Chapter 2
Smooth Oscillating Processes

Abstract. This chapter gives a brief overview of selected analytical methods for smooth oscillating processes. Most of such methods are indeed quasi-linear. In other words, the corresponding technical implementations employ harmonic oscillators as generating models. The description focuses only on the ideas and technical details that are further combined with non-smooth methods. As most effective way, procedures of asymptotic integration of the differential equations of motion bring original systems to such simple form that further solution becomes straightforward. In particular, the method of asymptotic integration of the differential equations of motion based on the Hausdorff equation for operators Lie is reproduced.

2.1 Linear and Weakly Non-linear Approaches

By both practical and theoretical reasons, the quantitative methods of dynamics were developed first for smooth processes. As a rule, smooth oscillations can be directly observed under no special conditions. For instance, projection of any fixed point of a body rotating with constant angular speed, makes a perfect impression about harmonic oscillations. Interestingly, in 1693, Leibniz derived the differential equation for sine geometrically by considering a circle. Much later, original analytical ideas of nonlinear vibrations emerged from the celestial mechanics considering perturbations of circular orbits of rigid-body motions rather than any mass-spring oscillators. Robert Hooke (1635-1703) was probably first who suggested the basic elastic mass-spring model, whereas Galileo and Huygens were investigating the pendulum. Later, d'Alambert, Daniel Bernoulli and Euler considered a one-dimensional continual model of a string. It was found that the vibrating string represents the infinity of harmonic oscillators corresponding to different mode shapes of the string. It is well known that a serious discussion arised about whether or not

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the sum of smooth functions, such as sines, can represent a non-smooth shape of the string. These discussions were finalized by the Fourier theorem.

Let us reproduce the result for a periodic function of time $f(t)$ of the period T in the complex form

$$f(t) = \sum_{k=-\infty}^{\infty} c_k \exp(i\omega_k t) \tag{2.1}$$

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\exp(-i\omega_k t)dt, \quad \omega_k = \frac{2\pi}{T}k$$

This relation generates a one-to-one mapping between function and its Fourier coefficients

$$f(t) \longleftrightarrow \{...c_{-2}, c_{-1}, c_1, c_2, ...\} \tag{2.2}$$

Note that mathematical expressions (2.1) do not necessarily imply that the periodic process $f(t)$ must be produced by linear systems even though the right-hand side of (2.1) combines free vibrations of linear oscillators, in other words - 'rigid-body rotations' as discussed in Chapter 1. Therefore, the Fourier analysis and associated analytical tools provides a '*linear language*' for *nonlinear systems* regardless specifics of algorithm implementations. Indeed, most quantitative methods for weakly nonlinear periodic motions, actually estimate Fourier coefficients of the corresponding solutions. As a result, on one hand, such tools possess a high level of generality. On the other hand, even 'elementary' strongly nonlinear phenomena (as qualified in Chapter 1) may become quite difficult to describe in terms of the 'linear language.' Nevertheless, the quantitative theory of nonlinear vibration has been advanced by new asymptotic techniques developed originally for solving nonlinear differential equations. Most traditional methods are essentially based on perturbation or averaging methods [50]. Similar results can be obtained within the theory of Poincare' normal forms [118], which retains resonance terms, whereas all non-resonance terms are eliminated by means of a coordinate transformation. Such a normal form is qualified as the simplest possible form of the equations of motion.

2.2 A Brief Overview of Smooth Methods

2.2.1 Periodic Motions of Quasi Linear Systems

Consider a weakly unharmonic oscillator of the form

$$\ddot{x} + \omega_0^2 x = \varepsilon f(x, \dot{x}) \tag{2.3}$$

where ε is a small parameter, $0 < \varepsilon \ll 1$, and $f(x, \dot{x})$ is smooth enough function.

Periodic solutions of equation (2.3) can be found by splitting the nonlinear system into the sequence of linear oscillators by means of the power series

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + ... \tag{2.4}$$

The perturbation on the right-hand side of equation (2.3) changes the principal frequency of the oscillator so that

$$\omega^2 = \omega_0^2(1 + \varepsilon\gamma_1 + \varepsilon^2\gamma_2 + ...) \tag{2.5}$$

The new frequency is introduced explicitly into the differential equation of motion by re-scaling the temporal argument

$$\varphi = \omega t \tag{2.6}$$

As a result, series (2.4) appears to be composed of trigonometric functions of multiple phases φ, 2φ, 3φ,... .

A similar idea was implemented by Lyapunov for systems of first-order equations, for instance

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + f_1(x_1, x_2) \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + f_2(x_1, x_2) \end{aligned} \tag{2.7}$$

where f_1 and f_2 are nonlinear functions.

It is assumed that system (2.7) admits first analytical integral and the corresponding linearized system has only periodic solutions. Then periodic solutions of (2.7) admit power series expansions with respect to the amplitude parameter.

There exist at least two extensions of Lyapunov's theory, such as local and global approaches to nonlinear normal modes, see for instance [100], [190], [119].

2.2.2 The Idea of Averaging

Let us illustrate different implementations of the idea of averaging by reproducing some technical details. The following description focuses on such tools that remains applicable to non-hamiltonian systems.

Following Van-der-Pol's approach, let us transform system (2.3) by changing the variables $\{x,\ \dot{x}\} \to \{a, \varphi\}$:

$$x = a\cos\varphi, \quad \dot{x} = -a\omega_0 \sin\varphi \tag{2.8}$$

As a result, one obtains

$$\dot{a} = -\frac{\varepsilon}{\omega_0} f(a\ \cos\varphi, -a\omega_0 \sin\varphi) \sin\varphi$$
$$\dot{\varphi} = \omega_0 - \frac{\varepsilon}{\omega_0 a} f(a\ \cos\varphi, -a\omega_0 \sin\varphi) \cos\varphi \tag{2.9}$$

Despite of a formal complexity, system (2.9) has essential advantage due to different time scales of the new variables. This enables one of eliminating the fast phase φ on the right-hand side of the system by applying the averaging

$$< \cdots >_\varphi \equiv \frac{1}{2\pi} \int_0^{2\pi} \cdots d\varphi$$

as follows

$$\dot{a} = -\frac{\varepsilon}{\omega_0} < f(a\ \cos\varphi, -a\omega_0 \sin\varphi) \sin\varphi >_\varphi$$
$$\dot{\varphi} = \omega_0 - \frac{\varepsilon}{\omega_0 a} < f(a\ \cos\varphi, -a\omega_0 \sin\varphi) \cos\varphi >_\varphi \tag{2.10}$$

Solutions of system (2.10) are considered then as approximate solutions of the original system (2.9).

This method was essentially generalized in thirties [28] by incorporating the Lindstedt-Poincare and Van-der-Pol's ideas as follows.

Let us consider the general system with one fast phase

$$\dot{x} = \varepsilon X(x, y)$$
$$\dot{y} = \omega(x) + \varepsilon Y(x, y) \tag{2.11}$$

where y and x are scalar and vector variables respectively.

In contrast to (2.9), the frequency in (2.11) depends on the slow vector-function x. Sometimes, such kind of systems is called essentially nonlinear since the condition $\varepsilon = 0$ does not make the frequency state independent. However, if $\varepsilon = 0$ then system (2.11) has no fast phase on the right-hand side. The problem is to find close to identical transformation

$$x = q + \varepsilon u(q, \psi) + O(\varepsilon^2)$$
$$y = \psi + \varepsilon v(q, \psi) + O(\varepsilon^2) \tag{2.12}$$

which eliminates the fast phase entirely from the system by bringing equation (2.11) to the form

$$\dot{q} = \varepsilon A(q) + O(\varepsilon^2)$$
$$\dot{\psi} = \omega_0(q) + \varepsilon\omega_1(q) + O(\varepsilon^2) \tag{2.13}$$

This problem is solved by substituting expansions (2.12) into equations (2.11) and enforcing then equations (2.13).

2.2.3 Averaging Algorithm for Essentially Nonlinear Systems

In order to illustrate the corresponding procedure, let us specify system (2.11) as follows

$$\begin{aligned} \dot{l} &= \mu R(\mu l, s, \varphi, \theta) \\ \dot{s} &= \mu^2 S(\mu l, s, \varphi, \theta) \\ \dot{\theta} &= \mu Q(l, s) + \mu^2 \Theta(\mu l, s, \varphi, \theta) \\ \dot{\varphi} &= \Omega(\mu l, s) + \mu^2 G(\mu l, s, \varphi, \theta) \end{aligned} \tag{2.14}$$

where all the coordinates and functions are scalars, and $\mu \ll 1$.

Such kind of equations may occur when considering 'essentially nonlinear' systems under different resonance conditions. This is the reason for using another notation for small parameter. In resonance cases, original small parameters are often modified as $\mu = \sqrt{\varepsilon}$ to capture specifics of the dynamics near resonance surfaces [9], [205].

The basic approximation is obtained from system (2.14) by applying the averaging procedure directly to the terms of order μ on the right hand side. This gives,

$$\begin{aligned} \dot{l} &= \mu < R(0, s, \varphi, \theta) >_{\varphi} \\ \dot{\theta} &= \mu Q(l, s) \end{aligned} \tag{2.15}$$

where s should be considered as a constant.

System (2.15) is easily integrated and the result is known to give an error of order μ on time intervals of order $1/\mu$. In many cases however, first approximation gives incomplete characterizations of systems.

In order to illustrate the basic stages of second approximation, consider the first equation only. It is sufficient for illustration of the procedure, which is sequentially applied in the same way to other equations.

Let us represent the first equation of system (2.14) in the form

$$\begin{aligned} \dot{l} = \mu < R(0, s, \varphi, \theta) >_{\varphi} + \mu[R(0, s, \varphi, \theta) - < R(0, s, \varphi, \theta) >_{\varphi}] \\ + \mu^2 l R'_{\mu l}(0, s, \varphi, \theta) + O(\mu^3) \end{aligned} \tag{2.16}$$

Following the idea of averaging, one eliminates the second term on the right-hand side by means of the coordinate transformation

$$l = q + \mu f(q, s, \varphi, \theta) \tag{2.17}$$

Then, substituting (2.17) into (2.16), gives

$$\begin{aligned}&\dot{q}+\mu\left(\frac{\partial f}{\partial q}\dot{q}+\frac{\partial f}{\partial s}\dot{s}+\frac{\partial f}{\partial\varphi}\dot{\varphi}+\frac{\partial f}{\partial\theta}\dot{\theta}\right)\\&=\mu<R(0,s,\varphi,\theta)>_{\varphi}+\mu[R(0,s,\varphi,\theta)-<R(0,s,\varphi,\theta)>_{\varphi}]\\&\quad+\mu^2qR'_{\mu l}(0,s,\varphi,\theta)+O(\mu^3)\end{aligned}\tag{2.18}$$

Now the fast phase φ is eliminated from the equation in first order of μ by taking into account (2.14) and imposing condition

$$\frac{\partial f}{\partial\varphi}\Omega(0,s)=R(0,s,\varphi,\theta)-<R(0,s,\varphi,\theta)>_{\varphi}\tag{2.19}$$

Further, $f(q,s,\varphi,\theta)$ is independent of q because the terms of order μ on the right-hand side of equation (2.18) are independent of q. As a result, equation (2.18) takes the form

$$\begin{aligned}\dot{q}=\mu<R(0,s,\varphi,\theta)>_{\varphi}+\mu^2qR'_{\mu l}(0,s,\varphi,\theta)\\-\mu^2\left(\frac{\partial f}{\partial\varphi}q\Omega'_{\mu l}(0,s)+\frac{\partial f}{\partial\theta}Q(q,s)\right)+O(\mu^3)\end{aligned}\tag{2.20}$$

Since the fast phase φ is eliminated from the terms of order μ then the averaging procedure is applied to the terms of order μ^2 analogously to the first stage of the method. As follows from (2.19), $<\partial f/\partial\varphi>_{\varphi}=<\partial f/\partial\theta>_{\varphi}=0$, therefore after the averaging, equation (2.20) takes the form

$$\dot{q}=\mu<R(0,s,\varphi,\theta)>_{\varphi}+\mu^2<qR'_{\mu l}(0,s,\varphi,\theta)>_{\varphi}+O(\mu^3)$$

or

$$\dot{q}=\mu<R(\mu q,s,\varphi,\theta)>_{\varphi}+O(\mu^3)\tag{2.21}$$

The second approximation therefore is obtained by applying the operator of averaging to original equation (2.14). Note however that the meaning of the coordinate q is now different. Namely, the original coordinate l is expressed through the new coordinate q by relationship (2.17), which due to (2.19) takes the form

$$l=q+\frac{\mu}{\Omega(\mu q,s)}\int_0^{\varphi}(R(\mu q,s,\varphi,\theta)-<R(\mu q,s,\varphi,\theta)>_{\varphi})d\varphi+O(\mu^2)\tag{2.22}$$

In this expression, the variable μq was put back into the expression R instead of zero. Although such a manipulation has no effect on the order of approximation, the new form is in a better match with the form of equation (2.21).

Expressions (2.21) and (2.22) summarize the averaging procedure in second order of μ.

Note that the case of multiple fast phases turns out to be more complicated in many respects due to the well known problem of small denominators.

2.2.4 Averaging in Complex Variables

In the physical literature, vibration problems are usually considered in terms of complex variables [89]. The idea of using the complex variables may be suggested by the standard manipulations of the variation of constants for oscillator (2.3) as follows.

If $\varepsilon = 0$ then general solution of equation (2.3) is represented in the complex form

$$x = \frac{1}{2}[A\exp(i\omega_0 t) + \bar{A}\exp(-i\omega_0 t)] \tag{2.23}$$

where A and $\bar{A}$ are arbitrary complex conjugate constants.

The velocity is

$$\dot{x} = \frac{i\omega_0}{2}[A\exp(i\omega_0 t) - \bar{A}\exp(-i\omega_0 t)] \tag{2.24}$$

If $\varepsilon \neq 0$ then the constants are assumed to be time dependent whereas expressions (2.23) and (2.24) are considered as a change of the state variables

$$\{x, \dot{x}\} \to \{A, \bar{A}\} \tag{2.25}$$

under the compatibility condition

$$\frac{dA}{dt}\exp(i\omega_0 t) + \frac{d\bar{A}}{dt}\exp(-i\omega_0 t) = 0 \tag{2.26}$$

By solving equations (2.23) and (2.24) with respect to A one obtains

$$A = \frac{1}{i\omega_0}\exp(-i\omega_0 t)(\dot{x} + i\omega_0 x) \tag{2.27}$$

Similar kind of complex amplitudes is used in both physics [89] and nonlinear mechanics [102].

Now equation (2.3) gives

$$\frac{dA}{dt} = \frac{\varepsilon}{i\omega_0}\exp(-i\omega_0 t)f \tag{2.28}$$

where $f = f(x, \dot{x})$ is expressed trough (2.23) and (2.24).

Equation (2.28) is still exactly equivalent to (2.3). If the parameter ε is small then the amplitude A is slow, and one can apply the averaging

$$\frac{dA}{dt} = \frac{\varepsilon}{2\pi i} \int_0^{2\pi/\omega_0} \exp(-i\omega_0 t) f dt \tag{2.29}$$

On theoretical point of view, complex amplitudes may bring some convenience compared to the traditional Van-der-Pol variables. Firstly, until the certain stage of manipulations, it is usually possible to keep only one equation since another one is its complex conjugate. Secondly, such a symmetry of the equations helps sometimes to reveal interesting features of the dynamics.

Note that the above manipulations remain valid in degenerated cases of multiple degrees of freedom systems. For instance, equation (2.3) can be interpreted as a vector equation with the scalar factor ω_0^2 .

2.2.5 Lie Group Approaches

The one-parameter Lie[1] group approaches are motivated by the idea of matching the tool and the object of study as explained in works [201] and [204]. Briefly, it is suggested to seek transformation (2.12) among solutions of dynamical systems rather than the class of the arbitrary nonlinear transformations.

Original materials and overviews of the mathematical structure of Lie groups, Lie algebras and Lie transforms with applications to nonlinear differential equations can be found in [38], [61], [22], [95], [35].

An essential ingredient of this version is the Hausdorff formula, which relates the Lie group operators of the original and new systems, and the operator of coordinate transformation. According to [202] and [204], most of the averaging techniques just reproduce this formula, each time implicitly, during the transformation process. But, there is no need of doing this, since it is reasonable to start the transformation using Hausdorff's relationship. The corresponding algorithms therefore enable one of optimizing the number of manipulations for high-order approximations of asymptotic integration.

The theory of Lie groups deals with a set of transformations. In other words, some dynamical system $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \varepsilon)$ is transformed into its simplest form $\dot{\mathbf{z}} = \mathbf{g}(\mathbf{z}, \varepsilon)$ by means of a coordinate transformation $\mathbf{y} \to \mathbf{z}$ produced by solution $\mathbf{z} = \mathbf{z}(\mathbf{y}, \varepsilon)$ of the third dynamical system

$$\frac{d\mathbf{z}}{d\varepsilon} = \mathbf{T}(\mathbf{z}, \varepsilon), \qquad \mathbf{z}\,|_{\varepsilon=0} = \mathbf{y}$$

where the choice for vector-function $\mathbf{T}(\mathbf{z}, \varepsilon)$ depends upon desired properties of the transformed system.

[1] Marius Sophus Lie (1842-1899), Norwegian mathematician; different mathematical objects are named after him, for instance, groups, operators, algebras, and series.

As mentioned, one of the advantages of the group formulation is that it specifies a general class of near identical transformations. Specifically, one should select the expression $\mathbf{z} = \mathbf{z}(\mathbf{y}, \varepsilon)$ among solutions of a dynamical system, but not among all classes of the near identical transformations. Another basic advantage is that all manipulations of the scheme can be done in linear terms of the monomial Lie group operators. Moreover, the result of transformation in general terms of operators is well-known and is given by the Hausdorff formula.

The description below presents all the stages starting with the traditional Newtonian form of the differential equations of motion as implemented in [147]. The original system will be reduced to its normal form by Poincare.

In terms of the principal coordinates q_k, a nonlinear dynamical system of n-degrees of freedom may be described by a set of $n+1$ autonomous differential equations written in the standard form

$$\ddot{q}_k + \omega_k^2 q_k = \varepsilon F_k(q_1, ..., q_{n+1}, \dot{q}_1, ..., \dot{q}_{n+1}); \qquad k = 1, ..., n+1 \tag{2.30}$$

where an overdot denotes differentiation with respect to time t, ε is a small parameter, and an external excitation has been replaced by the coordinate q_{n+1}. The functions F_k include all nonlinear terms and possibly parametric excitation terms, and ω_k are the principal mode frequencies. It is assumed that the functions F_k admit Taylor expansions near zero.

The Poincare normal form theory deals with sets of first-order differential equations written in terms of normal form coordinates. In this case it is convenient to transform the $n+1$ second-order differential equations (2.30) into $n+1$ first-order differential equations plus their conjugate set. This can be done by introducing the complex coordinates

$$y_k = \dot{q}_k + i\omega_k q_k \tag{2.31}$$

$$q_k = \frac{1}{2i\omega_k}(y_k - \bar{y}_k), \qquad \dot{q}_k = \frac{1}{2}(y_k + \bar{y}_k) \tag{2.32}$$

Introducing the transformation (2.31) into the equations of motion (2.30), gives

$$\begin{aligned} \ddot{q}_k + \omega_k^2 q_k &= \frac{d}{dt}(\dot{q}_k + i\omega_k q_k) - i\omega_k(\dot{q}_k + i\omega_k q_k) = \\ &= \frac{dy_k}{dt} - i\omega_k y_k = \varepsilon F_k(y_1, ..., y_{n+1}; \bar{y}_1, ..., \bar{y}_{n+1}) \end{aligned}$$

or

$$\dot{y}_k = i\omega_k y_k + \varepsilon F_k(y_1, ..., y_{n+1}; \bar{y}_1, ..., \bar{y}_{n+1}) \tag{2.33}$$

and the corresponding complex conjugate (cc) set of equations, where the functions $F_k(y_1, ..., y_{n+1}; \bar{y}_1, ..., \bar{y}_{n+1})$ are obtained by substituting (2.32) in the right hand side of equation (2.30). These terms can be represented in the polynomial form

$$F_k = \sum_{|\sigma|=2,3,\ldots} F_k^{\sigma} y_1^{m_1} \cdots y_{n+1}^{m_{n+1}} \bar{y}_1^{l_1} \cdots \bar{y}_{n+1}^{l_{n+1}} \tag{2.34}$$

where the Taylor coefficients are

$$F_k^{\sigma} = \frac{1}{\sigma!} \frac{\partial^{|\sigma|} F_k}{\partial y_1^{m_1} \ldots \partial y_{n+1}^{m_{n+1}} \partial \bar{y}_1^{l_1} \ldots \partial \bar{y}_{n+1}^{l_{n+1}}} \mid_{\mathbf{y}=\mathbf{0}}$$

and multiple-index notations have been introduced as follows

$$\begin{aligned} \sigma &= \{m_1, ..., m_{n+1}, l_1, ..., l_{n+1}\} \\ |\sigma| &= m_1 + \cdots + m_{n+1} + l_1 + \cdots + l_{n+1} \\ \sigma! &= m_1! \cdots m_{n+1}! l_1 \cdots l_{n+1}! \end{aligned}$$

Equations (2.33) correspond to the standard form, which is ready for analysis in terms of Lie group operators.

To apply the theory of the Lie groups we rewrite equations (2.33) in the form

$$\dot{\mathbf{y}} = A\mathbf{y}, \qquad A = A_0 + \varepsilon A_1 \tag{2.35}$$

where $\mathbf{y} = (y_1, ..., y_{n+1}; \bar{y}_1, ..., \bar{y}_{n+1})^T$, and

$$A_0 = \sum_{k=1}^{n+1} i\omega_k y_k \frac{\partial}{\partial y_k} + \text{cc} \qquad \text{and} \qquad A_1 = \sum_{k=1}^{n+1} F_k \frac{\partial}{\partial y_k} + \text{cc} \tag{2.36}$$

are operators of linear and nonlinear components of the system, respectively.

In order to bring the equations of motion to their simplest (Poincare) form, we introduce the coordinate transformation $\mathbf{y} \to \mathbf{z}$ in the Lie series form

$$\mathbf{y} = e^{-\varepsilon U}\mathbf{z} = \mathbf{z} - \varepsilon U\mathbf{z} + \frac{\varepsilon^2}{2!}U^2\mathbf{z} - ... \tag{2.37}$$

where $\mathbf{z} = (z_1, ..., z_{n+1}; \bar{z}_1, ..., \bar{z}_{n+1})^T$, and the operator of transformation U is represented in the power series form with respect to the small parameter ε

$$U = U_0 + \varepsilon U_1 + \cdots \tag{2.38}$$

The coefficients of this series are

$$U_j = \sum_{k=1}^{n+1} T_{j,k} \frac{\partial}{\partial z_k} + \text{cc} \tag{2.39}$$

where

$$T_{j,k} = T_{j,k}\,(z_1, ..., z_{n+1}; \bar{z}_1, ..., \bar{z}_{n+1}) \tag{2.40}$$

are unknown functions to be determined.

One of the advantages of this process is that the inverse coordinate transformation to the form (2.37) can be easily written as

$$e^{\varepsilon U}\mathbf{y} = \mathbf{z} \tag{2.41}$$

where one should simply replaces $\mathbf{z}$ with $\mathbf{y}$ in the operator of transformation, U.

If $\varepsilon = 0$, transformation (2.37) becomes identical, $\mathbf{y} = \exp(0)\,\mathbf{z} = \mathbf{z}$. In this case, equation (2.35) has already the simplest linear form and there is no need to transform the system. For $\varepsilon \neq 0$ transformation (2.37) converts the system (2.35) into the following one:

$$\dot{\mathbf{z}} = B\mathbf{z} \tag{2.42}$$

where the new operator B is given by the Hausdorff formula [22]:

$$B = A + \varepsilon\,[A, U] + \frac{\varepsilon^2}{2!}\,[[A, U], U] + ... \tag{2.43}$$

where $[A, U] = AU - UA$ is the commutator of operators A and U.

An optimized iterative algorithm for high-order solutions of equation (2.43) was suggested in [202] and [204]. In order to illustrate just the leading order terms of asymptotic expansions, let us follow the direct procedure though. Substituting the power series expansions for A and U given by relations (2.35) and (2.38) into (2.43) gives

$$\begin{aligned} B = {} & A_0 + \varepsilon\,(A_1 + [A_0, U_0]) \\ & + \varepsilon^2 \left([A_0, U_1] + [A_1, U_0] + \frac{1}{2!}\,[[A_0, U_0], U_0] \right) + ... \end{aligned} \tag{2.44}$$

A simple calculation gives

$$B = \sum_{k=1}^{n+1} \{ i\omega_k z_k + \varepsilon\,[F_k + (A_0 - i\omega_k)\,T_{0,k}] \}\,\frac{\partial}{\partial z_k} + O\left(\varepsilon^2\right) + \text{cc} \tag{2.45}$$

where the terms of order ε^2 have been ignored, $F_k = F_k\,|_{\mathbf{y}\to\mathbf{z}}$and $A_0 = A_0\,|_{\mathbf{y}\to\mathbf{z}}$.

The above relationships show that a transformation of the system

$$\dot{\mathbf{y}} = A\mathbf{y} \to \dot{\mathbf{z}} = B\mathbf{z}$$

can be considered in terms of operators $A \to B$. In order to bring the system into its normal form, one must eliminate as many nonlinear terms as possible from the transformed system such that the system dynamic characteristics are preserved. It follows from (2.45) that all nonlinear terms of order ε could be eliminated under the condition

$$F_k + (A_0 - i\omega_k)\, T_{0,k} = 0$$

Representing the unknown functions in the polynomial form

$$T_{0,k} = \sum_{|\sigma|=2,3,\ldots} T^{\sigma}_{0,k} z_1^{m_1} \cdots z_{n+1}^{m_{n+1}} \bar{z}_1^{l_1} \cdots \bar{z}_{n+1}^{l_{n+1}} \tag{2.46}$$

and taking into account (2.34), gives

$$F_k + (A_0 - i\omega_k)\, T_{0,k} = \sum_{|\sigma|=2,3,\ldots} \left(F^{\sigma}_k + i\Delta^{\sigma}_k T^{\sigma}_{0,k}\right) z_1^{m_1} \cdots z_{n+1}^{m_{n+1}} \bar{z}_1^{l_1} \cdots \bar{z}_{n+1}^{l_{n+1}}$$

where

$$\Delta^{\sigma}_k = (m_1 - l_1 - \delta_{1k})\,\omega_1 + \ldots + (m_{n+1} - l_{n+1} - \delta_{n+1,k})\,\omega_{n+1} \tag{2.47}$$

To reach zero-th coefficient of the monomial $z_1^{m_1} \cdots z_{n+1}^{m_{n+1}} \bar{z}_1^{l_1} \cdots \bar{z}_{n+1}^{l_{n+1}}$, one must put

$$T^{\sigma}_{0,k} = i\frac{F^{\sigma}_k}{\Delta^{\sigma}_k}$$

under the condition that $\Delta^{\sigma}_k \neq 0$.

If $\Delta^{\sigma}_k = 0$ for some k and σ then the corresponding nonlinear term cannot be eliminated from the transformed equation since it is qualified as a resonance term.

Finally, the result of transformation is summarized as follows.

The original set:

$$\dot{y}_k = i\omega_k y_k \; + \varepsilon \sum_{|\sigma|=2,3,\ldots} F^{\sigma}_k y_1^{m_1} \cdots y_{n+1}^{m_{n+1}} \bar{y}_1^{l_1} \cdots \bar{y}_{n+1}^{l_{n+1}} \tag{2.48}$$

The transformation of coordinates:

$$y_k = z_k - \varepsilon \sum_{\substack{|\sigma|=2,3,\ldots \\ \Delta^{\sigma}_k \neq 0}} i\frac{F^{\sigma}_k}{\Delta^{\sigma}_k} z_1^{m_1} \cdots z_{n+1}^{m_{n+1}} \bar{z}_1^{l_1} \cdots \bar{z}_{n+1}^{l_{n+1}} + O\left(\varepsilon^2\right) \tag{2.49}$$

The transformed set:

$$\dot{z}_k = i\omega_k z_k + \varepsilon \sum_{\substack{|\sigma|=2,3,\ldots \\ \Delta^{\sigma}_k = 0}} F^{\sigma}_k z_1^{m_1} \cdots z_{n+1}^{m_{n+1}} \bar{z}_1^{l_1} \cdots \bar{z}_{n+1}^{l_{n+1}} + O\left(\varepsilon^2\right) \tag{2.50}$$

Equations (2.50) represent the normal form of the system, where the summation is much simpler than that in the original set (2.48). Namely, the summation in (2.50) contains only those terms that give rise to resonance while the first term on the right hand side stands for the fast component of the

motion. The fast component of the motion can be extracted by introducing the complex amplitudes $a_k(t)$ as follows

$$z_k = a_k(t)\exp(i\omega_k t) \tag{2.51}$$

Substituting (2.51) into (2.50) and taking into account the resonance condition, $\Delta_k^\sigma = 0$, gives

$$\dot{a}_k = \varepsilon \sum_{\substack{|\sigma|=2,3,\ldots \\ \Delta_k^\sigma = 0}} F_k^\sigma a_1^{m_1} \ldots a_{n+1}^{m_{n+1}} \bar{a}_1^{l_1} \ldots \bar{a}_{n+1}^{l_{n+1}} + O\left(\varepsilon^2\right) \tag{2.52}$$

System (2.52) describes the dynamics in terms of slowly varying amplitudes and, as a result, reveals global properties of the dynamics in a much easier way then the original system. After solution of system (2.52) is obtained, the coordinate transformations (2.51) and (2.49) can interpret the result in terms of the original coordinates.