Modeling Space by Stereographic Rejection

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Abstract 3D geo-information analyses topological and metrical relationships between spatial objects. This analysis needs a suitable representation of the three-dimensional world. This paper proposes to use the 4D unit sphere as a model. In essence this model is already present in mathematical theories like Lie sphere geometry, Moebius geometry and Geometric Algebra. The forementioned theories use the stereographic projection implicitely to build the model. This paper explicitely uses this geometric transformation to introduce the model as simply as possible following both an intuitive geometric and a formal algebraic self-contained way. The calculation in a CAD-environment of 3D Voronoi cells around given 3D points gives a straightforward example of the topological and metrical capabilities of this model. The addition of geometrical meaningful algebraic operations to the model will increase its computational power.

1 Introduction

1.1 Statement and Significance of the Problem

3D geo-information analyses topological and metrical relationships between spatial objects. A first order analysis is a linear one: planar forms represent the spatial objects and linear algebra describes the relations. This paper proposes a second order approach: the addition of spherical forms to the representation of spatial objects while retaining efficient computations. The stereographic projection is the geometric transformation that does the work.

Earlier work described the linearized representation of 3D-spheres on the 4D unit sphere to facilitate geodetic calculations (Bil 1992).

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1.2 Sources

A representation of the stereographic projection on the unit sphere in terms of homogeneous coordinates was found by Hestenes and Sobczyk in a study of the spinor representation of the conformal group (Hestenes and Sobczyk 1984, Sect. 8.3). For efficient computation in manipulating geometric objects Hestenes et al. developed a unified algebraic framework and mathematical tools called Geometric Algebra (Hestenes et al. 1999). Dorst et al. implemented Geometric Algebra as a high-level language for geometric programming (Dorst et al. 2007).

Lie sphere geometry represents points as spheres with radius zero (Blaschke 1929; Cecil 1992). The idea to use stereographic projection for the determination of Voronoi cells from the convex hull is found in Brown (1979). The construction of the Voronoi Diagram in the model is pointed out in Dorst et al. (2007). Ledoux mentioned the problem of using a dynamic Voronoi Diagram in a geographical information system (Ledoux 2008).

1.3 Historic Notes

The use of the stereographic projection to perform spherical calculations and to make maps has a long history. Already the Greek astronomer Hipparchus (180–128 BC) was aware of the projection. Around the first century the Alexandrian Claudius Ptolemaeus wrote on it in the book *Planisphaerium*. The construction of ancient Greek and medieval arab astrolabes uses its properties. Around the millennium in the Arab world Al-Bīrūnī applied the stereographic projection to the making of maps. In 1587 the cartographer Gerhard Mercator invented conformality in his mappings of the eastern and western half spheres in stereographic projection. The Jesuit Aguilonius (1566–1617) is the first to use the name *stereographic projection* in his books on optics (Rosenfeld 1988; Grafarend and Krumm 2006).

1.4 Overview

For a first introduction the standard description of the (conformal) model in the literature on Geometric Algebra is too abstract, defining algebraic operations on coordinate free geometric objects on a null cone in five dimensions. The entrance to the model on the 4D unit sphere by the stereographic projection with just linear algebra seems simpler and because of the spherical symmetry has also an aesthetic appeal. The central thought is the idea to consider 3D linear space to be the stereographic projection of a four-dimensional unit sphere.

This elementary introduction to the model is both on an analytic and a synthetic level. In the first part the analysis of drawings stimulates geometric intuition.

With some concepts from projective and inversive geometry the stereographic projection of points, planes, and real and imaginary spheres is studied. One can look at the figures in two ways: as plane drawings on paper of the stereographic projection of a circle on a line, but also more in general as a mnemonic device showing the properties of the stereographic projection of a sphere on a linear space one dimension lower.

In the second part vectors (i.e. coordinates) describe the position of geometric objects in the vector spaces R^n and R^{n+1}, and linear algebra deduces the metrical and topological properties of the objects and their relations.

In the end the construction of Voronoi cells around points in space exemplifies the use of the model.

2 Mathematical Preliminaries

The explanation relies in the first place on the intuitive interpretation of drawings. Therefore this is not a mathematical text, although the goal is to be as precise as possible and to get acquainted with mathematical concepts. In mathematical terms the stereographic projection is an inversion, i.e. a special projective transformation. See for inversive geometrical concepts for instance (Pedoe 1979) and (Brannan et al. 1999). The core of projective geometry is the duality between point and plane, and the crux of inversive geometry is the concept of harmonic conjugacy.

Figure 1 illustrates in the plane some inversive concepts to be used. For example the points P_1 and I are inverses, i.e. $OI \times OP_1 = 1$. Indeed the triangles OTP_1 and IOT are similar.

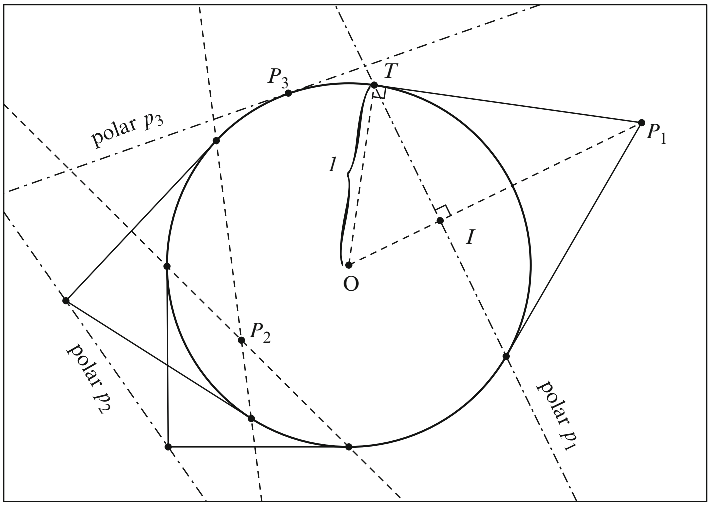


Fig. 1 Construction of pole and polar with regard to the unit sphere

So $\frac{OI}{OT} = \frac{OT}{OP_1} \Leftrightarrow \frac{OI}{1} = \frac{1}{OP_1} \Leftrightarrow OI \times OP_1 = 1$. In the following the inversive concepts pole and polar play a central role. A point P is the pole of a linear space (its polar) with regard to a conic.

Recipes for the geometric construction of the polar of a point P_i with regard to the unit sphere are:

- P_1 outside the unit sphere: construct the tangents from P_1 to the unit sphere. The polar is the linear space containing the tangent points on the unit sphere
- P_2 inside the unit sphere: consider P_2 as bundle of polars. Each polar has a pole. The polar of P_2 is the collection of these poles
- P_3 on the unit sphere: the polar is the tangent to P_3

To keep the figure readable tangents are only drawn between the pole and the point of tangency.

Homogeneous coordinates are suited to describe the duality between point and plane. See for this concept an elementary text on projective geometry, for instance (Ayres 1967). Homogeneous coordinates of a point or plane are the set of numbers that fulfill a homogeneous linear equation. Introduction of extra dimensions (coordinates) makes equations homogeneous. Consider for instance the equation $au_1 + bu_2 = c$. With the substitution $u_i = \frac{v_i}{v_3}, i = 1,2$ this equation becomes homogeneous: $a\frac{v_1}{v_3} + b\frac{v_2}{v_3} = c \Leftrightarrow av_1 + bv_2 - cv_3 = 0$, and after substitution the quadratic equation $u_1{}^2 + u_2{}^2 = 1$ becomes $\left(\frac{v_1}{v_3}\right)^2 + \left(\frac{v_2}{v_3}\right)^2 = 1 \Leftrightarrow v_1{}^2 + v_2{}^2 - v_3{}^2 = 0$.

If a triple of coordinates (v_1, v_1, v_3) fulfils a homogeneous equation, so does the triple $(\lambda v_1, \lambda v_2, \lambda v_3), \lambda \in R$. The equivalence class of the relation $(\lambda v_1, \lambda v_2, \lambda v_3) \approx (v_1, v_2, v_3), \lambda \neq 0$, with not every $v_i = 0$, constitutes the homogeneous coordinates of a point or plane.

The symbolism $\vec{x}^2 = \vec{x} \bullet \vec{x} = \|\vec{x}\|^2$ stems from Geometric Algebra: the square denotes the inner product of a vector with itself and is equal to the square of the length of the vector.

3 Geometric Analysis of the Stereographic Rejection

To get a better grasp on concepts such as stereographic projection, pole and polar it is instructive to first study some drawings. The notation S^n signifies the n-dimensional unit sphere in the $(n + 1)$-dimensional vector space R^{n+1}.

3.1 Points

Consider the south pole of the unit sphere S^n to be a bundle of lines (see Fig. 2). Every vector $\vec{x}$ in the vector space R^n is on a line in the bundle. This line contains

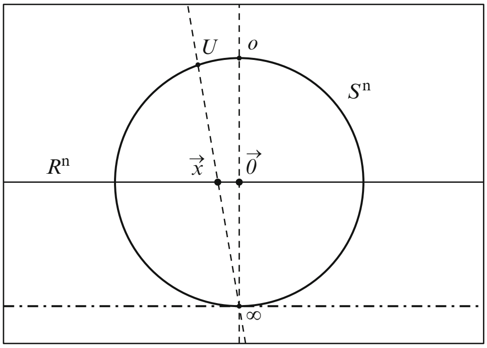


Fig. 2 The stereographic rejection U on S^n of vector $\vec{x}$ in R^n

another point U of S^n. Point U is called the stereographic rejection[1] of $\vec{x}$, and $\vec{x}$ the stereographic projection of U.[2] Beside the point $\vec{x}$ on S^n other points of R^{n+1} are on the line in the bundle through the center of projection and point $\vec{x}$ in R^n. $\vec{x}$ is called the stereographic projection of all these other points.

The north pole is the stereographic rejection of the origin. The center of projection is a representation of infinity, called the point at infinity and denoted by ∞: the greater the distance of a point in R^n to the origin, the closer the stereographic rejection of the point on the unit sphere to the center of projection.

In homogeneous coordinates: $o = \begin{pmatrix} \vec{0} \\ 1 \\ 1 \end{pmatrix}, \infty = \begin{pmatrix} \vec{0} \\ -1 \\ 1 \end{pmatrix}.$

Note that the lines on the polar of the center of projection, parallel to R^n, have neither an intersection with R^n nor a second intersection point with S^n.

3.2 Real Spheres

A n-sphere in R^n is stereographically rejected on a $(n+1)$-sphere on S^n that is part of a $(n+1)$-plane in R^{n+1} (see Fig. 3). The stereographic rejections of points inside the n-sphere are all on the same side of the $(n+1)$-plane. This $(n+1)$-plane is polar of a point P. Pole P is geometrically determined as the intersection point of all tangent planes to S^n at the rejection of the n-sphere. The stereographic projection of P gives back the center of the n-sphere in R^n. All points outside S^n can be considered as

[1]Here rejection is used in the sense of 'back' projection. In Geometric Algebra the rejection of a vector has a different meaning: the component complementary to its projection on another vector.

[2]For a better symbolic discrimination of the elements in R^{n+1} in stead of the letter X the letter U is used for a general element, and the letter P denotes a pole. Consequently coordinates in R^{n+1} are denoted by u_i.

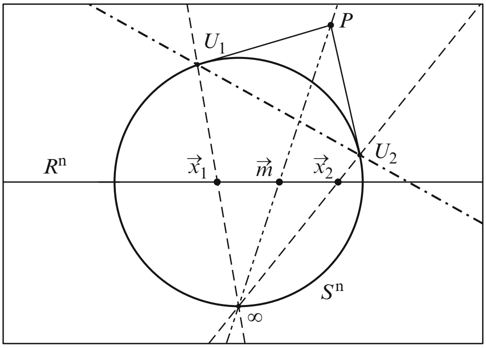


Fig. 3 Stereographic rejection to S^n of a sphere in R^n around $\vec{m}$

poles, corresponding to polars cutting S^n in $(n+1)$-spheres that are stereographic rejections of n-spheres.

3.3 Planes

If point P is on the polar of ∞, it can not be the stereographic rejection of a point in R^n, as shown in Sect. 3.1. The center of projection ∞ is on the polar. The polar is a $(n+1)$-plane and its intersection with R^n a n-plane. Thus pole P dually represents a n-plane. Figure 4 is a two dimensional cross section through the axis of the unit sphere. Figure 5a is a section through R^n. $\vec{n}$ is the normal vector of the plane. The equation of the plane is $\vec{n} \bullet \vec{x} = \vec{n} \bullet \vec{n}$. From the inversive relation, shown also in Fig. 1, follows: $\|\vec{n}\|\|\lambda\vec{n}\| = 1 \Leftrightarrow \lambda = \frac{1}{\vec{n}^2}$.

So the pole has coordinates $\begin{pmatrix} \frac{\vec{n}}{\vec{n}^2} \\ -1 \end{pmatrix}$.

The more the distance of pole P to the origin increases, the more $\vec{n}$ tends to $\vec{0}$ (Fig. 5b) and in the end the polar will contain the axis of the unit sphere. The normal vector of the plane in R^n will then also be the normal vector of the plane in R^{n+1}.

3.4 Position of Pole and Length of Radius

A n-point is the stereographic projection of all poles corresponding to n-spheres around this n-point (see Fig. 6). The greater the radius of the n-sphere, the greater the distance of its corresponding pole to the stereographic rejection of the n-point on S^n. The n-sphere with radius zero corresponds to the stereographic rejection of the n-point on S^n, equal to the pole of the tangent. So all points in R^n, considered as circles with radius zero, are stereographic projected on S.

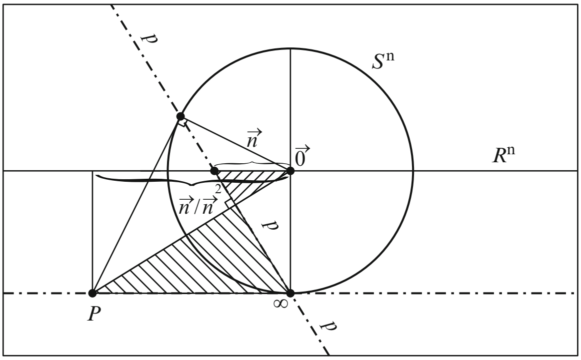


Fig. 4 Pole P on the polar of ∞ dually represents a plane in R^n

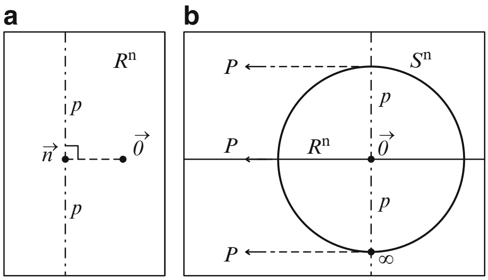


Fig. 5 (**a**) Cross section looking from the north pole. (**b**) The limiting case: pole P infinitely far from the origin, $\vec{n}$ becoming $\vec{0}$, polar p containing north and south pole

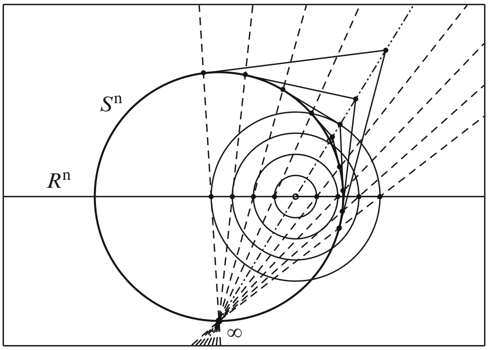


Fig. 6 The poles corresponding to spheres around a fixed point with different positive radii

3.5 Imaginary Spheres

Thus far points in R^{n+1} on and outside S^n are seen to represent points and spheres in R^n (see Fig. 7). What about points of R^{n+1} inside S^n? Point P is dually defined as

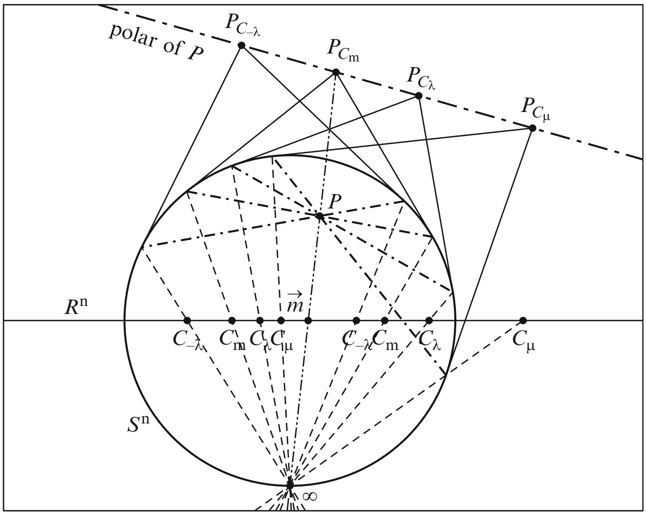


Fig. 7 Point P defined as a bundle of $(n+1)$-planes corresponding to a collection of real n-spheres C_i

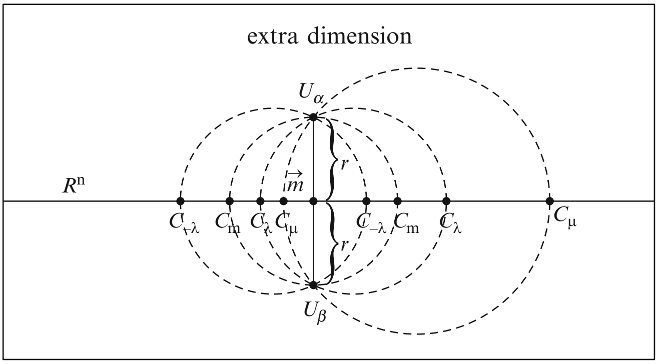


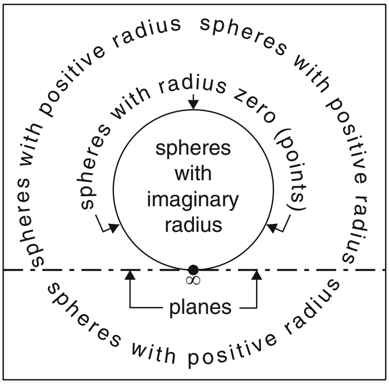
Fig. 8 Intersections, represented in the figure as $U_\alpha - U_\beta$, that the n-spheres in the collection shown in Fig. 7, have in common in the extra dimension

a bundle of planes cutting S^n in $(n + 1)$-spheres, with the P_{C_i} as the poles of these planes, together forming the polar of P. The $(n + 1)$-spheres on S^n stereographically project on n-spheres.

Embedding R^n in R^{n+1} all the extended n-spheres intersect in points in R^{n+1} at a certain distance of the stereographic projection $\vec{m}$ of P (see Figs. 8 and 9).[3]

[3]See also Sect. 4.5.

Fig. 9 Points in R^{n+1} dually represent points, planes and real and imaginary spheres in R^n depending on their position with regard to the unit sphere



3.6 $\mathbf{R^{n+1}}$ *Representing Spheres in* $\mathbf{R^n}$

Summarizing our analysis of the polar representation of spheres in R^n as points in R^{n+1}:

- The center of the stereographic projection represents infinity: ∞
- Points on the polar of ∞represent planes
- Other points outside the unit sphere represent real spheres
- Other points on the unit sphere represent points
- Points inside the unit sphere represent imaginary spheres

4 Algebraic Synthesis of the Stereographic Rejection

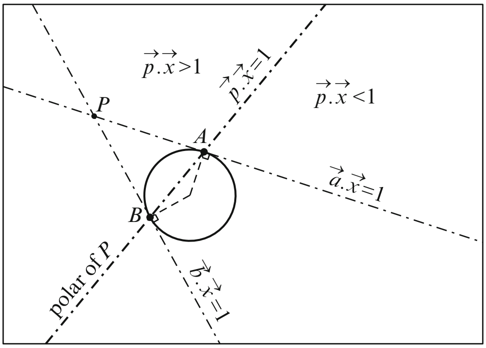
Practical applications need quantities. In the following sections linear algebra revisits the results found earlier and quantifies the relations between the coordinates of the points in the vector spaces R^n and R^{n+1}.

4.1 Pole and Polar

The polar relationship can be expressed in terms of linear algebra as follows. Point A is on the unit sphere, so $\vec{a} \bullet \vec{a} = 1$ (see Fig. 10). If a point is on the tangent to A, then $\vec{x} - \vec{a}$ is perpendicular to $\vec{a}$, i.e. $(\vec{x} - \vec{a}) \bullet \vec{a} = 0$. So $\vec{x} \bullet \vec{a} = \vec{a} \bullet \vec{a} = 1$.

In R^2: Let P be the polar of the line AB. P is on the tangent through A, so $\vec{p} \bullet \vec{a} = 1$, and P is on the tangent through B, so also $\vec{p} \bullet \vec{b} = 1$. Let X be another point of the line AB. If $\vec{x} = \vec{a} + \lambda(\vec{b} - \vec{a}), \lambda \in R$ then $\vec{p} \bullet \vec{x} = \vec{p} \bullet \vec{a} + \lambda(\vec{p} \bullet \vec{b} - \vec{p} \bullet \vec{a}) = 1$. So the equation of the polar of P is $\vec{p} \bullet \vec{x} = 1$.

Fig. 10 Pole P and its polar $\vec{p} \bullet \vec{x} = 1$



This result generalizes to R^n. If P is the intersection of the tangents to the points V_i, the linear space with equation $\vec{p} \bullet \vec{x} = 1$is the polar of P. This polar $\vec{p} \bullet \vec{x} = 1$ divides space in two: $\vec{p} \bullet \vec{x} > 1$ and $\vec{p} \bullet \vec{x} < 1$.

4.2 Points

The line of projection from the south pole has equation (see Fig. 11):

$$\begin{pmatrix} \vec{u} \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} \vec{x} \\ 0 \end{pmatrix} + \lambda\left(\begin{pmatrix} \vec{0} \\ -1 \end{pmatrix} - \begin{pmatrix} \vec{x} \\ 0 \end{pmatrix}\right) = \begin{pmatrix} (1-\lambda)\vec{x} \\ -\lambda \end{pmatrix}, \lambda \in R.$$

Now $\vec{u}^2 + {u_{n+1}}^2 = 1$, so $(1-\lambda)^2\vec{x}^2 + \lambda^2 = 1 \Leftrightarrow$

$$\lambda^2\left(1+\vec{x}^2\right) - 2\lambda\vec{x}^2 + \left(\vec{x}^2 - 1\right) = 0 \Leftrightarrow \lambda_1 = \frac{2\vec{x}^2+2}{2\left(1+\vec{x}^2\right)} = 1.$$

This gives the south pole, and the other solution is:

$$\lambda_2 = \frac{2\vec{x}^2-2}{2\left(1+\vec{x}^2\right)} = \left(\frac{\vec{x}^2-1}{1+\vec{x}^2}\right).$$

Using this value gives for the coordinates of the stereographic rejection U of $\vec{x}$:

$$\begin{pmatrix} \vec{u} \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} \left(1 - \left(\frac{\vec{x}^2-1}{1+\vec{x}^2}\right)\right)\vec{x} \\ -\left(\frac{\vec{x}^2-1}{1+\vec{x}^2}\right) \end{pmatrix} = \begin{pmatrix} \left(\frac{2\vec{x}}{1+\vec{x}^2}\right) \\ \left(\frac{1-\vec{x}^2}{1+\vec{x}^2}\right) \end{pmatrix}.$$

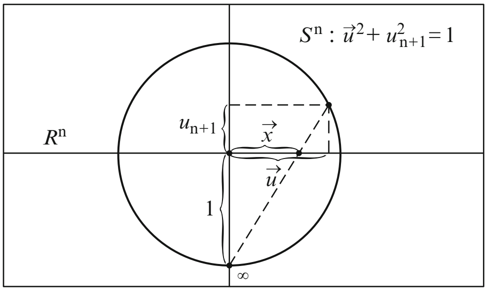


Fig. 11 The stereographic rejection in coordinates

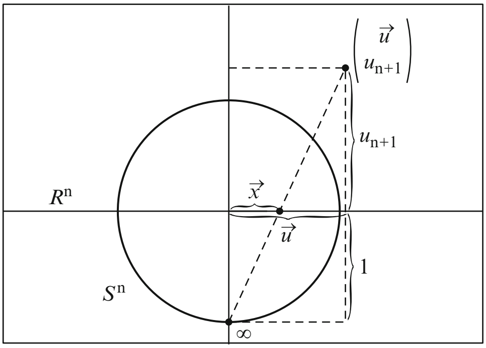


Fig. 12 The vector representation of the stereographic projection

In homogeneous coordinates this is[4]:

$$\begin{pmatrix} \left(\dfrac{-2\vec{x}}{1+\vec{x}^2}\right) \\ \left(\dfrac{1-\vec{x}^2}{1+\vec{x}^2}\right) \\ 1 \end{pmatrix} \approx \begin{pmatrix} \vec{x} \\ \left(\dfrac{1-\vec{x}^2}{2}\right) \\ \left(\dfrac{1+\vec{x}^2}{2}\right) \end{pmatrix} = S(\vec{x}), \tag{1}$$

in which the function $S\colon R^n \to R^{n+2}$ defined by (1) is the stereographic rejection expressed in homogeneous coordinates.

See Fig. 12, the stereographic projection of $\begin{pmatrix}\vec{u}\\ u_{n+1}\end{pmatrix} \in R^{n+1}$ on R^n is:

[4] See Sect. 2: substitute $u_i = \dfrac{v_i}{v_{n+2}}$ and scale to get a suitable representation.

$$\vec{x} = \frac{\left(\frac{2}{1+\vec{x}^2}\right)\vec{x}}{\left(\frac{2}{1+\vec{x}^2}\right)} = \frac{\left(\frac{2\vec{x}}{1+\vec{x}^2}\right)}{\left(\frac{1-\vec{x}^2}{1+\vec{x}^2}\right)+1} = \frac{\vec{u}}{u_{n+1}+1}. \tag{2}$$

4.3 Real Spheres

In R^n the equation of a sphere around a point M is $(\vec{x}-\vec{m})^2 = r^2$.

Using (2), substitution of $\vec{x} = \dfrac{\vec{u}}{1+u_{n+1}}$ gives:

$$\left(\frac{\vec{u}}{1+u_{n+1}} - \vec{m}\right)^2 = r^2 \Leftrightarrow \frac{\vec{u}^2}{(1+u_{n+1})^2} - \frac{2\vec{m}\bullet\vec{u}}{1+u_{n+1}} + \vec{m}^2 = r^2 \Leftrightarrow$$

$$\frac{1-u_{n+1}{}^2}{(1+u_{n+1})(1+u_{n+1})} - \frac{2\vec{m}\bullet\vec{u}}{1+u_{n+1}} + \vec{m}^2 = r^2 \Leftrightarrow$$

$$\frac{(1+u_{n+1})(1-u_{n+1})}{(1+u_{n+1})(1+u_{n+1})} - \frac{2\vec{m}\bullet\vec{u}}{1+u_{n+1}} + \vec{m}^2 = r^2 \Leftrightarrow$$

$$-2\vec{m}\bullet\vec{u} + \left(\vec{m}^2 - 1 - r^2\right)u_{n+1} = -\vec{m}^2 - 1 + r^2 \Leftrightarrow \tag{3}$$

$$\frac{-2\vec{m}\bullet\vec{u}}{-\vec{m}^2-1+r^2} + \frac{\vec{m}^2-r^2-1}{-\vec{m}^2-1+r^2}u_{n+1} = 1, \text{ with } \vec{m}^2 = \vec{m}\bullet\vec{m} \in R \tag{4}$$

and $\vec{u}^2 + u^2{}_{n+1} = 1$. According to Sect. 4 we read off from this equation of the plane in R^{n+1} that $\left(\dfrac{\left(\dfrac{-2\vec{m}}{-\vec{m}^2-1+r^2}\right)}{\left(\dfrac{\vec{m}^2-1-r^2}{-\vec{m}^2-1+r^2}\right)}\right)$ is pole of it.

The stereographic projection of this pole on R^n gives (2):

$$\frac{\vec{u}}{1+u_{n+1}} = \frac{\dfrac{-2\vec{m}}{-\vec{m}^2-1+r^2}}{\dfrac{\left(\vec{m}^2-1-r^2\right)+\left(-\vec{m}^2-1+r^2\right)}{-\vec{m}^2-1+r^2}} = \vec{m}.$$

Starting from the equation $(\vec{x}-\vec{m})^2 < r^2$, in the same way:

$$-2\vec{m}\bullet\vec{u} + \left(\vec{m}^2-1-r^2\right)u_{n+1} < -\vec{m}^2 - 1 + r^2. \tag{5}$$

Equation (5) is always valid, because $u_{n+1} + 1 \geq 0$. However, the sign of the equation corresponding to (4) is depending on the values of the parameters in the expression $-\vec{m}^2 - 1 + r^2$.

4.4 Planes

Consider a plane in R^n with normal vector $\vec{n}: \vec{n} \bullet \vec{x} = \vec{n}^2$. The equation of the plane in R^{n+1} containing stereographic rejected points is:

$$\vec{n} \bullet \frac{\vec{u}}{1 + u_{n+1}} = \vec{n}^2 \Leftrightarrow \vec{n} \bullet \vec{x} - \vec{n}^2 x_{n+1} = \vec{n}^2 \Leftrightarrow \frac{\vec{n}}{\vec{n}^2} \bullet \vec{u} - u_{n+1} = 1.$$

The pole is $\begin{pmatrix} \frac{\vec{n}}{\vec{n}^2} \\ -1 \end{pmatrix}$, and the center of projection $\begin{pmatrix} \vec{0} \\ 1 \end{pmatrix}$ is on the $(n + 1)$-plane.

4.5 Position of Pole and Length of Radius

It has been made clear the position of the pole depends on the radius of the n-sphere (see Fig. 6). The homogeneous coordinates of the pole of the $(n+1)$-plane that contains the stereographic rejection are deduced from (4):

$$\begin{pmatrix} \left(\frac{-2\vec{m}}{-\vec{m}^2 - 1 + r^2}\right) \\ \left(\frac{\vec{m}^2 - 1 - r^2}{-\vec{m}^2 - 1 + r^2}\right) \\ 1 \end{pmatrix} \approx \begin{pmatrix} -2\vec{m} \\ \vec{m}^2 - 1 - r^2 \\ -\vec{m}^2 - 1 + r^2 \end{pmatrix}$$

$$\approx \begin{pmatrix} \vec{m} \\ \frac{1 - \vec{m}^2}{2} \\ \frac{1 + \vec{m}^2}{2} \end{pmatrix} - \frac{1}{2} r^2 \begin{pmatrix} \vec{0} \\ -1 \\ 1 \end{pmatrix} = S(\vec{m}) - \frac{1}{2} r^2 \infty. \tag{6}$$

In (6) the equivalence class of homogeneous coordinates of the pole is represented by the coordinate tuple that encodes the stereographic rejection of the midpoint [see (1)]. This way the homogeneous vector of the pole, a $(n + 2)$-vector, carries the quantitative information about the corresponding n-sphere. Because on the model no metric and geometric algebra operations are defined, it is not possible to exploit this result here.

4.6 Imaginary Spheres

Having found an algebraic relation between points on S^n and points in R^n, and between points outside S^n and n-planes and n-spheres with positive radius, motivated by the geometric analysis in Sect. 3.5 the next search is for the algebraic

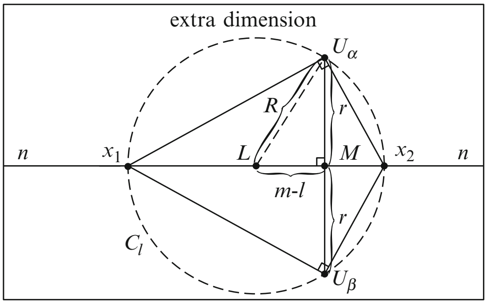


Fig. 13 The simplest case: an imaginary 1-sphere (point pair) U_α–U_β at imaginary distance r from M is part of the imaginary 2-sphere C_l with midpoint L and radius R containing the 1-sphere (point pair) $x_1 - x_2$ on the line n

relationship between points inside S^n and n-spheres with an imaginary radius (see Fig. 9).

Consider the line n in R^1 as simplest case (see Figs. 8 and 13). Embed this line in a space one dimension higher. If a midpoint M is given, determine the points with orthogonal distance r to M: the two-sphere with imaginary radius r. For all other point on 2D line n: draw the two-sphere (circle) containing the imaginary points U_α and U_β. This means on line n there are points $x_1$and x_2 fulfilling the equation: $(m - x)^2 = R^2 = (m - l)^2 + r^2$.[5] Take such a one-sphere with point L, having coordinate value λ, as center. According to (3) the equation of the two-line that is the stereographic rejection of the one-sphere is:

$$-2\lambda u_1 + ((\lambda^2 - (\lambda - m)^2 + r^2) - 1)u_2 = (\lambda - m)^2 + r^2 - \lambda^2 - 1. \quad (7)$$

The intersection point of this two-line with the two-line that is corresponding with the two-sphere through the imaginary points U_α and U_β with center $-\lambda$ gives as coordinates $u_1 = \dfrac{2m}{1 + m^2 + r^2}$ and $u_2 = \dfrac{1 - m^2 - r^2}{1 + m^2 + r^2}$. This expression is independent of λ. Repeat the construction in R^n by considering the points related to $\vec{m}$ by the equation: $(\vec{m} - \vec{x})^2 = (\vec{m} - \vec{l})^2 + r^2$ fulfilling the equation equivalent to (7):

$$-2\vec{l}\vec{u} + (\vec{l}^{\,2} - (\vec{l} - \vec{m})^2 + r^2) - 1)u_{n+1} = ((\vec{l} - \vec{m})^2 + r^2)^2 - \vec{l}^2 - 1. \quad (8)$$

The conjecture all planes contain $P = \begin{pmatrix} \dfrac{2\vec{m}}{1 + \vec{m}^2 + r^2} \\ \dfrac{1 - \vec{m}^2 - r^2}{1 + \vec{m}^2 + r^2} \end{pmatrix}$ is true, for P fulfils (7), independent of $\vec{l}$. So if a sphere C_i in R^n cuts the imaginary sphere with center M, P is on the plane in R^{n+1} that contains the stereographic rejection of C_i (see also

[5]Historic aside: this is in fact the ancient construction of the mean proportional from Proposition 13 of book VI in *The Elements* of Euclid.

Figs. 7 and 8). The representant of the homogeneous coordinates of pole P [compare this with (6)] is:

$$\begin{pmatrix} \left(\frac{2\vec{m}}{1+\vec{m}^2+r^2}\right) \\ \left(\frac{1-\vec{m}^2-r^2}{1+\vec{m}^2+r^2}\right) \\ 1 \end{pmatrix} \approx \begin{pmatrix} 2\vec{m} \\ 1-\vec{m}^2-r^2 \\ 1+\vec{m}^2+r^2 \end{pmatrix}$$

$$\approx \begin{pmatrix} \vec{m} \\ \frac{1-\vec{m}^2}{2} \\ \frac{1+\vec{m}^2}{2} \end{pmatrix} + \frac{1}{2}r^2 \begin{pmatrix} \vec{0} \\ -1 \\ 1 \end{pmatrix} = S(\vec{m}) - \frac{1}{2}(ir)^2\infty$$

$(n+1)$-point P lies inside S^n. Indeed, given the fact that

$$(1+\vec{m}^2+r^2)^2 - (1-\vec{m}^2-r^2)^2 = (2\vec{m}^2+2r^2)(1+1) = 4\vec{m}^2+4r^2,$$

it follows that $4\vec{m}^2 + (1-\vec{m}^2-r^2)^2 + 4r^2 = (1+\vec{m}^2+r^2)^2$.

So, because $4r^2 > 0$, $P^2 = \left(\frac{2\vec{m}}{1+\vec{m}^2+r^2}\right)^2 + \left(\frac{1-\vec{m}^2-r^2}{1+\vec{m}^2+r^2}\right)^2 < 1$.

Again this pole P is projected on $\vec{m}$, for from (2) follows:

$$\frac{\dfrac{2\vec{m}}{1+\vec{m}^2+r^2}}{1-\dfrac{1-\vec{m}^2-r^2}{1+\vec{m}^2+r^2}} = \vec{m}.$$

5 Example: Spatial Partitioning by Voronoi Cells

5.1 Introduction

The calculation of a partition of nD with the help of the topological and metrical properties of its rejection on the $(n+1)$D unit sphere shows the model at work.

In a number of random points $\vec{p} \in S \subset R^n$ a quantity is measured. A natural approach is to attribute to a point $\vec{q} \in R^n$ the measurement of the nearest point $\vec{p}$. The Voronoi cell is the region of points that are closest to $\vec{p}$. The mathematical definition of the Voronoi cell is:

$$V_p = \{\vec{x} \in R^n | \forall \vec{q} \in S : ||\vec{x}-\vec{p}|| \leq ||\vec{x}-\vec{q}||\}.$$

5.2 Analysis of the Voronoi Diagram in Two Dimensions

A planar analysis shows this to be a spherical construction.

The black points are randomly given (see Fig. 14). The edges of the Voronoi cells around the black points consist of points with equal distance to two nearest black points. The branch points (open circles in the drawing) consist of points with equal distance to three nearest black points. That is to say the branch point is the center of the circle to the three nearest neighbouring black points. These three black points form triangle with the property that no other black point lies inside the circumcircle, a Delaunay triangle. The plane is triangulated. The edges connecting the midpoints of the triangles around a black point make up the convex Voronoi cell (Fig. 15).

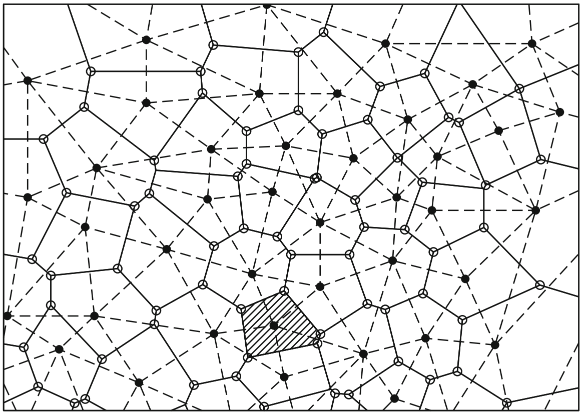


Fig. 14 Voronoi Diagram (of *white points*) and Delaunay triangulation (of *black points*)

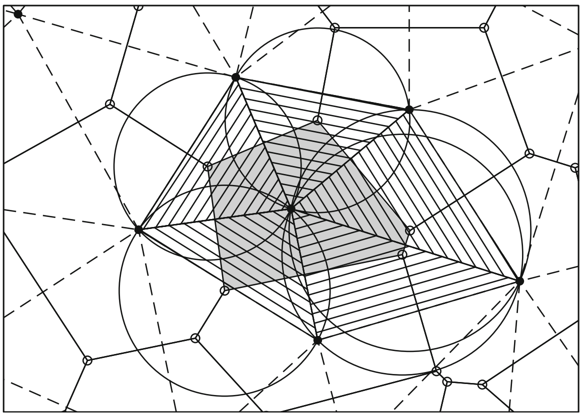


Fig. 15 Detail around a black given point. Five Delaunay triangles meet in this point. The circumcenters of the triangles are the vertices of the convex Voronoi cell around the point

5.3 The 2D Voronoi Diagram on the 3D Unit Sphere

See the scheme in Fig. 16. The stereographic rejections of the given points are on the 3D unit sphere. A Delaunay triangle of these points has the topological property that its circumcircle does not contain another given point. This property is equivalent to the topological property on the unit sphere that all stereographic rejections of the given points lie on the same side of the plane formed by the three points. In other words the Delaunay triangulation in the plane corresponds to the convex hull on the 3D unit sphere. The pole of the plane formed by the three points on the 3D sphere gives the stereographic rejection of the center of the circumcircle of these points. The mouthful formulation of the conclusion of the plane analysis is: *the Voronoi cell of a given point is the convex hull of the stereographic projection of the poles of the facets around the stereographic rejection of the given point of the convex hull of the stereographic rejection of all given points.*

5.4 Calculation of the 3D Voronoi Diagram

The abovementioned formulation is valid for the creation of a Voronoi Diagram in all dimensions. The following focuses on the construction in R^3.

The algorithm for the construction of Voronoi cells in R^3 is:

1. Input random points in space
2. Stereographically reject these points on the 4D unit sphere
3. Calculate the convex hull of these points in 4D. In non degenerate cases the convex hull consists of v facets of four vertices. The number v depends on the configuration of the given points in space

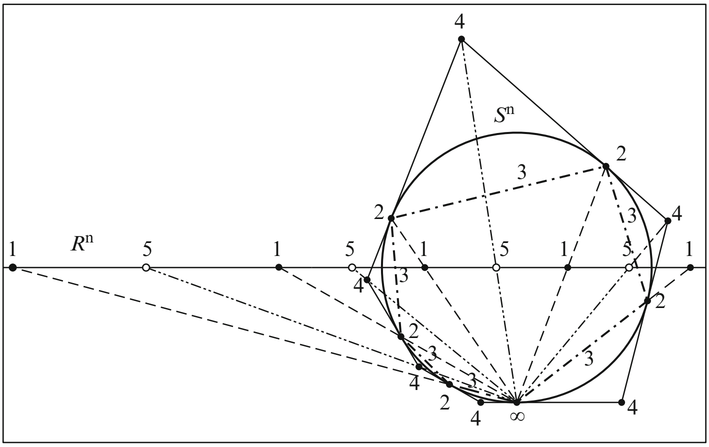


Fig. 16 The relation between the Voronoi and Delaunay tessellation, both in R^n and in R^{n+1}

4. Calculate the poles of the *v* planes incident with these facets
5. The stereographic projections of these *v* poles are the midpoints of the circumspheres of four given points (tetraeders), building blocks of the Voronoi cell around a given point in 3D
6. Calculate the convex hull of the stereographic projection of these *v* points around the given point in 3D. This gives the 3D facets of the Voronoi cell around a given 3D point

5.5 Neighbouring 3D Voronoi Cells

Neigbouring 3D Voronoi cells share a 2D facet. To find the connection of a Voronoi cell, first compare for all facets the number of vertices between all other calculated facets. In case of equality next compare the equality of the vertices of the facet with equal number of vertices. The order of the vertices of the matching facet is reversed. The connectivity of the Voronoi cells follows.

5.6 Moving Points

See Fig. 17 and also Sect. 4.1. If the given points move, so do their stereographic rejections, the facets/polars they are part of, the corresponding poles and the 3D Voronoi vertices. Sometimes the topology of the configuration changes. Having

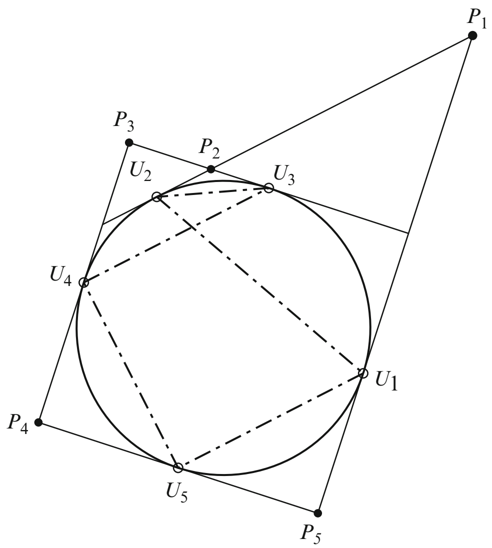


Fig. 17 Check of consistency on S^1: the vertices U_1, U_2, U_3 and U_4, on the polars of P_1 and P_3, disturb the convex hull structure

available the positions of the moving points in real time, the following test checks the consistency of the configuration.

Given the fact the configuration must be a convex hull, no point on the unit sphere should lie outside the polars formed by the facets of this hull. So for all poles all coordinate vectors of points U_i on the unit sphere ought to satisfy either the equations $\vec{p}_j \bullet \vec{u}_i \leq 1$, or $\vec{p}_j \bullet \vec{u}_i \geq 1$. For example in Fig. 17. the movements of points U_2 and U_3 have distorted the original convex hull U_1-U_2-U_3-U_4-U_5-U_1. For pole P_1 the sign of points U_3 is different from U_4 and U_5, and for pole P_3 the sign of U_2 is different from the points U_1 and U_5. In this case not only the position of the points and poles has to be updated, but also (part of) the topology.

6 Implementation

As proof of principle this theory has been implemented in software.

6.1 CAD

Given input is an ASCII-file with coordinates of points. The given points are then represented in a 3D Microstation designfile. *Visual Basic for Applications for Microstation* is the programming environment. The coordinates are read from the designfile and stored in the array of vertex structures that is kept in memory to speed up calculations. All coordinates are translated and scaled towards the 3D unit sphere. In this process the maximum distance of the vertices to the origin is determined. After the coordinates of the given points are read the 3D coordinates of an icosahedron are added to enclose all the given points. The 12 vertices of the icosahedron are: $(0, \pm\tau R_{\max}, \pm R_{\max})$, $(\pm\tau R_{\max}, 0, \pm R_{\max})$ and $(\pm R_{\max}, \pm\tau R_{\max}, 0)$, with $\tau = \frac{\sqrt{5}+1}{2}$ the divine proportion (Coxeter 1969). The addition of the south pole of the 4D unit sphere to the stereographically rejected 3D points closes the set of 4D points on the sphere. Pointers from within the vertex structures address the Voronoi coordinates (derived from the data in the array of facet structures).

6.2 Convex Hull Software

An introduction into the convex hull computation is given in (de Berg et al. 1998). In the application the calculations of the convex hull are done using QHull. Information on QHull can be found on http://www.qhull.org/.

The input for the *qconvex* executable is formed by writing 4D or 3D coordinates from the Microstation 3D designfile to an ASCII-file. After execution of the command the outputted ASCII-file is read. The following command lines were used:

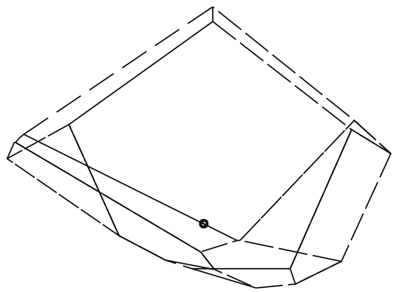


Fig. 18 CAD drawing of a Voronoi cell around a 3D point

1. *qconvex o* to calculate the 3D and 4D convex hulls. It outputs on the first line the dimension, on the second line the number of vertices, facets and ridges, then lines with vertex coordinates and finally the facet lines. Each facet line starts with the number of vertices. The vertex indices follow. In general for the convex hull on the 4D unit sphere the facets are simplices and made up by four vertices, corresponding with the 3D counterpart: four vertices determine a sphere in 3D. If more vertices are encountered this signals a degenerate configuration of the points and calculation measures can be taken, for instance by a little shift of a point concerned.
2. *qconvex FN* to list the number of facets on the first line and the neighbouring facet indices for each facet on the next lines. The line starts with the number of neighbouring facets (Fig. 18).

7 Conclusion, Caveat and Developments

The paper proposes to model space on the four-dimensional unit sphere, described by five homogeneous coordinates. Space is considered to be the stereographic projection of the 4D unit sphere. Points on the 4D unit sphere thus represent 3D points, i.e. spheres with radius zero. The 4D center of projection is the representation of the point at infinity and on its polar lie the 4D points that represent 3D planes. Points outside the 4D unit sphere dually represent 3D spheres with positive radius, and 4D points inside represent 3D spheres with imaginary radius, which are given a clear geometric interpretation. The expression of the 4D points in 5D homogeneous coordinates contains quantitative information on the center and radius of the dually represented 3D spheres and planes.

The first part of the paper introduces the concepts in an informal way analyzing the geometry in an intuitive manner. Next the conjectures are deduced in explicit formulae of elementary linear algebra.

As a straightforward example, useful in geographical information systems, Voronoi cells around given points in space are determined from the convex hull in the model.

As a proof of principle the theory has been implemented in software. Because the main objective of the author is to introduce the model, and because he is not fully aware of the state-of-the-art of the construction of Voronoi Diagrams and use of data structures, the software was not developed to the point it processes dynamic input. Nevertheless algorithms to calculate the connectivity of the Voronoi Diagram and to validate the configuration were indicated. *Microstation* offers the possibility for event-driven programming. So it is possible to monitor the position of the given points and calculate the stereographic rejections of the given points and values depending on them, in real time. The validation test of the configuration of the Voronoi cells could trigger the recalculation of the topology while running a program.

Although, as we have seen in the example of the calculation of the Voronoi Diagram, spheres and planes are represented on an equal footing in the model and we have linear algebra at our disposal, the model is not yet fully operational. At present the obtained methods are of limited use, for geometric meaningful algebraic operations are still lacking. To have the model at one's disposal without operations defined on it, is like having hardware without software. Now that 3D space is beamed up along the rays of the stereographic projection to the four-dimensional unit sphere "*one must so to speak throw away the ladder, after he has climbed up on it*"[6] and wander around in this copy of the world.

Geometric Algebra provides a unified coordinate-free algebraic framework for both multidimensional geometric objects and geometric operations in this model. The conformal model of Geometric Algebra goes one dimension up and models the 4D unit sphere in 5D as a null cone, isometrically embedding 3D. As an algebra Geometric Algebra is closed, i.e. every geometric product of (geometric) elements gives another element in the algebra. A main advantage is that only one formula is sufficient to describe a geometric situation. No "special case" processing is needed. For instance two spheres will always intersect. Imaginary spheres have, as is shown in this paper, a clear geometric interpretation.

To learn about the foundation of Geometric Algebra see (Hestenes and Sobczyk 1984) and about the conformal model see for instance (Hestenes et al. 1999), (Dorst et al. 2007), (Perwass 2008). Some communities have adapted the conformal model of Geometric Algebra as a computational tool. New applications of the model are frequently a topic at the *International Workshops on Computer Graphics, Vision and Mathematics* (GraVisMa) and the conferences on *Applied Geometric Algebras in Computer Science and Engineering* (AGACSE).

It remains to be seen whether or not the GA conformal model of Euclidean 3D geometry has killer applications in 3D geo-information.

[6] Satz 6.54 from the *Tractatus logico-philosophicus* of Ludwig Wittgenstein.

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