
Fixed-Order H-infinity Optimization of Time-Delay Systems

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Summary. H-infinity controllers are frequently used in control theory due to their robust performance and stabilization. Classical H-infinity controller synthesis methods for finite dimensional LTI MIMO plants result in high-order controllers for high-order plants whereas low-order controllers are desired in practice. We design fixed-order H-infinity controllers for a class of time-delay systems based on a non-smooth, non-convex optimization method and a recently developed numerical method for H-infinity norm computations.

Robust control techniques are effective to achieve stability and performance requirements under model uncertainties and exogenous disturbances [16]. In robust control of linear systems, stability and performance criteria are often expressed by H-infinity norms of appropriately defined closed-loop functions including the plant, the controller and weights for uncertainties and disturbances. The optimal H-infinity controller minimizing the H-infinity norm of the closed-loop functions for finite dimensional multi-input-multi-output (MIMO) systems is computed by Riccati and linear matrix inequality (LMI) based methods [8, 9]. The order of the resulting controller is equal to the order of the plant and this is a restrictive condition for high-order plants. In practical implementations, fixed-order controllers are desired since they are cheap and easy to implement in hardware and non-restrictive in sampling rate and bandwidth. The fixed-order optimal H-infinity controller synthesis problem leads to a non-convex optimization problem. For certain closed-loop functions, this problem is converted to an interpolation problem and the interpolation function is computed based on continuation methods [1]. Recently fixed-order H-infinity controllers are successfully designed for finite dimensional LTI MIMO plants using a non-smooth, non-convex optimization method [10]. This approach allows the user to choose the controller order and tunes the parameters of the controller to minimize the H-infinity norm of the objective function using the norm value and its derivatives with respect to the controller parameters. In our work, we design fixed-order H-infinity controllers for a class of time-delay systems based on a non-smooth, non-convex

optimization method and a recently developed H-infinity norm computation method [13]. H-infinity optimization control!fixed-order

1 Problem Formulation

We consider time-delay plant G determined by equations of the form,

$$\dot{x}(t) = A_0x(t) + \sum_{i=1}^m A_ix(t - \tau_i) + B_1w(t) + B_2u(t - \tau_{m+1}) \quad (1)$$

$$z(t) = C_1x(t) + D_{11}w(t) + D_{12}u(t) \quad (2)$$

$$y(t) = C_2x(t) + D_{21}w(t) + D_{22}u(t - \tau_{m+2}). \quad (3)$$

where all system matrices are real with compatible dimensions and $A_0 \in \mathbb{R}^{n \times n}$. The input signals are the exogenous disturbances w and the control signals u . The output signals are the controlled signals z and the measured signals y . All system matrices are real and the time-delays are positive real numbers. In robust control design, many design objectives can be expressed in terms of norms of closed-loop transfer functions between appropriately chosen signals w to z .

The controller K has a fixed-structure and its order n_K is chosen by the user *a priori* depending on design requirements,

$$\dot{x}_K(t) = A_Kx_K(t) + B_Ky(t) \quad (4)$$

$$u(t) = C_Kx_K(t) \quad (5)$$

where all controller matrices are real with compatible dimensions and $A_K \in \mathbb{R}^{n_K \times n_K}$.

By connecting the plant G and the controller K , the equations of the closed-loop system from w to z are written as,

$$\begin{aligned} \dot{x}_{cl}(t) &= A_{cl,0}x_{cl}(t) + \sum_{i=1}^{m+2} A_{cl,i}x_{cl}(t - \tau_i) + B_{cl}w(t) \\ z(t) &= C_{cl}x_{cl}(t) + D_{cl}w(t) \end{aligned} \quad (6)$$

where

$$\begin{aligned} A_{cl,0} &= \begin{pmatrix} A_0 & 0 \\ B_K C_2 & A_K \end{pmatrix}, \quad A_{cl,i} = \begin{pmatrix} A_i & 0 \\ 0 & 0 \end{pmatrix} \text{ for } i = 1, \dots, m, \\ A_{cl,m+1} &= \begin{pmatrix} 0 & B_2 C_K \\ 0 & 0 \end{pmatrix}, \quad A_{cl,m+2} = \begin{pmatrix} 0 & 0 \\ 0 & B_K D_{22} C_K \end{pmatrix}, \\ B_{cl} &= \begin{pmatrix} B_1 \\ B_K D_{21} \end{pmatrix}, \quad C_{cl} = (C_1 \ D_{12} C_K), \quad D_{cl} = D_{11}. \end{aligned} \quad (7)$$

The closed-loop matrices contain the controller matrices (A_K, B_K, C_K) and these matrices can be tuned to achieve desired closed-loop characteristics.

The transfer function from w to z is,

$$T_{zw}(s) = C_{cl} \left(sI - A_{cl,0} - \sum_{i=1}^{m+2} A_{cl,i} e^{-\tau_i s} \right)^{-1} B_{cl} + D_{cl} \quad (8)$$

and we define fixed-order H-infinity optimization problem as the following.

Problem Given a controller order n_K , find the controller matrices (A_K, B_K, C_K) stabilizing the system and minimizing the H-infinity norm of the transfer function T_{zw} .

2 Optimization Problem

2.1 Algorithm

The optimization algorithm consists of two steps:

1. **Stabilization:** minimizing the spectral abscissa, the maximum real part of the characteristic roots of the closed-loop system. The optimization process can be stopped when the controller parameters are found that stabilizes T_{zw} and these parameters are the feasible points for the H-infinity optimization of T_{zw} .
2. **H-infinity optimization:** minimizing the H-infinity norm of T_{zw} using the starting points from the stabilization step.

If the first step is successful, then a feasible point for the H-infinity optimization is found, i.e., a point where the closed-loop system is stable. If in the second step the H-infinity norm is reduced in a quasi-continuous way, then the feasible set cannot be left under mild controllability/observability conditions.

Both objective functions, the spectral abscissa and the H-infinity norm, are non-convex and not everywhere differentiable but smooth almost everywhere [15]. Therefore we choose a hybrid optimization method to solve a non-smooth and non-convex optimization problem, which has been successfully applied to design fixed-order controllers for the finite dimensional MIMO systems [10].

The optimization algorithm searches for the local minimizer of the objective function in three steps [5]:

1. A quasi-Newton algorithm (in particular, BFGS) provides a fast way to approximate a local minimizer [12],
2. A local bundle method attempts to verify local optimality for the best point found by BFGS,
3. If this does not succeed, gradient sampling [6] attempts to refine the approximation of the local minimizer, returning a rough optimality measure.

The non-smooth, non-convex optimization method requires the evaluation of the objective function -in the second step this is the H-infinity norm of T_{zw} - and the gradient of the objective function with respect to controller parameters where it exists. Recently a predictor-corrector algorithm has been developed to compute the H-infinity norm of time-delay systems [13]. We computed the gradients using the derivatives of singular values at frequencies where the H-infinity norm is achieved. Based on the evaluation of the objective function and its gradients, we apply the optimization method to compute fixed-order controllers. The computation of H-infinity norm of time-delay systems (8) is discussed in the following section. non-smooth optimization

2.2 Computation of the H-infinity Norm

We implemented a predictor-corrector type method to evaluate the H-infinity norm of T_{zw} in two steps (for details we refer to [13]): H-infinity!norm computationnorm!H-infinity

- **Prediction step:** we calculate the approximate H-infinity norm and corresponding frequencies where the highest peak values in the singular value plot occur.
- **Correction step:** we correct the approximate results from the prediction step.

Theoretical Foundation

The following theorem generalizes the well-known relation between the existence of singular values of the transfer function equal to a fixed value and the presence of imaginary axis eigenvalues of a corresponding Hamiltonian matrix [7] to time-delay systems:

Theorem 1. [13] *Let $\xi > 0$ be such that the matrix*

$$D_\xi := D_{cl}^T D_{cl} - \xi^2 I$$

is non-singular and define τ_{\max} as the maximum of the delays $(\tau_1, \dots, \tau_{m+2})$. For $\omega \geq 0$, the matrix $T_{zw}(j\omega)$ has a singular value equal to $\xi > 0$ if and only if $\lambda = j\omega$ is an eigenvalue of the linear infinite dimensional operator \mathcal{L}_ξ on $X := \mathcal{C}([-\tau_{\max}, \tau_{\max}], \mathbb{C}^{2n})$ which is defined by

$$\mathcal{D}(\mathcal{L}_\xi) = \{ \phi \in X : \phi' \in X, \phi'(0) = M_0 \phi(0) + \sum_{i=1}^{m+2} (M_i \phi(-\tau_i) + M_{-i} \phi(\tau_i)) \}, \quad (9)$$

$$\mathcal{L}_\xi \phi = \phi', \quad \phi \in \mathcal{D}(\mathcal{L}_\xi) \quad (10)$$

with

$$M_0 = \begin{bmatrix} A_{cl,0} - B_{cl} D_\xi^{-1} D_{cl}^T C_{cl} & -B_{cl} D_\xi^{-1} B_{cl}^T \\ \xi^2 C_{cl}^T D_\xi^{-T} C_{cl} & -A_{cl,0}^T + C_{cl}^T D_{cl} D_\xi^{-1} B_{cl}^T \end{bmatrix},$$

$$M_i = \begin{bmatrix} A_{cl,i} & 0 \\ 0 & 0 \end{bmatrix}, \quad M_{-i} = \begin{bmatrix} 0 & 0 \\ 0 & -A_{cl,i}^T \end{bmatrix}, \quad 1 \leq i \leq m+2.$$

By Theorem 1, the computation of H-infinity norm of T_{zw} can be formulated as an eigenvalue problem for the linear operator \mathcal{L}_ξ .

Corollary 1.

$\|T_{zw}\|_\infty = \sup\{\xi > 0 : \text{operator } \mathcal{L}_\xi \text{ has an eigenvalue on the imaginary axis}\}$

Conceptually Theorem 1 allows the computation of H-infinity norm via the well-known level set method [2, 4]. However, \mathcal{L}_ξ is an infinite dimensional operator. Therefore, we compute the H-infinity norm of the transfer function T_{zw} in two steps:

- 1) The prediction step is based on a matrix approximation of \mathcal{L}_ξ .

- 2) The correction step is based on reformulation of the eigenvalue problem of \mathcal{L}_ξ as a nonlinear eigenvalue problem of a finite dimension.

The approximation of the linear operator \mathcal{L}_ξ and the corresponding standard eigenvalue problem for Corollary 1 is given in Section 2.3. The correction algorithm of the approximate results in the second step is explained in Section 2.4.

2.3 Prediction Step

The infinite dimensional operator \mathcal{L}_ξ is approximated by a matrix \mathcal{L}_ξ^N . Based on the numerical methods for finite dimensional systems [2, 4], the \bar{H} -infinity norm of the transfer function T_{zw} can be computed approximately as

Corollary 2.

$\|T_{zw}\|_\infty \approx \sup\{\xi > 0 : \text{operator } \mathcal{L}_\xi^N \text{ has an eigenvalue on the imaginary axis}\}$.

The infinite-dimensional operator \mathcal{L}_ξ is approximated by a matrix using a *spectral method* (see, e.g. [3]). Given a positive integer N , we consider a mesh Ω_N of $2N + 1$ distinct points in the interval $[-\tau_{\max}, \tau_{\max}]$:

$$\Omega_N = \{\theta_{N,i}, i = -N, \dots, N\}, \tag{11}$$

where

$$-\tau_{\max} \leq \theta_{N,-N} < \dots < \theta_{N,0} = 0 < \dots < \theta_{N,N} \leq \tau_{\max}.$$

This allows to replace the continuous space X with the space X_N of discrete functions defined over the mesh Ω_N , i.e. any function $\phi \in X$ is discretized into a block vector $x = [x_{-N}^T \dots x_N^T]^T \in X_N$ with components

$$x_i = \phi(\theta_{N,i}) \in \mathbb{C}^{2n}, \quad i = -N, \dots, N.$$

Let $\mathcal{P}_N x$, $x \in X_N$ be the unique \mathbb{C}^{2n} valued interpolating polynomial of degree $\leq 2N$ satisfying

$$\mathcal{P}_N x(\theta_{N,i}) = x_i, \quad i = -N, \dots, N.$$

In this way, the operator \mathcal{L}_ξ over X can be approximated with the matrix $\mathcal{L}_\xi^N : X_N \rightarrow X_N$, defined as

$$\begin{aligned} (\mathcal{L}_\xi^N x)_i &= (\mathcal{P}_N x)'(\theta_{N,i}), \quad i = -N, \dots, -1, 1, \dots, N, \\ (\mathcal{L}_\xi^N x)_0 &= M_0 \mathcal{P}_N x(0) + \sum_{i=1}^{m+2} (M_i \mathcal{P}_N x(-\tau_i) + M_{-i} \mathcal{P}_N x(\tau_i)). \end{aligned}$$

Using the Lagrange representation of $\mathcal{P}_N x$,

$$\mathcal{P}_N x = \sum_{k=-N}^N l_{N,k} x_k,$$

where the Lagrange polynomials $l_{N,k}$ are real valued polynomials of degree $2N$ satisfying

$$l_{N,k}(\theta_{N,i}) = \begin{cases} 1 & i = k, \\ 0 & i \neq k, \end{cases}$$

we obtain the explicit form

$$\mathcal{L}_\xi^N = \begin{bmatrix} d_{-N,-N} & \dots & d_{-N,N} \\ \vdots & & \vdots \\ d_{-1,-N} & \dots & d_{-1,N} \\ a_{-N} & \dots & a_N \\ d_{1,-N} & \dots & d_{1,N} \\ \vdots & & \vdots \\ d_{N,-N} & \dots & d_{N,N} \end{bmatrix} \in \mathbb{R}^{(2N+1)(2n) \times (2N+1)2n},$$

where

$$\begin{aligned} d_{i,k} &= l'_{N,k}(\theta_{N,i})I, \quad i, k \in \{-N, \dots, N\}, i \neq 0, \\ a_0 &= M_0 x_0 + \sum_{k=1}^{m+2} (M_k l_{N,0}(-\tau_k) + M_{-k} l_{N,0}(\tau_k)), \\ a_i &= \sum_{k=1}^{m+2} (M_k l_{N,i}(-\tau_k) + M_{-k} l_{N,i}(\tau_k)), \quad k \in \{-N, \dots, N\}, k \neq 0. \end{aligned}$$

2.4 Correction Step

By using the finite dimensional level set methods, the largest level set ξ where \mathcal{L}_ξ^N has imaginary axis eigenvalues and their corresponding frequencies are computed. In the correction step, these approximate results are corrected by using the property that the eigenvalues of the \mathcal{L}_ξ appear as solutions of a finite dimensional nonlinear eigenvalue problem. The following theorem establishes the link between the linear infinite dimensional eigenvalue problem for \mathcal{L}_ξ and the nonlinear eigenvalue problem.

Theorem 2. [13] *Let $\xi > 0$ be such that the matrix*

$$D_\xi := D_{cl}^T D_{cl} - \xi^2 I$$

is non-singular. Then, λ is an eigenvalue of linear operator \mathcal{L}_ξ if and only if

$$\det H_\xi(\lambda) = 0, \tag{12}$$

where

$$H_\xi(\lambda) := \lambda I - M_0 - \sum_{i=1}^{m+2} (M_i e^{-\lambda\tau_i} + M_{-i} e^{\lambda\tau_i}) \tag{13}$$

and the matrices M_0, M_i, M_{-i} are defined in Theorem 1.

The correction method is based on the property that if $\hat{\xi} = \|T_{zw}(j\omega)\|_\infty$, then (13) has a multiple non-semisimple eigenvalue. If $\hat{\xi} \geq 0$ and $\hat{\omega} \geq 0$ are such that

$$\|T_{zw}(j\omega)\|_{\mathcal{H}_\infty} = \hat{\xi} = \sigma_1(T_{zw}(j\hat{\omega})), \tag{14}$$

then setting

$$h_\xi(\lambda) = \det H_\xi(\lambda),$$

the pair $(\hat{\omega}, \hat{\xi})$ satisfies

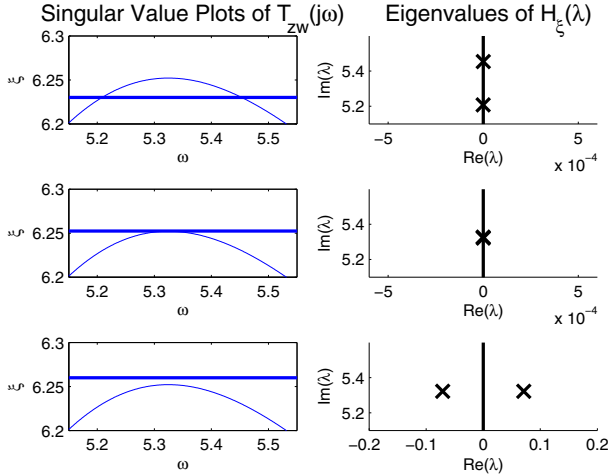


Fig. 1. (left) Intersections of the singular value plot of T_{zw} with the horizontal line $\xi = c$, for $c < \hat{\xi}$ (top), $c = \hat{\xi}$ (middle) and $c > \hat{\xi}$ (bottom). (right) Corresponding eigenvalues of $H_{\xi}(\lambda)$ (13).

$$h_{\xi}(j\omega) = 0, \quad h'_{\xi}(j\omega) = 0. \quad (15)$$

This property is clarified in Figure 1.

The drawback of working directly with (15) is that an explicit expression for the determinant of H_{ξ} is required. This scalar-valued conditions can be equivalently expressed in a matrix-based formulation.

$$\begin{cases} H(j\omega, \xi) \begin{bmatrix} u \\ v \end{bmatrix} = 0, & n(u, v) = 0, \\ \Im \{ v^* (I + \sum_{i=1}^{m+1} A_{cl,i} \tau_i e^{-j\omega \tau_i}) u \} = 0 \end{cases} \quad (16)$$

where $n(u, v) = 0$ is a normalizing condition. The approximate H-infinity norm and its corresponding frequencies can be corrected by solving (16). For further details, see [13].

2.5 Computing the Gradients

The optimization algorithm requires the derivatives of H-infinity norm of the transfer function T_{zw} with respect to the controller matrices whenever it is differentiable. Define the H-infinity norm of the function T_{zw} as

$$f(A_{cl,0}, \dots, A_{cl,m+2}, B_{cl}, C_{cl}, D_{cl}) = \|T_{zw}(j\omega)\|_{\infty}.$$

These derivatives exist whenever there is a unique frequency $\hat{\omega}$ such that (14) holds, and, in addition, the largest singular value $\hat{\xi}$ of $T_{zw}(j\hat{\omega})$ has multiplicity one. Let w_l and w_r be the corresponding left and right singular vector, i.e.

$$\begin{aligned} T_{zw}(j\hat{\omega}) w_r &= \hat{\xi} w_l, \\ w_l^* T_{zw}(j\hat{\omega}) &= \hat{\xi} w_r^*. \end{aligned} \quad (17)$$

When defining $\frac{\partial f}{\partial A_{cl,0}}$ as a n -by- n matrix whose (k, l) -th element is the derivative of f with respect to the (k, l) -th element of $A_{cl,0}$, and defining the other derivatives in a similar way, the following expressions are obtained [14]:

$$\begin{aligned} \frac{\partial f}{\partial A_{cl,0}} &= \frac{\Re(M(j\hat{\omega})^* C_{cl}^T w_l w_r^* B_{cl}^T M(j\hat{\omega})^*)}{w_r^* w_r}, \\ \frac{\partial f}{\partial A_{cl,i}} &= \frac{\Re(M(j\hat{\omega})^* C_{cl}^T w_l w_r^* B_{cl}^T M(j\hat{\omega})^* e^{j\omega\tau_i})}{w_r^* w_r} \text{ for } i = 1, \dots, m+2, \\ \frac{\partial f}{\partial B_{cl}} &= \frac{\Re(M(j\hat{\omega})^* C_{cl}^T w_l w_r^*)}{w_r^* w_r}, \quad \frac{\partial f}{\partial C_{cl}} = \frac{\Re(w_l w_r^* B_{cl}^T M(j\hat{\omega})^*)}{w_r^* w_r}, \\ \frac{\partial f}{\partial D_{cl}} &= \frac{\Re(w_l w_r^*)}{w_r^* w_r} \end{aligned}$$

where $M(j\omega) = (j\omega I - A_{cl,0} - \sum_{i=1}^{m+2} A_{cl,i} e^{-j\omega\tau_i})^{-1}$.

We compute the gradients with respect to the controller matrices as

$$\begin{aligned} \frac{\partial f}{\partial A_K} &= [0_{n_K \times n} \quad I_{n_K}] \frac{\partial f}{\partial A_{cl,0}} \begin{bmatrix} 0_{n \times n_K} \\ I_{n_K} \end{bmatrix}, \\ \frac{\partial f}{\partial B_K} &= [0_{n_K \times n} \quad I_{n_K}] \frac{\partial f}{\partial A_{cl,0}} \begin{bmatrix} I_n \\ 0_{n_K \times n} \end{bmatrix} C_2^T \\ &\quad + [0_{n_K \times n} \quad I_{n_K}] \frac{\partial f}{\partial A_{cl,m+2}} \begin{bmatrix} 0_{n \times n_K} \\ I_{n_K} \end{bmatrix} C_K^T D_{22}^T + [0_{n_K \times n} \quad I_{n_K}] \frac{\partial f}{\partial B_{cl}} D_{21}^T, \\ \frac{\partial f}{\partial C_K} &= B_2^T [I_n \quad 0_{n \times n_K}] \frac{\partial f}{\partial A_{cl,m+1}} \begin{bmatrix} 0_{n \times n_K} \\ I_{n_K} \end{bmatrix} \\ &\quad + D_{22}^T B_K^T [0_{n_K \times n} \quad I_{n_K}] \frac{\partial f}{\partial A_{cl,m+2}} \begin{bmatrix} 0_{n \times n_K} \\ I_{n_K} \end{bmatrix} + D_{12}^T \frac{\partial f}{\partial C_{cl}} \begin{bmatrix} 0_{n \times n_K} \\ I_{n_K} \end{bmatrix} \end{aligned}$$

where the matrices I_n , I_{n_K} and $0_{n \times n_K}$, $0_{n_K \times n}$ are identity and zero matrices.

3 Examples

We consider the time-delay system with the following state-space representation,

$$\begin{aligned} \dot{x}(t) &= -x(t) - 0.5x(t-1) + w(t) + u(t), \\ z(t) &= x(t) + u(t), \\ y(t) &= x(t) + w(t). \end{aligned}$$

We designed the first-order controller, $n_K = 1$,

$$\begin{aligned} \dot{x}_K(t) &= 3.61x_K(t) + 1.39y(t), \\ u(t) &= -0.83x_K(t), \end{aligned}$$

achieving the closed-loop H-infinity norm 0.064. The closed-loop H-infinity norms of fixed-order controllers for $n_K = 2$ and $n_K = 3$ are 0.021 and 0.020 respectively.

Our second example is a 4th-order time-delay system. The system contains 4 delays and has the following state-space representation,

$$\begin{aligned} \dot{x}(t) = & \begin{pmatrix} -4.4656 & -0.4271 & 0.4427 & -0.1854 \\ -0.8601 & -5.6257 & 0.8577 & -0.5210 \\ 0.9001 & -0.7177 & -6.5358 & 0.0417 \\ -0.6836 & 0.0242 & 0.4997 & -3.5618 \end{pmatrix} x(t) + \begin{pmatrix} 0.6848 & -0.0618 & 0.5399 & 0.5057 \\ 0.3259 & -0.3810 & 0.6592 & -0.0066 \\ 0.6325 & 0.3752 & 0.4122 & 0.7303 \\ 0.5878 & 0.9737 & 0.1907 & -0.8639 \end{pmatrix} x(t-3.2) \\ & + \begin{pmatrix} 0.9371 & -0.7859 & 0.1332 & 0.7429 \\ -0.8025 & 0.4483 & 0.6226 & 0.0152 \\ 0.0940 & 0.2274 & 0.1536 & 0.5776 \\ -0.1941 & 0.5659 & 0.8881 & -0.0539 \end{pmatrix} x(t-3.4) + \begin{pmatrix} 0.6576 & -0.8543 & -0.3460 & 0.6415 \\ -0.3550 & 0.5024 & 0.6081 & 0.9038 \\ 0.9523 & 0.6624 & 0.0765 & -0.8475 \\ -0.4436 & 0.8447 & -0.0734 & 0.4173 \end{pmatrix} x(t-3.9) \\ & + \begin{pmatrix} 1 & 0 \\ -1.6 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} w(t) + \begin{pmatrix} 0.2 \\ -1 \\ 0.1 \\ -0.4 \end{pmatrix} u(t-0.2) \\ z(t) = & \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0.1 & 1 \\ -1 & 0.2 \end{pmatrix} w(t) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} u(t) \\ y(t) = & (1 \ 0 \ -1 \ 0) x(t) + (-2 \ 0.1) w(t) + 0.4u(t-0.2) \end{aligned}$$

When $n_K = 1$, our method finds the controller achieving the closed-loop H-infinity norm 1.2606,

$$\begin{aligned} \dot{x}_K(t) &= -0.712x_K(t) - 0.1639y(t), \\ u(t) &= -0.2858x_K(t) \end{aligned}$$

and the results for $n_K = 2$ and $n_K = 3$ are 1.2573 and 1.2505 respectively.

4 Concluding Remarks

We successfully designed fixed-order H-infinity controllers for a class of time-delay systems. The method is based on non-smooth, non-convex optimization techniques and allows the user to choose the controller order as desired. Our approach can be extended to general time-delay systems. Although we illustrated our method for a dynamic controller, it can be applied to more general controller structures. The only requirement is that the closed-loop matrices should depend smoothly on the controller parameters. On the contrary, the existing controller design methods optimizing the closed-loop H-infinity norm are based on Lyapunov theory and linear matrix inequalities. These methods are conservative if the form of the Lyapunov functionals is restricted, and they require full state information.

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