Chapter 11
Principal Trajectories of Forced Vibrations

Abstract. As shown earlier by Zhuravlev (1992) that harmonically loaded linear conservative systems possess an alternative physically reasonable basis, which is generally different from that associated with conventional principal coordinates. Briefly, such a basis determines directions of harmonic loads along which the system response is equivalent to a single oscillator. The corresponding definition (principal *directions* of forced vibrations) is loosing sense in nonlinear case, when the linear tool of eigen vectors becomes inapplicable. However, it will be shown in this chapter that nonlinear formulation is still possible in terms of eigen vector-functions of time given by NSTT boundary value problems. Physical meaning of the corresponding nonlinear definitions for both discrete and continual models is discussed.

11.1 Introductory Remarks

The theory of linear normal modes defines a natural basis in the configuration space of linear conservative systems. The corresponding directions are associated with a set of independent harmonic oscillators. The number of such oscillators is infinite, if the original system is continuous. In the later case, the modal analysis provides reduction of a continuous system to the related discrete set of harmonic oscillators. As it is known, the normal modes are defined for a class of unforced systems, therefore only initial conditions select those oscillators that will be excited during the dynamical process. Practically, a normal mode regime must be supported by some external loading due to inevitable energy dissipation. However, the theory does not identify directly such external forces. Let $\psi_j(y)$ be, for instance, the jth mode shape of a beam. Generally speaking, the external loading of the same profile, $\psi_j(y)$, will excite not only the jth mode but also some others, unless the mass per unit length of the beam is constant. From the mathematical viewpoint, this is due to the mass density, say $\rho(y)$, participating as a weighting factor in the orthogonality condition

V.N. Pilipchuk: Nonlinear Dynamics, LNACM 52, pp. 265–273, 2010.
springerlink.com

$$< \psi_i(y)\rho(y)\psi_j(y) >= 0, \qquad i \neq j \tag{11.1}$$

The question therefore is what kind of external force must be applied to a mechanical system in order to generate a normal mode type of motion when all the system particles coherently vibrate with the same frequency?

Following reference [203], let us consider first the linear case assuming that the linear n-degree-of-freedom forced system oscillates as a single harmonic oscillator in such a manner that the coordinates vector $\mathbf{x}(t)$ and the force vector $\mathbf{p}(t)$ are collinear to the same constant vector $\mathbf{q}$ with a constant length ratio μ as follows

$$\mathbf{x} = \mathbf{q}\sin\omega t, \quad \mathbf{p} = \mu\mathbf{q}\sin\omega t \tag{11.2}$$

In the case of forced vibration, the frequency ω is rather predetermined by the external loading and therefore should not play the role of eigen-value. It was shown in [203] that the coefficient of proportionality μ can play such a role instead. In a regular case, the coefficient μ has exactly n eigen-values, whereas the vector $\mathbf{q}$ determines the corresponding 'principal directions' according to the definition of reference [203].

Note that the principal directions are always orthogonal regardless the mass matrix of the system. Such an approach therefore determines a new natural basis for external forces from the standpoint of system considered. This, of-course, should not be viewed as a substitute for the theory of normal modes, however, some non-autonomous problems can be naturally solved by making use of the above complementary basis.

In nonlinear cases, definition (11.2) is unapplicable and the above notion of principal directions loses its sense. However, it was shown in [136] that the basic idea still can be generalized by considering *trajectories* instead of *directions.* Also a mixed spatio-temporal consideration must be applied since spatial and temporal coordinates are not separable in nonlinear cases and the related vibration and forcing are generally neither harmonic in time not similar in space.

There are some practically important formulations of the problem for the case of nonlinear forced vibration, which could be qualified as inverse or semi-inverse approaches. The related methods select practically reasonable external forces that generates simple enough dynamics. For example, Harvey [57] considered 'natural forcing functions' proportional to the non-linear restoring force of the forced Duffing oscillator.

The notion of 'exact steady state' was defined by Rosenberg [168] for a strongly nonlinear single degree of freedom system as a vibration with the cosine-wave temporal shape of the period of external force. The corresponding forcing function is determined under some initial conditions. Kinney and Rosenberg [80] considered systems with many degrees of freedom.

11.2 Principal Directions of Linear Forced Systems

Let us illustrate first the basic idea of reference [203] by considering the linear n-degree-of-freedom forced system

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \mathbf{p}(\omega t), \quad \mathbf{x}(t) \in R^n \tag{11.3}$$

where M and K are constant mass and stiffness $n \times n$ -matrixes, respectively; $\mathbf{p}(\omega t)$ is a periodic vector-force of the period $T = 2\pi$ with respect to ωt, and the upper dot means differentiation with respect to time, t.

Substituting (11.2) in (11.3), gives the eigen-value problem with respect to the parameter μ and vector $\mathbf{q}$ in the form

$$-\omega^2 M\mathbf{q} + K\mathbf{q} = \mu\mathbf{q} \tag{11.4}$$

Let $\mathbf{q} = \mathbf{v}_s$ and $\mu = \mu_s$ be the sth eigen-vector and eigen-value respectively, $s = 1, ..., n$. The eigen-vectors $\mathbf{v}_s$ are orthogonal and can be normalized by condition

$$\mathbf{v}_i^T \mathbf{v}_j = \delta_{ij} \tag{11.5}$$

where δ_{ij} is the Kronecker symbol.

Therefore, the set of vectors $\mathbf{v}_s$ determine a natural basis for the case of forced vibrations. Let, for instance, the external force be $\mathbf{p} = \mathbf{Q}\sin\omega t$, where $\mathbf{Q} \in R^n$ is an arbitrary constant vector. In this case, the corresponding steady-state (particular) solution is written as

$$\mathbf{x} = \sum_s \frac{\left(\mathbf{v}_s^T \mathbf{Q}\right)}{\mu_s} \mathbf{v}_s \sin\omega t \tag{11.6}$$

Now, let $\mathbf{e}_s$ and ω_s be conventional linear normal modes and natural frequencies of the system. (The related eigen-value problem is obtained from (11.4) by setting $\mu = 0$.) As follows from the linear theory, the normal mode vectors are orthogonal with respect to the mass matrix M so that the normalization condition can be represented in the form

$$\mathbf{e}_i^T M \mathbf{e}_j = \delta_{ij} \tag{11.7}$$

Using the normal mode basis for the above steady-state, gives

$$\mathbf{x} = \sum_s \frac{\left(\mathbf{e}_s^T \mathbf{Q}\right)}{\omega_s^2 - \omega^2} \mathbf{e}_s \sin\omega t \tag{11.8}$$

Since the uniqueness theorem holds, expansions (11.6) and (11.8) must represent the same solution, and therefore,

$$\sum_s \frac{\left(\mathbf{v}_s^T \mathbf{Q}\right)}{\mu_s} \mathbf{v}_s = \sum_s \frac{\left(\mathbf{e}_s^T \mathbf{Q}\right)}{\omega_s^2 - \omega^2} \mathbf{e}_s \tag{11.9}$$

Let the external force amplitude vector $\mathbf{Q}$ be directed along one of the *principal directions*. Then, expansion (11.6) will include only one term, whereas expansion (11.8) still includes all n terms.

Now, let us consider the case, when the mass matrix is equal to the identity matrix, $M = E$. In this particular case, expression (11.4) takes the standard form of the eigen-value problem for normal modes with respect to the eigen-value parameter $\omega^2 + \mu$,

$$-\left(\omega^2 + \mu\right) E\mathbf{q} + K\mathbf{q} = \mathbf{0} \tag{11.10}$$

As follows from (11.10), the eigen-values of free and forced vibration are coupled by expression

$$\omega^2 + \mu_s = \omega_s^2, \quad s = 1, ..., n \tag{11.11}$$

It is seen that each eigen-value of forced vibration, $\mu_s = \omega_s^2 - \omega^2$, is a monotonically decreasing functions of the external frequency ω with only one zero at $\omega = \omega_s$.

11.3 Definition for Principal Trajectories of Nonlinear Discrete Systems

Let us consider the nonlinear case

$$M\ddot{\mathbf{x}} + K\mathbf{x} + \varepsilon\mathbf{f}(\mathbf{x}) = \mathbf{p}(\omega t), \qquad \mathbf{x}(t) \in R^n \tag{11.12}$$

where $\mathbf{f}(\mathbf{x})$ is an analytic nonlinear vector-function such that $\mathbf{f}(-\mathbf{x}) = -\mathbf{f}(\mathbf{x})$, ε is a small positive parameter, and the forcing function and matrixes are defined in equation (11.3).

If $\varepsilon \neq 0$, then the concept of *principal directions* of forced vibrations is not applicable any more, however it is still possible to consider *principal trajectories* instead based on the following

Definition 1. Trajectories of periodic motions of the period $T = 2\pi/\omega$ on which mechanical system (11.12) behaves as a Newtonian particle in R^n, namely the external force and acceleration vectors are coupled by the Newton second law,

$$m\ddot{\mathbf{x}}(t) = \mathbf{p}(\omega t) \tag{11.13}$$

will be called *principal trajectories* of forced vibrations.

In equation (11.13), m is a priory unknown effective mass parameter. The effective mass m and the force $\mathbf{p}(\omega t)$ must be chosen in order to make equations (11.12) and (11.13) compatible.

Note that, in the linear case, the above definition still gives principal *directions* of forced vibrations (11.2) after representing the mass parameter as follows

$$m = -\frac{\mu}{\omega^2} \tag{11.14}$$

Indeed, substituting expression $\mathbf{x}(t) = \mathbf{q}\sin\omega t$ in equation (11.13) and taking into account expression (11.14), gives definition (11.2) in the form $\mathbf{p} = \mu\mathbf{x}$. In contrast to linear case (11.2), however, definition (11.13) allows non-harmonic temporal shapes.

Current definition itself does not imply that the system is weakly nonlinear. However, if the parameter ε is small then explicit solutions can be obtained in terms of conventional asymptotic expansions as described in the next section.

As mentioned, the notion of principal trajectories seems to relate to the idea of 'natural forcing functions' introduced in [57] for the Duffing oscillator. Let us consider now a multidimensional case from that point of view.

Applying definition (11.13) to the general nonlinear system

$$M\ddot{\mathbf{x}} + \mathbf{F}(\mathbf{x}) = \mathbf{p}(\omega t) \tag{11.15}$$

and eliminating the acceleration, gives the external forcing vector-function as a linear transformation of the restoring force in the form,

$$\mathbf{p}(\omega t) = \left(E - \frac{1}{m}M\right)^{-1}\mathbf{F}(\mathbf{x}) \tag{11.16}$$

where the matrix of the transformation includes the effective mass parameter m.

Relationship (11.16) can be viewed as a vector version of the concept of natural forcing functions.

On the other hand, using the definition for principal trajectories and excluding the external forcing vector $\mathbf{p}(\omega t)$ from the equation of motion, gives an auxiliary free system described by the differential equation of motion

$$(M - mE)\ddot{\mathbf{x}} + \mathbf{F}(\mathbf{x}) = \mathbf{0}$$

The idea of transforming the forced problem to a free vibration problem by imposing the form of excitation was used also in [31] with illustrations on two degrees of freedom systems based on an essentially different methodology though.

11.4 Asymptotic Expansions for Principal Trajectories

In order to make equations (11.12) and (11.13) compatible, let us eliminate the forcing vector-function $\mathbf{p}(\omega t)$ and thus consider equation

$$M\ddot{\mathbf{x}} + K\mathbf{x} + \varepsilon\mathbf{f}(\mathbf{x}) = m\ddot{\mathbf{x}}(t) \tag{11.17}$$

A family of periodic solutions, that give principal directions of linearized system as $\varepsilon \to 0$, will be considered. Let us represent such solutions (principal trajectories) in the following parametric form

$$\mathbf{x} = \mathbf{X}\,(\tau) \tag{11.18}$$

where is $\tau = \tau((2\omega/\pi)t)$ is the triangular sine wave of the period of external loading, $T = 2\pi/\omega$.

Substituting (11.18) into (11.17), gives

$$L\mathbf{X}+\varepsilon f\,(\mathbf{X}) = \left(\frac{2\omega}{\pi}\right)^2 m\mathbf{X}'' \tag{11.19}$$

$$L \equiv \left(\frac{2\omega}{\pi}\right)^2 M\frac{d^2}{d\tau^2} + K$$

under the boundary condition

$$\mathbf{X}'\,(\tau)\,|_{\tau=\pm 1} = 0 \tag{11.20}$$

As mentioned above, the temporal and spatial variables generally are not separable any more in nonlinear cases, therefore it is impossible to obtain an exact nonlinear version of the eigenvector problem (11.4). As a result, both temporal and spatial mode shapes must be corrected on each step of the related asymptotic process as described below.

Remind that the differential operator L in equation (11.19) includes the frequency parameter ω fixed, whereas the mass m is an eigen value to be determined.

Let m_a and $\mathbf{e}_a(\tau)$ be the eigen value and eigen vector of the linearized problem, $\varepsilon = 0$, respectively,

$$L\mathbf{e}_a = m_a \left(\frac{2\omega}{\pi}\right)^2 \mathbf{e}_a'' \tag{11.21}$$

$$\mathbf{e}_a' \;|\; _{\tau=\pm 1} = 0$$

where the index $a = \{s, j\}$ consists of spatial and temporal mode shape numbers, $s = 1, ..., n$ and $j = 1, ...,$ respectively.

The scalar product of any two vector-functions $\mathbf{x} = \mathbf{x}\,(\tau)$ and $\mathbf{y} = \mathbf{y}\,(\tau)$ will be defined as follows

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{2}\int_{-1}^{1} \mathbf{x}^T \mathbf{y} d\tau$$

Let us represent solution of the weakly nonlinear eigen value problem (11.19) and (11.20) in the following form of asymptotic expansions

$$\begin{aligned} \mathbf{X}(\tau) &= A\mathbf{e}_a(\tau) + \varepsilon \mathbf{X}^{(1)}(\tau) + O\left(\varepsilon^2\right) \\ m &= m_a + \varepsilon\eta_1 + O\left(\varepsilon^2\right) \end{aligned} \tag{11.22}$$

Then substituting (11.22) in (11.19) and (11.20), and matching the coefficients of the first order of ε, gives equation

$$L\mathbf{X}^{(1)} - \left(\frac{2\omega}{\pi}\right)^2 m_a \mathbf{X}^{(1)\prime\prime} = -f\left(A\mathbf{e}_a\right) + \left(\frac{2\omega}{\pi}\right)^2 \eta_1 A\mathbf{e}_a^{\prime\prime} \tag{11.23}$$

and boundary condition

$$\mathbf{X}^{(1)\prime} \mid_{\tau=\pm 1} = 0 \tag{11.24}$$

Following the idea of perturbations for eigen-value problems [83], let us represent solution of equation (11.23)

$$\mathbf{X}^{(1)} = \sum_{b\neq a} a_b^{(1)} \mathbf{e}_b(\tau) \tag{11.25}$$

where $b = \{r, i\}$ is a double index, $a_b^{(1)}$ are yet unknown constant coefficients, and boundary condition (11.24) is automatically satisfied.

Let us assume the following normalization condition for the eigen vector-functions

$$\langle \mathbf{e}_a'(\tau), \mathbf{e}_b'(\tau)\rangle = \begin{cases} 0, & b \neq a \\ 1, & b = a \end{cases} \tag{11.26}$$

Substituting (11.25) in (11.23) and taking into account (11.26), determines the coefficients $a_b^{(1)}$ and η_1. As a result, expansions (11.22) give first-order asymptotic solution

$$\begin{aligned} \mathbf{X} &= A\mathbf{e}_a + \varepsilon\left(\frac{\pi}{2\omega}\right)^2 \sum_{b\neq a} \frac{\langle \mathbf{e}_b, f\left(A\mathbf{e}_a\right)\rangle \mathbf{e}_b}{m_b - m_a} + O\left(\varepsilon^2\right) \\ m &= m_a - \varepsilon\left(\frac{\pi}{2\omega}\right)^2 \frac{\langle \mathbf{e}_a, f\left(A\mathbf{e}_a\right)\rangle}{A} + O\left(\varepsilon^2\right) \end{aligned} \tag{11.27}$$

As follows from the form of solution (11.27), all the coefficients are uniquely determined under the condition that $m_a \neq m_b$ for $a \neq b$. The possibility of degeneration, namely $m_a = m_b$ for $a \neq b$, depends on the inner properties of the system and the frequency parameter ω. The related examples were considered earlier [136], [141].

11.5 Definition for Principal Modes of Continuous Systems

Let us consider a one-dimensional elastic system whose vibration is described by some function $u = u(t, y)$. For certainty reason, let us consider a non-linear string of the length l under external distributed loading described by the partial differential equation and boundary conditions

$$Lu+\varepsilon f[u] = p(\omega t, y), \qquad 0 < y < l \tag{11.28}$$

$$u(t,0) = u(t,l) = 0 \tag{11.29}$$

$$L \equiv \rho(y)\frac{\partial^2}{\partial t^2} - T\frac{\partial^2}{\partial y^2} \tag{11.30}$$

where L is the differential self-adjoint operator of linear string, $\rho(y)$ is a mass per unit length parameter, T is a constant tensile force, $f[u]$ is a nonlinear operator acting in the corresponding function space of configurations, ε is a small parameter, and $p(\omega t, y)$ is the external forcing function, which is assumed to be 2π-periodic with respect to ωt.

Now keeping in mind expressions (11.28) through (11.30), let us introduce

Definition 2. Periodic forced vibrations of a continuous system, in which the system motion is equivalent to a particle in the function space of configurations described by the second Newton law,

$$\sigma\frac{\partial^2 u(t,y)}{\partial t^2} = p(\omega t, y) \tag{11.31}$$

will be called a *principal mode of forced vibration.*

In one-dimensional cases, σ is a priory unknown effective mass per unit length.

Substituting (11.31) in (11.28), gives the following partial differential equation for principal modes of forced vibrations

$$Lu+\varepsilon f[u] = \sigma\frac{\partial^2 u}{\partial t^2} \tag{11.32}$$

Introducing the triangular wave time substitution as $\tau = \tau((2\omega/\pi)t)$ and $u(t,y) = U(\tau, y)$, gives

$$\begin{aligned} LU+\varepsilon f(U) &= \left(\frac{2\omega}{\pi}\right)^2 \sigma\frac{\partial^2 U}{\partial \tau^2} \\ L &\equiv \left(\frac{2\omega}{\pi}\right)^2 \rho(y)\frac{\partial^2}{\partial \tau^2} - T\frac{\partial^2}{\partial y^2} \end{aligned} \tag{11.33}$$

The boundary conditions are formulated for both temporal and spatial variables as

$$U(\tau, 0) = U(\tau, l) = 0 \tag{11.34}$$

and,

$$\frac{\partial U(\tau, l)}{\partial \tau}\,|_{\tau=\pm 1} = 0 \tag{11.35}$$

respectively.

In this case, the scalar product of two functions $U = U(\tau, y)$ and $V = V(\tau, y)$ from the configuration space can be defined as

$$\langle U,V\rangle = \frac{1}{2l}\int_{-1}^{1}\int_{0}^{l} UV d\tau dy \tag{11.36}$$

Further, a weakly nonlinear asymptotic procedure can be developed analogously to the above discrete case.