Barrier Methods for a Control Problem from Hyperthermia Treatment Planning

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Summary. We consider an optimal control problem from hyperthermia treatment planning and its barrier regularization. We derive basic results, which lay the groundwork for the computation of optimal solutions via an interior point path-following method in function space. Further, we report on a numerical implementation of such a method and its performance at an example problem.

1 Hyperthermia Treatment Planning

Regional hyperthermia is a cancer therapy that aims at heating up deeply seated tumors in order to make them more susceptible to an accompanying chemo or radio therapy [12]. We consider a treatment modality where heat is induced by a phased array microwave ring-applicator containing 12 antennas. Each antenna emits a time-harmonic electromagnetic field the amplitude and phase of which can be controlled individually. The linearly superposed field acts as a heat source inside the tissue. We are interested in controlling the resulting stationary heat distribution, which is governed by a semi-linear elliptic partial differential equation, the bio-heat transfer equation (BHTE), see [7]. The aim is to heat up the tumor as much as possible, without damaging healthy tissue. We thus have to impose constraints on the temperature, and mathematically, we have to solve an optimization problem subject to a PDE as equality constraint and pointwise inequality constraints on the state.

We consider an interior point path-following algorithm that has been applied to this problem. In order to treat the state constraints, the inequality constraints are replaced by a sequence of barrier functionals, which turn the inequality constrained problem into a sequence of equality constrained problems. We will show existence of barrier minimizers and derive first and second order optimality conditions, as well as as local existence and differentiability of the path, and local convergence of Newtons method. Our work extends the results of [10], which covers the case of linear PDE constraints, to a problem with a non-linear control-to-state mapping, governed by a semi-linear PDE.

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1.1 The Bio-Heat Transfer Equation

The stationary bio-heat transfer equation was first introduced in [7] to model the heat-distribution T in human tissue. This partial differential equation is a semi-linear equation of elliptic type, which can be written as $A(T) - B(u) = 0$, where $A(T)$ is a differential operator, applied to the temperature distribution, and $B(u)$ is a source term, which can be influenced by complex antenna parameters $u \in \mathbb{C}^{12}$.

More concretely, we set $v := (T, u)$ and consider the following equation in the weak form on a domain $\Omega \subset \mathbb{R}^3$, which is an individual model of a patient:

$$\begin{aligned}
\langle A(T), \varphi \rangle &:= \int_\Omega \langle \kappa \nabla T, \nabla \varphi \rangle_{\mathbb{R}^3} + w(T)(T - T_0)\varphi \, dx + \int_{\partial\Omega} h(T - T_{out})\varphi \, dS, \\
\langle B(u), \varphi \rangle &:= \int_\Omega \frac{\sigma}{2} \, |E(u)|^2_{\mathbb{C}^3} \, \varphi \, dx \\
\langle c(v), \varphi \rangle &:= \langle A(T) - B(u), \varphi \rangle = 0 \quad \forall \varphi \in C^\infty(\Omega),
\end{aligned}$$

where all coefficients may depend of the spacial variable x, and $E(u) = \sum_{k=1}^{12} E_k u_k$ is the superposition of complex time-harmonic electro-magnetic fields, and u_k are the complex coefficients of the control. Further, κ is the temperature diffusion coefficient, σ is the electric conductivity and $w(T)$ denotes the blood perfusion. By T_0, we denote the temperature of the unheated blood, e.g. 37°C. The domain Ω consists of a number of subdomains Ω_i, corresponding to various types of tissue. All coefficients may vary significantly from tissue type to tissue type. For a more detailed description of the parameters we refer to [2].

Assumption 4 *Assume that $\kappa, \sigma \in L_\infty(\Omega)$ are strictly positive on Ω. Similarly, let $h \in L_\infty(\partial\Omega)$ be strictly positive on $\partial\Omega$. Further, assume that $w(T, x)(T - T_0)$ is strictly monotone, bounded and measurable for bounded T, and twice continuously differentiable in T. Assume also that each electric field E_k is contained in $L_{q_E}(\Omega, \mathbb{C}^3)$ for some $q_E > 3$.*

Remark 1. Our assumptions are chosen in a way that that the temperature distribution inside the body is bounded and continuous, while still covering the case of jumping coefficients due to different tissue properties inside the patient models. Also the assumptions on the regularity of the fields $E_k \in L_{q_E}, q_E > 3$ are necessary for guaranteeing continuity of the temperature distribution (cf. e.g. [4, Thm. 6.6]). For the generic regularity $E_k \in L_2$ this cannot be guaranteed a-priori. In clinical practice, of course, pointwise unbounded temperature profiles do not occur. Overly large intensity peaks are avoided by construction of the applicator. However, it is observed that near tissue boundaries so called *hot spots* occur: small regions, where the temperature is significantly higher than in the surrounding tissue due to singularities in the electro-magnetic fields at tissue boundaries. One of the challenges of optimization is to eliminate these hot spots.

Under these assumption we can fix our functional analytic framework. As usual in state constrained optimal control, we have to impose an $\|\cdot\|_\infty$-topology on the space of temperature distributions. To this end, let q be in the range $q_E > q > 3$, and $q' = q/(q-1)$ its dual exponent. We define $V = C(\overline{\Omega}) \times \mathbb{C}^{12}$ and

$$c : (C(\overline{\Omega}) \supset D_q) \times \mathbb{C}^{12} \to (W^{1,q'})^*,$$

where D_q is the set of all T, such that $A(T) \in (W^{1,q'})^*$, i.e. $\langle A(T), \varphi\rangle \leq M\|\varphi\|_{W^{1,q'}}\ \forall \varphi \in C^\infty(\overline{\Omega})$. By suitable regularity assumptions $D_q = W^{1,q}(\Omega)$, a result, which we will, however, not need.

It is well known (cf. e.g. [11, 4]) that A has a continuous inverse $A^{-1} : (W^{1,q'})^* \to C(\overline{\Omega})$, and even $\|T\|_{C^\beta} \leq c\|A(T)\|_{(W^{1,q'})^*}$ for some $\beta > 0$ locally, where C^β is the space of Hölder continuous functions. Moreover, it is straightforward to show that D_q only depends on the main part of A, and is thus independent of T.

Lemma 1. *The mapping* $c(v) : (C(\overline{\Omega}) \supset D_q) \times \mathbb{C}^{12} \to (W^{1,q'}(\Omega))^*$ *is twice continuously Fréchet differentiable. Its derivatives are given by*

$$\langle c'(v)\delta v, \varphi\rangle = \langle A'(T)\delta T - B'(u)\delta u, \varphi\rangle$$

$$\langle A'(T)\delta T, \varphi\rangle = \int_\Omega \langle \kappa\nabla\delta T, \nabla\varphi\rangle_{\mathbb{R}^3} + (w'(T)(T-T_0)+w(T))\delta T\varphi\, dx + \int_{\partial\Omega} h\delta T\varphi dS$$

$$\langle B'(u)\delta u, \varphi\rangle = \int_\Omega \sigma \mathrm{Re}\left\langle \sum_{k=1}^{12} E_k u_k, \sum_{k=1}^{12} E_k \delta u_k\right\rangle_{\mathbb{C}^3} \varphi\, dx$$

$$\langle c''(v)(\delta v)^2, \varphi\rangle = \langle A''(T)(\delta T)^2 - B''(u)(\delta u)^2, \varphi\rangle =$$

$$= \int_\Omega (w''(T)(T-T_0) + 2w'(T))\delta T^2\varphi - \sigma\mathrm{Re}\left\langle \sum_{k=1}^{12} E_k\delta u_k, \sum_{k=1}^{12} E_k\delta u_k\right\rangle_{\mathbb{C}^3} \varphi\, dx.$$

Proof. Since all other parts are linear in T, it suffices to show Fréchet differentiability of $T \to w(T,x)(T-T_0)$ and $u \to |E(u,x)|^2$. Since by assumption, $w(T,\cdot) \in C^1(\Omega)$, differentiability of $T \to w(T,x)(T-T_0) : C(\overline{\Omega}) \to L_t(\Omega)$ for every $t < \infty$ follows from standard results of Nemyckii operators (cf. e.g. [3, Prop. IV.1.1], applied to remainder terms). By the dual Sobolev embedding $L_t(\Omega) \hookrightarrow (W^{1,q'}(\Omega))^*$ for sufficiently large t, differentiability of $T \to w(T,x)(T-T_0) : C(\overline{\Omega}) \supset D_q \to (W^{1,q'}(\Omega))^*$ is shown.

Similarly, differentiability of the mapping $u \to |E(u,x)|^2 : \mathbb{C}^{12} \to L_s(\Omega)$ for some $s > 3/2$ follows by the chain rule from the linearity of the mapping $u \to E(u,x) : \mathbb{C}^{12} \to L_{q_E}(\Omega, \mathbb{C}^3)$ and the differentiability of the mapping $w \to |w|^2 : L_{q_E}(\Omega,\mathbb{C}^3) \to L_{q_E/2}(\Omega,\mathbb{C}^3)$ with $q_E/2 = s > 3/2$. Again, by the dual Sobolev embedding $L_s(\Omega) \hookrightarrow (W^{1,q'}(\Omega))^*$ we obtain the desired result.

Similarly, one can discuss the second derivatives. We note that $(|E(u,x)|^2)'$ is linear in u, and thus it coincides with its linearization.

Remark 2. Note that $A' : C(\overline{\Omega}) \supset D_q \to (W^{1,q'}(\Omega))^*$ is not a continuous linear operator, but since it has a continuous inverse, it is a closed operator. Moreover, since the main part of A is linear, $A'(T)-A'(\tilde{T})$ contains no differential operator. Hence $\|\tilde{T}-T\|_\infty \to 0$ implies $\|A'(T)-A'(\tilde{T})\|_{C(\overline{\Omega})\to(W^{1,q'})^*} \to 0$. These facts allow us to apply results, such as the open mapping theorem and the inverse function theorem to A.

Lemma 2. *For each $v \in D_q \times \mathbb{C}^{12}$ the linearization*

$$c'(v) = A'(T) - B'(u) : D_q \times \mathbb{C}^{12} \to (W^{1,q'}(\Omega))^*$$

is surjective and has a finite dimensional kernel.

For each v with $c(v) = 0$ there is a neighborhood $U(v)$ and a local diffeomorphism

$$\psi_v : \ker c'(v) \leftrightarrow U(v) \cap \{v : c(v) = 0\},$$

satisfying $\psi_v'(0) = Id$ and $c'(v)\psi_v''(0) = -c''(v)$.

Proof. It follows from the results in [4] that $A'(T)$ has a continuous inverse $A'(T)^{-1} : (W^{1,q'}(\Omega))^* \to C(\overline{\Omega})$. Since A' is bijective, also $c'(v) = (A'(T), -B'(u))$ is surjective, and each element $\delta v = (\delta T, \delta u)$ of $\ker c'$ can be written in the form $(A'(T)^{-1}B'(u)\delta u, \delta u)$. Since $\delta u \in \mathbb{C}^{12}$, $\ker c'(v)$ is finite dimensional. Via the inverse function theorem we can now conclude local continuous invertibility of A, and also that A^{-1} is twice differentiable.

Let $(\delta T, \delta u) = \delta v \in \ker c'(v)$. Then we define

$$\psi_v(\delta v) := \begin{pmatrix} (A^{-1} \circ B)(u + \delta u) \\ u + \delta u \end{pmatrix}$$

and compute

$$\begin{aligned} (A^{-1} \circ B)'(v)\delta u &= A'(T)^{-1}B'(u)\delta u = \delta T \\ (A^{-1} \circ B)''(v)(\delta u)^2 &= -A'(T)^{-1}A''(T)A'(T)^{-1}(B'(u)\delta u)^2 + A'(T)^{-1}B''(u)(\delta u)^2 \\ &= -A'(T)^{-1}\left(A''(T)(\delta T)^2 - B''(u)(\delta u)^2\right). \end{aligned}$$

It follows

$$\begin{aligned} \psi_v'(0)\delta v &= (\delta T, \delta u) = \delta v \\ c'(v)\psi_v''(0)(\delta v)^2 &= (A'(T), -B'(u))\psi_v''(0)(\delta v)^2 \\ &= -(A''(T)(\delta T)^2 - B''(u)(\delta u)^2) = -c''(v)(\delta v)^2. \end{aligned}$$

1.2 Inequality constraints and objective

As for inequality constraints, we impose upper bounds on the amplitudes of the controls to model the limited power of the microwave applicator:

$$|u_k| \leq u_{\max}, \quad k = 1 \ldots 12.$$

Moreover, crucially, we impose upper bounds on the temperature inside the healthy tissue. These are state constraints, which pose significant practical and theoretical difficulties. These constraints are necessary to avoid excessive heating of healthy tissue, which would result in injuries of the patient. We have

$$T \leq T_{\max}(x),$$

where $T_{\max}$ is chosen as a piecewise constant function on each tissue type, depending on the sensitivity of the tissue with respect to heat.

Algorithmically, we treat the inequality constrained optimization problem in function space by a barrier approach (cf. [10]) and replace the inequality constraints by a sequence of barrier functionals, depending on a parameter μ (setting again $v = (T, u)$):

$$b(v;\mu) = \int_\Omega l(T_{\max} - T;\mu)\,dx - \mu \sum_{i=1}^{12} \ln(u_{\max} - |u_k|)$$

here l may be a sum of logarithmic and rational barrier functionals:

$$\begin{aligned} &l_k(\cdot;\mu) : \mathbb{R}_+ \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\} \\ &\qquad l_1(t;\mu) := -\mu \ln(t), \qquad l_k(t;\mu) := \mu^k/((k-1)t^{k-1}) \ (k > 1) \end{aligned}$$

A straightforward computation shows that $b(v;\mu)$ is a convex function (as a composition of convex and convex, monotone functions), and it is also clear that for strictly feasible v, $b : C(\overline{\Omega}) \times \mathbb{C}^{12}$ is twice continuously differentiable near v, and thus locally Lipschitz continuous there. It has been shown in [10] that b is also lower semi-continuous.

Finally, we consider an objective functional $J : C(\overline{\Omega}) \times \mathbb{C}^{12} \to \mathbb{R}$, which we assume to be twice continuously differentiable, and thus locally Lipschitz continuous. For our numerical experiments, below, we will choose a simple objective of the form $J(v) = \|T - T_{des}\|^2_{L_2}$ (recall that the control is finite dimensional), but more sophisticated functionals are under consideration, which more directly model the damage caused in the tumor.

Summarizing, we can write down regularized optimal control problem:

$$\min_{v \in V} J_\mu(v) := J(v) + b(v;\mu) \text{ s.t. } c(v) = 0. \tag{1}$$

2 Barrier Minimizers and their Optimality Conditions

Next we study existence and basic properties of solutions of the barrier problems. For this purpose we impose the assumption that there is at least one strictly feasible solution. This is fulfilled, for example by $u = 0$, if the upper bounds $T_{\max}$ are chosen reasonably.

Theorem 1. *For every $\mu > 0$ the barrier problem* (1) *has an optimal solution, which is strictly feasible with respect to the inequality constraints.*

Proof. Since the set of feasible controls is finite dimensional, closed, and bounded and by our assumptions the control-to-space mapping $u \to T$ is continuous (cf. e.g. [4, Thm. 6.6] and the discussion after that theorem), the set of all feasible pairs (T, u) is compact in $C(\overline{\Omega}) \times \mathbb{C}^{12}$. By assumption, there is at least one strictly feasible solution, for which $J + b$ takes a finite value. Hence, existence of an optimal solution follows immediately from the Theorem of Weierstraß (its generalization for lower semi-continuous functions).

Since all solutions of our PDE are Hölder continuous, strict feasibility for sufficiently high order of the barrier functional follows from [10, Lemma 7.1].

Lemma 3. *If v_μ is a locally optimal solution of* (1)*, then $\delta v = 0$ is a minimizer of the following convex problem:*

$$\min_{\delta v} J'(v_\mu)\delta v + b(v_\mu + \delta v; \mu) \;\; s.t. \;\; c'(v_\mu)\delta v = 0 \tag{2}$$

Proof. For given, $\delta v \in \ker c'(v_\mu)$, and $t > 0$ let $\tilde{v} = v_\mu + t\delta v$. By Lemma 2 there are $\hat{v} = \psi_{v_\mu}(\delta v)$, such that $c(\hat{v}) = 0$ and $\hat{v} - \tilde{v} = o(t)$. Further, by strict feasibility of v_μ, $J + b$ is locally Lipschitz continuous near v_μ with Lipschitz constant L_{J+b}. We compute

$$\begin{aligned} J'(v_\mu)(t\delta v) + b'(v_\mu;\mu)(t\delta v) &= (J+b)(\tilde{v};\mu) - (J+b)(v_\mu;\mu) + o(t) \\ &= (J+b)(\hat{v};\mu) - (J+b)(v_\mu;\mu) + (J+b)(\tilde{v};\mu) - (J+b)(\hat{v};\mu) + o(t) \\ &\geq 0 + L_{J+b}o(t) + o(t). \end{aligned}$$

it follows $J'(v_\mu)\delta v + b'(v_\mu;\mu)\delta v \geq 0$, and by linearity $J'(v_\mu)\delta v + b'(v_\mu;\mu)\delta v = 0$. By convexity of b we have $b'(v_\mu;\mu)\delta v \leq b(v_\mu + \delta v;\mu) - b(v_\mu;\mu)$ and thus

$$J'(v_\mu)\delta v + b(v_\mu + \delta v;\mu) - b(v_\mu;\mu) \geq 0$$

which proofs our assertion.

Theorem 2. *If v_μ is a locally optimal solution of* (1)*, then there exists a unique $p \in H^1(\Omega)$, such that*

$$0 = F(v, p; \mu) := \begin{cases} J'_\mu(v_\mu) + c'(v_\mu)^* p, \\ c(v_\mu). \end{cases} \tag{3}$$

Proof. Clearly, the second row of (3) holds by feasibility of v_μ. By Lemma 3 $\delta v = 0$ is a minimizer of the convex program (2). Hence, we can apply [10, Thm. 5.4] to obtain first order optimality conditions for this barrier problem with $p \in W^{1,p'}(\Omega)$. Taking into account strict feasibility of v_μ with respect to the inequality constraints, all elements of subdifferentials in [10, Thm. 5.4] can be replaced by Fréchet derivatives, so (3) follows. In particular, p satisifies the adjoint equation $\partial_y J_\mu(v_\mu) + A'(T)^* p = 0$, which can be interpreted as a PDE in variational form with $\partial_y J_\mu(v_\mu) \in L_\infty(\Omega)$, and thus $p \in H^1(\Omega)$ follows.

Before we turn to second order conditions we perform a realification of the complex vector $u \in \mathbb{C}^{12}$. Since $|E(u, x)|$ only depends on the the relative phase shifts of the antenna parameters, optimal controls of our problem are non-unique. This difficulty can be overcome easily by fixing $\mathrm{Im}(u_1) = 0$. After that, realification $(x + iy \to (x, y))$ yields a new control vector $u \in \mathbb{R}^{23}$ (dropping the component that corresponds to $\mathrm{Im}(u_1)$), which we will use in the following. We define the Hessian of the Lagrangian $H(v; p)$ by

$$H(v, p)\delta v^2 = J''_\mu(v)\delta v^2 + \langle p, c''(v)\delta v^2\rangle$$

Theorem 3. *Let* (v_μ, p_μ) *be a solution of* (3)*. Then,*

$$\frac{1}{2}H(v_\mu, p_\mu)\delta v^2 = J_\mu(\psi_{v_\mu}(\delta v)) - J_\mu(v_\mu) + o(\|\delta v\|^2). \tag{4}$$

(i) $H(v_\mu, p_\mu)$ *is positive semi-definite on* $\ker c'(v_\mu)$*, if* v_μ *is a local minimizer of* (1)*.*

(ii) $H(v_\mu; p_\mu)$ *is positive definite on* $\ker c'(v_\mu)$*, if and only if* v_μ *is a local minimizer of* (1) *and* J_μ *satisfies a local quadratic growth condition.*
Then for each $(r_1, r_2) \in ((H^1(\Omega))^* \times \mathbb{R}^{23}) \times (W^{1,q'}(\Omega))^*$ *the linear system*

$$\begin{pmatrix} H(v_\mu, p_\mu) & c'(v_\mu)^* \\ c'(v_\mu) & 0 \end{pmatrix} \begin{pmatrix} \delta v \\ \delta p \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \tag{5}$$

has a unique solution $(\delta v, \delta p) \in V \times H^1(\Omega)$*, depending continuously on* (r_1, r_2)*.*

Proof. Let $\delta v \in \ker c'(v_\mu)$, and ψ_{v_μ} be defined as in Lemma 2. We show (4) by Taylor expansion:

$$\begin{aligned} J_\mu(\psi_{v_\mu}(\delta v)) - J_\mu(v_\mu) =& J'_\mu(v_\mu)\psi'_{v_\mu}(0)\delta v \\ &+ 0.5\left(J''_\mu(v_\mu)(\psi'_{v_\mu}(0)\delta v)^2 + J'_\mu(v_\mu)\psi''_{v_\mu}(0)(\delta v)^2\right) + o(\|\delta v\|^2). \end{aligned} \tag{6}$$

Since $J'_\mu(v_\mu)\delta v = 0\,\forall \delta v \in \ker c'(v_\mu)$, $\psi'_{v_\mu}(0) = Id$, it follows $J'_\mu(v_\mu)\psi'_{v_\mu}(0)\delta v = 0$. Further, by $J'_\mu(v_\mu)\delta v + \langle p_\mu, c'(v_\mu)\delta v\rangle = 0\,\forall \delta v \in V$ and $c'(v_\mu)\psi''_{v_\mu}(0) = -c''(v_\mu)$ we deduce

$$J'_\mu(v_\mu)\psi''_{v_\mu}(0)(\delta v)^2 = -\langle p_\mu, c'(v_\mu)\psi''_{v_\mu}(0)(\delta v)^2\rangle = \langle p_\mu, c''(v_\mu)(\delta v)^2\rangle.$$

Inserting these two results into (6) yields (4).

All other assertions, except for solvability of (5) then follow directly, using the fact that $|\|\delta v\| - \|\psi_{v_\mu}(\delta v) - v_\mu\|| \le \|v_\mu + \delta v - \psi_{v_\mu}(\delta v)\| = o(\|\delta v\|)$.

Let us turn to (5). If $H(v_\mu; p_\mu)$ is positive definite on $\ker c'(v_\mu)$ (which is finite dimensional), then the minimization problem

$$\min_{c'(v_\mu)\delta v = r_2} -\langle r_1, \delta v\rangle + H(v_\mu; p_\mu)\delta v^2$$

is strictly convex and has a unique solution δv. The first order optimality conditions for this problem yield solvability of the system (5) at (v_μ, p_μ). Since we have assumed $r_1 \in (H^1)^* \times \mathbb{R}^{23}$ and $A'(T_\mu)^* : H^1 \to H^{-1}$ is an isomorphism, we obtain $\delta p \in H^1$. Thus, the matrix in (5) is surjective, and we may deduce its continuous invertibility by the open mapping theorem.

Corollary 1. *If $H(v_\mu, p_\mu)$ is positive definite on $\ker c'(v_\mu)$, then, locally, there is a differentiable path $\mu \to z_\mu$ of local minimizers of the barrier problems, defined in some open interval $]\overline{\mu}, \underline{\mu}[\supset \mu$. Further, Newton's method, applied to $F(v, p; \mu)$ converges locally superlinearly to (v_μ, p_μ).*

Proof. We note that $F(v, p; \mu)$ is differentiable w.r.t. μ, and w.r.t. (v, p). Since $F' = dF/d(v, p)$, given by (5) is continuously invertible, local existence and differentiability follows from the implicit function theorem. Since $F'(v, p; \mu)$ depends continuously on (v, p), we can use a standard local convergence result for Newton's method (cf. e.g. [6, Thm. 10.2.2]).

Remark 3. Since all these results depend on the positive definiteness of H, we cannot expect to obtain global convergence results for barrier homotopy paths. From a global point of view, several branches may exist, and if H is only positive semi-definite at a point of one such branch, it may cease to exist or bifurcate. As a consequence, a local Newton path-following scheme should be augmented by a globalization scheme. for non-convex optimization in the spirit of trust-region methods. This is subject to current reasearch.

3 Numerical results

For the optimization of the antenna parameters we use an interior point path-following method, applying Newton's method to the system (3). As barrier functional we use the sum of rational barrier functionals, and the reduction of the barrier parameter is chosen adaptively in the spirit of [1, Chapt. 5] by an affine covariant estimation of the non-linearity of the barrier subproblems. Further, Newton's method is augmented by a pointwise damping step. A more detailed description of this algorithm can be found in [9]. This algorithm can be applied safely in a neighborhood of the barrier homotopy path, as long as positive definiteness of $H(v_\mu, p_\mu)$ holds. In practice, this works well, as long as a reasonable starting guess is available for the antenna parameters. Just as predicted by the theory in the convex case (cf. [10]) the error in the function value decreases linearly with μ (cf. Figure 1, right).

The discretization of the Newton steps was performed via linear finite element spaces X_h for T and p (cf. [5]). Discretization and assembly were performed with the library KASKADE7. In view of Newton's method this gives rise to the following block matrix, which has to be factorized at each Newton step:

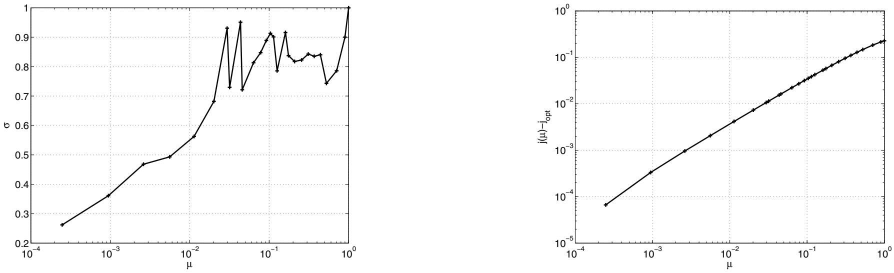

Fig. 1. Left: μ-reduction factors $\sigma_k = \mu_{k+1}/\mu_k$. Right: error in functional values.

$$F'(v,p;\mu) = \begin{pmatrix} H_1(T,p;\mu) & 0 & A'(T)^* \\ 0 & H_2(u,p;\mu) & B'(u)^* \\ A'(T) & B'(u) & 0 \end{pmatrix},$$

where

$$\begin{aligned} H_1(T,p;\mu)(v,w) &= J''(T)(v,w) + b''(T;\mu)(v,w) + \langle p, A''(T)(v,w)\rangle_{L_2(\Omega)} \\ H_2(u,p;\mu)(v,w) &= b''(u;\mu)(v,w) + \langle p, B''(u)(v,w)\rangle_{L_2(\Omega)}. \end{aligned}$$

Note that $H_2 : \mathbb{R}^{23} \to \mathbb{R}^{23}$, and $B' : \mathbb{R}^{23} \to X_h^*$ are dense matrices, while $A', H_1 : X_h \to X_h^*$ are sparse. The factorization of this matrix is performed via building a Schur complement for the $(2,2)$-block, so that essentially only a sparse factorization of A' and a couple of back-solves have to be performed via a direct sparse solver. As an alternative one can use an iterative solver, preconditioned by incomplete factorizations as proposed in [8].

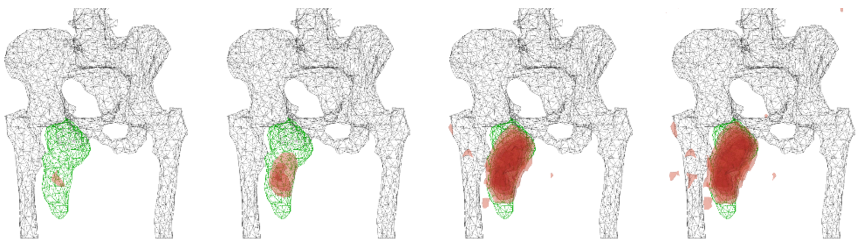

Fig. 2. Heat distribution inside body for $\mu = 1.0, 0.7, 0.1, 10^{-4}$ (left to right).

Let us consider the development of the stationary heat distribution during the algorithm in Figure 2. We observe the effect of the barrier regularization. The algorithm starts with a very conservative choice of antenna parameters, an tends to a more and more aggressive configuration, as μ decreases. This may be of practical value for clinicians. Further, it is interesting to observe that already at a relatively large value of $\mu = 0.1$, we are rather close to the optimal solution. This is reflected by the choice of steps (cf. Figure 1).

4 Conclusion and Outlook

In this work basic results in function space for barrier methods applied to a hyperthermia planning problem with state constraints were established. The theory extends known results from the convex case. While the set of assumptions is taylored for hyperthermia, it is clear that the theory also applies to a wider class of optimal control problems, as long as appropriate regularity results for the involved differential equation are at hand. Subject of current research is the extension of our algorithm by a globalization scheme in the spirit of non-linear programming, in order to increase its robustness in the presence of non-convexity.

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