Chapter 6
NSTT for Linear and Piecewise-Linear Systems

Abstract. Remind that the tool of nonsmooth argument substitutions was introduced first to describe strongly nonlinear vibrations whose temporal mode shapes are asymptotically close to non-smooth ones. Such cases are known to be most difficult for analyses because different quasi-harmonic methods are already ineffective whereas nonsmooth mapping tools are still inapplicable. It is quite clear however that the non-smooth arguments can be introduced regardless the strength of nonlinearity or the form of dynamical systems in general. For instance, it is shown in this chapter that the non-smooth substitutions can essentially simplify analyses of different linear models with non-smooth or discontinuous inputs. It is also shown that, in piecewise-linear cases, the nonsmooth temporal transformation provides an automatic matching the motions from different subspaces of constant stiffness and justifies quasi-linear asymptotic solutions for the specific nonsmooth case of piece-wise linear characteristics.

6.1 Free Harmonic Oscillator: Temporal Quantization of Solutions

Introducing the sawtooth temporal argument into the differential equations of motion may bring some specific features into the corresponding solutions. For illustrating purposes, let us consider the harmonic oscillator

$$\ddot{x} + \omega_0^2 x = 0 \tag{6.1}$$

First, let us obtain exact general solution of the oscillator (6.1) in terms of the sawtooth temporal argument by using the substitution

$$x = X(\tau) + Y(\tau)\, e \tag{6.2}$$

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where $\tau = \tau(t/a)$ and $e = e(t/a)$ are the standard triangular and rectangular wave functions, respectively.

Substituting (6.2) in (6.1), gives the boundary value problem

$$a^{-2}X''(\tau) + \omega_0^2 X(\tau) = 0 \tag{6.3}$$

$$a^{-2}Y''(\tau) + \omega_0^2 Y(\tau) = 0 \tag{6.4}$$

$$X'(\pm 1) = 0,\ Y(\pm 1) = 0 \tag{6.5}$$

By considering the parameter a as an eigen value of the problem, one obtains the set of eigen values and the corresponding solutions as, respectively,

$$a_j = \frac{j\pi}{2\omega_0} \tag{6.6}$$

and

$$X_j = \sin\left(\frac{j\pi\tau}{2} + \varphi_j\right), \quad Y_j = \cos\left(\frac{j\pi\tau}{2} - \varphi_j\right) \tag{6.7}$$

where $\varphi_j = (\pi/4)\left[1 + (-1)^j\right]$, $\tau = \tau(t/a_j)$, and j is any positive real integer.

Therefore, introducing the sawtooth oscillating time produced the discrete family of solutions for harmonic oscillator (6.1).

The nature of such kind of quantization is due to the specific temporal symmetry of periodic motions. In other words, the quantization is associated with a multiple choice for the period

$$T_j = 4a_j = jT \tag{6.8}$$

where $T = 2\pi/\omega_0$ is the natural period of oscillator (6.1).

In terms of the original temporal variable t, the number j plays no role for the temporal mode shape, given by

$$\begin{aligned} x(t) = A\sin\left[\frac{j\pi}{2}\tau\left(\frac{2\omega_0 t}{j\pi}\right) + \varphi_j\right] \\ +B\cos\left[\frac{j\pi}{2}\tau\left(\frac{2\omega_0 t}{j\pi}\right) - \varphi_j\right] e\left(\frac{2\omega_0 t}{j\pi}\right) \end{aligned} \tag{6.9}$$

where A and B are arbitrary constants, and $x(t)$ is the same harmonic wave regardless the number j.

In this section, the free linear oscillator was considered for illustrating purposes. Of course, there is no other pragmatic reason for introducing the sawtooth time into equation (6.1). The situation drastically changes however in non-autonomous cases of non-smooth or discontinuous inputs. It is shown below that, in such cases, the sawtooth time variable can help to facilitate determining particular solutions. The effect of 'temporal quantization' represented by expression (6.9), which seems to be just identical transformation in the autonomous case, acquires helpful meaning at the presence of external excitations. For instance, according to (6.9) the so-called combination resonances will appear to be an inherent property of oscillators.

6.2 Non-autonomous Case

6.2.1 Standard Basis

Consider the linear harmonic oscillator under the external forcing described by the linear combination of triangular and rectangular wave functions

$$\ddot{x} + \omega_0^2 x = F\tau\left(\frac{t}{a}\right) + Ge\left(\frac{t}{a}\right) \tag{6.10}$$

where F and G are constant amplitudes, and a is a quarter of the period.

Substituting (6.2) in (6.10), leads to the boundary value problem

$$a^{-2}X''(\tau) + \omega_0^2 X(\tau) = F\tau \tag{6.11}$$

$$a^{-2}Y''(\tau) + \omega_0^2 Y(\tau) = G \tag{6.12}$$

under the boundary conditions (6.5).

In contrast to autonomous case (6.1), the parameter a is known. However, the equations (6.11) and (6.12) are non-homogeneous, and thus a non-zero solution exists for any a and can be found in few elementary steps. As a result, the particular periodic solution of the original equation (6.10) takes the form

$$\begin{aligned} x_p(t) = X(\tau) + Y(\tau)e = \frac{F}{\omega_0^2}\left\{\tau\left(\frac{t}{a}\right) - \frac{\sin\left[a\omega_0\tau(t/a)\right]}{a\omega_0\cos a\omega_0}\right\} \\ + \frac{G}{\omega_0^2}\left\{1 - \frac{\cos\left[a\omega_0\tau(t/a)\right]}{\cos a\omega_0}\right\}e\left(\frac{t}{a}\right) \end{aligned} \tag{6.13}$$

The corresponding general solution is $x(t) = A\cos(\omega_0 t - \varphi) + x_p(t)$, where A and φ are arbitrary amplitude and phase parameters. Note that solution (6.13) immediately shows all possible resonance combinations $a\omega_0 = (2k+1)\pi/2$ or

$$\frac{\omega_0}{\Omega} = 2k + 1 \tag{6.14}$$

where $k = 1, 2, 3...$, and $\Omega = 2\pi/T = \pi/(2a)$ is the principal circular frequency of the external forcing.

It is interesting to compare the solution (6.13) with those obtained by the conventional methods such as Fourier series. So, taking into account expansion,

$$\tau\left(\frac{t}{a}\right) = \frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sin\frac{(2k+1)\pi t}{2a} \tag{6.15}$$

gives the particular solution of equation (6.10) in the form

$$x_p(t) = \sum_{k=0}^{\infty} \frac{1}{\omega_0^2 - \left[\frac{(2k+1)\pi}{2a}\right]^2} \times \tag{6.16}$$
$$\times \left[\frac{8F(-1)^k}{\pi^2(2k+1)^2} \sin \frac{(2k+1)\pi t}{2a} + \frac{4G(-1)^k}{\pi(2k+1)} \cos \frac{(2k+1)\pi t}{2a} \right]$$

Solution (6.16) indicates the same resonance conditions, (6.14). However, infinite trigonometric series are less convenient for calculations, especially when dealing with derivatives of the solutions; indeed, differentiation slows down convergence of series (6.16).

6.2.2 Idempotent Basis

Consider the linear oscillator including viscous damping under the rectangular wave external loading

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = pe\left(\frac{t}{a}\right) \tag{6.17}$$

The purpose is to obtain periodic (particular) solution with the period of external loading $T = 4a$. Recall that the idempotent basis is introduced by means of the linear transformation (see Chapters 1 and 4)

$$\{1, e\} \longrightarrow \{e_+, e_-\}: \qquad e_\pm = \frac{1}{2}(1 \pm e) \tag{6.18}$$

or, inversely, $1 = e_+ + e_-$ and $e = e_+ - e_-$, where $e_\pm^2 = e_\pm$ and $e_+e_- = 0$.

Now, the periodic solution and external loading are represented in the form

$$\begin{aligned} x(t) &= U(\tau)e_+ + V(\tau)e_- \\ pe &= p(e_+ - e_-) \end{aligned} \tag{6.19}$$

where $e_\pm = e_\pm(t/a)$, and $U(\tau)$ and $V(\tau)$ are unknown functions of the triangular wave $\tau = \tau(t/a)$.

Substituting (6.19) in (6.17), and sequentially eliminating derivatives of the rectangular wave $e(t/a)$ as described in Chapter 4, gives equations

$$\begin{aligned} U'' + 2\zeta\omega a U' + (\omega a)^2 U &= pa^2 \\ V'' - 2\zeta\omega a V' + (\omega a)^2 V &= -pa^2 \end{aligned} \tag{6.20}$$

and boundary conditions

$$\begin{aligned} (U - V)|_{\tau=\pm 1} &= 0 \\ (U' + V')|_{\tau=\pm 1} &= 0 \end{aligned} \tag{6.21}$$

All the coefficients and right-hand sides of both equations in (6.20) are constant, and the equations are decoupled. As a result, solution of boundary value problem (6.20) and (6.21) is easily obtained in the form

$$U(\tau) = \frac{p}{\omega^2} - \frac{2p\exp(-\alpha\tau)}{\beta\omega^2(\cos 2\beta + \cosh 2\alpha)} \times [\cos\beta\cosh\alpha(\beta\cos\beta\tau + \alpha\sin\beta\tau) + \sin\beta\sinh\alpha(\alpha\cos\beta\tau - \beta\sin\beta\tau)] \tag{6.22}$$

$$V(\tau) = -\frac{p}{\omega^2} + \frac{2p\exp(\alpha\tau)}{\beta\omega^2(\cos 2\beta + \cosh 2\alpha)} \times [\cos\beta\cosh\alpha(\beta\cos\beta\tau - \alpha\sin\beta\tau) + \sin\beta\sinh\alpha(\alpha\cos\beta\tau + \beta\sin\beta\tau)] \tag{6.23}$$

where $\alpha = \omega a\zeta$ and $\beta = \omega a\sqrt{1-\zeta^2}$.

Substituting (6.22) and (6.23) in (6.19), gives closed form particular solution of original equation (6.17). Transition to the original temporal variable is given by the functions $\tau(\varphi) = (2/\pi)\arcsin[\sin(\pi t/2)]$ and $e(\varphi) = \mathrm{sgn}[\cos(\pi t/2)]$. Since the system under consideration is linear, the general solution of equation (6.17) can be obtained by adding general equation of the corresponding equation with zero right-hand side. Finally, note that neither trigonometric expansions nor any integral transforms were involved into the solution procedure.

6.3 Systems under Periodic Pulsed Excitation

Instantaneous impulses acting on a mechanical system can be modeled either by imposing specific matching conditions on the system state vector at pulse times or by introducing Dirac's functions into the differential equations of motion. The first approach deals with the differential equations of a free system separately between the impulses, therefore a sequence of systems under the matching conditions are considered. The second method gives a single set of equations over the whole time interval without any conditions of matching. In this case however the analysis can be carried out correctly in terms of distributions, which unfortunately requires additional mathematical justifications in non-linear cases. Both of the above approaches are actually employed for different quantitative and qualitative analyses. The analytical tool, which is described below, on the one hand, eliminates the singular terms from the equations and, on the other hand, brings solutions to the unit-form of a single analytic expression for the whole time interval.

6.3.1 Regular Periodic Impulses

Introducing the sawtooth temporal argument may significantly simplify solutions whenever loading functions are combined of the triangular wave and its derivatives. For instance, let us seek a particular solution of the first order differential equation

$$\dot{v} + \lambda v = \mu \sum_{k=-\infty}^{\infty} \left[\delta(t + 1 - 4k) - \delta(t - 1 - 4k) \right] \tag{6.24}$$

where λ and μ are constant parameters.

For positive λ, equation (6.24) describes the velocity of a particle moving in a viscous media under the periodic impulsive force. The corresponding physical model is shown in Fig. 6.1, where the freely moving massive tank experiences perfectly elastic reflections from the stiff obstacles. By scaling the variables, one can bring the differential equation of motion of the particle to the form (6.24), where $v(t) = \dot{x}(t)$.

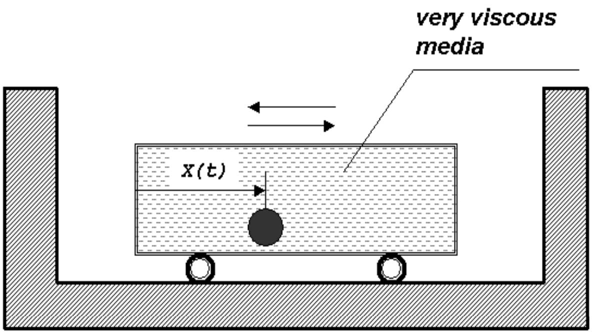

Fig. 6.1 If the particle' mass is very small compared to the total mass of the tank then the inertia force applied to the particle inside the tank has the periodic pulse-wise character.

First, note that the right-hand side of equation (6.24) can be expressed through the generalized derivative of the rectangular wave function as follows

$$\dot{v} + \lambda v = \frac{\mu}{2} \dot{e}(t) \tag{6.25}$$

Now let us represent the particular solution in the form

$$v(t) = X(\tau(t)) + Y(\tau(t))\, e(t) \tag{6.26}$$

Substituting (6.26) in (6.25), gives

$$Y' + \lambda X + (X' + \lambda Y)\, e(t) + \underline{\left(Y - \frac{\mu}{2}\right) \dot{e}(t)} = 0 \tag{6.27}$$

Apparently, the elements $\{1, e\}$ and $\dot{e}$ in combination (6.27) are linearly independent as functions of different classes of smoothness. Therefore,

$$Y' + \lambda X = 0, \qquad X' + \lambda Y = 0, \qquad Y|_{\tau=\pm 1} = \frac{\mu}{2} \tag{6.28}$$

In contrast to equation (6.24) or (6.25), boundary value problem (6.28) includes no discontinuities whereas the new independent variable belongs to the standard interval, $-1 \leq \tau \leq 1$.

Solving the boundary value problem (6.28) and taking into account substitution (6.26), gives periodic solution of equation (6.24) in the form

$$v = X + Ye = \frac{\mu}{2\cosh\lambda}\left(-\sinh\lambda\tau + e\cosh\lambda\tau\right)$$

or

$$v = \frac{\mu}{2\cosh\lambda}\exp\left[-\lambda\tau\left(t\right)e\left(t\right)\right]e\left(t\right) \tag{6.29}$$

Fig. 6.2 illustrates solution (6.29) for $\mu = 0.2$ and different magnitudes of λ.

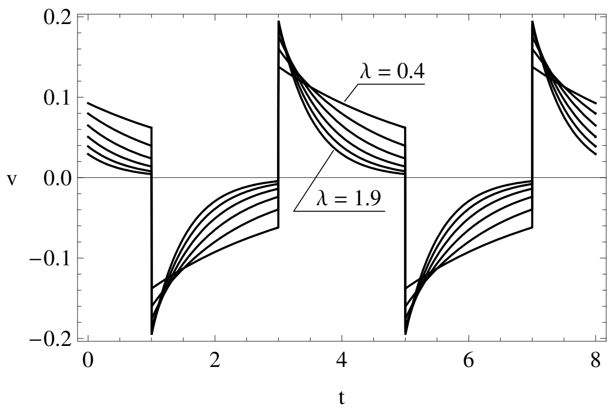

Fig. 6.2 The family of discontinuos periodic solutions.

Note that the discontinuous solution $v\left(t\right)$ is described by the unit-form expression (6.29) through the two elementary functions $\tau\left(t\right)$ and $e\left(t\right)$.

6.3.2 Harmonic Oscillator under the Periodic Impulsive Loading

Let us consider the harmonic oscillator subjected to periodic pulses

$$\ddot{x} + \omega_0^2 x = 2p\sum_{k=-\infty}^{\infty}\left[\delta\left(\omega t + 1 - 4k\right) - \delta\left(\omega t - 1 - 4k\right)\right] \tag{6.30}$$

where p, ω_0 and ω are constant parameters.

The right-hand side of equation (6.30) can be expressed through first derivative of the rectangular wave as follows

$$\ddot{x} + \omega_0^2 x = p \frac{de(\omega t)}{d(\omega t)} \tag{6.31}$$

Let us seek a periodic solution of the period $T = 4/\omega$ in the form

$$x(t) = X(\tau(\omega t)) + Y(\tau(\omega t))\, e(\omega t) \tag{6.32}$$

Substituting (6.32) in (6.31) under the necessary condition of continuity for $x(t)$, gives

$$\omega^2 X'' + \omega_0^2 X + \left(\omega^2 Y'' + \omega_0^2 Y\right) e + \underline{\left(\omega^2 X' - p\right) \frac{de(\omega t)}{d(\omega t)}} = 0 \tag{6.33}$$

Analogously to the previous subsection, equation (6.33) gives the boundary value problem

$$X'' + \left(\frac{\omega_0}{\omega}\right)^2 X = 0, \qquad Y'' + \left(\frac{\omega_0}{\omega}\right)^2 Y = 0 \tag{6.34}$$
$$X'|_{\tau=\pm 1} = \frac{p}{\omega^2}, \qquad Y|_{\tau=\pm 1} = 0$$

Solving boundary value problem (6.34) and taking into account (6.32), gives the periodic solution of the original equation (6.30) in the form

$$x = X(\tau(\omega t)) = \frac{p}{\omega\omega_0} \frac{\sin\left[(\omega_0/\omega)\, \tau(\omega t)\right]}{\cos(\omega_0/\omega)} \tag{6.35}$$

where $Y \equiv 0$.

Solution (6.35) is continuous, but nonsmooth at those times t where $\tau(\omega t) = \pm 1$. All possible resonances are given by

$$\omega = \frac{2}{\pi} \frac{\omega_0}{k}; \qquad k = 1, 3, 5, ... \tag{6.36}$$

where the factor $2/\pi$ is due to different normalization of the periods for trigonometric and sawtooth sines.

Now let us consider the case of viscous damping described by the differential equation of motion

$$\ddot{x} + 2\zeta \dot{x} + \omega_0^2 x = p \frac{de(\omega t)}{d(\omega t)} \tag{6.37}$$

where ζ is the damping factor.

In this case, the boundary value problem becomes coupled

$$X'' + 2\frac{\zeta}{\omega}Y' + \left(\frac{\omega_0}{\omega}\right)^2 X = 0$$
$$Y'' + 2\frac{\zeta}{\omega}X' + \left(\frac{\omega_0}{\omega}\right)^2 Y = 0 \tag{6.38}$$

$$X'|_{\tau=\pm 1} = \frac{p}{\omega^2}, \qquad Y|_{\tau=\pm 1} = 0$$

As a result, the periodic solution has both X and Y components

$$\begin{aligned} x = X + Ye = & \frac{p}{\beta\omega^2\left(\cos^2\beta\cosh^2\alpha + \sin^2\beta\sinh^2\alpha\right)} \\ & \times[\cosh\alpha\cos\beta\cosh\alpha\tau\sin\beta\tau - \sinh\alpha\sin\beta\sinh\alpha\tau\cos\beta\tau \\ & + (\sinh\alpha\cos\beta\tau\cosh\alpha\tau\sin\beta - \sinh\alpha\tau\sin\beta\tau\cosh\alpha\cos\beta)\,e] \end{aligned} \tag{6.39}$$

where $\tau = \tau(\omega t)$, $e = e(\omega t)$; $\alpha = \zeta/\omega$ and $\beta = \sqrt{\omega_0^2 - \zeta^2}/\omega$.

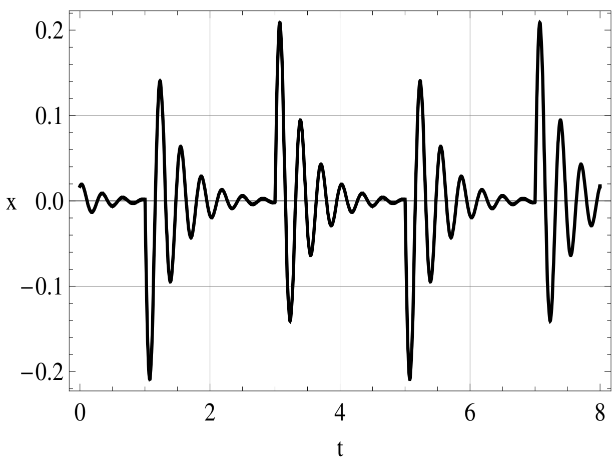

Fig. 6.3 Response of the damped harmonic oscillator under the periodic impulsive excitation for $p = 0.1$, $\zeta = 0.5$, $\omega_0 = 4$ and $\omega = 0.2$ (low-frequency pulses.)

Figs. 6.3 through 6.5 illustrate qualitatively different responses of the system when varying the input frequency. In different proportions, the responses combine properties of the harmonic damped motion and the non-smooth motion due to the impulsive loading. For instance, when $\omega >> \omega_0$ and $\omega >> \zeta$, the system is near the limit of a free particle under the periodic impulsive force. In this case, the boundary value problem is reduced to

$$X'' = 0, \qquad Y'' = 0; \qquad X' \,|_{\tau=\pm 1} = \frac{p}{\omega^2}, \qquad Y\,|_{\tau=\pm 1} = 0 \tag{6.40}$$

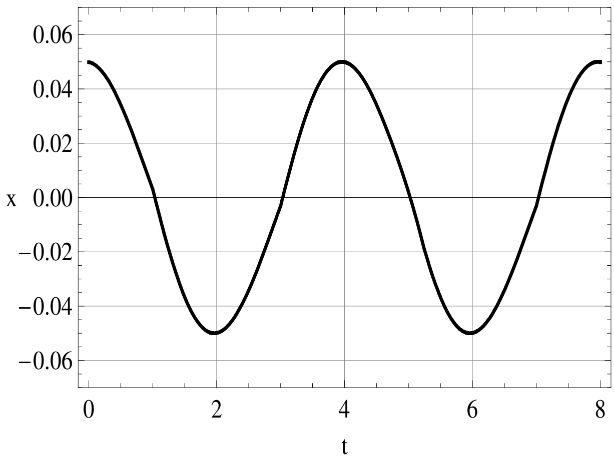

Fig. 6.4 System response on 'resonance' pulses $\omega = (2/\pi)\omega_0 = 2.5465$.

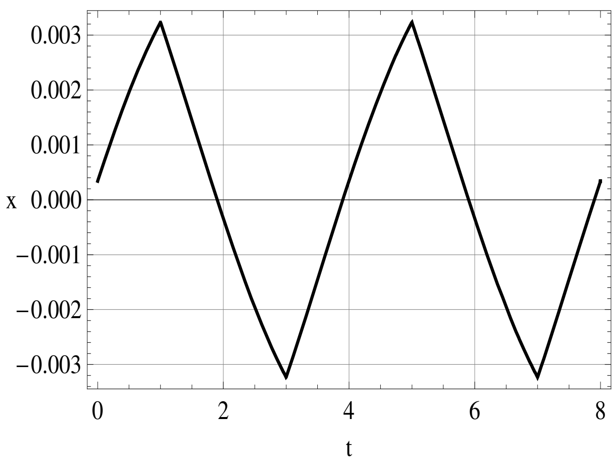

Fig. 6.5 Response on high-frequency pulses; $\omega = 6$.

This gives the triangular temporal shape of the motion, $x = p\tau(\omega t)/\omega^2$, which is approached by the time history record on Fig. 6.5.

Finally, let us consider N-degrees-of-freedom system

$$M\ddot{\mathbf{y}} + K\mathbf{y} = \mathbf{p}\frac{de(\omega t)}{d(\omega t)} \tag{6.41}$$

where $\mathbf{y}(t)$ is N-dimensional vector-function, $\mathbf{p}$ is a constant vector, M and K are constant $N \times N$ mass and stiffness matrixes respectively.

Let $\{\mathbf{e}_1, ..., \mathbf{e}_N\}$ and ω_1,...,ω_N be the normal mode basis vectors and the corresponding natural frequencies, respectively, such that

$$K\mathbf{e}_j = \omega_j^2 M\mathbf{e}_j, \qquad \mathbf{e}_k^T M\mathbf{e}_j = \delta_{kj}$$

for any $k = 1, ..., N$ and $j = 1, ..., N$.

Introducing the principal coordinates $x^j(t)$,

$$\mathbf{y} = \sum_{j=1}^{N} x^j(t)\,\mathbf{e}_j \tag{6.42}$$

gives a decoupled set of impulsively forced harmonic oscillators of the form (6.31),

$$\ddot{x}^j + \omega_j^2 x^j = p^j \frac{de(\omega t)}{d(\omega t)} \tag{6.43}$$

where $p^j = \mathbf{e}_j^T \mathbf{p}$.

Therefore, making use of solution (6.35) for each of the oscillators (6.43) and taking into account (6.42), gives

$$\mathbf{y} = \sum_{j=1}^{N} \frac{(\mathbf{e}_j^T \mathbf{p})\mathbf{e}_j}{\omega\omega_j} \frac{\sin[(\omega_j/\omega)\,\tau(\omega t)]}{\cos(\omega_j/\omega)} \tag{6.44}$$

The corresponding resonances are determined by the condition

$$\omega = \frac{2}{\pi}\frac{\omega_j}{k}$$

where $k = 1, 3, 5, ...$ and $j = 1, ..., N$.

6.3.3 Periodic Impulses with a Temporal 'Dipole' Shift

Let us consider the impulsive excitation with a dipole shift of pulse times. In this case, the right-hand side of equation (6.25) can be expressed by second derivative of the saw-tooth function with some incline described the parameter γ as shown in Fig. 6.6

$$\begin{aligned}\dot{v} + \lambda v &= p\frac{\partial^2 \tau(\omega t, \gamma)}{\partial(\omega t)^2} = p\frac{\partial e(\omega t, \gamma)}{\partial(\omega t)} \\ &= \frac{2p}{1-\gamma^2} \sum_{k=-\infty}^{\infty} [\delta(\omega t + 1 - \gamma - 4k) - \delta(\omega t - 1 + \gamma - 4k)]\end{aligned} \tag{6.45}$$

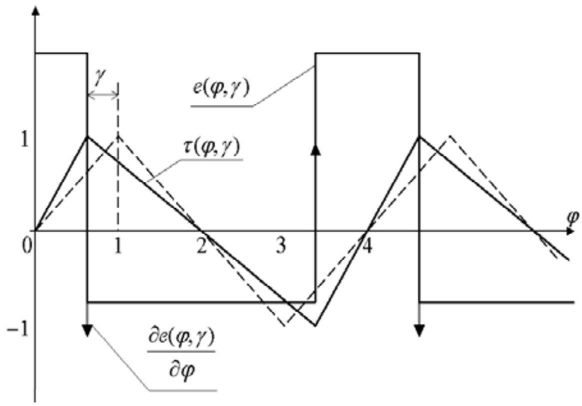

Fig. 6.6 Basic NSTT asymmetric wave functions.

Based on the NSTT identities introduced in Chapter 4, periodic solutions of equation (6.45) still can be represented in the form

$$v = X(\tau) + Y(\tau)\, e \tag{6.46}$$

where $\tau = \tau(\omega t, \gamma)$ and $e = e(\omega t, \gamma)$; see Fig. 6.6 for graphic illustrations.
Substituting (6.46) in equation (6.45), gives

$$\omega\alpha Y' + \lambda X + [\omega(X' + \beta Y') + \lambda Y]\, e + (\omega Y - p)\frac{\partial e(\omega t,\gamma)}{\partial(\omega t)} = 0 \tag{6.47}$$

where $\alpha = 1/\left(1-\gamma^2\right)$, $\beta = 2\gamma\alpha$, and the identity $e^2 = \alpha + \beta e$ has been taken into account.

Equation (6.47) is equivalent to the boundary-value problem

$$\begin{aligned} \omega(X' + \beta Y') &= -\lambda Y \\ \omega\alpha Y' &= -\lambda X \\ \omega Y|_{\tau=\pm1} &= p \end{aligned} \tag{6.48}$$

The corresponding solution is

$$\begin{aligned} Y &= \frac{p}{\omega}\left[\cosh\left(\gamma\frac{\lambda}{\omega}\right)\frac{\cosh\left(\frac{\lambda}{\omega}\tau\right)}{\cosh\frac{\lambda}{\omega}} + \sinh\left(\gamma\frac{\lambda}{\omega}\right)\frac{\sinh\left(\frac{\lambda}{\omega}\tau\right)}{\sinh\frac{\lambda}{\omega}}\right]\exp\left(\gamma\frac{\lambda}{\omega}\tau\right) \\ X &= -\frac{\omega\alpha}{\lambda}Y' \end{aligned} \tag{6.49}$$

where the X-component is defined by differentiation due to the second equation in (6.48).

6.4 Parametric Excitation

In this section, two different cases of parametric excitation are considered based on relatively simple linear models. Piecewise-constant and impulsive excitations are described by means of the functions $e(\omega t, \gamma)$ and $\partial e(\omega t, \gamma)/\partial(\omega t)$, respectively. There are at least two reasons for using NSTT as a preliminary analytical step. First, NSTT automatically gives conditions for matching solutions at discontinuity points. Second, due to the automatic matching through the NSTT functions, the corresponding solutions appear to be in the closed form that is important feature when further manipulations with the solutions are required by problem formulations.

6.4.1 Piecewise-Constant Excitation

Let us consider the linear oscillator under periodic piecewise-constant excitation

$$\ddot{x} + \omega_0^2[1 + \varepsilon e(\omega t, \gamma)]x = 0 \tag{6.50}$$

where ω_0, ω, γ and ε are constant parameters.

We will seek periodic solutions with the period of excitation $T = 4/\omega$ in the form

$$x = X(\tau) + Y(\tau)\, e \tag{6.51}$$

where $\tau = \tau(\omega t, \gamma)$ and $e = e(\omega t, \gamma)$.

As follows from the form of equation (6.50), the acceleration $\ddot{x}$ may have step-wise discontinuities due to the presence of the function $e(\omega t, \gamma)$, whereas the coordinate $x(t)$ and the velocity $\dot{x}(t)$ must be continuous. So neither velocity $\dot{x}(t)$ nor acceleration $\ddot{x}(t)$ can include Dirac δ-functions.

Taking first derivative of (6.51), gives

$$\dot{x}(t) = \left[\alpha Y' + (X' + \beta Y')e + \underline{Y\frac{\partial e(\omega t, \gamma)}{\partial(\omega t)}}\right]\omega \tag{6.52}$$

where the last term, that consists of the periodic sequence of δ-functions, must be excluded by imposing the boundary condition for Y-component

$$Y\,|_{\tau=\pm 1} = 0 \tag{6.53}$$

Under condition (6.53), the second derivative takes the form

$$\begin{aligned} \ddot{x}(t) = {} & \omega^2[\alpha(X'' + \beta Y'')] + \omega^2[\beta X'' + (\alpha + \beta^2)Y'']e \\ & \underline{+\omega^2(X' + \beta Y')\frac{\partial e(\omega t, \gamma)}{\partial(\omega t)}} \end{aligned} \tag{6.54}$$

In this case, the singular term, which is underlined in (6.54), is eliminated by condition

$$(X' + \beta Y')\mid_{\tau=\pm 1} = 0 \tag{6.55}$$

Substituting (6.51) and (6.54) in the differential equation of motion (6.50) and taking into account the algebraic properties, brings the left-hand side of the equation to the algebraic form $\{\cdots\} + \{\cdots\}e$. Then, setting separately each of the two algebraic components to zero, gives the set of differential equations for $X(\tau)$ and $Y(\tau)$ in the following matrix form

$$\begin{bmatrix} \alpha & \alpha\beta \\ \beta & \alpha+\beta^2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}'' + r^2 \begin{bmatrix} 1 & \alpha\varepsilon \\ \varepsilon & 1+\beta\varepsilon \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = 0 \tag{6.56}$$

where $r = \omega_0/\omega$.

Further, any particular solution of linear differential equations with constant coefficients (6.56) can be represented in the exponential form

$$\begin{bmatrix} X \\ Y \end{bmatrix} = B \begin{bmatrix} 1 \\ \mu \end{bmatrix} \exp(\lambda\tau) \tag{6.57}$$

where B, μ and λ are constant parameters.

Substituting (6.57) in (6.56), leads to a characteristic equation which two pairs of roots determined by the relationships

$$\begin{aligned} \lambda^2 &= \left[-(1-\gamma)\varepsilon - (1-\gamma)^2\right] r^2 \equiv \pm k^2 \\ \lambda^2 &= \left[(1+\gamma)\varepsilon - (1+\gamma)^2\right] r^2 \equiv \pm l^2 \end{aligned} \tag{6.58}$$

where signs of the notations $\pm k^2$ and $\pm l^2$ depend on the parameters ε and γ.

Let us consider the case of negative signs, when the following condition holds

$$-(1-\gamma) < \varepsilon < (1+\gamma) \tag{6.59}$$

Due to condition (6.59), the stiffness coefficient in equation (6.50) is always positive, whereas (6.58) gives $\lambda = \pm ki$ and $\lambda = \pm li$. As a result, the general solution of equations (6.56) takes the form

$$\begin{aligned} X &= B_1 \sin k\tau + B_2 \cos k\tau + B_3 \sin l\tau + B_4 \cos l\tau \\ Y &= \mu_1 (B_1 \sin k\tau + B_2 \cos k\tau) + \mu_2 (B_3 \sin l\tau + B_4 \cos l\tau) \end{aligned} \tag{6.60}$$

where B_1,...,B_4 are arbitrary constants, and

$$\mu_1 = -\frac{1}{\alpha}\frac{\alpha k^2 - r^2}{\beta k^2 - \varepsilon r^2} \quad \text{and} \quad \mu_2 = -\frac{1}{\alpha}\frac{\alpha l^2 - r^2}{\beta l^2 - \varepsilon r^2}$$

Substituting (6.60) in boundary conditions (6.53) and (6.55), gives the homogeneous set of four linear algebraic equations with respect to the arbitrary constants. Setting the corresponding determinant to zero, gives condition for non-zero solutions in the form

$$[\mu_1 (1+\beta\mu_2)\, l\cos k \sin l - \mu_2 (1+\beta\mu_1)\, k \cos l \sin k]$$
$$\times [\mu_1 (1+\beta\mu_2)\, l\cos l \sin k - \mu_2 (1+\beta\mu_1)\, k \cos k \sin l] = 0 \qquad (6.61)$$

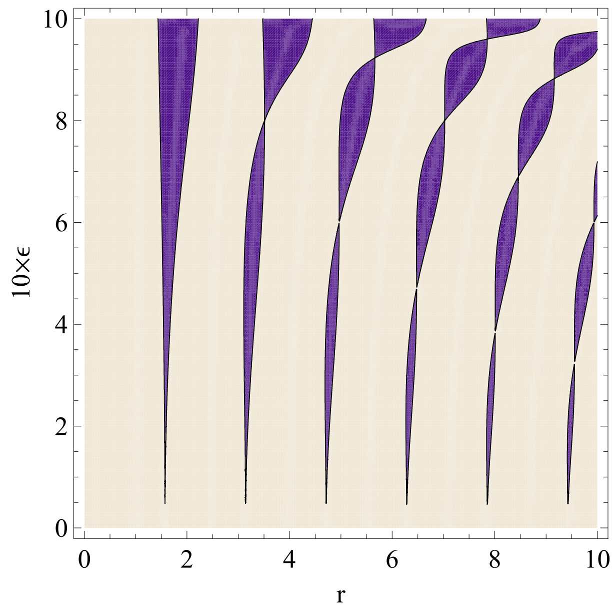

Fig. 6.7 Instability zones for piecewise constant parametric excitation when $\gamma = 0.7$

One the parameter plane, $\varepsilon - r$, equation (6.61) describes the family of curves separating stability and instability zones as shown in Fig. 6.7, where the instability zones are shadowed.

6.4.2 Parametric Impulsive Excitation

Let us consider the case of parametric impulsive excitation whose temporal shape is given by first derivative of the basic function, $e(\omega t, \gamma)$,

$$\ddot{x} + \omega_0^2 \left[1 + \varepsilon \frac{\partial e(\omega t, \gamma)}{\partial (\omega t)}\right] x = 0 \qquad (6.62)$$

This case was considered in [138] based on the saw-tooth transformation of time. In particular, it was shown that the periodic solutions of the period $T = 4/\omega$ exists under the condition

$$p^2 = \frac{2r^2\left(1-\gamma^2\right)^2 \sin^2 2r}{\cos 4r - \cos 4\gamma r} \tag{6.63}$$

where $r = \omega_0/\omega$ and $p = \varepsilon r^2$.

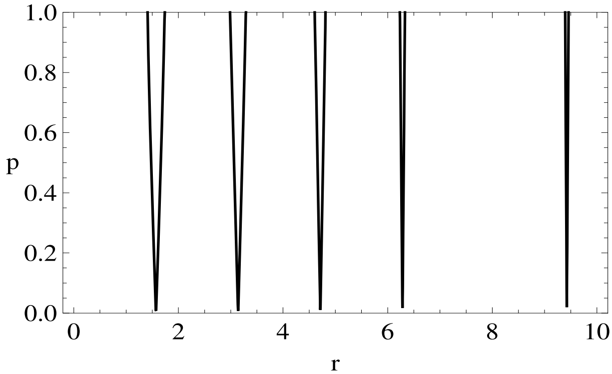

Fig. 6.8 'Collapse' of the instability zones at $\gamma = 1/5$: each fifth zone is missing; here and below, only the upper half-plane is shown due to the symmetry.

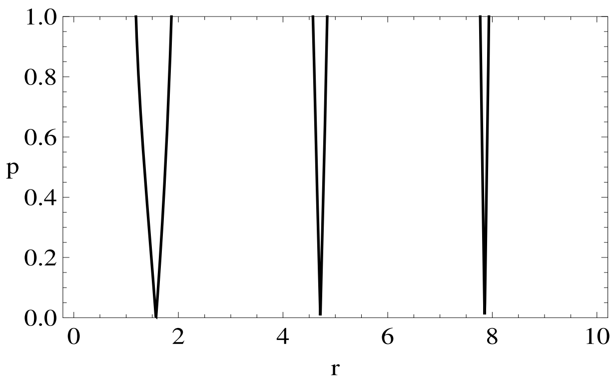

Fig. 6.9 $\gamma = 1/2$: each second zone is missing.

The dependence of p on r for fixed γ has the branched zone-like structure which is typical for different cases of parametrically excited oscillators.

Interestingly enough, different subsequences of zones may disappear as the parameter γ varies. For instance, if $\gamma = 1/5$ then each fifth zone is missing and, if $\gamma = 1/2$ then each second zone is missing; see Figs. 6.8 and 6.9, respectively. Such an effect was discussed in [138].

6.4.3 General Case of Periodic Parametric Excitation

Below, the problem formulation only is discussed for the case of periodic parametric loading with both regular and singular components. It is assumed that there are two discontinuities and singularities on each period located at the same points. The differential equation of motion is represented in the vector form

$$\ddot{x} + \left[Q\left(\tau\right) + P\left(\tau\right)e + p\frac{\partial e}{\partial \varphi} \right] x = 0 \tag{6.64}$$

where $x(t) \in R^n$ is the coordinates vector-column,$\tau = \tau\left(\varphi, \gamma\right)$, $e = e\left(\varphi, \gamma\right)$, $\varphi = \omega t$ is the phase variable, p is a constant $n \times n$ matrix, and $Q\left(\tau\left(\varphi, \gamma\right)\right)$ and $P\left(\tau\left(\varphi, \gamma\right)\right)$ are periodic matrixes of the period $T = 4$ with respect to the phase φ.

In equation (6.64), the first two terms of the coefficient can represent any periodic function $q\left(\varphi\right)$ with step-wise discontinuities on $\Lambda = \{t : \tau\left(\varphi, \gamma\right) = \pm 1\}$. In case the original function $q\left(\varphi\right)$ is continuous, one has $P = 0$ on Λ.

Let us represent periodic solutions of the period $T = 4$ in the form (6.51).

Substituting (6.51) in equations (6.64), taking into account the equality $e^2 = \alpha + \beta e$, the necessary condition of continuity of the vector function $x\left(t\right)$, (6.53), and using (6.52) and (6.54) gives equations

$$\begin{aligned} \omega^2\left(\alpha X'' + \alpha\beta Y''\right) + QX + \alpha PY = 0 \\ \omega^2\left[\left(\alpha + \beta^2\right)Y'' + \beta X''\right] + PX + QY + \beta PY = 0 \end{aligned} \tag{6.65}$$

and the boundary condition

$$\left[\omega^2\left(X' + \beta Y'\right) + pX\right]|_{\tau=\pm 1} = 0 \tag{6.66}$$

where, in the case of fixed sign of impulses, the matrix p should be provided with the factor $\mathrm{sgn}(\tau)$.

Together with (6.53), relations (6.65) and (6.66) represent a boundary-value problem for determining the vector functions X and Y and the corresponding conditions for existence of periodic solutions.

Note that substitution (6.51) in equation (6.64) generates the specific term $e\partial e/\partial\varphi$. Let us show that, within the theory of distributions, this terms can be interpreted as follows

$$e\frac{\partial e}{\partial \varphi} = \frac{1}{2}\beta\frac{\partial e}{\partial \varphi} \tag{6.67}$$

First, note that, at this point, the relationship (6.67) is a result of formal differentiation of both sides of the relation $e^2 = \alpha + \beta e$ with respect to the phase φ. To justify (6.67), let us assume that $\omega = 1$ so that $\varphi \equiv t$ and consider expression (6.53) locally, near the point $t = 1 - \gamma$, which is a typical point of the entire set of discontinuity points $\Lambda = \{t : \tau\left(t\right) = \pm 1\}$.

Generally speaking, the 'product' $f(t)\delta(t)$ requires the function $f(t)$ to be at least continuous at $t = 0$. However, it is possible to provide the left-hand side of (6.67) with a certain meaning due to the fact that both terms of the product are generated by the same sequence of smooth functions.

In order to illustrate the above remark and prove equality (6.67), let us consider a family of smooth functions $\{\delta_\varepsilon(t)\}$ such that

$$\int_{-\varepsilon}^{\varepsilon} \delta_\varepsilon(t)\, dt = 1 \qquad (6.68)$$

for all positive ε, and $\delta_\varepsilon(t) = 0$ outside the interval $-\varepsilon < t < \varepsilon$.

Therefore, in terms of weak limits, $\delta_\varepsilon(t) \to \delta(t)$ as $\varepsilon \to 0$.

Now, sequences of smooth functions approximating e and $\partial e/\partial t$ in the neighborhood of point $t = 1 - \gamma$ can be chosen as, respectively,

$$e_\varepsilon = \frac{1}{1-\gamma} - \frac{\beta}{\gamma}\theta_\varepsilon(t - 1 + \gamma) \quad \text{and} \quad \frac{\partial e_\varepsilon}{\partial t} = -\frac{\beta}{\gamma}\delta_\varepsilon(t - 1 + \gamma) \qquad (6.69)$$

where $\theta_\varepsilon(t) = \int_{-\infty}^{t} \delta_\varepsilon(\xi)\, d\xi$ and $-1 + \gamma < t < 3 + \gamma$.

Based on the above definitions for e_ε and $\partial e_\varepsilon/\partial t$, one has $e_\varepsilon \to e$ and $\partial e_\varepsilon/\partial t \to \partial e/\partial t$ as $\varepsilon \to 0$ in the interval $-1 + \gamma < t < 3 + \gamma$.

Substituting (6.69) in equality (6.67) instead of e and $\partial e/\partial \varphi$, reduces the problem to the proof of identity

$$\theta_\varepsilon \delta_\varepsilon = \frac{1}{2}\delta_\varepsilon \qquad (6.70)$$

as $\varepsilon \to 0$.

For simplicity reason, let us move the origin to the point $t = 1 - \gamma$ and show that the left-hand side of (6.70) gives $\delta(t)/2$ as $\varepsilon \to 0$ in the sense of weak limit.

First, the area bounded by $\theta_\varepsilon \delta_\varepsilon$ is

$$\int_{-\varepsilon}^{\varepsilon} \theta_\varepsilon \delta_\varepsilon dt = \int_{-\varepsilon}^{\varepsilon} \theta_\varepsilon \frac{d\theta_\varepsilon}{dt} dt = \frac{1}{2}\theta_\varepsilon^2 \mid_{-\varepsilon}^{\varepsilon} = \frac{1}{2}$$

Then, let $\phi(t)$ belongs to the class of continuous testing functions, which is usually considered in the theory of distributions. By definition, in some ε-neighborhood of the point $t = 0$, one has $\mid \phi(t) - \phi(0) \mid < 2\eta$, where η is as small as needed whenever ε is sufficiently small. Therefore,

$$\mid \int_{-\varepsilon}^{\varepsilon} \theta_\varepsilon(t)\, \delta_\varepsilon(t)\, \phi(t)\, dt - \frac{1}{2}\phi(0) \mid \leq \int_{-\varepsilon}^{\varepsilon} \theta_\varepsilon(t)\, \delta_\varepsilon(t) \mid \phi(t) - \phi(0) \mid dt \leq \eta$$

In other words,

$$\int_{-\varepsilon}^{\varepsilon} \theta_{\varepsilon}(t)\, \delta_{\varepsilon}(t)\, \phi(t)\, dt \to \frac{1}{2}\phi(0)$$

as $\varepsilon \to 0$.

This completes the proof.

6.5 Input-Output Systems

The input-output form of dynamical systems may be convenient for different reasons, for instance, when dealing with control problems. In many linear cases, input-output systems are represented in the form of a single high order equation

$$a_n \frac{d^n y}{dt^n} + ... + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m u}{dt^m} + ... + b_1 \frac{du}{dt} + b_0 u \tag{6.71}$$

where $u = u(t)$ and $y = y(t)$ are input and output, respectively, and a_n, ... , a_1, a_0, b_m, ... ,b_1, b_0 are constant coefficients.

For illustration purposes, a two-degrees-of-freedom model as shown in Fig. 6.10 is considered, although the general case (6.71) can be handled in the same way.

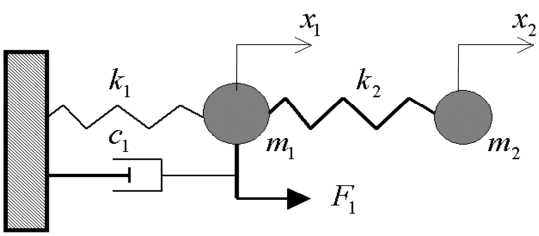

Fig. 6.10 Two mass-spring model.

Eliminating $x_2(t)$ from the system, gives a single higher-order equation with respect to the another coordinate, $x_1(t)$, in the form

$$m_1 \frac{d^4 x_1}{dt^4} + c_1 \frac{d^3 x_1}{dt^3} + (k_1 + k_2 + \frac{m_1}{m_2} k_2) \frac{d^2 x_1}{dt^2} + \frac{c_1}{m_2} k_2 \frac{dx_1}{dt} + \frac{k_1 k_2}{m_2} x_1$$
$$= \frac{d^2 F_1}{dt^2} + \frac{k_2}{m_2} F_1 \tag{6.72}$$

System (6.72) is a particular case of (6.71), where $n = 4$ and $m = 2$.

Let us consider the step-wise discontinuous periodic function $F_1(t) = u(t) = e(\omega t)$ and represent equation (6.72) in the form

$$a_4 \frac{d^4 y}{dt^4} + ... + a_1 \frac{dy}{dt} + a_0 y = b_2 \omega^2 e'' + b_1 \omega e' + b_0 e \tag{6.73}$$

where $' \equiv d/d(\omega t)$, and all the coefficients and variables are identified by comparing (6.72) to (6.73).

The right-hand side of equation (6.73) contains discontinuous and singular functions, therefore equation (6.73) must be treated in terms of distributions. Nevertheless, let us show that, on the manifold of periodic solutions, equation (6.73) is equivalent to some classic boundary-value problem.

Let us represent the output in the form

$$y(t) = X(\tau) + Y(\tau)e \tag{6.74}$$

where $\tau = \tau(\omega t)$ and $e = e(\omega t)$.

When differentiating expression (6.74) step-by-step one should eliminate the singular term e' in the first two derivatives by sequentially setting boundary conditions as follows

$$\begin{aligned} \frac{dy}{dt} &= (Y' + X'e)\omega, \qquad Y|_{\tau=\pm 1} = 0 \\ \frac{d^2 y}{dt^2} &= (X'' + Y''e)\omega^2, \qquad X'|_{\tau=\pm 1} = 0 \end{aligned} \tag{6.75}$$

However, it is dictated by the form of the input in (6.73), that the singular terms e' and e'' must be preserved on the next two steps given by

$$\begin{aligned} \frac{d^3 y}{dt^3} &= (Y''' + X'''e + Y''e')\omega^3 \\ \frac{d^4 y}{dt^4} &= (X^{(4)} + Y^{(4)}e + X'''e' + Y''e'')\omega^4 \end{aligned} \tag{6.76}$$

The fourth-order derivative in (6.76) takes into account the equality $ee' = 0$, which easily follows from (6.53) in the symmetric case $\beta = 0$.

Substituting (6.75) and (6.76) in (6.73), and considering $\{1, e, e', e''\}$ as a linearly independent basis, gives equations

$$\begin{aligned} a_4 \omega^4 X^{IV} + a_3 \omega^3 Y''' + a_2 \omega^2 X'' + a_1 \omega Y' + a_0 X = 0 \\ a_4 \omega^4 Y^{IV} + a_3 \omega^3 X''' + a_2 \omega^2 Y'' + a_1 \omega X' + a_0 Y = b_0 \end{aligned} \tag{6.77}$$

under the boundary conditions at $\tau = \pm 1$:

$$\begin{aligned} Y = 0, \qquad X' = 0 \\ \omega^2 Y'' = \frac{b_2}{a_4}, \qquad \omega^3 X''' = \frac{1}{a_4}\left(b_1 - \frac{a_3}{a_4} b_2\right) \end{aligned} \tag{6.78}$$

In contrast to equation (6.73), the boundary value problem (6.77) and (6.78) does not include discontinuous terms any more.

Although the number of equations in (6.77) is doubled as compared to (6.73), such a complication is rather formal due to the symmetry of the equations. Indeed, introducing the new variables, $U = X+Y$ and $V = X-Y$, decouples system (6.77) in such a way that the corresponding roots of the characteristic equations differ just by signs. (Besides, this fact reveals the possibility of using the idempotent basis for decoupling the resultant set of equations as discussed in Chapter 4 and will be discussed later in this chapter.) In addition, the type of the symmetry suggests that $X(\tau)$ and $Y(\tau)$ are odd and even functions, respectively. This enables one of reducing the general form of solution to a family of solutions with four arbitrary constants

$$X = \sum_{j=1}^{2} \left[A_j \cosh\left(\frac{\alpha_j}{\omega}\tau\right) \sin\left(\frac{\beta_j}{\omega}\tau\right) + B_j \sinh\left(\frac{\alpha_j}{\omega}\tau\right) \cos\left(\frac{\beta_j}{\omega}\tau\right) \right] \quad (6.79)$$

$$Y = \sum_{j=1}^{2} \left[A_j \sinh\left(\frac{\alpha_j}{\omega}\tau\right) \sin\left(\frac{\beta_j}{\omega}\tau\right) + B_j \cosh\left(\frac{\alpha_j}{\omega}\tau\right) \cos\left(\frac{\beta_j}{\omega}\tau\right) \right] + \frac{b_0}{a_0}$$

where $\alpha_j \pm \beta_j i$ are complex conjugate roots of the characteristic equation

$$a_4 p^4 + ... + a_1 p + a_0 = 0 \quad (6.80)$$

The assumption that both of the roots are complex reflects the physical meaning of the example, however other cases would lead to even less complicated expressions.

Finally, substituting (6.79) in (6.78) gives a linear algebraic set of four independent equations with respect to four constants: A_1, A_2, B_1 and B_2. Although the corresponding analytical solution is easy to obtain by using the standard *Mathematica*® commands, the result is somewhat complicated for reproduction. Practically, it may be reasonable to determine the constants by setting the system parameters to their numerical values moreover that only numerical solution are often possible for characteristic equations.

6.6 Piecewise-Linear Oscillators with Asymmetric Characteristics

Piecewise-linear oscillators are often considered as finite degrees-of-freedom models of cracked elastic structures [32],[2],[192], but may occur also due to specific design solutions. In many cases, the corresponded periodic solutions can be combined of different pieces of linear solutions valid for two different subspaces of the configuration space [33], [75], [192]. In this section, it will be shown that the nonsmooth transformation of time results in a closed form analytical solution matching both pieces of the solution automatically by means of elementary functions.

6.6.1 Amplitude-Phase Equations

Let us consider a piece-wise linear oscillator of the form

$$m\ddot{q} + k[1 - \varepsilon H(q)]q = 0 \tag{6.81}$$

where $H(q)$ is Heaviside unit-step function, m and k are mass and stiffness parameters, respectively, and $|\varepsilon| \ll 1$.

Therefore, $k_- = k$ and $k_+ = k(1-\varepsilon)$ are elastic stiffness of the oscillator for $q < 0$ and $q > 0$, respectively.

The exact general solution of oscillator (6.81) can be obtained by satisfying the continuity conditions for q and $\dot{q}$ at the matching point $q = 0$, where the characteristic has a break. Such approaches are often facing quite challenging algebraic problems, however, as the number of degrees of freedom increases or external forces are involved. This is mainly due to the fact that times of crossing the point $q = 0$ are a priory unknown.

In this section, it will be shown that, applying a combination of asymptotic expansions with respect to ε and nonsmooth temporal transformations, gives a unit-form solution for oscillator (6.81) with a possibility of generalization on the normal mode motions of multiple degrees-of-freedom systems. In particular, the nonsmooth temporal transformation:

1) provides an automatic matching the motions from different subspaces of constant stiffness, and

2) justifies quasi-linear asymptotic solutions for the specific nonsmooth case of piece-wise linear characteristics.

Let us clarify the above two remarks. Introducing the notation $\omega^2 = k/m$, brings equation (6.81) to the standard form of a weakly non-linear oscillator

$$\ddot{q} + \omega^2 q = \varepsilon\omega^2 H(q)q \tag{6.82}$$

The non-linear perturbation on the right-hand side of oscillator (6.82) is a continuous but non-smooth function of the coordinate q. Since the major algorithms of quasi-linear theory assume smoothness of non-linear perturbations, then such algorithms are not applicable in this case unless appropriate modifications and extensions have been made. Even though deriving first-order asymptotic solutions usually require no differentiation of characteristics, dealing with two pieces of the solution may complicate any further stages.

Let us show that combining quasi-linear methods of asymptotic integration, such as Krylov-Bogolyubov averaging, with nonsmooth temporal transformations results in a closed form analytical solution for piece-wise linear oscillator (6.81). Note that oscillator (6.81) plays an illustrative role for the approach developed below. Then a more complicated case will be considered.

At this stage, let us introduce the amplitude-phase coordinates $\{A(t), \varphi(t)\}$ on the phase plane of oscillator (6.81) through relationships

$$\begin{aligned} q &= A\cos\varphi \\ \dot{q} &= -\omega A\sin\varphi \end{aligned} \tag{6.83}$$

The following compatibility condition is imposed on transformation (6.83)

$$\dot{A}\cos\varphi - A\sin\varphi\dot{\varphi} = -\omega A\sin\varphi \tag{6.84}$$

Substituting (6.83) in (6.82) and taking into account (6.84), gives

$$\begin{aligned} \dot{A} &= -\frac{1}{2}\varepsilon\omega AH(A\cos\varphi)\sin 2\varphi \\ \dot{\varphi} &= \omega - \varepsilon\omega H(A\cos\varphi)\cos^2\varphi \end{aligned} \tag{6.85}$$

The right-hand sides of equations (6.85) are 2π-periodic with respect to the phase variable, φ. Therefore, nonsmooth transformation of the phase variable applies through the couple of functions

$$\tau = \tau(2\varphi/\pi) \qquad \text{and} \qquad e = e(2\varphi/\pi) \tag{6.86}$$

Assuming that $A \geq 0$ and taking into account the obvious identities,

$$\begin{aligned} \sin\varphi &= \sin(\pi\tau/2) \\ \cos\varphi &= \cos(\pi\tau/2)e \\ H(A\cos\varphi) &= (1+e)/2 \\ e^2 &= 1 \end{aligned} \tag{6.87}$$

brings (6.85) to the form

$$\dot{A} = -\frac{1}{4}\varepsilon\omega(1+e)A\sin\pi\tau \tag{6.88}$$

$$\dot{\varphi} = \omega - \frac{1}{2}\varepsilon\omega(1+e)\cos^2\frac{\pi\tau}{2} \tag{6.89}$$

Note that the right-hand sides of (6.88) and (6.89) are nonsmooth but continuous with respect to the phase φ since the step-wise discontinuities of the rectangular cosine $e(2\varphi/\pi)$ are suppressed by the factors $\sin\pi\tau$ and $\cos^2(\pi\tau/2)$, respectively.

6.6.2 Amplitude Solution

Let us show that equation (6.88) has an exact 2π-periodic solution with respect to the phase variable, φ.

According to the idea of NSTT, any periodic solution can be represented in the form

$$A = X(\tau) + Y(\tau)e \tag{6.90}$$

where τ and e are defined by (6.86).

Substituting (6.90) in (6.88) and taking into account (6.89), gives boundary-value problem

$$\begin{aligned} &(X - Y)' = 0 \\ &\frac{(X + Y)'}{X + Y} = -\frac{\varepsilon\pi}{4}\frac{\sin \pi\tau}{1 - \varepsilon \cos^2 \frac{\pi\tau}{2}} \end{aligned} \tag{6.91}$$

$$Y|_{\tau=\pm 1} = 0 \tag{6.92}$$

where $' \equiv d/d\tau$.

Solution of the boundary value problem, (6.91) and (6.92), is obtained by elementary integration. Then representation (6.90) gives

$$\begin{aligned} A(\varphi) &= \alpha[1 + \zeta(\tau)] - \alpha[1 - \zeta(\tau)]e \\ \zeta(\tau) &= (1 - \varepsilon \cos^2 \frac{\pi\tau}{2})^{-1/2} \end{aligned} \tag{6.93}$$

where $\tau = \tau(2\varphi/\pi)$, $e = e(2\varphi/\pi)$, and α is an arbitrary positive constant.

Note that solution (6.93) exactly captures the amplitude in both subspaces $q < 0$ and $q > 0$. However, the temporal mode shape and the period essentially depend on the phase variable φ described by equation (6.89).

Generally speaking, the phase equation (6.89) admits exact integration, but the result would appear to have implicit form. Alternatively, it is shown below that solution for the phase variable can be approximated by asymptotic series in the explicit form

$$\begin{aligned} \varphi = \phi &- \frac{1}{8}\varepsilon[\pi\tau + (1 + e)\sin \pi\tau] \\ &- \frac{1}{128}\varepsilon^2\{4(2 - \cos \pi\tau)(\pi\tau + \sin \pi\tau) \\ &- [4\pi\tau(1 + \cos \pi\tau) - 8 \sin \pi\tau + \sin 2\pi\tau]e\} + O(\varepsilon^3) \end{aligned} \tag{6.94}$$

where $\tau = \tau(2\phi/\pi)$, $e = e(2\phi/\pi)$, and

$$\phi = \omega[1 - \frac{1}{4}\varepsilon - \frac{3}{32}\varepsilon^2 + O(\varepsilon^3)]t \tag{6.95}$$

Note that the functions τ and e in (6.93) and (6.94) depend on the different arguments.

6.6.3 Phase Solution

In this subsection, a second-order asymptotic procedure for phase equations with non-smooth periodic perturbations is introduced. If applied to equation (6.89), the developed algorithm gives solution (6.94).

Let us consider some phase equation of the general form

$$\dot{\varphi} = \omega[1 + \varepsilon f(\varphi)] \tag{6.96}$$

where $f(\varphi)$ is a 2π-periodic, nonsmooth or even step-wise discontinuous function, and ε is a small parameter, $|\varepsilon| \ll 1$.

Using the basic NSTT identity for $f(\varphi)$, brings equation (6.96) to the form

$$\dot{\varphi} = \omega + \varepsilon\omega\{G[\tau(2\varphi/\pi)] + M[\tau(2\varphi/\pi)]e\} \tag{6.97}$$

where the functions $G(\tau)$ and $M(\tau)$ are expressed through $f(\varphi)$.

Note that the class of smoothness of the periodic perturbation in equation (6.97) depends on the behavior of functions $G(\tau)$ and $M(\tau)$ and their derivatives at the boundaries $\tau = \pm 1$. If, for instance, $M(\pm 1) \neq 0$ then the perturbation is step-wise discontinuous whenever $\tau(2\varphi/\pi) = \pm 1$.

Let us introduce the asymptotic procedure for equation (6.97). Note that, in case $\varepsilon = 0$, the right-hand side of equation (6.97) is constant. So, following the idea of asymptotic integration, let us find phase transformation

$$\varphi = \phi + \varepsilon F_1(\phi) + \varepsilon^2 F_2(\phi) + ... \tag{6.98}$$

where functions $F_i(\phi)$ are such that the new phase variable also has a constant temporal rate even though $\varepsilon \neq 0$.

In other words, transformation (6.98) should bring equation (6.97) to the form

$$\dot{\phi} = \omega(1 + \varepsilon\gamma_1 + \varepsilon^2\gamma_2 + ...) \tag{6.99}$$

where γ_i are constant coefficients to be determined together with $F_i(\phi)$ during the asymptotic procedure.

Note that the procedure, which is described below, has several specific features due to the presence of nonsmooth periodic functions. In particular, high-order approximations require a non-conventional interpretation for power series expansions; see the next subsection for the related remarks. Other modifications occur already in the leading order approximation.

Substituting (6.98) into equation (6.97), then enforcing equation (6.99) and collecting the terms of order ε, gives

$$F_1'(\phi) = G(\tau) + eM(\tau) - \gamma_1 \tag{6.100}$$

where the triangular and rectangular waves depend now on the new phase variable ϕ as $\tau = \tau(2\phi/\pi)$ and $e = e(2\phi/\pi)$, respectively.

According to the conventional averaging procedure, the constant γ_1 is selected to achieve zero mean on the right-hand side of equation (6.100) and thus provide periodicity of solution, $F_1(\phi)$. In the algorithm below, the periodicity is due to the form of representation for periodic solutions, whereas the operator of averaging occurs automatically from the corresponding conditions of smoothness that is boundary conditions for the solution components.

So we seek solution of equation (6.100) in the form

$$F_1(\phi) = U_1(\tau) + eV_1(\tau) \tag{6.101}$$

Substituting (6.101) in (6.100) and following the NSTT procedure, gives the boundary-value problem

$$\begin{aligned} U_1'(\tau) &= \frac{\pi}{2} M(\tau) \\ V_1'(\tau) &= \frac{\pi}{2} [G(\tau) - \gamma_1] \\ V_1(\pm 1) &= 0 \end{aligned} \tag{6.102}$$

Note that there are two conditions on the function $V_1(\tau)$ described by the first-order differential equation in (6.102). However, there is a choice for γ_1, which is to satisfy one of the two conditions. As a result, solution of boundary-value problem (6.102) is obtained by integration in the form

$$\begin{aligned} U_1(\tau) &= \frac{\pi}{2} \int_0^\tau M(z)dz \\ V_1(\tau) &= \frac{\pi}{2} \int_{-1}^\tau [G(z) - \gamma_1]dz \\ \gamma_1 &= \frac{1}{2} \int_{-1}^1 G(\tau)d\tau \end{aligned} \tag{6.103}$$

Further, collecting the terms of order ε^2, gives

$$F_2'(\phi) = G_2(\tau) + eM_2(\tau) + P_2(\tau)e' - \gamma_2 \tag{6.104}$$

where

$$\begin{aligned} M_2(\tau) &= \frac{2}{\pi} [U_1(\tau)G'(\tau) + V_1(\tau)M'(\tau)] - M(\tau)\gamma_1 \\ G_2(\tau) &= \frac{2}{\pi} U_1(\tau)M'(\tau) - G(\tau)\gamma_1 + \gamma_1^2 \\ P_2(\tau) &= \frac{2}{\pi} U_1(\tau)M(\tau) \\ e' &\equiv de(2\phi/\pi)/d(2\phi/\pi) \end{aligned} \tag{6.105}$$

In contrast to first-order equation (6.100), equation (6.104) includes the singular term $P_2(\tau)e'$ produced by the power series expansion of the perturbation in equation (6.97). If the perturbation is smooth then $P_2(\pm 1) = 0$ and such singular term disappear; see the example below for illustration. Nevertheless, the second-order approximation remains valid even in discontinuous case, when $P_2(\pm 1) \neq 0$.

So let us represent solution of equation (6.104) in the form

$$F_2(\phi) = U_2(\tau) + eV_2(\tau) \tag{6.106}$$

Then, substituting (6.106) in (6.104), gives boundary-value problem

$$\begin{aligned} U_2'(\tau) &= \frac{\pi}{2} M_2(\tau) \\ V_2'(\tau) &= \frac{\pi}{2} [G_2(\tau) - \gamma_2] \\ V_2(\pm 1) &= \frac{\pi}{2} P_2(\pm 1) \end{aligned} \tag{6.107}$$

In contrast to (6.102), boundary-value problem (6.107) has, generally speaking, non-homogeneous boundary conditions for V_2. These conditions compensate the singular term e' from differential equation (6.104). As a result equations (6.107) are free of any singularities and admit solution analogously to first-order equations (6.102),

$$\begin{aligned} U_2(\tau) &= \frac{\pi}{2} \int_0^\tau M_2(z) dz \\ V_2(\tau) &= \frac{\pi}{2} \int_{-1}^\tau [G_2(z) - \gamma_2] dz + \frac{\pi}{2} P_2(-1) \\ \gamma_2 &= \frac{1}{2} \int_{-1}^1 G_2(\tau) d\tau + \frac{1}{2} [P_2(-1) - P_2(1)] \end{aligned} \tag{6.108}$$

Now, we return to the illustrating model. In particular case (6.89), one has

$$G(\tau) \equiv M(\tau) \equiv -\frac{1}{2} \cos^2 \frac{\pi\tau}{2} \tag{6.109}$$

and

$$\begin{aligned} G(\pm 1) &= M(\pm 1) = 0 \\ G'(\pm 1) &= M'(\pm 1) = 0 \\ G''(\pm 1) &= M''(\pm 1) = -\pi^2/4 \end{aligned} \tag{6.110}$$

where $' \equiv d/d\tau$.

First two lines of conditions (6.110) provide continuity for the right hand side of (6.97) and its first derivative at those φ where $\tau(2\varphi/\pi) = \pm 1$. As follows from (6.105), for this class of smoothness one has $P_2(\pm 1) = 0$ and thus no singular terms occur in the first two steps of asymptotic procedure. Finally, taking into account (6.109) and (6.110) and conducting integration in (6.103) and (6.108), brings solution (6.98) to the form (6.94) and (6.95).

6.6.4 Remarks on Generalized Taylor Expansions

Nonsmoothness of the triangular sine is similar to that function $|t|$ has at zero. So let us consider its formal power series

$$|t+\varepsilon| = |t| + |t|'\varepsilon + \frac{1}{2!}|t|''\varepsilon^2 + ... \tag{6.111}$$

where $\varepsilon > 0$ and $-\infty < t < \infty$, and prime indicates Schwartz derivative.

It is clear that equality (6.111) has no regular point-wise meaning. For instance, equality (6.111) is obviously not true on the interval $-\varepsilon < t < 0$. In addition, the right-hand side of (6.111) is uncertain at $t = 0$, whereas the left-hand side gives ε. Nevertheless, let us show that equality (6.111) admits a generalized interpretation and holds in terms of distributions. Let $\psi(t)$ be a *test function* in terms of the distribution theory, more precisely, $\psi(t)$ is infinitely differentiable with compact support that is identically zero outside of some bounded interval. Integrating by parts and then shifting the variable of integration, gives

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left(|t| + |t|'\varepsilon + \frac{1}{2!}|t|''\varepsilon^2 + ...\right)\psi(t)dt \\
&= \int_{-\infty}^{\infty}|t|\left[\psi(t) - \psi'(t)\varepsilon + \frac{1}{2!}\psi''(t)\varepsilon^2 - ...\right]dt \\
&= \int_{-\infty}^{\infty}|t|\psi(t-\varepsilon)dt = \int_{-\infty}^{\infty}|t+\varepsilon|\psi(t)dt
\end{aligned} \tag{6.112}$$

Therefore, equality (6.111) holds in the integral sense of distributions.

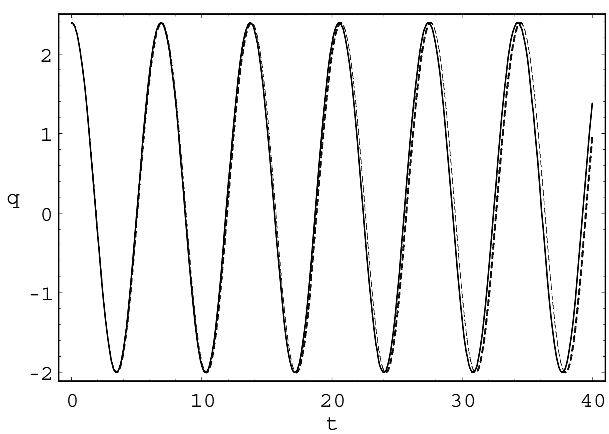

Fig. 6.11 Second-order asymptotic and numerical solutions shown by solid and dashed lines, respectively

Fig. 6.11 compares analytical solution (6.83), (6.93) and (6.94) shown by the solid line and numerical solution shown by the dashed line. As expected, the amplitude show the perfect match, whereas some phase shift develops after several cycles.

6.7 Multiple Degrees-of-Freedom Case

Let us consider a multiple degrees-of-freedom piecewise-linear system of the form

$$M\ddot{x} + Kx = \varepsilon H(Sx)Bx \tag{6.113}$$

where $x(t) \in R^n$ is a vector-function of the system coordinates, M is a mass matrix, H denotes the Heaviside unit-step function, S is a normal vector to the plane splitting the configuration space into two parts with different elastic properties, so that the stiffness matrix is K when $Sx < 0$ and $K - \varepsilon B$ when $Sx > 0$. It is assumed that the stiffness jump is small, $|\varepsilon| \ll 1$.

The number of possible iterations of the classic perturbation tools usually depends on a class of smoothness of the perturbation. The perturbation term on the right-hand side of (6.113) is continuous but nonsmooth. Therefore, only first-order asymptotic solution can be obtained within the classic theory of differential equations. Moreover, the piecewise character of the perturbation complicates the form of the solution due to the necessity of matching the different pieces of the solution.

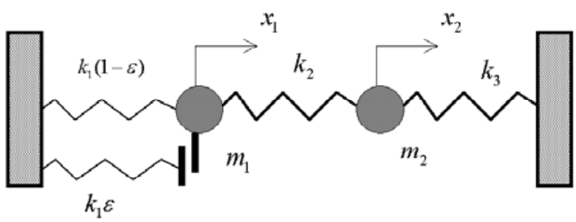

Fig. 6.12 Two degrees-of-freedom piecewise-linear system as a model of a rod with a small crack.

However, we show that the idea of nonsmooth time transformation gives a unit-form solution by automatically matching the pieces of solution in two different configuration subspaces with different stiffness properties.

Let seek a 2π-periodic normal mode solution of (6.113) with respect to the phase φ in the form

$$x(\varphi) = A_j \cos\varphi + \varepsilon x^{(1)}(\varphi) + O(\varepsilon^2)$$
$$\varphi = \omega_j \sqrt{1 + \varepsilon\gamma^{(1)} + O(\varepsilon^2)}t \tag{6.114}$$

where ω_j and A_j are arbitrary eigen frequency and eigen vector (normal mode) of the linearized system:

$$(-\omega_j^2 M + K)A_j = 0; \qquad j = 1,...,n \tag{6.115}$$

Substituting (6.114) in (6.113), taking into account (6.87) and (6.115) in the first order of ε, gives

$$\omega_j^2 M \frac{d^2 x^{(1)}}{d\varphi^2} + Kx^{(1)} = \left[\frac{1}{2}BA_j + (\frac{1}{2}BA_j + \gamma^{(1)}KA_j)e\right]\cos\frac{\pi\tau}{2} \tag{6.116}$$

where $\tau = \tau(2\varphi/\pi)$, $e = e(2\varphi/\pi)$, and the relationship $(1 + \varepsilon\gamma^{(1)})^{-1} = 1 - \varepsilon\gamma^{(1)} + O(\varepsilon^2)$ has been used.

Since the function $x^{(1)}(\varphi)$ is sought to be 2π-periodic with respect to φ, we represent it in the form

$$x^{(1)} = X(\tau) + Y(\tau)e \tag{6.117}$$

This gives the boundary-value problem

$$\left(\frac{2\omega_j}{\pi}\right)^2 MX'' + KX = \frac{1}{2}BA_j\cos\frac{\pi\tau}{2}, \qquad X'|_{\tau=\pm1} = 0 \tag{6.118}$$

$$\left(\frac{2\omega_j}{\pi}\right)^2 MY'' + KY = \left(\frac{1}{2}BA_j + \gamma^{(1)}KA_j\right)\cos\frac{\pi\tau}{2} \tag{6.119}$$

$$Y|_{\tau=\pm1} = 0$$

Representing the corresponding solution in terms of the normal mode coordinates

$$X = \sum_{i=1}^{n} A_i X_i(\tau), \qquad Y = \sum_{i=1}^{n} A_i Y_i(\tau) \tag{6.120}$$

and taking into account M-orthogonality of the set of eigen vectors, gives

$$\left(\frac{2\omega_j}{\pi}\right)^2 X_i'' + \omega_i^2 X_i = \beta_{ij}\cos\frac{\pi\tau}{2}, \qquad X_i'|_{\tau=\pm1} = 0 \tag{6.121}$$

$$\left(\frac{2\omega_j}{\pi}\right)^2 Y_i'' + \omega_i^2 Y_i = (\beta_{ij} + \gamma^{(1)}\varkappa_{ij})\cos\frac{\pi\tau}{2}, \quad Y_i|_{\tau=\pm1} = 0 \tag{6.122}$$

where

$$\beta_{ij} = \frac{1}{2}\frac{A_i BA_j}{A_i MA_i}, \qquad \varkappa_{ij} = \frac{A_i KA_j}{A_i MA_i} \tag{6.123}$$

are dimensionless coefficients.

Note that despite of the similar representation for solution (6.114), there is a noticeable difference between the classic Poincare-Lindshtedt method and current procedure due to (6.117). Namely, according to the

Poincare-Lindshtedt method, the frequency correction term $\gamma^{(1)}$ is to kill the so called secular terms in the asymptotic expansions. In our case, the secular terms appear to be periodic due to the 'built in' periodicity of the new temporal argument. However, periodicity of solutions is provided by the existence of solutions of the boundary-value problems, such as (6.121) and (6.122). Due to the linearity, the existence of solutions allows the direct verification. So if $i \neq j$ then both problems (6.121) and (6.122) are solved in the standard way with no presence of $\gamma^{(1)}$ because $\varkappa_{ij} = 0$. The corresponding solution is given by

$$X_i = \frac{\beta_{ij}}{\omega_i^2 - \omega_j^2}\left(\cos\frac{\pi\tau}{2} - \frac{\omega_j}{\omega_i}\cos\frac{\pi\omega_i\tau}{2\omega_j}\csc\frac{\pi\omega_i}{2\omega_j}\right) \tag{6.124}$$

$$Y_i = \frac{\beta_{ij}}{\omega_i^2 - \omega_j^2}\cos\frac{\pi\tau}{2} \tag{6.125}$$

In the particular case $i = j$, problem (6.121) still has a solution, but problem (6.122) generally speaking does not. Fortunately, in this case, we have $\varkappa_{jj} \neq 0$ and thus the problem is set to have the trivial solution by condition

$$\gamma^{(1)} = -\frac{\beta_{jj}}{\varkappa_{jj}} \tag{6.126}$$

Therefore,

$$X_j = \frac{\pi\beta_{jj}}{4\omega_j^2}\left(\tau\sin\frac{\pi\tau}{2} + \frac{2}{\pi}\cos\frac{\pi\tau}{2}\right) \tag{6.127}$$

$$Y_j = 0 \tag{6.128}$$

So expressions (6.117), (6.120), and (6.124) through (6.128) completely determine the first order approximation $x^{(1)}(\varphi)$.

Let us consider the example of mass-spring model

$$\begin{aligned} m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 &= \varepsilon k_1 H(x_1)x_1 \\ m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 &= 0 \end{aligned} \tag{6.129}$$

Equations (6.111) can be represented in the form (6.113), where

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \qquad K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}, \qquad B = \begin{bmatrix} k_1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad S = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

In this case, the first-order asymptotic solution for the in-phase ($j = 1$) and out-of-phase ($j = 2$) takes the form, respectively,

$$
\begin{aligned}
x_1 &= e\cos\frac{\pi\tau}{2} + \frac{\varepsilon\pi}{16}\left(\frac{2}{\pi}\cos\frac{\pi\tau}{2} + \tau\sin\frac{\pi\tau}{2}\right) + O(\varepsilon^2) \\
x_2 &= e\cos\frac{\pi\tau}{2} - \frac{\varepsilon k_1}{8k_2}\left[e\cos\frac{\pi\tau}{2} + \cos\frac{\pi\tau}{2}\right. \\
&\quad \left. - \left(1 + 2\frac{k_2}{k1}\right)^{-1/2}\cos\left(\sqrt{1+2\frac{k_2}{k_1}}\frac{\pi\tau}{2}\right)/\sin\left(\sqrt{1+2\frac{k_2}{k_1}}\frac{\pi}{2}\right)\right] + O(\varepsilon^2)
\end{aligned}
\tag{6.130}
$$

$$\varphi = \sqrt{\frac{k_1}{m}}\sqrt{1 - \frac{\varepsilon}{4} + O(\varepsilon^2)t}$$

and

$$
\begin{aligned}
x_1 &= -e\cos\frac{\pi\tau}{2} + \frac{\varepsilon k_1}{8k_2}\left[e\cos\frac{\pi\tau}{2} + \cos\frac{\pi\tau}{2} - \left(1 + 2\frac{k_2}{k1}\right)\right. \\
&\quad \left. \times\cos\left(\frac{\pi\tau}{2}/\sqrt{1+2\frac{k_2}{k_1}}\right)/\sin\left(\frac{\pi}{2}/\sqrt{1+2\frac{k_2}{k_1}}\right)\right] + O(\varepsilon^2) \\
x_2 &= e\cos\frac{\pi\tau}{2} + \frac{\varepsilon k_1\pi}{16(k_1+2k_2)}\left(\frac{2}{\pi}\cos\frac{\pi\tau}{2} + \tau\sin\frac{\pi\tau}{2}\right) + O(\varepsilon^2)
\end{aligned}
\tag{6.131}
$$

$$\varphi = \sqrt{\frac{k_1+2k_2}{m}}\sqrt{1 - \frac{\varepsilon k_1}{4(k_1+2k_2)} + O(\varepsilon^2)t}$$

where it is assumed that $m_1 = m_2 = m$. Solutions (6.130) and (6.100) show that a bi-linearity may have quite different effect on different modes. In particular, solution (6.130) reveals the possibility of internal resonances, when

$$\sin\left(\frac{\pi\omega_2}{2\omega_1}\right) = 0, \qquad \frac{\omega_2}{\omega_1} = \sqrt{1+2\frac{k_2}{k_1}} \tag{6.132}$$

If, for instance, the system is close to the frequency ratio $\omega_2/\omega_1 = 2$ then the in phase mode may be affected significantly by a crack even under very small magnitudes of the parameter ε. In contrary, solution (6.100) has the denominator $\sin[(\pi/2)\omega_1/\omega_2]$, which is never close to zero because $0 < \omega_1/\omega_2 < 1$. Therefore, in current asymptotic approximation, the influence of crack on the out-of phase mode is always of order ε provided that $k_2/k_1 = O(1)$.

The influence of the bilinear stiffness on inphase mode trajectories in the closed to internal resonance case is seen from Fig. 6.13, where both analytical and numerical solutions are shown for comparison reasons. The frequency ratio $\omega_2/\omega_1 = 2.0025$ is achieved by conditioning the spring stiffness parameters as follows $k_2 = (3/2)k_1 + 0.005$.

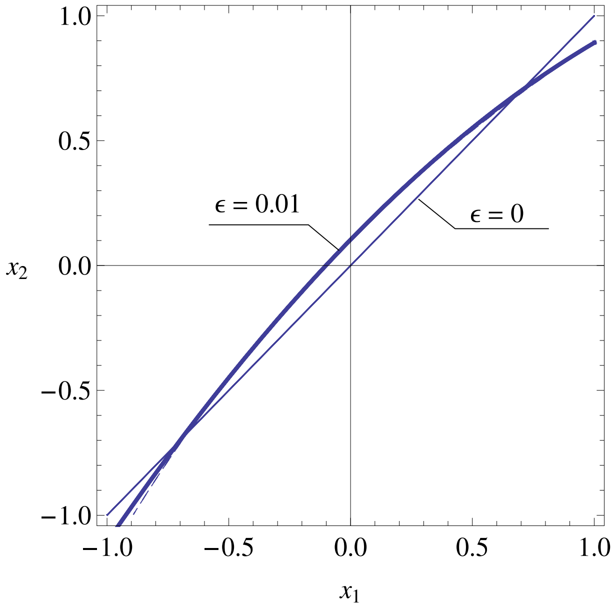

Fig. 6.13 The influence of a small "crack" $\varepsilon = 0.01$ on the in-phase mode trajectory near the frequency ratio $\omega_2/\omega_1 = 2$; the dashed line shows the numerical solution, and the thin solid line corresponds to the perfect (linear) case $\varepsilon = 0$.

6.8 The Amplitude-Phase Problem in the Idempotent Basis

Recall that the idempotent basis is given by $e_+ = (1+e)/2$ and $e_- = (1-e)/2$ so that $e_+^2 = e_+$, $e_-^2 = e_-$ and $e_+e_- = 0$. Equations (6.88) and (6.89) therefore take the form

$$\dot{A} = -\frac{1}{2}\varepsilon\omega e_+ A \sin \pi\tau \tag{6.133}$$

$$\dot{\varphi} = \omega - \varepsilon\omega e_+ \cos^2 \frac{\pi\tau}{2} \tag{6.134}$$

Let us represent the amplitude as a function of φ in the form

$$A(\varphi) = X_+(\tau)e_+ + X_-(\tau)e_- \tag{6.135}$$

where $e_+ = e_+(2\varphi/\pi)$, $e_- = e_-(2\varphi/\pi)$ and $\tau = \tau(2\varphi/\pi)$.

Substituting (6.135) in (6.133) and taking into account (6.134), gives

$$\frac{2}{\pi}(X'_+e_+ - X'_-e_-)(\omega - \varepsilon\omega e_+ \cos^2 \frac{\pi\tau}{2}) = -\frac{1}{2}\varepsilon\omega e_+(X_+e_+ + X_-e_-) \sin \pi\tau$$

or

$$
\begin{aligned}
(1-\varepsilon\cos^2\frac{\pi\tau}{2})X_+' &= -\frac{\pi}{4}\varepsilon X_+ \sin\pi\tau \\
X_-' &= 0
\end{aligned}
\tag{6.136}
$$

under the boundary condition

$$(X_+ - X_-)|_{\tau=\pm1} = 0 \tag{6.137}$$

The boundary-value problem (6.136) and (6.137) admits exact solution so that (6.135) gives finally

$$A(\varphi) = \alpha[(1-\varepsilon\cos^2\frac{\pi\tau}{2})^{-1/2}e_+ + e_-] \tag{6.138}$$

where α is an arbitrary positive constant.