0 = A^0\tau \tag{8.23}$$

where $A^0 \in R^n$ is an arbitrary constant vector, and the oddness condition has been enforced in order to set to zero another constant vector.

In line with the discussion at the end of the previous section, solution (8.23) describes a multi-dimensional vibro-impact oscillator between two absolutely stiff and perfectly elastic barriers such that A^0 is the normal vector to both barriers. Direction of the vector A^0 will be defined on the next step of successive approximations, whereas its length will appear to be coupled with the parameter h_0 by some relationship due to boundary condition (8.21).

So substituting (8.23) in (8.19) and integrating, gives solution

$$X^1 = A^1\tau - h_0 \int_0^\tau (\tau - \xi) f(A^0\xi) d\xi \tag{8.24}$$

where A^1 is another arbitrary constant vector.

Note that the first term in expression (8.24) is similar to generating solution (8.23) and thus contributes nothing new into the entire solution within the first two steps of the procedure. Therefore, we take $A^1 = 0$ and then substitute the combination $X^0 + X^1$ in the boundary condition (8.21). This gives a nonlinear eigen value problem with respect to the vector A^0 in the form

$$\int_0^1 f(A^0\tau) d\tau = \frac{1}{h_0} A^0 \tag{8.25}$$

Equation (8.25) represents a set of n scalar equations relating the components of vector A^0 and the parameter h_0. The combination A^0 and $1/h_0$ will be interpreted as an eigen vector and eigen value of the nonlinear eigen vector problem (8.25).

Taking scalar product of both sides of equation (8.25) with A^{0T}, gives

$$h_0 = \frac{A^{0T} A^0}{A^{0T} \int_0^1 f(A^0\tau) d\tau} \tag{8.26}$$

where the upper index T stays for transpose operation.

In order to clarify the meaning of expressions (8.25) and (8.26), let us consider the linear case $f(x) \equiv Kx$, where K is an $n \times n$ stiffness matrix. The corresponding relationships will differ from those of the exact linear theory by specific constant factors because the temporal mode shape of vibrations is not exact but approximated by the triangular sine wave. Nevertheless, in nonlinear cases, expression (8.26) can provide estimates for amplitude-frequency response characteristics.

Further, integrating equation (8.20), gives

$$X^2 = A^2\tau - h_0 \int_0^\tau (\tau - \xi)[\gamma_1 f(A^0\xi) + f'_x(A^0\xi) X^1(\xi)] d\xi \tag{8.27}$$

where A^2 is an arbitrary constant vector, and $f'_x(\ A^0\xi\ )$ is the $n \times n -$ matrix of first partial derivatives (Jacobian).

Then boundary condition (8.22) gives

$$A^2 = h_0 \int_0^1 [\gamma_1 f(A^0\tau) + f'_x(A^0\tau)X^1(\tau)]d\tau \tag{8.28}$$

where the coefficient γ_1 is yet unknown.

In order to determine the coefficient γ_1, an additional condition for the vector A^2 can be imposed, for instance, as follows

$$A^{0T}A^2 = 0 \tag{8.29}$$

This condition means that the vector A^2 must be orthogonal to the corresponding vector of the generating solution A^0 in order to keep the amplitude fixed.

Substituting (8.28) in (8.29), gives

$$\gamma_1 = -\frac{A^{0T}\int_0^1 f'_x(A^0\tau)X^1 d\tau}{A^{0T}\int_0^1 f(A^0\tau)d\tau} \tag{8.30}$$

This completes the second step of successive approximations. All the further steps can be passed in the same way.

In general terms, convergence properties of the above procedure are due to the following integral operator

$$F[X] \equiv a^2 \left\{ \tau \int_\tau^1 f(X(\xi))d\xi + \int_0^\tau \xi f(X(\xi))d\xi \right\} \tag{8.31}$$

where

$$a^2 = h_0 \frac{A^{0T}\int_0^1 f(A^0\tau)d\tau}{A^{0T}\int_0^1 f(X)d\tau} \tag{8.32}$$

Based on definition (8.31), the original boundary value problem (8.14) admits representation in the form $X = F[X]$. Therefore, the convergence condition is

$$\frac{\|F'_X[X^0]\delta X\|}{\|\delta X\|} < 1 \tag{8.33}$$

where δX is an arbitrary vector-function from a small enough neighborhood of X^0.

In the case of linearized system, condition (8.33) leads to the set of inequalities $\omega_i/\omega_j < 1$ for all $i \neq j$, where ω_j is the eigen frequency of the linear normal mode, which is chosen to be a generating solution.

8.3.2 One Degree-of-Freedom General Conservative Oscillator

In the one-degree-of-freedom case with odd characteristic, the boundary condition at $\tau = 1$ is reduced to a single equation, which is sequentially satisfied by the factor h_0 and terms γ_1, γ_2,... of series (8.16). As a result, the process of successive approximations therefore eases by setting $A^0 = A$ and $A^i = 0$ for $i = 1$, 2,... .

Let us introduce notations $h_i = h_0\gamma_i$ and represent series (8.15) and (8.16) in the form

$$\begin{aligned} X &= X_0(\tau) + X_1(\tau) + X_2(\tau) + ... \\ a^2 &= h_0 + h_1 + h_2 + ... \end{aligned} \tag{8.34}$$

where $X_i(\tau)$ are scalar functions of the triangular sine wave $\tau = \tau(t/a)$.

Due to the reduction of one-dimensional case, all terms of the expansions are iteratively determined by the explicit relationships. First two steps of the iterative procedure are coupled by the smoothing boundary condition (8.21) that provides the leading order smooth estimate for the temporal mode shape by coupling the parameters, $h_0 = h_0(A)$, as follows

$$X_0 = A\tau \tag{8.35}$$

$$\begin{aligned} X_1 &= -h_0 \int_0^{\tau} (\tau - \xi) f(A\xi) d\xi \\ h_0 &= A / \int_0^1 f(A\xi) d\xi \end{aligned} \tag{8.36}$$

All the next steps of the procedure are passed then in a similar way based on relationships

$$\begin{aligned} X_i &= -\sum_{j=1}^{i} h_{j-1} \int_0^{\tau} (\tau - \xi) R_{i-j} d\xi \\ h_{i-1} &= -\sum_{j=1}^{i-1} \alpha_{i-j} h_{j-1} \\ &(i = 2, 3, ...) \end{aligned} \tag{8.37}$$

where the coefficients and integrands are generated by means of the formal auxiliary parameter, ε, as follows

$$\begin{aligned} \alpha_i &= \int_0^1 R_i d\xi / \int_0^1 R_0 d\xi \\ R_i &= \frac{1}{i!}\frac{d^i f(X^0 + \varepsilon X^1 + \varepsilon^2 X^2 + \cdots)}{d\varepsilon^i}|_{\varepsilon=0} \\ (i &= 0, 1, 2, \ldots) \end{aligned} \tag{8.38}$$

The way of using the parameter ε is in compliance with assumptions (8.17). Respectively, such parameter splits the restoring force according to (8.38) and then disappears from expressions. The convergence of such iterative series of successive approximations is illustrated by the following example.

Example 12. Let us consider the oscillator

$$\ddot{x} + x^m = 0$$

where m is an odd positive integer. This oscillator was already discussed in Chapter 3 under the notation $m = 2n - 1$. Now, applying two iterations according to the above scheme, (8.36) and (8.37), gives solution

$$X = A\left[\tau - \frac{\tau^{m+2}}{m+2} + \frac{m}{2(m+2)}\left(\frac{\tau^{2m+3}}{2m+3} - \frac{\tau^{m+2}}{m+2}\right) - R_3 - R_4 - \cdots\right] \tag{8.39}$$

$$a^2 = \frac{m+1}{A^{m-1}}\left\{1 + \frac{m}{2(m+2)}\left[1 + \frac{(m+1)^2}{(m+2)(2m+3)}\right] + r_3 + r_4 + \cdots\right\} \tag{8.40}$$

where expressions

$$\begin{aligned} 0 &< R_i(m,\tau) < \frac{m\,|\tau|^{m+2}}{2^{i-1}(m+2)^2} \\ 0 &< r_i(m) < \frac{m}{2^i(m+2)} \end{aligned} \tag{8.41}$$

provide estimates for high order terms of the successive approximations. In particular, expressions (8.41) indicate that series (8.39) and (8.40) may converge quite slowly. However, the asymptotic of large exponents m essentially improves precision of the truncated series even though first few terms of the series are included. The temporal mode shapes of different iterations are shown in Fig. 8.1, whereas Fig. 8.2 illustrates the period as a function of the number m under the fixed energy in the first iteration only. For comparison reason, the exact result and first order approximation according to the harmonic balance method are also shown in the diagram. In particular, Fig. 8.1 shows that high-order iterations are localized near the time points corresponding to amplitude positions of the oscillator. For instance, the first

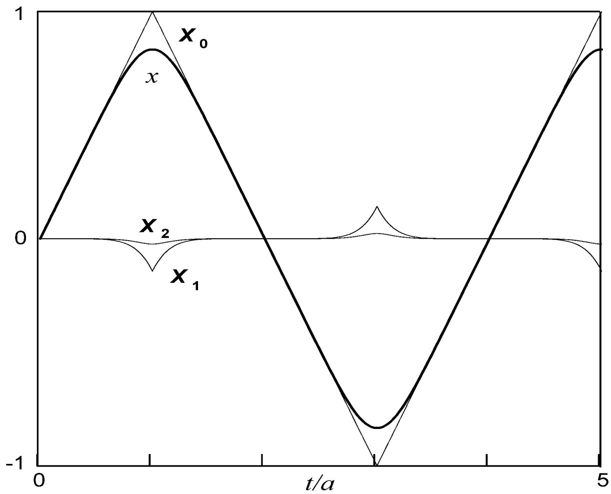

Fig. 8.1 The first three terms of iteration (thin lines) and their sum (solid line) for the temporal mode shape of the oscillator $\ddot{x} + x^5 = 0$.

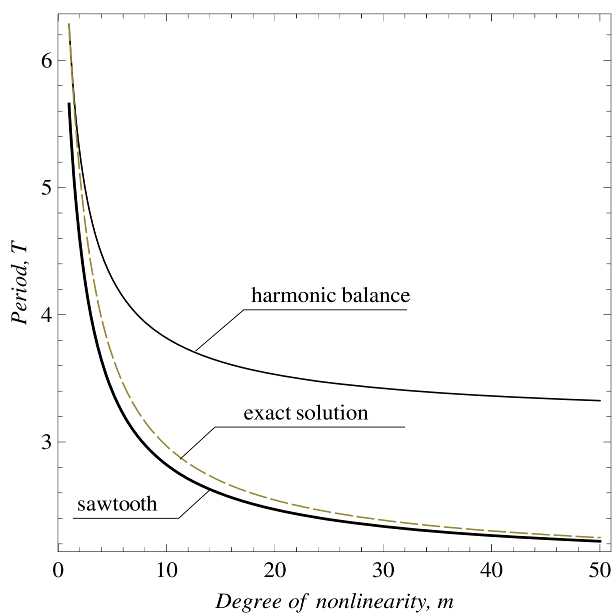

Fig. 8.2 The period of the oscillator with power characteristic at different exponents m obtained by three different methods under the total energy $E = 2$.

iteration X_1 just compensates the discontinuities of slope of the generating solution X_0 with a minor effect on the rest of the triangular sine wave. Further, Fig. 8.2 confirms that expansion (8.40) gives a better estimate for the period than the harmonic balance as the exponent m increases. Note that the entire series (8.39) and (8.40) are not asymptotic with respect to m or $1/m$ in the sense of Poincare, however the series perfectly capture the asymptotic of impact oscillator as $m \to \infty$.

8.3.3 A Nonlinear Mass-Spring Model That Becomes Linear at High Amplitudes

As another example of conservative oscillator, we consider a single mass vibrating system illustrated by Fig. 8.3 and described by the Lagrangian

$$L = \frac{m\dot{w}^2}{2} - kl^2 \left(\sqrt{1 + \frac{w^2}{l^2}} - 1 \right)^2$$

Here, m is mass, k is the linear stiffness of each spring, l is the length of each spring at the equilibrium position at which the springs are horizontal, and w is the particle vertical coordinate.

In terms of the dimensionless coordinate $x = w/l$ and phase $\varphi = (2k/m)^{1/2}t$, the corresponding differential equation of motion takes the form

$$\frac{d^2x}{d\varphi^2} + x - \frac{x}{\sqrt{1+x^2}} = 0 \tag{8.42}$$

Then, applying substitution (8.13) as $x = X(\tau)$ and $\tau = \tau(\varphi/a)$, leads to the boundary problem

$$X'' = -h \left(X - \frac{X}{\sqrt{1+X^2}} \right) \equiv -hf(X)$$
$$X'|_{\tau=1} = 0 \tag{8.43}$$

where $h = a^2$.

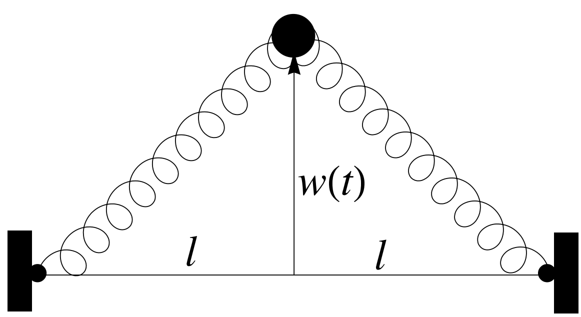

Fig. 8.3 The system which becomes weakly non-linear at large amplitudes.

First two steps of the successive approximation procedure give

$$X_0 = A\tau$$
$$h_0 = \left(\frac{1}{2} - \frac{\sqrt{1+A^2}-1}{A^2}\right)^{-1}$$

and

$$X_1(\tau) = -\frac{h_0}{2A^2}\left[\frac{1}{3}(A\tau)^3 + 2A\tau - A\tau\sqrt{1+(A\tau)^2} - \operatorname{arcsinh}(A\tau)\right] \quad (8.44)$$
$$\gamma_1 = \frac{h_0}{A^3}\left[\frac{6A+A^3}{12} + \frac{9A-A^3}{6\sqrt{1+A^2}} - \left(1+\frac{1}{\sqrt{1+A^2}}\right)\operatorname{arcsinh}(A)\right]$$

Interestingly enough, this model is essentially nonlinear at small amplitudes, but it becomes linear as the amplitudes are infinitely large. Indeed, taking the corresponding limits, shows that

$$\frac{X_1}{A} \to -\frac{1}{5}\tau^5, \quad h_0A^2 \to 8, \quad \gamma_1 \to \frac{3}{10} \quad \text{as} \quad A \to 0 \quad (8.45)$$

and

$$\frac{X_1}{A} \to -\frac{1}{3}\tau^3, \quad h_0 \to 2, \quad \gamma_1 \to \frac{1}{6} \quad \text{as} \quad A \to \infty \quad (8.46)$$

The asymptotic (8.45) obviously corresponds the nonlinear oscillator, whereas the limit case (8.46) associates with the harmonic oscillator. Nevertheless, solution (8.44) is valid for both large and small amplitudes. Both limit cases follow from equation (8.42). In the case of small amplitudes, $|x| \ll 1$, by using the estimate $(1+x^2)^{-1/2} \sim 1 - x^2/2$ in equation (8.42), we obtain

$$d^2x/d\varphi^2 + x^3/2 = 0 \quad (8.47)$$

In the case of large amplitudes, it follows even from Fig. 8.3 that the distance between the spring fixed ends becomes negligible if as compared to l. As a result, the mechanical model becomes effectively close to a mass-spring oscillator of mass m with a single spring of stiffness $2k$. In terms of the differential equations of motion, we also obtain the corresponding limit from exact equation (8.42) by assuming that, during 'most of the time of vibration cycle,' $|x| \gg 1$ and thus $\sqrt{1+x^2} \sim |x|$. As a result, equation (8.42) is replaced by

$$d^2x/d\varphi^2 + x - \operatorname{sgn}(x) = 0 \quad (8.48)$$

where the term $\operatorname{sgn}(x)$ has to be neglected due to the same condition $|x| \gg 1$ that gives the standard linear oscillator.

Alternatively, the discontinuous term in equation (8.48) can be saved and then considered as a perturbation of the harmonic oscillator. Note that equation (8.48) admits another form as follows

$$d^2x/d\varphi^2 + \mathrm{sgn}(x)(|x| - 1) = 0 \tag{8.49}$$

The restoring force characteristic of oscillator (8.49) represents a particular case of the characteristic, $p(x) = \mathrm{sgn}(x)f(|x|)$, which is considered later in this chapter.

8.3.4 Strongly Non-linear Characteristic with a Step-Wise Discontinuity at Zero

Let us consider the case of symmetric exponentially growing restoring force characteristic with a step-wise discontinuity at zero such that

$$f(x) = \begin{cases} \exp(x) & \text{for } x > 0 \\ -\exp(-x) & \text{for } x < 0 \end{cases} \tag{8.50}$$

Although force (8.50) has no certain value at the point $x = 0$, this still can play the role of equilibrium position. From the physical standpoint, this is equilibrium of a small bead at the bottom of V-shaped potential well. The local dynamics in a small neighborhood of such type of equilibria is considered later in this chapter; see the text to Fig. 8.13.

It follows from (8.50) that $f(-x) = -f(x)$. Therefore, periodic motions of the corresponding oscillator can be described by the function $x = X(\tau)$, where $X(-\tau) = -X(\tau)$ and $\tau = \tau(t/a)$. In terms of these NSTT variables, the oscillator of a unit mass is described by the boundary value problem

$$\begin{array}{ll} X'' + h\exp(X) = 0, & X'|_{\tau=1} = 0 \quad \text{for } \tau \in (0,1] \\ X'' - h\exp(-X) = 0, & X'|_{\tau=-1} = 0 \text{ for } \tau \in [-1,0) \end{array} \tag{8.51}$$

where $h = a^2$.

This problem is exactly solvable, and solution that satisfies the continuity of state condition at $\tau = 0$ has the form

$$\begin{aligned} X_{\pm}(\tau) \equiv\ & A\tau \pm 2\ln(1 + \exp(-A)) \\ & \mp 2\ln\left(1 + \frac{h}{2h_0}\exp(\pm A\tau - A)\right) \end{aligned} \tag{8.52}$$

where X_+ and X_- are taken for positive and negative subintervals of τ, respectively, and

$$h = 2h_0 = \frac{2A^2}{\exp(A)[1 + \exp(-A)]^2} \tag{8.53}$$

Note that both the differential equation of oscillator and its solution admit unit form representations as, respectively,

$$\ddot{x} + \mathrm{sgn}(x)\exp(|x|) = 0 \tag{8.54}$$

and

$$X = \operatorname{sgn}(\tau)\left(A|\tau| + 2\ln\left(\frac{1+\exp(-A)}{1+(h/(2h_0))\exp(A|\tau|-A)}\right)\right) \tag{8.55}$$

The parameter h is obtained from equation

$$X'|_{\tau=1} = 0 \tag{8.56}$$

or

$$1 - \frac{h}{h_0}\left(1+\frac{h}{2h_0}\right)^{-1} = 0 \tag{8.57}$$

This exactly solvable case can play the role of a majorant for evaluation of convergence properties of successive approximations. For that reason, we introduce a formal 'small' parameter, $\varepsilon = 1$, and represent solution of equation (8.57) in the form

$$h = 2h_0 = \varepsilon h_0\left(1-\frac{\varepsilon}{2}\right)^{-1} \tag{8.58}$$

Taking into account (8.58), brings (8.55) to the form

$$X = \operatorname{sgn}(\tau)\left(A|\tau| + 2\ln\left(\frac{1-(\varepsilon/2)\,(1-\exp(-A))}{1-(\varepsilon/2)\,(1-\exp(A|\tau|-A))}\right)\right) \tag{8.59}$$

It can be shown by direct calculations that the power series expansions of (8.58) and (8.59) with respect to ε lead to the same series as those obtained by means of the iterative procedure introduced in this section for a general one-degree-of-freedom oscillator. Moreover, the structure of expression (8.58) suggests that considering the modified series,

$$h = \frac{\varepsilon h_0}{1-\varepsilon\lambda_1-\varepsilon^2\lambda_2-...} \tag{8.60}$$

leads to the exact value h already on the second step of the procedure. This fact can be employed for other cases in order to improve efficiency of the successive approximation series (8.16). For instance, according to the idea of Padè transform [19], the following equality must hold in every order of ε

$$\frac{\varepsilon h_0}{1-\varepsilon\lambda_1-\varepsilon^2\lambda_2-...} = \varepsilon h_0(1+\varepsilon\gamma_1+\varepsilon^2\gamma_2-...) \tag{8.61}$$

This is equivalent to

$$(1+\varepsilon\gamma_1+\varepsilon^2\gamma_2-...)(1-\varepsilon\lambda_1-\varepsilon^2\lambda_2-...) = 1 \tag{8.62}$$

Taking the product of series on the left-hand side of (8.62) and considering different orders of ε, generates a sequence of equations for the coefficients

λ_1, λ_2, ... Then, substituting the corresponding solutions in (8.60), gives a particular case of Padè transform of (8.16) in the form

$$h = \frac{\varepsilon h_0}{1 - \varepsilon\gamma_1 - \varepsilon^2(\gamma_2 - \gamma_1^2) - ...} \tag{8.63}$$

In many cases, expansion (8.63) appears to be more effective than (8.16). Note that it is also possible to organize the successive approximations procedure by using expansion (8.60) instead of (8.16).

8.3.5 A Generalized Case of Odd Characteristics

This subsection deals with some generalization of the standard one-degree-of-freedom conservative oscillator

$$\ddot{x} + f(x) = 0 \tag{8.64}$$

where $f(x)$ is a smooth odd characteristic,

$$f(-x) = -f(x) \tag{8.65}$$

It was shown in this chapter that periodic solutions of the oscillator (8.64) admit the form $x = X(\tau)$, where

$$X(-\tau) = -X(\tau) \tag{8.66}$$

and, in addition, $X(\tau)\tau \geq 0$ for $-1 \leq \tau \leq 1$.

We consider now the following class of oscillators

$$\ddot{x} + \text{sgn}(x)f(|x|) = 0 \tag{8.67}$$

In the case of odd characteristic, $f(x)$, oscillator (8.67) is equivalent to the original one (8.64). The extension is due to the fact that the oscillator (8.67) always has an odd characteristic regardless whether or not the function $f(x)$ itself is odd. In general case, however, the characteristic $\text{sgn}(x)f(|x|)$ may appear to be nonsmooth at the equilibrium point $x = 0$. As a result, direct implementation of iterative procedures with high-order derivatives of oscillator' characteristics becomes quite limited. Nevertheless, as illustrated below, the group properties of equation (8.67) can help to effectively build solution of equation (8.67) based on solution of equation $\ddot{x} + f(x) = 0$ for $x > 0$ by ignoring the point $x = 0$.

Obviously, if $P(x)$ is the potential energy of oscillator (8.64), then $P(|x|)$ is the potential energy corresponding to oscillator (8.67). The following example explains why equation (8.67) covers a broader class of oscillators than (8.64).

Example 13. $\ddot{x}+\text{sgn}(x)|x|^{3/2} = 0$ is an oscillator, but $\ddot{x} + x^{3/2} = 0$ is not; see also Chapter 3 for the related discussion.

Based on the transition from equation (8.64) to equation (8.67) and the general symmetry properties (8.65) and (8.66), we introduce the following representation for periodic solutions of equation (8.67)

$$x = \operatorname{sgn}(\tau)X(|\tau|) \tag{8.68}$$

Such an extension enables one to obtain ‘closed form’ analytical solutions for a large number of oscillators by a simple adaptation of already known solutions, $X(\tau)$, for different cases of smooth characteristics.

Example 14. Applying transformation (8.68) to solution (8.39), which was derived for the power form characteristic x^α with an odd positive exponent $\alpha = m$, gives

$$X = A\operatorname{sgn}(\tau)\left[|\tau| - \frac{|\tau|^{\alpha+2}}{\alpha+2} + \frac{\alpha}{2(\alpha+2)}\left(\frac{|\tau|^{2\alpha+3}}{2\alpha+3} - \frac{|\tau|^{\alpha+2}}{\alpha+2}\right)\right] \tag{8.69}$$

where $\tau = \tau(t/a)$ and the expansion (8.40) for a^2 requires only the replacement $m \to \alpha$. Expansion (8.69) represents an approximate solution of the equation

$$\ddot{x} + \operatorname{sgn}(x)|x|^\alpha = 0 \tag{8.70}$$

where the notation α substitutes m in order to emphasize that the new exponent can take any positive real value, such as even, odd, rational or irrational. Figs. 8.4 and 8.5 illustrate solution (8.69) compared to numerical solution for two different exponents α and the same parameter $A = 1$. As both figures show, the analytical and numerical solutions are in a better match under the large exponent α due to the influence of vibroimpact asymptotic, $\alpha \to \infty$.

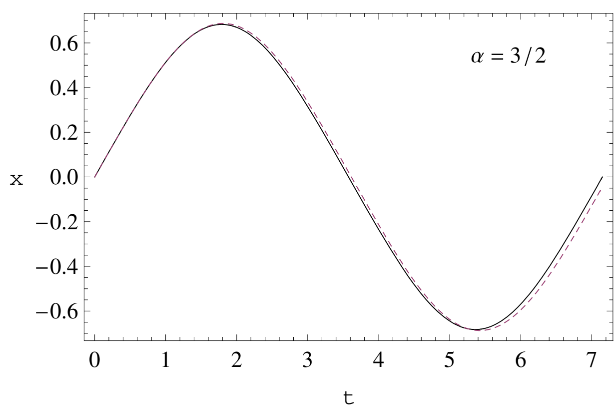

Fig. 8.4 Analytical and numerical solutions of the modified oscillator shown by continuous and dashed lines respectively.

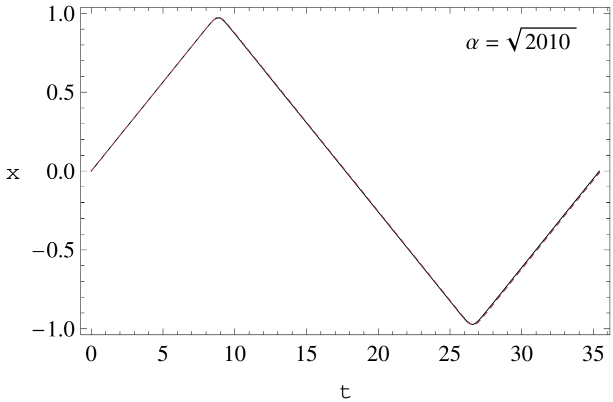

Fig. 8.5 Analytical and numerical solutions of the generalized oscillator under the large exponent α.

The common feature of the algorithms and examples of this section is that generating solutions for successive approximations are represented by triangular sine waves of proper amplitudes and periods. Such generating solutions belong to the 'real' component of the hyperbolic 'number,' $x = X + Ye$. In contrast, the next section introduces algorithms of successive approximations based on the 'imaginary' component. It will be seen that these two approaches have different physical contents.

8.4 Periodic Motions Close to Separatrix Loop

In this section, the classic mathematical pendulum is considered as an example, although the developed algorithm may be applicable to other cases of one degree-of-freedom systems with multiple equilibrium positions. So we illustrate the algorithm based on the differential equation of motion

$$\ddot{x} + \sin x = 0 \tag{8.71}$$

It is assumed that the pendulum oscillates inside the separatrix loop around the stable equilibrium $(x, \dot{x}) = (0, 0)$ in between two physically identical unstable saddle-points $(x, \dot{x}) = (\pm\pi, 0)$. The separatrix loop also represents trajectory of the system with the total energy

$$E_s \equiv \int_0^{\pi} \sin x dx = 2 \tag{8.72}$$

The pendulum remains close to the separatrix loop in the area of periodic motions if the total energy E is within the range

$$0 < 1 - \frac{E}{E_s} << 1 \tag{8.73}$$

Let us show that, under condition (8.73), a successive approximation solution can be derived from the particular case of boundary value problem (8.7) and (8.8). Such particular case is given by setting $X \equiv 0$ so that

$$x(t) = Y(\tau(t/a))e(t/a) \tag{8.74}$$

Substituting (8.74) in equation (8.71) and following the NSTT procedure, yields

$$Y'' = -a^2 \sin Y \tag{8.75}$$

and

$$Y|_{\tau=1} = 0, \quad Y(-\tau) \equiv Y(\tau) \tag{8.76}$$

We seek solution of the boundary value problem, (8.75) and (8.76), in the form of series of successive approximations

$$Y = \pi + \varepsilon Y_1(\tau) + \varepsilon^3 Y_3(\tau) + \varepsilon^5 Y_5(\tau) + ... \tag{8.77}$$

$$a^2 = p^2 / \left(1 - \varepsilon^2 \lambda_2 - \varepsilon^4 \lambda_4 - ...\right) \tag{8.78}$$

where $\varepsilon = 1$ is an auxiliary "parameter" that helps to organize the iterative process.

According to the formal expansion (8.77), the generating solution is represented by the rectangular cosine of the amplitude π, $x(t) = \pi e(t/a)$, which is a step-wise discontinuous function. Such temporal mode shapes occur near the separatrix loop in the natural time scale of the pendulum because the system spends most of the time during one period near the unstable equilibrium positions $x = \pi$ and its physically identical, $x = -\pi$. Therefore, expansion (8.77) is designed to be a high-energy expansion near the unstable equilibrium, rather than around the stable equilibrium position, $x = 0$.

Substituting expansions (8.77) and (8.78) into the equation (8.75) and collecting terms with the same power of ε, leads to the sequence of equations

$$\frac{d^2 Y_1}{d\tau^2} - p^2 Y_1 = 0 \tag{8.79}$$

$$\frac{d^2 Y_3}{d\tau^2} - p^2 Y_3 = p^2 \left(\lambda_2 Y_1 - \frac{1}{6} Y_1^3\right) \tag{8.80}$$

$$\frac{d^2 Y_5}{d\tau^2} - p^2 Y_5 = p^2 \left[\left(\lambda_2^2 + \lambda_4\right) Y_1 - \frac{\lambda_2}{6} Y_1^3 + \frac{1}{120} Y_1^5 + \lambda_2 Y_3 - \frac{1}{2} Y_1^2 Y_3\right] \tag{8.81}$$

$$...$$

Further, the family even solutions of equation (8.79) can be represented in the form

$$Y_1 = -A\frac{\cosh p\tau}{\cosh p} \tag{8.82}$$

where A is an arbitrary constant accompanied by the factor $-\cosh^{-1} p$, which is convenient for further calculations due to the relationship $Y_1(1) = -A$. In particular, this provides the same order of magnitude for the arbitrary constant A as the parameter p and period $T = 4a$ both go to infinity.

Further procedure is formally similar to the standard Poincare-Lindstedt algorithm for nonlinear conservative oscillators with positive linear stiffness. For example, substituting (8.82) into the right part of the equation (8.80), gives a 'resonance term' on the right-hand side of the equation, which is proportional to $\cosh p\tau$. This generates 'hyperbolic secular terms' of the form $\tau\cosh p\tau$ and $\tau\cosh p\tau$ in the particular solution of equation (8.80). Occurrence of such terms can be prevented, however, analogously to the Poincare-Lindstedt method by setting

$$\lambda_2 = \frac{A^2}{8\cosh^2 p} \tag{8.83}$$

As a result, the particular solution of equation (8.80) takes the form

$$Y_3 = \frac{A^3\cosh 3p\tau}{192\cosh^3 p} \tag{8.84}$$

At the next stage, equation (8.81) gives solution

$$Y_5 = \frac{A^5}{4096\cosh^5 p}\left(\cosh 3p\tau - \frac{1}{5}\cosh 5p\tau\right) \tag{8.85}$$

under the condition

$$\lambda_4 = -\frac{3A^4}{512\cosh^4 p} \tag{8.86}$$

Substituting (8.82) through (8.83) in (8.77) and (8.78), and setting $\varepsilon = 1$, gives approximate solution

$$Y = \pi - \frac{A\cosh p\tau}{\cosh p} + \frac{A^3\cosh 3p\tau}{192\cosh^3 p} + \frac{A^5}{4096\cosh^5 p}\left(\cosh 3p\tau - \frac{1}{5}\cosh 5p\tau\right) \tag{8.87}$$

and

$$h = p^2\left(1 - \frac{A^2}{8\cosh^2 p} + \frac{3A^4}{512\cosh^4 p}\right)^{-1} \tag{8.88}$$

where $\tau = \tau\,(t/a)$ and $a = \sqrt{h}$.

The truncated series of successive approximations (8.87) and (8.88) depend upon two parameters, A and p, coupled by the boundary (continuity) condition (8.76) as follows

$$A = \pi + \frac{A^3 \cosh 3p}{192 \cosh^3 p} + \frac{A^5}{4096 \cosh^5 p}\left(\cosh 3p - \frac{1}{5}\cosh 5p\right) \tag{8.89}$$

Equation (8.89) should be interpreted as implicit function $A = A(p)$ near the point $A = \pi$. Therefore, expansions (8.87) and (8.88) represent a one-parameter family of periodic solutions with the parameter p.

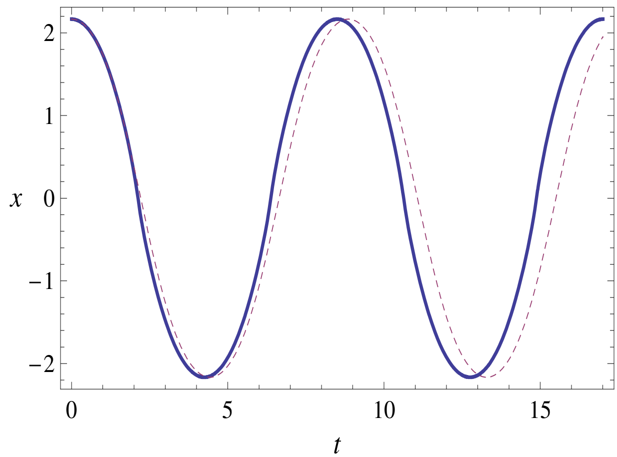

Fig. 8.6 Analytical and numerical solutions of the period T=8.5 (p=2) shown by solid and thin lines, respectively.

Note that the equation (8.71) admits the group of time translations. As a result, another arbitrary parameter, say t_0, is introduced by substitution $t \rightarrow t + t_0$.

Figures 8.6 and 8.7 show that the analytical and numerical solutions are matching better for larger periods as the system trajectory becomes closer to the separatrix loop.

8.5 Self-excited Oscillator

This section illustrates the case when both X and Y components of solutions participate in the iterative process.

In particular, we consider periodic self-sustained vibrations described the differential equation of motion

$$\ddot{x} + g(x)\dot{x} + f(x) = 0 \tag{8.90}$$

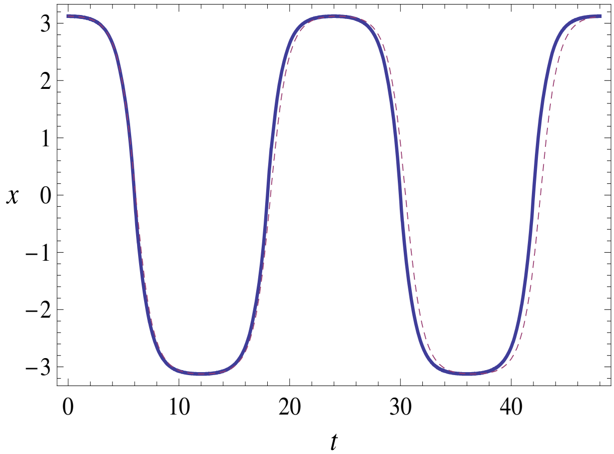

Fig. 8.7 Analytical and numerical solutions of the period T=24.5 (p=6) shown by the solid and thin lines, respectively.

where $f(x)$ and $g(x)$ are analytic functions, such that (Lienard' conditions)

a) $G(x) = \int\limits_0^x g(u)du$ is an odd function such that $G(0) = G(\pm\mu) = 0$ for some $\mu > 0$,
b) $G(x) \to \infty$ if $x \to \infty$, and $G(x)$ is a monotonously increasing function for $x > \mu$,
c) $f(x)$ is an odd function such that $f(x) > 0$ for $x > 0$.

The above conditions guarantee that system (8.90) has a single stable limit cycle. In this case, the boundary-value problem (8.7) and (8.8) takes the form

$$\begin{aligned} X'' &= -a^2 R_f - a\left(R_g Y' + I_g X'\right) \equiv -\varepsilon F_X, \quad X'|_{\tau=\pm1} = 0 \\ Y'' &= -a^2 I_f - a\left(I_g Y' + R_g X'\right) \equiv -\varepsilon F_Y, \quad Y\,|_{\tau=\pm1} = 0 \end{aligned} \tag{8.91}$$

where the period of limit cycle $T = 4a$ is unknown, expressions R_f and I_f as well as R_g and I_g are obtained by applying (8.5) and (8.6) to each of the functions $f(x)$ and $g(x)$, and notations εF_X and εF_Y are introduced with the formal factor $\varepsilon = 1$ for further convenience.

We seek solution of the boundary value problem (8.91) in the form of series of successive approximations

$$\begin{aligned} X &= X_0(\tau) + \varepsilon X_1(\tau) + \varepsilon^2 X_2(\tau) + \cdots \\ Y &= Y_0(\tau) + \varepsilon Y_1(\tau) + \varepsilon^2 Y_2(\tau) + \cdots \end{aligned} \tag{8.92}$$

$$a = q_0 + \varepsilon q_1 + \varepsilon^2 q_2 + \cdots \tag{8.93}$$

Further solution procedure can be simplified by taking into account the symmetry properties $X(-\tau) \equiv -X(\tau)$ and $Y(-\tau) \equiv Y(\tau)$ due to the above conditions (a) through (c). Substituting (8.92) and (8.93) into (8.91) and matching the coefficients of the respective powers of ε, gives the following sequence of boundary value problems

$$\begin{aligned} X_0'' &= 0 \\ Y_0'' &= 0, \quad Y_0|_{\tau=1} = 0 \end{aligned} \tag{8.94}$$

$$\begin{aligned} X_1'' &= -F_{X,0}, \quad (X_0 + X_1)'|_{\tau=1} = 0 \\ Y_1'' &= -F_{Y,0}, \quad Y_1|_{\tau=1} = 0 \end{aligned} \tag{8.95}$$

$$\begin{aligned} X_{i+1}'' &= -F_{X,i}, \quad X_{i+1}'|_{\tau=1} = 0 \\ Y_{i+1}'' &= -F_{Y,i}, \quad Y_{i+1}|_{\tau=1} = 0 \\ &(i = 1, 2, ...) \end{aligned} \tag{8.96}$$

where

$$F_{X,i} = \frac{1}{i!}\frac{d^i F_X}{d\varepsilon^i}|_{\varepsilon=0}, \quad F_{Y,i} = \frac{1}{i!}\frac{d^i F_Y}{d\varepsilon^i}|_{\varepsilon=0} \tag{8.97}$$

Note that zero-order and first-order approximations are coupled through the boundary condition for X-component in (8.95), whereas no boundary condition is imposed on X_0 in (8.94). This specific represents a formalization of the physical assumption regarding the dominating component in the temporal mode shape of vibration, which is assumed to be close to the sawtooth sine rather than rectangular cosine. As a result, the generating system (8.94) gives solution

$$X_0 = A\tau, \;\; Y_0 \equiv 0 \tag{8.98}$$

where A is an arbitrary constant.

Substituting (8.98) in the right-hand side of equations (8.95) and integrating, yields

$$X_1 = -q_0^2 \int_0^\tau (\tau - \xi) f(A\xi) d\xi, \quad Y_1 = -Aq_0 \int_0^\tau (\tau - \xi) g(A\xi) d\xi \tag{8.99}$$

Then, substituting (8.99) in the boundary conditions in (8.95), gives the following two equations for parameters q_0 and A

$$q_0^2 \int_0^1 f(A\xi) d\xi = A, \quad \int_0^1 (1 - \xi) g(A\xi) d\xi = 0 \tag{8.100}$$

Relationships (8.98) through (8.100) complete first basic steps of the iterative procedure. Further iterations are organized then in a similar way as follows

$$X_{i+1} = -\int_0^\tau (\tau - \xi) F_{X,i}(\xi) d\xi, \quad Y_{i+1} = -\int_1^\tau d\zeta \int_0^\zeta F_{Y,i}(\xi) d\xi \tag{8.101}$$

$$\int_0^1 F_{X,i}(\xi) d\xi = 0; \quad i = 1, 2, ... \tag{8.102}$$

Note that the boundary conditions for Y_{i+1} are satisfied automatically due to the lower limit of the outer integral in (8.101), whereas the boundary conditions for X_{i+1} generates equations (8.102) for determining the coefficients of series (8.93). Practically, high-order approximations can be obtained by using computer systems of symbolic manipulations.

Example 15. Consider the self-excited oscillator with the power form stiffness of the degree $m = 3$,

$$\ddot{x} + (bx^2 - 1)\dot{x} + x^3 = 0$$

In this case, $g(x) = bx^2 - 1$ and $f(x) = x^3$. Conducting elementary integrations in (8.100), gives the algebraic system

$$\frac{1}{4} q_0^2 A^3 = A, \quad \frac{1}{12} q_0 A \left(6 - bA^2\right) = 0$$

with non-trivial solution

$$q_0 = \sqrt{\frac{2b}{3}}, \quad A = \sqrt{\frac{6}{b}}$$

As a result, integrating (8.99), yields

$$X_1 = -\sqrt{\frac{6}{b}} \frac{\tau^5}{5}, \quad Y_1 = \tau^2 - \tau^4$$

All further steps of the procedure are conducted according to the same scheme (8.101) and (8.102). For instance, first two steps of the procedure give approximate solution

$$x = \sqrt{\frac{6}{b}} \{\tau - \frac{\tau^5}{5} + \frac{1}{3150}[105\tau^9 + 900\tau^7 b - 21\tau^5(70b + 9) + 350\tau^3 b]\}$$
$$+(1 - \tau^2)\{\tau^2 - \frac{1}{420}[20 - 43\tau^2 + 20\tau^4 + 216\tau^6]\}e$$

and the period

$$T = 4a = 8\sqrt{\frac{b}{6}} \left(1 + \frac{3}{20}\right)$$

Two more steps of the procedure correct the above expression for the period as follows

$$T = 4a = 8\sqrt{\frac{b}{6}}\left(1 + \frac{3}{20} + \frac{2960b + 2121}{50400} + \frac{7367360b^2 + 45554992b + 8659035}{605404800}\right)$$

Figs.8.8 and 8.9 show limit cycle trajectories described by the analytical solutions in one and two iterations, respectively. For comparison, the numerical solution for transition to the limit cycle is also presented. Then, Fig. 8.10 illustrates dependence of the quarter of period parameter, $a = T/4$, versus the quantity $b^{-1/2}$, which can be viewed as an estimate for the amplitude of limit cycle.

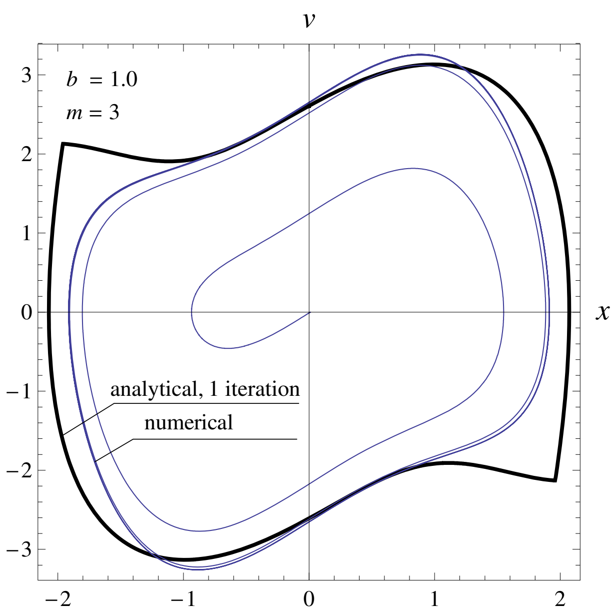

Fig. 8.8 Trajectories of numerical solution and the one iteration analytical limit-cycle solution.

8.6 Strongly Nonlinear Oscillator with Viscous Damping

This section describes the successive approximation procedure combined with the asymptotic of small energy dissipation that leads to a slow amplitude decay. The scheme of the algorithm is closed to that was introduced earlier in [137].

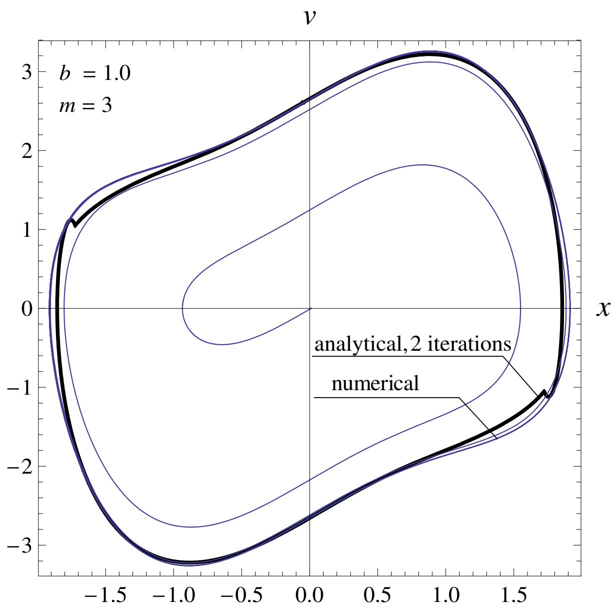

Fig. 8.9 Trajectories of numerical solution and approximate (two iterations) analytical limit-cycle solution.

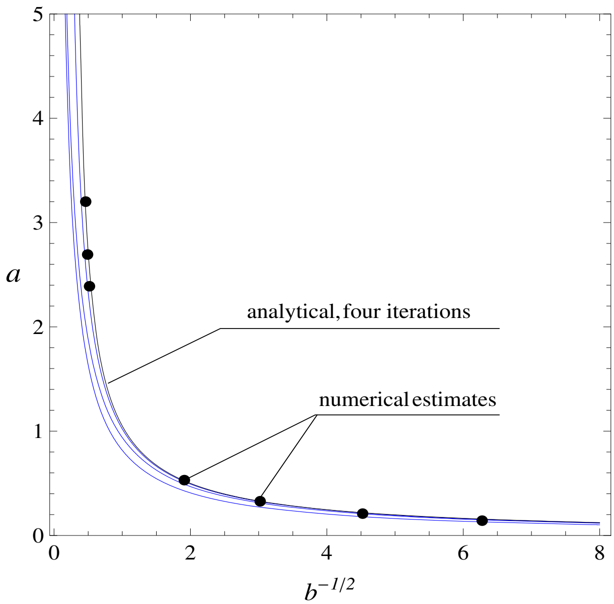

Fig. 8.10 Illustration of convergence of the iterative procedure on the parameter plane.

So consider a strongly nonlinear oscillator under the viscous damping

$$\ddot{x} + 2\mu\dot{x} + f(x) = 0 \tag{8.103}$$

where $f(x)$ is an odd function such that $xf(x) \geq 0$, and $0 < \mu << 1$.

The idea of two variable expansions will be employed below in combination with the sawtooth time substitution. Let us assume that $\tau(\varphi)$ is a 'fast' time scale, whose phase φ depends on the 'slow' time scale $t^0 = \mu t$ according to the following differential equation

$$\dot{\varphi} = \omega(t^0) \tag{8.104}$$

where the right-hand side is a priory unknown.

Let us represent unknown solution of equation (8.103) in the form

$$x = x(\varphi, t^0) = X(\tau(\varphi), t^0) + Y(\tau(\varphi), t^0)e(\varphi) \tag{8.105}$$

Substituting (8.105) in equation (8.103), and imposing 'smoothness conditions,'

$$Y|_{\tau=\pm 1} = 0, \quad \frac{\partial X}{\partial \tau}|_{\tau=\pm 1} = 0 \tag{8.106}$$

gives two partial differential equations

$$\begin{aligned} \omega^2 \frac{\partial^2 X}{\partial \tau^2} &= -R_f - \mu H \frac{\partial Y}{\partial \tau} - \mu^2 L X \\ \omega^2 \frac{\partial^2 Y}{\partial \tau^2} &= -I_f - \mu H \frac{\partial X}{\partial \tau} - \mu^2 L Y \end{aligned} \tag{8.107}$$

where, as follows from (8.5) and (8.6),

$$\begin{aligned} R_f &= R_f(X, Y) = \frac{1}{2}[f(X+Y) + f(X-Y)] \\ I_f &= I_f(X, Y) = \frac{1}{2}[f(X+Y) - f(X-Y)] \end{aligned}$$

and two linear differential operators are introduced

$$\begin{aligned} H &\equiv 2\omega \left(1 + \frac{\partial}{\partial t^0}\right) + \frac{d\omega}{dt^0} \\ L &\equiv \frac{\partial^2}{\partial t^{02}} + 2\frac{\partial}{\partial t^0} \end{aligned} \tag{8.108}$$

Note that the fast and slow temporal scales are associated with different physical processes developed in the system. The slow energy dissipation process is represented by the explicit small parameter μ, but there is no explicit parameter associated with perturbations of the triangular sine wave, which is supposed to be a generating solution of the iterative process. However, as discussed earlier in this section, introducing the sawtooth temporal argument, implies that the entire right-hand side of system (8.107) is small. Otherwise, the triangular sine wave cannot be considered as a dominating component of the temporal mode shape of oscillations. Recall that, in a similar way, selecting the harmonic wave as a dominating solution in quasi harmonic approaches implies that nonlinearities are small regardless system' parameters. Therefore, iterative procedure for boundary value problem (8.106) and (8.107) should incorporate two different procedures, as those described above in this section, and a proper asymptotic procedure related to the dissipation process. Once again, the quasi-harmonic methods face similar situation when dealing with weakly nonlinear systems under small damping conditions. For instance, if being applied to such cases, the method of multiple scales accounts for both unharmonicity and dissipation, after appropriate assumption regarding the relation between non-linearity and damping parameters has been made. Very often though, such parameters are assumed to be of the same order of magnitude. As to the boundary value problem (8.106) and (8.107), similar assumption can be introduced by providing the terms R_f and I_f, and the parameter μ with the same formal 'small factor' $\varepsilon = 1$. Then, the multiple scales or two variables expansions can be organized by using the auxiliary parameter ε [137]. In the case of linear oscillator, such an algorithm recovers the exact solution of the linear differential equation of motion however in the specific form

$$x = X(\tau, t^0) = C\frac{\omega \exp(-t^0)}{\sqrt{\varepsilon(1-\varepsilon\mu^2)}}\sin\left[\frac{\sqrt{\varepsilon(1-\varepsilon\mu^2)}}{\omega}\tau(\omega t)\right] \tag{8.109}$$

and

$$\omega^2 = \frac{\varepsilon(1-\varepsilon\mu^2)}{4\arcsin^2\sqrt{\varepsilon/2}}$$

where C is an arbitrary constant, and another arbitrary constant can be introduced through the arbitrary time shift.

Note that solution (8.109) includes no Y-component because, at every stage of the iterative process, it appears to be possible to satisfy condition

$$H(\partial X/\partial\tau) = 0 \tag{8.110}$$

Therefore, the second equation of system (8.107) is satisfied by setting $Y \equiv 0$. Practically, condition (8.110) generates the common factor $\exp(-t^0)$ for all successive approximations.

In general nonlinear case, however, it is rather impossible to satisfy condition (8.110) at every stage of the process, but it still works for leading order approximate solutions.

Example 16. Consider the weakly damped oscillator of the m degree power form restoring force characteristic

$$\ddot{x} + 2\mu\dot{x} + x^m = 0 \tag{8.111}$$

At this stage, the exponent m is an odd positive number. (It will be shown later that a broader class of power characteristics can also be considered.) Note that, for this kind of oscillators, whether or not the damping is small depends on the level of amplitude and the exponent m. This is due to the fact that, under small enough amplitudes, the elastic force becomes negligible regardless the magnitude of damping. By assuming that the influence of damping is negligible during one cycle of vibration, one can use expressions (8.40) for estimations of magnitudes of damping and elastic forces. As a result, the condition of 'small damping' derives in the form

$$\mu^2 \ll \frac{1}{4}(m+1)A^{m-1} \tag{8.112}$$

One step of the procedure gives approximate solution [137]

$$x = C\exp\left(\frac{-4\mu t}{m+3}\right)\left(\tau - \frac{\tau^{m+2}}{m+2}\right) \tag{8.113}$$

where $\tau = \tau(\varphi)$ and the phase variable is approximated by

$$\varphi = \varphi_\infty\left[1 - \exp\left(-2\mu\frac{m-1}{m+3}t\right)\right] \tag{8.114}$$

$$\varphi_\infty = \frac{1}{2\mu}\frac{m+3}{m-1}\frac{C^{(m-1)/2}}{\sqrt{2(m+1)}}$$

Interestingly enough, the above approximate solution predicts that the oscillator makes only a finite number of waves as $m > 1$ and $t \to \infty$. Figs. 8.11 and 8.12 illustrate damped responses of the oscillator with two different degrees of nonlinearity, $m = 3$ and $m = 7$, respectively. As follows from the diagrams, the approximate analytical solution and numerical one are matching relatively well, especially at higher exponent, $m = 7$. In particular, this justifies the idea of using the sawtooth wave in strongly nonlinear cases, when the oscillator becomes close to the standard vibroimpact model.

8.6.1 Remark on NSTT Combined with Two Variables Expansion

In general, the iterative process of sawtooth expansions and the averaging procedure can be separated. Moreover, the stage of sawtooth time substitution does not impose any specific method of analyses. So let us apply the two variables method directly to the nonlinear boundary value problem (8.106) and (8.107) by means of the asymptotic series

$$\begin{aligned} X &= X_0(\tau, t^0) + \mu X_1(\tau, t^0) + \mu^2 X_2(\tau, t^0) + \cdots \\ Y &= Y_0(\tau, t^0) + \mu Y_1(\tau, t^0) + \mu^2 Y_2(\tau, t^0) + \cdots \end{aligned} \tag{8.115}$$

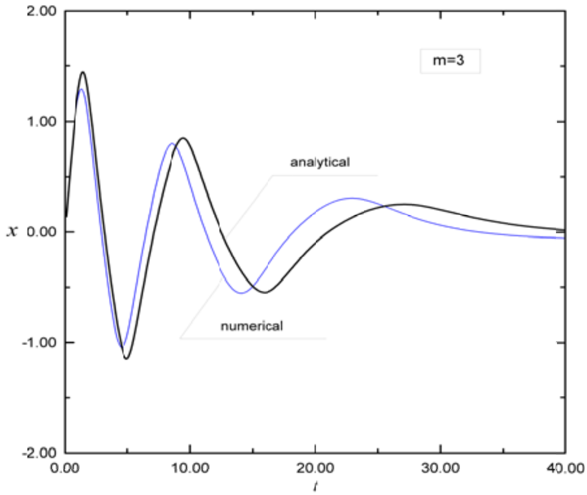

Fig. 8.11 Approximate analytical and numerical solutions of the damped oscillator with cubic power form characteristic.

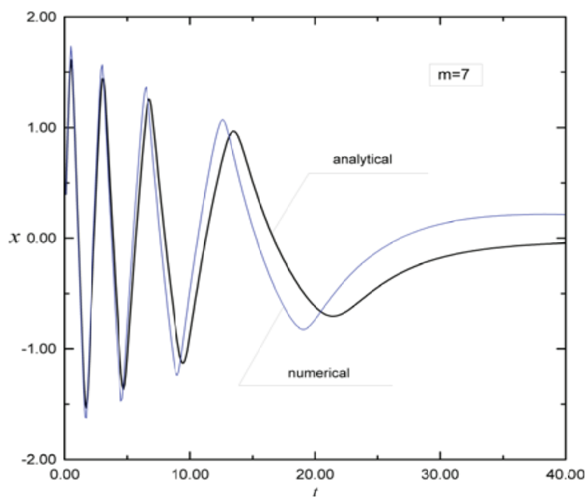

Fig. 8.12 Approximate analytical and numerical solutions of the damped oscillator with the seven-th degree power form characteristic.

and

$$\begin{aligned} \omega &= \omega_0(t^0) + \mu\omega_1(t^0) + \mu^2\omega_2(t^0) + \cdots \\ H &= H_0 + \mu H_1 + \mu^2 H_2 + ... \end{aligned} \tag{8.116}$$

where $H_i = 2\omega_i \left(1 + \partial/\partial t^0\right) + d\omega_i/dt^0$.

Substituting (8.115) and (8.116) into (8.106) and (8.107) and matching coefficients of like powers of μ, gives, in particular,

$$\begin{aligned} \omega_0^2 \frac{\partial^2 X_0}{\partial \tau^2} &= -R_f^{(0)}(X_0, Y_0), \quad \frac{\partial X_0}{\partial \tau}|_{\tau=\pm 1} = 0 \\ \omega_0^2 \frac{\partial^2 Y_0}{\partial \tau^2} &= -I_f^{(0)}(X_0, Y_0), \quad Y_0|_{\tau=\pm 1} = 0 \end{aligned} \tag{8.117}$$

and

$$\begin{aligned} \omega_0^2 \frac{\partial^2 X_1}{\partial \tau^2} &= -R_f^{(1)}(X_0, Y_0, X_1, Y_1) - 2\omega_0\omega_1 \frac{\partial^2 X_0}{\partial \tau^2} - H_0 \frac{\partial Y_0}{\partial \tau} \\ \frac{\partial X_1}{\partial \tau}|_{\tau=\pm 1} &= 0 \\ \omega_0^2 \frac{\partial^2 Y_1}{\partial \tau^2} &= -I_f^{(1)}(X_0, Y_0, X_1, Y_1) - 2\omega_0\omega_1 \frac{\partial^2 Y_0}{\partial \tau^2} - H_0 \frac{\partial X_0}{\partial \tau} \\ Y_1|_{\tau=\pm 1} &= 0 \end{aligned} \tag{8.118}$$

where $R_f^{(0)}$, $R_f^{(1)}$, ... and $I_f^{(0)}$,$I_f^{(1)}$, ... are determined by the expansions

$$\begin{aligned} R &= R_f^{(0)} + \mu R_f^{(1)} + \mu^2 R_f^{(2)} + ... \\ I &= I_f^{(0)} + \mu I_f^{(1)} + \mu^2 I_f^{(2)} + ... \end{aligned}$$

By taking into account the assumptions on $f(x)$ in equation (8.103), one can represent solution of problem (8.117) in the following general form

$$X_0 = X_0(\tau, A, \omega_0), \quad Y_0 \equiv 0 \tag{8.119}$$

where $A = A(t^0)$ is an arbitrary function of the slow time scale, which is coupled with the frequency ω_0 through the boundary condition

$$\frac{\partial X_0(\tau, A, \omega_0)}{\partial \tau}|_{\tau=1} = 0 \tag{8.120}$$

In general, this relationship determines the implicit function $\omega_0 = \omega_0(t^0)$.

Now substituting solution (8.119) in (8.118), gives equations

$$\omega_0^2 \frac{\partial^2 X_1}{\partial \tau^2} + f'(X_0)X_1 = -2\omega_0\omega_1 \frac{\partial^2 X_0}{\partial \tau^2} \tag{8.121}$$

and

$$\omega_0^2 \frac{\partial^2 Y_1}{\partial \tau^2} + f'(X_0) Y_1 = -H_0 \frac{\partial X_0}{\partial \tau} \tag{8.122}$$

Let us consider equation (8.122). The best choice would be achieved by setting the right-hand side to zero and therefore making possible the solution $Y_1 \equiv 0$ which is consistent with zero-order solution (8.119) and provides a better smoothness property of the corresponding solution at this stage; see condition (8.110). Note that the right-hand side cannot be always made zero for any τ unless the zero-order solution admits separation of the variables t^0 and τ. However, it is still possible to 'minimize' the right-hand side of equation (8.122) by making it orthogonal to solution of the corresponding homogeneous equation, $\partial X_0 / \partial \tau$, in other words,

$$\frac{1}{2} \int_{-1}^{1} \frac{\partial X_0}{\partial \tau} H_0 \frac{\partial X_0}{\partial \tau} d\tau \equiv \left\langle \frac{\partial X_0}{\partial \tau} H_0 \frac{\partial X_0}{\partial \tau} \right\rangle = 0 \tag{8.123}$$

Taking into account the expression $H_0 = 2\omega_0 \left(1 + \partial / \partial t^0\right) + d\omega_0 / dt^0$ and condition (8.123), gives

$$\omega_0 \left\langle \left(\frac{\partial X_0}{\partial \tau} \right)^2 \right\rangle = C \exp(-2t^0) \tag{8.124}$$

where C is an arbitrary constant.

It can be shown that the 'minimization condition' (8.123) occurs also in a rigorous mathematical way based on the boundary conditions for Y_1; at least, this can be easily verified in the linear case $f(x) \equiv x$.

8.6.2 Oscillator with Two Nonsmooth Limits

Consider the following generalization of equation (8.111)

$$\ddot{x} + 2\mu \dot{x} + \operatorname{sgn}(x)|x|^\alpha = 0 \tag{8.125}$$

where α is a non-negative real number; see the comments to equation (8.70).

In this case, zero-order solution (8.119) can be obtained in the form (8.69),

$$\begin{aligned} x(t) &= A(t^0)\operatorname{sgn}(\tau(\varphi)) \\ &\times \left[|\tau(\varphi)| - \frac{|\tau(\varphi)|^{\alpha+2}}{\alpha+2} + \frac{\alpha}{2(\alpha+2)} \left(\frac{|\tau(\varphi)|^{2\alpha+3}}{2\alpha+3} - \frac{|\tau(\varphi)|^{\alpha+2}}{\alpha+2} \right) \right] \\ \dot{\varphi}(t) &= \omega_0(t^0), \quad t^0 = \mu t \end{aligned} \tag{8.126}$$

where the functions $A(t^0)$ and $\omega_0(t^0)$ are coupled by relation (8.40) as follows

$$\omega_0 = \frac{1}{a} = \frac{A^{(\alpha-1)/2}}{\sqrt{\alpha+1}}\left\{1+\frac{\alpha}{2(\alpha+2)}\left[1+\frac{(\alpha+1)^2}{(\alpha+2)(2\alpha+3)}\right]\right\}^{-1/2} \tag{8.127}$$

Equations (8.127) and (8.124) admit exact solution

$$A = C\exp\left(-\frac{4\mu t}{3+\alpha}\right), \quad \varphi = \varphi_\infty\left[1-\exp\left(-2\mu\frac{\alpha-1}{\alpha+3}t\right)\right] \tag{8.128}$$

where C is a new arbitrary constant and

$$\varphi_\infty = \frac{1}{2\mu}\frac{\alpha+3}{\alpha-1}\frac{C^{(\alpha-1)/2}(2+\alpha)\sqrt{2(3+2\alpha)}}{\sqrt{(\alpha+1)\left(7\alpha^3+31\alpha^2+47\alpha+24\right)}} \tag{8.129}$$

It follows from expressions (8.128) and (8.129) that the linear system $\alpha = 1$ plays the role of a boundary between the two strongly nonlinear areas

$$N_0 = \{\alpha : 0 \le \alpha << 1\} \quad \text{and} \quad N_\infty = \{\alpha : 1 << \alpha < \infty\} \tag{8.130}$$

In other words, we show that $\alpha = 1$ separates two qualitatively different regions of the dynamics determined by the influence of different nonsmooth limits of the potential well; see Fig. 8.13 for illustration. In particular, if $\alpha > 1$ then the phase variable φ has the finite limit φ_∞ as $t \to \infty$. In contrast, if $\alpha < 1$ then the phase with its temporal rate are exponentially growing, as the amplitude decays and the system approaches the bottom of the potential well. The physical meaning of this effect is most clear from the limit case $\alpha \to 0$, which is discussed below.

Figs. 8.14, 8.15 and 8.16, 8.17 illustrate solution (8.126) through (8.129) for large and small exponents α, respectively. The diagrams suggest quite a good match with numerical solution in *both* branches of the exponent (8.130). The numerical solutions shown by dashed lines were obtained by the standard solver *NDSolve* built in *Mathematica*®. Fig. 8.14 also shows that some divergence between the curves occurs when the amplitude is decreased to the level about $A = 0.6$. Below this level, the condition of small damping (8.112) is not guaranteed any more. In contrast, the curves are in a better match for smaller amplitudes if $\alpha < 1$, see Fig. 8.16. In this case, the amplitude decay just strengthens condition (8.112). The phase plane diagrams shown in Figs. 8.15 and 8.17 have qualitatively different shapes as dictated by the influence of different nonsmooth limits of the potential well, see Fig. 8.13. Let us show now that solution (8.126) captures both nonsmooth limits $\alpha \to 0$ and $\alpha \to \infty$.

For a physically meaningful transition to the limits, let us express the arbitrary parameter C through the initial velocity $v_0 = \dot{x}|_{t=0}$,

$$C = \left[\frac{v_0^2 (\alpha + 1) \left(7\alpha^3 + 31\alpha^2 + 47\alpha + 24\right)}{2(\alpha + 2)^2 (2\alpha + 3)} \right]^{1/(\alpha+1)}$$

and consider the two different cases.

1) As $\alpha \to \infty$, the solution (8.126) through (8.129) gives

$$x = \tau(\varphi) \qquad (8.131)$$
$$\varphi = \frac{v_0}{2\mu} \left[1 - \exp(-2\mu t)\right]$$

Solution (8.131) *exactly* describes the system motion in the square potential well.

2) When $\alpha \to 0$, expressions (8.126) through (8.129) are reduced to

$$x = v_0^2 \exp\left(-\frac{4}{3}\mu t\right) \tau(\varphi) \left(1 - \frac{|\tau(\varphi)|}{2}\right) \qquad (8.132)$$
$$\varphi = \frac{3}{2\mu v_0} \left[\exp\left(\frac{2}{3}\mu t\right) - 1\right]$$

where the identity $\mathrm{sgn}[\tau(\varphi)]|\tau(\varphi)| \equiv \tau(\varphi)$ has been taken into account.

If, in addition $\mu = 0$, then solution (8.132) also *exactly* describes the system dynamics with another nonsmooth limit of the potential energy, $|x|$, as shown in Fig. 8.13. However, if $\mu \neq 0$ then substituting solution (8.132) into the differential equation of motion gives an error $O(\mu^2)$. In terms of first-order asymptotic solutions, the error of order μ occurs on the time period of order $1/\mu$. Therefore, solution (8.132) exactly captures the carrying shape of the vibration, but gives only asymptotic estimate for the exponential decay.

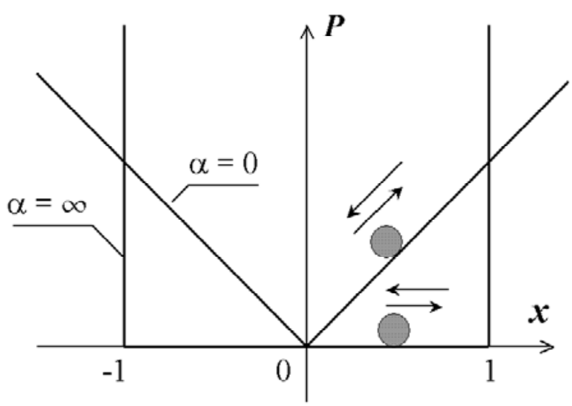

Fig. 8.13 Potential energy representation for the two limit oscillators.

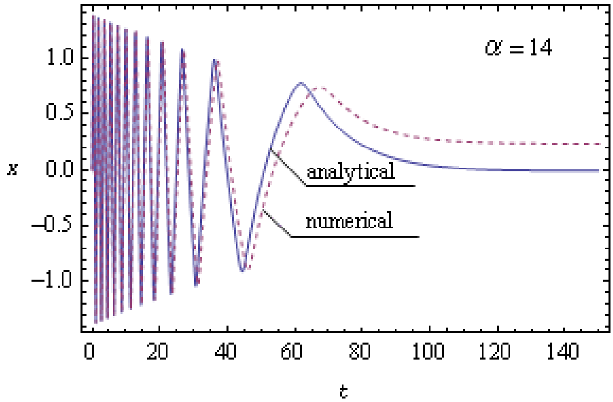

Fig. 8.14 Temporal mode shape of the vibration for $\alpha \in N_\infty$, $C = 1.5$ and $\mu = 0.04$; here and below, the dashed line represents numerical solutions.

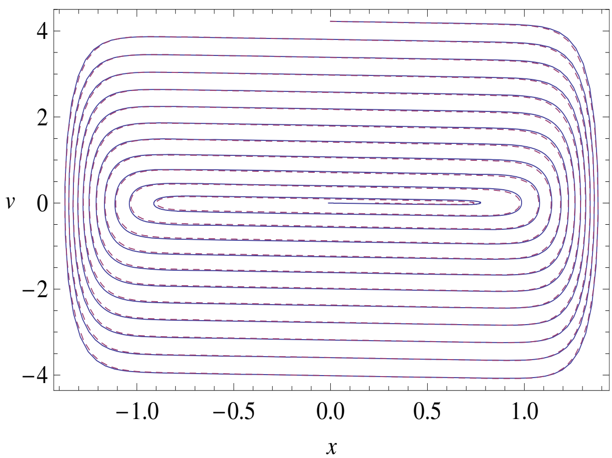

Fig. 8.15 Phase plane diagram for $\alpha \in N_\infty$.

Note that the error of solution (8.126) is due to the error of the iterative procedure for elastic vibrations and the error of asymptotic for energy dissipation. As shown above, the error of successive approximations vanishes as either $\alpha \to \infty$ or $\alpha \to 0$, but the error of asymptotic vanishes only as $\alpha \to \infty$.

Finally, let us discuss the qualitative difference of the dynamics in the parameter intervals N_0 and N_∞. As follows from equation (8.128), for $\alpha \in N_0$, the phase of vibration and the corresponding frequency are exponentially increasing in the slow time scale μt . The physical meaning of this phenomenon

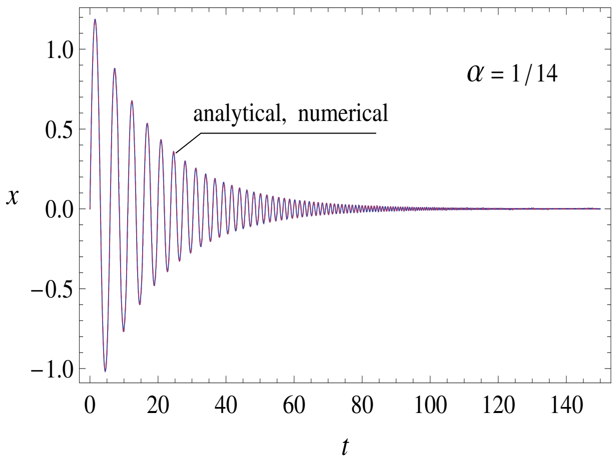

Fig. 8.16 Temporal mode shape of the vibration for $\alpha \in N_0$, $C = 2.5$ and $\mu = 0.04$.

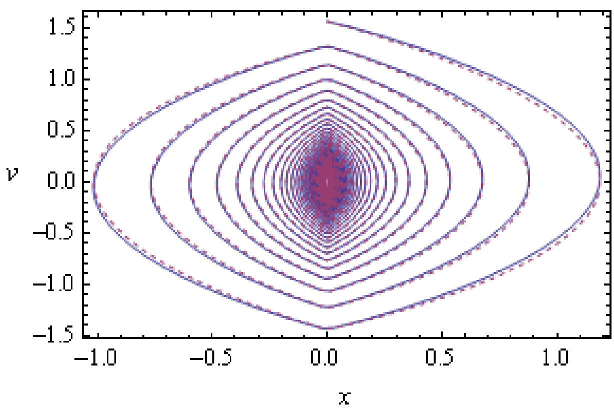

Fig. 8.17 Phase plane diagram for $\alpha \in N_0$.

becomes most clear in the limit case $\alpha = 0$. In this case, according to solution (8.132), the amplitude and frequency are, respectively

$$A(\mu t) = \frac{v_0^2}{2} \exp\left(-\frac{4}{3}\mu t\right) \quad \text{and} \quad \dot{\varphi} = \frac{1}{v_0} \exp\left(\frac{2}{3}\mu t\right) = \frac{1}{\sqrt{2A(\mu t)}} \tag{8.133}$$

Expressions (8.133) describe increasingly rapid vibrations -'dither'- near the corner of the potential energy $|x|$ as the amplitude approaches zero. This result is confirmed by the much earlier analysis of the corresponding conservative case [78]. In contrast, when $\alpha \in N_\infty$, the oscillator makes a limited number of cycles such that the phase φ remains bounded for any time t. Again, the most clear interpretation is obtained in the limit case $\alpha \to \infty$, when, as follows from (8.131), the phase variable $\varphi(t)$ represents the total distance passed by the particle by time t, and $\dot{\varphi} = v$ is the absolute value of the velocity. Since the barriers are perfectly elastic the particle reflects with no energy loss, the velocity $v\,(t)$ remains continuous function of time described by the linear differential equation $\dot{v} + 2\mu v = 0$ or $\ddot{\varphi} + 2\mu\dot{\varphi} = 0$. Under the initial conditions $\varphi(0) = 0$ and $\dot{\varphi}(0) = v_0$, one obtains exactly solution (8.131).

In conclusion, the explicit analytical solution for a class of strongly nonlinear oscillators with viscous damping is introduced. Two different nonsmooth functions involved into the solution are associated with two different nonsmooth limits of the oscillator. As a result, the solution is drastically simplified to give the best match with numerical tests if approaching any of the two limits.

8.7 Bouncing Ball

In this section, we consider a small ball of mass m falling under the gravity force onto a horizontal plane. In addition to the gravity, the ball is subjected to the linear damping with the coefficient c. Impacts with the plane are inelastic with the restitution coefficient κ. The vertical coordinate $z(t)$ therefore is described by the following equations of motion

$$\begin{aligned} m\ddot{z} &= -mg - c\dot{z} \quad (z \neq 0) \\ \dot{z}_+ &= -\kappa \dot{z}_- \quad (z = 0) \end{aligned} \tag{8.134}$$

where $\dot{z}_-$ and $\dot{z}_+$ are velocities right before and immediately after the impact, respectively.

Let h be a natural spatial scale of the system. This can be, for instance, the maximal height that has been reached by the ball during the very first cycle. Introducing the coordinate transformation $z = h|x|$ and re-scaling the time as $t = \sqrt{h/gp}$, brings equations (8.134) to the form [204]

$$\begin{aligned} &\frac{d^2x}{dp^2} + 2\mu\frac{dx}{dp} + \mathrm{sgn}x = 0 \\ &\left(\frac{dx}{dp}\right)_+ - \left(\frac{dx}{dp}\right)_- = (1-\kappa)\left(\frac{dx}{dp}\right)_- \end{aligned} \tag{8.135}$$

where

$$\mu = \frac{1}{2}\frac{c}{m}\sqrt{\frac{h}{g}}$$

In the particular case $\kappa = 1$, solution (8.132) becomes applicable to equation (8.135). Then, returning back to the original notations of equation (8.134), gives

$$z(t) = C \exp\left(-\frac{2c}{3m}t\right)\left[|\tau(\varphi)| - \frac{1}{2}\tau^2(\varphi)\right] \qquad (8.136)$$
$$\varphi(t) = \sqrt{\frac{g}{C}}\frac{3m}{c}\left[\exp\left(\frac{c}{3m}t\right) - 1\right]$$

where C is a new arbitrary constant.

If $C = \dot{z}_0^2/g$ then solution (8.136) satisfies the specific initial conditions $z(0) = 0$ and $\dot{z}(0) = \dot{z}_0$. One more arbitrary constant can be introduced by shifting the time, $t- > t + t_0$, that would allow to consider non-zero initial height of the ball.

In order to compare solution (8.136) with numerical solution, let us represent equation (8.134) in the form

$$\dot{z} = u$$
$$\dot{u} = -g - \frac{c}{m}u \qquad (8.137)$$

where $u_+ = -\kappa u_-$ whenever $z = 0$.

Further, introduce new unknown state variables $\{s, v\}$ according to [73]

$$z = s\text{sgn}(s)$$
$$u = \text{sgn}(s)[1 - k\text{sgn}(sv)]v \qquad (8.138)$$
$$k = \frac{1-\kappa}{1+\kappa}$$

Applying (8.138) to (8.137), gives[1]

$$\dot{s} = [1 - k\text{sgn}(sv)]v \qquad (8.139)$$
$$\dot{v} = -\frac{c}{m}v - \frac{g}{1-k^2}[\text{sgn}(s) + k\text{sgn}(v)]$$

As compared to (8.137), system (8.138) automatically accounts for the velocity jump condition at $z = 0$. In other words, transformation (8.138) makes the strong nonlinearity (due to non-elastic impacts) explicitly present in (8.139). From the standpoint of numerical procedures, such a transformation enables one of using built-in solvers of different packages with no impact conditioning at $z = 0$.

Note that, in the particular case $\kappa = 1$ $(k = 0)$, system (8.139) becomes equivalent to (8.135). In this particular case, the direct numerical integration of equations (8.139), using the NDSolve procedure built in *Mathematica*®

[1] Note that differentiation of sgn-functions will produce Dirac's delta-functions with effectively zero factors however.

package, gives solution in a perfect match with analytical solution (8.136); see Fig.8.18.

Let us assume now that the energy loss happens only due to impact interactions of the ball with the plane $z = 0$. The question is whether or not solution (8.136) can still be adapted by interpreting the parameter c as some "effective damping coefficient," such that the energy loss between two impacts is equal to that happens in one impact.

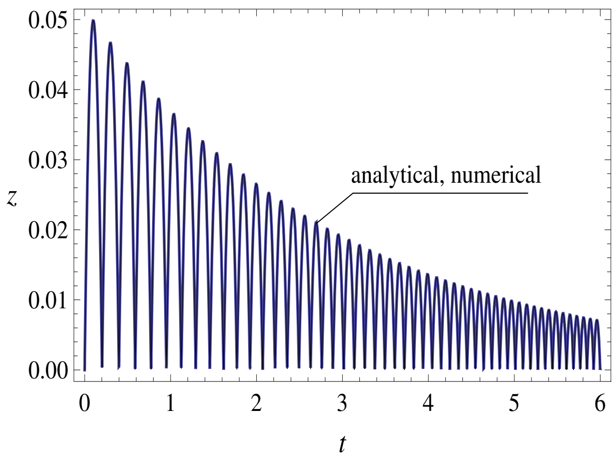

Fig. 8.18 NSTT analytical and direct numerical solutions for the height of bouncing ball under the linear dissipation condition; $m = 1.0$, $c = 0.5$, $z(0) = 0$ and $\dot{z}(0) = 1.0$.

Assuming that the damping is small enough and using the classical parabolic approximation for the ball height during one cycle, $z(t) = -gt^2/2 + v_0 t$, yields

$$c_{eff} \int_0^{2v_0/g} \dot{z}^2 dt = \frac{2c_{eff} v_0^3}{3g} = \frac{1}{2} m(1 - \kappa^2) v_0^2$$

or

$$c_{eff} = \frac{3mg}{4v_0}(1 - \kappa^2) \tag{8.140}$$

Expression (8.140) shows that, in this case, effective linear damping cannot be introduced on the entire time interval since the "initial velocity" v_0 decreases from cycle to cycle. Choosing $v_0 = \dot{z}(0)$ in (8.140), provides a good enough match between analytical (8.136) and numerical solutions during first few cycles of the motion, as follows from Fig. 8.19. Then, the divergence between the curves accelerates. The result can be improved by using some

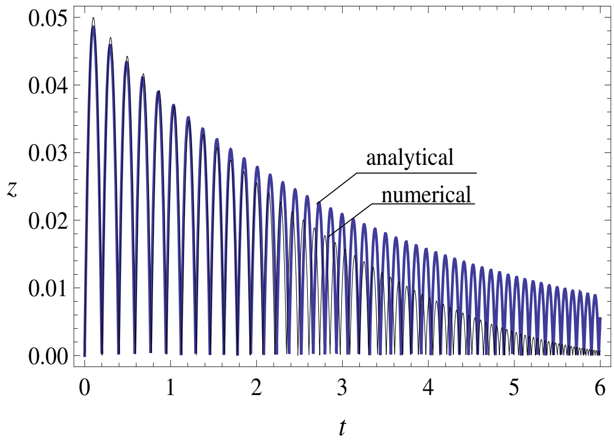

Fig. 8.19 NSTT analytical solution with effective damping corresponding to the coefficient of restitution $\kappa = 0.97$, and direct numerical solutions for the height of bouncing ball; the parameters are: $m = 1.0$, $c = 0.0$, $z(0) = 0$ and $\dot{z}(0) = 1.0$.

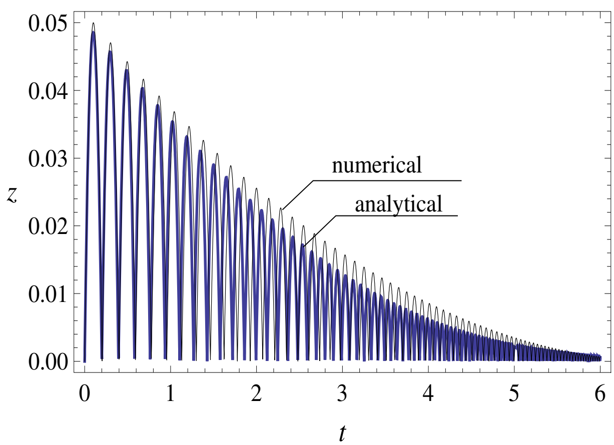

Fig. 8.20 NSTT analytical solution with effective damping adjusted by "variable initial velocity," and direct numerical solutions for the height of bouncing ball; the parameters are: $m = 1.0$, $c = 0.0$, $\kappa = 0.97$, $z(0) = 0$ and $\dot{z}(0) = 1.0$.

estimate for the "initial velocity" decay based on solution (8.136), where the effective damping is still constant, however. This one-step iteration gives

$$\tilde{c}_{eff} = c_{eff} \exp\left(\frac{c_{eff}}{3m}t\right) \tag{8.141}$$

Fig. 8.20 shows a better match between numerical and analytical solutions achieved due to modification (8.141).

8.8 The Kicked Rotor Model

The so-called kicked rotor model is introduced in physics as a relatively simple essentially nonlinear model for chaotic behavior of systems, where one variable may be either bounded or unbounded in phase space [95], [20], [71]. The kicked rotor is described, in some units, by the Hamiltonian

$$H = \frac{1}{2}I^2 + K\cos\theta \sum_{n=-\infty}^{\infty} \delta(t-n) \tag{8.142}$$

where I is the angular momentum, θ is the conjugate angle, and K is the stochasticity parameter that determines qualitative features of the dynamics.

The sequence of pulses in (8.142) can be expressed through the sawtooth sine $\tau = \tau(2t-1)$ in the form,

$$\sum_{n=-\infty}^{\infty} \delta(t-n) = -\frac{1}{2}\tau(2t-1)\,\tau''(2t-1) \tag{8.143}$$

where primes denotes differentiation with respect to the entire argument of a function, $2t-1$.

Note that the only role of the first multiplier, $\tau(2t-1)$, on the right-hand side is to provide pulses with the same sign.

Therefore, (8.142) takes the form

$$H = \frac{1}{2}I^2 - \frac{1}{2}K\tau(2t-1)\,\tau''(2t-1)\cos\theta \tag{8.144}$$

The corresponding differential equation of motion is

$$\ddot{\theta} = -\frac{1}{2}K\tau\tau''\sin\theta \tag{8.145}$$

We seek a family of solutions with the period $T = 2$ by introducing the sawtooth time argument τ,

$$\theta = \theta(\tau) \tag{8.146}$$

Substituting (8.146) in (8.145), gives

$$\frac{d^2\theta}{d\tau^2} = -\left(\frac{1}{8}K\tau\sin\theta + \frac{d\theta}{d\tau}\right)\tau''$$

Eliminating the singular term τ'', leads to the boundary condition

$$\frac{d\theta}{d\tau}|_{\tau=\pm1} = -\frac{1}{8}K\tau\sin\theta|_{\tau=\pm1} \tag{8.147}$$

and the differential equation

$$\frac{d^2\theta}{d\tau^2} = 0 \tag{8.148}$$

The boundary-value problem (8.147) and (8.148) admits solution

$$\theta = A\tau\,(2t - 1) + B \tag{8.149}$$

where A and B appear to be coupled by the set of equations

$$A = -\frac{1}{8}K\,\sin(A \pm B) \tag{8.150}$$

or

$$A = -\frac{1}{8}K \sin A \cos B \tag{8.151}$$

and

$$\cos A \sin B = 0 \tag{8.152}$$

In particular, equation (8.151) show that the number of periodic solutions of the period $T = 2$ is growing as the parameter K increases.

8.9 Oscillators with Piece-Wise Nonlinear Restoring Force Characteristics

Consider the case of asymmetric restoring force characteristics described by two pieces of smooth monotone increasing functions $f(x)$ and $g(x)$ of which one or both may be nonlinear (Fig. 8.21)

$$F(x) = \begin{cases} f(x), & -\infty < x < x_1 \\ g(x), & x_1 < x < \infty \end{cases} \tag{8.153}$$

It is assumed that the entire characteristic $F(x)$ is at least continuous in the interval $-\infty < x < \infty$, in other words, the following matching condition holds

$$f(x_1) = g(x_1) \tag{8.154}$$

and $f(0) = 0$.

Expression (8.153) admits the unit form

$$F(x) = f(x) + [g(x) - f(x)]H(x - x_1) \tag{8.155}$$

where H indicates the unit-step Heaviside function.

Let us outline analytical procedure for periodic solutions of equation

$$\ddot{x} + F(x) = 0 \tag{8.156}$$

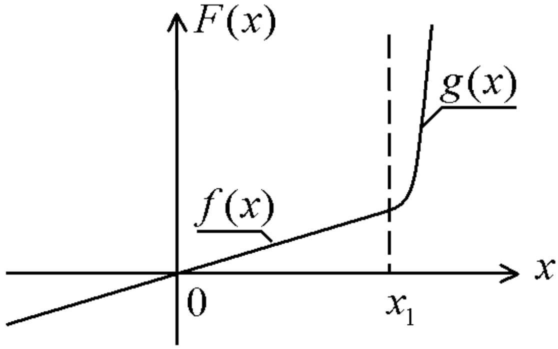

Fig. 8.21 Asymmetric piece-wise nonlinear restoring force characteristic.

Introducing the triangular wave temporal argument $\tau = \tau(t/a)$ with unknown scaling parameter a and making the substitution $x = X(\tau)$ in equation (8.156), gives boundary value problem

$$X'' + hF(X) = 0 \tag{8.157}$$

$$X'|_{\tau=\pm 1} = 0 \tag{8.158}$$

where, as usually, $h = a^2 = (T/4)^2$.

Following the physical reasoning discussed in Introduction and the second section of this chapter, we seek solution in the form of successive approximation series with no explicit small parameter

$$\begin{aligned} X &= X_0(\tau) + X_1(\tau) + X_2(\tau) + ... \\ h &= h_0 + h_1 + ... \end{aligned} \tag{8.159}$$

where the generating solution $X_0(\tau)$ obeys the differential equation $X_0''(\tau) = 0$ and therefore describes the dynamics of a two-parameter family of free impact oscillators with arbitrary amplitudes A_0 and the origin shift as follows

$$X_0(\tau, \tau_1, A_0) = x_1 + A_0(\tau - \tau_1) \tag{8.160}$$

The idea behind approximation (8.160) is to choose its parameters in order to make the motion $x(t) = X_0(\tau(t/a))$ in 'some sense'[2] close to that of oscillator (8.156). The form of (8.160) implies that the system is at the matching point whenever $\tau = \tau_1$.

Next term of series (8.159) is found then from the equation $X_1'' = -h_0 F(X_0)$ in the integral form

[2] Note that the harmonic balance analogy of generating solution (8.160) would be $x(t) = A \sin \omega t + B$.

$$X_1(\tau,\tau_1,A_0,A_1) = A_1(\tau-\tau_1) - h_0\int_{\tau_1}^{\tau}(\tau-\xi)F(X_0(\xi,\tau_1,A_0))d\xi \quad (8.161)$$

where the combination of arbitrary constants is similar to that in generating solution (8.160).

Since $X_1(\tau_1,\tau_1,A_0,A_1) = 0$ then $\tau = \tau_1$ corresponds to the matching point $x = x_1$ in first two terms of the successive approximations, (8.160) and (8.161),

$$X = x_1 + (A_0+A_1)(\tau-\tau_1) - h_0\int_{\tau_1}^{\tau}(\tau-\xi)F(X_0(\xi,\tau_1,A_0))d\xi \quad (8.162)$$

Therefore, after the first correction to generating solution (8.160), we have three arbitrary constants A_0, $A_1$and τ_1, and one still unknown parameter h_0 related to the period of free vibration. Now, substituting (8.162) in boundary conditions (8.158), gives

$$A_0 + A_1 - h_0\int_{\tau_1}^{1}F(X_0(\tau,\tau_1,A_0))d\tau = 0$$

$$A_0 + A_1 + h_0\int_{-1}^{\tau_1}F(X_0(\tau,\tau_1,A_0))d\tau = 0 \quad (8.163)$$

where the variable of integration ξ has been formally replaced by τ.

Equations (8.163) are equivalent to

$$\int_{-1}^{1}F(X_0(\tau,\tau_1,A_0))d\tau = 0 \quad (8.164)$$

$$\int_{\tau_1}^{1}F(X_0(\tau,\tau_1,A_0))d\tau = h_0^{-1}(A_0+A_1) \quad (8.165)$$

If no more iterations are planned then we set $A_1 = 0$, because the corresponding term contributes nothing qualitatively new into the approximate solution. In this case, equation (8.164) is used to express τ_1 through another constant A_0 and the matching point coordinate x_1. Then, equation (8.165) provides the link between the parameters of amplitude A_0 and period $T = 4\sqrt{h_0}$.

The next step of iteration employs the parameter A_1, however. The form of differential equation for the next step of procedure is analogous to (8.20), where $\gamma_1 h_0 = h_1$. Therefore, on the next step, the general solution is given by

$$X_2 = -h_1 \int_{\tau_1}^{\tau} (\tau - \eta) F(X_0(\eta, \tau_1, A_0)) d\eta$$
$$-h_0 \int_{\tau_1}^{\tau} (\tau - \eta) F'(X_0(\eta, \tau_1, A_0)) X_1(\eta, \tau_1, A_0, A_1) d\eta + A_2(\tau - \tau_1) \tag{8.166}$$

where zero lower limit of integration provides $X_2(\tau_1) = 0$, and the prime indicates derivative with respect to the entire argument, $F'(X_0) \equiv dF(X_0)/dX_0$.

Note that the characteristic $F(x)$ is, generally speaking, nonsmooth at the matching point x_1 so that the derivative $F'(x)$ may not exist at $x = x_1$. Although integration of step-wise discontinuous functions is still possible, the current procedure is designed to avoid calculating derivatives of the characteristic at the point x_1. For that reason, the lower limit of integration in (8.166) is associated with the non-smoothness point by expression (8.160). In addition, as follows from (8.161), the uncertainty $F'(x_1)$ in the integrand is suppressed by zero factor $X_1(\tau_1) = 0$. Obviously, on the next step of iteration, this factor will accompany the second derivative $F''(x_1)$, whereas the first derivative acquires the factor $X_2(\tau_1)$.

Now, applying boundary conditions (8.158) to (8.166), yields

$$-h_1 \int_{\tau_1}^{1} F(X_0(\tau, \tau_1, A_0)) d\tau$$
$$- h_0 \int_{\tau_1}^{1} F'(X_0(\tau, \tau_1, A_0)) X_1(\tau, \tau_1, A_0, A_1) d\tau + A_2 = 0 \tag{8.167}$$

$$h_1 \int_{-1}^{\tau_1} F(X_0(\tau, \tau_1, A_0)) d\tau$$
$$+ h_0 \int_{-1}^{\tau_1} F'(X_0(\tau, \tau_1, A_0)) X_1(\tau, \tau_1, A_0, A_1) d\tau + A_2 = 0 \tag{8.168}$$

Substracting the both sides of equation (8.167) from (8.168) and taking into account equation (8.164), gives

$$\int_{-1}^{1} F'(X_0(\tau, \tau_1, A_0)) X_1(\tau, \tau_1, A_0, A_1) d\tau = 0 \tag{8.169}$$

Then, taking into account (8.165), brings equation (8.167) to the form

$$\int_{\tau_1}^{1} F'(X_0(\tau, \tau_1, A_0))X_1(\tau)d\tau = -h_1 h_0^{-2}(A_0 + A_1) + h_0^{-1} A_2 \qquad (8.170)$$

If no more iterations needed, then we set $A_2 = 0$ in expression (8.166) and equation (8.170). Further, substituting (8.161) in (8.169), gives equation for A_1, whereas (8.170) gives equation for h_1. This completes two steps of successive approximations, although the algebraic problem still persists, and its complexity depends on the functions $f(x)$ and $g(x)$.