The MUSSA II Land Use Auction Equilibrium Model

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Abstract In this chapter the description of a new version of the MUSSA model is presented. The supply side of the model leading to new equilibrium problems and a solution algorithm that enhances the model performance has been significantly improved. The model is designed to forecast the expected location of agents, residents and firms, in an urban area. The model stands upon the paradigm of static market equilibrium.

1 Introduction

MUSSA is a model designed to forecast the expected location of agents, residents and firms, in the urban area, originally presented by Martínez (1996) and improved in Martínez and Donoso (2001).[1] This paper describes a new version of MUSSA model, which has significantly improved the supply side of the model leading to new equilibrium problem and a solution algorithm that enhances the model performance significantly.

The model stands upon the paradigm of static market equilibrium. The location problem assumes that real estates are allocated to the highest bidder by auctions and that market equilibrium is attained by the condition that all agents are located somewhere, therefore, supply satisfies demand. This auctioning process produces rents for each real estate in the market and simultaneously defines levels of satisfaction (benefits) to located agents at equilibrium. A discrete approach is followed for all units of demand and supply: households and firms are clustered

[1] The model is commercially available as a Windows based software since 2002 and currently distributed by Citilabs Inc. under the name of Cube Land. The software license belongs to the Chilean Government (SECTRA).

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into categories, while land is divided into zones and dwellings into types; the number of discrete units is defined by the modeller. Consumers' agents, households and firms, are assumed rational and their idiosyncratic differences are modelled by a stochastic behaviour.

The place of MUSSA in the context of other land use models can be defined from a theoretical and historical perspective. A first generation of these models was designed under the assumption that agents locate as to minimize the travel cost to other activities, which may be called the maximum access model, where the transport system has a predominant role. Several models of this class where developed following either the Alonso's (1964) bid-rent approach or the Lowry's (1964) gravity – then entropy – approach, or even a combination of these two. A second generation introduced market elements into the location problem by including rents and good prices, what we call the linear market model. Rents have been introduced in two ways, using an hedonic rent function based on average zone attraction indices, or by assuming the location options are quasi-unique so rents are the result of simulating an auction process known as the bid-auction approach. In this case, input–output tables have been used to incorporate spatially differentiated prices on goods. The third generation introduces an important amount of complexity into the model buy incorporating an explicit representation of the direct interaction between agents decisions, that is the interaction that affect behaviour in addition to the price effects. These interactions describe the fact that location options are valued, by all agents and in a significant degree, by their built environment and the location pattern, usually called zone attributes. In the economic literature (see Mas-Colell et al. 1995, p 350) this type of interactions are defined as a multilateral public externality, because it involves all agents and public or non-rivalrous goods, which we call the location externality. The significance of this phenomenon to the model formulation is that the built environment is generated by the solution of the location problem itself, then zone attributes are endogenous variables. This describes a non-linear location equilibrium problem, with a large number of endogenous variables whose solution requires more sophisticated mathematical techniques than previous generations of models. The advantage of modelling location externalities explicitly is that they describe the real inherent dynamic of the location process.

A significant difference with other land use models is that in MUSSA the interaction between consumer agents – households and firms – is explicitly described in agents' behaviour and solved to attain equilibrium. We call these interactions location externalities, which represent local attributes of a location that depends on the agents' valuation of neighbour residents and firms' valuations of agglomeration economies. These interactions are complex, so normally ignored or simplified, because they makes each agents' location choice dependant on all other agents choices, thus making the calculation of equilibrium a highly complex mathematical problem, which is solved in MUSSA by ad-hoc algorithms. It is worth noting the tremendous dynamic in the land use pattern introduced by location externalities, because each choice affects all other choices and, in theory, the whole location pattern. We shall see however, that in MUSSA this dynamic is as smooth as it is observed in reality, but it reflects a real phenomenon.

A second key difference provides a useful tool for urban planning. The model represents explicitly the whole set of physical constraints (e.g. land capacity) and planning regulations that supply must comply with. Additionally, the model allows the direct simulation of the effects of pricing incentives (taxes or subsidies) introduced by the modeller. These features provide MUSSA with tools to assess a large number of issues in urban planning, like the economic impact, i.e. social benefits/costs, of regulatory and/or pricing scenarios.

Another feature of MUSSA is that all the model parameters are calibrated by econometric methods, which provides the set of parameters required by functions that describes the behaviour of demand and supply agents. This procedure maximizes the likelihood that choices actually made by agents and observed by the modeller, are best reproduced by the set of parameters obtained conditional on the functions specified and the data used. Stated and revealed preferences data may be used. The main advantage of this methodology is that parameters can be defined as mutually consistent, considering correlation dependencies.

The improvements made in MUSSA II are significant in order to handle efficiently the non-linearity issue in the context of highly constrained space. The supply of real state units has changed, from the aggregate deterministic econometric model of the previous version to a stochastic behaviourally based logit model. Since the demand side of the original MUSSA was already based on logit models, in the new version the demand–supply equilibrium is specified as system of logit equations.

Additionally, agents' behaviour in the land use context is subject to a large number of constrains, for example households have an income, suppliers produce real state units subject to non negative profit and to comply with a large number of planning regulations. In the previous versions of MUSSA these constraints were modelled as deterministic within a constrained optimization procedure, but because of the large number of constraints this process meant an unbearable computing burden in large cities, so the solution algorithm was based on a two levels heuristic. In the new version the behaviour of agents is constrained as to comply with all constraints, such that no choice of supply or demand violates constraints. This approach is implemented in MUSSA II using the constrained logit model (Martínez et al. 2005), where a binomial logit restrains behaviour of agents to their specific feasible space.[2]

2 The Consumers' Behaviour

The goods traded in the market modelled by MUSSA II are real estate properties differentiated by type ($v \in V$) –detached, semidetached, back-to-back house, department, etc. – by the location zone ($i \in I$) and by a vector of attributes

[2]MUSSA II is specified using the RB&SM model approach of fixed points presented in Martínez and Henríquez (2007), except for the constraints on consumers' and producers' behaviour, which has been modified.

$(z = (z_{vik}, k \in K))$ –number of rooms, neighbourhood, access, land lot size, etc. The model does not restrain the dimensions $\#V$, $\#I$ or $\#K$. Consumers are households and firms classified into clusters $(h \in H)$, households are classified by socioeconomic characteristics and firms by industry. Real estate suppliers are also classified into clusters $(j \in J)$ according to their differences in production costs.

A fundamental theoretical assumption in MUSSA is that a location in the urban context is a highly scarce resource because the right to use it (by renting or buying) provides restricted access to enjoy the neighbour amenities generated by the built and natural environment. This makes each location a quasi-unique or a differentiable good, which yields a monopoly power to the landowner who obtains maximum benefit by an auction process that extract the maximum willingness-to-pay from consumers, as proposed by Alonso (1964). Consumers play in the auction game by making bids for location options, where bids represent their willingness-to-pay. Since Solow (1973) and Rosen (1974), the willingness-to-pay is a function analytically obtained as the inverse in land rents of the correspondent indirect utility function, conditional on the location choice.

We denote by V_{hvi} the indirect utility function conditional on the location option vi. Assuming that each agent "consumes" only one location and has an income y_h, this function can be expressed as $V_{hvi} = V_h(y_h - r_{vi}, p, z_{vi})$. Then, the willingness-to-pay or bid function, conditional on obtaining a given utility level U_h, is:

$$B_{hvi} \equiv I_h - f_h(p, z_{vi}, U_h), \tag{1}$$

which represents the maximum value the agent is willing-to-pay for a location described by z_{vi}, to obtain a utility level U_h given the exogenous y_h and prices p. One can understand this function in the context of choice models by thinking that the agent considers a constant utility level (taken from market conditions), and assesses her/his monetary value for each available location option in the city using this bid function. Thus, it represents the price that would make the agent indifferent on choosing any alternative location, since the utility level is assumed fixed across space. An important observation is that from (1), $e_h = B_{hvi} + f_{hvi}$ represents the expenditure function in all goods plus location cost if the consumer pays B; this is relevant for evaluating different land use patterns and it is explained in Martínez (2003). Another observation is that similar bid functions can be derived for firms directly from their differential profits obtained at different locations.

3 The Equilibrium Problem

In MUSSA II all agents maximize their individual utility (called profit for producers) subject to a set of constraints in a static context. Consumers are constrained by an exogenous income and producers by a number of regulations. The equilibrium conditions are twofold: locations are assigned to the best bidder by an absent auctioneer, and all households are located somewhere, with total demand equal to total supply of real estate options in the city.

The economic static equilibrium problem analyzed in MUSSA II is the following:

$$CP)\quad \begin{array}{l} \underset{v,i}{Max}\ (B_{hvi} - r_{vi}) \quad \forall h \\ s.t.\ \ r_{vi} \leqslant y_h - px \end{array} \qquad PP)\quad \begin{array}{l} \underset{S_{vij}}{Max} \quad S_{vij}\ (r_{vi} - c_{vij}) \quad \forall j \\ s.t.\ \ \sum\limits_{j} S_{vij} \in R_i \end{array} \quad ,$$

$$EQ1)\ l_{hvi} = \arg\max_{h \in H}(B_{hvi}) \quad \forall v, i \quad EQ2) \sum_{v \in V, i \in I} S_{vi}\ l_{hvi} = \bar{N}_h \quad \forall h.$$

The CP problem represents the consumer h's problem, which chooses a location that maximizes surplus defined by the difference between the annual willingness to pay, or bid, (B_h), minus rents (r_{vi}), subject to an available budget $y_h - px$. Here px is the expenditure in a set of goods x and p is the vector of goods' prices, assumed exogenous in the model.[3] The real estate annual rent is exogenous for the consumer but endogenous to the equilibrium yield from an auction.

The PP problem represents the supplier's j behaviour, who decides how many real estate options to offer in each submarket (v,i), denoted S_{vi}, by maximizing profit. Profits are equal to the annual rent charged minus the building plus maintenance annualized costs (c_{vij}). The supply in each zone i is subject to a set of regulations $R_i = (R_{im}, m \in M_i)$, which makes all suppliers' behaviour interdependent.

The first equilibrium condition (EQ1) represents the auction. Consumers' bids at each location (B_{hvi}) are presented to the auctioneer who assigns each allocation to the best bidder.[4] Thus, l_{hvi} is one is h is the best bidder in location (v,i), zero otherwise. This best bidder rule is sufficient to assure simultaneously that suppliers maximize profit from available supply and consumer agents maximize utility or consumers' surplus (Martínez 1992, 2000). Thus, the location that satisfies condition EQ1 implies that CP is maximized for all consumers and that real estate suppliers obtain the highest rent for their stock.

The second equilibrium condition (EQ2) relates to the whole market. Unlike markets of products where consumers decide how much –if any – they buy of each good; in this market we assume that all agents consume only one, but also not less than one location. This means that every agent has to be located somewhere at equilibrium, provided that there are sufficient location options.

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[3]Note that, theoretically, the consumption vector x is optimal if the willingness to pay function is derived from the indirect utility function conditional on location choice.

[4]In the urban land market building properties usually have known common values, for example provided by real estate agents, so we expect that the auctioneer receives several similar bids, nevertheless, inevitably the final value is only defined by the auction. On the issue of auctions with common values see for example the review by McAffe and McMillan (1987).

The equilibrium yields the land use described by two vectors. The allocation of agents at alternative real estate options, denoted by vector $N = (N_{hvi}, h \in H, v \in V, i \in I)$, such that $\sum_{v,i} N_{hvi} = \bar{N}_h$, where $\bar{N}_h$[5] denotes the number of agents in cluster h exogenously defined; and by the vector of real estate supply $S = (S_{vi}, v \in V, i \in I)$, such that $\sum_{v,i} S_{vi} = \sum_{h} \bar{N}_h = \bar{N}$.

Equilibrium condition (EQ2) imply that participants in auctions outbid each other on real estate auctions, up to an equilibrium state that defines the maximum utility level attainable by each consumer in the market, represented by $U_h = U_h^*$ in (1), which yields bids at equilibrium.

The introduction of externalities represents a phenomenon that induces inherent instability in the model outcomes. This phenomenon has been widely described in social sciences (see Schelling 1978) and it is well recognized in game theory that it leads to complex non-linear mathematical formulations: small changes in initial conditions may cause dramatic differences in the location pattern and rents. Thus, it is important to note that bids functions theoretically embed location externalities, that is, the interaction between activities, by means of vector z in (1). Because the allocation of residents (neighbourhood quality) and firms (agglomeration economies) define neighbourhoods' attributes, then $z = z(N,S)$, it follows that this interaction is analytically represented by:

$$B_{hvi} \equiv I_h - f_h(p, z_{vi}(N,S), U_h) \tag{2}$$

This dependency represents a technological externality between agents, defined directly in their utility function, which operates in the urban system in addition to the pecuniary interaction through land rents.

The equilibrium presented above leads to a complex model of the city economics, extremely difficult to use for predictions. First, the facts that supply is discrete (zone system) and differentiable (location externalities), makes the location equilibrium problem mathematically untreatable for large cities in the deterministic context presented. Mathematically, it generates a complex non-linear fixed-point problem that describes a relevant and real urban dynamics introduced by the explicit representation of location externalities.

In MUSSA II, the approach to tackle this difficulty is to introduce continuity by transforming the deterministic problem into its probabilistic equivalent, which smoothes out the discontinuity associated to the agents' choice process. This approach has also the advantage of a more realistic model because idiosyncratic differences among agents within a cluster are represented by stochastic behaviour. The idiosyncratic variability on agents' behaviour takes into account the usual socio-economic and cultural differences of consumers considered in random utility theory and the variability in information and speculative behaviour in auction

[5]Overlined variables denote exogenous information required by the model.

processes; additionally, the idiosyncratic behaviour of real state suppliers describes differences in information and on strategic behaviour. The model uses the Gumbel distribution for consumers' and suppliers' behaviour, because it provides several relevant properties that help to solve the equilibrium problem.

4 The Probabilistic Bid-Auction Sub-model

In MUSSAII we conveniently assume the following bid function $B_{hvi} = I_h - f^1(U_h) - f^2(z_{vi})$, with good prices assumed as fix parameters and function f_h in (1) assumed additive. This latter assumption implies that the underpinning utility function is quasi-linear, which imposes relevant theoretical constraints into the model but with limited effect in real contexts.[6] But, it also introduces significant benefits in calculating the equilibrium, allowing the model to deal with the complex non-linearities. The assumption yields:

$$B_{hvi} = b_h^1 + b_{hvi}^2(N,S) + b^3 \tag{3}$$

where the bid function components are: b_h^1: adjusts utility U_h levels to attain equilibrium. b_{hvi}^2: describes the valuation of property attributes. Some attributes are exogenous to the location and land use distribution, like rivers, parks, hills, etc., and then they are represented by zone attractive parameters in this term. The complex attributes are those endogenous, which describe location externalities and are defined by two types of variables. First the distribution of agents' clusters in the zone, given by $N_{\bullet\bullet i}$[7] that describes attributes like neighbourhood quality by combining the characteristics of agents located in the zone (all building types) with the number of agents there located. Second, the building stock supplied in the zone (all buildings). $\overrightarrow{S}_{\bullet i}$, which describes the building environment in the zone. b^3: is a term independent of consumers and supply options, which adjusts bids to absolute levels in the whole market. This component is relevant only in the calculation of absolute value of rents and bids.

In the case of firms (non-residential activities), their bid function is derived from the profit function for each economic sector or industry. In this case, it is also assumed that the bid function is additive, like in (3).

In order to include idiosyncratic variability among consumers within a cluster, bids are assumed to be random variables: $\tilde{B}_{hvi} = B_{hvi} + \varepsilon_{hvi}$, with random terms ε_{hvi} distributed Gumbel, identical and independent (IID), justified by Ellickson (1981). From these assumptions, the (multinomial) probability that one of the $\bar{N}_h$ agents

[6] It merely requires that the utility function is linear in at least one good of the consumption bundle.

[7] Notation: $x_{\bullet k}$ denotes the vector of all elements of x whose second component is k.

type h is the highest bidder in (v,i), is yield – conditional on the real estate option being available – by:

$$P_{h/vi} = \frac{\bar{N}_h \exp(\mu B_{hvi})}{\sum_{g \in H} \bar{N}_g \exp(\mu B_{gvi})}, \tag{4}$$

where the parameter μ is inversely proportional to the variance of the bids. Here the aggregated version of the multinomial logit probability is utilized, which includes the correction for different sizes between agents' clusters, as proposed by McFadden (1978). Thus, the expected number of agents h located at (v,i) is given by $N_{hvi} = S_{vi} P_{h/vi}$

Then, using (3) in (4), the demand model is:

$$N_{hvi} = S_{vi} \frac{\overline{N}_h \exp(\mu(b_h^1 + b_{hvi}^2(N_{\bullet\bullet i}, S_{\bullet i})))}{\sum_{g \in H} \overline{N}_g \exp(\mu(b_g^1 + b_{gvi}^2(N_{\bullet\bullet i}, S_{\bullet i})))}, \tag{5}$$

where b^3 is cancelled out. In a synthetic form this is written as:

$$N_{hvi} = N(b_\bullet^1, N_{\bullet\bullet i}, S_{\bullet i}) \, \forall h, v, i. \tag{6}$$

This equation represents the *location fixed point*, with the probability variable both in the right and left hands of an unsolvable equation. It mathematically describes the interdependence between consumer decisions, i.e. location externalities, in which the location of an agent depends on locations of other agents (households and firms) in the same zone.

Additionally, as a direct result of the auction, the rent of a real estate (v,i) is determined by the expected value of the highest bid, which thanks to the Gumbel distribution is the known logsum or – implicit value function – given by:

$$r_{vi} = \frac{1}{\mu} \ln\left(\sum_{g \in H} \overline{N}_g \exp(\mu B_{gvi}) \right) + \frac{\gamma}{\mu} \tag{7}$$

with γ the Euler's constant. Notice that the rent depends on bids B_{hvi} and these in turn on the all other variables.

Equation (5) represents the solution of the consumer's maximizing problem (CP) and the auction condition (EQ1) simultaneously in a stochastic context. The solution of the fixed-point (5) yields the agents spatial location pattern conditional on two state variables: consumers' utilities (b^1) and supply (S). Note that the complexity of externalities remains in (5) and (5), but unlike the deterministic case discusses above, the fixed-point problem these equations is treatable, what is analyzed in detail by Martínez and Henríquez (2007).

5 Real Estate Supply Sub-model

The behaviour of real estate suppliers is twofold. First, they seek to obtain the maximum rents of their real estate stock. Second, for new development, they decide what combination of building and zone (v,i) would generate the maximum profit, subject to prevailing market regulations and rents. The first condition is already fulfilled by the auction mechanism.

There are some theoretical aspects to discuss. An important feature of the supply market is that it is highly regulated by zoning regulations, affecting both zone and building type, hence it is plausible that profit may be different by sub-markets defined by (v,i). A second issue is the heterogeneity of the suppliers, which occurs when suppliers have different profit functions. This function may be different depending on various sources of heterogeneity, for example the size of the firm that may imply different access to technology and generate different supply costs.

Another theoretical aspect is the potential for the presence of scale economies, within the same firm and zone sub-market, or scope economies (or diseconomies) across firms and zones. The most general case includes full interdependency reflected in costs functions denoted as $c_{vij} = c_j(S_{\bullet\bullet})$, where production cost depends on what is supplied everywhere by every developer. Less complex interdependencies are of course likely to occur in real markets. In any case, the supplier must determine the optimum amount to produce in each sub-market (v,i), which requires that he/she has to define an optimal vector ($S_{\cdot\cdot j}$).

In order to develop an operational supply model notice that rents are random variables, hence profits are also random. Moreover, by the property of conservation of the Gumbel distribution under maximization, rents are random variables with a Gumbel distribution that preserves the same scale parameter μ of the bids functions defined above. Thus, assuming that costs are deterministic, profits would be Gumbel distributed IID with the same scale factor as the bids. However, MUSSA II assumes that profits have a different scale parameter λ, thus allowing a more flexible model that adjusts better to the real behaviour.

In MUSSA II Hence, the expected number of residential supply units type (v,i), S_{vi}, is given by the aggregated production of each developer, which is given by the developer share of total production (S_j) times the probability that this unit type is the maximum profit option for that developer:$S_{vi} = \sum S_j P_{vi/j}$. Assuming the market share given by P_j, and the conditional probability $P^j_{vi/j}$ a multinomial logit, then:

$$S_{vi} = \bar{N} \sum_j P_j \frac{\exp(\lambda(r_{vi} - c_{vij}))}{\sum\limits_{v'i'j} \exp(\lambda(r_{v'i'} - c_{v'i'j}))}, \tag{8}$$

where λ is inversely proportional to the profit variance and $\bar{N}$ is the total number of supply units in the urban area, which is given exogenously by the total number of agents demanding locations.

Notice that, from the rent equation (7) it can be seen that b^3 is cancelled out in (8). More importantly, rents are functions of bids, then of vector N and S, and scale and scope economies make costs a function of S; therefore, the reduced form of the supply model is:

$$S_{vi} = S(b^1_{\bullet}, N_{\bullet\bullet\bullet}, S_{\bullet\bullet}), \tag{9}$$

which represents the *fixed-point equation* of the non-linear supply model. Equation (8) represents the solution of the producer's maximizing problem (PP) in a stochastic context and without regulations. The solution of the fixed-point (9) yields the spatial distribution of real estate supply conditional on two state variables: consumers' utilities (b^1) and consumers' allocation (N).

Notice that, despite the fact that the fixed-point problems (6) and (9) are written upon multinomial logit probabilities, they different because the state variables N and S in (8) are embedded in the rent logsum function (additionally, S is argument of the cost function). Thus, (9) has a more complex functional form than (6). Nevertheless, again the probability approach makes the fixed-point problem (8 and 9) treatable (see Martínez and Henríquez 2007).

6 Equilibrium

Here we study the supply–demand auction equilibrium (EQ2). This condition in the stochastic context of our model is expressed by:

$$\sum_{v \in V, i \in I} S_{vi} P_{h/vi} = \overline{N}_h \qquad \forall h \tag{10}$$

in which equilibrium is verified for each consumer category h and for all of them simultaneously. This condition is met if b^1 verifies that:

$$b^1_h = -\frac{1}{\mu} \ln\left(\frac{1}{\overline{N}_h} \sum_{vi} S_{vi} \exp(\mu(b^2_{hvi} - r_{vi})) \right), \tag{11}$$

which is obtained solving (11) for b^1_h. As $b^2(N,S)$ and $r_{vi}(b^1,N,S)$, this equation can be written in a reduced form as:

$$b^1_h = b_h(b^1_{\bullet}, P_{\bullet/\bullet\bullet}, S_{\bullet\bullet}) \tag{12}$$

and constitutes another fixed point, this time in vector b^1, whose solution verifies equilibrium conditions.

The adjustment of b^1 represents de adjustment of the utility levels that yield equilibrium: Under the additive assumption on bids, b^1 is negatively related with utility: the higher the bid for a location the lower the utility obtained (all location attributes held constant). Then the values obtained from (11) represent an index of the utilities attained by agents' clusters at equilibrium. As expected, ceteris paribus and neglecting second order effects caused by non-linearities, this index increases with $\overline{N}_h$, so utility decreases with population because supply is more demanded; more supply increases utility while higher rents have the opposite effect.

7 Modelling Constraints on Behaviour

The above models do not yet include the constraints on agents' behaviour: budget for consumers and zone regulations (R) for suppliers. The number of constrains that define a feasible domain for equilibrium variables are very large in real cities, which constitutes perhaps the highest computational burden for operational models.

MUSSA II uses a novel technique to model all constraints in the system with high computing performance. The approach is based on the Constrained Multinomial Logit Model (CML), proposed by Martínez et al. (2005). The approach consist on introducing a cut-off factor in the behaviour function, bid or profits, which has the property of making these functions tending to minus infinity as long as the attribute (or attributes) makes the constraint active. The new behaviour function is called the constrained behaviour function.

To resume how this technique works, consider for example the constrain on bids of the CP problem, which restrain bids to be positive and not greater than available income. We define the constrained bid as:

$$\tilde{B}_{hvi} = B_{hvi} + \frac{1}{\mu} \ln \varphi_{ni} + \varepsilon_{ni} \tag{13}$$

with B the unconstrained bid function. We also define φ, a cut-off factor that makes bids beyond the feasible range to have an extremely high negative value, thus reducing the probability of making that bid the best bid in the auction. Replacing the unconstrained by the constrained bid function in (4), yields the feasible probability:

$$\tilde{P}_{h/vi} = \frac{\bar{N}_h \phi_{hvi} \exp(\mu B_{hvi})}{\sum\limits_{g \in H} \bar{N}_g \phi_{gvi} \exp(\mu B_{gvi})}. \tag{14}$$

This new probability function approach zero when is evaluated at bids values out of the bid domain.

Furthermore, it is also possible to represent a set of constraints. For that the cut-off factor may be composed by a large number of factors that confine state variables to live in a multidimensional domain, with upper and lower bounds. That is $\varphi_{hvi} = \prod\limits_{k=1}^{K} \varphi^{L}_{hvik} \cdot \varphi^{U}_{nvik}$, with K the number of regulations applying to choosier h

on options v,i. This extended approach zero whenever one constraint is hit and it is useful for example to accommodate the large number of zone regulations that normally apply to each zone, representing upper and lower bounds.

In MUSSA II the cut-off factor is defined as a binomial logit function. For example, the mentioned constraint on budget represents an upper bound for rents affordable for the consumer. In our example the cut-off factor is defined by:

$$\varphi_{hvi}^{U} = \frac{1}{1+\exp(\omega_k(B_{hvi} - y_h + \rho))} \begin{cases} \to 1 & \textit{if} \quad (B_{hvi} - y_h) \to -\infty \\ = \eta & \textit{if} \quad B_{hvi} = y_h \\ \to 0 & \textit{if} \quad (B_{hvi} - y_h) \to +\infty \end{cases}, \tag{15}$$

where the parameter ω defines the speed of decay of the choice probability (14) as rents approach budget; ρ and $\eta$are factors that defines the tolerance allowed by the modeller for violating the bound. Note that this tolerance is strictly positive, because binomial probabilities are positive and non-zero (except at infinite). This implies that constraints are complied only up to a minimum probability value η, which is consistent with the theory of stochastic behaviour since bounds are always subject to the agent's perception and choice on whether to comply or not.

Modelling zoning regulations is a fundamental feature of a land use model, especially in order to make it applicable as a design tool for zoning plans. The cut-off approach allows the analysis of linear and non-linear constraints, without limiting the number of regulations included. More details on how the cut-off technique is formulated and applied can be seen in Martínez et al. (2005).

Thus, using appropriate cut-off functions to the above logit functions (5) and (8) yields consumers' locations ($\tilde{N}$) that comply with the budget constraints and real estate supply ($\tilde{S}$) that comply with planning regulations. Once rents are calculated using constrained bids ($\tilde{B}$), they also internalize the effect of all constraints, both on consumers' and suppliers' behaviour. It is worth noting that, following Martínez et al. (2005), it is possible to isolate the impact of each individual regulation on rents, which represents its economic value in the urban market; thus, it allows us to make an economic assessment for each regulation.

8 The Equilibrium Equation Systems

The MUSSA II's equilibrium is represented by the simultaneous solution of the previous set of equations, which together can be written like a macro fixed-point problem such as:

$$\begin{aligned} \tilde{N}_{h/vi} &= N_{h/vi}(b^1_\bullet, \tilde{N}_{\bullet/\bullet i}, \tilde{S}_{\bullet i}) \qquad \forall h, v, i, \\ \tilde{S}_{vi} &= S_{vi}(b^1_\bullet, \tilde{N}_{\bullet/\bullet\bullet}, \tilde{S}_{\bullet\bullet}) \qquad \forall v, i, \\ b^1_h &= b_h(b^1_\bullet, \tilde{N}_{\bullet/\bullet\bullet}, \tilde{S}_{\bullet\bullet}) \qquad \forall h, \end{aligned} \tag{16}$$

which is a system of dependent non-linear equations with dimension [(#H+1)(#V#I)+(#H)], with the same number of unknown variables. This system yields the solution vector $(b^1*, \tilde{N}*, \tilde{S}*)$, that is, the location of consumers, supply of real state units and relative values for rents and bids; both $\tilde{N}*$ and $\tilde{S}*$ comply with all constrains in the system. Note that each fix point is associated with either, interaction between agents (externalities the first one and economies of scale and scope the second one), or a market clearance condition (the third one).

The solution for equilibrium exists if the logit scale parameters μ and λ belong to a real range defined by Martínez and Henríquez (2007). They show that uniqueness of the solution is guarantied is scale parameters μ and λ are sufficiently low, i.e. that bids and profits do have a minimum idiosyncratic variability. Extensive numerical search indicates that such variability is usually present in real studies. When such minimum variability condition does not hold, the solution depends on the initial point because in this is the case agents' behaviour tends to be deterministic and probabilities tend to 0 or 1. In real cases studied dispersion is high enough to guarantee uniqueness.

Outputs may be used to perform rigorous economic assessments of land use scenarios. Benefits may be calculated using income compensated variations as the variation of bid values associated with scenario's changes (Martínez 2003). Thus, MUSSA II is a useful tool to obtain economic assessment of planning options such as: land use regulatory schemes, location pricing policies, transport-accessibility projects, etc. A particularly interesting application is to assess the social benefit (or cost) of each planning regulation, therefore dressing the planning process with an economic viewpoint. It can also be used by the private sector to assess expected profits from real state investments, land acquisition, location of retail and services, etc.

Some other remarks. First, it may seem that all agents, residents and non residents are allocated at each forecasting period without dependency to the previous period allocation. This interpretation is wrong, since it is possible to write the probability distributions as an incremental multinomial logit (similarly for rents), which makes evident the inter-temporal dependency in the model. Second, land use variables are endogenously updated within the equilibrium algorithm, which makes that zone attraction attributes are endogenous to the model and modified in each forecasting period. Third, residents' location behaviour consider multiple attributes, including access to most relevant activities (work, education, shopping, etc.), which are defined and calculated as transport users benefits. The trade off between attributes is defined through the bid functions by their calibrated parameters.

9 Application

The implementation of these algorithms imposes the additional challenge of obtaining a solution for a case of large dimensions. The model was applied to Santiago city, with 409 zones, 12 building types, 65 household clusters, the latter categorized by income, family size and car ownership and five firm types. This generates probability matrices with 145,000 elements that are updated in the fixed-point problem.

Additionally, the model handles 150,000 regulation constraints. The running time was 40 min, in a computer with 1.4 GHz and 1 Gb Ram memory. Therefore, despite the complexity of the interaction involved, including externalities among consumer's agents, economies of scale and scope in real state production, equilibrium conditions and regulation constraints, the model performs highly efficiently.

The model is calibrated using observed cross-section data of residential and non-residential location, rents and accessibility indices. It is also possible to use stated preference data or a mix with stated and revealed preferences. The methodology may be the standard maximum likelihood method for logit models. It is also possible to calibrate the model considering all the system equations simultaneously. The best practice will depend on the quality of the data associated to rents, consumers' location and real estate supply.

The model forecasts for any future year the urban equilibrium. The basic output includes:

- Property monthly rents, by building type and zone
- Location distribution of agents or the land use pattern, by cluster
- Buildings' distribution by zone, including houses of different sizes and types and building blocks by height levels, which defines density and average heights
- Benefits by agent clusters, which define distribution of welfare across agents
- Planning regulations and their slackness, and an index of their impact on rents

The model requires the following inputs for each forecasting year:

- Accessibility indexes by zone and preferably by cluster.
- Total city households' population by cluster and total activity for non-residential agents, also by cluster.
- The initial observation of the land use variables to initiate attributes of the built environment (by zone): average income of residents, commercial, education and services floor space.
- The initial observed distribution of real estate supply.
- The set of planning regulations and pricing (tax/subsidy) policies.
- The variables' set that describes dwelling types, e.g. lot size, floor space and building type (house or building).

The theoretical formulation of MUSSA II based on fixed point problems has also been used as component in two new tolls for planning. One tool performs optimization of regulation schemes and subsidies, which is presented in Martínez and Aguila (2004). In this case, the modeller provides the objective function and the model searches the optimal combination of regulations and subsidies. The technique was first developed to maximize social benefits in the land use market and then applied to minimize travel CO_2 emissions for Santiago city (Donoso et al. 2006).

Another tool is a dynamic model, where time is made explicit in the model as delays in the production process; the model is discussed in Martínez and Hurtubia (2006). Here auctions perform as in MUSSA II, but supply is assumed to take some time to be actually offered after the building decision is made; moreover, suppliers decide facing uncertain future. In this model excess of supply performs cycles because building is made in lump sums, which is the typical profile observed in stock economics.

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