Chapter 3
Dynamical (Deterministic) Models of Evolution

3.1 Terminology

3.1.1 Operator, Map, Equation, Evolution Operator

Dynamical modelling requires specification of a D-dimensional state vector $\mathbf{x} = (x_1, x_2, \ldots, x_D)$, where x_i are dynamical variables, and some rule Φ_t allowing unique determination of future states $\mathbf{x}(t)$ based on an initial state $\mathbf{x}(0)$:

$$\mathbf{x}(t) = \Phi_t(\mathbf{x}(0)). \tag{3.1}$$

The rule Φ_t is called an *evolution operator*. "*Operator* is the same as a *mapping*... Mapping is a law according to which an every element x of a given set X is confronted with a uniquely determined element y of another given set Y. At that, X may coincide with Y. The latter situation is called *self-mapping*" (Mathematical dictionary, 1988) (Fig. 3.1a, b). In application to an evolution of a dynamical system state (motion of a representative point in a phase space), one often uses the term "*point map*".

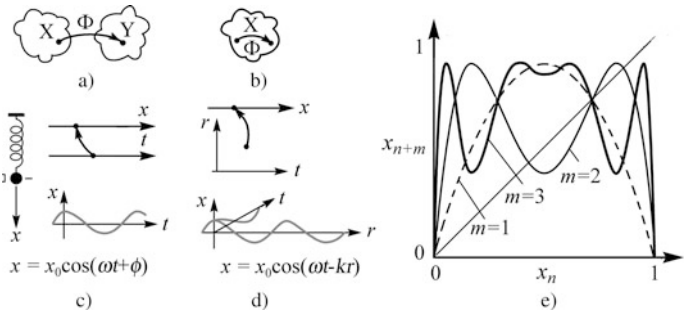

Fig. 3.1 Different kinds of maps: (**a**) from one set into another one; (**b**) self-mapping; (**c**) a function of time describing friction-free oscillations of a pendulum; (**d**) a function of two variables describing a harmonic wave; (**e**) iterates of a quadratic map $x_{n+1} = rx_n(1 - x_n)$ at $r = 3.5$

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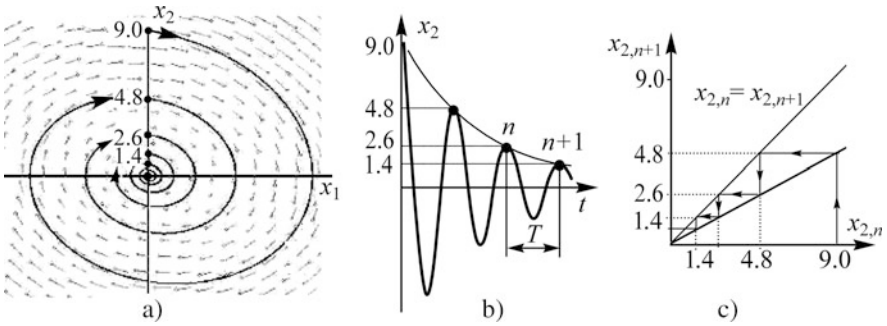

Fig. 3.2 Phase space of the linear dissipative oscillator (3.2) and its discrete description: (**a**) a velocity field specified by equation (3.4), *arrows* denote velocities of a state change; (**b**) a time realisation $x_2(t)$, *filled squares* are the points corresponding to the Poincare section $x_1 = 0$. They are separated by a time interval $\tau = T$. Their relationship is described with a map presented in the panel *c*; (**c**) a one-dimensional return map for the Poincare section $x_1 = 0$; an evolution can be studied conveniently with Lamerey's diagram (*arrows*)

An evolution operator can be specified directly as a map from a set of initial states $\mathbf{x}(0)$ into a set of future states $\mathbf{x}(t)$. However, it is more often determined indirectly with the aid of equations. "*Equation* is a way to write down a problem of looking for such elements a in a set A which satisfy an equality $F(a) = G(a)$, where F and G are given maps from a set A into a set B"[1] (Mathematical dictionary, 1988). If an equation is given, an evolution operator can be obtained via its solution. Thus, for an ordinary differential equation the theorem about unique existence of a solution assures the existence and one-oneness of a map Φ_t in equation (3.1) under some general conditions. If an exact solution of an equation is impossible, one searches for an approximate solution in the form of a numerical algorithm simulating a representative point motion in a phase space (Fig. 3.2a).

3.1.2 Functions, Continuous and Discrete time

Functions of independent variables (of a single variable $\mathbf{x} = \mathbf{F}(t)$ or of several ones $\mathbf{x} = \mathbf{F}(t, \mathbf{r})$) map a set of the values of the independent variables into a set of the values of the dependent (dynamical) variables. In Fig. 3.1c, d, time t and the vector of spatial coordinates $\mathbf{r}$ are independent variables, while a deviation x from an equilibrium state is a dynamical variable. If a function $\mathbf{F}$ depends explicitly on the initial values of dynamical variables, it can represent an evolution operator, see, e.g., equation (3.3).

A state of an object may be traced either *continuously* in time or *discretely*, i.e. at certain instants t_n separated from each other with a step Δt. In the latter

[1] If A and B are number sets, one gets algebraic or transcendental equations. If they are function sets, one gets differential, integral and other equations depending on the kind of the maps.

case, the order number of a time instant $n = 0, 1, 2, 3, \ldots$ is called *discrete time*. If observations are separated by an equal time interval Δt, then the relationship between continuous time t and discrete time n is linear: $t_n = n\Delta t$. For unequal intervals, the dependency can be more complicated. Similarly, one can use discrete versions of spatial coordinates, e.g. a number of steps along a chosen direction, a number of elements in a chain or a lattice.

3.1.3 Discrete Map, Iterate

In "discrete modelling" the values of dynamical variables $\mathbf{x}_n$ at different discrete-time instants n are related to each other via a map from a phase space X into itself ($X \to X$): $\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n, \mathbf{c})$, where $\mathbf{c}$ is a parameter vector. Such a *recurrent formula*[2] for an evolution operator is also called *discrete map*. To study a map, one uses its iterates. *Iterate* (from a Latin word "iteratio", i.e. repetition) is a result of a repeated application of some mathematical operation. Thus, if $\mathbf{F}(\mathbf{x}) \equiv \mathbf{F}^{(1)}(\mathbf{x})$ is a certain function of $\mathbf{x}$ mapping its domain into itself, then functions $\mathbf{F}^{(2)}(\mathbf{x}) \equiv \mathbf{F}[\mathbf{F}(\mathbf{x})]$, $\mathbf{F}^{(3)}(\mathbf{x}) \equiv \mathbf{F}[\mathbf{F}^{(2)}(\mathbf{x})]$, ..., and $\mathbf{F}^{(m)}(\mathbf{x}) \equiv \mathbf{F}[\mathbf{F}^{(m-1)}(\mathbf{x})]$ are called the second, the third, ..., and the mth iterates of $\mathbf{F}(\mathbf{x})$, respectively. The index m is the order number of an iterate. For instance, Fig. 3.1e shows three iterates of a quadratic map $x_{n+1} = rx_n(1 - x_n)$, where r is a parameter.

3.1.4 Flows and Cascades, Poincare Section and Poincare Map

In a DS whose evolution operator is specified via differential equations, time is continuous. In a phase space of such a DS, motions starting from close initial points correspond to a beam of phase orbits resembling lines of flow in a fluid (Fig. 3.2a). Such DSs are called *flows* in contrast to *cascades*, i.e. to DSs described with discrete maps, $\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n, \mathbf{c})$.

The term "Poincare section" denotes a section of a phase space of a flow with a set of dimension $D - 1$, e.g. a section of a three-dimensional space with a surface or a two-dimensional space with a curve. The term "Poincare map" is used for mapping of a set of unidirectional "punctures" of a Poincare section with a phase orbit into itself. It relates a current "puncture" to the next one.

3.1.5 Illustrative Example

Let us illustrate the above-mentioned terms with a model of the oscillations of a load on a spring in a viscous medium. An etalon model of low-amplitude oscillations in a

[2] A recurrent formula is the relationship of the form $x_{n+p} = f(x_n, x_{n+1}, \ldots, x_{n+p-1})$ allowing calculation of any element in a sequence if its p starting elements are specified.

viscous fluid under the influence of a restoring force, proportional to the deviation x from an equilibrium state, is an ordinary differential equation of "a linear oscillator":

$$\mathrm{d}^2x/\mathrm{d}t^2 + 2\delta \mathrm{d}x/\mathrm{d}t + \omega_0^2 x = 0. \tag{3.2}$$

Similar to system (2.1), one must provide initial conditions $x(0) = x_0$ and $\mathrm{d}x(0)/\mathrm{d}t = v_0$ to specify an orbit of the two-dimensional system uniquely. An analytic solution to equation (3.2) reads as

$$\begin{aligned} x(t) &= \left[x_0 \cdot \cos \omega t + \tfrac{v_0+\delta \cdot x_0}{\omega} \sin \omega t\right] \mathrm{e}^{-\delta \cdot t}, \\ v(t) &= \left[v_0 \cdot \cos \omega t - \tfrac{\delta \cdot v_0+\omega_0^2 x_0}{\omega} \sin \omega t\right] \mathrm{e}^{-\delta \cdot t}, \end{aligned} \tag{3.3}$$

where $\omega = \sqrt{\omega_0^2 - \delta^2}$ and $v(t) = \mathrm{d}x(t)/\mathrm{d}t$. The formula (3.3) determines the relationship between an initial state x_0, v_0 and a future state $x(t)$, $v(t)$. Thus, it gives explicitly an evolution operator of the system (3.2).

Another way to write down evolution equations for the same system is a set of two first-order ordinary differential equations:

$$\begin{aligned} \mathrm{d}x_1/\mathrm{d}t &= x_2, \\ \mathrm{d}x_2/\mathrm{d}t &= -2\delta \cdot x_2 - \omega_0^2 x_1, \end{aligned} \tag{3.4}$$

where $x_1 = x$, $x_2 = \mathrm{d}x/\mathrm{d}t$. It is convenient for graphical representations, since it specifies explicitly a velocity field on the phase plane (Fig. 3.2a). Roughly speaking, one can move from an initial state to subsequent ones by doing small steps in the directions of arrows. It is realised in different algorithms for numerical solution of differential equations. To construct a discrete analogue of equation (3.4), one must convert to the discrete time $n = t/\Delta t$. In the simplest case, one can approximately replace the derivatives with finite differences $\mathrm{d}x(t)/\mathrm{d}t \approx (x_{n+1} - x_n)/\Delta t$ and $\mathrm{d}v(t)/\mathrm{d}t \approx (v_{n+1} - v_n)/\Delta t$ and get difference equations which can be rewritten in the form of a two-dimensional discrete map

$$\begin{aligned} x_{n+1} &= x_n + v_n \Delta t, \\ v_{n+1} &= v_n(1 - 2\delta \cdot \Delta t) - \omega_0^2 x_n \Delta t. \end{aligned} \tag{3.5}$$

At sufficiently small Δt, an orbit of the map approximates well a solution to equation (3.4), i.e. the map (3.5) is a sufficiently *accurate difference scheme*.

In a Poincare section of the phase plane with a straight line $x_1 = 0$ (an ordinate axis), it is possible to establish the relationship between subsequent "punctures" of the axis by a phase orbit (Fig. 3.2). The resulting Poincare map takes the form

$$v_{n+1} = v_n \mathrm{e}^{-\delta \cdot T}, \tag{3.6}$$

where $T = 2\pi/\omega$. One can get more detailed information about a modelled object motion from the map (3.5) than from the map (3.6), since the latter describes only the decay of an amplitude. On the other hand, one can use vivid Lamerey's diagram in the one-dimensional case. To construct that diagram on the plane v_n, v_{n+1}, one passes a vertical straight line to the plot of the map, then a horizontal line to the diagonal $v_n = v_{n+1}$, etc., as shown in Fig. 3.2c.

3.2 Systematisation of Model Equations

Mathematicians have developed a rich arsenal of tools for dynamical description of motions. Here, we present their systematisations according to different principles. Firstly, we consider descriptive capabilities in application to objects with various complexity of their spatial structure. Any real-world object is somewhat "spatially extended". Depending on the number and sizes of composing elements, intensity and speed of their interaction, one can model an object as *concentrated* at a single spatial point or at several ones. The latter is the simplest kind of "spatially extended" configuration. A "completely smeared" (continuous) spatial distribution of an object characteristic is also possible. Such an object is also called a "distributed system". Further, we use the term "spatially extended system" more often, since it is more general.

If an object is characterised by a uniform spatial distribution of variables and one can consider only their temporal variations, it is regarded as *concentrated* at a single spatial point. Such a representation is appropriate if a perturbation at a certain spatial point reaches other parts of a system in a time interval much less than time scales of the processes under consideration. In the language of the theory of oscillations and waves, a perturbation wavelength is much greater than the size of an object. Concentrated systems are described with finite-dimensional models such as difference or ordinary differential equations.

If one has to provide a continuous set of values to specify a system state uniquely, then the system is distributed. Classical tools to model such a system are partial differential equations (PDEs), integro-differential equations (IDEs) and delay differential equations (DDEs). For instance, in description of a fluid motion, one refuses consideration of the molecular structure. The properties are regarded uniformly "smeared" within "elementary volumes" which are sufficiently big as compared with a molecule size, but small as compared with macro-scales of a system. This is the so-called *mesoscopic level*.[3] Such "volumes" play a role of elementary particles whose properties vary in space and time according to the Navier–Stokes equations. These famous partial differential equations represent an etalon infinite-dimensional model in hydrodynamics.

[3] It is intermediate between a microscopic level, when one studies elements of a system separately (e.g. molecules of a fluid), and a macroscopic one, when an entire system is considered as a whole (e.g. in terms of some averaged characteristics).

Spatially extended systems can be thought of as separated into parts (elements). Each of the parts is a system concentrated at a certain spatial point. Models of such systems are typically multidimensional. One often uses PDEs or a set of coupled maps or ODEs. Depending on the intensity of coupling between elements, a model dimension required for the description of motion and relevant mathematical tools can vary significantly. Thus, if a liquid freezes, one does no longer need PDEs to describe motion of a resulting ice floe and is satisfied with a set of several ODEs for rotational and progressive motions of a solid. If only progressive motions take place, then even a model of a material point suffices.

When a signal with sufficiently broad power spectrum (Sect. 6.4.2), e.g. a short pulse, propagates in a system, variations in its power spectrum and phase shifts at some frequencies may induce a time delay and smearing of the signal. Smearing occurs if a system bandwidth is insufficient to pass all components of a signal, e.g. due to sluggishness. Thus, if one uses a δ-function input, sluggishness of a system leads to a finite width of a response signal waveform. The stronger the sluggishness, the wider the response waveform. A shift of the time instant when a response signal appears relative to the time instant of an input pulse is an estimate of the delay time (Fig. 6.4c). Both sluggishness and delay are often modelled with finite-dimensional models, but the phenomenon of time delay is more naturally described with a DDE. The latter is an infinite-dimensional system, since it requires an *initial curve* over a time-delay interval as an initial state, i.e. a continuous set of values of a dynamical variable.

In Fig. 3.3, mathematical tools for modelling of temporal evolution are systematised according to their *level of generality*, their capability to describe more diverse objects and kinds of motion. As a rule, model equations of greater generality require greater computational efforts for their investigation.

The simplest kind of models is *explicit functions of time* $\mathbf{x} = \mathbf{F}(t)$. In linear problems or special cases, such models can be obtained as *analytic* solutions to evolution equations. Despite an enormous number of functions used in practice (Sect. 3.3), their capabilities for the description of complex (especially, chaotic) time realisations are quite restricted. A somewhat more general case is represented by algebraic or transcendental equations

$$\mathbf{F}(\mathbf{x}, t) = 0. \tag{3.7}$$

If equation (3.7) has no analytic solution, then one says that it defines a dependency $\mathbf{x}(t)$ implicitly.

A "left column" of the scheme consists of various differential equations (DEs). These are equations involving derivatives of dynamical variables in respect of independent variables (time t and spatial coordinates $\mathbf{r}$). For instance, a general first-order DE reads as

$$\mathbf{F}(\mathbf{x}(t, \mathbf{r}), \partial\mathbf{x}(t, \mathbf{r})/\partial t,\ \partial\mathbf{x}(t, \mathbf{r})/\partial\mathbf{r}, t, \mathbf{r,c}) = 0, \tag{3.8}$$

where $\mathbf{x}$ is a vector of dynamical variables. ODEs were the first differential equations used in scientific practice

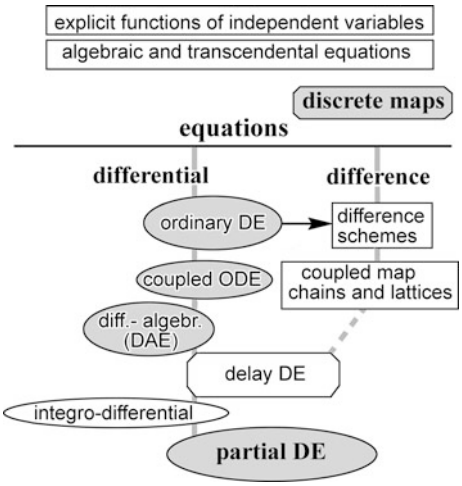

Fig. 3.3 A conventional scheme of dynamical model kinds extending the scheme given in Horbelt (2001). Descriptive capabilities and computational efforts required for investigation increase from top to bottom

$$\mathbf{F}(\mathbf{x}(t), \mathrm{d}\mathbf{x}(t)/\mathrm{d}t, \ldots, \mathrm{d}^n\mathbf{x}(t)/\mathrm{d}t^n, t, \mathbf{c}) = 0. \tag{3.9}$$

ODEs of the form $\mathrm{d}\mathbf{x}/\mathrm{d}t = \mathbf{F}(\mathbf{x},\mathbf{c})$ allow a clear geometric interpretation. They specify *velocity field*: a direction and an absolute value of a state change velocity $\mathbf{v} = \mathrm{d}\mathbf{x}/\mathrm{d}t$ at each point of a finite-dimensional phase space. A non-zero vector $\mathbf{v}$ is tangent to a phase orbit at any point. Specification of the velocity field provides a unique prediction of a phase orbit starting from any initial state, i.e. description of all possible motions in the phase space (Fig. 3.2a).

Derivatives of dynamical variables are used in equations of several kinds which differ essentially in the properties of their solutions and the methods of getting the solutions. They are united with a wide vertical line in Fig. 3.3 as branches with a tree stem. ODEs located at the top of a "stem" describe dynamics of concentrated (finite-dimensional) systems, where one does not need to take into account *continuous spatial distribution* of object properties. PDEs also involve spatial coordinates as independent variables and are located at the very bottom of the scheme. They are the most general tool, since they also describe infinite-dimensional motions of spatially distributed systems. However, solving PDEs requires much greater computational efforts compared to solving ODEs. Besides, PDEs loose a vivid geometric interpretation peculiar to ODEs.

Differential algebraic equations (DAEs) are just a union of ODEs and algebraic equations:

$$\begin{aligned} &\mathbf{F}(\mathrm{d}\mathbf{x}(t)/\mathrm{d}t,\ \mathbf{x}(t), \mathbf{y}(t), t, \mathbf{c}) = 0, \\ &\mathbf{G}(\mathbf{x}(t), \mathbf{y}(t), t, \mathbf{c}) = 0, \end{aligned} \tag{3.10}$$

where $\mathbf{x}$ is a D-dimensional state vector, $\mathbf{y}$ is a K-dimensional vector which does not add new degrees of freedom, $\mathbf{F}$ is a vector-valued function of dimension D, $\mathbf{G}$ is a vector-valued function of dimension K. The second equation is algebraic and determines (implicitly) a dependence of $\mathbf{y}(t)$ on $\mathbf{x}(t)$. Methods to solve such equations are very similar to those for ODEs.

Delay differential equations can, for instance, read as

$$\mathbf{F}(\mathbf{x}(t), \mathrm{d}\mathbf{x}(t)/\mathrm{d}t, \mathbf{x}(t-\tau), \mathbf{c}) = 0. \tag{3.11}$$

Distinction from ODEs consists in that the values of dynamical variables at a separated past time instant enter the equations along with their current values. ODEs can be regarded as a particular case of DDEs for a zero time delay τ.

Integro-differential equations (IDEs) do not, strictly speaking, belong to the class of DEs. Along with derivatives, they involve integrals of dynamical variables, e.g., as

$$\mathbf{F}\left(\mathbf{x}(t), \mathrm{d}\mathbf{x}(t)/\mathrm{d}t, \ldots, \mathrm{d}^n\mathbf{x}(t)/\mathrm{d}t^n, \int_{-\infty}^{\infty} k(t,t')\mathbf{x}(t')\mathrm{d}t', t, \mathbf{c}\right) = 0, \tag{3.12}$$

where $k(t,t')$ is a kernel of the linear integral transform. If no derivatives enter an IDE, it is called just an integral equation.

DDEs and IDEs also provide an infinite-dimensional description. DDEs can often be considered as a particular case of IDEs. For instance, an IDE $\mathrm{d}\mathbf{x}(t)/\mathrm{d}t = \mathbf{F}(\mathbf{x}(t)) + \int_{-\infty}^{\infty} k(t,t')\mathbf{x}(t')\mathrm{d}t'$ in the case of $k(t,t') = \delta(t-t'-\tau)$ turns into a DDE $\mathrm{d}\mathbf{x}(t)/\mathrm{d}t = \mathbf{F}(\mathbf{x}(t)) + \mathbf{x}(t-\tau)$.

To construct a discrete analogue of equation (3.4), one turns to the discrete time $n = t/\Delta t$ and finite differences. At sufficiently small Δt, the difference equation (3.5) have a solution close to that of equation (3.4). With increase in Δt, a difference equation (a discrete map) stops to reflect properties of the original ODEs properly. However, one can construct discrete models exhibiting good correspondence to the original system for large time steps as well. In the example of an oscillator considered above (Fig. 3.2), subsequent values v_n corresponding to the marked points in Fig. 3.2a, b are related at $\Delta t = T$ strictly via the one-dimensional map (3.6) (Fig. 3.2c). The latter map has a dimension smaller than the dimension of the original system and reflects only the monotonous decay of an oscillation amplitude and the transition to an equilibrium state. Here, the loss of information about a system behaviour between observation instants is a payment for a model simplicity.

Both discrete and continuous systems are valuable by themselves so that one could avoid speaking of any priorities. However, modelling practice and recognition of specialists are historically in favour of DEs. It is due to the fact that until the middle of twentieth century, physics was a "scientific prime" and relied mainly on DEs, in particular, on PDEs. To study them, physicists used various analytic

techniques. Computers and digital methods, which can now efficiently cope with difference equations, were not yet widely available. Therefore, an arsenal of maps used in modelling was much poorer at that time, than a collection of exemplary flows. However, contemporary tendencies of wider usage of non-linear equations and development of numerical techniques for investigation of multidimensional systems with complex spatial and temporal behaviours seem favourable to the progress of discrete approaches. Currently, popular tools are discrete *ensemble models* called *coupled maps lattices*, which combine a large number of maps with non-trivial temporal dynamics (Sect. 3.7). As models of spatially extended systems, they yield to PDEs in generality but are much simpler for numerical investigation. A specific kind of multidimensional maps or ODEs is represented by artificial neural networks which have recently become a widespread tool, in particular, in the field of function approximation (Sect. 3.8).

3.3 Explicit Functional Dependencies

Dynamical models of evolution in the form of *explicit functions* of time $\mathbf{x} = \mathbf{F}(t)$ can be specified analytically, graphically or as tables and can be obtained in any of the ways described in Sect. 1.5, e.g. by solving a DE or approximating experimental data (Sect. 7.2.1). It is impossible to list all explicit functions used by mathematicians. Yet, it is possible to distinguish some classes of functions. A practically important class of *elementary functions* includes algebraic polynomials, power, rational, exponential, trigonometric and inverse trigonometric functions. As well, it includes functions obtained via a finite number of arithmetical operations and compositions[4] of the listed ones. Let us consider several elementary functions and DEs, whose solutions they represent.

(1) Linear function $x(t) = x_0 + v_0 t$ is a solution to an equation

$$\mathrm{d}x/\mathrm{d}t = v_0, \tag{3.13}$$

which describes a progressive motion with a constant velocity v_0 and an initial condition $x(0) = x_0$. Its plot is a straight line (Fig. 3.4a).

(2) Algebraic polynomial of an order K reads as

$$x(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_K t^K, \tag{3.14}$$

[4] "Superposition (composition) of functions is arranging a composite function (function of function) from two functions" (Mathematical dictionary, 1988). Here, the terms "superposition" and "composition" are synonyms. However, physicists often call superposition of functions f_1 and f_2 their linear combination $af_1 + bf_2$, where a and b are constants. Then, the meanings of the terms "superposition" and "composition" become different. To avoid misunderstanding, we use only the term "composition" in application to composite functions.

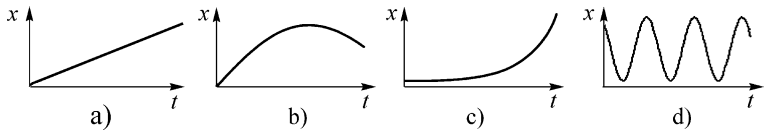

Fig. 3.4 Plots of some elementary functions: (**a**) linear function; (**b**) power function; (**c**) exponential function with $\alpha > 0$; (**d**) sinusoid

where c_i are constant coefficients. It is a solution to an equation $\mathrm{d}^K x/\mathrm{d}t^K = \text{const}$. A linear function is a particular case of equation (3.14) for $K = 1$. In the case of uniformly accelerated motion of a body thrown up from a height h with an initial velocity v_0, an equation of motion obtained from the Newton's second law and the law of gravity takes the form $\mathrm{d}^2x/\mathrm{d}t^2 = -g/m$, where an x-axis is directed upward, m is the mass of a body, g is the gravitational acceleration. The solution is $x(t) = h + v_0 t - gt^2/2$ (Fig. 3.4b). It is valid in a friction-free case and until a body falls down on a land.

(3) Fractional rational function is a ratio of two algebraic polynomials $x(t) = P(t)/Q(t)$. Its particular case for $Q(t) = \text{const}$ is an algebraic polynomial.

(4) Power function $x(t) = t^\alpha$, where α is an arbitrary real number. If α is a non-integer, only the domain $t > 0$ is considered. For an integer α, it is a particular case of an algebraic polynomial or a fractional rational function.

(5) Exponential function $x(t) = x_0 e^{\alpha t}$ (Fig. 3.4c) is famous due to the property that the speed of its change at a given point t is proportional to its value at the same point. It is the solution to the equation $\mathrm{d}x/\mathrm{d}t = \alpha x$ with an initial condition $x(0) = x_0$, which describes, for instance, dynamics of a biological population, where α is a constant parameter meaning birth rate.[5]

(6) A harmonic function $x(t) = x_0 \cos(\omega t + \phi_0)$ is one of the trigonometric functions (Fig. 3.4d). It is a solution to an equation of the harmonic oscillator $\mathrm{d}^2x/\mathrm{d}t^2 + \omega^2 x = 0$, which is an *exemplary model* of friction-free oscillations of a material point under the influence of a restoring force, proportional to a deviation x from an equilibrium. Its constant parameters are an amplitude of oscillations x_0, angular frequency ω and an initial phase ϕ_0. A bivariate harmonic function $x(t, r) = x_0 \cos(\omega t - kr + \phi_0)$ describes a monochromatic wave of length $\lambda = 2\pi/k$ travelling along the r-axis, which is a solution to the simple wave equation $\partial x/\partial t + V\ \partial x/\partial r = 0$.

Wide usage of trigonometric functions is to a significant extent due to the fact that according to *Weierstrass' theorem,* any continuous periodic function $x(t)$ can be arbitrarily accurately approximated with a *trigonometric polynomial*

[5] Exponential rise of a population observed at $\alpha > 0$ is called the Malthusian rise, since a catholic monk Malthus in the sixteenth century was the first who got this result. It is valid until population gets too large so that there is no longer enough food for everybody.

$$x(t) = \sum_{i=0}^{K} c_k \cos(2\pi k/T + \phi_k), \tag{3.15}$$

where K is a polynomial order. A non-periodic function can be approximated with such a polynomial over a finite interval.

An analogous theorem was proved by Weierstrass for the approximation of functions with an algebraic polynomial (3.14). Algebraic and trigonometric polynomials are often used for approximation of dependencies. This is the subject of the theory of approximation (constructive theory of functions), see Sect. 7.2. In the recent decades, artificial neural networks (Sects. 3.8 and 10.2.1), radial basis functions (Sect. 10.2.1) and wavelets compete with polynomials in the approximation practice. Wavelets have become quite popular and are considered in more detail in Sect. 6.4.2. Here, we just note that they are well-localised functions with zero mean, e.g. $x(t) = \mathrm{e}^{-t^2/2} - (1/2)\mathrm{e}^{-t^2/8}$.

We will consider non-elementary functions and extensions to the class of elementary functions in Sect. 3.5.

3.4 Linearity and Non-linearity

"Nonlinearity is omnipresent, many-sided and inexhaustibly diverse. It is everywhere, in large and small, in phenomena fleeting and lasting for epochs... Nonlinearity is a capacious concept with many tinges and gradations. Nonlinearity of an effect or a phenomenon means one thing, while nonlinearity of a theory means something different" (Danilov, 1982).

3.4.1 Linearity and Non-linearity of Functions and Equations

The word "linear" at a sensory level is close to "rectilinear". It is associated with a straight line, proportional variations of a cause and an effect, a permanent course, as in Fig. 3.4a. However, according to the terminology used in mathematics and non-linear dynamics, all the dynamical systems mentioned in Sect. 3.3 are linear though the plots of their solutions are by no means straight lines (Fig. 3.4b–d). Evolution operators of those dynamical systems (i.e. differential or difference equations and discrete maps) are linear *rather than their solutions* (i.e. functions of time representing time realisations).

What is common in all the evolution equations presented in Sect. 3.3? All of them obey the *superposition principle*: If functions $x_1(t)$ and $x_2(t)$ of an independent variable t are solutions to an equation, then their linear combination $ax_1(t) + bx_2(t)$ is also a solution, i.e. being substituted instead of $x(t)$, it turns an equation into identity. Only the first powers of a dynamical variable and its derivatives (x, $\mathrm{d}x/\mathrm{d}t$, ..., $\mathrm{d}^n x/\mathrm{d}t^n$) may enter a linear DE. No higher powers and products of the derivatives may be present. Accordingly, linear difference equations

may include only the first powers of finite differences or a dynamical variable values at discrete-time instants. Equations of any kind are linear if their right-hand and left-hand sides are linear functions of a dynamical variable and its derivatives. Violation of this property means non-linearity of an equation. For instance, equations (3.2), (3.4), (3.5), (3.6) and (3.13) are linear, while equation (2.1) is non-linear. However, linear non-autonomous (involving an explicit time dependence) equations may include non-linear functions of an independent variable (time), e.g. a non-autonomous linear oscillator reads as $\mathrm{d}^2x/\mathrm{d}t^2 + 2\delta \mathrm{d}x/\mathrm{d}t + \omega_0^2 x = A\ \cos\ \omega t$.

A linear function "behaves" quite simply: it monotonously decreases or increases with an argument or remains constant. But linearity of a dynamical system does not mean that its motion is inevitably primitive, which can be seen even from several examples presented in Fig. 3.4. Taking into account the superposition principle, one may find a solution for a multidimensional linear equation as a combination of power, exponential and trigonometric functions (each of them being a solution) which demonstrates quite a complicated temporal dependence, indistinguishable in its outward appearance from an irregular, chaotic behaviour over a finite time interval. However, linear systems "cannot afford many things": changes in a waveform determined by the arousal of higher harmonics, dynamical chaos (irregular solutions with exponential sensitivity to small perturbations), multistability (coexistence of several kinds of established motions), etc.

Systems, processes, effects, phenomena are classified as *linear* or *non-linear* depending on whether they are adequately described with linear *equations* or non-linear ones. The world of non-linear operators is far richer than that of linear ones. Further, there are much more kinds of behaviour in non-linear dynamical systems. The place of "linear things" in a predominantly non-linear environment, "particularly" of linear representations, follows already from the fact that non-linear systems can be linearised (i.e. reduced to linear ones) only for low oscillation amplitudes. For that, one replaces dynamical variables x_k in the equations by the sums of their stationary and variable parts as $x_k = x_{0,k} + \tilde{x}_k (x_{0,k} >> \tilde{x}_k)$ and neglects small terms (higher powers of $\tilde{x}_k$, their products, etc.).

Historically, linear equations in a precomputer epoch had incontestable advantage over non-linear ones in scientific practice, since the former could be investigated rigorously and solved analytically. For a long time, one confidently thought that linear phenomena are more important and widespread in nature and linear approaches are all-sufficient (see discussion in Danilov, 1982). Development of computers, numerical techniques for solving non-linear equations and tools for their graphical representation along with the emergence of contemporary mathematical conceptions, including dynamical chaos theory, shifted an attitude of scientists more in favour of non-linear methods and ideas. At that, the linear viewpoint is regarded an important but a special case.

3.4.2 The Nature of Non-linearity

Non-linearity is natural and organically inherent in the world where we live. Its origin can be different and determined by specific properties of objects. One should

speak of conditions for linearity to be observed, rather than for non-linearity. However, according to existing traditions one often explains appearance of non-linearity by a competition between eigenmodes of a linearised system or by a parameter dependence on a dynamical variable. The latter dependence is often vitally necessary and can be realised via a feedback. If sensitivity of vision or hearing organs were constant, independent of an influence level (luminosity or sound volume), one might not successfully orient oneself in gloom and avoid becoming blind during a sunny day, hear a rustle of a creeping snake and avoid getting deaf from thunder. "Biological systems which could not capture enormous range of vitally important influences from environment have just died out loosing the struggle for existence. One could write down on their graves: They were too linear for this world" (Danilov, 1982).

Thus, if a coefficient of reproduction k for a population were constant, independent of the number of individuals x_n (n is discrete time), then at $k > 1$ one would observe its unbounded rise with time according to a linear evolution law:

$$x_{n+1} = kx_n. \tag{3.16}$$

In such a case, overpopulation would be inevitable, while at $k < 1$ a total disappearance of a population would come. A more realistic is a dependence of the parameter k on the variable x_n, e.g. $k = r(1 - x_n)$ leading to non-linearity of an evolution operator $x_{n+1} = rx_n(1 - x_n)$. Non-trivial properties of this exemplary one-dimensional dynamical system called the logistic map, including its chaotic behaviour, are well studied (see also Sect. 3.6.2).

3.4.3 Illustration with Pendulums

Widely accepted exemplary objects for illustrations of linear and non-linear oscillatory phenomena are *pendulums*, i.e. systems oscillating near a stable equilibrium state. Their simplest mechanical representatives are a massive load suspended with a thread or a rod (Fig. 3.5a), a load on a spring (Fig. 3.5c), a ball rolling in a pit, a bottle swimming in water, liquid in a U-shaped vessel, and many others. An electric pendulum is the name for a circuit consisting of a capacitor and inductance, an oscillatory circuit (Fig. 3.5b). One speaks of a chemical pendulum (mixture of chemicals reacting in an oscillatory manner) and an ecological pendulum (two interacting populations of predators and preys) (Trubetskov, 1997).

A free real-world pendulum reaches finally a stable equilibrium state (free motions, Fig. 3.5). Depending on initial conditions (a deviation from a stable equilibrium state x and a velocity $\mathrm{d}x/\mathrm{d}t$) and object properties, different motions may precede it. In Fig. 3.5 we illustrate two areas of qualitatively different motions: the left one corresponds to relatively large x, when non-linearity is essential, while the right one corresponds to small, "linear" ones. Time realisations of low-amplitude oscillations are identical for all the pendulums considered. The oscillations are *isochronous*, i.e. their quasi-period T_1 does not depend on a current state. They represent a decaying sinusoid which is a solution to the linear

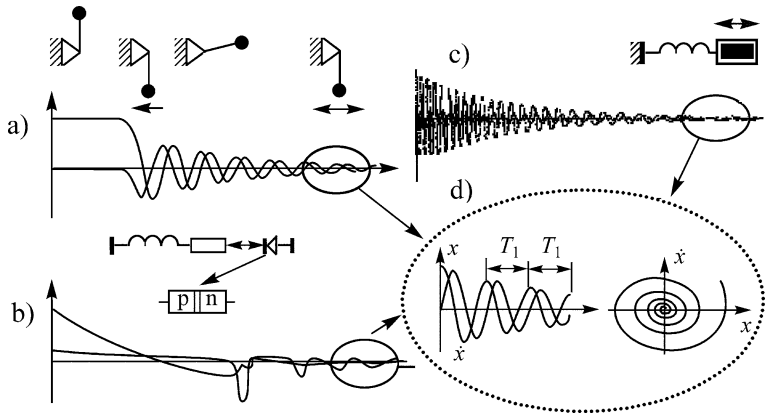

Fig. 3.5 Qualitative outlook of time realisations of x and $\mathrm{d}x/\mathrm{d}t$ for different pendulums: (**a**) a load on a rod; (**b**) an oscillatory circuit with a diode; (**c**) a load on a spring; (**d**) a linear stage of oscillations (the same for all the examples) is magnified and the corresponding phase portrait on the plane $(x, \mathrm{d}x/\mathrm{d}t)$ is shown. A representative point moves along an intertwining spiral to an attractor, i.e. to a stable equilibrium point at the origin. Phase portraits of the pendulums are more complicated and diverse in a region of large values of coordinates and velocities

equation (3.2) describing low-amplitude oscillations of all the systems considered up to the coefficients of proportionality. This circumstance is a legal reason to call those oscillations linear. Monotonous decay of oscillations can be modelled with the linear one-dimensional map $x_{n+1} = ax_n$, where $a = \exp(-\delta T_1) < 1$ (Fig. 3.2c).

As compared with a common "standard" for linear oscillations (3.2), types of non-linear behaviour are quite diverse and determined by the properties of each concrete pendulum. Thus, a character of non-linear behaviour differs essentially for the three examples in Fig. 3.5, while their linear stages are identical (Fig. 3.5d). This is related to the peculiarities of each pendulum and to the kinds of their non-linearity (dependency of the parameters on the dynamical variables). For instance, a load on a rod (Fig. 3.5a) exhibits non-linearity due to sinusoidal dependence of a gravitational force moment about a rotation axis on a rotation angle. In an electric pendulum with a semiconductor capacitor (a varactor diode, Fig. 3.5b), non-linearity is related to the properties of a $p - n$ junction, injection and finiteness of charge carrier lifetime. Non-linearity of a spring pendulum (Fig. 3.5c) is determined by the dependence of an elastic force on a spring deformation. For instance, spring coils close up under compression so that an elastic force rises abruptly as compared with the force expected from Hooke's law, i.e. the spring "gets harder". At that, a period of oscillations decreases with their amplitude. In analogy, non-linearity of any oscillator leading to decrease (increase) in a period with an amplitude is called *hard* (*soft*) *spring* non-linearity.

3.5 Models in the form of Ordinary Differential Equations

3.5.1 Kinds of Solutions

Emergence of ordinary differential equations and their history is related to the names of Newton and Leibniz (seventeenth to eighteenth centuries). Afterwards, general procedures to obtain model equations and to find their solutions were developed within analytic mechanics and the theory of differential equations. Here, we describe possible kinds of solutions following the review of Rapp et al. (1999).

3.5.1.1 Elementary Solutions

A solution to a differential equation in the form of an elementary function is called an elementary solution. We confine ourselves with examples from Sect. 3.3. In all of them functions–solutions give exhaustive information about a model dynamics. Interestingly, understanding the behaviour of a dynamical system at Newton's time was tantamount to writing down a formula for a solution $x = F(t)$. That approach even got the name of a *Newtonian paradigm* (Rapp et al., 1999). One spoke of a finite (preferably short) expression consisting of radicals (nth roots), fractional rational, exponential, logarithmic and trigonometric functions. All the solutions considered in Sect. 3.3 are of such a form.

The class of elementary functions (and elementary solutions) is often extended with *algebraic functions*, i.e. solutions to the algebraic equations

$$a_n(t)x^n(t) + a_{n-1}(t)x^{n-1}(t) + \ldots + a_1(t)x(t) + a_0(t) = 0, \tag{3.17}$$

where n is an integer, $a_i(t)$ are algebraic polynomials. All fractional rational functions and radicals are algebraic functions. The reverse is not true: algebraic functions can be defined by equation (3.17) implicitly.

3.5.1.2 Closed-Form Solutions

Not all differential equations have elementary solutions. There are elementary functions whose integrals are not elementary functions. One of the simplest examples is an elliptic integral

$$\int_0^t \frac{\mathrm{d}\tau}{\sqrt{1+\tau^4}}.$$

The integral exists but is not an elementary function. However, even if an integral of an elementary function is not elementary, one can efficiently cope with it by evaluating it approximately with the aid of available numerical methods.

Expression of a solution via formulas containing integrals of elementary functions is also regarded a complete solution to an equation, so-called integration in finite terms. Thus, a solution to the equation

$$\mathrm{d}x/\mathrm{d}t + \mathrm{e}^{t^2} x = 0 \tag{3.18}$$

given $x(0) = x_0$, reads

$$x(t) = x_0 \exp\left(-\int_0^t \mathrm{e}^{\tau^2} \mathrm{d}\tau \right).$$

Such a result is called a *closed-form solution*. An elementary solution is its particular case.

Liouville showed that some DEs have no closed-form solutions. For instance, an equation

$$\mathrm{d}x/\mathrm{d}t + x^2 = t \tag{3.19}$$

which at the first glance seems very simple, cannot be solved in finite terms. A solution exists but cannot be expressed in a closed form. There is no general procedure to get closed-form solutions, though there are many special techniques. In practice, it is often very difficult or even impossible to obtain a closed-form solution.

3.5.1.3 Analytic Solutions

When a closed-form solution is lacking, one can further complicate a technique and try to find a solution in the form of an infinite power series. For instance, let us search for a solution to an equation

$$\mathrm{d}^2x/\mathrm{d}t^2 - 2t\,\mathrm{d}x/\mathrm{d}t - 2x = 0 \tag{3.20}$$

in the form

$$x(t) = a_0 + a_1 t + a_2 t^2 + \ldots = \sum_{i=0}^{\infty} a_i t^i. \tag{3.21}$$

Let us substitute the latter formula into the original equation and combine the terms with the same powers of t. Each such combination must be equal to zero. Finally, one gets the following recurrent relationship for the coefficients: $a_{n+2} = 2a_n/(n+2)$. The coefficients a_0 and a_1 are determined by initial conditions. Thus, for $a_0 = 1$ and $a_1 = 0$, one gets

$$x(t) = 1 + t^2 + t^4/2! + t^6/3! + \ldots\ . \tag{3.22}$$

In this particular case, one gets a Taylor expansion for the function $x(t) = e^{t^2}$ as the answer. If an obtained power series converges, which is not always the case, and one has derived a formula for its coefficients, then such a solution is called an *analytic solution* or a series solution. It is the second-best to the closed-form solution. If an obtained series converges slowly, then its practical application is unfeasible. In particular, such a situation takes place in a famous three-body problem which has a practically useless analytic solution in the form of a very slowly converging series (Wang, 1991).

3.5.1.4 Numerical solutions

Above considered equations with explicit time dependence and elementary non-linearities are relatively simple. In a general case, when a problem cannot be reduced to a linear one or to a certain specific class of equations, one searches for an approximate solution with numerical techniques, given initial and/or boundary conditions. The oldest and simplest one is the Euler technique. However, more accurate and complicated modern methods rely to a significant extent on the same idea. In particular, Runge–Kutta techniques are very popular. Adams integrator and Bulirsch and Stoer technique have their own advantages and shortcomings, they are often superior to Runge–Kutta techniques in terms of both computation time and accuracy (Kalitkin, 1978; Press et al., 1988; Samarsky, 1982).

According to the above-mentioned Newton's paradigm, a numerical solution was by no means satisfactory since it did not allow understanding qualitative features of dynamics and could be useful only for the prediction of future behaviour. The viewpoint changed since efficient computers and rich computer graphical tools arose, which currently allows one both to get qualitative ideas about a model behaviour and to compute a sufficiently accurate approximate solution. Since one can now investigate numerically a very broad class of non-linear equations, researchers pay more attention to the problem of how to get a model DE.

The use of any of the four ways mentioned in Sect. 1.5 is possible for that. Still, the most popular method is a way *from general to particular* since majority of known physical laws take the form of DEs. Besides, the entire apparatus of DEs was created to describe basic mechanical phenomena. Most of the models considered by physicists are asymptotic ones; they are obtained via restrictions imposed on universal formulas by a specific problem. Sometimes, one says that a model is obtained from "the *first principles*" implying some general relationships for a considered range of phenomena, from which one deduces concrete models (yet, such a use of the term "first principles" is criticised from a philosophical viewpoint). These are conservation laws and Newton's laws in mechanics, continuity equations and Navier–Stokes equations in hydrodynamics, Maxwell's equations in electrodynamics, derived special rules like Kirchhoff's laws in the theory of electric circuits, etc. Many non-standard examples of an asymptotic modelling of important physical and biological objects are given by a mathematician Yu.I. Neimark (1994–1997).

Modelling *from simple to complex*, e.g. creation of ensembles, is also typical when DEs are used. It is widely exploited in the description of spatially extended

systems. The greater the number of elements included into an ensemble, the wider the class of phenomena covered by the model. Classical models are ensembles of coupled oscillators which represent an approved way of sequential complication of phenomena under consideration in tutorials on the theory of oscillations. An *empirical approach* to obtaining model DEs (reconstruction from time series) is considered in Part II.

3.5.2 Oscillators, a Popular Class of Model Equations

To illustrate possibilities of DE-based models, we select again the class of oscillators. Why is our choice from an "ocean" of models so monotonous? The point is that any number of diverse examples cannot capture all specific descriptive capabilities of DEs. Thus, any example would give just a fragment of a general picture while really general things would be lacking. Therefore, it is reasonable to consider an example whose prehistory and some basic properties are known to a wide audience. Many people have met oscillators for the first time already at the lessons of school physics.

One calls "oscillators" both *objects* capable of oscillating about an equilibrium state and *equations* modelling such motions. Motion of an oscillator occurs within some potential profile either with friction or without it. An etalon *oscillator equation* is a second-order DE

$$\mathrm{d}^2x/\mathrm{d}t^2 + \gamma(x, \mathrm{d}x/\mathrm{d}t)\mathrm{d}x/\mathrm{d}t + f(x) = F(t), \tag{3.23}$$

where the second term on the left-hand side corresponds to dissipation (friction forces), the third term is determined by a potential U (a restoring force is $-\partial U/\partial x = -f(x)$) and the right-hand side represents an external force. A number of research papers, reviews and dissertations are devoted to different kinds of oscillators (Scheffczyk et al., 1991; http://sgtnd.narod.ru/eng/index.htm).

Linear oscillators correspond to the case of $\gamma = \text{const}$ and $f(x) = \omega_0^2 x$. The latter means the quadratic potential $U(x) \sim x^2$. An autonomous oscillator ($F = 0$) is a two-dimensional ($D = 2$) dynamical system. It demonstrates either decaying ($\gamma > 0$, a dissipative oscillator) or diverging ($\gamma < 0$) oscillations. The autonomous dissipative oscillator has a stable fixed point as an attractor. Dimension of this attractor is zero and both Lyapunov exponents are negative. This is one of the simplest dynamical systems in terms of possible kinds of behaviour.

Under a periodic external driving, the dissipative oscillator can be rewritten as a three-dimensional ($D = 3$) autonomous dynamical system (Sect. 3.5.3). It exhibits periodic oscillations with a period of the external force, i.e. has a limit cycle as an attractor in a three-dimensional phase space, and demonstrates a phenomenon of resonance. The dimension of the attractor is one and the largest Lyapunov exponent is equal to zero. Thus, the driven linear dissipative oscillator represents a more complex dynamical system compared to the autonomous one.

If a dissipative term involves non-linearity, e.g. like in *van der Pol equation*

$$\mathrm{d}^2x/\mathrm{d}t^2 - \alpha(1 - \beta x^2)\mathrm{d}x/\mathrm{d}t + \omega_0^2 x = 0, \tag{3.24}$$

then the oscillator becomes non-linear and is capable of demonstrating its own periodic oscillations (a regime of self-sustained oscillations). The system is two dimensional and its attractor is a limit cycle in a two-dimensional phase space for $\beta > 0$. The dimension of the attractor is then equal to one and its largest Lyapunov exponent is zero. In terms of the complexity of the dynamics (Lyapunov exponents and dimension of an attractor), the system (3.24) is more complex than the autonomous linear dissipative oscillator. In terms of the above dynamical characteristics, it exhibits at fixed values of α, β approximately the same complexity as the harmonically driven linear oscillator. However, the shape of time realisations can be more diverse for the system (3.24), depending on α, β. For instance, it exhibits almost sinusoidal waveform like the driven linear oscillator for small values of α, β and periodic "relaxation" oscillations, where the plot $x(t)$ resembles a saw, for big values of α, β.

In the non-autonomous case, the oscillator (3.24) exhibits much richer dynamics. Even harmonic driving may induce such kinds of behaviour as quasi-periodic oscillations, synchronisation of self-sustained oscillations by an external signal or even chaotic sets in the phase space if the driving amplitude is large enough. Thus, harmonically driven van der Pol oscillator is a considerably more complex system than the above linear oscillators and the autonomous van der Pol oscillator.

A non-quadratic potential profile $U(x)$ also means non-linearity of equation (3.23). Accordingly, its possible solutions get essentially more diverse. Even under a harmonic influence, a non-linear oscillator (which can be rewritten as a three-dimensional autonomous dynamical system) may exhibit a hierarchy of oscillatory regimes and non-linear phenomena including transition to chaos, multistability and hysteresis in a region of resonance. Thus, it can exhibit attractors with fractal dimensions greater than one and a positive largest Lyapunov exponent. Therefore, it is a more complex system than the linear oscillators or the autonomous van der Pol oscillator.

It is not straightforward to decide whether the driven van der Pol oscillator or the driven non-linear dissipative oscillator is more complex. The latter exhibits more diverse dynamical regimes than does the former due to different possible forms of $U(x)$ as described below, but in the case of strong dissipation, it cannot exhibit stable quasi-periodic regimes (where an attractor is a torus, the dimension is equal to two and two largest Lyapunov exponents are equal to zero) which are typical of the driven van der Pol oscillator.

The general non-autonomous oscillator (3.23) with arbitrary non-linear dissipation and arbitrary potential profile includes both the case of the driven non-linear dissipative oscillator and the driven van der Pol oscillator and, hence, may exhibit all the dynamical regimes mentioned above. Specific properties of different non-linear oscillators are determined by the concrete functions entering equation (3.23).

To select general features, one systematises oscillators in respect of (i) the dependency of a period of oscillations on their amplitude ("hard spring" and "soft spring") (Neimark and Landa, 1987; Scheffczyk et al., 1991; Thompson and Stewart, 2002), (ii) an order of the polynomial specifying the potential profile like in the theory of catastrophes (Kuznetsov and Potapova, 2000; Kuznetsova et al., 2004), etc. Below, we characterise complexity of the dynamics of several oscillators on the basis of their parameter space configurations.

When dissipation is too strong so that self-sustained oscillations are impossible, a parameter space of an oscillator is typically characterised by bifurcation sets called "cross-road area" (Carcasses et al., 1991; Mira and Carcasses, 1991) and "spring area" (Fig. 3.6). Cross-road area is a situation where domains of two cycles intersect, their boundaries representing period-doubling lines are stretched along boundaries of a "tongue" formed by saddle-node bifurcation lines and bistability takes place inside the tongue. Spring area is the case where a period-doubling line stretched along the above-mentioned tongue makes a characteristic turn around a "vertex" of the tongue, a point of the "cusp" catastrophe. Those universal configurations fill a parameter space in a self-similar manner (Parlitz, 1991; Schreiber, 1997). A fragment of a typical picture is seen already in Fig. 3.6: the right structure of the "spring area" (born on the basis of a double-period cycle) is built into an analogous upper structure (born on the basis of a "mother" cycle whose period is doubled when one moves to bottom along a parameter plane). A chart of regimes in Fig. 3.11 gives additional illustrations.

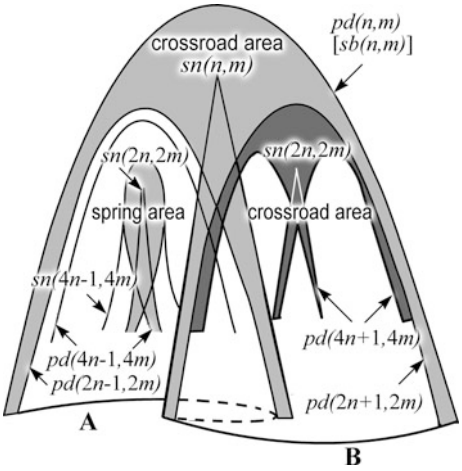

Fig. 3.6 A typical configuration of bifurcation lines "cross-road area" and "spring area" on a parameter plane. Domains of cycle stability are shown in greyscale. Lines of a saddle-node bifurcation are denoted as *sn*, lines of a period-doubling bifurcation are *pd*, lines of a symmetry-breaking bifurcation are *sb*. In parentheses, we show the number of an external force periods n and the number of the own periods m contained in a single period of a cycle loosing stability on a given line. A and B are conventional sheets used to illustrate bistability

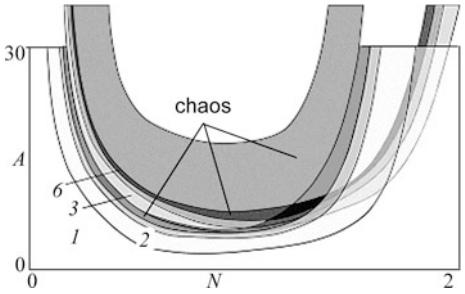

Fig. 3.7 Parameter plane A–N for the Toda oscillator (3.25): A is the driving amplitude, N is the normalised driving frequency. Greyscales show domains of existence of different regimes. Numbers indicate periods of oscillations in units of the driving period

Those self-similar typical configurations do not exhaust a diversity of possible bifurcation structures in parameter spaces of oscillators. For instance, oscillators with strong dissipation and a potential profile essentially different from the quadratic one exhibit a specific configuration of a domain where an arbitrary oscillatory regime exists and evolves to chaos. The domain has the form of a narrow bent strip resembling an "ear" (Fig. 3.7). An equation of the *Toda oscillator* driven by a sinusoidal external force demonstrating the described structure of the parameter space reads as

$$\mathrm{d}^2x/\mathrm{d}t^2 + \gamma\,\mathrm{d}x/\mathrm{d}t + \mathrm{e}^x - 1 = A\sin\omega t. \tag{3.25}$$

Let us denote the normalised frequency of driving $N = \omega/\omega_0$, where ω_0 is the frequency of low-amplitude free oscillations, $\omega_0 = 1$ for the system (3.25).

One more universal configuration of bifurcation sets for non-autonomous oscillators is presented in Sect. 3.6, where Fig. 3.9 illustrates a parameter plane of a circle map. It corresponds to a periodic driving applied to a system capable of exhibiting self-sustained oscillations. A universal configuration on the plane of driving parameters represents a hierarchy of so-called *Arnold's tongues*, i.e. domains where synchronisation takes place. Bifurcation lines inside a tongue exhibit the "cross-road area" structure.

As a solid residual from the current subsection, we stress (i) diversity of evolutionary phenomena which can be modelled with equations of oscillators and with DE-based models in general; (ii) complexity of observed pictures which can be systematised and interpreted in different ways, in particular, on the basis of typical "charts of dynamical regimes" and scaling properties (Kuznetsov and Kuznetsov, 1991, 1993b; Kuznetsov and Potapova, 2000; Neimark and Landa, 1987; Parlitz, 1991; Scheffczyk et al., 1991; http://sgtnd.narod.ru/eng/index.htm); (iii) an opportunity to observe different bifurcation sets and other specific features for different kinds of non-linearity.

3.5.3 "Standard form" of Ordinary Differential Equations

Despite multitude of ODE forms, the following one is the most popular among them and allows a clear geometrical interpretation:

$$\begin{aligned} \mathrm{d}x_1/\mathrm{d}t &= F_1(x_1, x_2, \ldots, x_n), \\ \mathrm{d}x_2/\mathrm{d}t &= F_2(x_1, x_2, \ldots, x_n), \\ &\ldots, \\ \mathrm{d}x_n/\mathrm{d}t &= F_n(x_1, x_2, \ldots, x_n). \end{aligned} \tag{3.26}$$

Any set of autonomous ODEs can be reduced to such a form, solved in respect of the highest derivatives. A system (3.26) via a change of variables (probably, at the expense of the dimension increase, i.e. $D > n$) can be rewritten in the form

$$\begin{aligned} \mathrm{d}y_1/\mathrm{d}t &= y_2, \\ \mathrm{d}y_2/\mathrm{d}t &= y_3, \\ &\ldots, \\ \mathrm{d}y_\mathrm{D}/\mathrm{d}t &= F(y_1, y_2, \ldots, y_\mathrm{D}), \end{aligned} \tag{3.27}$$

where y_1 is an arbitrary smooth function of the vector $\mathbf{x}$: $y_1 = h(x_1, x_2, \ldots, x_n)$, e.g. $y_1 = x_1$. Equation (3.27) is sometimes called *standard* (Gouesbet and Letellier, 1994; Gouesbet et al., 2003b). It is widely used in empirical modelling when a model state vector is reconstructed from a scalar observable via sequential differentiation (Sect. 10.2.2). However, it is not always possible to derive the function F in equation (3.27) explicitly. Possibility of reduction of any set of ODEs to the form (3.27) was proven by Dutch mathematician Floris Takens. Formulations of the theorems and some comments are given in Sect. 10.2.1 below.

A simple example is an equation of a dissipative oscillator under an additive harmonic driving:

$$\mathrm{d}^2x/\mathrm{d}t^2 + \gamma \mathrm{d}x/\mathrm{d}t + f(x) = A\ \cos\,(\omega t) \tag{3.28}$$

with $\gamma = \mathrm{const}$ which can be rewritten as a non-autonomous set of two equations:

$$\begin{aligned} \mathrm{d}x_1/\mathrm{d}t &= x_2, \\ \mathrm{d}x_2/\mathrm{d}t &= -\gamma x_2 - f(x_1) + A\ \cos\,(\omega t), \end{aligned} \tag{3.29}$$

where $x_1 = x$, or as a three-dimensional autonomous system

$$\begin{aligned} \mathrm{d}x_1/\mathrm{d}t &= x_2, \\ \mathrm{d}x_2/\mathrm{d}t &= -\gamma x_2 - f(x_1) + A \cdot \cos\ x_3, \\ \mathrm{d}x_3/\mathrm{d}t &= \omega, \end{aligned} \tag{3.30}$$

where $x_1 = x$ and $x_3 = \omega t$, or as a four-dimensional "standard" system

$$\begin{aligned} \mathrm{d}x_1/\mathrm{d}t &= x_2, \\ \mathrm{d}x_2/\mathrm{d}t &= x_3, \\ \mathrm{d}x_3/\mathrm{d}t &= x_4, \\ \mathrm{d}x_4/\mathrm{d}t &= -\gamma x_4 - \left(\frac{\mathrm{d}f(x_1)}{\mathrm{d}x_1} + \omega_2\right) x_3 - \frac{\mathrm{d}^2 f(x_1)}{\mathrm{d}x_1^2} x_2^2 - \omega^2 \gamma x_2 - \omega^2 f(x_1), \end{aligned} \tag{3.31}$$

where $x_1 = x$. To derive the formula (3.31), one differentiates twice equation (3.28) in respect of time and substitutes the left-hand side of (3.28) instead of $A\ \cos\ (\omega t)$ into an obtained equation. Apart from increasing number of dynamical variables (four ones instead of the two), the conversion leads to complication of the right-hand side of the last equation in (3.31) as compared with the original form (3.28). However, all the dynamical variables are related only to the variable x (they are its derivatives) that gives an advantage in the construction of such a model from a time realisation of x.

3.6 Models in the Form of Discrete Maps

3.6.1 Introduction

Similar to DE-based models, *discrete maps* represent a whole "stratum" of mathematical culture with its own history and specific features (Neimark, 1972). This section is a specialised introduction oriented to applications of discrete maps to empirical modelling.

Quite a widespread approach to obtain a model map is to approximate experimental data. In asymptotic modelling, maps are most often derived through a conversion from a DE to a finite-difference scheme or a Poincare section (Sect. 3.1). Creation of an ensemble of maps is a popular way to model a spatially extended system. Usually, such models take the form of coupled map chains and lattices with different coupling architectures: local coupling (only between "neighbours"), global coupling (all-to-all connections), random connections, networks with complex topology (Sect. 3.7.3), etc.

Simplicity of numerical investigation, diversity of dynamical regimes ranging from an equilibrium to chaos exhibited even by one-dimensional maps and the ease of constructing ensembles from simple basic elements have made discrete maps a dominating mathematical tool in non-linear dynamics. Currently, they are a full-fledged "competitor" of flows. Let us discuss several examples.

3.6.2 Exemplary Non-linear Maps

3.6.2.1 Piecewise Linear Maps

Seemingly, piecewise linear maps have the second simplest form after the linear map (3.16) capable of demonstrating simple rise or decay of a variable. Different kinds of such maps were studied, in particular:

(i) The "saw tooth" is the map $x_{n+1} = \{2x_n\}$, where braces denote fractional part of a number. Its plot is shown in Fig. 3.8a. The map is remarkable since it allows strictly and clearly prove an existence of dynamical chaos in simple non-linear systems. In a binary system, the map in a single iteration shifts a binary point one position to the right (Bernoulli's shift) and throws away an integer part of the resulting number. To illustrate irregularity and high sensitivity to small perturbations inherent in chaotic motions, let us specify an irrational number as an initial condition and write it down as an infinite non-periodic binary fraction $x_0 = 0.0100101010001010010001011010....$ Then, a sequence x_n generated by the map is also non-periodic: whether x_n belongs to the interval (0,0.5) or (0.5,1) is determined by the first figure after the decimal point which behaves according to the sequence of "0" and "1" in the binary fraction x_0. Moreover, a variation in any figure in the fraction x_0, even arbitrarily far from the decimal point (i.e. arbitrarily small), leads to a change in x_n of the order of 1 in a finite number of steps.
Thus, the saw tooth is an example of a one-dimensional system. Its chaotic "attractor" contains all irrational numbers; therefore, this is a set of full measure. Thus, its fractal dimension is equal to one. Its only Lyapunov exponent is positive: it equals ln 2 as can be readily shown. Hence, in terms of Lyapunov exponents, complexity of the dynamics is greater than that for the above-mentioned two- and three-dimensional continuous-time systems like the autonomous and harmonically driven linear oscillators and the autonomous van der Pol oscillator (Sect. 3.5) which cannot have attractors with positive Lyapunov exponents. In this respect, the saw tooth is as complex as driven non-linear oscillators in chaotic regimes. In terms of the attractor geometry, the saw tooth is simpler since its "attractor" does not exhibit any fractal structure.

(ii) Models of neurons. Modelling a neuron dynamics is a problem topical both in biophysics and non-linear dynamics (see, e.g., Izhikevich, 2000; Kazantsev, 2004; Kazantsev and Nekorkin, 2003; 2005; Kazantsev et al., 2005; Nekorkin et al., 2005) where one considers mainly ODE-based models. However, discrete map models are also developed in the last years, since investigation of their dynamics requires less computational efforts and extends possibilities of modelling large ensembles of coupled neurons. Simple model maps capable of generating "spikes" and "bursts" (i.e. short pulses and "packets" of pulses) have been suggested. A pioneering work considering a two-dimensional piecewise smooth map is Rulkov (2001). The piecewise linear map illustrated in

Fig. 3.8b (Andreev and Krasichkov, 2003) can also exhibit those dynamical features. Since it is two-dimensional, the plotted dependence x_{n+1} versus x_n is non-unique. The choice of a branch is governed by the second dynamical variable y (we do not present the entire cumbersome equations). Complexity of this model is greater than that of the saw tooth since it exhibits different periodic and chaotic regimes depending on the parameter values. The two-dimensional phase space of the neuron model map allows richer possibilities of dynamics than does the one-dimensional phase space of the saw tooth.

(iii) Maps for information storage and processing (Fig. 3.8c) illustrate a practical application to information recording with the use of the multitude of generated cycles (Andreev and Dmitriev, 1994; Dmitriev, 1991). Their authors have created a special software allowing to store and selectively process amounts of information compared to the contents of big libraries with such maps (http://www.cplire.ru/win/InformChaosLab/index.html).

3.6.2.2 One-Dimensional Quadratic Map

Non-linearity which seems the most natural and widespread in real-world systems is the quadratic non-linearity. Its properties are reflected by the class of one-

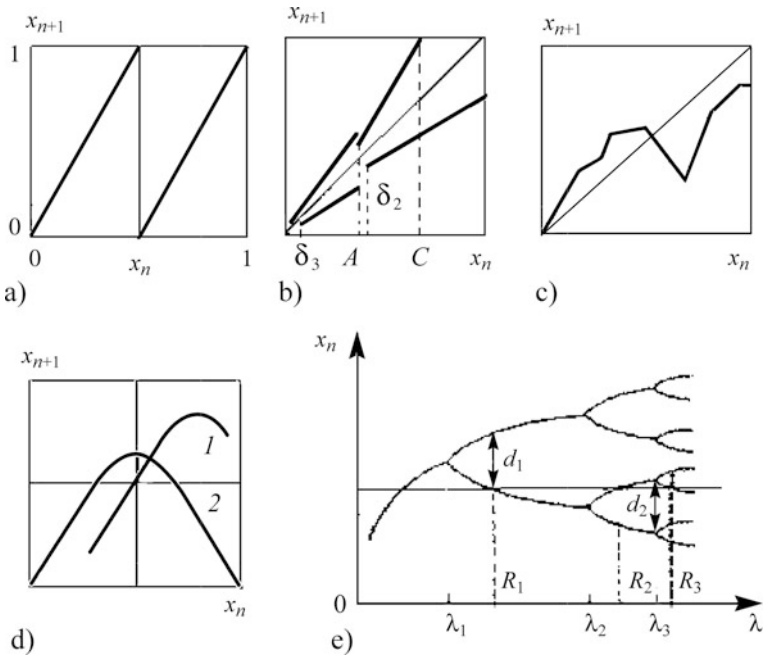

Fig. 3.8 One-dimensional maps: (**a**) the "saw tooth"; (**b**) a neuron model; (**c**) a map for information recording; (**d**) quadratic maps with different locations of the maximum; (**e**) Feigenbaum's "tree" for a quadratic map

dimensional maps $x_{n+1} = f(x_n)$, where the function f exhibits a quadratic maximum. The most eminent representative of this class is the *logistic map* (Fig. 3.8d, curve 1):

$$x_{n+1} = rx_n(1 - x_n). \tag{3.32}$$

The parameter r plays a role of the birth rate in population dynamics. As well, savings in bank account with "floating" bank interest would rise according to the same rule if the interest were introduced so as to restrict infinite enrichment of depositors (Schuster, 1984). Namely, if x_n is a value in account at an nth year and interest per annum is $\varepsilon = \text{const}$, then a simple interest gives a sum of $x_{n+1} = (1+\varepsilon)x_n$ at the next year and the sum rises infinitely. At that, a small deposit does not promise essential change in a depositor's prosperity within nearest years as compared with good prospects of a person having a big initial sum of money. If one introduced a floating interest from the "considerations of justice", then one would get a map $x_{n+1} = \varepsilon_0(1 - x_n/x_{\max})x_n$ which is reduced to the logistic map with a parameter $r = x_{\max}(1+\varepsilon_0)^2/\varepsilon_0$ via the change of variable $z_n = x_n\varepsilon_0/x_{\max}(1+\varepsilon_0)$. It is possible to list more examples from diverse fields. Any map $x_{n+1} = f(x_n)$ with the second-order polynomial f can be rewritten in the form (3.32) or in another often used form $x_{n+1} = \lambda - x_n^2$ (Fig. 3.8d, curve 2). Among "services" of the quadratic map, the following ones can be distinguished:

(1) M. Feigenbaum detected transition to chaos via a period-doubling sequence and described its universal regularities at the chaos boundary using this map as an example (Feigenbaum, 1980; Kuznetsov and Kuznetsov, 1993a). Figure 3.8e shows famous Feigenbaum's "tree", "established" values of the dynamical variable x_n versus the parameter λ. Universal quantities are, for instance, the ratios of the parameter bifurcation values near a point of transition to chaos λ_∞: $(\lambda_\infty - \lambda_n)/(\lambda_\infty - \lambda_{n+1}) = \text{const} = \delta$ or, in another form, $\lambda_n = \lambda_\infty - \text{const}\cdot\delta^{-n}$, where $\delta = 4,6692016091\ldots$ and $n >> 1$.
(2) It is a basic element for the construction of non-linear models in the form of chains and lattices (Kuznetsov and Kuznetsov, 1991) and for illustration of non-linear phenomena under periodic and quasi-periodic external driving (Bezruchko et al., 1997b).
(3) It was used to demonstrate the phenomena of hysteresis and symmetry breaking under fast change of a parameter value across a bifurcation point (Butkovsky et al., 1998).

In terms of Lyapunov exponents and fractal dimensions, complexity of the logistic map (3.32) is greater than that of the saw tooth. At $r = 4$, its attractor is a full measure set similar to the saw tooth dynamics. However, the logistic map exhibits dynamics with different fractal dimensions less than one at different parameter values. Thus, it has richer dynamical properties compared to the saw tooth.

Circle map. This is a one-dimensional map

$$\theta_{n+1} = \theta_n + \Delta + (k/2\pi)\sin\theta_n \pmod{2\pi}, \tag{3.33}$$

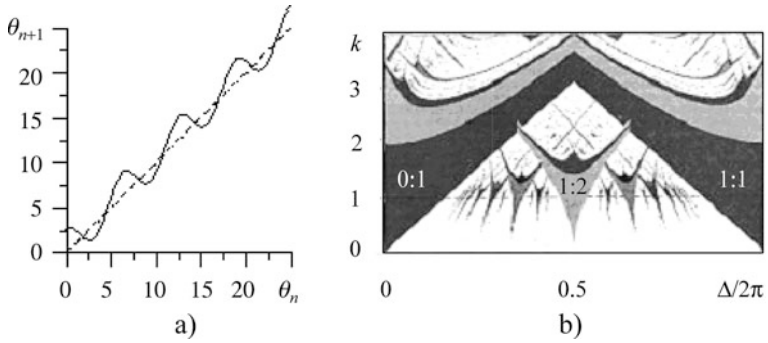

Fig. 3.9 The circle map (3.33): (**a**) its plot without taking modulo 2π; (**b**) its parameter plane $(k,\ \Delta)$ where domains of periodic regimes are shown in greyscale, while domains of quasi-periodic dynamics and chaos are shown in white

whose plot is shown in Fig. 3.9a. It can be interpreted from a physical viewpoint. Under certain assumptions, one can reduce model DEs for a self-sustained oscillator driven by a periodic sequence of pulses to such a map. An attractor of an original system can be a torus, while the map (3.33) can be considered as a Poincare map in a plane cross section of the torus (Kuznetsov, 2001).

In a cross section of a torus, a representative point under subsequent "punctures" draws a closed curve whose points can be described with an angular coordinate θ_n, where n is the order number of a puncture. The parameter Δ is determined by the ratio of periods of rotation along "big" and "small" circumferences, i.e. the ratio of frequencies of autonomous self-sustained oscillations and driving. The parameter k characterises the driving amplitude. Structure of the parameter plane for the system (3.33) is shown in Fig. 3.9b. Different greyscale tones mark domains of stable periodic regimes. Periodic regimes corresponding to the synchronisation of self-sustained oscillations by an external signal exist in domains resembling beaks. These domains are called *Arnold's tongues* by the name of a soviet mathematician V.I. Arnold. At that, an orbit on a torus becomes a closed curve in the cross section. Different tongues correspond to different values of the rotation number, i.e. the number of revolutions of a representative point along a small circumference during a single revolution along a big circumference. The dynamics of the circle map has been studied in detail, in particular, a characteristic dependence of the total width of synchronisation intervals versus k is described, regularities of chaos domain location are established, etc.

In terms of the Lyapunov exponents and fractal dimensions, the circle map complexity is similar to that of the logistic map. Both systems can exhibit periodic and chaotic regimes at different parameter values. However, the circle map can also exhibit quasi-periodic regimes with zero Lyapunov exponents which are not observed in the logistic map. Accordingly, it exhibits additional bifurcation mechanisms and the corresponding structures on the parameter plane. Thus, the circle map is, in some sense, a more complex object than the logistic map.

3.6.2.3 A Model Map for a *Non-isochronous Non-linear Oscillator* Under Dissipative Pulse Driving

A plot of a one-dimensional multi-parametric map

$$x_{n+1} = x_n \mathrm{e}^{-d/N} \cos(2\pi/(N(1+\beta x_n))) + A. \tag{3.34}$$

is shown in Fig. 3.10a (Bezruchko et al., 1995).

The idea behind this map and meaning of its four parameters are illustrated in Fig. 3.14c with a time realisation of a dissipative oscillator, e.g. a mathematical pendulum, driven periodically in a specific manner. Namely, a load is taken aside by the same value A along the x-axis. After that, it starts to oscillate with the same initial phase. For instance, one can take a load by hand and leave it with zero initial velocity. In the case of an electric pendulum, an *RL* diode circuit shown in Fig. 3.5b, such driving is realised via pulses of current with direct polarity for the diode. At that, big active conductance of a diode quickly cancels free oscillations so that an initial phase of free oscillations does not vary between pulses (Fig. 3.14a). If a quasi-period T during exponentially decaying free oscillations $x(t) = x_n \mathrm{e}^{-\delta \cdot t} \cos(2\pi t/T)$ is regarded constant between two pulses and non-isochronism is taken into account in a simplified manner as a dependence of T on an initial amplitude $T = T_0(1 + \beta x_n)$, where x_n is a starting value in the nth train of free oscillations (i.e. after the nth pulse), then a model map takes the form (3.34). Here, A is the amplitude of a driving pulse, $N = T_0/T$ is a normalised driving frequency, $d = \delta \cdot T_0$ is a damping coefficient, β is a coefficient of non-linearity, which is positive for a "soft spring" and negative for a "hard" one.

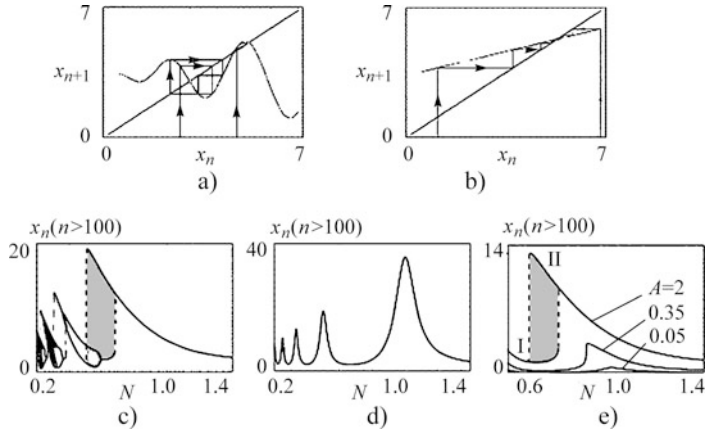

Fig. 3.10 Investigations of the map (3.34): (**a**, **b**) its plots and Lamerey's diagrams at $A = 3.4$, $N = 0.1$, $d = 0.1$ and $\beta = 0.05$ (**a**) or $\beta = 0$ (**b**); (**c**, **d**) bifurcation diagrams $x_n(N)$ for $\beta = 0.05$ (**c**) or $\beta = 0$ (**d**) which can be interpreted as resonance curves. Intervals of single valuedness correspond to period-1 oscillations, divarication of a curve means period doubling, "smeared" intervals show chaos. Resonance curves at different values of A exhibit a transition from a linear resonance to a non-linear one (e). Domains of bistability and chaos are shaded

Despite being one-dimensional and relatively simple in its mathematical form, the map exhibits practically all basic non-linear phenomena inherent in low-dimensional non-linear dynamical systems: a multitude of oscillatory regimes on the basis of different kinds of oscillations (modes), linear and non-linear resonance, bi- and multistability, complexity and fractal character of basins of attraction, hysteresis and dynamical chaos (Prokhorov and Smirnov, 1996). Thus, its dynamical complexity is similar to that of the circle map and logistic map and is even greater in some respects (Fig. 3.11).

Higher dimensional maps are capable of demonstrating even more diverse dynamics and, hence, greater complexity in terms of the number of positive Lyapunov exponents and big value of fractal dimension. However, the phenomena illustrated above with one-dimensional examples can already convince a reader that discrete maps represent a very fruitful and efficient research tool.

3.6.3 Role of Discrete Models

We consider the role of discrete models with a concrete example. In 1981, Lindsay reported an observation of dynamical chaos in quite an accessible (cheap) and popular system, a circuit with an inductance coil and a varactor diode driven by a harmonic electromotive force (Linsay, 1981). Since then a piece of wire convoluted in a coil and a piece of semiconductor supplied with contacts are actively used for experimental demonstrations of non-linear phenomena. A paper in "Scientific American" even recommended to have such systems "on a windowsill in each house".[6] Below, we demonstrate capabilities of discrete models of this object.

3.6.3.1 "Ancestors" of the Object

A circuit consisting of an inductance coil and a capacitor (an oscillatory circuit) is an electric analogue of a mechanical pendulum. Similar to how mechanical pendulum properties are determined by its shape and parameters, processes in a circuit depend on the construction of its elements. For the simplest case when plates of an air capacitor are connected with a wire coils (Fig. 3.12a), a conceptual model (an equivalent scheme) takes the form shown in Fig. 3.12b. Given the parameters L, C, R of the scheme,[7] one can readily derive a model of the circuit in the form of the linear dissipative oscillator (3.2) from Kirchhoff's laws, where x is a dynamical variable

[6] This is a diode with a $p - n$ junction whose capacity depends on voltage, i.e. an electrically controlled capacitor. Circuits with such diodes are used in radioengineering for more than half a century. They were even suggested as memory elements for computers. Different kinds of such circuits are widely presented in contemporary radio sets and TV sets.

[7] When a charge is accumulated on the capacitor plates and a current flows in the wires, electric and magnetic forces appear and tend to compress or stretch the wires. Therefore, if substances of the coil and capacitor are not hard enough, their size (and, hence, C and L) can depend on the current and voltage (dynamical variables) implying emergence of nonlinearity.

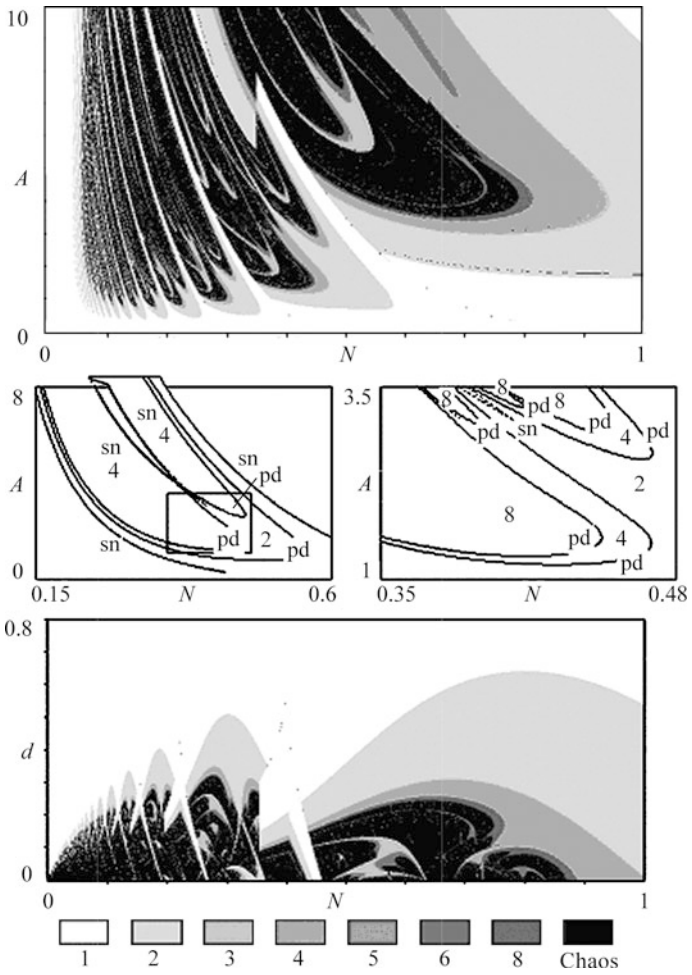

Fig. 3.11 Chart of the dynamical regimes for the map (3.34) on the parameter planes A–N and d–N. Greyscale tones show domains of oscillations whose periods are reported at the bottom. The same tone may correspond to different motions with the same period or to chaotic regimes developed on the basis of different cycles. Bifurcations occur at the boundaries of the domains. The fragment in the middle shows domains of bi- and multistability. A domain of existence and evolution of a certain cycle is shown with a separate sheet: *sn* are boundaries of the sheets, *pd* are period-doubling curves. Overlap of two sheets corresponds to bistability and hysteresis

(charge) and $\delta = R/2L$ is a damping coefficient. Free oscillations of the dissipative oscillator decay, while oscillations driven by a periodic signal are periodic with a period of driving T. The only oscillatory effect demonstrated by the system is a *resonance*, an increase in the driven oscillation amplitude when natural and driving frequencies get closer.

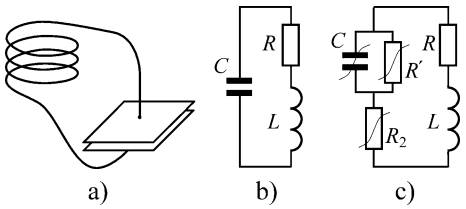

Fig. 3.12 Electric pendulums: (**a**) the simplest oscillatory circuit; (**b**) its equivalent scheme; (**c**) an equivalent scheme where the role of a capacitor is played by a diode represented by a combination of non-linear capacitor and resistors

3.6.3.2 Consequences of a Capacitor Replacement

Inclusion of a diode whose equivalent parameters R and C depend on the current and voltage into the circuit (Fig. 3.12c) leads to a striking extension of the range of observed oscillatory phenomena. Even under the simplest harmonic driving, the "electric pendulum" demonstrates a hierarchy of driven motions of various complexity: harmonic, more complex periodic and chaotic ones. Similar picture is observed under a pulse driving. Bifurcation sets (surfaces in three-dimensional spaces, curves on two-dimensional cross sections) bound domains of existence of different oscillatory regimes in a parameter space forming the structures presented in Fig. 3.13. The driving amplitude V and the normalised frequency $N = \omega/\omega_0$ are shown along the horizontal axes of the three-dimensional picture and the linear resistance R along the vertical axis. The structure can be understood better by considering different plane cross sections of the parameter space. Oscillation type within a domain can be illustrated with a respective time realisation.

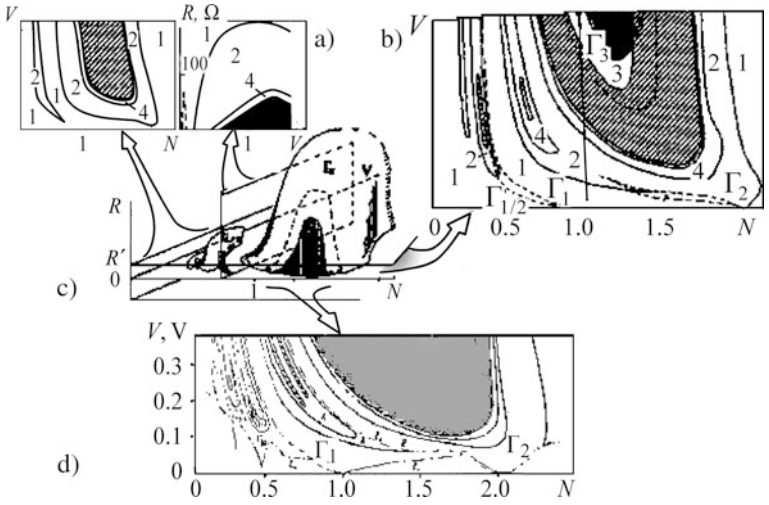

Fig. 3.13 Parameter space of an *RL*-diode circuit under harmonic external driving obtained from experimental investigations. *Dashes* show curves of hysteresis (hard) transitions. Chaos domains are shaded. Numbers denote period of oscillations in the respective domains

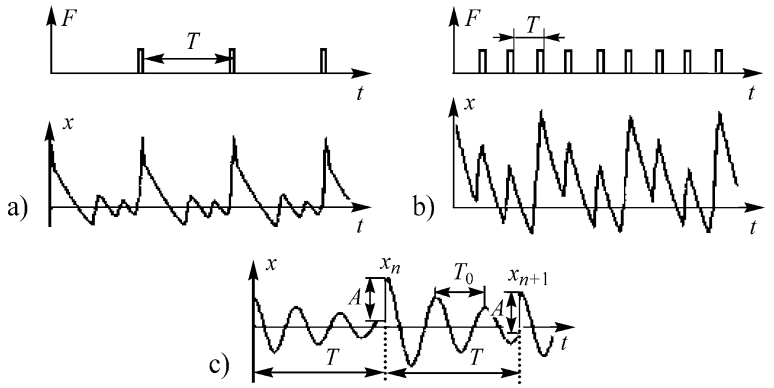

Fig. 3.14 Time realisation of a current in an *RL*-diode circuit under periodic pulse driving of direct polarity $F(t)$: (**a**) a cycle belonging to the class of subharmonic oscillations, a driving period T is three times as big as a quasi-period of free oscillations ($\Gamma_{1/3}$); (**b**) a cycle belonging to the class of "period adding sequence", a driving period T is three times as small as a quasi-period of free oscillations ($\Gamma_{3/1} = \Gamma_3$); (**c**) a time realisation-based model of subharmonic oscillations, where a quasi-period of decaying oscillation within a train is constant and depends only on an initial deviation

Figure 3.14 shows typical time realisations of a current in the case of pulse driving emf whose polarity is "direct" for the diode. Pulses come with a period T, $\omega = 2\pi/T$. Despite small duration of pulses, free oscillations quickly decay during a pulse since an equivalent capacity of a diode for a direct current (Fig. 3.12c) is shunted by its low active resistance. When a pulse ends, oscillations start almost with the same initial phase (Fig. 3.14a, b), while a time realisation between two pulses represents decaying free oscillations. Depending on the driving amplitude and period, the damping rate, the kind of non-linearity and initial conditions, different repeated motions (*cycles*) can be established in the system. *Periods of the cycles* are equal to the driving period or divisible by it, i.e. kT, where k is an integer. Possible variants of periodic motions are diverse but can be systematised as follows. All cycles can be conventionally divided into two groups based on the similarity property. Each of the groups preserves some peculiarities of the waveform of time realisations and the shape of limit cycles in the phase space.

The first group is formed by cycles whose period is equal to the driving period $1T$ and exists in the low-frequency domain $N < 1$. Such cycles are usually called *subharmonic cycles*. Since the driving period is big as compared with the time scale of free motions, there are generally several maxima in a time realisation within a train (Fig. 3.14a). The second group consists of cycles with periods kT, where $k = 2, 3, \ldots$, which are observed for bigger driving frequencies $0.5 < N < 2$. Examples of such cycles are shown in Fig. 3.14b. Since a change of such a regime under the increase in amplitude is accompanied by subsequent increase in k by 1, they are called cycles of "period adding sequence". A conventional notation of the cycles is $\Gamma_{m/k}$. Here, k corresponds to the ratio of the driving period to the quasi-period of free oscillations. It can be estimated as the number of maxima within an

interval T in an oscillogram. The value of m is a period of cycle measured in units of the driving period.

3.6.3.3 Mathematical Models

Processes in semiconductor diodes whose properties determine non-linearity of a system are analysed most strictly with the use of partial differential equations. However, for sufficiently slow motions, a diode can be considered as a bipole with some equivalent properties reflecting relationship between the voltage on its contacts and the current in connecting wires so that one can use ODEs. Even simpler models can be obtained in the form of maps, if one restricts the consideration only with a part of possible motions. Further, we consider models capable of describing fragments of the above-mentioned (Fig. 3.13) complex picture.

A Continuous-Time Model

Let us represent a semiconductor diode as a non-linear capacitor, whose capacity C depends on the voltage as $C = C_0/(1-U/\varphi)$, where C_0 is the initial diode capacity, U is the voltage on the diode, φ is the contact potential. Then, a model equation for the circuit derived from Kirchhoff's laws takes the form of Toda oscillator (3.25):

$$\mathrm{d}^2x/\mathrm{d}\tau^2 + \gamma \mathrm{d}x/\mathrm{d}\tau + \mathrm{e}^x - 1 = A \ \sin \ N\tau,$$

where x is the dimensionless charge on the capacitor plates, γ is the damping coefficient, A is the dimensionless driving amplitude, $N = \omega/\omega_0$ is the normalised driving frequency, $\tau = \omega_0 t$ is dimensionless time. Results of numerical simulations presented in Fig. 3.7 demonstrate good qualitative description of an object in the *entire parameter space*.

Discrete-Time Models

(i) One can successfully use one-dimensional multimodal map (3.34) as a discrete-time model for subharmonic oscillations, i.e. in the low-frequency domain $N = \omega/\omega_0 \leq 1$. A model is adequate to the real-world system in the parameter space domains where motions on the basis of the cycles $\Gamma_{1/2}$, $\Gamma_{1/3}$ and so on take place (Fig. 3.11). Those domains have qualitatively the same structure. They are similar to each other and self-similar. Self-similarity means a configuration like in "matreshka" (a set of nesting dolls): a basic constructive element is reproduced at smaller and smaller scales. However, in contrast to matreshka, the domains of existence of various oscillation kinds on the parameter plane at sufficiently low levels of dissipation can overlap forming domains of multistability (Fig. 3.11, bottom panel)

(ii) A *two-dimensional map* modelling driven dynamics of the circuit in a higher-frequency domain $0.8 \leq N \leq 2$ is suggested in Bezruchko et al. (1997a) based on the characteristic waveform of time realisation of the cycles belonging to

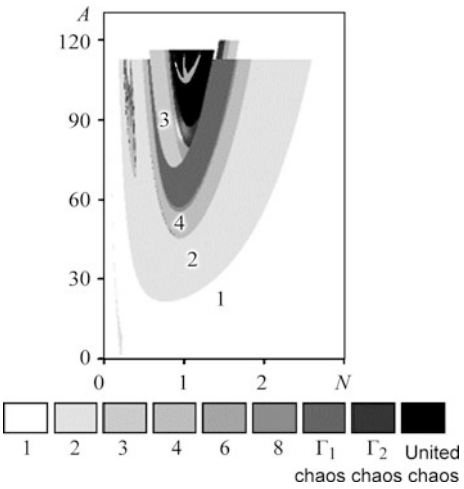

Fig. 3.15 Chart of the dynamical regimes of the map suggested in Bezruchko et al. (1997a), which describes the cycles of "period adding sequence"

the "period adding sequence" (Fig. 3.14b). It is more complicated than map (3.34) and reproduces well the structure of the parameter plane of an original circuit (Kipchatov, 1990) for large driving frequencies and amplitudes, where the cycles of the "period adding sequence" exist (cf. Figs. 3.13b and 3.15). At that, it does not reflect diversity of the basis cycles and other peculiarities of the circuit dynamics described above.

(iii) In the domains where any of the basis cycles demonstrate period-doubling sequence under a parameter change, a good model of the circuit is the one-dimensional quadratic map $x_{n+1} = \lambda - x_n^2$.

(iv) In a domain of negative resistance where an object demonstrates self-sustained oscillations, its dynamics is modelled well with the circle map (3.33) which exhibits the phenomena of synchronisation by a weak periodic driving and of suppression of the oscillations by a strong periodic driving.

Thus, the Toda oscillator equation (3.25) describes the circuit dynamics in the most complete way among all the considered models. It reflects all families of the characteristic cycles of the *RL*-diode circuit and peculiarities of its parameter space structure. The discrete-time model (3.34) and the two-dimensional model map describe only one of the two existing families of cycles, either "subharmonic" or "period adding" one. In particular, map (3.34) reflects such phenomena as linear and non-linear resonance, multistability and hysteresis. The quadratic map is universal but does not capture specificity of the object. The same holds true for the circle map. Is it possible to create a model map which could compete with the differential equation of the Toda oscillator? Currently, we could not say how complex a formula for such a map might be.

3.7 Models of Spatially Extended Systems

To model spatially extended objects, one often uses ensembles of coupled ODEs or coupled maps (e.g. Afraimovich et al., 1989; Nekorkin and Velarde, 2002; Shalfeev and Matrosov, 2005). Spatial properties of such systems manifest themselves in solutions with different spatial profiles of characterising variables. For instance, oscillations of two coupled linear oscillators can be represented as a superposition of two basic sinusoidal regimes with different frequencies. One of them corresponds to in-phase oscillations, when the elements move in a completely identical manner, while the other one reflects anti-phase oscillation, when there is a constant phase shift between the oscillators by π. This peculiarity of a spatially extended system, consisting of concentrated elements, can be considered as an analogue of spatial modes in a bounded continuously distributed system (Fig. 3.16).

A property of *multistability* resembling multitude of spatial modes is ubiquitous in ensembles of oscillatory systems. Such a principal multimodality and the corresponding sensitivity to weak parameter variations (when possible kinds of motions are numerous and their basins of attraction form complicated and even fractal structures) is a typical property of spatially extended non-linear systems. Capabilities of relatively simple discrete models to describe this basic phenomenon are illustrated in Sect. 3.7.1, while more complicated tools are briefly considered after that.

3.7.1 Coupled Map Lattices

In chains and lattices, identical basis maps $x_{n+1} = f(x_n)$ are usually coupled to each other in a certain manner: locally (only nearest neighbours), globally (all to all) or within groups. Complexity of these models rises with the number of coupled maps, i.e. with the dimension of a model. In general, the greater is the model dimension, the greater can be the number of coexisting attractors, their fractal dimension and the number of positive Lyapunov exponents.

3.7.1.1 Two Dissipatively Coupled Quadratic Maps

A symmetric coupling, when elements influence each other in the same way, is shown by a rectangle in Fig. 3.17. A triangle marks a unidirectional coupling, when

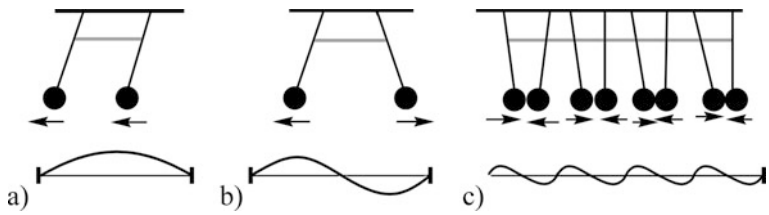

Fig. 3.16 Oscillatory modes in an ensemble of two (**a**, **b**) and several (**c**) pendulums. *Top panels* illustrate the systems, bottom ones illustrate their spatial modes

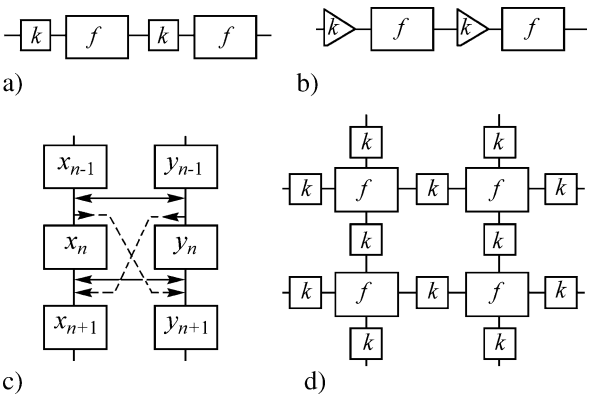

Fig. 3.17 Coupled map lattices: (**a**, **b**) one-dimensional lattices; (**c**) a space-and-time diagram for interacting populations to illustrate symmetric coupling kinds; (**d**) a two-dimensional lattice

only one element affects another one. An intermediate case of asymmetric coupling is also possible. A systematisation of the *coupling kinds* is given in Kuznetsov (1986), where symmetric couplings between maps are reduced to the following types: *dissipative* coupling

$$\begin{aligned} x_{n+1} &= f(x_n) + k(f(y_n) - f(x_n)), \\ y_{n+1} &= f(y_n) + k(f(x_n) - f(y_n)), \end{aligned} \tag{3.35}$$

inertial coupling

$$\begin{aligned} x_{n+1} &= f(x_n) + k(y_n - x_n), \\ y_{n+1} &= f(y_n) + k(x_n - y_n), \end{aligned} \tag{3.36}$$

or their combination. Here, x and y are dynamical variables, k is the coupling coefficient, f is the non-linear function of the basis map.

The systematisation allows an interesting interpretation in the language of the population biology. One can assume that individuals first breed in their population and then get an opportunity to migrate to another population. "First breed, then creep away". The cycle is repeated next year. Solid lines on a space – time diagram (Fig. 3.17c) correspond to such a case. Such coupling tends to make simultaneous states of subsystems equal to each other so that it can be naturally called *dissipative coupling*. Dashed lines in Fig. 3.17c correspond to a situation when individuals may migrate before the cycle of breeding and death within their population. Such coupling can be reasonably called *inertial coupling* since it promotes memorising a previous-step state. A combined coupling is also possible.

The choice of coupling type in practical modelling is non-trivial. In particular, it is illustrated by experimental investigations of a set of coupled non-linear electric circuits (Sect. 3.6.3) in the domain of parameter space, where each system transits to chaos via period doublings. It appears that coupling via a resistor (a dissipative

element) is adequately described as dissipative, while coupling via a capacitor (a purely reactive element) as combined one, rather than purely inertial (Astakhov et al., 1991b).

The choice of the basis map and of the kind of coupling introduce specific features into a model behaviour, but the phenomenon of *multistability* in ensembles of coupled maps is always determinative. It is illustrated by the simplest set of two quadratic maps $f(x_n) = \lambda - x_n^2$ with the dissipative coupling (3.36):

$$\begin{aligned} x_{n+1} &= \lambda - x_n^2 + k\left(x_n^2 - y_n^2\right), \\ y_{n+1} &= \lambda - y_n^2 + k\left(y_n^2 - x_n^2\right). \end{aligned} \tag{3.37}$$

For the same value of λ in both subsystems, we introduce the following systematisation of oscillatory modes. In the limit of zero coupling ($k = 0$), each regime of a period N can be realised in N ways differing by the shifts between the subsystems oscillations in time by $m = 0, 1, 2, \ldots, N-1$ steps as shown in Fig. 3.18 for $N = 2$ and 4. We call those N ways the *oscillation kinds* and use them to describe a hierarchy of the oscillatory regimes in the presence of coupling when interaction leads to different variants of mutual synchronisation. We denote periodic regimes as N_m. Despite the lack of repeatability, the same classification principle can be maintained for chaotic regimes N^m if one interprets N as the number of the attractor strips and m as a time shift between maximal values of x_n and y_n. Regimes with $m = 0$ are called in-phase.

By showing the domains of existence and evolution of each oscillation kind on a separate sheet, one can get a vivid multi-sheet scheme of the domains of existence (stability) of oscillatory regimes on the plane (k, λ). Figure 3.19a shows the domains for all oscillatory regimes of the periods 1, 2, 4 and 8 at $k < 0.5$. Figure 3.19b represents a cross section of the multi-sheet picture shown in Fig. 3.19a with a plane $k = 0.05$ and qualitatively illustrates an evolution of motion in system (3.37) under the variation of the parameter λ at a fixed weak coupling. Solid lines correspond to stable regimes and dashed ones to unstable regimes. Points indicate bifurcation transitions. The letters A, B, C and D mark branches combining certain groups of regimes[8]: they start with periodic regimes whose number rises with λ and end with chaotic ones.

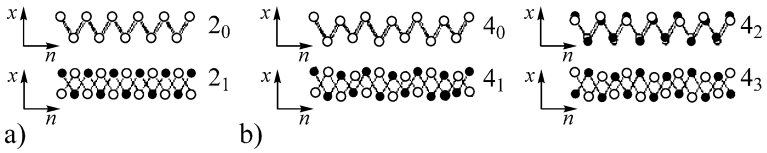

Fig. 3.18 Coupled map dynamics: time realisations (**a**) for the period $N = 2$; (**b**) for the period $N = 4$. Dynamical variables of the first and second subsystems are shown by filled and open circles, respectively. Notations of the oscillation kinds N_m are shown to the right

[8] The branch A corresponds to the evolution of in-phase regimes ($m = 0$), $B - D$ to the others.

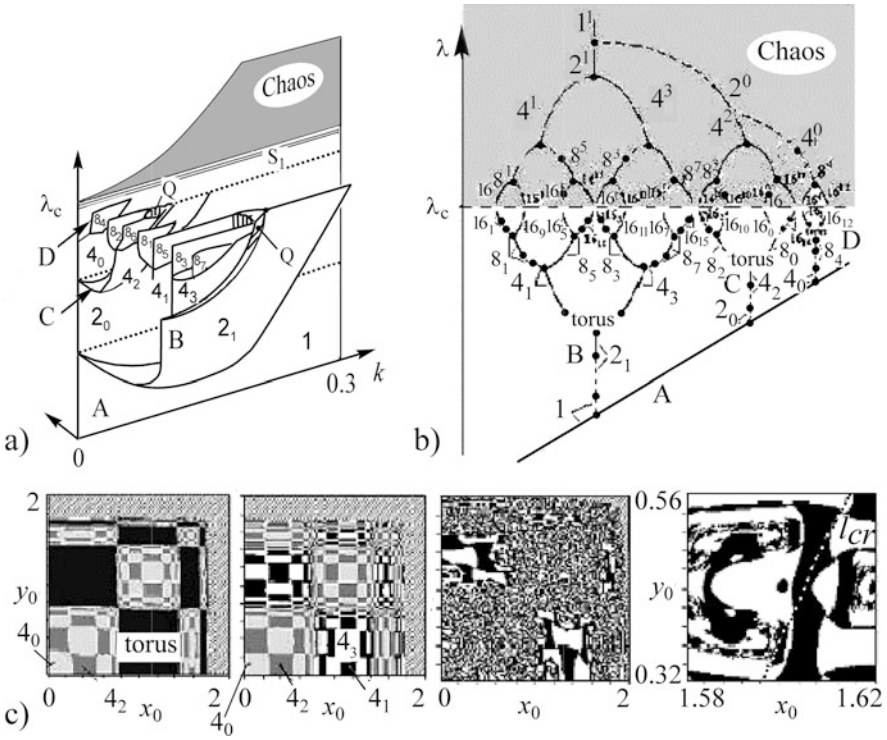

Fig. 3.19 Dynamics of the systems (3.37): (**a**) a scheme of the evolution of oscillation kinds on a parameter plane; (**b**) its section for $k = 0.05$; (**c**) phase space division into basins of attractors in cases of multistability (they change each other when one moves along the parameter plane in the panel **b**)

Domains of chaotic regimes are shaded. A critical value of the non-linearity parameter at which a transition to chaos occurs (an accumulation point of a period-doubling sequence) is denoted λ_c. Domains denoted by a letter Q or a word *torus* correspond to quasi-periodic oscillations and transition to chaos via their breaking. Different oscillation kinds can divide phase space into *basins of attraction* with a fractal structure (Fig. 3.19c). Increase in the dissipative coupling strength k is accompanied by reduction in the number of coexisting states so that only in-phase motion is stable at large k, i.e. the system becomes in effect one-dimensional.[9]

3.7.1.2 Complex Dynamics of a Chain: Consequences of the Increase in the Number of Elements

It is not surprising that the increase in the number of elements in an ensemble leads to even more complicated oscillatory picture. Indeed, the longer the chain,

[9] Complex dynamics of this non-linear system is illustrated by a computer program available at http://www.nonlinmod.sgu.ru and in research papers Astakhov et al. (1989, 1991a).

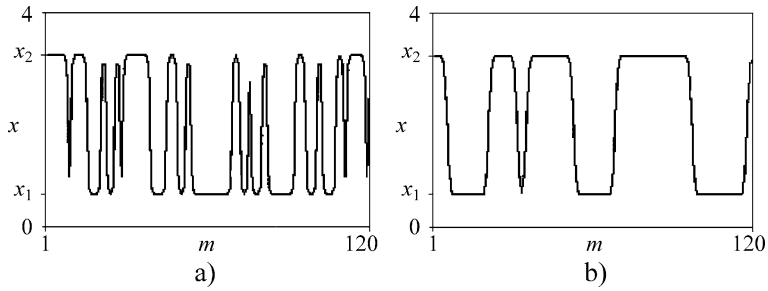

Fig. 3.20 A period-1 regime with a non-uniform spatial distribution in the chain (3.38) of 120 elements, $A = 0.965,\ N = 0.64,\ d = 0.2,\ \beta = 0.2$: (**a**) $k = 0.1$; (**b**) $k = 0.35$. The quantities x_1 and x_2 are equilibrium states of a bistable elementary cell

the more the kinds of motions with different temporal and spatial profiles possible. Figure 3.20 illustrates it with numerical results for the chain of dissipatively coupled pendulum maps

$$x_{n+1}^m = (1-k)\, f\left(x_n^m\right) + (k/2)\left(f\left(x_n^{m+1}\right) + f\left(x_n^{m-1}\right)\right), \tag{3.38}$$

where n is the discrete time, m is the number of a chain element, k is the coupling coefficient, f is the multimodal map (3.34). A number of an element is shown along the horizontal axis and its instantaneous state along the vertical one. There are flat intervals in the pattern (*domains*) and fast transitions between them (*kinks*).[10]

The structures evolve under the parameter changes: temporal and spatial periods double; periodic, quasi-periodic and chaotic configurations arise. Domains widen with the coupling coefficient rise, while kinks get flatter. Finally, a very large k provides a spatially uniform regime for any initial conditions, an analogue to the emergence of an ice floe in the above-mentioned example with cooled water (Sect. 3.2). Details on the dynamical properties of the chain are given in Bezruchko and Prokhorov (1999).

3.7.1.3 Two-Dimensional Map Lattice

Further complication of the model (3.38) in the sense of its spatial development can be performed both via the increase in the number of elements and via changes in the coupling architecture. In the next example, the same multimodal maps constitute a lattice, where each element interacts with its four nearest neighbours. Coupling is local and dissipative:

[10] Map (3.34) is taken for simplicity. Thus, multistability in a set of quadratic maps is formed on the basis of a period-doubled cycle, while in a set of maps (3.34) it is observed already for the period-1 cycles. When an isolated map (3.34) has two period-1 states, there are four period-1 oscillation kinds in a set of two maps (Bezruchko and Prokhorov, 1999).

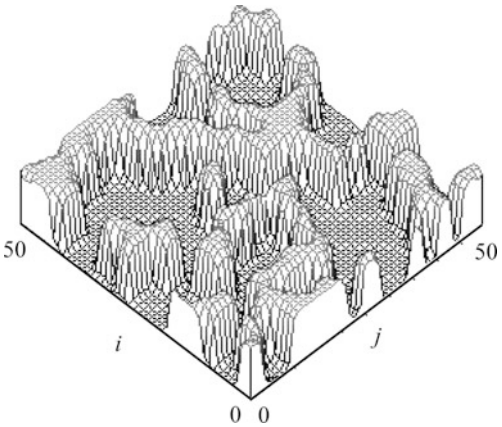

Fig. 3.21 A dynamical regime with a non-uniform spatial distribution in the two-dimensional lattice (3.39) consisting of 50×50 period-1 elements, $A = 0.965,\ N = 0.64,\ d = 0.2,\ \beta = 0.2,$ $k = 0.2$. Boundary conditions are periodic

$$x_{n+1}^{i,j} = (1-k) f\left(x_n^{i,j}\right) + (k/4)\left(f\left(x_n^{i+1,j}\right) + f\left(x_n^{i-1,j}\right) + f\left(x_n^{i,j+1}\right) + f\left(x_n^{i,j-1}\right)\right), \tag{3.39}$$

where i and j determine spatial location of a lattice element. An example of instantaneous snapshot of such a lattice having the same numbers of elements along each direction is shown in Fig. 3.21. This is a stationary structure achieved from random initial conditions. At weak couplings, one can get almost any required stationary distribution by specifying different initial conditions. Under the increase in the coupling coefficient, the number of possible structures reduces. Above some threshold value of k, the only attractor is a spatially uniform distribution.

3.7.2 Cellular Automata

A *cellular automaton* is a discrete dynamical system representing a set of identical cells coupled with each other in the same way. All the cells form a cellular automaton lattice. Lattices may be of various types differing both in dimension and shape of the cells (Minsky, 1967). Cellular automata were suggested in the work of von Neumann (1966) and became a universal model of parallel computations like Turing's machine for sequential computations. Any cell computes its new state at each step from the states of its nearest neighbours. Thus, the laws in a system are local and everywhere the same. "Local" means that it is sufficient to look at the neighbourhood state to learn what will happen at a future instant; no long-range interactions are allowed. "Sameness" means that one can distinguish one place from

another one by a landscape,[11] not by any difference in the laws (Margolus and Toffoli, 1990). Based on this description, one can single out the following characteristic properties of cellular automata:

(i) A lattice is uniform and evolution law for the cells is everywhere the same.
(ii) Changes in the states of all cells occur simultaneously, after calculation of a new state of each cell in a lattice.
(iii) Interactions are local, only neighbouring cells can affect a given cell.
(iv) A set of cell states is finite.

Usually, one illustrates a cellular automaton with an example of a model called the *game "Life"* created by D. Conway, a mathematician from Cambridge University, in 1970. It is widely presented in the Internet (see, e.g., http://www.famlife.narod.ru). Rules of functioning of that automaton somewhat mimic real-world processes observed in birth, development and death of a colony of living organisms. One considers an infinite flat lattice of square cells (Fig. 3.22). Living cells are shown by dark colour. Time is discrete ($n = 0,\ 1,\ 2,\ \ldots$) and a situation at the next time step $n + 1$ is determined by the presence of living neighbours for each living cell. Neighbouring cells are those having common edges. Evolution is governed by the following laws:

(i) *Survival*. Each cell having two or three neighbouring living cells survives and transits to the next generation.
(ii) *Death*. Each cell with more than three neighbours dies due to overpopulation. Each cell with less than two neighbours dies due to solitude.
(iii) *Birth*. If the number of living cells neighbouring to a certain empty cell is equal exactly to three, then a new organism is born in that empty cell.

Thus, if an initial distribution of living cells (a landscape) has the form shown in Fig. 3.22a, then a configuration shown in Fig. 3.22b appears in a single time step.

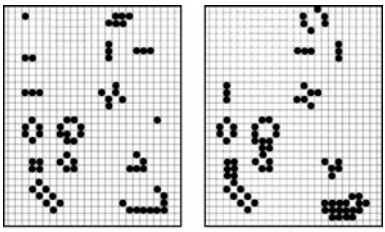

Fig. 3.22 Examples of landscapes for the cellular automaton "Life": (**a**) an initial profile, $n = 0$; (**b**) a situation after the first step $n = 1$

[11] A distribution of the values of a characterising quantity over an automaton workspace.

Furthermore, some initial structures may die out, while the others survive and get stationary, or are repeated periodically, or move in space, and so on. Basic properties of this automaton are as follows: structures separated by two empty cells do not affect each other at once; a configuration at a time instant n completely determines the future (states at time steps $n+1$, $n+2$, etc.); one cannot restore the past of the system from its present (the dynamics is non-invertible); stable forms are, as a rule, symmetric, etc. The larger the area occupied by a population, the more complicated its behaviour.

Currently, the game "Life" has got further development. Thus, in modern versions the automaton is three dimensional and capable of modelling several populations like interacting "herbivores" and "predators". However, even more sophisticated versions of this simple system do not represent a limit complexity level for the problems which can be solved with cellular automata.

A cellular automaton can be equivalently described with a set of coupled maps with discrete states. Its peculiarity is the simplicity of construction and convenience of computer investigation. Cellular automata are used to model hydrodynamic and gas-dynamic flows, electric circuits, heat propagation, movement of a crowd, etc. (Loskutov and Mikhailov, 2007; Malinetsky and Stepantsev, 1997; Shalizi, 2003). They are applied to create genetic algorithms, to find a shortest way on a graph and so forth (see, e.g., Margolus and Toffoli, 1990; http://is.ifmo.ru).

3.7.3 Networks with Complex Topology

The above coupled map lattices and cellular automata describe spatially extended systems with local interactions between the elements. More complicated coupling architectures involve various non-local interactions. In such a case, one speaks of a *network of coupled maps* rather than a coupled map lattice. Coupling architecture is often called *topology* of a network. Topology is said to be regular if it is described by a simple regular law, e.g. the above locally coupled maps (where only the nearest neighbours interact) or globally coupled maps (where every element is connected to every other element, all-to-all coupling).

It is easy to imagine other topologies which are not described as simply as the above regular topologies. For example, one can artificially generate a completely random coupling architecture. Typically, one speaks of a complex topology if it looks rather complicated and irregular and, simultaneously, exhibits some non-trivial statistical properties different from completely random networks. During the last years *networks with complex topology* are actively studied in different fields as reflected by the reviews (Albert and Barabasi, 2002; Arenas et al., 2008; Boccaletti et al., 2006; Dorogovtsev and Mendes, 2003; Kurths et al., 2009; Osipov et al., 2007; Strogatz, 2001; Watts, 1999). A special attention is paid to the so-called "small-world" and "scale-free" properties of a network. To define them, one must introduce the concepts of node, link, degree and path. Each element of a network is called a *node*. If two nodes are coupled (interact with each other), then one says that there exists a *link* between them. These two nodes are called *vertices* of the link. If

a node is a vertex of M links ($0 \leq M \leq N-1$, where N is the size of the network), then the number M is called a *degree* of that node. A *path* connecting two nodes A and B is the sequence of vertices which one has to pass by (via the existing links) to reach the node B from the node A. The shortest path between two nodes is a path consisting of the smallest number of vertices.

The *small-world property* means that any two elements of a network are connected by a sufficiently short path. To give a quantitative formulation, one can notice that a mean (over all pairs of nodes) shortest path length in a regular hypercubic lattice with d dimensions grows with the lattice size as $N^{1/d}$. Small-world property is defined as follows: a mean shortest path length grows at most logarithmically with N. This notion was first introduced in Watts and Strogatz (1998). The small-world property has been observed in a variety of real-world networks, including biological and technological ones (see Sects. 2.2.1 and 2.2.3 in Boccaletti et al., 2006 and references therein).

The *scale-free property* concerns heterogeneity of couplings. Homogeneity in coupling structure means that all nodes are topologically equivalent, e.g. regular lattices or random networks. In regular lattices, each node has the same degree except for the edges. In random networks, each of the $N(N-1)/2$ possible links is present with equal probability. Therefore, a degree distribution is binomial or Poisson in the limit of large network size. However, it was found that many real complex networks exhibit a degree distribution $p(M) \sim M^{-\gamma}$, where $2 < \gamma < 3$. Such networks were introduced in Barabasi and Albert (1999) and called scale-free, since the power law has the same functional form at all scales. There are many examples of technical, biological and social networks characterised as scale free (see Sects. 2.2.2 and 2.2.3 in Boccaletti et al., 2006 and references therein).

As for the dynamical aspect, many studies have been devoted to studying synchronisation in complex networks of coupled dynamical systems (see, e.g., Arenas et al., 2008; Osipov et al., 2007). It was found that the small-world property often enhances synchronisation as compared with regular lattices. Under some conditions, the scale-free property may lead to similar enhancement, see e.g. Motter et al. (2005).

Finally, we note that a network with complex topology consisting of N coupled one-dimensional maps is an N-dimensional dynamical system, similar to a coupled map lattice consisting of N one-dimensional maps. Thus, both models are equivalent in terms of their state vector dimension. However, a network with complex topology is a much more complicated object in terms of coupling structure. Thus, a network with complex topology can be considered as a more complex model of spatially extended systems.

3.7.4 Delay Differential Equations

Delay differential equations are typically used to model systems whose behaviour at present is determined not only by a present state but also by the values of dynamical

variables at previous time instants. Such objects are widely presented in nature. They are studied in physics, biology, physiology and chemistry. Causes of a time delay can be different. Thus, in a population dynamics, a delay is connected with the fact that individuals participate in a reproduction process only after becoming adult. In spatially extended radio-physical systems, a delay is determined by a finite speed of signal propagation. A delay time τ is related to the time necessary for a signal to overpass a distance between elements. In a sufficiently general case, a time-delay system is described with the equation

$$\varepsilon_n \frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n} + \varepsilon_{n-1} \frac{\mathrm{d}^{n-1} x(t)}{\mathrm{d}t^{n-1}} + \ldots + \varepsilon_1 \frac{\mathrm{d}x(t)}{\mathrm{d}t} = F(x(t), x(t-\tau_1), \ldots, x(t-\tau_k)), \tag{3.40}$$

where $\tau_1, \ldots, \tau_k$ stands for several possible time delays caused by different factors. Particular cases of equation (3.40) are as follows: *Ikeda equation* $\dot{x}(t) = -x(t) + \mu \cdot \sin(x(t-\tau) - x_0)$ describing the dynamics of a passive optical resonator; *Mackey – Glass equation* $\dot{x}(t) = -b \cdot x(t) + a \cdot x(t-\tau)/(1 + x^c(t-\tau))$ describing the process of red corpuscle generation in living organisms; the delayed feedback generator[12] $\varepsilon \cdot \dot{x}(t) = -x(t) + f(x(t-\tau))$, which is a very popular model in radio-physics.

Despite the only scalar dynamical variable, all the listed dynamical systems are infinite-dimensional, since one must specify a distribution of a dynamical variable over the interval $[0, \tau]$ as an initial condition. Even a first-order non-linear DDE can exhibit complex motions corresponding to attractors of very high dimensionality, chaos, multistability and other non-linear phenomena. In general, infinite dimensionality of the phase space leads to the possibility of observing attractors of arbitrary high dimension and with arbitrarily many positive Lyapunov exponents. In this sense, DDEs are more complex systems than previously described finite-dimensional model maps and ODEs. For more detailed information about DDE-based models, we refer to the review on complex dynamics of the feedback generator (Kuznetsov, 1982), the monograph Dmitriev and Kislov (1989), the research paper Kislov et al. (1979) and the website http://www.cplire.ru/win/InformChaosLab/index.html.

3.7.5 Partial Differential Equations

This is probably the most extensively studied mathematical tool developed specially for modelling of spatially extended systems. PDEs are used in very different scientific disciplines ranging from physics, chemistry and biology to ecology and economics. It is sufficient to recall famous Maxwell's equations in electrodynamics, Schrödinger's equation in quantum mechanics, reaction – diffusion equations in

[12] It is a ring consisting of a non-linear amplifier (characterised by a function f), an inertial element (a filter with a response time determined by ε) and a delay line (with a delay time τ).

chemistry and biology, and Ginzburg – Landau equation everywhere. Many classical models of the wave theory take the form of PDEs:

Simple wave equation $\partial x/\partial t + v(x)\partial x/\partial z = 0$, where x is the characterising quantity, v is the velocity of a perturbation propagation (depending on the perturbation value, in general); z is the spatial coordinate. The model can describe steepening and turnover of a wave profile.

Corteveg – de Vries equation $\partial x/\partial t + v(x)\partial x/\partial z + \beta\partial^3 x/\partial z^3 = 0$ is the simplest model exhibiting soliton-like solutions. Roughly, the latter ones are localised perturbations propagating with a constant waveform and velocity and preserving these characteristics after collision with each other.

Burgers' equation $\partial x/\partial t + v(x)\partial x/\partial z - \alpha\partial^2 x/\partial z^2 = 0$ is the simplest model describing waves in a medium with dissipation, in particular, shock waves (i.e. movements of a region of fast change in the value of x).

A dynamical system described with a PDE is infinite-dimensional even for a single spatial coordinate. To specify its state, one must provide an initial function $x(0, z)$. If a system without spatial boundaries is considered (such an idealisation is convenient if a system is very lengthy so that any phenomena at its boundaries do not significantly affect the dynamics under study and are not of interest for a researcher), then an initial function must be defined over an entire axis $-\infty < z < \infty$. If a system is bounded, then an initial function must be defined only over a corresponding interval $0 < z < L$, while boundary conditions are specified at its edges (e.g. fixed values $x(t, 0) = x(t, L) = 0$). In the latter case, one speaks of a boundary problem.

PDEs can exhibit both such attractors as fixed points, limit cycles, other kinds of low-dimensional behaviour and a very high-dimensional dynamics. This mathematical tool is even richer with properties and more complex for investigation compared to all the above-mentioned model equations. Of basic interest is the question about conditions of existence and uniqueness of a solution to a PDE. In part due to it, recently researchers have paid much attention to *regimes with sharpening* (when a solution exists only over a finite time interval) which are quite typical (Malinetsky and Potapov, 2000, pp. 148–170).

A huge body of literature is devoted to PDEs, (e.g. Loskutov and Mikhailov, 2007; Mikhailov and Loskutov, 1989; Sveshnikov et al., 1993; Tikhonov and Samarsky, 1972; Vladimirov, 1976).

3.8 Artificial Neural Networks

Artificial neural network (ANN) is a kind of mathematical model whose construction mimics some principles of organisation and functioning of networks of brain nerve cells (neurons). The idea is that each neuron can be modelled with a sufficiently simple automaton (an artificial neuron), while the entire brain complexity,

flexibility and other important properties are determined by the couplings between neurons. The term "neural networks" was established in the middle of the 1940s (McCulloc and Pitts, 1943). Very active investigations in this field were carried out until the 1970s. After that, a significant decrease in the attention of researchers took place. In the 1980s, the interest reappeared due to problems of associative memory and neurocomputers so that the number of international conferences on ANNs and neurocomputers has reached a hundred by the end of twentieth century.

If an artificial neuron represents a function relating input and output values and a signal can propagate in a network only in one direction (no feedbacks), then an ANN is also just a function transforming an input signal into an output value. Below, we briefly consider mainly such a simple version. If feedbacks are present and/or a neuron is a system with its own dynamics (namely, a discrete map), then an ANN is a multidimensional map, i.e. a set of coupled maps with specific properties of the elements and couplings (see, e.g., Ivanchenko et al., 2004). Analogously, if a neuron is described with ordinary differential equations, then the respective ANN is a set of coupled ODEs (see, e.g., Kazantsev, 2004; Kazantsev and Nekorkin, 2003; 2005). Thus, complexity of an ANN dynamics depends on the kind of basic elements, the number of basic elements and couplings between them.

3.8.1 Standard Formal Neuron

Such an artificial neuron consists of an adaptive integrator and a non-linear converter (Fig. 3.23a). A vector of values $\{x_i\}$ is fed to its inputs. Each input x_i is supplied with a certain weight w_i. The integrator performs weighted (adaptive) summation of inputs

$$S = \sum_{i=1}^{n} w_i x_i . \tag{3.41}$$

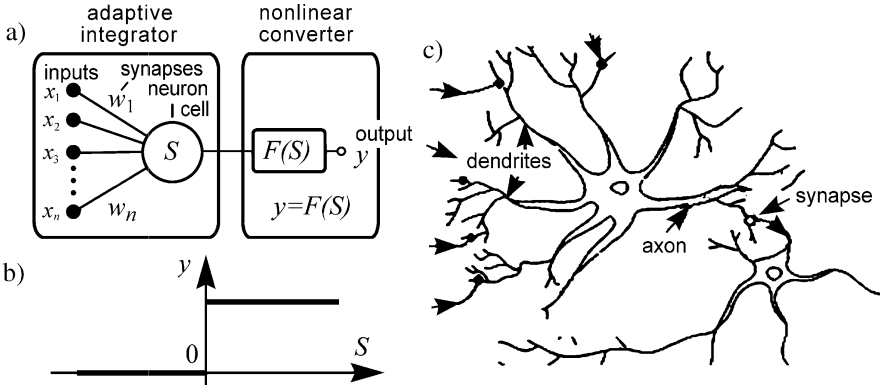

Fig. 3.23 Formal and biological neurons: (**a**) a scheme of an artificial neuron; (**b**) a plot of the unit step function; (**c**) a biological neuron (*filled circles* mark input synapses, *open ones* mark an output, triangles mark direction of excitation propagation)

The non-linear converter forms an output signal of a neuron as

$$y = F(S). \tag{3.42}$$

The choice of the neuron activation function F is determined by (i) specificity of a problem; (ii) convenience of realisation with a computer, an electric scheme or another tool; (iii) a "learning" algorithm (some learning algorithms impose constraints on the activation function properties, Sect. 3.8.4). Most often, the kind of non-linearity does not principally affect a problem solution. However, a successful choice can reduce duration of learning several times. Initially, one used the "unit step" as a function F:

$$F(S) = \begin{cases} 0, & S < 0, \\ 1, & S \geq 0, \end{cases} \tag{3.43}$$

whose plot is shown in Fig. 3.23b. Currently, a list of possible activation functions would occupy a huge space (e.g. Gorban' et al., 1998; http://www.neuropower.de). In particular, a widespread version is a non-linear function with saturation, the so-called logistic function or classical sigmoid:

$$F(S) = \frac{1}{1 + \mathrm{e}^{-\alpha S}}. \tag{3.44}$$

With decrease in α, sigmoid gets flatter tending to a horizontal straight line at the level of 0.5 in the limit of $\alpha \to 0$. With increase in α, sigmoid tends to the unit step function (3.43).

Input values of the variable x can be likened to excitations of a real-world (biological) neuron (Fig. 3.23c) coming from dendrites of surrounding neurons via synapses (connection places). A real-world neuron can have the number of dendrites ranging from units to dozens of thousands. They provide information about the states of surrounding cells coupled to the neuron. Coupling strengths in a model are reflected by the weight coefficients w_i. Carrier of information in nerve cells is a jump of a membrane potential (a neural pulse, a *spike*). It is formed in a cell after a joint action of dendrites exceeds some critical value that is modelled in a formal neuron by the summation and the non-linear function. A spike propagates via an axon as a wave of membrane polarisation. Coming to a synapse, such a wave induces secretion of substances (neurotransmitters) which diffuse into dendrites of the neurons coupled to a given axon and are converted by receptors into an electric excitation pulse.[13] After generation of a pulse, a cell turns out unreceptive to external influences for a certain time interval. Such a state is called *refractory*. In other words, one deals with an excitable system which can be in the *resting phase* (before generation), *excitation phase* (during conduction of a pulse) and *refractory*

[13] As well, there are purely electric mechanisms of neuron coupling.

phase (during a certain interval after a pulse). The refractory period determines a limit possible frequency of pulse generation (less than 200 Hz).

3.8.2 Architecture and Classification of Neural Networks

To construct an ANN, one usually selects one of several standard architectures and removes superfluous elements or adds (more rarely) new ones. Two architectures are regarded as basic ones: fully connected and (multi)layered networks. In *fully connected neural networks* each neuron sends its output signal to all the neurons including itself. All input signals are sent to all neurons. Output signals of a network can be defined as all or some of neuron output signals after several steps of network functioning.

In *multi-layer neural networks*, neurons are combined in layers (Fig. 3.24). A layer contains neurons with the same input signals. The number of neurons in a layer is arbitrary and does not depend on the number of neurons in other layers. In general, a network consists of several layers which are enumerated from left to right in Fig. 3.24. External input signals are fed to inputs of neurons of an input layer (it is often enumerated as 0th), while the network outputs are output signals of the last layer. Apart from an input and an output layer, a multi-layered network may contain one or several *hidden layers*. Depending on whether the next layers send their signals to previous ones, one distinguishes between feed-forward networks (without feedbacks) and recurrent networks (with feedbacks). We note that after introduction of feedbacks a network is no longer a simple mapping from a set of input vectors to a set of output vectors. It becomes a dynamical system of high dimensionality and the question about its stability arises. Besides, neural networks can be divided into the following:

(i) Homogeneous and heterogeneous (i.e. with the same activation function for all neurons or with different activation functions);
(ii) Binary (operate with binary signals consisting of 0s and 1s) and analogous (operate with real-valued numbers);
(iii) Synchronous and asynchronous.

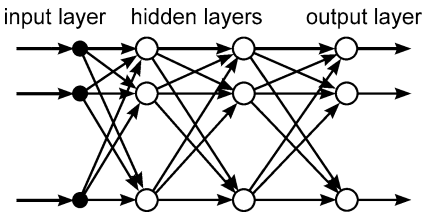

Fig. 3.24 A three-layer feed-forward network (a perceptron)

As well, ANNs differ in the number of layers. Theoretically, the number of layers and the number of neurons in each layer can be arbitrary. However, they are bounded in fact by computational resources realising a neural network. The more complex a network is, the more complicated tasks it can solve.

3.8.3 Basic Properties and Problems

Despite primitivism in comparison with biological systems, even multi-layer feed-forward ANNs possess a number of useful properties and are capable of solving quite important tasks. Those properties are as follows.

(i) *Learning ability*. After selection of an ANN architecture and neuron properties, one can "train" an ANN to solve some problem with the aid of a certain learning algorithm. There are no guarantees that it is always possible but in many cases learning appears successful.
(ii) *Generalisation capability*. After the learning stage, a network becomes insensitive to small variations in an input signal (noise) and gives a correct result at its output.
(iii) *Abstraction capability*. If several distorted variants of an input image are presented to a network, the latter can itself create at its output an ideal image, which has never been met by it previously.

Among the tasks solved with ANNs, we note pattern (e.g. visual or auditory images) recognition, associative memory[14] realisation, clustering (division of an investigated set of objects into groups of similar ones), approximation of functions, time series prediction (Sect. 10.2.1), automatic control, decision making, diagnostics.

Many of the listed tasks are reduced to the following mathematical formulation. It is necessary to construct a map $X \to Y$ such that a correct output signal Y is formed in response to each possible input X. The map is specified by a finite number of pairs (an input, a correct output). The number of those pairs (learning examples) is significantly less than the total number of possible input signals. A set of all learning examples is called a *learning sample*. For instance, in image recognition, an input X is some representation of an image (a figure, a vector), an output Y is the number of a class to which an input image belongs. In automatic control, X is a set of values of the control parameters of an object, Y is a code determining an action appropriate for the current values of control parameters. In forecast, an input signal is a set of

[14] In von Neumann's model of computations (realised in a usual computer), memory access is possible only via an address, which does not depend on the memory contents. Associative memory is accessible based on the current contents. Memory contents can be called even by partial or distorted contents.

values of an observable quantity until a current time instant and an output is a set of the next values of an observable.

All these and many other applied problems can be reduced to a problem of construction of some multivariate function. What are capabilities of ANNs in this respect? As it was illustrated, they compute *univariate* linear and non-linear functions and their compositions obtained due to cascade connection of neurons. What can one get with the use of such operations? What functions can be accurately approximated with ANNs? As a result of long-lasted polemics between Kolmogorov and Arnold, a possibility of exact representation of a continuous multivariate function via a composition of univariate continuous functions and summation was shown (Arnold, 1959; Kolmogorov, 1957). The most complete answer to the question about approximating properties of neural networks is given by Stone's theorem (Stone, 1948) stating universal approximating capabilities of an arbitrary non-linearity: linear operations and cascade connection allow to get a device approximating any continuous multivariate function to any required accuracy on the basis of an arbitrary non-linear element. A popular exposition of the theorems of Kolmogorov and Stone in application to ANNs is given in Gorban' (1998). Thus, neurons in a network may have practically any non-linear activation function, only the fact of its non-linearity is important. In principle, ANNs are capable of doing "very many things". Yet, an open question is: How to teach them to do it?

3.8.4 Learning

During its functioning, a neural network forms an output signal Y corresponding to an input signal X, i.e. realises a certain function $Y = g(X)$. If a network architecture is specified, then the values of g are determined by synaptic weights. The choice of their optimal values is called *network learning*. There are various approaches to learning.

Learning by instruction. Here, one uses a learning sample, i.e. pairs of known input and output values $(X_1, Y_1), \ldots, (X_N, Y_N)$.

Let the values of vectors X and Y be related via $Y = g(X)$, in particular, $Y_i = g(X_i)$, $i = 1, \ldots, N$. A function g is unknown. We denote E as an error function assessing deviation of an arbitrary function f from the function g. Solving a problem with an ANN of a given architecture means to construct a function f by selecting synaptic weights so as to minimise the error function. In the simplest case, learning consists of searching for a function f which minimises E over a learning sample. Given a learning sample and the form of function E, learning of a network turns into a multidimensional non-linear optimisation problem (Dennis and Schnabel, 1983), which is often very complicated in practice (see also Sect. 10.2.1). It requires time-consuming computations and represents an iterative procedure; the number of iterations ranges typically from 10^3 to 10^8.

Since creation of intellectual schemes is based to a significant extent on biological prototypes, researchers still discuss whether the algorithms of learning by instruction can be considered as analogues to natural learning processes or they are

completely artificial. It is known that, for instance, neurons of visual cortex learn to react on light pulses only under the influence of the pulses themselves without an external teacher. In particular, we are able to solve such a complicated task as image recognition. However, higher stages of learning (e.g. for children) are impossible without a teacher (their parents). Besides, some brain areas are quite able to play a role of "teacher" for other areas by controlling their activity. Therefore, it is not possible to claim uniquely which type of learning (with a teacher or without it) is more biologically plausible.

Learning without a teacher. In a widespread version it is as follows. There is a set of input vectors. A set of output vectors is absent. Learning a network means selecting its parameter values so that it would classify input vectors in some "optimal" way. An ANN must divide a set of input vectors into groups (classes) so that each class contains vectors close to each other while differences between classes are relatively big. This is done via optimisation of a cost function involving the two mentioned factors. When a new input vector is presented, a learned network attributes it to one of the classes which have been formed by it previously (without a teacher). One of the most well-known examples of such a way to solve classification problems is the learning of Cohonen network (see, e.g., Gorban' et al., 1998).

Currently, there is a huge body of literature on neural networks highlighting very different questions ranging from the choice of the ANN architecture to its learning and practical applications. In particular, there are many works accessible to a wide readership (Gorban', 1998; Gorban' et al., 1998; Loskutov and Mikhailov, 2007; Malinetsky and Potapov, 2000, pp. 171–203; Wasserman, 1989; http://www.neuropower.de). Some additional details and examples of ANN applications to modelling from time series are given in Sect. 10.2.1.

Thus, several representative classes of deterministic models are discussed in Sects. 3.5, 3.6, 3.7 and 3.8. Roughly speaking, we have described them in the order of increasing complexity in terms of the phase space dimension, fractal dimension of possible attractors, the number of positive Lyapunov exponents, the diversity of possible dynamical regimes and configurations of the parameter space. Yet, linear ordering most often appears impossible; therefore, we have presented more specific discussion of the complexity for each example separately. To summarise, the presented deterministic models are capable of describing huge number of phenomena observed in real-world systems ranging from quite simple ones (e.g. an equilibrium state, a limit cycle and a linear resonance) to very complex (e.g. high-dimensional chaotic motions, transition to chaos and diverse bifurcations in multi-parametric non-linear systems).

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