Chapter 8
Nonlinear Beams and Rods

In this chapter, nonlinear theories for rods and beams will be discussed in the Cartesian coordinate frame and the curvilinear frame of the initial configuration. Without torsion, the theory for in-plane beams will be presented. The traditional treatises of nonlinear rods were based on the Cosserat's theory (e.g., E. and F. Cosserat, 1896) or the Kirchhoff assumptions (e.g., Kirchhoff, 1859; Love, 1944). This chapter will extend the ideas of Galerkin (1915), and the nonlinear theory of rods and beams will be developed from the general theory of the 3-dimensional deformable body. The definitions for beams and rods are given as follows.

Definition 8.1. If a 1-D deformable body on the three directions of fibers resists internal forces, *bending* and *twisting* moments, the 1-D deformable body is called a *deformable* rod.

Definition 8.2. If a 1-D deformable body on the three directions of fibers resists internal forces and *bending* moments, the 1-D deformable body is called a *deformable* beam.

8.1. Differential geometry of curves

Consider an initial configuration of a nonlinear rod as shown in Fig.8.1. The unit vectors $\mathbf{I}_I$ ($I = 1, 2, 3$) are the base vectors for the Cartesian coordinates and the based vectors $\mathbf{G}_\alpha$ ($\alpha = 1, 2, 3$) for the curvilinear coordinates are defined later. To present the nonlinear rod theory, it is assumed that the base vector $\mathbf{G}_1$ is normal to the surface formed by the other base vectors $\mathbf{G}_2$ and $\mathbf{G}_3$. The surface formed by the vectors $\mathbf{G}_2$ and $\mathbf{G}_3$ is called the *cross section* of the rod. The material particle on the central curves of the intersections of two neutral surfaces in the initial configuration is

A. C. J. Luo, *Nonlinear Deformable-body Dynamics*

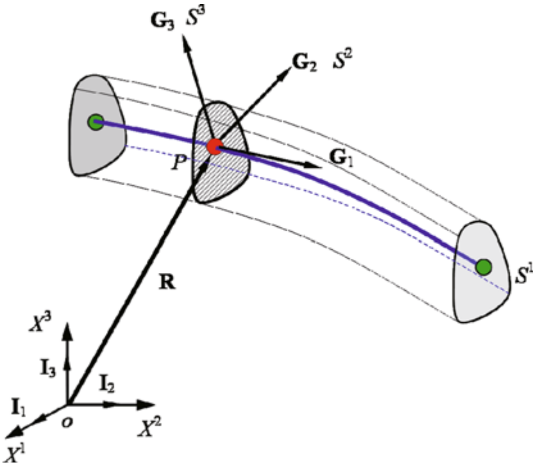


Fig. 8.1 A material particle *P* on an initial configuration of a nonlinear rod.

$$\mathbf{R} = X^I(S)\mathbf{I}_I, \tag{8.1}$$

where $S^1 = S$ and $S^2 = S^3 = 0$. From Eq.(8.1), the base vectors for the rod can be obtained. The base vector in the tangential direction of the rod is defined by

$$\mathbf{G}_1 = \frac{d\mathbf{R}}{dS} = \frac{\partial X^I}{\partial S}\mathbf{I}_I = X^I_{,S}\mathbf{I}_I \equiv G^I_1\mathbf{I}_I. \tag{8.2}$$

Note that $(\cdot)_{,S} = (\cdot)_{,1}$ and the metric measure is given by

$$G_{11} = \frac{\partial X^I}{\partial S}\frac{\partial X^I}{\partial S} = X^I_{,1}X^I_{,1} \equiv G^I_1 G^I_1 \text{ (summation on } I\text{)}. \tag{8.3}$$

and

$$\mathbf{N}_1 = \frac{\mathbf{G}_1}{\sqrt{G_{11}}} = \frac{X^I_{,1}}{\sqrt{G_{11}}}\mathbf{I}_I, \tag{8.4}$$

while an arc length variable s is defined by

$$ds = \sqrt{G_{11}}dS, \tag{8.5}$$

$$\mathbf{N}_1 = X^I_{,s}\mathbf{I}_I \text{ and } \mathbf{G}_1 = \sqrt{G_{11}}\ X^I_{,s}\mathbf{I}_I \equiv G^I_1\mathbf{I}_I, \tag{8.6}$$

with $\sqrt{X^I_{,s}X^I_{,s}} = 1$. The direction of $\mathbf{G}_1$ is the tangential direction of the initial configuration of the rod curve. From differential geometry in Kreyszig (1968), the curvature vector can be determined by

$$\mathbf{G}_2 \equiv G_2^I \mathbf{I}_I = \frac{d\mathbf{N}_1}{ds} = X_{,ss}^I \mathbf{I}_I$$
$$= \frac{d\mathbf{N}_1}{dS}\frac{dS}{ds} = \frac{1}{G_{11}^2}\left[X_{,11}^I(X_{,1}^K X_{,1}^K) - X_{,1}^I X_{,1}^K X_{,11}^K\right]\mathbf{I}_I \tag{8.7}$$

and

$$\begin{aligned} G_2^I &\equiv \frac{1}{G_{11}^2}\left[X_{,11}^I(X_{,1}^K X_{,1}^K) - X_{,1}^I X_{,1}^K X_{,11}^K\right], \\ G_{22} &\equiv G_2^I G_2^I = \frac{1}{G_{11}^3}\left[(X_{,11}^I X_{,11}^I)(X_{,1}^K X_{,1}^K) - (X_{,1}^I X_{,11}^I)^2\right]. \end{aligned} \tag{8.8}$$

The *curvature of the rod* in the initial configuration is

$$\kappa = |\mathbf{G}_2| = \sqrt{G_{22}} = \sqrt{X_{,ss}^I X_{,ss}^I}$$
$$= \frac{\sqrt{(X_{,11}^I X_{,11}^I)(X_{,1}^K X_{,1}^K) - (X_{,1}^I X_{,11}^I)^2}}{G_{11}^{3/2}}. \tag{8.9}$$

The *unit principal normal* vector is given by

$$\mathbf{N}_2 = \frac{\mathbf{G}_2}{\sqrt{G_{22}}} = \frac{\mathbf{G}_2}{\kappa} = \frac{G_2^I}{\kappa}\mathbf{I}_I, \quad \frac{d\mathbf{N}_1}{ds} = \mathbf{G}_2 = \kappa\mathbf{N}_2. \tag{8.10}$$

The *unit bi-normal vector* is defined by

$$\mathbf{N}_3 = \mathbf{N}_1 \times \mathbf{N}_2 \tag{8.11}$$

and let

$$\mathbf{N}_3 = G_3^I \mathbf{I}_I \text{ with } G_3^I = e_{IJK}\frac{G_1^J}{\sqrt{G_{11}}}\frac{G_2^K}{\kappa(S)}, \tag{8.12}$$

where e_{IJK} is the Ricci symbol in Eq.(2.105). Therefore, $G_{33} = G_3^I G_3^I = 1$ (*summation on I*)

Consider the change rate of the unit bi-normal direction with respect to the arc length (s), which gives

$$\frac{d\mathbf{N}_3}{ds} = -\tau\mathbf{N}_2 \;\Rightarrow\; \tau = -\mathbf{N}_2 \cdot \frac{d\mathbf{N}_3}{ds}. \tag{8.13}$$

The *torsional curvature of the rod* (or torsion of the curve called in mathematics) is

$$\tau = \frac{1}{\kappa^2}[\mathbf{R}_{,s}\mathbf{R}_{,ss}\mathbf{R}_{,sss}]$$

$$= \frac{[\mathbf{R}_{,1}\mathbf{R}_{,11}\mathbf{R}_{,111}]}{(\mathbf{R}_{,11}\cdot\mathbf{R}_{,11})(\mathbf{R}_{,1}\cdot\mathbf{R}_{,1})-(\mathbf{R}_{,1}\cdot\mathbf{R}_{,11})^2}$$

$$= \frac{e_{IJK}X^I_{,1}X^J_{,11}X^K_{,111}}{(X^I_{,11}X^I_{,11})(X^K_{,1}X^K_{,1})-(X^I_{,1}X^I_{,11})^2}. \tag{8.14}$$

Based on the definition of unit based vector, the vector product gives

$$\mathbf{N}_2 = \mathbf{N}_3\times\mathbf{N}_1,\ \mathbf{N}_3 = -\mathbf{N}_2\times\mathbf{N}_1,\ \mathbf{N}_1 = -\mathbf{N}_3\times\mathbf{N}_2. \tag{8.15}$$

With Eqs.(8.10) and (8.13),

$$\begin{aligned}\frac{d\mathbf{N}_2}{ds} &= \frac{d\mathbf{N}_3}{ds}\times\mathbf{N}_1 + \mathbf{N}_3\times\frac{d\mathbf{N}_1}{ds}\\ &= -\tau\mathbf{N}_2\times\mathbf{N}_1 + \kappa\mathbf{N}_3\times\mathbf{N}_2\\ &= -\kappa\mathbf{N}_1 + \tau\mathbf{N}_3.\end{aligned} \tag{8.16}$$

Thus, from the formulae of Frenet (1847),

$$\begin{bmatrix}\frac{d\mathbf{N}_1}{ds}\\ \frac{d\mathbf{N}_2}{ds}\\ \frac{d\mathbf{N}_3}{ds}\end{bmatrix} = \begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\mathbf{N}_1\\ \mathbf{N}_2\\ \mathbf{N}_3\end{bmatrix}, \tag{8.17}$$

$$\begin{bmatrix}\frac{d\mathbf{N}_1}{dS}\\ \frac{d\mathbf{N}_2}{dS}\\ \frac{d\mathbf{N}_3}{dS}\end{bmatrix} = \begin{bmatrix}0 & \sqrt{G_{11}}\kappa & 0\\ -\sqrt{G_{11}}\kappa & 0 & \sqrt{G_{11}}\tau\\ 0 & -\sqrt{G_{11}}\tau & 0\end{bmatrix}\begin{bmatrix}\mathbf{N}_1\\ \mathbf{N}_2\\ \mathbf{N}_3\end{bmatrix}. \tag{8.18}$$

Consider a rotation vector (or the vector of Darboux)

$$\boldsymbol{\omega} = \tau\mathbf{N}_1 + \kappa\mathbf{N}_3 \tag{8.19}$$

Equations (8.17) and (8.18) become

$$\frac{d\mathbf{N}_1}{ds} = \boldsymbol{\omega}\times\mathbf{N}_1,\ \frac{d\mathbf{N}_2}{ds} = \boldsymbol{\omega}\times\mathbf{N}_2, \frac{d\mathbf{N}_3}{ds} = \boldsymbol{\omega}\times\mathbf{N}_3, \tag{8.20}$$

$$\frac{d\mathbf{N}_1}{dS} = \sqrt{G_{11}}\boldsymbol{\omega}\times\mathbf{N}_1,\ \frac{d\mathbf{N}_2}{dS} = \sqrt{G_{11}}\boldsymbol{\omega}\times\mathbf{N}_2,\ \frac{d\mathbf{N}_3}{dS} = \sqrt{G_{11}}\boldsymbol{\omega}\times\mathbf{N}_3. \tag{8.21}$$

Consider a material point $\mathbf{R}$ on the cross section of the rod

$$\mathbf{R} = X^I(S^1, S^2, S^2)\mathbf{I}_I. \tag{8.22}$$

Without loss of generality, S^1, S^2 and S^3 are collinear to the directions of $\mathbf{N}_1$, $\mathbf{N}_2$ and $\mathbf{N}_3$, respectively. On the cross section of $\mathbf{N}_1$, any variable can be converted on the two directions of $\mathbf{N}_2$ and $\mathbf{N}_3$.

Consider a displacement vector field at the point $\mathbf{R}$ to be

$$\mathbf{u} = u^I(S^1, S^2, S^3)\mathbf{I}_I \quad \text{or} \quad \mathbf{u} = u^\Lambda(S^1, S^2, S^3)\mathbf{G}_\Lambda. \tag{8.23}$$

From the previous definitions,

$$\mathbf{N}_\Lambda = \frac{\mathbf{G}_\Lambda}{\sqrt{G_{\Lambda\Lambda}}} = \frac{G_\Lambda^I}{\sqrt{G_{\Lambda\Lambda}}}\mathbf{I}_I. \tag{8.24}$$

The particle in the deformed configuration is expressed by the location and displacement vectors, i.e.,

$$\begin{aligned} \mathbf{r} &= \mathbf{R} + \mathbf{u} = (X^I + u^I(S^1, S^2, S^3))\mathbf{I}_I, \text{ or} \\ \mathbf{r} &= \mathbf{R} + \mathbf{u} = (S^\Lambda + u^\Lambda(S^1, S^2, S^3))\mathbf{G}_\Lambda \end{aligned} \tag{8.25}$$

and the corresponding infinitesimal line element of the deformed rod is

$$\begin{aligned} d\mathbf{r} &= d\mathbf{R} + d\mathbf{u} = (X^I_{,\alpha} + u^I_{,\alpha})dS^\alpha \mathbf{I}_I \\ &= (\delta^\Lambda_\alpha + u^\Lambda_{;\alpha})dS^\alpha \mathbf{G}_\Lambda. \end{aligned} \tag{8.26}$$

The base vector for the deformed rod becomes

$$\mathbf{g}_\alpha = (X^I_{,\alpha} + u^I_{,\alpha})\mathbf{I}_I = (\delta^\Lambda_\alpha + u^\Lambda_{;\alpha})\mathbf{G}_\Lambda \tag{8.27}$$

and the corresponding unit vector is

$$\begin{aligned} \mathbf{n}_\alpha &= \frac{X^I_{,\alpha} + u^I_{,\alpha}}{\sqrt{(X^K_{,\alpha} + u^K_{,\alpha})(X^K_{,\alpha} + u^K_{,\alpha})}}\mathbf{I}_I \\ &= \frac{\delta^\Lambda_\alpha + u^\Lambda_{;\alpha}}{\sqrt{G_{\Gamma\Gamma}}\sqrt{(\delta^\Gamma_\alpha + u^\Gamma_{;\alpha})(\delta^\Gamma_\alpha + u^\Gamma_{;\alpha})}}\mathbf{G}_\Lambda. \end{aligned} \tag{8.28}$$

8.2. A nonlinear theory of straight beams

Consider a *beam* in the initial configuration to be straight. This requires that the curvature and torsion should be zero ($\kappa(S) = 0$ and $\tau(S) = 0$). Thus, $S^I = X^I$, $G_{\alpha\beta} = 0$ and $G_{\alpha\alpha} = 1$ ($\alpha, \beta = 1, 2, 3$).

$$d\mathbf{R} = dX^I \mathbf{I}_I,\ d\mathbf{r} = d\mathbf{R} + d\mathbf{u} = (\delta^I_\alpha + u^I_{,\alpha})dX^\alpha \mathbf{I}_I. \tag{8.29}$$

The strain based on the change in length of $d\mathbf{R}$ per unit length gives

$$\begin{aligned}\varepsilon_\alpha &= \frac{|\underset{\alpha}{d\mathbf{r}}| - |\underset{\alpha}{d\mathbf{R}}|}{|\underset{\alpha}{d\mathbf{R}}|} = \sqrt{1+2E_{\alpha\alpha}} - 1 \\ &= \sqrt{(\delta^I_\alpha + u^I_{,\alpha})(\delta^I_\alpha + u^I_{,\alpha})} - 1.\end{aligned} \tag{8.30}$$

The Lagrangian strain tensor $E_{\alpha\beta}$ referred to the initial configuration is

$$E_{\alpha\beta} = \frac{1}{2}(\delta^I_\alpha u^I_{,\beta} + \delta^I_\beta u^I_{,\alpha} + u^I_{,\alpha}u^I_{,\beta}). \tag{8.31}$$

In the similar fashion, the angles between $\mathbf{n}_\alpha$ and $\mathbf{n}_\beta$ *before* and *after* deformation are expressed by $\Theta_{(\mathbf{N}_\alpha,\mathbf{N}_\beta)} = \pi/2$ and $\theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)}$, i.e.,

$$\begin{aligned}\cos\theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)} &\equiv \cos(\Theta_{(\mathbf{N}_\alpha,\mathbf{N}_\beta)} - \gamma_{\alpha\beta}) \\ &= \frac{\underset{\alpha}{d\mathbf{r}} \cdot \underset{\beta}{d\mathbf{r}}}{|\underset{\alpha}{d\mathbf{r}}||\underset{\beta}{d\mathbf{r}}|} = \frac{(\delta^I_\alpha + u^I_{,\alpha})(\delta^I_\beta + u^I_{,\beta})}{\sqrt{(1+2E_{\alpha\alpha})(1+2E_{\beta\beta})}} \\ &= \frac{(\delta^I_\alpha + u^I_{,\alpha})(\delta^I_\beta + u^I_{,\beta})}{(1+\varepsilon_\alpha)(1+\varepsilon_\beta)}\end{aligned} \tag{8.32}$$

and the corresponding shear strain is defined by

$$\begin{aligned}\gamma_{\alpha\beta} &\equiv \Theta_{(\mathbf{N}_\alpha,\mathbf{N}_\beta)} - \theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)} \\ &= \sin^{-1}\frac{(\delta^I_\alpha + u^I_{,\alpha})(\delta^I_\beta + u^I_{,\beta})}{\sqrt{(1+2E_{\alpha\alpha})(1+2E_{\beta\beta})}} \\ &= \sin^{-1}\frac{(\delta^I_\alpha + u^I_{,\alpha})(\delta^I_\beta + u^I_{,\beta})}{(1+\varepsilon_\alpha)(1+\varepsilon_\beta)}.\end{aligned} \tag{8.33}$$

From Eq.(8.28), the direction cosine of the rotation is

$$\cos\theta_{(\mathbf{N}_\alpha,\mathbf{n}_\beta)} = \frac{\underset{\alpha}{d\mathbf{R}} \cdot \underset{\beta}{d\mathbf{r}}}{|\underset{\alpha}{d\mathbf{R}}||\underset{\beta}{d\mathbf{r}}|} = \frac{\delta^\alpha_\beta + u^\alpha_{,\beta}}{\sqrt{1+2E_{\beta\beta}}} = \frac{\delta^\alpha_\beta + u^\alpha_{,\beta}}{1+\varepsilon_\beta}. \tag{8.34}$$

In addition, the area changes before and after deformation are given by

$$\frac{\underset{\alpha\beta}{da}}{\underset{\alpha\beta}{dA}} = (1+\varepsilon_\alpha)(1+\varepsilon_\beta)\sin\theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)}, \tag{8.35}$$

where $\underset{\alpha\beta}{da} = |\underset{\alpha}{d\mathbf{r}} \times \underset{\beta}{d\mathbf{r}}|$ and $\underset{\alpha\beta}{dA} = |\underset{\alpha}{d\mathbf{R}} \times \underset{\beta}{d\mathbf{R}}|$.

Consider the coordinate X^1 to be along the longitudinal direction of the beam and the other two coordinates X^2 and X^3 on the cross section of beam on the direction of X^1. The coordinates for the deformed straight beam are (s^1, s^2, s^3). Because the initial configuration of the beam is a straight beam, under external force, the deformed configuration of the beam does not experience any torsion ($\tau(s^1) \equiv 0$). Thus, the deformed configuration of the beam is a plane curve. Without loss of generality, the curvature direction of the deformed configuration can be assumed to be collinear to X^2. Because the widths of beam in two directions of X^2 and X^3 are very small compared to the length of the beam in direction of X^1, the elongation in the two directions of X^2 and X^3 should be very small, which can be neglected. From the aforementioned discussions, the following assumptions are adopted:

(i) The deformed configuration of the beam does not experience any torsion ($\tau(s^1) \equiv 0$).
(ii) The curvature direction of the deformed configuration is collinear to s^2.
(iii) The elongations in the two directions of X^2 and X^3 are very small (i.e., $\varepsilon_2 \approx 0$ and $\varepsilon_3 = 0$).
(iv) Under bending only, the neutral surface is not deformed (i.e., $\gamma_{12} \approx 0$ and $\gamma_{13} = 0$).

Consider an arbitrary coordinates as (X^1, Y, Z) at the *centroid* on the cross section of the beam. Under the resultant forces, the bending of beam is in the curvature direction of s^2. The deformed curve of the beam is on the plane of (X^1, X^2), as shown in Fig.8.2. In other words, the neutral surface of the deformed beam is on the plane of (X^1, X^3). When the transversal forces act at a point on the cross section of the beam and if the beam will not be twisted, such a point on the cross section of the beam is called the *shear center* (*or flexural center*). From Assumption (i), no torsion exists. In addition, the transversal forces should be placed to the shear center. Because the transversal forces are applied to the beam off the shear center, the beam will be twisted and bent. To explain this case, consider external distributed forces and moments at *the shear center* on the initial configuration to be

$$\mathbf{q} = q^I \mathbf{I}_I \text{ and } \mathbf{m} = m^I \mathbf{I}_I \ (I = 1, 2, 3) \tag{8.36}$$

and concentrated forces on the initial configuration at a point $X^1 = S_k$,

$${}^k\mathbf{F} = {}^kF^I \mathbf{I}_I \text{ and } {}^k\mathbf{M} = {}^kM^I \mathbf{I}_I \ (I = 1, 2, 3). \tag{8.37}$$

The displacement vectors on the initial configuration are

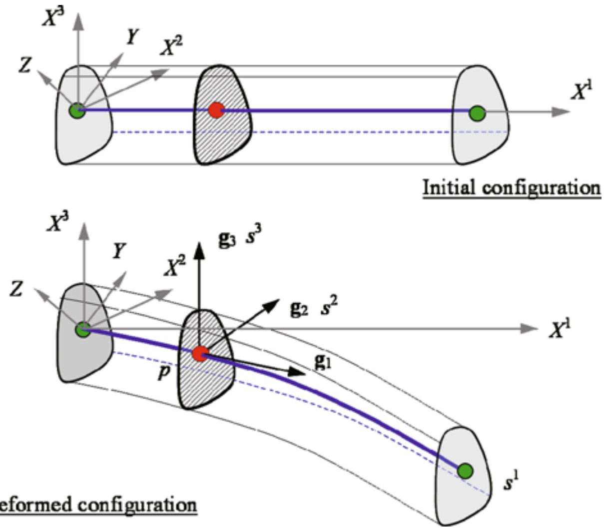


Fig. 8.2 A straight beam with initial and deformed configuration.

$$\mathbf{R} = X^1\mathbf{I}_1,\ {}^S\mathbf{R} = S\mathbf{I}_1 \text{ and } {}^k\mathbf{R} = X_k^1\mathbf{I}_1. \tag{8.38}$$

The internal forces and moments for $S > X_k^1$ are

$$\begin{aligned}\mathbf{F}\,|_{X^1=S} &= \sum_{k=1} {}^k\mathbf{F} + \int_0^S \mathbf{q}dX^1,\\ \mathbf{M}\,|_{X^1=S} &= \sum_{k=1} {}^k\mathbf{M} + \int_0^S \mathbf{m}dX^1,\\ &\quad + \sum_{k=1} ({}^S\mathbf{R} - {}^k\mathbf{R}) \times {}^k\mathbf{F} + \int_0^S ({}^S\mathbf{R} - \mathbf{R}) \times \mathbf{q}dX^1;\end{aligned} \tag{8.39}$$

or for $I = 1, 2, 3$,

$$\begin{aligned}F^I\,|_{X^1=S} &= \sum_{k=1} {}^kF^I + \int_0^S q^I dX^1,\\ M^1\,|_{X^1=S} &= \sum_{k=1} {}^kM^1 + \int_0^S m^1 dX^1,\\ M^2\,|_{X^1=S} &= \sum_{k=1} {}^kM^2 + \int_0^S m^2 dX^1\\ &\quad - \sum_{k=1} {}^kF^3(S - X_k^1) - \int_0^S X^1 q^3 dX^1,\\ M^3\,|_{X^1=S} &= \sum_{k=1} {}^kM^3 + \int_0^S m^3 dX^1\\ &\quad + \sum_{k=1} {}^kF^2(S - X_k^1) + \int_0^S (S - X^1) q^2 dX^1.\end{aligned} \tag{8.40}$$

From assumptions (i) and (ii), the following conditions exist:

$$
\begin{aligned}
F^3 |_{X^1=S} &= \sum_{k=1} {}^kF^3 + \int_0^S q^3 dX^1 = 0, \\
M^1 |_{X^1=S} &= \sum_{k=1} {}^kM^1 + \int_0^S m^1 dX^1 = 0, \\
M^2 |_{X^1=S} &= \sum_{k=1} {}^kM^2 + \int_0^S m^2 dX^1 \\
&\quad - \sum_{k=1} {}^kF^3 (S - X_k^1) - \int_0^S (S - X^1) q^3 dX^1 = 0.
\end{aligned}
\tag{8.41}
$$

For all points on the beam to satisfy Eq.(8.41),

$$
\begin{aligned}
{}^kF^3 &= 0 \quad \text{and} \quad q^3 = 0, \\
{}^kM^1 &= 0 \quad \text{and} \quad m^1 = 0, \\
{}^kM^2 &= 0 \quad \text{and} \quad m^2 = 0.
\end{aligned}
\tag{8.42}
$$

If the external forcing exerts on the three directions of (X^1, Y, Z), the resultant forces and moments on three directions of (X^1, X^2, X^3) should satisfy Eq.(8.42). Such projection of the forces can be done through the rotation angle between the two coordinates (X^1, Y, Z) and (X^1, X^2, X^3).

From assumption (ii),

$$
u^I = u_0^I(S,t) + \sum_{n=1}^{\infty} (X^2)^n \varphi_n^{(I)}(S,t) \quad \text{for } I=1, 2,
\tag{8.43}
$$

where $X^2 = \sqrt{(Y)^2 + (Z)^2}$ is a distance to the neutral surface along the direction of curvature. From assumption (ii), no displacements exist in the direction of X^3 (i.e., $u^3 = 0$). From *Kirchhoff's assumptions,* under bending only, if the cross section is normal to the neutral surface before deformation, then after deformation, the deformed cross section is still normal to the deformed neutral surface. Thus, equation (8.43) becomes

$$
u^I = u_0^I(S,t) + X^2 \varphi^{(I)}(S,t) \quad (I = 1, 2).
\tag{8.44}
$$

From assumptions (iii) and (iv),

$$
\begin{aligned}
(\delta_2^I + u_{,2}^I)(\delta_2^I + u_{,2}^I) &= 1, \\
(\delta_1^I + u_{,1}^I)(\delta_2^I + u_{,2}^I) &= 0.
\end{aligned}
\tag{8.45}
$$

With $u^3 = 0$ and Eq.(8.43), the Taylor series expansion of Eq.(8.45) give for the zero-order of X^2,

$$
\begin{aligned}
(\delta_2^I + \varphi_1^{(I)})(\delta_2^I + \varphi_1^{(I)}) &= 1, \\
(\delta_1^I + u_{0,1}^I)(\delta_2^I + \varphi_1^{(I)}) &= 0.
\end{aligned}
\tag{8.46}
$$

From Eq.(8.46),

$$\varphi_1^{(1)} = \mp \frac{u_{0,1}^2}{\sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^2)^2}},$$
$$\varphi_1^{(2)} = \pm \frac{1+u_{0,1}^1}{\sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^2)^2}} - 1. \tag{8.47}$$

From the sign convention, the positive "+" in the second equation of Eq.(8.47) will be adopted. Following the similar fashion, one can obtain $\varphi_n^{(I)}$ ($n = 1, 2, \ldots$ and $I = 1, 2$). Further, using the Taylor series expansion, the approximations of three strains on the cross section of the deformed beam are

$$\begin{aligned}\varepsilon_1 &\approx \varepsilon_1^{(0)} + \left.\frac{d\varepsilon_1}{dX^2}\right|_{X^2=0} X^2 + \frac{1}{2}\left.\frac{d^2\varepsilon_1}{d(X^2)^2}\right|_{X^2=0} (X^2)^2 + \cdots \\ &= \varepsilon_1^{(0)} + \frac{(\delta_1^I + u_{0,1}^I)\varphi_{1,1}^{(I)}}{1+\varepsilon_1^{(0)}} X^2 + \frac{1}{2}\left\{\frac{[2(\delta_1^I + u_{0,1}^I)\varphi_{2,1}^{(I)}] + \varphi_{1,1}^{(I)}\varphi_{1,1}^{(I)}}{1+\varepsilon_1^{(0)}}\right. \\ &\quad \left. - \frac{[(\delta_1^I + u_{0,1}^I)\varphi_{1,1}^{(I)}]^2}{(1+\varepsilon_1^{(0)})^3}\right\}(X^2)^2 + \cdots,\end{aligned} \tag{8.48}$$

$$\begin{aligned}\varepsilon_2 &\approx \varepsilon_2^{(0)} + \left.\frac{d\varepsilon_2}{dX^2}\right|_{X^2=0} X^2 + \frac{1}{2}\left.\frac{d^2\varepsilon_2}{d(X^2)^2}\right|_{X^2=0} (X^2)^2 + \cdots \\ &= \varepsilon_2^{(0)} + \frac{2(\delta_2^I + \varphi_1^{(I)})\varphi_2^{(I)}}{1+\varepsilon_2^{(0)}} X^2 + \left\{\frac{[2\varphi_2^{(I)}\varphi_2^{(I)} + 3(\delta_2^I + \varphi_1^{(I)})\varphi_3^{(I)}]}{1+\varepsilon_2^{(0)}}\right. \\ &\quad \left. - \frac{2[(\delta_2^I + \varphi_1^{(I)})\varphi_2^{(I)}]^2}{(1+\varepsilon_2^{(0)})^3}\right\}(X^2)^2 + \cdots;\end{aligned} \tag{8.49}$$

$$\begin{aligned}\gamma_{12} &\approx \gamma_{12}^{(0)} + \left.\frac{d\gamma_{12}}{dX^2}\right|_{X^2=0} X^2 + \frac{1}{2}\left.\frac{d^2\gamma_{12}}{d(X^2)^2}\right|_{X^2=0} (X^2)^2 + \cdots \\ &= \gamma_{12}^{(0)} + \frac{1}{\cos\gamma_{12}^{(0)}}\left\{\frac{2(\delta_1^I + u_{0,1}^{(I)})\varphi_2^{(I)} + (\delta_2^I + \varphi_1^{(I)})\varphi_{1,1}^{(I)}}{(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})}\right. \\ &\quad \left. - \sin\gamma_{12}^0\left[\frac{(\delta_1^I + u_{0,1}^{(I)})\varphi_{1,1}^{(I)}}{(1+\varepsilon_1^{(0)})^2} + \frac{2(\delta_2^I + \varphi_1^{(I)})\varphi_2^{(I)}}{(1+\varepsilon_2^{(0)})^2}\right]\right\} X^2 + \cdots,\end{aligned} \tag{8.50}$$

where for $I = 1, 2$,

$$\begin{aligned}\varepsilon_1^{(0)} &= \sqrt{(\delta_1^I + u_{0,1}^I)(\delta_1^I + u_{0,1}^I)} - 1 \\ &= \sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^2)^2} - 1,\end{aligned} \tag{8.51}$$

$$\begin{aligned}\varepsilon_2^{(0)} &= \sqrt{(\delta_2^I + \varphi_1^{(I)})(\delta_2^I + \varphi_1^{(I)})} - 1\\ &= \sqrt{(\varphi_1^{(1)})^2 + (1+\varphi_1^2)^2} - 1,\end{aligned} \tag{8.52}$$

$$\begin{aligned}\gamma_{12}^{(0)} &= \sin^{-1}\frac{(\delta_1^I + u_{0,1}^I)(\delta_2^I + \varphi_1^{(I)})}{(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})}\\ &= \sin^{-1}\frac{(1+u_{0,1}^1)\varphi_1^{(1)} + u_{0,1}^2(1+\varphi_1^{(2)})}{(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})}.\end{aligned} \tag{8.53}$$

The constitutive laws give the stresses on the deformed configuration as

$$\sigma_1 = f(\varepsilon_1, t) \text{ and } \sigma_{12} = g(\gamma_{12}, t). \tag{8.54}$$

The internal forces and moments in the deformed beam are defined as

$$\begin{aligned}N_1 &= \int_A \sigma_1 \left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23}\right] dA,\\ M_3 &= \int_A \sigma_1 \frac{X^2}{1+\varphi_1^{(2)}}\left[(1+\varepsilon_3)(1+\varepsilon_2)^2 \cos\gamma_{23}\right] dA,\\ Q_1 &= \int_A \sigma_{12}\left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23}\right] dA.\end{aligned} \tag{8.55}$$

For convenience, the subscripts of the internal forces can be dropped. The internal force vectors can be defined as

$$\begin{aligned}\mathbf{M} &\equiv M^3\mathbf{I}_3 = M\mathbf{n}_3,\\ \boldsymbol{\mathcal{N}} &\equiv N^I\mathbf{I}_I = N\mathbf{n}_1 + Q\mathbf{n}_2,\\ {}^{\mathrm{N}}\mathbf{M} &\equiv {}^{\mathrm{N}}M^I\mathbf{I}_I = \mathbf{g}_1 \times \boldsymbol{\mathcal{N}},\end{aligned} \tag{8.56}$$

where

$$\mathbf{g}_1 \equiv \frac{d\mathbf{r}}{dS} = (\delta_1^I + u_{0,1}^I)\mathbf{I}_I \text{ and } {}^{\mathrm{N}}\mathbf{M} \equiv \frac{1}{dS} d\mathbf{r} \times \boldsymbol{\mathcal{N}}. \tag{8.57}$$

The components of the internal forces in the $\mathbf{I}_I$ -direction are

$$\begin{aligned}N^I &= N\mathbf{n}_1 \cdot \mathbf{I}_I + Q\mathbf{n}_2 \cdot \mathbf{I}_I = N\cos\theta_{(\mathbf{n}_1,\mathbf{I}_I)} + Q\cos\theta_{(\mathbf{n}_2,\mathbf{I}_I)}\\ &= \frac{N(\delta_1^I + u_{0,1}^I)}{1+\varepsilon_1^{(0)}} + \frac{Q(\delta_2^I + \varphi_1^I)}{1+\varepsilon_2^{(0)}},\end{aligned} \tag{8.58}$$

$$\begin{aligned}M^I &= M\mathbf{n}_3 \cdot \mathbf{I}_I = M\cos\theta_{(\mathbf{n}_3,\mathbf{I}_I)} = \frac{M(\delta_3^I + u_{0,3}^I)}{1+\varepsilon_3^{(0)}},\\ {}^{\mathrm{N}}M^3 &= (\mathbf{g}_1 \times \boldsymbol{\mathcal{N}})\cdot\mathbf{I}_3 = (1+u_{0,1}^1)N^2 - u_{0,1}^2 N^1.\end{aligned} \tag{8.59}$$

Because of $u^I_{0,3}=0$ and $\varepsilon^{(0)}_3=0$, one obtains

$$
\begin{aligned}
{}^{\mathrm{N}}M^3 &= Q(1+\varepsilon^{(0)}_1),\ {}^{\mathrm{N}}M^1 = {}^{\mathrm{N}}M^2 = 0,\\
M^1 &= M^2 = 0 \text{ and } M^3 = M.
\end{aligned}
\tag{8.60}
$$

Equations of motion on the deformed beam are given by

$$
\begin{aligned}
\boldsymbol{\mathcal{N}}_{,1} + \mathbf{q} &= \rho \mathbf{u}_{0,tt} + I_3 \boldsymbol{\varphi}_{1,tt},\\
\mathbf{M}_{,1} + {}^{\mathrm{N}}\mathbf{M} + \mathbf{m} &= I_3 \mathbf{u}_{0,tt} + J_3 \boldsymbol{\varphi}_{1,tt};
\end{aligned}
\tag{8.61}
$$

and the corresponding scalar expressions are for $I=1,2$,

$$
\begin{aligned}
N^I_{,1} + q^I &= \rho u^I_{(0),tt} + I_3 \varphi^{(I)}_{1,tt},\\
M^3_{,1} + {}^{\mathrm{N}}M^3 + m^3 &= I_3 u^1_{(0),tt} + J_3 \varphi^{(1)}_{1,tt};
\end{aligned}
\tag{8.62}
$$

or

$$
\begin{aligned}
&\left[\frac{N(1+u^1_{0,1})}{1+\varepsilon^{(0)}_1} - \frac{Q u^2_{0,1}}{1+\varepsilon^{(0)}_1}\right]_{,1} + q^1 = \rho u^1_{(0),tt} + I_3 \varphi^{(1)}_{1,tt};\\
&\left[\frac{N u^2_{0,1}}{1+\varepsilon^{(0)}_1} + \frac{Q(1+u^1_{0,1})}{1+\varepsilon^{(0)}_1}\right]_{,1} + q^2 = \rho u^2_{(0),tt} + I_3 \varphi^{(2)}_{1,tt};\\
&M_{,1} + Q(1+\varepsilon^{(0)}_1) + m^3 = I_3 u^1_{(0),tt} + J_3 \varphi^{(1)}_{1,tt},
\end{aligned}
\tag{8.63}
$$

where $\rho = \int_A \rho_0 dA$, $I_3 = \int_A \rho_0 X^2 dA$ and $J_3 = \int_A \rho_0 (X^2)^2 dA$.

The force condition at a point $\mathscr{P}_k$ with $X^1 = X^1_k$ is

$$
\begin{aligned}
{}^{-}\mathbf{N}(X^1_k) + {}^{+}\mathbf{N}(X^1_k) + \mathbf{F}_k &= 0,\\
{}^{-}N^I(X^1_k) = {}^{+}N^I(X^1_k) + F^I_k &\quad (I=1,2).
\end{aligned}
\tag{8.64}
$$

The force boundary condition at the boundary point $\mathscr{P}_r$ is

$$
\mathbf{N}(X^1_r) + \mathbf{F}_r = 0 \ \text{ or } \ N^I(X^1_r) + F^I_r = 0 \quad (I=1,2).
\tag{8.65}
$$

If there is a concentrated moment at a point $\mathscr{P}_k$ with $X^1 = X^1_k$, the corresponding moment boundary condition is

$$
\begin{aligned}
{}^{-}\mathbf{M}(X^1_k) + {}^{+}\mathbf{M}(X^1_k) + \mathbf{M}_k &= 0,\\
{}^{-}M^3(X^1_k) = {}^{+}M^3(X^1_k) + M^3_k &.
\end{aligned}
\tag{8.66}
$$

The moment boundary condition at the boundary point $\mathscr{P}_k$ is

$$
\mathbf{M}(X^1_r) + \mathbf{M}_r = 0 \ \text{ or } \ M^3(X^1_r) + M^3_r = 0.
\tag{8.67}
$$

The displacement continuity and boundary conditions are

$$u_{k-}^{I} = u_{k+}^{I} \text{ and } u_{r}^{I} = B_{r}^{I}. \tag{8.68}$$

The afore-developed beam theory can be reduced to the beam theory given by Reissner (1972). The nonlinear vibration and chaos of a beam were extensively investigated (e.g., Verma, 1972; Luo and Han, 1999).

8.3. Nonlinear curved beams

Consider an arbitrary coordinate system as (X^1, Y, Z) at the *centroid* on the cross section of the beam. The central curve of the deformed beam is on the plane of (X^1, X^2), as shown in Fig.8.3. In other words, the neutral surface of the deformed beam is on the plane of (X^1, X^3). Let the coordinate S^1 be along the longitudinal direction of beam and the other two coordinates S^2 and S^3 be on the cross section of beam with the direction of S^1. The coordinates for the deformed, curved beam are (s^1, s^2, s^3). Because the initial configuration of the beam is a curved beam, under external force, the deformed configuration of the beam to the initial configuration does not experience any torsion ($\tau(s^1) \equiv 0$). In other words, under the resultant forces, the bending of beam is in the plane of (S^1, S^2). Thus, the configuration of the deformed beam is still a plane curve. Without loss of generality, the curvature direction of the deformed configuration can be assumed to be collinear to S^2. Because the widths of beam in two directions of S^2 and S^3 are very small compared to the length of the beam in direction of S^1, the elongation in the two directions of S^2 and S^3 should be very small, which can be neglected. Thus, as in the straight beam, the following assumptions are enforced.

(i) The configuration of the deformed beam to the initial curved beam does not experience any torsion ($\tau(s^1) \equiv 0$).
(ii) The curvature direction of the deformed beam is collinear to s^2.
(iii) The elongations in the two directions of S^2 and S^3 are very small (i.e., $\varepsilon_2 \approx 0$ and $\varepsilon_3 = 0$).
(iv) For bending only, the neutral surface is not deformed (i.e., $\gamma_{12} \approx 0$ and $\gamma_{13} = 0$).

From Assumption (i), no torque exists, and the transversal external forces should be added at *the shear center*. Similar to Eqs.(8.36) and (8.37), the external distributed forces and moments on the initial configuration are for ($I, \Lambda = 1, 2, 3$)

$$\mathbf{q} = q^I \mathbf{I}_I = q^\Lambda \mathbf{N}_\Lambda \text{ and } \mathbf{m} = m^I \mathbf{I}_I = m^\Lambda \mathbf{N}_\Lambda \tag{8.69}$$

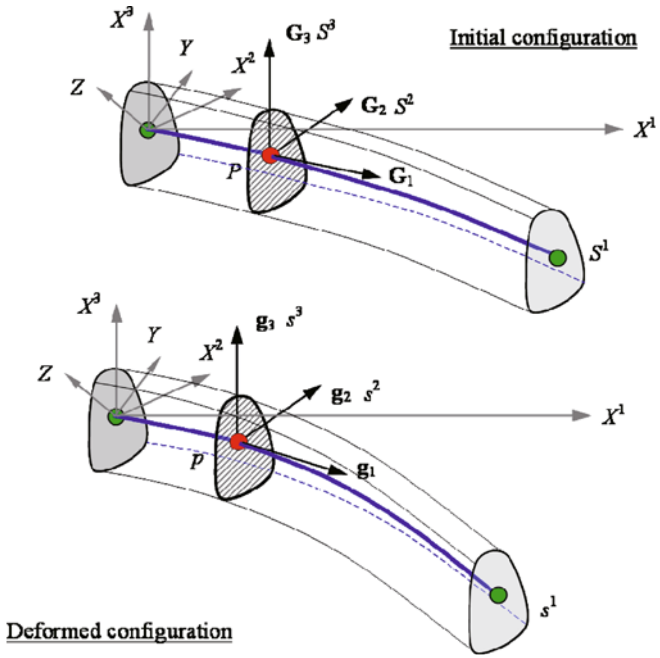


Fig. 8.3 A curved beam with initial and deformed configuration.

and concentrated forces on the initial configuration at a point $S^1 = S_k$,

$$\mathbf{F}_k = F_k^I \mathbf{I}_I = F_k^\Lambda \mathbf{N}_\Lambda \text{ and } \mathbf{M}_k = M_k^I \mathbf{I}_I = M_k^\Lambda \mathbf{N}_\Lambda. \tag{8.70}$$

Thus,

$$\begin{aligned} F^\Lambda |_{S^1=S} &= F^I \mathbf{I}_I \cdot \mathbf{N}_\Lambda = F^I \cos\theta_{(\mathbf{I}_I,\mathbf{N}_\Lambda)}, \\ M^\Lambda |_{S^1=S} &= M^I \mathbf{I}_I \cdot \mathbf{N}_\Lambda = M^I \cos\theta_{(\mathbf{I}_I,\mathbf{N}_\Lambda)}. \end{aligned} \tag{8.71}$$

The displacement vectors on the initial configuration are

$$\mathbf{R}(S^1) = X^I(S^1)\mathbf{I}_I, \mathbf{R}(S) = X_S^I \mathbf{I}_I \text{ and } \mathbf{R}_k = X_k^I \mathbf{I}_I. \tag{8.72}$$

The internal forces and moments for $(S^1 > S_k^1)$ are

$$\begin{aligned} \mathbf{F}|_{S^1=S} &= \sum_{k=1} \mathbf{F}_k + \int_0^S \mathbf{q}dS^1, \\ \mathbf{M}|_{S^1=S} &= \sum_{k=1} \mathbf{M}_k + \int_0^S \mathbf{m}dS^1 \\ &\quad + \sum_{k=1} (\mathbf{R}(S) - \mathbf{R}_k) \times {}^k\mathbf{F} + \int_0^S (\mathbf{R}(S) - \mathbf{R}(S^1)) \times \mathbf{q}dS^1; \end{aligned} \tag{8.73}$$

or for $I = 1, 2, 3$,

$$
\begin{aligned}
F^I|_{S^1=S} &= \sum_{k=1} {}^kF^I + \int_0^S q^I dS^1, \\
M^I|_{S^1=S} &= \sum_{k=1} M_k^I + \int_0^S m^I dS^1 \\
&\quad + \sum_{k=1} e_{IJK}(X_S^J - X_k^J)F_k^K + \int_0^S e_{IJK}(X_S^J - X_k^J)q^K dS^1.
\end{aligned} \tag{8.74}
$$

Assumptions (i) and (ii) requires the following conditions:

$$
F^\Lambda|_{S^1=S} = 0 \ \text{ for } \Lambda=3 \ \text{ and } \ M^\Lambda|_{S^1=S} = 0 \ \text{ for } \Lambda=1, 2. \tag{8.75}
$$

Since the vectors $\mathbf{I}_3$ and $\mathbf{N}_3$ ($\mathbf{G}_3$) are collinear and all points on the beam satisfy Eq.(8.75), one obtains

$$
\begin{aligned}
&{}^kF^I = 0 \ \text{ and } \ q^I = 0 \ \text{ for } I = 3, \\
&M^1 \cos\theta_{(\mathbf{I}_1,\mathbf{N}_\Lambda)} + M^2 \cos\theta_{(\mathbf{I}_2,\mathbf{N}_\Lambda)} = 0 \ \text{for } \Lambda = 1, 2.
\end{aligned} \tag{8.76}
$$

Because

$$
\begin{vmatrix} \cos\theta_{(\mathbf{I}_1,\mathbf{N}_1)} & \cos\theta_{(\mathbf{I}_2,\mathbf{N}_1)} \\ \cos\theta_{(\mathbf{I}_1,\mathbf{N}_2)} & \cos\theta_{(\mathbf{I}_2,\mathbf{N}_2)} \end{vmatrix} \neq 0, \tag{8.77}
$$

the second equation of Eq.(8.76) gives

$$
M^1 = M^2 = 0. \tag{8.78}
$$

Thus, the external force conditions for the curved beam without twisting are given by the first equation of Eqs.(8.76) and (8.78). In other words, no external distributed and concentrated forces are in the direction of X^3 and the resultant external moments in the directions of X^1 and X^3 are zero.

8.3.1. A nonlinear theory based on the Cartesian coordinates

The strain based on the change in length of $d\mathbf{R}$ per unit length for a curved beam in the Cartesian coordinates gives

$$
\begin{aligned}
\varepsilon_\alpha &= \frac{|\underset{\alpha}{d\mathbf{r}}| - |\underset{\alpha}{d\mathbf{R}}|}{|\underset{\alpha}{d\mathbf{R}}|} = \frac{1}{\sqrt{G_{\alpha\alpha}}}\sqrt{G_{\alpha\alpha} + 2E_{\alpha\alpha}} - 1 \\
&= \frac{1}{\sqrt{G_{\alpha\alpha}}}\sqrt{(X_{,\alpha}^I + u_{,\alpha}^I)(X_{,\alpha}^I + u_{,\alpha}^I)} - 1
\end{aligned} \tag{8.79}
$$

in which no summation on α can be completed. The Lagrangian strain tensor $E_{\alpha\beta}$ referred to the initial configuration is

$$E_{\alpha\beta} = \frac{1}{2}(X^I_{,\alpha}u^I_{,\beta} + X^I_{,\beta}u^I_{,\alpha} + u^I_{,\alpha}u^I_{,\beta}). \tag{8.80}$$

In the similar fashion, the angles between $\mathbf{n}_\alpha$ and $\mathbf{n}_\beta$ *before* and *after* deformation are expressed by $\Theta_{(\mathbf{N}_\alpha,\mathbf{N}_\beta)} = \pi/2$ and $\theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)}$, i.e.,

$$\begin{aligned}\cos\theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)} &= \frac{d\underset{\alpha}{\mathbf{r}} \cdot d\underset{\beta}{\mathbf{r}}}{|d\underset{\alpha}{\mathbf{r}}||d\underset{\beta}{\mathbf{r}}|} \\ &= \frac{(X^I_{,\alpha} + u^I_{,\alpha})(X^I_{,\beta} + u^I_{,\beta})}{\sqrt{(G_{\alpha\alpha} + 2E_{\alpha\alpha})(G_{\beta\beta} + 2E_{\beta\beta})}} \\ &= \frac{(X^I_{,\alpha} + u^I_{,\alpha})(X^I_{,\beta} + u^I_{,\beta})}{\sqrt{G_{\alpha\alpha}G_{\beta\beta}}(1+\varepsilon_\alpha)(1+\varepsilon_\beta)},\end{aligned} \tag{8.81}$$

and the corresponding shear strain is defined as

$$\begin{aligned}\gamma_{\alpha\beta} &\equiv \Theta_{(\mathbf{N}_\alpha,\mathbf{N}_\beta)} - \theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)} \\ &= \sin^{-1}\frac{(X^I_{,\alpha} + u^I_{,\alpha})(X^I_{,\beta} + u^I_{,\beta})}{\sqrt{(G_{\alpha\alpha} + 2E_{\alpha\alpha})(G_{\beta\beta} + 2E_{\beta\beta})}} \\ &= \sin^{-1}\frac{(X^I_{,\alpha} + u^I_{,\alpha})(X^I_{,\beta} + u^I_{,\beta})}{\sqrt{G_{\alpha\alpha}G_{\beta\beta}}(1+\varepsilon_\alpha)(1+\varepsilon_\beta)}.\end{aligned} \tag{8.82}$$

From Eqs.(8.24) and (8.28), the direction cosine of the rotation without summation on α and β is

$$\begin{aligned}\cos\theta_{(\mathbf{N}_\alpha,\mathbf{n}_\beta)} &= \frac{d\underset{\alpha}{\mathbf{R}} \cdot d\underset{\beta}{\mathbf{r}}}{|d\underset{\alpha}{\mathbf{R}}||d\underset{\beta}{\mathbf{r}}|} = \frac{X^I_{,\alpha}(X^I_{,\beta} + u^I_{,\beta})}{\sqrt{G_{\alpha\alpha}}\sqrt{G_{\beta\beta} + 2E_{\beta\beta}}} \\ &= \frac{X^I_{,\alpha}(X^I_{,\beta} + u^I_{,\beta})}{\sqrt{G_{\alpha\alpha}G_{\beta\beta}}(1+\varepsilon_\beta)}.\end{aligned} \tag{8.83}$$

Finally, the change ratio of areas before and after deformation is

$$\frac{d\underset{\alpha\beta}{a}}{d\underset{\alpha\beta}{A}} = (1+\varepsilon_\alpha)(1+\varepsilon_\beta)\sin\theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)}, \tag{8.84}$$

where $d\underset{\alpha\beta}{a} = |d\underset{\alpha}{\mathbf{r}} \times d\underset{\beta}{\mathbf{r}}|$ and $d\underset{\alpha\beta}{A} = |d\underset{\alpha}{\mathbf{R}} \times d\underset{\beta}{\mathbf{R}}|$.

From assumption (ii),

$$u^I = u_0^I(S,t) + \sum_{n=1}^{\infty} (S^2)^n \varphi_n^{(I)}(S,t) \ \text{ for } I=1,2. \tag{8.85}$$

From Assumption (ii), no displacements exist in the direction of S^3 (i.e., $u^3 = 0$). From the Kirchhoff's assumptions, under bending only, if the cross section is normal to the neutral surface before deformation, then after deformation, the deformed cross section is still normal to the deformed neutral surface. Thus, equation (8.85) can be assumed as

$$u^I = u_0^I(S,t) + S^2 \varphi^{(I)}(S,t) \quad (\alpha = 1, 2). \tag{8.86}$$

From assumptions (iii) and (iv),

$$\begin{aligned} &\frac{1}{G_{22}}(X_{,2}^I + u_{,2}^I)(X_{,2}^I + u_{,2}^I) = 1, \\ &(X_{,1}^I + u_{,1}^I)(X_{,2}^I + u_{,2}^I) = 0. \end{aligned} \tag{8.87}$$

With $u^3 = 0$ and Eq.(8.85), the Taylor series expansion of Eq.(8.87) give for the zero-order of S^2,

$$\begin{aligned} &\frac{1}{G_{22}}(X_{,2}^I + \varphi_1^{(I)})(X_{,2}^I + \varphi_1^{(I)}) = 1, \\ &(X_{,1}^I + u_{0,1}^I)(X_{,2}^I + \varphi_1^{(I)}) = 0, \end{aligned} \tag{8.88}$$

where $X_{,2}^I = G_2^I$. From Eq.(8.88),

$$\begin{aligned} \varphi_1^{(1)} &= \mp \frac{(X_{,1}^2 + u_{0,1}^2)\sqrt{G_{22}}}{\sqrt{(X_{,1}^1 + u_{0,1}^1)^2 + (X_{,1}^2 + u_{0,1}^2)^2}} - X_{,2}^1, \\ \varphi_1^{(2)} &= \pm \frac{(X_{,1}^1 + u_{0,1}^1)\sqrt{G_{22}}}{\sqrt{(X_{,1}^1 + u_{0,1}^1)^2 + (X_{,1}^2 + u_{0,1}^2)^2}} - X_{,2}^2. \end{aligned} \tag{8.89}$$

Similarly, one can obtain $\varphi_n^{(I)}$ ($n = 1, 2, \dots$ and $I = 1, 2$). Using the Taylor series expansion gives the approximate strains, i.e.,

$$\begin{aligned} \varepsilon_1 &\approx \varepsilon_1^{(0)} + \left.\frac{d\varepsilon_1}{dS^2}\right|_{S^2=0} S^2 + \frac{1}{2}\left.\frac{d^2\varepsilon_1}{d(S^2)^2}\right|_{S^2=0} (S^2)^2 + \cdots \\ &= \varepsilon_1^{(0)} + \frac{(X_{,1}^I + u_{0,1}^I)\varphi_{1,1}^{(I)}}{G_{11}(1+\varepsilon_1^{(0)})} S^2 \\ &\quad + \frac{1}{2}\frac{[2(X_{,1}^I + u_{0,1}^I)\varphi_{2,1}^{(I)}] + \varphi_{1,1}^{(I)}\varphi_{1,1}^{(I)}}{G_{11}(1+\varepsilon_1^{(0)})}(S^2)^2 \\ &\quad - \frac{1}{2}\frac{[(X_{,1}^I + u_{0,1}^I)\varphi_{1,1}^{(I)}]^2}{G_{11}^2(1+\varepsilon_1^{(0)})^3}(S^2)^2 + \cdots, \end{aligned} \tag{8.90}$$

$$\begin{aligned}\varepsilon_2 &\approx \varepsilon_2^{(0)} + \left.\frac{d\varepsilon_2}{dS^2}\right|_{S^2=0} S^2 + \frac{1}{2}\left.\frac{d^2\varepsilon_2}{d(S^2)^2}\right|_{S^2=0}(S^2)^2 + \cdots \\ &= \varepsilon_2^{(0)} + \frac{2(X^I_{,2} + \varphi_1^{(I)})\varphi_2^{(I)}}{G_{22}(1+\varepsilon_2^{(0)})} S^2 \\ &\quad + \frac{[2\varphi_2^{(I)}\varphi_2^{(I)} + 3(\delta_2^I + \varphi_1^{(I)})\varphi_3^{(I)}]}{G_{22}(1+\varepsilon_2^{(0)})}(S^2)^2 \\ &\quad - \frac{2[(X^I_{,2} + \varphi_1^{(I)})\varphi_2^{(I)}]^2}{G_{22}^2(1+\varepsilon_2^{(0)})^3}(S^2)^2 + \cdots;\end{aligned} \tag{8.91}$$

$$\begin{aligned}\gamma_{12} &\approx \gamma_{12}^{(0)} + \left.\frac{d\gamma_{12}}{dS^2}\right|_{S^2=0} S^2 + \frac{1}{2}\left.\frac{d^2\gamma_{12}}{d(S^2)^2}\right|_{S^2=0}(S^2)^2 + \cdots \\ &= \gamma_{12}^{(0)} + \frac{1}{\cos\gamma_{12}^{(0)}}\left\{\frac{2(X^I_{,1} + u_{0,1}^{(I)})\varphi_2^{(I)} + (X^I_{,2} + \varphi_1^{(I)})\varphi_{1,1}^{(I)}}{\sqrt{G_{11}}\sqrt{G_{22}}(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})}\right. \\ &\quad \left. - \sin\gamma_{12}^{(0)}\left[\frac{(X^I_{,1} + u_{0,1}^{(I)})\varphi_{1,1}^{(I)}}{G_{11}(1+\varepsilon_1^{(0)})^2} + \frac{2(X^I_{,2} + \varphi_1^{(I)})\varphi_2^{(I)}}{G_{22}(1+\varepsilon_2^{(0)})^2}\right]\right\} S^2 + \cdots,\end{aligned} \tag{8.92}$$

where for $I = 1, 2$

$$\begin{aligned}\varepsilon_1^{(0)} &= \frac{1}{\sqrt{G_{11}}}\sqrt{(X^I_{,1} + u^I_{0,1})(X^I_{,1} + u^I_{0,1})} - 1 \\ &= \frac{1}{\sqrt{G_{11}}}\sqrt{(X^1_{,1} + u^1_{0,1})^2 + (X^2_{,1} + u^2_{0,1})^2} - 1,\end{aligned} \tag{8.93}$$

$$\begin{aligned}\varepsilon_2^{(0)} &= \frac{1}{\sqrt{G_{22}}}\sqrt{(X^I_{,2} + \varphi_1^{(I)})(X^I_{,2} + \varphi_1^{(I)})} - 1 \\ &= \frac{1}{\sqrt{G_{22}}}\sqrt{(X^1_{,2} + \varphi_1^{(1)})^2 + (X^2_{,2} + \varphi_1^{(2)})^2} - 1,\end{aligned} \tag{8.94}$$

$$\begin{aligned}\gamma_{12}^{(0)} &= \sin^{-1}\frac{(X^I_{,1} + u^I_{0,1})(X^I_{,2} + \varphi_1^{(I)})}{\sqrt{G_{11}G_{22}}(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})} \\ &= \sin^{-1}\frac{(X^1_{,1} + u^1_{,1})(X^1_{,2} + \varphi_1^{(1)}) + (X^2_{,1} + u^2_{,1})(X^2_{,2} + \varphi_1^{(2)})}{\sqrt{G_{11}G_{22}}(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})}.\end{aligned} \tag{8.95}$$

The constitutive laws give the stresses on the deformed configuration, i.e.,

$$\sigma_1 = f(\varepsilon_1, \gamma_{12}, t) \text{ and } \sigma_{12} = g(\varepsilon_1, \gamma_{12}, t). \tag{8.96}$$

The internal forces and moments in the deformed beam are defined as

$$
\begin{aligned}
N_1 &= \int_A \sigma_1 \left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23}\right] dA,\\
M_3 &= \int_A \frac{\sigma_1 S^2}{1+\varphi_1^{(2)}}\left[\sqrt{G_{22}}(1+\varepsilon_3)(1+\varepsilon_2)^2\cos\gamma_{23}\right] dA,\\
Q_1 &= \int_A \sigma_{12}\left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23}\right] dA.
\end{aligned}
\tag{8.97}
$$

For convenience, the subscripts of the internal forces can be dropped. The internal force vectors are defined as

$$
\begin{aligned}
\mathbf{M} &\equiv M^I \mathbf{I}_I = M\mathbf{n}_3,\\
\boldsymbol{\mathcal{N}} &\equiv N^I \mathbf{I}_I = N\mathbf{n}_1 + Q\mathbf{n}_2,\\
{}^{\mathrm{N}}\mathbf{M} &\equiv {}^{\mathrm{N}}M^I \mathbf{I}_I = \mathbf{g}_1 \times \boldsymbol{\mathcal{N}};
\end{aligned}
\tag{8.98}
$$

$$
\mathbf{g}_1 \equiv \frac{d\mathbf{r}}{dS} = (X^I_{,1} + u^I_{0,1})\mathbf{I}_I \text{ and } {}^{\mathrm{N}}\mathbf{M} \equiv \frac{1}{dS} d\mathbf{r} \times \boldsymbol{\mathcal{N}}.
\tag{8.99}
$$

The components of the internal forces in the $\mathbf{I}_I$ -direction are

$$
\begin{aligned}
N^I &= N\mathbf{n}_1 \cdot \mathbf{I}_I + Q\mathbf{n}_2 \cdot \mathbf{I}_I = N\cos\theta_{(\mathbf{n}_1,\mathbf{I}_I)} + Q\cos\theta_{(\mathbf{n}_2,\mathbf{I}_I)}\\
&= \frac{N(X^I_{,1} + u^I_{0,1})}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})} + \frac{Q(X^I_{,2} + \varphi_1^I)}{\sqrt{G_{22}}(1+\varepsilon_2^{(0)})},
\end{aligned}
\tag{8.100}
$$

$$
\begin{aligned}
M^I &= M\mathbf{n}_3 \cdot \mathbf{I}_I = M\cos\theta_{(\mathbf{n}_3,\mathbf{I}_I)} = \frac{M(X^I_{,3} + u^I_{(0),3})}{1+\varepsilon_3^{(0)}},\\
{}^{\mathrm{N}}M^1 &= (\mathbf{g}_1 \times \boldsymbol{\mathcal{N}}) \cdot \mathbf{I}_1 = (X^2_{,1} + u^2_{0,1})N^3 - (X^3_{,1} + u^3_{0,1})N^2,\\
{}^{\mathrm{N}}M^2 &= (\mathbf{g}_1 \times \boldsymbol{\mathcal{N}}) \cdot \mathbf{I}_2 = (X^3_{,1} + u^3_{0,1})N^1 - (X^1_{,1} + u^1_{0,1})N^3,\\
{}^{\mathrm{N}}M^3 &= (\mathbf{g}_1 \times \boldsymbol{\mathcal{N}}) \cdot \mathbf{I}_3 = (X^1_{,1} + u^1_{0,1})N^2 - (X^2_{,1} + u^2_{0,1})N^1.
\end{aligned}
\tag{8.101}
$$

Due to $u^I_{0,3} = 0,\ X^3_{,\alpha} = 0,\ X^1_{,3} = X^2_{,3} = 0,\ u^3 = 0$ and $\varepsilon_3^{(0)} = 0$, the following equations are achieved, i.e.,

$$
\begin{aligned}
&{}^{\mathrm{N}}M^3 = Q\frac{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})}{\sqrt{G_{22}}(1+\varepsilon_2^{(0)})}, \text{ and } {}^{\mathrm{N}}M^1 = {}^{\mathrm{N}}M^2 = 0.\\
&M^1 = M^2 = 0 \text{ and } M^3 = M.
\end{aligned}
\tag{8.102}
$$

Based on the deformed middle surface in the Lagrangian coordinates, the equations of motion for the deformed beam are

$$
\begin{aligned}
\boldsymbol{\mathcal{N}}_{,1} + \mathbf{q} &= \rho\mathbf{u}_{0,tt} + I_3\boldsymbol{\varphi}_{1,tt},\\
\mathbf{M}_{,1} + {}^{\mathrm{N}}\mathbf{M} + \mathbf{m} &= I_3\mathbf{u}_{0,tt} + J_3\boldsymbol{\varphi}_{1,tt};
\end{aligned}
\tag{8.103}
$$

and the scalar expressions are for $I = 1, 2$

$$\begin{aligned} N^I_{,1} + q^I &= \rho u^I_{0,tt} + I_3 \varphi^{(I)}_{1,tt}, \\ M^3_{,1} + {}^{\mathrm{N}}M^3 + m^3 &= I_3 u^1_{0,tt} + J_3 \varphi^{(1)}_{1,tt} \end{aligned} \tag{8.104}$$

where $\rho = \int_A \rho_0 dA$, $I_3 = \int_A \rho_0 X^2 dA$ and $J_3 = \int_A \rho_0 (X^2)^2 dA$. With Eqs.(8.89) and (8.100)–(8.102), the foregoing equation gives

$$\begin{aligned} &\left[\frac{N(X^1_{,1} + u^1_{0,1})}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)} - \frac{Q(X^2_{,1} + u^2_{0,1})}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)} \right]_{,1} + q^1 = \rho u^1_{0,tt} + I_3 \varphi^{(1)}_{1,tt}, \\ &\left[\frac{N(X^2_{,1} + u^2_{0,1})}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)} + \frac{Q(X^1_{,1} + u^1_{0,1})}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)} \right]_{,1} + q^2 = \rho u^2_{0,tt} + I_3 \varphi^{(2)}_{1,tt}, \end{aligned} \tag{8.105}$$

$$M_{,1} + Q \frac{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)}{\sqrt{G_{22}}} + m^3 = I_3 u^1_{0,tt} + J_3 \varphi^{(1)}_{1,tt}. \tag{8.106}$$

As in Eqs.(8.61)–(8.65), the force and displacement continuity and boundary conditions can be given as follows.

The force condition at a point $\mathscr{P}_k$ with $S^1 = S^1_k$ is

$$\begin{aligned} &{}^{-}\mathbf{N}(S^1_k) + {}^{+}\mathbf{N}(S^1_k) + \mathbf{F}_k = 0, \\ &{}^{-}N^I(S^1_k) = {}^{+}N^I(S^1_k) + F^I_k \quad (I = 1, 2). \end{aligned} \tag{8.107}$$

The force boundary condition at the boundary point $\mathscr{P}_r$ is

$$\mathbf{N}(S^1_r) + \mathbf{F}_r = 0 \ \text{ or } \ N^I(S^1_r) + F^I_r = 0 \quad (I = 1, 2). \tag{8.108}$$

If there is a concentrated moment at a point $\mathscr{P}_k$ with $S^1 = S_k$, the corresponding moment boundary condition is

$$\begin{aligned} &{}^{-}\mathbf{M}(S^1_k) + {}^{+}\mathbf{M}(S^1_k) + \mathbf{M}_k = 0, \\ &{}^{-}M^I(S^1_k) = {}^{+}M^I(S^1_k) + M^I_k \quad (I = 3). \end{aligned} \tag{8.109}$$

The moment boundary condition at the boundary point $\mathscr{P}_k$ is

$$\mathbf{M}(S^1_r) + \mathbf{M}_r = 0 \text{ or } M^I(S^1_r) + M^I_r = 0 \quad (I = 3). \tag{8.110}$$

The displacement continuity and boundary conditions are the same as in Eq.(8.68). From the sign convention, the positive "+" in the second equation of Eq.(8.89) was adopted.

8.3.2. A nonlinear theory based on the curvilinear coordinates

The strain based on the change in length of $d\mathbf{R}$ per unit length gives

$$\varepsilon_\alpha = \frac{|\underset{\alpha}{d\mathbf{r}}| - |\underset{\alpha}{d\mathbf{R}}|}{|\underset{\alpha}{d\mathbf{R}}|} = \sqrt{1+\frac{2E_{\alpha\alpha}}{G_{\alpha\alpha}}} - 1 = \frac{1}{\sqrt{G_{\alpha\alpha}}}\sqrt{G_{\alpha\alpha}+2E_{\alpha\alpha}} - 1$$

$$= \frac{\sqrt{G_{\beta\beta}}}{\sqrt{G_{\alpha\alpha}}}\sqrt{(\delta_\alpha^\beta + u^\beta_{;\alpha})(\delta_\alpha^\beta + u^\beta_{;\alpha})} - 1, \tag{8.111}$$

where the Lagrangian strain tensor $E_{\alpha\beta}$ to the initial configuration is

$$E_{\alpha\beta} = \frac{1}{2}(u_{\alpha;\beta} + u_{\alpha;\beta} + u^\gamma_{;\alpha} u_{\gamma;\beta})$$

$$= \frac{1}{2}\left[(\delta_\alpha^\gamma + u^\gamma_{;\alpha})(\delta_\beta^\delta + u^\delta_{;\beta})G_{\gamma\delta} - G_{\alpha\beta}\right]. \tag{8.112}$$

Similarly, the angles between $\mathbf{n}_\alpha$ and $\mathbf{n}_\beta$ *before* and *after* deformation are expressed by $\Theta_{(\mathbf{N}_\alpha,\mathbf{N}_\beta)} = \pi/2$ and $\theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)}$, i.e.,

$$\cos\theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)} = \frac{\underset{\alpha}{d\mathbf{r}} \cdot \underset{\beta}{d\mathbf{r}}}{|\underset{\alpha}{d\mathbf{r}}||\underset{\beta}{d\mathbf{r}}|} = \frac{G_{\alpha\beta}+2E_{\alpha\beta}}{\sqrt{(G_{\alpha\alpha}+2E_{\alpha\alpha})(G_{\beta\beta}+2E_{\beta\beta})}}$$

$$= \frac{(\delta_\alpha^\gamma + u^\gamma_{;\alpha})(\delta_\beta^\delta + u^\delta_{;\beta})G_{\gamma\delta}}{\sqrt{G_{\alpha\alpha}G_{\beta\beta}}(1+\varepsilon_\alpha)(1+\varepsilon_\beta)}, \tag{8.113}$$

and the shear strain is

$$\gamma_{\alpha\beta} \equiv \Theta_{(\mathbf{N}_\alpha,\mathbf{N}_\beta)} - \theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)}$$

$$= \sin^{-1}\frac{G_{\alpha\beta}+2E_{\alpha\beta}}{\sqrt{(G_{\alpha\alpha}+2E_{\alpha\alpha})(G_{\beta\beta}+2E_{\beta\beta})}},$$

$$= \sin^{-1}\frac{(\delta_\alpha^\gamma + u^\gamma_{;\alpha})(\delta_\beta^\delta + u^\delta_{;\beta})G_{\gamma\delta}}{\sqrt{G_{\alpha\alpha}G_{\beta\beta}}(1+\varepsilon_\beta)(1+\varepsilon_\beta)}. \tag{8.114}$$

The direction cosine of the rotation without summation on α and β is

$$\cos\theta_{(\mathbf{N}_\alpha,\mathbf{n}_\beta)} = \frac{\underset{\alpha}{d\mathbf{R}} \cdot \underset{\beta}{d\mathbf{r}}}{|\underset{\alpha}{d\mathbf{R}}||\underset{\beta}{d\mathbf{r}}|} = \frac{G_{\alpha\beta}+u_{\alpha;\beta}}{\sqrt{G_{\alpha\alpha}}\sqrt{G_{\beta\beta}+2E_{\beta\beta}}}$$

$$= \frac{G_{\alpha\beta}+u_{\alpha;\beta}}{\sqrt{G_{\alpha\alpha}G_{\gamma\gamma}}\sqrt{(\delta_\beta^\gamma + u^\gamma_{;\beta})(\delta_\beta^\gamma + u^\gamma_{;\beta})}}$$

$$= \frac{G_{\alpha\beta} + u_{\alpha;\beta}}{\sqrt{G_{\alpha\alpha} G_{\beta\beta}}(1+\varepsilon_\beta)} = \frac{(\delta_\beta^\gamma + u_{;\alpha}^\gamma) G_{\gamma\alpha}}{\sqrt{G_{\alpha\alpha} G_{\beta\beta}}(1+\varepsilon_\beta)}. \tag{8.115}$$

In addition, the change ratio of areas *before* and *after* deformation are given by

$$\frac{\underset{\alpha\beta}{da}}{\underset{\alpha\beta}{dA}} = (1+\varepsilon_\alpha)(1+\varepsilon_\beta)\sin\theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)} \tag{8.116}$$

where $\underset{\alpha\beta}{da} = |\, \underset{\alpha}{d\mathbf{r}} \times \underset{\beta}{d\mathbf{r}} \,|$ and $\underset{\alpha\beta}{dA} = |\, \underset{\alpha}{d\mathbf{R}} \times \underset{\beta}{d\mathbf{R}} \,|$.

From assumption (ii),

$$u^\Lambda = u_0^\Lambda(S,t) + \sum\nolimits_{n=1}^{\infty} (S^2)^n \varphi_n^{(\Lambda)}(S,t) \quad \text{for } \Lambda=1,\,2. \tag{8.117}$$

No displacements exist in the direction of S^3 (i.e., $u^3 = 0$). From the Kirchhoff's assumptions, under bending only, if the cross section is normal to the neutral surface before deformation, then after deformation, the deformed cross section is still normal to the deformed neutral surface. Thus, equation (8.117) becomes

$$u^\Lambda = u_0^\Lambda(S,t) + S^2 \varphi^{(\Lambda)}(S,t) \quad (\alpha = 1, 2\,). \tag{8.118}$$

From Assumptions (iii) and (iv),

$$\begin{aligned} (\delta_2^\Lambda + u_{;2}^\Lambda)(\delta_2^\Lambda + u_{;2}^\Lambda) G_{\Lambda\Lambda} &= G_{22}, \\ (\delta_1^\Lambda + u_{;1}^\Lambda)(\delta_2^\Gamma + u_{;2}^\Gamma) G_{\Lambda\Gamma} &= 0. \end{aligned} \tag{8.119}$$

With $u^3 = 0$ and Eq.(8.117), the Taylor series expansion of Eq.(8.119) gives for the zero-order of S^2:

$$\begin{aligned} G_{\Lambda\Lambda}(\delta_2^{(\Lambda)} + \varphi_1^{(\Lambda)})(\delta_2^{(\Lambda)} + \varphi_1^{(\Lambda)}) &= G_{22}, \\ (\delta_1^\Lambda + u_{0;1}^\Lambda)(\delta_2^\Gamma + \varphi_1^{(\Gamma)}) G_{\Lambda\Gamma} &= 0. \end{aligned} \tag{8.120}$$

Form the foregoing equations,

$$\begin{aligned} \varphi_1^{(1)} &= \mp \frac{u_{0;1}^2 G_{22}}{\sqrt{G_{11}}\sqrt{(1+u_{0;1}^1)^2 G_{11} + (u_{0;1}^2)^2 G_{22}}}, \\ \varphi_1^{(2)} &= \pm \frac{(1+u_{0;1}^1)\sqrt{G_{11}}}{\sqrt{(1+u_{0;1}^1)^2 G_{11} + (u_{0;1}^2)^2 G_{22}}} - 1. \end{aligned} \tag{8.121}$$

From the sign convention, the positive "+" in the second equation of Eq.(8.121) will be adopted. Similarly, one obtains $\varphi_n^{(\alpha)}$ ($n = 1, 2, \ldots$ and $\alpha = 1, 2$). The approximate strains for the curved beam in the curvilinear coordinates are:

$$\varepsilon_1 \approx \varepsilon_1^{(0)} + \left.\frac{d\varepsilon_1}{dS^2}\right|_{S^2=0} S^2 + \frac{1}{2}\left.\frac{d^2\varepsilon_1}{d(S^2)^2}\right|_{S^2=0} (S^2)^2 + \cdots$$

$$= \varepsilon_1^{(0)} + \frac{(\delta_1^\Lambda + u_{0;1}^\Lambda)\varphi_{1;1}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{11}(1+\varepsilon_1^{(0)})} S^2$$

$$+ \frac{1}{2}\left\{ \frac{[2(\delta_1^\Lambda + u_{0;1}^\Lambda)\varphi_{2;1}^{(\Lambda)}] + \varphi_{1;1}^{(\Lambda)}\varphi_{1;1}^{(\Lambda)}}{G_{11}(1+\varepsilon_1^{(0)})} G_{\Lambda\Lambda} \right.$$

$$\left. - \frac{[(\delta_1^\Lambda + u_{0;1}^\Lambda)\varphi_{1;1}^{(\Lambda)}]^2 G_{\Lambda\Lambda}^2}{G_{11}^2(1+\varepsilon_1^{(0)})^3} \right\} (S^2)^2 + \cdots, \tag{8.122}$$

$$\varepsilon_2 \approx \varepsilon_2^{(0)} + \left.\frac{d\varepsilon_2}{dS^2}\right|_{S^2=0} S^2 + \frac{1}{2}\left.\frac{d^2\varepsilon_2}{d(S^2)^2}\right|_{S^2=0} (S^2)^2 + \cdots$$

$$= \varepsilon_2^{(0)} + \frac{2(X_{,2}^\Lambda + \varphi_1^{(\Lambda)})\varphi_2^{(\Lambda)} G_{\Lambda\Lambda}}{G_{22}(1+\varepsilon_2^{(0)})} S^2$$

$$+ \left\{ \frac{[2\varphi_2^{(\Lambda)}\varphi_2^{(\Lambda)} + 3(\delta_2^\Lambda + \varphi_1^{(\Lambda)})\varphi_3^{(\Lambda)}] G_{\Lambda\Lambda}}{G_{22}(1+\varepsilon_2^{(0)})} \right.$$

$$\left. - \frac{2[(X_{,2}^\Lambda + \varphi_1^{(\Lambda)})\varphi_2^{(\Lambda)}]^2 G_{\Lambda\Lambda}^2}{G_{22}^2(1+\varepsilon_2^{(0)})^3} \right\} (S^2)^2 + \cdots, \tag{8.123}$$

$$\gamma_{12} \approx \gamma_{12}^{(0)} + \left.\frac{d\gamma_{12}}{dS^2}\right|_{S^2=0} S^2 + \frac{1}{2}\left.\frac{d^2\gamma_{12}}{d(S^2)^2}\right|_{S^2=0} (S^2)^2 + \cdots$$

$$= \gamma_{12}^{(0)} + \frac{1}{\cos\gamma_{12}^{(0)}} \left\{ \frac{(\delta_1^\Lambda + u_{0;1}^{(\Lambda)})\varphi_1^{(\Lambda)} + (\delta_2^\Lambda + \varphi_1^{(\Lambda)})\varphi_{1;1}^{(\Lambda)}}{\sqrt{G_{11}}\sqrt{G_{22}}(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})} G_{\Lambda\Lambda} \right.$$

$$\left. - \sin\gamma_{12}^{(0)} \left[\frac{(\delta_1^\Lambda + u_{0;1}^{(\Lambda)})\varphi_{1;1}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{11}(1+\varepsilon_1^{(0)})^2} + \frac{2(\delta_2^\Lambda + \varphi_1^{(\Lambda)})\varphi_2^{(\Lambda)} G_{\Lambda\Lambda}}{G_{22}(1+\varepsilon_2^{(0)})^2} \right] \right\} S^2 + \cdots, \tag{8.124}$$

$$\varepsilon_1^{(0)} = \frac{1}{\sqrt{G_{11}}} \sqrt{(\delta_1^\Lambda + u_{0;1}^\Lambda)(\delta_1^\Lambda + u_{0;1}^\Lambda) G_{\Lambda\Lambda}} - 1$$

$$= \frac{1}{\sqrt{G_{11}}} \sqrt{(1+u_{0;1}^1)^2 G_{11} + (u_{0;1}^2)^2 G_{22}} - 1, \tag{8.125}$$

$$\varepsilon_2^{(0)} = \frac{1}{\sqrt{G_{22}}} \sqrt{(\delta_2^\Lambda + \varphi_1^{(\Lambda)})(\delta_2^\Lambda + \varphi_1^{(\Lambda)}) G_{\Lambda\Lambda}} - 1$$

$$= \frac{1}{\sqrt{G_{22}}} \sqrt{(\varphi_1^{(1)})^2 G_{11} + (1+\varphi_1^{(2)})^2 G_{22}} - 1, \tag{8.126}$$

$$\gamma_{12}^{(0)} = \sin^{-1}\frac{(\delta_1^\Lambda + u_{0;1}^\Lambda)(\delta_2^\Gamma + \varphi_1^{(\Gamma)})G_{\Lambda\Gamma}}{\sqrt{G_{11}G_{22}}\,(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})}$$

$$= \sin^{-1}\frac{(1+u_{0;1}^1)\varphi_1^{(1)}G_{11} + u_{0;1}^2(1+\varphi_1^{(2)})G_{22}}{\sqrt{G_{11}G_{22}}\,(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})}. \tag{8.127}$$

As in Eq.(8.96), the stresses on the deformed configuration can be defined by the constitutive laws, i.e.,

$$\sigma_1 = f(\varepsilon_1, \gamma_{12}, t) \text{ and } \sigma_{12} = g(\varepsilon_1, \gamma_{12}, t). \tag{8.128}$$

The internal forces and moments in the deformed beam are defined as

$$\begin{aligned}
N_1 &= \int_A \sigma_1 \left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23}\right] dA,\\
M_3 &= \int_A \sigma_1 \frac{S^2}{1+\varphi_1^{(2)}}\left[(1+\varepsilon_3)(1+\varepsilon_2)^2\cos\gamma_{23}\right] dA,\\
Q_1 &= \int_A \sigma_{12} \left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23}\right] dA.
\end{aligned} \tag{8.129}$$

For convenience, the subscripts of the internal forces can be dropped again. The internal force vectors are defined as

$$\begin{aligned}
\mathbf{M} &\equiv M^3\mathbf{N}_3 = M\mathbf{n}_3,\\
\mathcal{N} &\equiv N^\Lambda \mathbf{N}_\Lambda = N\mathbf{n}_1 + Q\mathbf{n}_2,\\
{}^{\mathrm{N}}\mathbf{M} &\equiv {}^{\mathrm{N}}M^\Lambda \mathbf{N}_\Lambda = \mathbf{g}_1 \times \mathcal{N},
\end{aligned} \tag{8.130}$$

where

$$\mathbf{g}_1 \equiv \frac{d\mathbf{r}}{dS} = (\delta_1^\Lambda + u_{0;1}^\Lambda)\sqrt{G_{\Lambda\Lambda}}\,\mathbf{N}_\Lambda \text{ and } {}^{\mathrm{N}}\mathbf{M} \equiv \frac{1}{dS}d\mathbf{r}\times\mathcal{N}. \tag{8.131}$$

The components of the internal forces in the $\mathbf{G}_\Lambda$-direction are

$$\begin{aligned}
N^\Lambda &= N\mathbf{n}_1\cdot\mathbf{N}_\Lambda + Q\mathbf{n}_2\cdot\mathbf{N}_\Lambda = N\cos\theta_{(\mathbf{n}_1,\mathbf{N}_\Lambda)} + Q\cos\theta_{(\mathbf{n}_2,\mathbf{N}_\Lambda)}\\
&= \frac{N(\delta_1^\Gamma + u_{0;1}^\Gamma)G_{\Gamma\Lambda}}{\sqrt{G_{\Lambda\Lambda}G_{11}}\,(1+\varepsilon_1^{(0)})} + \frac{Q(\delta_2^\Gamma + \varphi_1^{(\Gamma)})G_{\Gamma\Lambda}}{\sqrt{G_{\Lambda\Lambda}G_{22}}\,(1+\varepsilon_2^{(0)})},
\end{aligned} \tag{8.132}$$

$$\begin{aligned}
M_1^\Lambda &= M\mathbf{n}_3\cdot\mathbf{N}_\Lambda = M\cos\theta_{(\mathbf{n}_3,\mathbf{N}_\Lambda)} = \frac{MG_{3\Lambda}}{\sqrt{G_{\Lambda\Lambda}G_{33}}},\\
{}^{\mathrm{N}}M^1 &= (\mathbf{g}_1\times\mathcal{N})\cdot\mathbf{N}_1 = u_{0;1}^2\sqrt{G_{22}}N^3 - u_{0;1}^3\sqrt{G_{33}}N^2,\\
{}^{\mathrm{N}}M^2 &= (\mathbf{g}_1\times\mathcal{N})\cdot\mathbf{N}_2 = u_{0;1}^3\sqrt{G_{33}}N^1 - (1+u_{0;1}^1)\sqrt{G_{11}}N^3,\\
{}^{\mathrm{N}}M^3 &= (\mathbf{g}_1\times\mathcal{N})\cdot\mathbf{N}_3 = (1+u_{0;1}^1)\sqrt{G_{11}}N^2 - u_{0;1}^2\sqrt{G_{22}}N^1.
\end{aligned} \tag{8.133}$$

If $u^{\Lambda}_{(0),3} = u^3 = 0$, $\varepsilon^{(0)}_3 = 0$, $G_{\Lambda 3} = 0$ ($\Lambda \neq 3$) and $G_{33} = 1$, then

$$
\begin{aligned}
&M^1_1 = M^2_1 = 0, \quad M^3_1 = M, \\
&{}^{\mathrm{N}}M^1 = {}^{\mathrm{N}}M^2 = 0, \\
&{}^{\mathrm{N}}M^3 = \frac{Q}{(1+\varepsilon^{(0)}_2)} \frac{\sqrt{G_{11}}}{\sqrt{G_{22}}} (1+\varepsilon^{(0)}_1).
\end{aligned}
\tag{8.134}
$$

Based on the deformed middle surface in the Lagrangian coordinates, the equations of motion for the deformed beam are

$$
\begin{aligned}
\mathcal{N}_{,1} + \mathbf{q} &= \rho \mathbf{u}_{0,tt} + I_3 \boldsymbol{\varphi}_{1,tt}, \\
\mathbf{M}_{,1} + {}^{\mathrm{N}}\mathbf{M} + \mathbf{m} &= I_3 \mathbf{u}_{0,tt} + J_3 \boldsymbol{\varphi}_{1,tt};
\end{aligned}
\tag{8.135}
$$

and for $(\Lambda = 1, 2)$,

$$
\begin{aligned}
N^{\Lambda}_{;1} + q^{\Lambda} &= \rho u^{\Lambda}_{0,tt} + I_3 \varphi^{(\Lambda)}_{1,tt}, \\
M^3_{;1} + {}^{\mathrm{N}}M^3 + m^3 &= I_3 u^1_{0,tt} + J_3 \varphi^{(1)}_{1,tt},
\end{aligned}
\tag{8.136}
$$

or

$$
\begin{aligned}
&\left[\frac{N(1+u^1_{0;1})}{1+\varepsilon^{(0)}_1} + \frac{Q\varphi^{(1)}_1 \sqrt{G_{11}}}{\sqrt{G_{22}}(1+\varepsilon^{(0)}_2)} \right]_{;1} + q^1 = \rho u^1_{0,tt} + I_3 \varphi^{(1)}_{1,tt}, \\
&\left[\frac{N u^2_{0;1} \sqrt{G_{22}}}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)} + \frac{Q(1+\varphi^{(2)}_1)}{1+\varepsilon^{(0)}_2} \right]_{;1} + q^2 = \rho u^2_{0,tt} + I_3 \varphi^{(2)}_{1,tt}, \\
&M_{;1} + Q \frac{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)}{\sqrt{G_{22}}} + m^3 = I_3 u^1_{0,tt} + J_3 \varphi^{(1)}_{1,tt},
\end{aligned}
\tag{8.137}
$$

where $\rho = \int_A \rho_0 dA$, $I_3 = \int_A \rho_0 S^2 dA$ and $J_3 = \int_A \rho_0 (S^2)^2 dA$.

As in Eqs.(8.106)–(8.109), the force and displacement continuity and boundary conditions can be given. The force condition at a point $\mathscr{P}_k$ with $S^1 = S^1_k$ is

$$
\begin{aligned}
&{}^{-}\mathbf{N}(S^1_k) + {}^{+}\mathbf{N}(S^1_k) + \mathbf{F}_k = 0, \\
&{}^{-}N^{\Lambda}(S^1_k) = {}^{+}N^{\Lambda}(S^1_k) + F^{\Lambda}_k \quad (\Lambda = 1, 2).
\end{aligned}
\tag{8.138}
$$

The force boundary condition at the boundary point $\mathscr{P}_r$ is

$$
\mathbf{N}(S^1_r) + \mathbf{F}_r = 0 \ \text{ or } \ N^{\Lambda}(S^1_r) + F^{\Lambda}_r = 0 \quad (\Lambda = 1, 2).
\tag{8.139}
$$

If there is a concentrated moment at $\mathscr{P}_k$ with $S^1 = S^1_k$, the corresponding moment boundary condition is

$$
\begin{aligned}
&{}^{-}\mathbf{M}(S_k^1) + {}^{+}\mathbf{M}(S_k^1) + \mathbf{M}_k = 0,\\
&{}^{-}M^{\Lambda}(S_k^1) = {}^{+}M^{\Lambda}(S_k^1) + M_k^{\Lambda} \quad (\Lambda = 3).
\end{aligned}
\tag{8.140}
$$

The moment boundary condition at the boundary point $\mathscr{P}_r$ is

$$
\mathbf{M}(S_r^1) + \mathbf{M}_r = 0 \ \text{ or } \ M^{\Lambda}(S_r^1) + M_r^{\Lambda} = 0 \quad (\Lambda = 3). \tag{8.141}
$$

The displacement continuity and boundary conditions are similar to Eq. (8.68). i.e., $u_{k-}^{\Lambda} = u_{k+}^{\Lambda}$ and $u_r^{\Lambda} = B_r^{\Lambda}$ ($\Lambda = 1, 2$).

8.4. A nonlinear theory of straight rods

Consider a nonlinear rod in the initial configuration to be straight, which requires that the initial curvature and torsion should be zero ($\kappa(S) = 0$ and $\tau(S) = 0$). Thus, let $S^I = X^I$, $G_{IJ} = 0$ and $G_{II} = 1$ ($I, J = 1, 2, 3$). The three dimensional displacements, strains, the directional cosine of rotation and the change rate of the area are given in Eqs.(8.29)−(8.35). It is assumed that the coordinate X^1 is along the longitudinal direction of rod and the other two coordinates X^2 and X^3 are on the cross section of the rod with the direction of X^1. The coordinates for the deformed straight rod are (s^1, s^2, s^3). As in the thin beam theory, the widths of rod in two directions of X^2 and X^3 are very small compared to the length of the rod in direction of X^1, the elongation in the two directions of X^2 and X^3 should be very small, which can be neglected. Based on the aforementioned reasons, the assumptions for thin rods are adopted.

(i) The elongations in the two directions of X^2 and X^3 are very small (i.e., $\varepsilon_2 \approx 0$ and $\varepsilon_3 \approx 0$).

(ii) Under bending only, the neutral surface is not deformed (i.e., $\gamma_{12} \approx 0$ and $\gamma_{13} \approx 0$).

Choose an arbitrary coordinate frame as (X^1, X^2, X^3) and the coordinate of X^1 goes through the *centroid* on the cross section of the rod. The centroid curve of the deformed rod is along the coordinate of s^1 in the coordinates of (s^1, s^2, s^3), as shown in Fig.8.4. The external forces on the rod can be given as in Eqs.(8.36)−(8.40). Under the torque, the rod possesses torsion $\tau(s^1) = \tau(s)$ in the direction of s^1. The transverse forces off the shear center produces the torques included in m^1 and ${}^kM^1$. Compared to the longitudinal length S, X^2 and X^3 on the cross section are very small. From assumption (i), three displacements $u^I = u^I(S, X^2, X^3)$ can be expressed by the Taylor series as

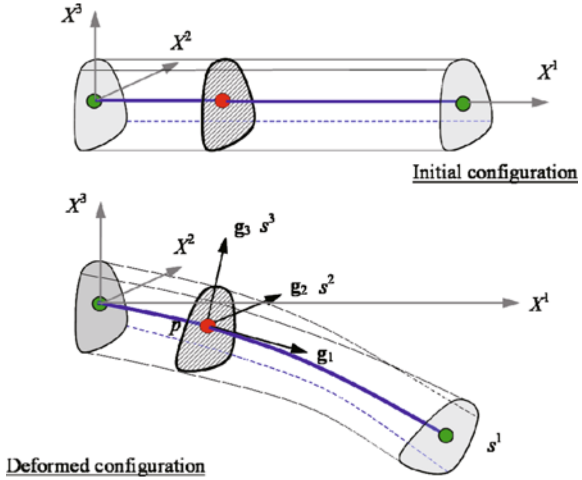


Fig. 8.4 A straight rod with initial and deformed configuration.

$$\begin{aligned} u^I &= u_0^I(S,t) + \sum_{n=1}^{\infty}(X^2)^n \varphi_n^{(I)}(S,t) + \sum_{n=1}^{\infty}(X^3)^n \theta_n^{(I)}(S,t) \\ &\quad + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(X^2)^m (X^3)^n \vartheta_{mn}^{(I)}(S,t), \end{aligned} \tag{8.142}$$

where

$$\begin{aligned} \varphi_n^{(I)} &= \frac{1}{n!}\frac{\partial^n u^I}{\partial (X^2)^n}\bigg|_{(X^2,X^3)=(0,0)}, \\ \theta_n^{(I)} &= \frac{1}{n!}\frac{\partial^n u^I}{\partial (X^3)^n}\bigg|_{(X^2,X^3)=(0,0)}, \\ \vartheta_{mn}^{(I)} &= \frac{1}{m!n!}\frac{\partial^{m+n} u^I}{\partial (X^2)^m \partial (X^3)^n}\bigg|_{(X^2,X^3)=(0,0)}. \end{aligned} \tag{8.143}$$

From Eqs.(8.30) and (8.33), the approximate six strains are

$$\begin{aligned} \varepsilon_1 &\approx \varepsilon_1^{(0)} + \frac{\partial \varepsilon_1}{\partial X^2}\bigg|_{(X^2,X^3)=(0,0)} X^2 + \frac{\partial \varepsilon_1}{\partial X^3}\bigg|_{(X^2,X^3)=(0,0)} X^3 \\ &\quad + \frac{1}{2!}\frac{\partial^2 \varepsilon_1}{\partial (X^2)^2}\bigg|_{(X^2,X^3)=(0,0)} (X^2)^2 + \frac{1}{2!}\frac{\partial^2 \varepsilon_1}{\partial (X^3)^2}\bigg|_{(X^2,X^3)=(0,0)} (X^3)^2 \\ &\quad + \frac{\partial^2 \varepsilon_1}{\partial X^2 \partial X^3}\bigg|_{(X^2,X^3)=(0,0)} X^2 X^3 + \cdots \\ &= \varepsilon_1^{(0)} + \frac{(\delta_1^I + u_{0,1}^I)\varphi_{1,1}^{(I)}}{1+\varepsilon_1^{(0)}} X^2 + \frac{(\delta_1^I + u_{0,1}^I)\theta_{1,1}^{(I)}}{1+\varepsilon_1^{(0)}} X^3 \end{aligned}$$

$$+\frac{1}{2}\left\{\frac{[2(\delta_1^I+u_{0,1}^I)\varphi_{2,1}^{(I)}]+\varphi_{1,1}^{(I)}\varphi_{1,1}^{(I)}}{1+\varepsilon_1^{(0)}}-\frac{[(\delta_1^I+u_{0,1}^I)\varphi_{1,1}^{(I)}]^2}{(1+\varepsilon_1^{(0)})^3}\right\}(X^2)^2$$
$$+\frac{1}{2}\left\{\frac{[2(\delta_1^I+u_{0,1}^I)\theta_{2,1}^{(I)}]+\theta_{1,1}^{(I)}\theta_{1,1}^{(I)}}{1+\varepsilon_1^{(0)}}-\frac{[(\delta_1^I+u_{0,1}^I)\theta_{1,1}^{(I)}]^2}{(1+\varepsilon_1^{(0)})^3}\right\}(X^3)^2$$
$$+\left[\frac{(\delta_1^I+u_{0,1}^I)\vartheta_{11,1}^{(I)}}{1+\varepsilon_1^{(0)}}+\frac{\varphi_{1,1}^{(I)}\theta_{1,1}^{(I)}}{1+\varepsilon_1^{(0)}}\right]X^2X^3+\cdots, \quad (8.144)$$

$$\varepsilon_2\approx\varepsilon_2^{(0)}+\left.\frac{\partial\varepsilon_2}{\partial X^2}\right|_{(X^2,X^3)=(0,0)}X^2+\left.\frac{\partial\varepsilon_2}{\partial X^3}\right|_{(X^2,X^3)=(0,0)}X^3$$
$$+\frac{1}{2!}\left.\frac{\partial^2\varepsilon_2}{\partial(X^2)^2}\right|_{(X^2,X^3)=(0,0)}(X^2)^2+\frac{1}{2!}\left.\frac{\partial^2\varepsilon_2}{\partial(X^3)^2}\right|_{(X^2,X^3)=(0,0)}(X^3)^2$$
$$+\left.\frac{\partial^2\varepsilon_2}{\partial X^2\partial X^3}\right|_{(X^2,X^3)=(0,0)}X^2X^3+\cdots$$
$$=\varepsilon_2^{(0)}+\frac{2(\delta_2^I+\varphi_1^{(I)})\varphi_2^{(I)}}{1+\varepsilon_2^{(0)}}X^2+\frac{(\delta_2^I+\varphi_1^{(I)})\vartheta_{11}^{(I)}}{1+\varepsilon_2^{(0)}}X^3$$
$$+\left\{\frac{[2\varphi_2^{(I)}\varphi_2^{(I)}+3(\delta_2^I+\varphi_1^{(I)})\varphi_3^{(I)}]}{1+\varepsilon_2^{(0)}}-\frac{2[(\delta_2^I+\varphi_1^{(I)})\varphi_2^{(I)}]^2}{(1+\varepsilon_2^{(0)})^3}\right\}(X^2)^2$$
$$+\frac{1}{2}\left\{\frac{\vartheta_{11}^{(I)}\vartheta_{11}^{(I)}}{1+\varepsilon_2^{(0)}}-\frac{[(\delta_2^I+\varphi_1^{(I)})\vartheta_{11}^{(I)}]^2}{2(1+\varepsilon_2^{(0)})^3}\right\}(X^3)^2$$
$$+\left[\frac{4(\delta_2^I+\varphi_1^{(I)})\vartheta_{21}^{(I)}}{1+\varepsilon_2^{(0)}}+\frac{2\varphi_2^{(I)}\vartheta_{11}^{(I)}}{1+\varepsilon_2^{(0)}}\right]X^2X^3+\cdots; \quad (8.145)$$

$$\varepsilon_3\approx\varepsilon_3^{(0)}+\left.\frac{\partial\varepsilon_3}{\partial X^2}\right|_{(X^2,X^3)=(0,0)}X^2+\left.\frac{\partial\varepsilon_3}{\partial X^3}\right|_{(X^2,X^3)=(0,0)}X^3$$
$$+\frac{1}{2!}\left.\frac{\partial^2\varepsilon_3}{\partial(X^2)^2}\right|_{(X^2,X^3)=(0,0)}(X^2)^2+\frac{1}{2!}\left.\frac{\partial^2\varepsilon_3}{\partial(X^3)^2}\right|_{(X^2,X^3)=(0,0)}(X^3)^2$$
$$+\left.\frac{\partial^2\varepsilon_3}{\partial X^2\partial X^3}\right|_{(X^2,X^3)=(0,0)}X^2X^3+\cdots$$
$$=\varepsilon_3^{(0)}+\frac{(\delta_3^I+\theta_1^{(I)})\vartheta_{11}^{(I)}}{1+\varepsilon_3^{(0)}}X^2+\frac{2(\delta_3^I+\theta_1^{(I)})\theta_2^{(I)}}{1+\varepsilon_3^{(0)}}X^3$$
$$+\frac{1}{2}\left\{\frac{\vartheta_{11}^{(I)}\vartheta_{11}^{(I)}}{1+\varepsilon_3^{(0)}}-\frac{[(\delta_2^I+\theta_1^{(I)})\vartheta_{11}^{(I)}]^2}{2(1+\varepsilon_3^{(0)})^3}\right\}(X^2)^2$$

$$+\left\{\frac{[2\theta_2^{(I)}\theta_2^{(I)}+3(\delta_3^I+\theta_1^{(I)})\theta_3^{(I)}]}{1+\varepsilon_3^{(0)}}-\frac{2[(\delta_3^I+\theta_1^{(I)})\theta_2^{(I)}]^2}{(1+\varepsilon_3^{(0)})^3}\right\}(X^3)^2$$
$$+\left[\frac{4(\delta_3^I+\theta_1^{(I)})\vartheta_{12}^{(I)}}{1+\varepsilon_3^{(0)}}+\frac{2\theta_2^{(I)}\vartheta_{11}^{(I)}}{1+\varepsilon_3^{(0)}}\right]X^2X^3+\cdots; \tag{8.146}$$

$$\gamma_{12}\approx\gamma_{12}^{(0)}+\left.\frac{\partial\gamma_{12}}{\partial X^2}\right|_{(X^2,X^3)=(0,0)}X^2+\left.\frac{\partial\gamma_{12}}{\partial X^3}\right|_{(X^2,X^3)=(0,0)}X^3$$
$$+\frac{1}{2!}\left.\frac{\partial^2\gamma_{12}}{\partial(X^2)^2}\right|_{(X^2,X^3)=(0,0)}(X^2)^2+\frac{1}{2!}\left.\frac{\partial^2\gamma_{12}}{\partial(X^3)^2}\right|_{(X^2,X^3)=(0,0)}(X^3)^2$$
$$+\left.\frac{\partial^2\gamma_{12}}{\partial X^2\partial X^3}\right|_{(X^2,X^3)=(0,0)}X^2X^3+\cdots$$
$$=\gamma_{12}^{(0)}+\frac{1}{\cos\gamma_{12}^{(0)}}\left\{\frac{2(\delta_1^I+u_{0,1}^{(I)})\varphi_2^{(I)}+(\delta_2^I+\varphi_1^{(I)})\varphi_{1,1}^{(I)}}{(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})}\right.$$
$$\left.-\sin\gamma_{12}^{(0)}\left[\frac{(\delta_1^I+u_{0,1}^{(I)})\varphi_{1,1}^{(I)}}{(1+\varepsilon_1^{(0)})^2}+\frac{2(\delta_2^I+\varphi_1^{(I)})\varphi_2^{(I)}}{(1+\varepsilon_2^{(0)})^2}\right]\right\}X^2$$
$$+\frac{1}{\cos\gamma_{12}^{(0)}}\left\{\frac{(\delta_2^I+\varphi_1^{(I)})\theta_{1,1}^{(I)}+(\delta_1^I+u_{0,1}^{(I)})\vartheta_{11}^{(\alpha)}}{(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})}\right.$$
$$\left.-\sin\gamma_{12}^{(0)}\left[\frac{(\delta_1^I+u_{0,1}^{(I)})\theta_{1,1}^{(I)}}{(1+\varepsilon_1^{(0)})^2}+\frac{(\delta_2^I+\varphi_1^{(I)})\vartheta_{11}^{(I)}}{(1+\varepsilon_2^{(0)})^2}\right]\right\}X^3+\cdots; \tag{8.147}$$

$$\gamma_{13}\approx\gamma_{13}^{(0)}+\left.\frac{\partial\gamma_{13}}{\partial X^2}\right|_{(X^2,X^3)=(0,0)}X^2+\left.\frac{\partial\gamma_{12}}{\partial X^3}\right|_{(X^2,X^3)=(0,0)}X^3$$
$$+\frac{1}{2!}\left.\frac{\partial^2\gamma_{13}}{\partial(X^2)^2}\right|_{(X^2,X^3)=(0,0)}(X^2)^2+\frac{1}{2!}\left.\frac{\partial^2\gamma_{13}}{\partial(X^3)^2}\right|_{(X^2,X^3)=(0,0)}(X^3)^2$$
$$+\left.\frac{\partial^2\gamma_{13}}{\partial X^2\partial X^3}\right|_{(X^2,X^3)=(0,0)}X^2X^3+\cdots$$
$$=\gamma_{13}^{(0)}+\frac{1}{\cos\gamma_{13}^{(0)}}\left\{\frac{(\delta_3^I+\theta_1^{(I)})\varphi_{1,1}^{(I)}+(\delta_1^I+u_{0,1}^{(I)})\vartheta_{11}^{(I)}}{(1+\varepsilon_1^{(0)})(1+\varepsilon_3^{(0)})}\right.$$
$$\left.-\sin\gamma_{13}^{(0)}\left[\frac{(\delta_1^I+u_{0,1}^{(I)})\varphi_{1,1}^{(I)}}{(1+\varepsilon_1^{(0)})^2}+\frac{(\delta_3^I+\theta_1^{(I)})\vartheta_{11}^{(I)}}{(1+\varepsilon_3^{(0)})^2}\right]\right\}X^2$$
$$+\frac{1}{\cos\gamma_{13}^{(0)}}\left\{\frac{2(\delta_1^I+u_{0,1}^{(I)})\theta_2^{(I)}+(\delta_3^I+\theta_1^{(I)})\vartheta_{1,1}^{(I)}}{(1+\varepsilon_1^{(0)})(1+\varepsilon_3^{(0)})}\right.$$

$$-\sin\gamma_{13}^{(0)}\left[\frac{(\delta_1^I+u_{0,1}^{(I)})\vartheta_{1,1}^{(I)}}{(1+\varepsilon_1^{(0)})^2}+\frac{2(\delta_3^I+\theta_1^{(I)})\theta_2^{(I)}}{(1+\varepsilon_3^{(0)})^2}\right]\Bigg\}X^3+\cdots; \tag{8.148}$$

$$\begin{aligned}\gamma_{23}&\approx\gamma_{23}^{(0)}+\frac{\partial\gamma_{23}}{\partial X^2}\bigg|_{(X^2,X^3)=(0,0)}X^2+\frac{\partial\gamma_{23}}{\partial X^3}\bigg|_{(X^2,X^3)=(0,0)}X^3\\&+\frac{1}{2!}\frac{\partial^2\gamma_{23}}{\partial(X^2)^2}\bigg|_{(X^2,X^3)=(0,0)}(X^2)^2+\frac{1}{2!}\frac{\partial^2\gamma_{23}}{\partial(X^3)^2}\bigg|_{(X^2,X^3)=(0,0)}(X^3)^2\\&+\frac{\partial^2\gamma_{23}}{\partial X^2\partial X^3}\bigg|_{(X^2,X^3)=(0,0)}X^2X^3+\cdots\\&=\gamma_{23}^{(0)}+\frac{1}{\cos\gamma_{23}^{(0)}}\Bigg\{\frac{2(\delta_3^I+\theta_1^{(I)})\varphi_2^{(I)}+(\delta_2^I+\varphi_1^{(I)})\vartheta_{11}^{(I)}}{(1+\varepsilon_2^{(0)})(1+\varepsilon_3^{(0)})}\\&-\sin\gamma_{23}^{(0)}\left[\frac{2(\delta_2^I+\varphi_1^{(I)})\varphi_2^{(I)}}{(1+\varepsilon_2^{(0)})^2}+\frac{(\delta_3^I+\theta_1^{(I)})\vartheta_{11}^{(I)}}{(1+\varepsilon_3^{(0)})^2}\right]\Bigg\}X^2\\&+\frac{1}{\cos\gamma_{23}^{(0)}}\Bigg\{\frac{2(\delta_2^I+\varphi_1^{(I)})\theta_2^{(I)}+(\delta_3^I+\theta_1^{(I)})\vartheta_{11}^{(I)}}{(1+\varepsilon_2^{(0)})(1+\varepsilon_3^{(0)})}\\&-\sin\gamma_{23}^{(0)}\left[\frac{(\delta_2^I+\varphi_1^{(I)})\vartheta_{11}^{(I)}}{(1+\varepsilon_2^{(0)})^2}+\frac{2(\delta_3^I+\theta_1^{(I)})\theta_2^{(I)}}{(1+\varepsilon_3^{(0)})^2}\right]\Bigg\}X^3+\cdots,\end{aligned} \tag{8.149}$$

where

$$\begin{aligned}\varepsilon_1^{(0)}&=\sqrt{(\delta_1^I+u_{0,1}^I)(\delta_1^I+u_{0,1}^I)}-1,\\\varepsilon_2^{(0)}&=\sqrt{(\delta_2^I+\varphi_1^{(I)})(\delta_2^I+\varphi_1^{(I)})}-1,\\\varepsilon_3^{(0)}&=\sqrt{(\delta_3^I+\theta_1^{(I)})(\delta_3^I+\theta_1^{(I)})}-1,\\\gamma_{12}^{(0)}&=\sin^{-1}\frac{(\delta_1^I+u_{0,1}^I)(\delta_2^I+\varphi_1^{(I)})}{(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})},\\\gamma_{13}^{(0)}&=\sin^{-1}\frac{(\delta_1^I+u_{0,1}^I)(\delta_2^I+\theta_1^{(I)})}{(1+\varepsilon_1^{(0)})(1+\varepsilon_3^{(0)})},\\\gamma_{23}^{(0)}&=\sin^{-1}\frac{(\delta_2^I+\varphi_1^{(I)})(\delta_3^I+\theta_1^{(I)})}{(1+\varepsilon_2^{(0)})(1+\varepsilon_3^{(0)})}.\end{aligned} \tag{8.150}$$

From Assumptions (i) and (ii), consider the zero order of the Taylor series of the six strains to give

$$\varepsilon_2^{(0)}=0,\,\varepsilon_3^{(0)}=0;\;\gamma_{12}^{(0)}=0,\,\gamma_{13}^{(0)}=0,\,\gamma_{23}^{(0)}=0. \tag{8.151}$$

The deformed rod for $X^2 = X^3 = 0$ satisfies the following relation:

$$(1+\varepsilon_1^{(0)})^2 = (\delta_1^I + u_{0,1}^I)(\delta_1^I + u_{0,1}^I). \tag{8.152}$$

Note that one assumed $\varepsilon_1^{(0)} = 0$, which is not adequate (e.g., Novozhilov, 1953). Equation (8.152) implies that only 1-dimenisonal membrane force in the rod is considered. From Eqs.(8.151) and (8.152),

$$\begin{aligned}
\frac{1}{(1+\varepsilon_1^{(0)})^2}(\delta_1^I + u_{0,1}^I)(\delta_1^I + u_{0,1}^I) = 1,\\
(\delta_2^I + \varphi_1^{(I)})(\delta_2^I + \varphi_1^{(I)}) = 1,\\
(\delta_3^I + \theta_1^{(I)})(\delta_3^I + \theta_1^{(I)}) = 1;\\
(\delta_1^I + u_{0,1}^I)(\delta_2^I + \varphi_1^{(I)}) = 0,\\
(\delta_1^I + u_{0,1}^I)(\delta_3^I + \theta_1^{(I)}) = 0,\\
(\delta_2^I + \varphi_1^{(I)})(\delta_3^I + \theta_1^{(I)}) = 0.
\end{aligned} \tag{8.153}$$

Using the zero order terms of X^2 and X^3 in Eq.(8.34), the direction cosine matrix $((l_{ij})_{3\times 3})$ is given by

$$\begin{aligned}
\cos\theta_{(\mathbf{n}_1,\mathbf{I}_I)} = l_{1I} = \frac{\delta_1^I + u_{0,1}^I}{1+\varepsilon_1^{(0)}},\\
\cos\theta_{(\mathbf{n}_2,\mathbf{I}_I)} = l_{2I} = \frac{\delta_2^I + \varphi_1^{(I)}}{1+\varepsilon_2^{(0)}},\\
\cos\theta_{(\mathbf{n}_3,\mathbf{I}_I)} = l_{3I} = \frac{\delta_3^I + \theta_1^{(I)}}{1+\varepsilon_3^{(0)}}.
\end{aligned} \tag{8.154}$$

From the geometrical relations, the nine directional cosines must satisfy the trigonometric relations without summation on $\alpha = 1, 2, 3$ as

$$l_{\alpha I} l_{\alpha I} = 1 \text{ for } I\text{=1, 2, 3} \tag{8.155}$$

and for $\alpha, \beta = 1, 2, 3$ and $\alpha \neq \beta$,

$$l_{\alpha I} l_{\beta I} = 0. \tag{8.156}$$

As aforesaid, only the three rotations of rod are considered. Thus, the unknowns $\varphi_1^{(I)}$ and $\theta_1^{(I)}$ ($I = 1, 2, 3$) can be determined by the three Euler angles (Φ, Ψ and Θ). The Euler angles Φ and Ψ rotates around the axes of X^2 and X^3, respectively, and the Euler angle Θ rotates around the axis of X^1, as sketched in Fig.8.5. Due to bending, the first rotation around the axis of X^2 is to form $(\bar{X}^1, \bar{X}^2, \bar{X}^3)$ in Fig.8.5(a). The second rotation around the axis of $\bar{X}^3$ gives

$(\bar{\bar{X}}^1, \bar{\bar{X}}^2, \bar{\bar{X}}^3)$, as shown in Fig.8.5(b). The last rotation around the axis of $\bar{\bar{X}}^1$ is because of the torsion, and the final state of the rod in the frame of $(\bar{\bar{\bar{X}}}^1, \bar{\bar{\bar{X}}}^2, \bar{\bar{\bar{X}}}^3)$ in Fig.8.5(c) gives the coordinates (s^1, s^2, s^3) for the deformed rod. The rotation deformation is the same as the rotation given by Eq.(8.153). The rotation matrices are

$$\mathbf{R}^1 = \begin{bmatrix} \cos\Phi & 0 & -\sin\Phi \\ 0 & 1 & 0 \\ \sin\Phi & 0 & \cos\Phi \end{bmatrix}, \tag{8.157}$$

$$\mathbf{R}^2 = \begin{bmatrix} \cos\Psi & \sin\Psi & 0 \\ -\sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{8.158}$$

$$\mathbf{R}^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Theta & \sin\Theta \\ 0 & -\sin\Theta & \cos\Theta \end{bmatrix}. \tag{8.159}$$

From the above rotations, the directional cosine matrix ($\boldsymbol{l} = (l_{ij})_{3\times3}$) is

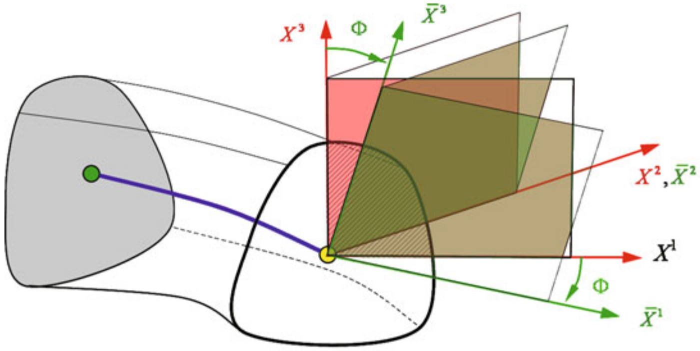
$$\boldsymbol{l} = \mathbf{R}^3\mathbf{R}^2\mathbf{R}^1 = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ l_{12} & l_{22} & l_{32} \\ l_{13} & l_{23} & l_{33} \end{bmatrix}, \tag{8.160}$$

where

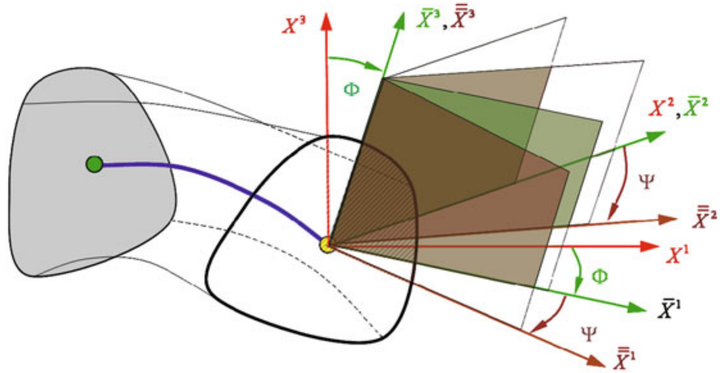
$$\begin{aligned} l_{11} &= \cos\Phi\cos\Psi, \\ l_{21} &= -\cos\Phi\sin\Psi\cos\Theta + \sin\Phi\sin\Theta, \\ l_{31} &= \cos\Phi\sin\Psi\sin\Theta + \sin\Phi\cos\Theta, \\ l_{12} &= \sin\Psi, \\ l_{22} &= \cos\Psi\cos\Theta, \\ l_{32} &= -\cos\Psi\sin\Theta, \\ l_{13} &= -\sin\Phi\cos\Psi, \\ l_{23} &= \sin\Phi\sin\Psi\cos\Theta + \cos\Phi\sin\Theta, \\ l_{33} &= -\sin\Phi\sin\Psi\sin\Theta + \cos\Phi\cos\Theta. \end{aligned} \tag{8.161}$$

Compared with Eq.(8.153), equations (8.156) and (8.161) give

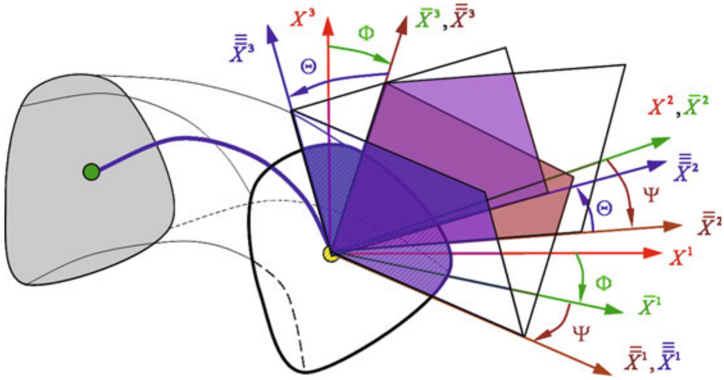
$$\begin{aligned} u^1_{0,1} &= (1 + \varepsilon_1^{(0)})\cos\Phi\cos\Psi - 1, \\ \varphi_1^{(1)} &= -\cos\Phi\sin\Psi\cos\Theta + \sin\Phi\sin\Theta, \end{aligned}$$



(a)



(b)



(c)

Fig. 8.5 Euler angles of rod rotation caused by bending and torsion: (a) the initial (red) to first rotation (green), (b) the first to second rotation (brown), (c) from the second to the last rotation (blue). (color plot in the book end)

$$
\begin{aligned}
\theta_1^{(1)} &= \cos\Phi \sin\Psi \sin\Theta + \sin\Phi \cos\Theta, \\
u_{0,1}^2 &= (1+\varepsilon_1^{(0)})\sin\Psi, \\
\varphi_1^{(2)} &= \cos\Psi \cos\Theta - 1, \\
\theta_1^{(1)} &= -\cos\Psi \sin\Theta, \\
u_{0,1}^3 &= -(1+\varepsilon_1^{(0)})\sin\Phi \cos\Psi, \\
\varphi_1^{(3)} &= \sin\Phi \sin\Psi \cos\Theta + \cos\Phi \sin\Theta, \\
\theta_1^{(1)} &= -\sin\Phi \sin\Psi \sin\Theta + \cos\Phi \cos\Theta - 1.
\end{aligned} \tag{8.162}
$$

The first, fourth and seventh equations of the foregoing equation give

$$
\begin{aligned}
\cos\Psi &= \pm\frac{\sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2}}{1+\varepsilon_1^{(0)}}, \\
\sin\Psi &= \frac{u_{0,1}^2}{1+\varepsilon_1^{(0)}}; \\
\cos\Phi &= \pm\frac{1+u_{0,1}^1}{\sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2}}, \\
\sin\Phi &= \mp\frac{u_{0,1}^3}{\sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2}},
\end{aligned} \tag{8.163}
$$

and

$$
\begin{aligned}
\varphi_1^{(1)} &= \mp\frac{1}{\sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2}}\left[\frac{u_{0,1}^2(1+u_{0,1}^1)}{1+\varepsilon_1^{(0)}}\cos\Theta + u_{0,1}^3 \sin\Theta\right], \\
\theta_1^{(1)} &= \pm\frac{1}{\sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2}}\left[\frac{u_{0,1}^2(1+u_{0,1}^1)}{1+\varepsilon_1^{(0)}}\sin\Theta - u_{0,1}^3 \cos\Theta\right], \\
\varphi_1^{(2)} &= \pm\frac{\sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2}}{1+\varepsilon_1^{(0)}}\cos\Theta - 1, \\
\theta_1^{(2)} &= \mp\frac{\sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2}}{1+\varepsilon_1^{(0)}}\sin\Theta, \\
\varphi_1^{(3)} &= \mp\frac{1}{\sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2}}\left[\frac{u_{0,1}^3 u_{0,1}^2}{1+\varepsilon_1^{(0)}}\cos\Theta - (1+u_{0,1}^1)\sin\Theta\right], \\
\theta_1^{(3)} &= \pm\frac{1}{\sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2}}\left[\frac{u_{0,1}^3 u_{0,1}^2}{1+\varepsilon_1^{(0)}}\sin\Theta + (1+u_{0,1}^1)\cos\Theta\right] - 1.
\end{aligned} \tag{8.164}
$$

If $\Theta = 0$, the rotation about the longitudinal axis disappears. So only the bending rotation exists. This case reduces to the pure bending of the rod as discussed in Section 8.2.

From Eq.(8.154), the directional cosine vectors are defined as

$$\boldsymbol{l}_\alpha = l_{\alpha I}\mathbf{I}_I \quad \text{for } \alpha = 1, 2, 3. \tag{8.165}$$

Thus, the change ratio of the directional cosines along the deformed rod is

$$\frac{d\boldsymbol{l}_\alpha}{ds} = \frac{dl_{\alpha I}}{ds}\mathbf{I}_I = \frac{dl_{\alpha I}}{(1+\varepsilon_1^{(0)})dS}\mathbf{I}_I \quad \text{for } \alpha = 1, 2, 3 \tag{8.166}$$

The three vectors form a instantaneous, rotational coordinate frame, and the rotation ratio vector about three axes are defined as

$$\boldsymbol{\omega} = \omega_I \mathbf{I}_I. \tag{8.167}$$

From rigid-body dynamics (e.g., Goldstein et al., 2002), the change ratio of the directional cosines along the deformed rod can be computed in an analogy way, i.e.,

$$\begin{aligned} \frac{d\boldsymbol{l}_\alpha}{ds} &= \frac{dl_{\alpha I}}{ds}\mathbf{I}_I = \frac{dl_{\alpha I}}{(1+\varepsilon_1^{(0)})dS}\mathbf{I}_I = \boldsymbol{\omega}\times\boldsymbol{l}_\alpha \\ &= \begin{vmatrix} \mathbf{I}_1 & \mathbf{I}_2 & \mathbf{I}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ l_{\alpha 1} & l_{\alpha 2} & l_{\alpha 3} \end{vmatrix} = e_{IJK}\,\omega_J l_{\alpha K}\mathbf{I}_I, \end{aligned} \tag{8.168}$$

for $\alpha = 1, 2, 3$ and $I, J, K \in \{1, 2, 3\}$ with $I \neq J \neq K \neq I$. From the foregoing equation, the rotation ratio components are given by

$$e_{IJK}\frac{dl_{\alpha I}}{ds}l_{\alpha K} = e_{IJK}\frac{dl_{\alpha I}}{(1+\varepsilon_1^{(0)})dS}l_{\alpha K} = \omega_J. \tag{8.169}$$

In other words, the foregoing equation is expressed by

$$\begin{aligned} \omega_1 &= e_{I1K}\frac{dl_{\alpha I}}{ds}l_{\alpha K} = -\frac{dl_{\alpha 2}}{ds}l_{\alpha 3} = -\frac{dl_{\alpha 2}}{(1+\varepsilon_1^{(0)})dS}l_{\alpha 3}, \\ \omega_2 &= e_{I2K}\frac{dl_{\alpha I}}{ds}l_{\alpha K} = -\frac{dl_{\alpha 3}}{ds}l_{\alpha 1} = -\frac{dl_{\alpha 3}}{(1+\varepsilon_1^{(0)})dS}l_{\alpha 1}, \\ \omega_3 &= e_{I3K}\frac{dl_{\alpha I}}{ds}l_{\alpha K} = -\frac{dl_{\alpha 1}}{ds}l_{\alpha 2} = -\frac{dl_{\alpha 1}}{(1+\varepsilon_1^{(0)})dS}l_{\alpha 2}. \end{aligned} \tag{8.170}$$

From Eq.(8.161), the foregoing equations gives

$$\omega_1 = \frac{1}{1+\varepsilon_1^{(0)}}\left(\sin\Phi\frac{d\Psi}{dS} + \cos\Psi\cos\Phi\frac{d\Theta}{dS}\right),$$

$$\omega_2 = \frac{1}{1+\varepsilon_1^{(0)}}(\frac{d\Psi}{dS} + \sin\Psi \frac{d\Theta}{dS}),$$
$$\omega_3 = \frac{1}{1+\varepsilon_1^{(0)}}(\cos\Phi \frac{d\Psi}{dS} - \sin\Phi\cos\Psi \frac{d\Theta}{dS}); \tag{8.171}$$

or

$$\omega_1 = \sin\Phi \frac{d\Psi}{ds} + \cos\Psi\cos\Phi \frac{d\Theta}{ds},$$
$$\omega_2 = \frac{d\Psi}{ds} + \sin\Psi \frac{d\Theta}{ds}, \tag{8.172}$$
$$\omega_3 = \cos\Phi \frac{d\Psi}{ds} - \sin\Phi\cos\Psi \frac{d\Theta}{ds}.$$

Notice that one often assumes $dS = ds$ in Love (1944), which is not adequate for large deformation.

On the other hand, using Eq.(8.25), the particle location on the deformed rod is expressed by

$$\mathbf{r} = (X^I + u^I)\mathbf{I}_I. \tag{8.173}$$

Because $X^1 = S$, X^2 and X^3 are independent of S. From Eq.(8.3), the base vector along the longitudinal direction of the deformed rod is given by

$$\tilde{\mathbf{g}}_1 = \tilde{g}_1^I \mathbf{I}_I = (\delta_1^I + u_{,1}^I)\mathbf{I}_I, \tag{8.174}$$

and the corresponding unit vector is

$$\tilde{\mathbf{n}}_1 = \frac{\tilde{\mathbf{g}}_1}{|\tilde{\mathbf{g}}_1|} = \frac{\delta_1^I + u_{,1}^I}{\sqrt{(\delta_1^K + u_{,1}^K)(\delta_1^K + u_{,1}^K)}}\mathbf{I}_I = \frac{\delta_1^I + u_{,1}^I}{1+\varepsilon_1}\mathbf{I}_I. \tag{8.175}$$

For $X^2 = X^3 = 0$,

$$\tilde{\mathbf{n}}_1 = \frac{\delta_1^I + u_{0,1}^I}{\sqrt{(\delta_1^K + u_{0,1}^K)(\delta_1^K + u_{0,1}^K)}}\mathbf{I}_I = \frac{\delta_1^I + u_{0,1}^I}{1+\varepsilon_1^{(0)}}\mathbf{I}_I. \tag{8.176}$$

Note that $\tilde{\mathbf{n}}_1 = \mathbf{n}_1$. The base vector in the principal normal direction of the deformed rod is

$$\tilde{\mathbf{g}}_2 = \frac{d\tilde{\mathbf{n}}_1}{ds} = \tilde{g}_2^I \mathbf{I}_1, \tag{8.177}$$

where

$$\tilde{g}_2^I \equiv \frac{1}{\tilde{g}_{11}^2}\left[u_{,11}^I(\delta_1^K + u_{,1}^K)(\delta_1^K + u_{,1}^K) - (\delta_1^I + u_{,1}^I)(\delta_1^K + u_{,1}^K)u_{,11}^K\right]. \tag{8.178}$$

The unit principal normal vector is

$$\tilde{\mathbf{n}}_2 = \frac{\tilde{\mathbf{g}}_2}{|\tilde{\mathbf{g}}_2|} = \frac{\tilde{\mathbf{g}}_2}{\sqrt{\tilde{g}_{22}}} = \frac{\tilde{\mathbf{g}}_2}{\tilde{\kappa}(S)} = \frac{\tilde{g}_2^I}{\tilde{\kappa}(S)}\mathbf{I}_I, \tag{8.179}$$

and from Eq.(8.9), the curvature of the deformed rod becomes

$$\begin{aligned}\tilde{\kappa}(S) &= |\tilde{\mathbf{g}}_2| = \sqrt{\tilde{g}_{22}} = \sqrt{\left(X_{,ss}^I + u_{,ss}^I\right)\left(X_{,ss}^I + u_{,ss}^I\right)} \\ &= \frac{\sqrt{(u_{,11}^I u_{,11}^I)(\delta_1^K + u_{,1}^K)(\delta_1^K + u_{,1}^K) - [(\delta_1^I + X_{,1}^I)u_{,11}^I]^2}}{\tilde{g}_{11}^{3/2}}.\end{aligned} \tag{8.180}$$

For $X^2 = X^3 = 0$,

$$\begin{aligned}\tilde{g}_2^I = \frac{1}{\tilde{g}_{11}^2}&\left[u_{0,11}^I(\delta_1^K + u_{0,1}^K)(\delta_1^K + u_{0,1}^K)\right. \\ &\left. - (\delta_1^I + u_{0,1}^I)(\delta_1^K + u_{0,1}^K)u_{0,11}^K\right].\end{aligned} \tag{8.181}$$

$$\tilde{\kappa}(S) = \frac{\sqrt{(u_{0,11}^I u_{0,11}^I)(\delta_1^K + u_{0,1}^K)(\delta_1^K + u_{0,1}^K) - [(\delta_1^I + u_{0,1}^I)u_{0,11}^I]^2}}{\tilde{g}_{11}^{3/2}}. \tag{8.182}$$

From Eq.(8.11), the unit bi-normal vector is obtained by

$$\tilde{\mathbf{n}}_3 = \tilde{\mathbf{n}}_1 \times \tilde{\mathbf{n}}_2 = \tilde{g}_3^I \mathbf{I}_I, \tag{8.183}$$

with

$$\tilde{g}_3^I = e_{IJK}\frac{\tilde{g}_1^J}{\sqrt{g_{11}}}\frac{\tilde{g}_2^K}{\tilde{\kappa}(S)} \text{ and } \tilde{g}_{33} = 1. \tag{8.184}$$

Because of the axial rotation, the rotation vector along the longitudinal arc length $s^1 = s$ of the deformed rod is

$$\boldsymbol{\omega} = \tilde{\kappa}(S)\mathbf{n}_3 + \tilde{\tau}(S)\mathbf{n}_1 = \omega_I \mathbf{I}_I, \tag{8.185}$$

where the torsion of the deformed rod is computed from Eq.(8.13), i.e.,

$$\tilde{\tau}(S) = \frac{e_{IJK}(\delta_1^I + u_{,1}^I)u_{,11}^J u_{,111}^K}{(u_{,11}^I u_{,11}^I)[(\delta_1^K + u_{,1}^K)(\delta_1^K + u_{,1}^K)] - [(\delta_1^I + u_{,1}^I)u_{,11}^I]^2}. \tag{8.186}$$

For $X^2 = X^3 = 0$, the foregoing equation is rewritten as

$$\tilde{\tau}(S) = \frac{e_{IJK}(\delta_1^I + u_{0,1}^I)u_{0,11}^J u_{0,111}^K}{(u_{0,11}^I u_{0,11}^I)[(\delta_1^K + u_{0,1}^K)(\delta_1^K + u_{0,1}^K)] - [(\delta_1^I + u_{0,1}^I)u_{0,11}^I]^2}. \tag{8.187}$$

From Eq.(8.185),

$$\begin{aligned} \tilde{\kappa}(S) &= \omega_I \mathbf{I}_I \cdot \tilde{\mathbf{n}}_3 = \omega_I \tilde{g}_3^I, \\ \tilde{\tau}(S) &= \omega_I \mathbf{I}_I \cdot \tilde{\mathbf{n}}_1 = \omega_I \tilde{g}_2^I, \end{aligned} \tag{8.188}$$

or

$$\omega_I = \tilde{\kappa}(S)\mathbf{n}_3 \cdot \mathbf{I}_I + \tilde{\tau}(S)\mathbf{n}_1 \cdot \mathbf{I}_I = \tilde{\kappa}(S)\tilde{g}_3^I + \tilde{\tau}(S)\tilde{g}_1^I. \tag{8.189}$$

However, from Eq.(8.163), $d\Psi / dS$ and $d\Phi / dS$ are determined by

$$\begin{aligned} \frac{d\Psi}{dS} &= \pm \frac{u_{0,11}^2[(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2] - u_{0,1}^2[(1+u_{0,1}^1)u_{0,11}^1 + (u_{0,1}^3)u_{0,11}^3]}{(1+\varepsilon_1^{(0)})^2\sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2}}, \\ \frac{d\Phi}{dS} &= -\frac{u_{0,11}^3(1+u_{0,1}^1) - u_{0,11}^1 u_{0,1}^3}{(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2}; \end{aligned} \tag{8.190}$$

or

$$\begin{aligned} \frac{d\Psi}{ds} &= \pm \frac{u_{0,11}^2[(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2] - u_{0,1}^2[(1+u_{0,1}^1)u_{0,11}^1 + (u_{0,1}^3)u_{0,11}^3]}{(1+\varepsilon_1^{(0)})^3\sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2}}, \\ \frac{d\Phi}{ds} &= -\frac{u_{0,11}^3(1+u_{0,1}^1) - u_{0,11}^1 u_{0,1}^3}{(1+\varepsilon_1^{(0)})[(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2]}. \end{aligned} \tag{8.191}$$

Substitution of Eqs.(8.163) and (8.190) into Eqs.(8.171) and (8.188) gives Θ and $d\Theta / dS$ when the initial twisting about the longitudinal direction of s is zero ($\Theta_0 = 0$).

As before, the constitutive laws for deformed rods give the corresponding resultant stresses for ($\alpha = 1, 2, 3$) as

$$\sigma_{1\alpha} = f_\alpha(\varepsilon_1, \gamma_{12}, \gamma_{13}, t). \tag{8.192}$$

The internal forces and moments in the deformed rod are defined as

$$\begin{aligned} N_1 &= \int_A \sigma_{11}\left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23}\right]dA, \\ Q_2 &= \int_A \sigma_{12}\left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23}\right]dA, \\ Q_3 &= \int_A \sigma_{13}\left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23}\right]dA, \\ M_3 &= \int_A \sigma_{11}\frac{X^2}{1+\varphi_1^{(2)}}\left[(1+\varepsilon_3)(1+\varepsilon_2)^2\cos\gamma_{23}\right]dA, \end{aligned}$$

$$M_2 = -\int_A \sigma_{11} \frac{X^3}{1+\theta_1^{(3)}}\left[(1+\varepsilon_3)(1+\varepsilon_2)^2 \cos\gamma_{23}\right] dA,$$

$$T_1 = \int_A \left[\sigma_{12}\frac{X^3(1+\varepsilon_3)}{1+\theta_1^{(3)}} - \sigma_{13}\frac{X^2(1+\varepsilon_2)}{1+\varphi_1^{(2)}}\right]\left[(1+\varepsilon_3)(1+\varepsilon_2)\cos\gamma_{23}\right] dA. \tag{8.193}$$

For convenience, the notations ($Q_2 \equiv N_2$, $Q_3 \equiv N_3$ and $T_1 \equiv M_1$) are used.

$$\begin{aligned}\mathbf{M} &\equiv M^I \mathbf{I}_I = M_\alpha \mathbf{n}_\alpha,\\ \boldsymbol{\mathcal{N}} &\equiv N^I \mathbf{I}_I = N_\alpha \mathbf{n}_\alpha,\\ {}^{\mathrm{N}}\mathbf{M} &\equiv {}^{\mathrm{N}}M^I \mathbf{I}_I = \mathbf{g}_1 \times \boldsymbol{\mathcal{N}},\end{aligned} \tag{8.194}$$

where

$$\mathbf{g}_1 \equiv \frac{d\mathbf{r}}{dS} = (\delta_1^I + u_{0,1}^I)\mathbf{I}_I \text{ and } {}^{\mathrm{N}}\mathbf{M} \equiv \frac{1}{dS} d\mathbf{r} \times \boldsymbol{\mathcal{N}}. \tag{8.195}$$

With Eq.(8.154), the components of the internal forces in the $\mathbf{I}_I$-direction are

$$\begin{aligned} N^I &= N_\alpha \mathbf{n}_\alpha \cdot \mathbf{I}_I = N_\alpha \cos\theta_{(\mathbf{n}_\alpha, \mathbf{I}_I)}\\ &= \frac{N_1(\delta_1^I + u_{0,1}^I)}{1+\varepsilon_1^{(0)}} + \frac{N_2(\delta_2^I + \varphi_1^I)}{1+\varepsilon_2^{(0)}} + \frac{N_3(\delta_3^I + \theta_1^I)}{1+\varepsilon_3^{(0)}},\end{aligned} \tag{8.196}$$

$$\begin{aligned} M^I &= M_\alpha \mathbf{n}_\alpha \cdot \mathbf{I}_I = M_\alpha \cos\theta_{(\mathbf{n}_\alpha, \mathbf{I}_I)}\\ &= \frac{M_1(\delta_1^I + u_{0;1}^I)}{1+\varepsilon_1^{(0)}} + \frac{M_2(\delta_2^I + \varphi_1^I)}{1+\varepsilon_2^{(0)}} + \frac{M_3(\delta_3^I + \theta_1^I)}{1+\varepsilon_3^{(0)}},\\ {}^{\mathrm{N}}M^I &= (\mathbf{g}_1 \times \boldsymbol{\mathcal{N}}) \cdot \mathbf{I}_I = e_{IJK}(\delta_1^J + u_{0,1}^J) N^K.\end{aligned} \tag{8.197}$$

Using the external forces as in Eqs.(8.34)–(8.38), equations of motion on the deformed rod are given by

$$\begin{aligned}\boldsymbol{\mathcal{N}}_{,1} + \mathbf{q} &= \rho \mathbf{u}_{0,tt} + I_3 \boldsymbol{\varphi}_{1,tt} + I_2 \boldsymbol{\theta}_{1,tt},\\ \mathbf{M}_{,1} + {}^{\mathrm{N}}\mathbf{M} + \mathbf{m} &= \mathcal{J}\boldsymbol{\Theta}_{,tt},\end{aligned} \tag{8.198}$$

where

$$\begin{aligned}\mathcal{J}\boldsymbol{\Theta}_{,tt} &= [(I_3 u_{0,tt}^3 - I_2 u_{0,tt}^2) + (J_{22}\varphi_{1,tt}^{(3)} - J_{23}\varphi_{1,tt}^{(2)})\\ &\quad + (J_{23}\theta_{1,tt}^{(3)} - J_{33}\theta_{1,tt}^{(2)})]\mathbf{I}_1 + (I_3 u_{0,tt}^1 + J_{23}\varphi_{1,tt}^{(1)} + J_{22}\theta_{1,tt}^{(1)})\mathbf{I}_2\\ &\quad - (I_2 u_{0,tt}^1 + J_{22}\varphi_{1,tt}^{(1)} + J_{23}\theta_{1,tt}^{(1)})\mathbf{I}_3,\end{aligned} \tag{8.199}$$

$$\rho = \int_A \rho_0 dA,\ \ I_2 = \int_A \rho_0 X^3 dA,\ \ I_3 = \int_A \rho_0 X^2 dA,$$

$$J_{22} = \int_A \rho_0 (X^3)^2 dA,$$
$$J_{33} = \int_A \rho_0 (X^2)^2 dA, \tag{8.200}$$
$$J_{23} = \int_A \rho_0 (X^2)(X^3) dA,$$

and the scalar expressions are for $I = 1, 2, 3$,

$$\begin{aligned} N^I_{,1} + q^I &= \rho u^I_{0,tt} + I_3 \varphi^{(I)}_{1,tt} + I_2 \theta^{(I)}_{1,tt}, \\ M^I_{,1} + {}^{\mathrm{N}}M^I + m^I &= \mathscr{J} \boldsymbol{\Theta}_{,tt} \cdot \mathbf{I}_I. \end{aligned} \tag{8.201}$$

or

$$\begin{aligned} &\left[\frac{N_1(\delta^I_1 + u^I_{0,1})}{1+\varepsilon^{(0)}_1} + \frac{N_2(\delta^I_2 + \varphi^I_1)}{1+\varepsilon^{(0)}_2} + \frac{N_3(\delta^I_3 + \theta^I_1)}{1+\varepsilon^{(0)}_3} \right]_{,1} \\ &+ q^I = \rho u^I_{0,tt} + I_3 \varphi^{(I)}_{1,tt} + I_2 \theta^{(I)}_{1,tt}; \\ &\left[\frac{M_1(\delta^I_1 + u^I_{0,1})}{1+\varepsilon^{(0)}_1} + \frac{M_2(\delta^I_2 + \varphi^I_1)}{1+\varepsilon^{(0)}_2} + \frac{M_3(\delta^I_3 + \theta^I_1)}{1+\varepsilon^{(0)}_3} \right]_{,1} \\ &+ e_{IJK}(\delta^J_1 + u^J_{0,1}) N^K + m^I = \mathscr{J} \boldsymbol{\Theta}_{,tt} \cdot \mathbf{I}_I. \end{aligned} \tag{8.202}$$

The force condition at a point $\mathscr{P}_k$ with $X^1 = X^1_k$ is

$$\begin{aligned} {}^{-}\mathbf{N}(X^1_k) + {}^{+}\mathbf{N}(X^1_k) + \mathbf{F}_k &= 0, \\ {}^{-}N^I(X^1_k) = {}^{+}N^I(X^1_k) + F^I_k \quad &(I = 1, 2, 3). \end{aligned} \tag{8.203}$$

The force boundary condition at the boundary point $\mathscr{P}_r$ is

$$\begin{aligned} \mathbf{N}(X^1_r) + \mathbf{F}_r &= 0, \\ N^I(X^1_r) + F^I_r = 0 \quad &(I = 1, 2, 3). \end{aligned} \tag{8.204}$$

If there is a concentrated moment at a point $\mathscr{P}_k$ with $X^1 = X^1_k$, the corresponding moment boundary condition is

$$\begin{aligned} {}^{-}\mathbf{M}(X^1_k) + {}^{+}\mathbf{M}(X^1_k) + \mathbf{M}_k &= 0, \\ {}^{-}M^I(X^1_k) = {}^{+}M^I(X^1_k) + M^I_k \quad &(I = 1, 2, 3). \end{aligned} \tag{8.205}$$

The moment boundary condition at the boundary point $\mathscr{P}_r$ is

$$\begin{aligned} \mathbf{M}(X^1_r) + \mathbf{M}_r &= 0, \\ M^I(X^1_r) + M^I_r = 0 \quad &(I = 1, 2, 3). \end{aligned} \tag{8.206}$$

The displacement continuity and boundary conditions are

$$u_{k-}^I = u_{k+}^I \text{ and } u_r^I = B_r^I \quad (I = 1, 2, 3). \tag{8.207}$$

The rod theory can be reduced to the Cosserat theory of rods (e.g., E. and F. Cosserat,1909; Ericksen and Truesdell, 1958; Whitman and DeSilva,1969).

8.5. Nonlinear curved rods

Consider an arbitrary coordinates as (S^1, S^2, S^3) on the cross section of the rod. The deformed curve of the rod is shown in Fig.8.6. The coordinate S^1 is along the longitudinal direction of rod and the other two coordinates S^2 and S^3 are on the cross section of the rod. Without loss of generality, S^2 and S^3 are collinear to $\mathbf{N}_2$ and $\mathbf{N}_3$ for the curvature and torsion directions of the curve, respectively. The coordinates for the deformed rod are (s^1, s^2, s^3). Since the widths of rod in two directions of S^2 and S^3 are very small compared to the length of the rod, the elongations in the two directions of S^2 and S^3 should be very small, which are ignorable. Thus, the following assumptions will be adopted.

(i) The elongations in the two directions of S^2 and S^3 are very small (i.e., $\varepsilon_2 \approx 0$ and $\varepsilon_3 \approx 0$).

(ii) Under bending only, the neutral surface is not deformed (i.e., $\gamma_{12} \approx 0$ and $\gamma_{13} \approx 0$).

As in Eqs.(8.69) and (8.74), consider external distributed forces and moments on the initial configuration for ($I, \Lambda = 1, 2, 3$) as

$$\mathbf{q} = q^I \mathbf{I}_I = q^\Lambda \mathbf{N}_\Lambda \quad \text{and} \quad \mathbf{m} = m^I \mathbf{I}_I = m^\Lambda \mathbf{N}_\Lambda \tag{8.208}$$

and concentrated forces on the initial configuration at a point $S^1 = S_k$,

$$\mathbf{F}_k = F_k^I \mathbf{I}_I = F_k^\Lambda \mathbf{N}_\Lambda \text{ and } \mathbf{M}_k = M_k^I \mathbf{I}_I = M_k^\Lambda \mathbf{N}_\Lambda. \tag{8.209}$$

Thus, one obtains the relations, i.e.,

$$\begin{aligned} F^\Lambda |_{S^1=S} &= F^I \mathbf{I}_I \cdot \mathbf{N}_\Lambda = F^I \cos\theta_{(\mathbf{I}_I, \mathbf{N}_\Lambda)}, \\ M^\Lambda |_{S^1=S} &= M^I \mathbf{I}_I \cdot \mathbf{N}_\Lambda = M^I \cos\theta_{(\mathbf{I}_I, \mathbf{N}_\Lambda)}. \end{aligned} \tag{8.210}$$

The displacement vectors on the initial configuration are

$$\mathbf{R}(S^1) = X^I(S^1)\mathbf{I}_I, \mathbf{R}(S) = X^I(S)\mathbf{I}_I \text{ and } \mathbf{R}_k = X_k^I \mathbf{I}_I. \tag{8.211}$$

The internal forces and moments for $(S^1 > S_k^1)$ are

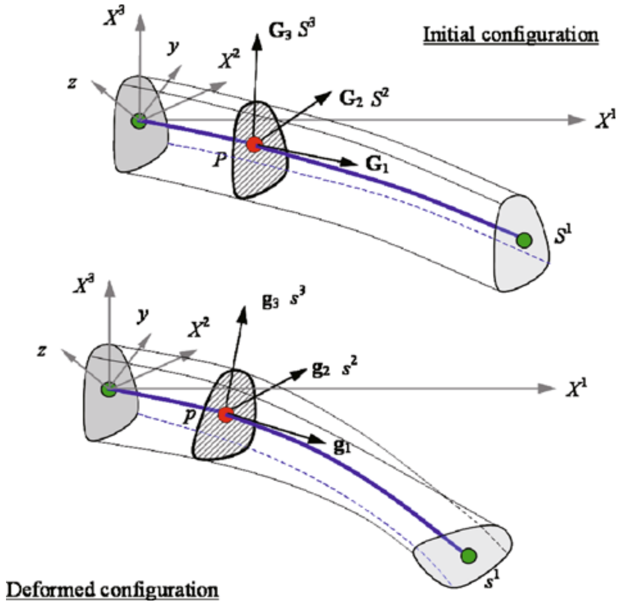


Fig. 8.6 A curved rod with initial and deformed configuration.

$$\begin{aligned} \mathbf{F}|_{S^1=S} &= \sum_{k=1} \mathbf{F}_k + \int_0^S \mathbf{q} dS^1, \\ \mathbf{M}|_{S^1=S} &= \sum_{k=1} \mathbf{M}_k + \int_0^S \mathbf{m} dS^1 \\ &\quad + \sum_{k=1} (\mathbf{R}(S) - \mathbf{R}_k) \times {}^k\mathbf{F} + \int_0^S (\mathbf{R}(S) - \mathbf{R}(S^1)) \times \mathbf{q} dS^1; \end{aligned} \tag{8.212}$$

or for $I, J, K = 1, 2, 3 \;\; (I \neq J \neq K \neq I)$,

$$\begin{aligned} F^I\Big|_{S^1=S} &= \sum_{k=1} {}^kF^I + \int_0^S q^I dS^1, \\ M^I\Big|_{S^1=S} &= \sum_{k=1} M_k^I + \int_0^S m^I dS^1 \\ &\quad + \sum_{k=1} e_{IJK} (X_S^J - X_k^J) F_k^K + \int_0^S e_{IJK} (X_S^J - X_k^J) q^K dS^1. \end{aligned} \tag{8.213}$$

8.5.1. A curved rod theory based on the Cartesian coordinates

As in Eqs.(8.79)–(8.84) for the strains of the 3-D deformed beam, the exact strain for the 3-D deformed rods can be obtained. Similar to Eq. (8.142), the displacement field for any fiber of the deformed rod at a position $\mathbf{R}$ is assumed by

$$u^I = u_0^I(S,t) + \sum_{n=1}^{\infty}(S^2)^n \varphi_n^{(I)}(S,t) + \sum_{n=1}^{\infty}(S^3)^n \theta_n^{(I)}(S,t)$$
$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(S^2)^m (S^3)^n \vartheta_{mn}^{(I)}(S,t), \tag{8.214}$$

where $S^1 = S$, and u_0^I is displacements of the centroid curve of the rod for $S^2 = S^3 = 0$, and $X^I = X^I(S^1)$. The coefficients of the higher order terms $\varphi_n^{(I)}$, $\theta_n^{(I)}$ and $\vartheta_{mn}^{(I)}(S)$ ($m, n = 1, 2, \cdots$) are from the Taylor series expansion, i.e.,

$$\varphi_n^{(I)} = \frac{1}{n!}\frac{\partial^n u^I}{\partial (S^2)^n}\bigg|_{(S^2,S^3)=(0,0)},$$
$$\theta_n^{(I)} = \frac{1}{n!}\frac{\partial^n u^I}{\partial (S^3)^n}\bigg|_{(S^2,S^3)=(0,0)}, \tag{8.215}$$
$$\vartheta_{mn}^{(I)} = \frac{1}{m!n!}\frac{\partial^{m+n} u^I}{\partial (S^2)^m \partial (S^3)^n}\bigg|_{(S^2,S^3)=(0=0)}.$$

Substitution of Eq.(8.214) into Eqs.(8.79)–(8.82) gives

$$\varepsilon_1 \approx \varepsilon_1^{(0)} + \frac{\partial \varepsilon_1}{\partial S^2}\bigg|_{(S^2,\, S^3)=(0,0)} S^2 + \frac{\partial \varepsilon_1}{\partial S^3}\bigg|_{(S^2,\, S^3)=(0,\,0)} S^3$$
$$+ \frac{1}{2!}\frac{\partial^2 \varepsilon_1}{\partial (S^2)^2}\bigg|_{(S^2,\, S^3)=(0,\,0)} (S^2)^2 + \frac{1}{2!}\frac{\partial^2 \varepsilon_1}{\partial (S^3)^2}\bigg|_{(S^2,\, S^3)=(0,\,0)} (S^3)^2$$
$$+ \frac{\partial^2 \varepsilon_1}{\partial S^2 \partial S^3}\bigg|_{(S^2,\, S^3)=(0,\,0)} S^2 S^3 + \cdots$$
$$= \varepsilon_1^{(0)} + \frac{(X_{,1}^I + u_{0,1}^I)\varphi_{1,1}^{(I)}}{G_{11}(1+\varepsilon_1^{(0)})} S^2 + \frac{(X_{,1}^I + u_{0,1}^I)\theta_{1,1}^{(I)}}{G_{11}(1+\varepsilon_1^{(0)})} S^3$$
$$+ \frac{1}{2}\left\{\frac{[2(X_{,1}^I + u_{0,1}^I)\varphi_{2,1}^{(I)}] + \varphi_{1,1}^{(I)}\varphi_{1,1}^{(I)}}{G_{11}(1+\varepsilon_1^{(0)})} - \frac{[(X_{,1}^I + u_{0,1}^I)\varphi_{1,1}^{(I)}]^2}{G_{11}^2[(1+\varepsilon_1^{(0)})]^3}\right\}(S^2)^2$$
$$+ \frac{1}{2}\left\{\frac{[2(X_{,1}^I + u_{0,1}^I)\theta_{2,1}^{(I)}] + \theta_{1,1}^{(I)}\theta_{1,1}^{(I)}}{G_{11}(1+\varepsilon_1^{(0)})} - \frac{[(X_{,1}^I + u_{0,1}^I)\theta_{1,1}^{(I)}]^2}{G_{11}^2[(1+\varepsilon_1^{(0)})]^3}\right\}(S^3)^2$$
$$+ \left[\frac{(X_{,1}^I + u_{0,1}^I)\vartheta_{11,1}^{(I)}}{G_{11}(1+\varepsilon_1^{(0)})} + \frac{\varphi_{1,1}^{(I)}\theta_{1,1}^{(I)}}{G_{11}(1+\varepsilon_1^{(0)})}\right] S^2 S^3 + \cdots, \tag{8.216}$$

$$\varepsilon_2 \approx \varepsilon_2^{(0)} + \frac{\partial \varepsilon_2}{\partial S^2}\bigg|_{(S^2,\, S^3)=(0,\,0)} S^2 + \frac{\partial \varepsilon_2}{\partial S^3}\bigg|_{(S^2,\, S^3)=(0,\,0)} S^3$$
$$+ \frac{1}{2!}\frac{\partial^2 \varepsilon_2}{\partial (S^2)^2}\bigg|_{(S^2,\, S^3)=(0,\,0)} (S^2)^2 + \frac{1}{2!}\frac{\partial^2 \varepsilon_2}{\partial (X^3)^2}\bigg|_{(S^2,\, S^3)=(0,\,0)} (S^3)^2$$

$$+\left.\frac{\partial^2\varepsilon_2}{\partial S^2\partial S^3}\right|_{(S^2,\,S^3)=(0,\,0)} S^2S^3+\cdots$$

$$=\varepsilon_2^{(0)}+\frac{2(X^I_{,2}+\varphi_1^{(I)})\varphi_2^{(I)}}{G_{22}(1+\varepsilon_2^{(0)})}S^2+\frac{(X^I_{,2}+\varphi_1^{(I)})\vartheta_{11}^{(I)}}{G_{22}(1+\varepsilon_2^{(0)})}S^3$$

$$+\left\{\frac{[2\varphi_2^{(I)}\varphi_2^{(I)}+3(X^I_{,2}+\varphi_1^{(I)})\varphi_3^{(I)}]}{G_{22}(1+\varepsilon_2^{(0)})}-\frac{2[(X^I_{,2}+\varphi_1^{(I)})\varphi_2^{(I)}]^2}{G_{22}^2(1+\varepsilon_2^{(0)})^3}\right\}(S^2)^2$$

$$+\frac{1}{4}\left\{\frac{2\vartheta_{11}^{(I)}\vartheta_{11}^{(I)}}{G_{22}(1+\varepsilon_2^{(0)})}-\frac{[(X^I_{,2}+\varphi_1^{(I)})\vartheta_{11}^{(I)}]^2}{G_{22}^2(1+\varepsilon_2^{(0)})^3}\right\}(S^3)^2$$

$$+\left[\frac{4(X^I_{,2}+\varphi_1^{(I)})\vartheta_{21}^{(I)}}{G_{22}(1+\varepsilon_2^{(0)})}+\frac{2\varphi_2^{(I)}\vartheta_{11}^{(I)}}{G_{22}(1+\varepsilon_2^{(0)})}\right]S^2S^3+\cdots; \tag{8.217}$$

$$\varepsilon_3\approx\varepsilon_3^{(0)}+\left.\frac{\partial\varepsilon_3}{\partial S^2}\right|_{(S^2,\,S^3)=(0,0)} S^3+\left.\frac{\partial\varepsilon_3}{\partial S^3}\right|_{(S^2,\,S^3)=(0,0)} S^3$$

$$+\frac{1}{2!}\left.\frac{\partial^2\varepsilon_3}{\partial(S^2)^2}\right|_{(S^2,\,S^3)=(0,0)}(S^2)^2+\frac{1}{2!}\left.\frac{\partial^2\varepsilon_3}{\partial(S^3)^2}\right|_{(S^2,\,S^3)=(0,0)}(S^3)^2$$

$$+\left.\frac{\partial^2\varepsilon_3}{\partial S^2\partial S^3}\right|_{(S^2,\,S^3)=(0,\,0)} S^2S^3+\cdots$$

$$=\varepsilon_3^{(0)}+\frac{(X^I_{,3}+\theta_1^{(I)})\vartheta_{11}^{(I)}}{G_{33}(1+\varepsilon_3^{(0)})}S^2+\frac{2(X^I_{,3}+\theta_1^{(I)})\theta_2^{(I)}}{G_{33}(1+\varepsilon_3^{(0)})}S^3$$

$$+\frac{1}{2}\left\{\frac{\vartheta_{11}^{(I)}\vartheta_{11}^{(I)}}{G_{33}(1+\varepsilon_3^{(0)})}-\frac{[(X^I_{,3}+\theta_1^{(I)})\vartheta_{11}^{(I)}]^2}{2G_{33}^2(1+\varepsilon_3^{(0)})^3}\right\}(S^2)^2$$

$$+\left\{\frac{[2\theta_2^{(I)}\theta_2^{(I)}+3(X^I_{,3}+\theta_1^{(I)})\theta_3^{(I)}]}{G_{33}(1+\varepsilon_3^{(0)})}-\frac{2[(X^I_{,3}+\theta_1^{(I)})\theta_2^{(I)}]^2}{G_{33}^2(1+\varepsilon_3^{(0)})^3}\right\}(S^3)^2$$

$$+\left[\frac{4(X^I_{,3}+\theta_1^{(I)})\vartheta_{12}^{(I)}}{G_{33}(1+\varepsilon_3^{(0)})}+\frac{2\theta_2^{(I)}\vartheta_{11}^{(I)}}{G_{33}(1+\varepsilon_3^{(0)})}\right]S^2S^3+\cdots; \tag{8.218}$$

$$\gamma_{12}\approx\gamma_{12}^{(0)}+\left.\frac{\partial\gamma_{12}}{\partial S^2}\right|_{(S^2,\,S^3)=(0,\,0)} S^2+\left.\frac{\partial\gamma_{12}}{\partial S^3}\right|_{(S^2,\,S^3)=(0,\,0)} S^3$$

$$+\frac{1}{2!}\left.\frac{\partial^2\gamma_{12}}{\partial(S^2)^2}\right|_{(S^2,\,S^3)=(0,\,0)}(S^2)^2+\frac{1}{2!}\left.\frac{\partial^2\gamma_{12}}{\partial(S^3)^2}\right|_{(S^2,\,S^3)=(0,\,0)}(S^3)^2$$

$$+\left.\frac{\partial^2\gamma_{12}}{\partial S^2\partial S^3}\right|_{(S^2,\,S^3)=(0,\,0)} S^2S^3+\cdots$$

$$= \gamma_{12}^{(0)} + \frac{1}{\cos\gamma_{12}^{(0)}} \left\{ \frac{2(X_{,1}^{I} + u_{0,1}^{(I)})\varphi_{2}^{(I)} + (X_{,2}^{I} + \varphi_{1}^{(I)})\varphi_{1,1}^{(I)}}{\sqrt{G_{11}G_{22}}\,(1+\varepsilon_{1}^{(0)})(1+\varepsilon_{2}^{(0)})} \right.$$

$$\left. - \sin\gamma_{12}^{(0)} \left[\frac{(X_{,1}^{I} + u_{0,1}^{(I)})\varphi_{1,1}^{(I)}}{G_{11}(1+\varepsilon_{1}^{(0)})^{2}} + \frac{2(X_{,2}^{I} + \varphi_{1}^{(I)})\varphi_{2}^{(I)}}{G_{22}(1+\varepsilon_{2}^{(0)})^{2}} \right] \right\} S^{2}$$

$$+ \frac{1}{\cos\gamma_{12}^{(0)}} \left\{ \frac{(X_{,2}^{I} + \varphi_{1}^{(I)})\theta_{1,1}^{(I)} + (X_{,1}^{I} + u_{0,1}^{(I)})\vartheta_{11}^{(I)}}{\sqrt{G_{11}G_{22}}\,(1+\varepsilon_{1}^{(0)})(1+\varepsilon_{2}^{(0)})} \right.$$

$$\left. - \sin\gamma_{12}^{(0)} \left[\frac{(X_{,1}^{I} + u_{0,1}^{(I)})\theta_{1,1}^{(I)}}{G_{11}(1+\varepsilon_{1}^{(0)})^{2}} + \frac{(X_{,2}^{I} + \varphi_{1}^{(I)})\vartheta_{11}^{(I)}}{G_{22}(1+\varepsilon_{2}^{(0)})^{2}} \right] \right\} S^{3} + \cdots; \tag{8.219}$$

$$\gamma_{13} \approx \gamma_{13}^{(0)} + \left.\frac{\partial\gamma_{13}}{\partial S^{2}}\right|_{(S^{2},\,S^{3})=(0,\,0)} S^{2} + \left.\frac{\partial\gamma_{13}}{\partial S^{3}}\right|_{(S^{2},\,S^{3})=(0,\,0)} S^{3}$$

$$+ \frac{1}{2!}\left.\frac{\partial^{2}\gamma_{13}}{\partial(S^{2})^{2}}\right|_{(S^{2},\,S^{3})=(0,\,0)} (S^{2})^{2} + \frac{1}{2!}\left.\frac{\partial^{2}\gamma_{13}}{\partial(S^{3})^{2}}\right|_{(S^{2},\,S^{3})=(0,\,0)} (S^{3})^{2}$$

$$+ \left.\frac{\partial^{2}\gamma_{13}}{\partial S^{2}\partial S^{3}}\right|_{(S^{2},\,S^{3})=(0,\,0)} S^{2}S^{3} + \cdots$$

$$= \gamma_{13}^{(0)} + \frac{1}{\cos\gamma_{13}^{(0)}} \left\{ \frac{(X_{,3}^{I} + \theta_{1}^{(I)})\varphi_{1,1}^{(I)} + (X_{,1}^{I} + u_{0,1}^{(I)})\vartheta_{11}^{(I)}}{\sqrt{G_{11}G_{33}}\,(1+\varepsilon_{1}^{(0)})(1+\varepsilon_{3}^{(0)})} \right.$$

$$\left. - \sin\gamma_{13}^{(0)} \left[\frac{(X_{,1}^{I} + u_{0,1}^{(I)})\varphi_{1,1}^{(I)}}{G_{11}(1+\varepsilon_{1}^{(0)})^{2}} + \frac{(X_{,3}^{I} + \theta_{1}^{(I)})\vartheta_{11}^{(I)}}{G_{33}(1+\varepsilon_{3}^{(0)})^{2}} \right] \right\} S^{2}$$

$$+ \frac{1}{\cos\gamma_{13}^{(0)}} \left\{ \frac{2(X_{,1}^{I} + u_{0,1}^{(I)})\theta_{2}^{(I)} + (X_{,3}^{I} + \theta_{1}^{(I)})\vartheta_{1,1}^{(I)}}{\sqrt{G_{11}G_{33}}\,(1+\varepsilon_{1}^{(0)})(1+\varepsilon_{3}^{(0)})} \right.$$

$$\left. - \sin\gamma_{13}^{(0)} \left[\frac{(X_{,1}^{I} + u_{0,1}^{(I)})\vartheta_{1,1}^{(I)}}{G_{11}(1+\varepsilon_{1}^{(0)})^{2}} + \frac{2(X_{,3}^{I} + \theta_{1}^{(I)})\theta_{2}^{(I)}}{G_{33}(1+\varepsilon_{3}^{(0)})^{2}} \right] \right\} S^{3} + \cdots; \tag{8.220}$$

$$\gamma_{23} \approx \gamma_{23}^{(0)} + \left.\frac{\partial\gamma_{23}}{\partial S^{2}}\right|_{(S^{2},\,S^{3})=(0,\,0)} S^{2} + \left.\frac{\partial\gamma_{23}}{\partial S^{3}}\right|_{(S^{2},\,S^{3})=(0,\,0)} S^{3}$$

$$+ \frac{1}{2}\left.\frac{\partial^{2}\gamma_{23}}{\partial(S^{2})^{2}}\right|_{(S^{2},\,S^{3})=(0,\,0)} (S^{2})^{2} + \frac{1}{2}\left.\frac{\partial^{2}\gamma_{23}}{\partial(S^{3})^{2}}\right|_{(S^{2},\,S^{3})=(0,\,0)} (S^{3})^{2}$$

$$+ \left.\frac{\partial^{2}\gamma_{23}}{\partial S^{2}\partial S^{3}}\right|_{(S^{2},\,S^{3})=(0,\,0)} S^{2}S^{3} + \cdots$$

$$
\begin{aligned}
&= \gamma_{23}^{(0)} + \frac{1}{\cos\gamma_{23}^{(0)}} \left\{ \frac{2(X_{,3}^I + \theta_1^{(I)})\varphi_2^{(I)} + (X_{,2}^I + \varphi_1^{(I)})\vartheta_{11}^{(I)}}{\sqrt{G_{22}G_{33}}(1+\varepsilon_2^{(0)})(1+\varepsilon_3^{(0)})} \right.\\
&\quad \left. - \sin\gamma_{23}^{(0)} \left[\frac{2(X_{,2}^I + \varphi_1^{(I)})\varphi_2^{(I)}}{G_{22}(1+\varepsilon_2^{(0)})^2} + \frac{(X_{,3}^I + \theta_1^{(I)})\vartheta_{11}^{(I)}}{G_{33}(1+\varepsilon_3^{(0)})^2} \right] \right\} S^2 \\
&\quad + \frac{1}{\cos\gamma_{23}^{(0)}} \left\{ \frac{2(X_{,2}^I + \varphi_1^{(I)})\theta_2^{(I)} + (X_{,3}^I + \theta_1^{(I)})\vartheta_{11}^{(I)}}{(1+\varepsilon_2^{(0)})(1+\varepsilon_3^{(0)})} \right.\\
&\quad \left. - \sin\gamma_{23}^{(0)} \left[\frac{(X_{,2}^I + \varphi_1^{(I)})\vartheta_{11}^{(I)}}{G_{22}(1+\varepsilon_2^{(0)})^2} + \frac{2(X_{,3}^I + \theta_1^{(I)})\theta_2^{(I)}}{G_{33}(1+\varepsilon_3^{(0)})^2} \right] \right\} S^3 + \cdots, \qquad (8.221)
\end{aligned}
$$

where

$$
\begin{aligned}
\varepsilon_1^{(0)} &= \frac{1}{\sqrt{G_{11}}}\sqrt{(X_{,1}^I + u_{0,1}^I)(X_{,1}^I + u_{0,1}^I)} - 1,\\
\varepsilon_2^{(0)} &= \frac{1}{\sqrt{G_{22}}}\sqrt{(X_{,2}^I + \varphi_1^{(I)})(X_{,2}^I + \varphi_1^{(I)})} - 1,\\
\varepsilon_3^{(0)} &= \frac{1}{\sqrt{G_{33}}}\sqrt{(X_{,3}^I + \theta_1^{(I)})(X_{,3}^I + \theta_1^{(I)})} - 1,\\
\gamma_{12}^{(0)} &= \sin^{-1}\frac{(X_{,1}^I + u_{0,1}^I)(X_{,2}^I + \varphi_1^{(I)})}{\sqrt{G_{11}G_{22}}(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})},\\
\gamma_{13}^{(0)} &= \sin^{-1}\frac{(X_{,1}^I + u_{0,1}^I)(X_{,3}^I + \theta_1^{(I)})}{\sqrt{G_{11}G_{33}}(1+\varepsilon_1^{(0)})(1+\varepsilon_3^{(0)})},\\
\gamma_{23}^{(0)} &= \sin^{-1}\frac{(X_{,2}^I + \varphi_1^{(I)})(X_{,3}^I + \theta_1^{(I)})}{\sqrt{G_{22}G_{33}}(1+\varepsilon_2^{(0)})(1+\varepsilon_3^{(0)})}.
\end{aligned}
\qquad (8.222)
$$

From Assumptions (i) and (ii), consider the zero order terms of the Taylor series of the five strains to give

$$
\varepsilon_2^{(0)} = 0,\ \varepsilon_3^{(0)} = 0;\ \gamma_{12}^{(0)} = 0,\ \gamma_{13}^{(0)} = 0,\ \gamma_{23}^{(0)} = 0. \qquad (8.223)
$$

The stretch of the deformed rod for $S^2 = S^3 = 0$ satisfies

$$
(1+\varepsilon_1^{(0)})^2 = \frac{1}{G_{11}}(X_{,1}^I + u_{0,1}^I)(X_{,1}^I + u_{0,1}^I). \qquad (8.224)
$$

Equation (8.224) implies that only 1-dimenisonal stretch is considered as in the cable. Similarly, from the higher order terms of the Taylor series of the six strains, the relations for the unknowns in displacement field can be obtained. From Eqs.(8.222) and (8.223),

$$\begin{aligned}
&\frac{1}{G_{11}(1+\varepsilon_1^{(0)})^2}(X_{,1}^I+u_{0,1}^I)(X_{,1}^I+u_{0,1}^I)=1,\\
&\frac{1}{G_{22}}(X_{,2}^I+\varphi_1^{(I)})(X_{,2}^I+\varphi_1^{(I)})=1,\\
&\frac{1}{G_{33}}(X_{,3}^I+\theta_1^{(I)})(X_{,3}^I+\theta_1^{(I)})=1;\\
&(X_{,1}^I+u_{0,1}^I)(X_{,2}^I+\varphi_1^{(I)})=0,\\
&(X_{,1}^I+u_{0,1}^I)(X_{,3}^I+\theta_1^{(I)})=0,\\
&(X_{,2}^I+\varphi_1^{(I)})(X_{,3}^I+\theta_1^{(I)})=0.
\end{aligned} \tag{8.225}$$

Using the zero order terms of S^2 and S^3 in Eq.(8.83), the directional cosine matrix ($(l_{ij})_{3\times3}$) is

$$\begin{aligned}
\cos\theta_{(\mathbf{n}_1,\mathbf{I}_I)}&=l_{1I}=\frac{X_{,1}^I+u_{0,1}^I}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})},\\
\cos\theta_{(\mathbf{n}_2,\mathbf{I}_I)}&=l_{2I}=\frac{X_{,2}^I+\varphi_1^{(I)}}{\sqrt{G_{22}}(1+\varepsilon_2^{(0)})},\\
\cos\theta_{(\mathbf{n}_3,\mathbf{I}_I)}&=l_{3I}=\frac{X_{,3}^I+\theta_1^{(I)}}{\sqrt{G_{33}}(1+\varepsilon_3^{(0)})}.
\end{aligned} \tag{8.226}$$

From the geometrical relations, the nine directional cosines must satisfy trigonometric relations in Eqs.(8.155) and (8.156). As in Fig.8.5, consider the initial Euler angles (Φ_0,Ψ_0 and Θ_0) rotating about the axes of X^1, X^2 and X^3, respectively. The Euler angles of the deformed rod are (Φ,Ψ and Θ). As same as in Eq.(8.157)-(8.159) gives the direction cosine matrix ($\boldsymbol{l}=(l_{ij})_{3\times3}$) in Eqs.(8.160) and (8.161) for the deformed rod. Compared with Eq.(8.235), with Eq.(8.156), equation (8.161) for the deformed rod gives

$$\begin{aligned}
&\frac{X_{,1}^1+u_{0,1}^1}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})}=\cos\Phi\cos\Psi,\\
&\frac{X_{,2}^1+\varphi_1^{(1)}}{\sqrt{G_{22}}}=-\cos\Phi\sin\Psi\cos\Theta+\sin\Phi\sin\Theta,\\
&\frac{X_{,3}^1+\theta_1^{(1)}}{\sqrt{G_{33}}}=\cos\Phi\sin\Psi\sin\Theta+\sin\Phi\cos\Theta,\\
&\frac{X_{,1}^2+u_{0,1}^2}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})}=\sin\Psi,
\end{aligned}$$

$$\frac{X^2_{,2}+\varphi^{(2)}_1}{\sqrt{G_{22}}}=\cos\Psi\cos\Theta,$$
$$\frac{X^2_{,3}+\theta^{(2)}_1}{\sqrt{G_{33}}}=-\cos\Psi\sin\Theta,$$
$$\frac{X^3_{,1}+u^3_{0,1}}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)}=-\sin\Phi\cos\Psi, \tag{8.227}$$
$$\frac{X^3_{,2}+\varphi^{(3)}_1}{\sqrt{G_{22}}}=\sin\Phi\sin\Psi\cos\Theta+\cos\Phi\sin\Theta,$$
$$\frac{X^3_{,3}+\theta^{(3)}_1}{\sqrt{G_{33}}}=-\sin\Phi\sin\Psi\sin\Theta+\cos\Phi\cos\Theta.$$

The first, fourth and seventh equations of Eq.(8.227) give

$$\cos\Psi=\pm\frac{\sqrt{(X^1_{,1}+u^1_{0,1})^2+(X^3_{,1}+u^3_{0,1})^2}}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)}=\pm\frac{\Delta}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)},$$
$$\sin\Psi=\frac{X^2_{,1}+u^2_{0,1}}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)};$$
$$\cos\Phi=\pm\frac{X^1_{,1}+u^1_{0,1}}{\sqrt{(X^1_{,1}+u^1_{0,1})^2+(X^3_{,1}+u^3_{0,1})^2}}=\pm\frac{X^1_{,1}+u^1_{0,1}}{\Delta}, \tag{8.228}$$
$$\sin\Phi=\mp\frac{X^3_{,1}+u^3_{0,1}}{\sqrt{(X^1_{,1}+u^1_{0,1})^2+(X^3_{,1}+u^3_{0,1})^2}}=\mp\frac{X^3_{,1}+u^3_{0,1}}{\Delta},$$

and

$$\frac{X^1_{,2}+\varphi^{(1)}_1}{\sqrt{G_{22}}}=\mp\frac{1}{\Delta}\Big(\frac{\Delta_{12}}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)}\cos\Theta+(X^1_{,1}+u^3_{0,1})\sin\Theta\Big),$$
$$\frac{X^1_{,3}+\theta^{(1)}_1}{\sqrt{G_{33}}}=\pm\frac{1}{\Delta}\Big(\frac{\Delta_{12}}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)}\sin\Theta-(X^3_{,1}+u^3_{0,1})\cos\Theta\Big),$$
$$\frac{X^2_{,2}+\varphi^{(2)}_1}{\sqrt{G_{22}}}=\pm\frac{\Delta}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)}\cos\Theta,$$
$$\frac{X^2_{,3}+\theta^{(2)}_1}{\sqrt{G_{33}}}=\mp\frac{\Delta}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)}\sin\Theta,$$
$$\frac{X^3_{,2}+\varphi^{(3)}_1}{\sqrt{G_{22}}}=\mp\frac{1}{\Delta}\Big(\frac{\Delta_{23}}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)}\cos\Theta-(X^1_{,1}+u^1_{0,1})\sin\Theta\Big),$$

$$\frac{X^3_{,3}+\theta^{(3)}_1}{\sqrt{G_{33}}}=\pm\frac{1}{\Delta}\left(\frac{\Delta_{23}}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)}\sin\Theta+(X^1_{,1}+u^1_{0,1})\cos\Theta\right), \tag{8.229}$$

where

$$\begin{aligned}\Delta&=\sqrt{(X^1_{,1}+u^1_{0,1})^2+(X^3_{,1}+u^3_{0,1})^2},\\ \Delta_{12}&=(X^2_{,1}+u^2_{0,1})(X^1_{,1}+u^1_{0,1}),\\ \Delta_{23}&=(X^3_{,1}+u^3_{0,1})(X^2_{,1}+u^2_{0,1}).\end{aligned} \tag{8.230}$$

If $\Theta=\Theta_0$, the rotation about the longitudinal axis disappears. So only the bending rotation exists, which is the pure bending of the rod as in Section 8.3.1.

From Eq.(8.165), the change ratio of the directional cosines along the deformed rod in Eq.(8.166) becomes

$$\frac{\boldsymbol{dl}_\alpha}{ds}=\frac{dl_{\alpha I}}{ds}\mathbf{I}_I=\frac{dl_{\alpha I}}{\sqrt{\mathrm{G}_{11}}(1+\varepsilon^{(0)}_1)dS}\mathbf{I}_I \quad \text{for } \alpha=1,2,3. \tag{8.231}$$

As in Eq.(8.168), the change ratio of the directional cosines along the deformed rod can be computed by

$$\frac{\boldsymbol{dl}_\alpha}{ds}=\varepsilon_{IJK}\omega_J l_{\alpha K}\mathbf{I}_I, \tag{8.232}$$

where the rotation ratio components in Eq.(8.232) are

$$\omega_J=e_{IJK}\frac{dl_{\alpha I}}{ds}l_{\alpha K}=e_{IJK}\frac{dl_{\alpha I}}{\sqrt{\mathrm{G}_{11}}(1+\varepsilon^{(0)}_1)dS}l_{\alpha K}. \tag{8.233}$$

From Eq.(8.161), the foregoing equations give

$$\begin{aligned}\omega_1&=\frac{1}{\sqrt{\mathrm{G}_{11}}(1+\varepsilon^{(0)}_1)}\left(\sin\Phi\frac{d\Psi}{dS}+\cos\Psi\cos\Phi\frac{d\Theta}{dS}\right),\\ \omega_2&=\frac{1}{\sqrt{\mathrm{G}_{11}}(1+\varepsilon^{(0)}_1)}\left(\frac{d\Psi}{dS}+\sin\Psi\frac{d\Theta}{dS}\right),\\ \omega_3&=\frac{1}{\sqrt{\mathrm{G}_{11}}(1+\varepsilon^{(0)}_1)}\left(\cos\Phi\frac{d\Psi}{dS}-\sin\Phi\cos\Psi\frac{d\Theta}{dS}\right);\end{aligned} \tag{8.234}$$

similar to Eq.(8.171).

In an alike fashion, using Eq.(8.25), the particle location on the deformed, curved rod can be expressed by

$$\mathbf{r}=(X^I+u^I)\mathbf{I}_I. \tag{8.235}$$

The corresponding base vector of the deformed, curved rod is

$$\tilde{\mathbf{g}}_1 = \tilde{g}_1^I \mathbf{I}_I = (X_{,1}^I + u_{,1}^I)\mathbf{I}_I, \tag{8.236}$$

and the unit vector for the deformed, curved rod is

$$\tilde{\mathbf{n}}_1 = \frac{\tilde{\mathbf{g}}_1}{|\tilde{\mathbf{g}}_1|} = \frac{X_{,1}^I + u_{,1}^I}{\sqrt{(X_{,1}^I + u_{,1}^I)(X_{,1}^I + u_{,1}^I)}}\mathbf{I}_I = \frac{X_{,1}^I + u_{,1}^I}{\sqrt{G_{11}}(1+\varepsilon_1)}\mathbf{I}_I. \tag{8.237}$$

For $S^2 = S^3 = 0$,

$$\tilde{\mathbf{n}}_1 = \frac{\tilde{\mathbf{g}}_1}{|\tilde{\mathbf{g}}_1|} = \frac{X_{,1}^I + u_{0,1}^I}{\sqrt{(X_{,1}^I + u_{0,1}^I)(X_{,1}^I + u_{0,1}^I)}}\mathbf{I}_I = \frac{X_{,1}^I + u_{0,1}^I}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})}\mathbf{I}_I. \tag{8.238}$$

The base vector in the principal normal direction is as in Eq.(8.177), i.e.,

$$\tilde{\mathbf{g}}_2 = \frac{d\tilde{\mathbf{n}}_1}{ds} = \tilde{g}_2^I \mathbf{I}_I, \tag{8.239}$$

where

$$\begin{aligned}\tilde{g}_2^I = \frac{1}{g_{11}^2}\Big[&(X_{,11}^I + u_{,11}^I)(X_{,1}^K + u_{,1}^K)(X_{,1}^K + u_{,1}^K)\\ &-(X_{,1}^I + u_{,1}^I)(X_{,1}^K + u_{,1}^K)(X_{,11}^K + u_{,11}^K)\Big].\end{aligned} \tag{8.240}$$

Thus, the unit principal normal vector in Eq.(8.179) can be rewritten, i.e.,

$$\tilde{\mathbf{n}}_2 = \frac{\tilde{\mathbf{g}}_2}{|\tilde{\mathbf{g}}_2|} = \frac{\tilde{\mathbf{g}}_2}{\sqrt{\tilde{g}_{22}}} = \frac{\tilde{\mathbf{g}}_2}{\tilde{\kappa}(S)} = \frac{\tilde{g}_2^I}{\tilde{\kappa}(S)}\mathbf{I}_I, \tag{8.241}$$

where the curvature of the deformed rod is

$$\begin{aligned}&\tilde{\kappa}(S) = |\tilde{\mathbf{g}}_2| = \sqrt{\tilde{g}_{22}} = \sqrt{(X_{,ss}^I + u_{,ss}^I)(X_{,ss}^I + u_{,ss}^I)} = \frac{1}{\tilde{g}_{11}^{3/2}}\sqrt{\Xi_1},\\ &\Xi_1 = (X_{,11}^I + u_{,11}^I)(X_{,11}^I + u_{,11}^I)(X_{,1}^K + u_{,1}^K)(X_{,1}^K + u_{,1}^K)\\ &\qquad -[(X_{,1}^I + X_{,1}^I)(X_{,11}^I + u_{,11}^I)]^2.\end{aligned} \tag{8.242}$$

For $S^2 = S^3 = 0$,

$$\begin{aligned}\tilde{g}_2^I = \frac{1}{\tilde{g}_{11}^2}\Big[&(X_{,11}^I + u_{0,11}^I)(X_{,1}^K + u_{0,1}^K)(X_{,1}^K + u_{0,1}^K)\\ &-(X_{,1}^I + u_{0,1}^I)(X_{,1}^K + u_{0,1}^K)(X_{,11}^K + u_{0,11}^K)\Big],\end{aligned} \tag{8.243}$$

$$\begin{aligned}\Xi_1 = {}&(X_{,11}^I + u_{0,11}^I)(X_{,11}^I + u_{0,11}^I)(X_{,1}^K + u_{0,1}^K)(X_{,1}^K + u_{0,1}^K)\\ &-[(X_{,1}^I + u_{0,1}^I)(X_{,11}^I + u_{0,11}^I)]^2.\end{aligned} \tag{8.244}$$

The rotation vector of the deformed, curved rod can be expressed as in Eq.(8.185), i.e.,

$$\boldsymbol{\omega} = \tilde{\kappa}(S)\tilde{\mathbf{n}}_3 + \tilde{\tau}(S)\tilde{\mathbf{n}}_1 = \omega_I \mathbf{I}_I \tag{8.245}$$

where the torsion of the deformed rod is

$$\tilde{\tau}(S) = \frac{e_{IJK}(X^I_{,1} + u^I_{,1})(X^I_{,11} + u^J_{,11})(X^K_{,111} + u^K_{,111})}{\Xi_1}. \tag{8.246}$$

The foregoing equation for $S^2 = S^3 = 0$ becomes

$$\tilde{\tau}(S) = \frac{e_{IJK}(X^I_{,1} + u^I_{0,1})(X^I_{,11} + u^J_{0,11})(X^K_{,111} + u^K_{0,111})}{\Xi_1}. \tag{8.247}$$

From Eq.(8.245), equations similar to Eqs.(8.188) and (8.189) are:

$$\begin{aligned} \tilde{\kappa}(S) &= \omega_I \mathbf{I}_I \cdot \mathbf{n}_3 = \omega_I \tilde{g}^I_3, \\ \tilde{\tau}(S) &= \omega_I \mathbf{I}_I \cdot \mathbf{n}_1 = \omega_I \tilde{g}^I_2, \end{aligned} \tag{8.248}$$

or

$$\begin{aligned} \omega_I &= \tilde{\kappa}(S)\mathbf{n}_3 \cdot \mathbf{I}_I + \tilde{\tau}(S)\mathbf{n}_1 \cdot \mathbf{I}_I \\ &= \tilde{\kappa}(S)\tilde{g}^I_3 + \tilde{\tau}(S)\tilde{g}^I_1. \end{aligned} \tag{8.249}$$

From Eq.(8.238), $d\Psi / dS$ and $d\Phi / dS$ are determined by

$$\begin{aligned} \frac{d\Psi}{dS} &= \pm \frac{(X^2_{,11} + u^2_{0,11})\Delta^2 - [\Delta_{12}(X^2_{,11} + u^1_{0,11}) + \Delta_{23}(X^3_{,11} + u^3_{0,11})]}{G_{11}(1+\varepsilon^{(0)}_1)^2 \Delta}, \\ \frac{d\Phi}{dS} &= -\frac{(X^3_{,11} + u^3_{0,11})(X^1_{,1} + u^1_{0,1}) - (X^1_{,11} + u^1_{0,11})(X^3_{,1} + u^3_{0,1})}{\Delta^2}; \end{aligned} \tag{8.250}$$

or

$$\begin{aligned} \frac{d\Psi}{ds} &= \pm \frac{(X^2_{,11} + u^2_{0,11})\Delta^2 - [\Delta_{12}(X^2_{,11} + u^1_{0,11}) + \Delta_{23}(X^3_{,11} + u^3_{0,11})]}{\left[\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)\right]^3 \Delta}, \\ \frac{d\Phi}{ds} &= -\frac{(X^3_{,11} + u^3_{0,11})(X^1_{,1} + u^1_{0,1}) - (X^1_{,11} + u^1_{0,11})(X^3_{,1} + u^3_{0,1})}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)\Delta^2}. \end{aligned} \tag{8.251}$$

Substitution of Eqs.(8.228) and (8.250) into Eqs.(8.234) and (8.248) gives Θ and $d\Theta / dS$.

As in Eq.(8.192), the constitutive laws for deformed rods give the corresponding resultant stresses, and the internal forces and moments in the deformed rod are in the form of Eq.(8.193), i.e.,

$$
\begin{aligned}
N_1 &= \int_A \sigma_{11}\left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23}\right]dA,\\
Q_2 &= \int_A \sigma_{12}\left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23}\right]dA,\\
Q_3 &= \int_A \sigma_{13}\left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23}\right]dA,\\
M_3 &= \int_A \sigma_{11}\frac{S^2\sqrt{G_{22}}}{1+\varphi_1^{(2)}}\left[(1+\varepsilon_3)(1+\varepsilon_2)^2\cos\gamma_{23}\right]dA,\\
M_2 &= -\int_A \sigma_{11}\frac{S^3\sqrt{G_{33}}}{1+\theta_1^{(3)}}\left[(1+\varepsilon_3)^2(1+\varepsilon_2)\cos\gamma_{23}\right]dA,\\
T_1 &= \int_A \left[\sigma_{12}\frac{S^3(1+\varepsilon_3)\sqrt{G_{33}}}{1+\theta_1^{(3)}}-\sigma_{13}\frac{S^2(1+\varepsilon_2)\sqrt{G_{22}}}{1+\varphi_1^{(2)}}\right]\\
&\quad\times\left[(1+\varepsilon_3)(1+\varepsilon_2)\cos\gamma_{23}\right]dA.
\end{aligned}
\tag{8.252}
$$

The notations $Q_2 \equiv N_2$, $Q_3 \equiv N_3$, and $T_1 \equiv M_1$ are used again. The internal forces are expressed as in Eq.(8.194), i.e.,

$$
\begin{aligned}
\mathbf{M} &\equiv M^I\mathbf{I}_I = M_\alpha \mathbf{n}_\alpha,\\
\boldsymbol{\mathcal{N}} &\equiv N^I\mathbf{I}_I = N_\alpha \mathbf{n}_\alpha,\\
{}^{\mathrm{N}}\mathbf{M} &\equiv {}^{\mathrm{N}}M^I\mathbf{I}_I = \mathbf{g}_1\times\boldsymbol{\mathcal{N}},
\end{aligned}
\tag{8.253}
$$

where

$$
\mathbf{g}_1 \equiv \frac{d\mathbf{r}}{dS} = (X^I_{,1}+u^I_{0,1})\mathbf{I}_I \text{ and } {}^{\mathrm{N}}\mathbf{M} \equiv \frac{1}{ds}d\mathbf{r}\times\boldsymbol{\mathcal{N}}.
\tag{8.254}
$$

With Eq.(8.225), the components of the internal forces in the $\mathbf{I}_I$-direction are

$$
\begin{aligned}
N^I &= N_\alpha \mathbf{n}_\alpha\cdot\mathbf{I}_I = N_\alpha\cos\theta_{(\mathbf{n}_\alpha,\mathbf{I}_I)}\\
&= \frac{N_1(X^I_{,1}+u^I_{0,1})}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})}+\frac{N_2(X^I_{,2}+\varphi^I_1)}{\sqrt{G_{22}}(1+\varepsilon_2^{(0)})}+\frac{N_3(X^I_{,3}+\theta^I_1)}{\sqrt{G_{33}}(1+\varepsilon_3^{(0)})},
\end{aligned}
\tag{8.255}
$$

$$
\begin{aligned}
M^I &= M_\alpha \mathbf{n}_\alpha\cdot\mathbf{I}_I = M_\alpha\cos\theta_{(\mathbf{n}_\alpha,\mathbf{I}_I)}\\
&= \frac{M_1(X^I_{,1}+u^I_{0,1})}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})}+\frac{M_2(X^I_{,2}+\varphi^I_1)}{\sqrt{G_{22}}(1+\varepsilon_2^{(0)})}+\frac{M_3(X^I_{,3}+\theta^I_1)}{\sqrt{G_{33}}(1+\varepsilon_3^{(0)})},\\
{}^{\mathrm{N}}M^I &= (\mathbf{g}_1\times\boldsymbol{\mathcal{N}})\cdot\mathbf{I}_I = e_{IJK}(X^J_{,1}+u^J_{0,1})N^K.
\end{aligned}
\tag{8.256}
$$

Using the external forces as in Eqs.(8.208)–(8.212), equations of motion on the deformed rod are given as in Eq.(8.198), i.e.,

$$\begin{aligned} &\mathcal{N}_{,1} + \mathbf{q} = \rho \mathbf{u}_{0,tt} + I_3 \boldsymbol{\varphi}_{1,tt} + I_2 \boldsymbol{\theta}_{1,tt}, \\ &\mathbf{M}_{,1} + {}^{\mathrm{N}}\mathbf{M} + \mathbf{m} = \mathscr{J}\boldsymbol{\Theta}_{,tt}, \end{aligned} \tag{8.257}$$

where

$$\begin{aligned} \mathscr{J}\boldsymbol{\Theta}_{,tt} = &[(I_3 u^3_{0,tt} - I_2 u^2_{0,tt}) + (J_{22}\varphi^{(3)}_{1,tt} - J_{23}\varphi^{(2)}_{1,tt}) \\ &+(J_{23}\theta^{(3)}_{1,tt} - J_{33}\theta^{(2)}_{1,tt})]\mathbf{I}_1 + (I_3 u^1_{0,tt} + J_{23}\varphi^{(1)}_{1,tt} + J_{22}\theta^{(1)}_{1,tt})\mathbf{I}_2 \\ &-(I_2 u^1_{0,tt} + J_{22}\varphi^{(1)}_{1,tt} + J_{23}\theta^{(1)}_{1,tt})\mathbf{I}_3, \end{aligned} \tag{8.258}$$

and the scalar expressions are for $I = 1, 2, 3$,

$$\begin{aligned} &N^I_{,1} + q_I = \rho u^I_{(0),tt} + I_3 \varphi^{(I)}_{1,tt} + I_2 \theta^{(I)}_{1,tt}; \\ &M^I_{,1} + {}^{\mathrm{N}}M^I + m^I = (\mathscr{J}\boldsymbol{\Theta}_{,tt}) \cdot \mathbf{I}_I. \end{aligned} \tag{8.259}$$

or

$$\begin{aligned} &\left[\frac{N_1(X^I_{,1} + u^I_{0,1})}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)} + \frac{N_2(X^I_{,2} + \varphi^I_1)}{\sqrt{G_{22}}(1+\varepsilon^{(0)}_2)} + \frac{N_3(X^I_{,3} + \theta^I_1)}{\sqrt{G_{33}}(1+\varepsilon^{(0)}_3)}\right]_{,1} \\ &+q^I = \rho u^I_{(0),tt} + I_3 \varphi^{(I)}_{1,tt} + I_2 \theta^{(I)}_{1,tt}; \\ &\left[\frac{M_1(X^I_{,1} + u^I_{0,1})}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)} + \frac{M_2(X^I_{,2} + \varphi^I_1)}{\sqrt{G_{22}}(1+\varepsilon^{(0)}_2)} + \frac{M_3(X^I_{,3} + \theta^I_1)}{\sqrt{G_{33}}(1+\varepsilon^{(0)}_3)}\right]_{,1} \\ &+e_{IJK}(X^J_{,1} + u^J_{0,1})N^K + m^I = (\mathscr{J}\boldsymbol{\Theta}_{,tt}) \cdot \mathbf{I}_I. \end{aligned} \tag{8.260}$$

From Assumption (i), one has $\varepsilon^{(0)}_2 = \varepsilon^{(0)}_3 = 0$. In addition, the force and moment balance conditions at any point $\mathscr{P}_k$, and the force boundary conditions are given in Eqs.(8.203)–(8.206), and the displacement continuity and boundary conditions are the same as in Eq.(8.207).

8.5.2. A curved rod theory based on the curvilinear coordinates

In this section, the curved rod theory on the curvilinear coordinates is discussed in an analogy way as in the Cartesian coordinates. The strains for 3-D deformed beam in Eqs.(8.110)–(8.116) can be used for the 3-D deformed rod. Similar to Eq.(8.214), the displacement field for any fiber of the deformed rod at a position $\mathbf{R}$ is assumed by

$$\begin{aligned} u^{\Lambda} = &u^{\Lambda}_0(S,t) + \sum\nolimits_{n=1}^{\infty}(S^2)^n \varphi^{(\Lambda)}_n(S,t) + \sum\nolimits_{n=1}^{\infty}(S^3)^n \theta^{(\Lambda)}_n(S,t) \\ &+ \sum\nolimits_{m=1}^{\infty}\sum\nolimits_{n=1}^{\infty}(S^2)^m (S^3)^n \vartheta^{(\Lambda)}_{mn}(S,t), \end{aligned} \tag{8.261}$$

where $S^1 = S$ and u_0^Λ ($\Lambda = 1, 2, 3$) are displacements of centroid curve of the rod for $S^2 = S^3 = 0$. The coefficients of the Taylor series expansion $\varphi_n^{(\Lambda)}$, $\theta_n^{(\Lambda)}$ and $\vartheta_{mn}^{(\Lambda)}$ ($m, n = 1, 2, \cdots$) are

$$\begin{aligned} \varphi_n^{(I)} &= \frac{1}{n!} \frac{\partial^n u^I}{\partial (S^2)^n} \bigg|_{(S^2,\, S^3)=(0,0)}, \\ \theta_n^{(I)} &= \frac{1}{n!} \frac{\partial^n u^I}{\partial (S^3)^n} \bigg|_{(S^2,\, S^3)=(0,0)}, \\ \vartheta_{mn}^{(I)} &= \frac{1}{m!n!} \frac{\partial^{m+n} u^I}{\partial (S^2)^m \partial (S^3)^n} \bigg|_{(S^2,\, S^3)=(0,0)}. \end{aligned} \tag{8.262}$$

Because $G_{\Lambda\Gamma} = 0$ ($\Lambda, \Gamma \in \{1, 2, 3\}$ and $\Lambda \neq \Gamma$), the Taylor series expansion of six strains are given as follows:

$$\begin{aligned} \varepsilon_1 &\approx \varepsilon_1^{(0)} + \frac{\partial \varepsilon_1}{\partial S^2}\bigg|_{(S^2,\, S^3)=(0,0)} S^2 + \frac{\partial \varepsilon_1}{\partial S^3}\bigg|_{(S^2,\, S^3)=(0,\, 0)} S^3 \\ &\quad + \frac{1}{2!} \frac{\partial^2 \varepsilon_1}{\partial (S^2)^2}\bigg|_{(S^2,\, S^3)=(0,0)} (S^2)^2 + \frac{1}{2!} \frac{\partial^2 \varepsilon_1}{\partial (S^3)^2}\bigg|_{(S^2,\, S^3)=(0,\, 0)} (S^3)^2 \\ &\quad + \frac{\partial^2 \varepsilon_1}{\partial S^2 \partial S^3}\bigg|_{(S^2,\, S^3)=(0,0)} S^2 S^3 + \cdots \\ &= \varepsilon_1^{(0)} + \frac{(\delta_1^\Lambda + u_{0;1}^\Lambda)\varphi_{1,1}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{11}(1+\varepsilon_1^{(0)})} S^2 + \frac{(\delta_1^\Lambda + u_{0;1}^\Lambda)\theta_{1;1}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{11}(1+\varepsilon_1^{(0)})} S^3 \\ &\quad + \frac{1}{2}\left\{ \frac{[2(\delta_1^\Lambda + u_{0;1}^\Lambda)\varphi_{2;1}^{(\Lambda)} + \varphi_{1;1}^{(\Lambda)}\varphi_{1;1}^{(\Lambda)}] G_{\Lambda\Lambda}}{G_{11}(1+\varepsilon_1^{(0)})} - \frac{[(\delta_1^\Lambda + u_{0;1}^\Lambda)\varphi_{1;1}^{(\Lambda)}]^2 G_{\Lambda\Lambda}^2}{G_{11}^2(1+\varepsilon_1^{(0)})^3} \right\} (S^2)^2 \\ &\quad + \frac{1}{2}\left\{ \frac{[2(\delta_1^\Lambda + u_{0;1}^\Lambda)\theta_{2;1}^{(\Lambda)} + \theta_{1;1}^{(\Lambda)}\theta_{1;1}^{(\Lambda)}] G_{\Lambda\Lambda}}{G_{11}(1+\varepsilon_1^{(0)})} - \frac{[(\delta_1^\Lambda + u_{0;1}^\Lambda)\theta_{1;1}^{(\Lambda)}]^2 G_{\Lambda\Lambda}^2}{G_{11}^2(1+\varepsilon_1^{(0)})^3} \right\} (S^3)^2 \\ &\quad + \left[\frac{(\delta_1^\Lambda + u_{0;1}^\Lambda)\vartheta_{11;1}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{11}(1+\varepsilon_1^{(0)})} + \frac{\varphi_{1;1}^{(\Lambda)}\theta_{1;1}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{11}(1+\varepsilon_1^{(0)})} \right] S^2 S^3 + \cdots, \end{aligned} \tag{8.263}$$

$$\begin{aligned} \varepsilon_2 &\approx \varepsilon_2^{(0)} + \frac{\partial \varepsilon_2}{\partial S^2}\bigg|_{(S^2,\, S^3)=(0,\, 0)} S^2 + \frac{\partial \varepsilon_2}{\partial S^3}\bigg|_{(S^2,\, S^3)=(0,\, 0)} S^3 \\ &\quad + \frac{1}{2!} \frac{\partial^2 \varepsilon_2}{\partial (S^2)^2}\bigg|_{(S^2,\, S^3)=(0,\, 0)} (S^2)^2 + \frac{1}{2!} \frac{\partial^2 \varepsilon_2}{\partial (X^3)^2}\bigg|_{(S^2,\, S^3)=(0,\, 0)} (S^3)^2 \\ &\quad + \frac{\partial^2 \varepsilon_2}{\partial S^2 \partial S^3}\bigg|_{(S^2,\, S^3)=(0,\, 0)} S^2 S^3 + \cdots \end{aligned}$$

$$= \varepsilon_2^{(0)} + \frac{2(\delta_2^\Lambda + \varphi_1^{(\Lambda)})\varphi_2^{(\Lambda)} G_{\Lambda\Lambda}}{G_{22}(1+\varepsilon_2^{(0)})} S^2 + \frac{(\delta_2^\Lambda + \varphi_1^{(\Lambda)})\vartheta_{11}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{22}(1+\varepsilon_2^{(0)})} S^3$$
$$+ \left\{ \frac{[2\varphi_2^{(\Lambda)}\varphi_2^{(\Lambda)} + 3(\delta_2^\Lambda + \varphi_1^{(\Lambda)})\varphi_3^{(\Lambda)}] G_{\Lambda\Lambda}}{G_{22}(1+\varepsilon_2^{(0)})} - \frac{2[(\delta_2^\Lambda + \varphi_1^{(\Lambda)})\varphi_2^{(\Lambda)}]^2 G_{\Lambda\Lambda}^2}{G_{22}^2(1+\varepsilon_2^{(0)})^3} \right\} (S^2)^2$$
$$+ \frac{1}{2} \left\{ \frac{\vartheta_{11}^{(\Lambda)}\vartheta_{11}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{22}(1+\varepsilon_2^{(0)})} - \frac{[(\delta_2^\Lambda + \varphi_1^{(\Lambda)})\vartheta_{11}^{(\Lambda)}]^2 G_{\Lambda\Lambda}^2}{2G_{22}^2(1+\varepsilon_2^{(0)})^3} \right\} (S^3)^2$$
$$+ \left[\frac{4(\delta_2^\Lambda + \varphi_1^{(\Lambda)})\vartheta_{21}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{22}(1+\varepsilon_2^{(0)})} + \frac{2\varphi_2^{(\Lambda)}\vartheta_{11}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{22}(1+\varepsilon_2^{(0)})} \right] S^2 S^3 + \cdots; \tag{8.264}$$

$$\varepsilon_3 \approx \varepsilon_3^{(0)} + \left.\frac{\partial \varepsilon_3}{\partial S^2}\right|_{(S^2,\,S^3)=(0,\,0)} S^2 + \left.\frac{\partial \varepsilon_3}{\partial S^3}\right|_{(S^2,\,S^3)=(0,\,0)} S^3$$
$$+ \frac{1}{2!} \left.\frac{\partial^2 \varepsilon_3}{\partial (S^2)^2}\right|_{(S^2,\,S^3)=(0,\,0)} (S^2)^2 + \frac{1}{2!} \left.\frac{\partial^2 \varepsilon_3}{\partial (S^3)^2}\right|_{(S^2,\,S^3)=(0,\,0)} (S^3)^2$$
$$+ \left.\frac{\partial^2 \varepsilon_3}{\partial S^2 \partial S^3}\right|_{(S^2,\,S^3)=(0,\,0)} S^2 S^3 + \cdots$$
$$= \varepsilon_3^{(0)} + \frac{(\delta_3^\Lambda + \theta_1^{(\Lambda)})\vartheta_{11}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{33}(1+\varepsilon_3^{(0)})} S^2 + \frac{2(\delta_3^\Lambda + \theta_1^{(\Lambda)})\theta_2^{(\Lambda)} G_{\Lambda\Lambda}}{G_{33}(1+\varepsilon_3^{(0)})} S^3$$
$$+ \frac{1}{2} \left\{ \frac{\vartheta_{11}^{(\Lambda)}\vartheta_{11}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{33}(1+\varepsilon_3^{(0)})} - \frac{[(\delta_2^\Lambda + \theta_1^{(\Lambda)})\vartheta_{11}^{(\Lambda)}]^2 G_{\Lambda\Lambda}^2}{2G_{33}^2(1+\varepsilon_3^{(0)})^3} \right\} (S^2)^2$$
$$+ \left\{ \frac{[2\theta_2^{(\Lambda)}\theta_2^{(\Lambda)} + 3(\delta_3^\Lambda + \theta_1^{(\Lambda)})\theta_3^{(\Lambda)}] G_{\Lambda\Lambda}}{G_{33}(1+\varepsilon_3^{(0)})} - \frac{2[(\delta_3^\Lambda + \theta_1^{(\Lambda)})\theta_2^{(\Lambda)}]^2 G_{\Lambda\Lambda}^2}{(1+\varepsilon_3^{(0)})^3} \right\} (S^3)^2$$
$$+ \left[\frac{4(\delta_3^\Lambda + \theta_1^{(\Lambda)})\vartheta_{12}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{33}(1+\varepsilon_3^{(0)})} + \frac{2\theta_2^{(\Lambda)}\vartheta_{11}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{33}(1+\varepsilon_3^{(0)})} \right] S^2 S^3 + \cdots; \tag{8.265}$$

$$\gamma_{12} \approx \gamma_{12}^{(0)} + \left.\frac{\partial \gamma_{12}}{\partial S^2}\right|_{(S^2,\,S^3)=(0,\,0)} S^2 + \left.\frac{\partial \gamma_{12}}{\partial S^3}\right|_{(S^2,\,S^3)=(0,\,0)} S^3$$
$$+ \frac{1}{2!} \left.\frac{\partial^2 \gamma_{12}}{\partial (S^2)^2}\right|_{(S^2,\,S^3)=(0,\,0)} (S^2)^2 + \frac{1}{2!} \left.\frac{\partial^2 \gamma_{12}}{\partial (S^3)^2}\right|_{(S^2,\,S^3)=(0,\,0)} (S^3)^2$$
$$+ \left.\frac{\partial^2 \gamma_{12}}{\partial S^2 \partial S^3}\right|_{(S^2,\,S^3)=(0,\,0)} S^2 S^3 + \cdots$$
$$= \gamma_{12}^{(0)} + \frac{1}{\cos \gamma_{12}^{(0)}} \left\{ \frac{2(\delta_1^\Lambda + u_{0;1}^{(\Lambda)})\varphi_2^{(\Lambda)} + (\delta_2^\Lambda + \varphi_1^{(\Lambda)})\varphi_{1;1}^{(\Lambda)}}{\sqrt{G_{11}G_{22}}(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})} G_{\Lambda\Lambda} \right.$$

$$
-\sin\gamma_{12}^{(0)}\left[\frac{(\delta_1^{\Lambda}+u_{0;1}^{(\Lambda)})\varphi_{1;1}^{(\Lambda)}}{G_{11}(1+\varepsilon_1^{(0)})^2}G_{\Lambda\Lambda}+\frac{2(\delta_2^{\Lambda}+\varphi_1^{(\Lambda)})\varphi_2^{(\Lambda)}}{G_{22}(1+\varepsilon_2^{(0)})^2}G_{\Lambda\Lambda}\right]\Bigg\}S^2
$$

$$
+\frac{1}{\cos\gamma_{12}^{(0)}}\left\{\frac{(\delta_2^{\Lambda}+\varphi_1^{(\Lambda)})\theta_{1;1}^{(\Lambda)}+(\delta_1^{\Lambda}+u_{0;1}^{(\Lambda)})\vartheta_{11}^{(\alpha)}}{\sqrt{G_{11}G_{22}}(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})}G_{\Lambda\Lambda}\right.
$$

$$
\left.-\sin\gamma_{12}^{(0)}\left[\frac{(\delta_1^{\Lambda}+u_{0;1}^{(\Lambda)})\theta_{1;1}^{(\Lambda)}G_{\Lambda\Lambda}}{G_{11}(1+\varepsilon_1^{(0)})^2}+\frac{(\delta_2^{\Lambda}+\varphi_1^{(\Lambda)})\vartheta_{11}^{(\Lambda)}}{G_{22}(1+\varepsilon_2^{(0)})^2}G_{\Lambda\Lambda}\right]\right\}S^3+\cdots; \quad (8.266)
$$

$$
\gamma_{13}\approx\gamma_{13}^{(0)}+\left.\frac{\partial\gamma_{13}}{\partial S^2}\right|_{(S^2,\,S^3)=(0,\,0)}S^2+\left.\frac{\partial\gamma_{12}}{\partial S^3}\right|_{(S^2,\,S^3)=(0,\,0)}S^3
$$

$$
+\frac{1}{2!}\left.\frac{\partial^2\gamma_{13}}{\partial(S^2)^2}\right|_{(S^2,\,S^3)=(0,\,0)}(S^2)^2+\frac{1}{2!}\left.\frac{\partial^2\gamma_{13}}{\partial(S^3)^2}\right|_{(S^2,\,S^3)=(0,\,0)}(S^3)^2
$$

$$
+\left.\frac{\partial^2\gamma_{13}}{\partial S^2\partial S^3}\right|_{(S^2,\,S^3)=(0,\,0)}S^2S^3+\cdots
$$

$$
=\gamma_{13}^{(0)}+\frac{1}{\cos\gamma_{13}^{(0)}}\left\{\frac{(\delta_3^{\Lambda}+\theta_1^{(\Lambda)})\varphi_{1;1}^{(\Lambda)}+(\delta_1^{\Lambda}+u_{0;1}^{(\Lambda)})\vartheta_{11}^{(\Lambda)}}{\sqrt{G_{11}G_{33}}(1+\varepsilon_1^{(0)})(1+\varepsilon_3^{(0)})}G_{\Lambda\Lambda}\right.
$$

$$
\left.-\sin\gamma_{13}^{(0)}\left[\frac{(\delta_1^{\Lambda}+u_{0;1}^{(\Lambda)})\varphi_{1;1}^{(\Lambda)}}{G_{11}(1+\varepsilon_1^{(0)})^2}+\frac{(\delta_3^{\Lambda}+\theta_1^{(\Lambda)})\vartheta_{11}^{(\Lambda)}}{G_{33}(1+\varepsilon_3^{(0)})^2}\right]G_{\Lambda\Lambda}\right\}S^2
$$

$$
+\frac{1}{\cos\gamma_{13}^{(0)}}\left\{\frac{2(\delta_1^{\Lambda}+u_{0;1}^{(\Lambda)})\theta_2^{(\Lambda)}+(\delta_3^{\Lambda}+\theta_1^{(\Lambda)})\vartheta_{1;1}^{(\Lambda)}}{\sqrt{G_{11}G_{33}}(1+\varepsilon_1^{(0)})(1+\varepsilon_3^{(0)})}G_{\Lambda\Lambda}\right.
$$

$$
\left.-\sin\gamma_{13}^{(0)}\left[\frac{(\delta_1^{\Lambda}+u_{0,1}^{(\Lambda)})\vartheta_{1,1}^{(\Lambda)}}{G_{11}(1+\varepsilon_1^{(0)})^2}+\frac{2(\delta_3^{\Lambda}+\theta_1^{(\Lambda)})\theta_2^{(\Lambda)}}{G_{33}(1+\varepsilon_3^{(0)})^2}\right]G_{\Lambda\Lambda}\right\}S^3+\cdots; \quad (8.267)
$$

$$
\gamma_{23}\approx\gamma_{23}^{(0)}+\left.\frac{\partial\gamma_{23}}{\partial S^2}\right|_{(S^2,\,S^3)=(0,\,0)}S^2+\left.\frac{\partial\gamma_{23}}{\partial S^3}\right|_{(S^2,\,S^3)=(0,\,0)}S^3
$$

$$
+\frac{1}{2!}\left.\frac{\partial^2\gamma_{23}}{\partial(S^2)^2}\right|_{(S^2,\,S^3)=(0,\,0)}(S^2)^2+\frac{1}{2!}\left.\frac{\partial^2\gamma_{23}}{\partial(S^3)^2}\right|_{(S^2,\,S^3)=(0,\,0)}(S^3)^2
$$

$$
+\left.\frac{\partial^2\gamma_{23}}{\partial S^2\partial S^3}\right|_{(S^2,\,S^3)=(0,\,0)}S^2S^3+\cdots
$$

$$
=\gamma_{23}^{(0)}+\frac{1}{\cos\gamma_{23}^{(0)}}\left\{\frac{2(\delta_3^{\Lambda}+\theta_1^{(\Lambda)})\varphi_2^{(\Lambda)}+(\delta_2^{\Lambda}+\varphi_1^{(\Lambda)})\vartheta_{11}^{(\Lambda)}}{\sqrt{G_{22}G_{33}}(1+\varepsilon_2^{(0)})(1+\varepsilon_3^{(0)})}G_{\Lambda\Lambda}\right.
$$

$$
-\sin\gamma_{23}^{(0)}\left[\frac{2(\delta_2^\Lambda+\varphi_1^{(\Lambda)})\varphi_2^{(\Lambda)}}{G_{22}(1+\varepsilon_2^{(0)})^2}+\frac{(\delta_3^\Lambda+\theta_1^{(\Lambda)})\vartheta_{11}^{(\Lambda)}}{G_{33}(1+\varepsilon_3^{(0)})^2}\right]G_{\Lambda\Lambda}\Bigg\}S^2
$$

$$
+\frac{1}{\cos\gamma_{23}^{(0)}}\Bigg\{\frac{2(\delta_2^\Lambda+\varphi_1^{(\Lambda)})\theta_2^{(\Lambda)}+(\delta_3^\Lambda+\theta_1^{(\Lambda)})\vartheta_{11}^{(\Lambda)}}{\sqrt{G_{22}G_{33}}(1+\varepsilon_2^{(0)})(1+\varepsilon_3^{(0)})}G_{\Lambda\Lambda}
$$

$$
-\sin\gamma_{23}^{(0)}\left[\frac{(\delta_2^\Lambda+\varphi_1^{(\Lambda)})\vartheta_{11}^{(\Lambda)}}{G_{22}(1+\varepsilon_2^{(0)})^2}+\frac{2(\delta_3^\Lambda+\theta_1^{(\Lambda)})\theta_2^{(\Lambda)}}{G_{33}(1+\varepsilon_3^{(0)})^2}\right]G_{\Lambda\Lambda}\Bigg\}S^3+\cdots, \tag{8.268}
$$

where

$$
\begin{aligned}
\varepsilon_1^{(0)}&=\frac{\sqrt{G_{\Lambda\Lambda}}}{\sqrt{G_{11}}}\sqrt{(\delta_1^\Lambda+u_{0;1}^\Lambda)(\delta_1^\Lambda+u_{0;1}^\Lambda)}-1,\\
\varepsilon_2^{(0)}&=\frac{\sqrt{G_{\Lambda\Lambda}}}{\sqrt{G_{22}}}\sqrt{(\delta_2^\Lambda+\varphi_1^{(\Lambda)})(\delta_2^\Lambda+\varphi_1^{(\Lambda)})}-1,\\
\varepsilon_3^{(0)}&=\frac{\sqrt{G_{\Lambda\Lambda}}}{\sqrt{G_{33}}}\sqrt{(\delta_3^\Lambda+\theta_1^{(\Lambda)})(\delta_3^\Lambda+\theta_1^{(\Lambda)})}-1,\\
\gamma_{12}^{(0)}&=\sin^{-1}\frac{(\delta_1^\Lambda+u_{0;1}^\Lambda)(\delta_2^\Lambda+\varphi_1^{(\Lambda)})G_{\Lambda\Lambda}}{\sqrt{G_{11}G_{22}}(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})},\\
\gamma_{13}^{(0)}&=\sin^{-1}\frac{(\delta_1^\Lambda+u_{0;1}^\Lambda)(\delta_3^\Lambda+\theta_1^{(\Lambda)})G_{\Lambda\Lambda}}{\sqrt{G_{11}G_{33}}(1+\varepsilon_1^{(0)})(1+\varepsilon_3^{(0)})},\\
\gamma_{23}^{(0)}&=\sin^{-1}\frac{(\delta_2^\Lambda+\varphi_1^{(\Lambda)})(\delta_3^\Lambda+\theta_1^{(\Lambda)})G_{\Lambda\Lambda}}{\sqrt{G_{22}G_{33}}(1+\varepsilon_2^{(0)})(1+\varepsilon_3^{(0)})}.
\end{aligned}
\tag{8.269}
$$

From Assumptions (i) and (ii), the zero order term of the Taylor series of the five strains gives

$$
\varepsilon_2^{(0)}=0,\ \varepsilon_3^{(0)}=0;\ \gamma_{12}^{(0)}=0,\ \gamma_{13}^{(0)}=0,\ \gamma_{23}^{(0)}=0. \tag{8.270}
$$

The stretch of the deformed rod for $S^2=S^3=0$ satisfies

$$
(1+\varepsilon_1^{(0)})^2=\frac{G_{\Lambda\Lambda}}{G_{11}}(\delta_1^\Lambda+u_{0;1}^\Lambda)(\delta_1^\Lambda+u_{0;1}^\Lambda), \tag{8.271}
$$

From Eqs.(8.269)-(8.271),

$$
\begin{aligned}
\frac{G_{\Lambda\Lambda}}{G_{11}(1+\varepsilon_1^{(0)})^2}(\delta_1^\Lambda+u_{0;1}^\Lambda)(\delta_1^\Lambda+u_{0;1}^\Lambda)&=1,\\
\frac{G_{\Lambda\Lambda}}{G_{22}}(\delta_1^\Lambda+\varphi_1^{(\Lambda)})(\delta_2^\Lambda+\varphi_1^{(\Lambda)})&=1,
\end{aligned}
$$

$$\frac{G_{\Lambda\Lambda}}{G_{33}}(\delta_3^\Lambda + \theta_1^{(\Lambda)})(\delta_3^\Lambda + \theta_1^{(\Lambda)}) = 1;$$
$$(\delta_1^\Lambda + u_{0;1}^\Lambda)(\delta_2^\Lambda + \varphi_1^{(\Lambda)})G_{\Lambda\Lambda} = 0,$$
$$(\delta_1^\Lambda + u_{0;1}^\Lambda)(\delta_3^\Lambda + \theta_1^{(\Lambda)})G_{\Lambda\Lambda} = 0, \tag{8.272}$$
$$(\delta_2^\Lambda + \varphi_1^{(\Lambda)})(\delta_3^\Lambda + \theta_1^{(\Lambda)})G_{\Lambda\Lambda} = 0.$$

Using the first order terms of S^2 and S^3 in Eq.(8.115), the direction cosine matrix $((l_{ij})_{3\times 3})$ is

$$\cos\theta_{(\mathbf{n}_1,\mathbf{N}_\Lambda)} = l_{1\Lambda} = \frac{(\delta_1^\Lambda + u_{0;1}^\Lambda)\sqrt{G_{\Lambda\Lambda}}}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})},$$
$$\cos\theta_{(\mathbf{n}_2,\mathbf{N}_\Lambda)} = l_{2\Lambda} = \frac{(\delta_2^\Lambda + \varphi_1^{(\Lambda)})\sqrt{G_{\Lambda\Lambda}}}{\sqrt{G_{22}}(1+\varepsilon_2^{(0)})}, \tag{8.273}$$
$$\cos\theta_{(\mathbf{n}_3,\mathbf{N}_\Lambda)} = l_{3\Lambda} = \frac{(\delta_3^\Lambda + \theta_1^{(\Lambda)})\sqrt{G_{\Lambda\Lambda}}}{\sqrt{G_{33}}(1+\varepsilon_3^{(0)})}.$$

As in Fig.8.5, the unknowns $\varphi_1^{(I)}$ and $\theta_1^{(I)}$ ($I = 1, 2, 3$) can be determined by the three Euler angles (Φ, Ψ and Θ). Similarly, the Euler angles Φ and Ψ rotates around the axes of S^2 and S^3, respectively, and the Euler angle Θ rotates around the axis of S^1. Due to bending, the first rotation around the axis of S^2 is to form $(\bar{S}^1, \bar{S}^2, \bar{S}^3)$. The second rotation around the axis of $\bar{S}^3$ gives $(\bar{\bar{S}}^1, \bar{\bar{S}}^2, \bar{\bar{S}}^3)$. The last rotation around the axis of $\bar{\bar{S}}^1$ gives a frame of $(\bar{\bar{\bar{S}}}^1, \bar{\bar{\bar{S}}}^2, \bar{\bar{\bar{S}}}^3)$ for the final state of the rod, which is the coordinates (s^1, s^2, s^3). The direction-cosines give

$$\frac{1+u_{0;1}^1}{(1+\varepsilon_1^{(0)})} = \cos\Phi\cos\Psi,$$
$$\frac{\varphi_1^{(1)}\sqrt{G_{11}}}{\sqrt{G_{22}}} = -\cos\Phi\sin\Psi\cos\Theta + \sin\Phi\sin\Theta,$$
$$\frac{\theta_1^{(1)}\sqrt{G_{11}}}{\sqrt{G_{33}}} = \cos\Phi\sin\Psi\sin\Theta + \sin\Phi\cos\Theta; \tag{8.247a}$$
$$\frac{u_{0;1}^2\sqrt{G_{22}}}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})} = \sin\Psi,$$
$$\varphi_1^{(2)} = \cos\Psi\cos\Theta - 1,$$
$$\frac{\theta_1^{(2)}\sqrt{G_{22}}}{\sqrt{G_{33}}} = -\cos\Psi\sin\Theta;$$

$$
\begin{aligned}
\frac{u^3_{0;1}\sqrt{G_{33}}}{\sqrt{G_{11}}\,(1+\varepsilon^{(0)}_1)} &= -\sin\Phi\cos\Psi, \\
\frac{\varphi^{(3)}_1\sqrt{G_{33}}}{\sqrt{G_{22}}} &= \sin\Phi\sin\Psi\cos\Theta+\cos\Phi\sin\Theta, \\
\theta^{(3)}_1 &= -\sin\Phi\sin\Psi\sin\Theta+\cos\Phi\cos\Theta-1.
\end{aligned}
\tag{8.274b}
$$

The first, fourth and seventh equations in Eq.(8.274) yield

$$
\begin{aligned}
\cos\Psi &= \pm\frac{\sqrt{(1+u^1_{0;1})^2G_{11}+(u^3_{0;1})^2G_{33}}}{\sqrt{G_{11}}\,(1+\varepsilon^{(0)}_1)} = \pm\frac{\Delta}{\sqrt{G_{11}}\,(1+\varepsilon^{(0)}_1)}, \\
\sin\Psi &= \frac{u^2_{0;1}\sqrt{G_{22}}}{\sqrt{G_{11}}\,(1+\varepsilon^{(0)}_1)}; \\
\cos\Phi &= \pm\frac{(1+u^1_{0;1})\sqrt{G_{11}}}{\sqrt{(1+u^1_{0,1})^2G_{11}+(u^3_{0,1})^2G_{33}}} = \pm\frac{(1+u^1_{0;1})\sqrt{G_{11}}}{\Delta}, \\
\sin\Phi &= \mp\frac{u^3_{0;1}\sqrt{G_{33}}}{\sqrt{(1+u^1_{0;1})^2G_{11}+(u^3_{0;1})^2G_{33}}} = \mp\frac{u^3_{0;1}\sqrt{G_{33}}}{\Delta},
\end{aligned}
\tag{8.275}
$$

and

$$
\begin{aligned}
\frac{\sqrt{G_{11}}}{\sqrt{G_{22}}}\varphi^{(1)}_1 &= \mp\frac{1}{\Delta}\left[\frac{\Delta_{12}}{\sqrt{G_{11}}\,(1+\varepsilon^{(0)}_1)}\cos\Theta+u^3_{0;1}\sqrt{G_{33}}\sin\Theta\right], \\
\frac{\sqrt{G_{11}}}{\sqrt{G_{33}}}\theta^{(1)}_1 &= \pm\frac{1}{\Delta}\left[\frac{\Delta_{12}}{\sqrt{G_{11}}\,(1+\varepsilon^{(0)}_1)}\sin\Theta-u^3_{0;1}\sqrt{G_{33}}\cos\Theta\right], \\
\varphi^{(2)}_1 &= \pm\frac{\Delta}{\sqrt{G_{11}}\,(1+\varepsilon^{(0)}_1)}\cos\Theta-1; \\
\frac{\sqrt{G_{22}}}{\sqrt{G_{33}}}\theta^{(2)}_1 &= \mp\frac{\Delta}{\sqrt{G_{11}}\,(1+\varepsilon^{(0)}_1)}\sin\Theta, \\
\frac{\sqrt{G_{33}}}{\sqrt{G_{22}}}\varphi^{(3)}_1 &= \mp\frac{1}{\Delta}\left[\frac{\Delta_{23}}{\sqrt{G_{11}}\,(1+\varepsilon^{(0)}_1)}\cos\Theta-(1+u^1_{0;1})\sqrt{G_{11}}\sin\Theta\right], \\
\theta^{(3)}_1 &= \pm\frac{1}{\Delta}\left[\frac{\Delta_{23}}{\sqrt{G_{11}}\,(1+\varepsilon^{(0)}_1)}\sin\Theta+(1+u^1_{0;1})\sqrt{G_{11}}\cos\Theta\right]-1,
\end{aligned}
\tag{8.276}
$$

where

$$\Delta = \sqrt{(1+u_{0;1}^{1})^2 G_{11} + (u_{0;1}^{3})^2 G_{33}},$$
$$\Delta_{12} = (u_{0;1}^{2})(1+u_{0;1}^{1})\sqrt{G_{11}G_{22}}, \tag{8.277}$$
$$\Delta_{23} = u_{0;1}^{3}u_{0;1}^{2}\sqrt{G_{22}G_{33}}.$$

If $\Theta = 0$, the rotation about the longitudinal axis disappears. So only the bending rotation exists, which is the pure bending of the rod as in Section 8.3.2.

From Eq.(8.165), the change ratio of the directional cosines along the deformed rod in Eq.(8.166) becomes

$$\frac{d\boldsymbol{l}_{\alpha}}{ds} = \frac{d(l_{\alpha\Lambda}\mathbf{G}_{\Lambda})}{ds} = \frac{l_{\alpha\Lambda;1}}{\sqrt{\mathrm{G}_{11}}(1+\varepsilon_1^{(0)})}\mathbf{G}_{\Lambda} \quad \text{for } \alpha=1, 2, 3. \tag{8.278}$$

Similar to Eq.(8.168), the change ratio of the directional cosines along the deformed rod can be computed by

$$\frac{d\boldsymbol{l}_{\alpha}}{ds} = \varepsilon_{\Lambda\Gamma\mathrm{K}}\,\omega_{\Gamma} l_{\alpha\mathrm{K}}\mathbf{G}_{\Lambda}, \tag{8.279}$$

where the rotation ratio components in Eq.(8.169) are computed by

$$\omega_{\Gamma} = \varepsilon_{\Lambda\Gamma\mathrm{K}}\frac{dl_{\alpha\Lambda;1}}{\sqrt{\mathrm{G}_{11}}(1+\varepsilon_1^{(0)})}l_{\alpha\mathrm{K}}. \tag{8.280}$$

From Eq.(8.161), the foregoing equations give

$$\omega_1 = \frac{1}{\sqrt{\mathrm{G}_{11}}(1+\varepsilon_1^{(0)})}\left[(\frac{d\Psi}{dS}\sin\Phi + \frac{d\Theta}{dS}\cos\Psi\cos\Phi) - \Gamma_{13}^{2}\right],$$
$$\omega_2 = \frac{1}{\sqrt{\mathrm{G}_{11}}(1+\varepsilon_1^{(0)})}\left[(\frac{d\Psi}{dS} + \frac{d\Theta}{dS}\sin\Psi) - \Gamma_{11}^{3}\right], \tag{8.281}$$
$$\omega_3 = \frac{1}{\sqrt{\mathrm{G}_{11}}(1+\varepsilon_1^{(0)})}\left[(\frac{d\Psi}{dS}\cos\Phi - \frac{d\Theta}{dS}\sin\Phi\cos\Psi) - \Gamma_{12}^{1}\right];$$

similar to Eq.(8.171). Note that for the orthogonal curvilinear coordinates, one has $\Gamma_{\Lambda\Gamma}^{\mathrm{K}} = 0$ ($\Lambda \neq \Gamma \neq \mathrm{K} \neq \Lambda$), $\Gamma_{\Lambda\Lambda}^{\Gamma} = -G_{\Lambda\Lambda,\Gamma}/2G_{\Gamma\Gamma}$ ($\Lambda \neq \Gamma$) and $\Gamma_{\Lambda\Gamma}^{\Lambda} = G_{\Lambda\Lambda,\Gamma}/2G_{\Lambda\Lambda}$ (no summation on Λ).

In an alike fashion, using Eq.(8.25), the particle location on the deformed, curved rod can be expressed by

$$\mathbf{r} = (S^{\Lambda} + u^{\Lambda})\mathbf{G}_{\Lambda}. \tag{8.282}$$

The corresponding base vector of the deformed, curved rod is

$$\tilde{\mathbf{g}}_1 = \tilde{g}_1^{\Lambda}\mathbf{G}_{\Lambda} = (\delta_1^{\Lambda} + u_{;1}^{\Lambda})\mathbf{G}_{\Lambda}, \tag{8.283}$$

and the unit vector for the deformed, curved rod is

$$\tilde{\mathbf{n}}_1 = \frac{\tilde{\mathbf{g}}_1}{|\tilde{\mathbf{g}}_1|} = \frac{\delta_1^\Lambda + u_{;1}^\Lambda}{\sqrt{(\delta_1^\Gamma + u_{;1}^\Gamma)(\delta_1^\Gamma + u_{;1}^\Gamma)}}\mathbf{G}_\Lambda = \frac{\delta_1^\Lambda + u_{;1}^\Lambda}{\sqrt{G_{11}}(1+\varepsilon_1)}\mathbf{G}_\Lambda. \tag{8.284}$$

For $S^2 = S^3 = 0$,

$$\tilde{\mathbf{n}}_1 = \frac{\tilde{\mathbf{g}}_1}{|\tilde{\mathbf{g}}_1|} = \frac{\delta_1^\Lambda + u_{0;1}^\Lambda}{\sqrt{(\delta_1^\Gamma + u_{0;1}^\Gamma)(\delta_1^\Gamma + u_{0;1}^\Gamma)}}\mathbf{G}_\Lambda = \frac{\delta_1^\Lambda + u_{0;1}^\Lambda}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})}\mathbf{G}_\Lambda. \tag{8.285}$$

The base vector in the principal normal direction of the deformed, curved rod is as in Eq.(8.177), i.e.,

$$\tilde{\mathbf{g}}_2 = \frac{d\tilde{\mathbf{n}}_1}{ds} = \tilde{g}_2^\Lambda \mathbf{G}_\Lambda, \tag{8.286}$$

where

$$\tilde{g}_2^\Lambda = \frac{1}{\sqrt{G_{11}}(1+\varepsilon_1)}\left[\frac{\delta_1^\Lambda + u_{;1}^\Lambda}{\sqrt{G_{11}}(1+\varepsilon_1)}\right]_{;1}. \tag{8.287}$$

Thus, the unit principal normal vector in Eq.(8.179) can be rewritten, i.e.,

$$\tilde{\mathbf{n}}_2 = \frac{\tilde{\mathbf{g}}_2}{|\tilde{\mathbf{g}}_2|} = \frac{\tilde{\mathbf{g}}_2}{\sqrt{\tilde{g}_{22}}} = \frac{\tilde{\mathbf{g}}_2}{\tilde{\kappa}(S)} = \frac{\tilde{g}_2^\Lambda}{\tilde{\kappa}(S)}\mathbf{G}_\Lambda, \tag{8.288}$$

where the curvature of the deformed rod is

$$\begin{aligned} &\tilde{\kappa}(S) = |\mathbf{g}_2| = \sqrt{\tilde{g}_{22}} = \sqrt{g_2^\Lambda g_2^\Lambda}, \\ &\tilde{g}_{22} = \frac{1}{G_{11}(1+\varepsilon_1)^2}\left[\frac{\delta_1^\Lambda + u_{;1}^\Lambda}{\sqrt{G_{11}}(1+\varepsilon_1)}\right]_{;1}\left[\frac{\delta_1^\Lambda + u_{;1}^\Lambda}{\sqrt{G_{11}}(1+\varepsilon_1)}\right]_{;1}. \end{aligned} \tag{8.289}$$

For $S^2 = S^3 = 0$,

$$\tilde{g}_2^\Lambda = \frac{1}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})}\left[\frac{\delta_1^\Lambda + u_{0;1}^\Lambda}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})}\right]_{;1}, \tag{8.290}$$

$$\tilde{g}_{22} = \frac{1}{G_{11}(1+\varepsilon_1^{(0)})^2}\left[\frac{\delta_1^\Lambda + u_{0;1}^\Lambda}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})}\right]_{;1}\left[\frac{\delta_1^\Lambda + u_{0;1}^\Lambda}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})}\right]_{;1}. \tag{8.291}$$

The rotation vector of the deformed curved rod can be expressed as in Eq.(8.186), i.e.,

$$\boldsymbol{\omega} = \tilde{\kappa}(S)\mathbf{n}_3 + \tilde{\tau}(S)\mathbf{n}_1 = \omega_\Lambda \mathbf{N}_\Lambda \tag{8.292}$$

where the torsion of the deformed rod is

$$\tilde{\tau}(S) = e_{IJK} \frac{(\delta_1^\Lambda + u_{;1}^\Lambda)}{\sqrt{G_{11}}(1+\varepsilon_1)} \frac{\tilde{g}_2^\Gamma}{\tilde{\kappa}(S)} \left[\frac{\tilde{g}_2^{\mathrm{K}}}{\tilde{\kappa}(S)} \right]_{;1} G_\Lambda^I G_\Gamma^J G_{\mathrm{K}}^K. \tag{8.293}$$

For $S^2 = S^3 = 0$, the foregoing equation becomes

$$\tilde{\tau}(S) = e_{IJK} \frac{(\delta_1^\Lambda + u_{0;1}^\Lambda)}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})} \frac{\tilde{g}_2^\Gamma}{\tilde{\kappa}(S)} \left[\frac{\tilde{g}_2^{\mathrm{K}}}{\tilde{\kappa}(S)} \right]_{;1} G_\Lambda^I G_\Gamma^J G_{\mathrm{K}}^K. \tag{8.294}$$

From Eq.(8.292),

$$\begin{aligned} \tilde{\kappa}(S) &= \omega_\Lambda \mathbf{N}_\Lambda \cdot \tilde{\mathbf{n}}_3 = \omega_\Lambda \tilde{g}_3^\Lambda, \\ \tilde{\tau}(S) &= \omega_\Lambda \mathbf{N}_\Lambda \cdot \tilde{\mathbf{n}}_1 = \omega_\Lambda \tilde{g}_2^\Lambda, \end{aligned} \tag{8.295}$$

or

$$\begin{aligned} \omega_\Lambda &= \tilde{\kappa}(S)\mathbf{n}_3 \cdot \mathbf{N}_\Lambda + \tilde{\tau}(S)\mathbf{n}_1 \cdot \mathbf{N}_\Lambda \\ &= \tilde{\kappa}(S)\tilde{g}_3^\Lambda + \tilde{\tau}(S)\tilde{g}_1^\Lambda. \end{aligned} \tag{8.296}$$

From Eq.(8.275), $d\Psi / dS$ and $d\Phi / dS$ are determined by

$$\begin{aligned} \frac{d\Psi}{dS} &= \pm \frac{1}{G_{11}(1+\varepsilon_1^{(0)})^2 \Delta} \left\{ \Delta^2 \frac{d}{dS}(u_{0;1}^2 \sqrt{G_{22}}) \right. \\ &\quad \left. - \Delta_{12} \frac{d}{dS}\left[(1+u_{0;1}^1)\sqrt{G_{11}} \right] - \Delta_{23} \frac{d}{dS}(u_{0;1}^3 \sqrt{G_{33}}) \right\}, \\ \frac{d\Phi}{dS} &= -\frac{1}{\Delta^2} \left\{ (1+u_{0;1}^1)\sqrt{G_{11}} \frac{d}{dS}(u_{0;1}^3 \sqrt{G_{33}}) \right. \\ &\quad \left. - (u_{0;1}^3 \sqrt{G_{33}}) \frac{d}{dS}\left[(1+u_{0;1}^1)\sqrt{G_{11}} \right] \right\}. \end{aligned} \tag{8.297}$$

Substitution of Eqs.(8.275) and (8.297) into Eqs.(8.281) and (8.295) gives Θ and $d\Theta / dS$.

As in Eq.(8.193), the constitutive laws for deformed rods give the corresponding resultant stresses. Similar to Eq.(8.194), the internal forces and moments in the deformed rod are defined by

$$\begin{aligned} N_1 &= \int_A \sigma_{11} \left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23} \right] dA, \\ Q_2 &= \int_A \sigma_{12} \left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23} \right] dA, \\ Q_3 &= \int_A \sigma_{13} \left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23} \right] dA, \\ M_3 &= \int_A \sigma_{11} \frac{S^2}{1+\varphi_1^{(2)}} \left[(1+\varepsilon_3)(1+\varepsilon_2)^2 \cos\gamma_{23} \right] dA, \end{aligned} \tag{8.298a}$$

$$M_2 = -\int_A \sigma_{11} \frac{S^3}{1+\theta_1^{(3)}} \left[(1+\varepsilon_3)^2(1+\varepsilon_2)\cos\gamma_{23}\right] dA,$$

$$T_1 = \int_A \left[\sigma_{12} \frac{S^3(1+\varepsilon_3)}{1+\theta_1^{(3)}} - \sigma_{13} \frac{S^2(1+\varepsilon_2)}{1+\varphi_1^{(2)}} \right] \times\left[(1+\varepsilon_3)(1+\varepsilon_2)\cos\gamma_{23}\right] dA. \tag{8.298b}$$

The notations $Q_2 \equiv N_2$, $Q_3 \equiv N_3$, and $T_1 \equiv M_1$ are used again. The internal forces are expressed as in Eq.(8.195), i.e.,

$$\begin{aligned} \mathbf{M} &\equiv M^\Lambda \mathbf{N}_\Lambda = M_\alpha \mathbf{n}_\alpha, \\ \boldsymbol{\mathcal{N}} &\equiv N^\Lambda \mathbf{N}_\Lambda = N_\alpha \mathbf{n}_\alpha, \\ {}^{\mathbf{N}}\mathbf{M} &\equiv {}^{\mathbf{N}}M^\Lambda \mathbf{N}_\Lambda = \mathbf{g}_1 \times \boldsymbol{\mathcal{N}}, \end{aligned} \tag{8.299}$$

where

$$\mathbf{g}_1 \equiv \frac{d\mathbf{r}}{dS} = (\delta_1^\Lambda + u_{0;1}^\Lambda) G_{\Lambda\Lambda} \mathbf{N}_\Lambda \text{ and } {}^{\mathbf{N}}\mathbf{M} \equiv \frac{1}{dS} d\mathbf{r} \times \boldsymbol{\mathcal{N}}. \tag{8.300}$$

With Eq.(8.273), the components of the internal forces in the $\mathbf{G}_\Lambda$-direction are

$$\begin{aligned} N^\Lambda &= N_\alpha \mathbf{n}_\alpha \cdot \mathbf{N}_\Lambda = N_\alpha \cos\theta_{(\mathbf{n}_\alpha, \mathbf{N}_\Lambda)} \\ &= \left[\frac{N_1(\delta_1^\Lambda + u_{0;1}^\Lambda)}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})} + \frac{N_2(\delta_2^\Lambda + \varphi_1^{(\Lambda)})}{\sqrt{G_{22}}(1+\varepsilon_2^{(0)})} + \frac{N_3(\delta_3^\Lambda + \theta_1^{(\Lambda)})}{\sqrt{G_{33}}(1+\varepsilon_3^{(0)})} \right] \sqrt{G_{\Lambda\Lambda}}, \end{aligned} \tag{8.301}$$

$$\begin{aligned} M^\Lambda &= M_\alpha \mathbf{n}_\alpha \cdot \mathbf{N}_\Lambda = M_\alpha \cos\theta_{(\mathbf{n}_\alpha, \mathbf{N}_\Lambda)} \\ &= \left[\frac{M_1(\delta_1^\Lambda + u_{0;1}^\Lambda)}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})} + \frac{M_2(\delta_2^\Lambda + \varphi_1^{(\Lambda)})}{\sqrt{G_{22}}(1+\varepsilon_2^{(0)})} + \frac{M_3(\delta_3^\Lambda + \theta_1^{(\Lambda)})}{\sqrt{G_{33}}(1+\varepsilon_3^{(0)})} \right] \sqrt{G_{\Lambda\Lambda}}, \end{aligned} \tag{8.302}$$

$${}^{\mathbf{N}}M^\Lambda = (\mathbf{g}_1 \times \boldsymbol{\mathcal{N}}) \cdot \mathbf{N}_\Lambda = e_{\Lambda\Gamma\mathrm{K}} (\delta_1^\Gamma + u_{0;1}^\Gamma) \sqrt{G_{\Gamma\Gamma}} N^{\mathrm{K}}.$$

Using the external forces as in Eqs.(8.209)–(8.213), equations of motion on the deformed rod are given as in Eq.(8.267), i.e.,

$$\begin{aligned} \boldsymbol{\mathcal{N}}_{,S} + \mathbf{q} &= \rho \mathbf{u}_{0,tt} + I_3 \boldsymbol{\varphi}_{1,tt} + I_2 \boldsymbol{\theta}_{1,tt}, \\ \mathbf{M}_{,S} + {}^{\mathbf{N}}\mathbf{M} + \mathbf{m} &= \mathscr{J}\boldsymbol{\Theta}_{,tt}, \end{aligned} \tag{8.303}$$

where

$$\begin{aligned} \mathscr{J}\boldsymbol{\Theta}_{,tt} = {} & [(I_3 u_{0,tt}^3 - I_2 u_{0,tt}^2) + (J_{22}\varphi_{1,tt}^{(3)} - J_{23}\varphi_{1,tt}^{(2)}) \\ & + (J_{23}\theta_{1,tt}^{(3)} - J_{33}\theta_{1,tt}^{(2)})]\mathbf{N}_1 + (I_3 u_{0,tt}^1 + J_{23}\varphi_{1,tt}^{(1)} + J_{22}\theta_{1,tt}^{(1)})\mathbf{N}_2 \\ & - (I_2 u_{0,tt}^1 + J_{22}\varphi_{1,tt}^{(1)} + J_{23}\theta_{1,tt}^{(1)})\mathbf{N}_3, \end{aligned} \tag{8.304}$$

and the scalar expressions are for $\Lambda = 1, 2, 3$,

$$\begin{aligned} &N^{\Lambda}_{;1} + q^{\Lambda} = \rho u^{\Lambda}_{(0),tt} + I_3 \varphi^{(\Lambda)}_{1,tt} + I_2 \theta^{(\Lambda)}_{1,tt}, \\ &M^{\Lambda}_{;1} + {}^{\mathbf{N}}M^{\Lambda} + m^{\Lambda} = (\mathscr{J}\boldsymbol{\Theta}_{,tt}) \cdot \mathbf{N}_{\Lambda}, \end{aligned} \tag{8.305}$$

or

$$\begin{aligned} &\left\{ \left[\frac{N_1(\delta^{\Lambda}_1 + u^{\Lambda}_{0;1})}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)} + \frac{N_2(\delta^{\Lambda}_2 + \varphi^{(\Lambda)}_1)}{\sqrt{G_{22}}(1+\varepsilon^{(0)}_2)} + \frac{N_3(\delta^{\Lambda}_3 + \theta^{(\Lambda)}_1)}{\sqrt{G_{33}}(1+\varepsilon^{(0)}_3)} \right] \sqrt{G_{\Lambda\Lambda}} \right\}_{;1} \\ &+ q^{\Lambda} = \rho u^{\Lambda}_{0,tt} + I_3 \varphi^{(\Lambda)}_{1,tt} + I_2 \theta^{(\Lambda)}_{1,tt}, \\ &\left\{ \left[\frac{M_1(\delta^{\Lambda}_1 + u^{\Lambda}_{0;1})}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)} + \frac{M_2(\delta^{\Lambda}_2 + \varphi^{(\Lambda)}_1)}{\sqrt{G_{22}}(1+\varepsilon^{(0)}_2)} + \frac{M_3(\delta^{\Lambda}_3 + \theta^{(\Lambda)}_1)}{\sqrt{G_{33}}(1+\varepsilon^{(0)}_3)} \right] \sqrt{G_{\Lambda\Lambda}} \right\}_{;1} \\ &+ e_{\Lambda\Gamma K}(\delta^{\Gamma}_1 + u^{\Gamma}_{0;1})\sqrt{G_{\Gamma\Gamma}} N^{K} + m^{\Lambda} = (\mathscr{J}\boldsymbol{\Theta}_{,tt}) \cdot \mathbf{N}_{\Lambda}. \end{aligned} \tag{8.306}$$

The force condition at a point $\mathscr{P}_k$ with $S^1 = S^1_k$ is

$$\begin{aligned} &{}^{-}\mathbf{N}(S^1_k) + {}^{+}\mathbf{N}(S^1_k) + \mathbf{F}_k = 0, \\ &{}^{-}N^{\Lambda}(S^1_k) = {}^{+}N^{\Lambda}(S^1_k) + F^{\Lambda}_k \quad (\Lambda = 1, 2, 3). \end{aligned} \tag{8.307}$$

The force boundary condition at the boundary point $\mathscr{P}_r$ is

$$\mathbf{N}(S^1_r) + \mathbf{F}_r = 0 \;\text{ or }\; N^{\Lambda}(S^1_r) + F^{\Lambda}_r = 0 \quad (\Lambda = 1, 2, 3). \tag{8.308}$$

If there is a concentrated moment at a point $\mathscr{P}_k$ with $S^1 = S^1_k$, the corresponding moment boundary condition is given by

$$\begin{aligned} &{}^{-}\mathbf{M}(S^1_k) + {}^{+}\mathbf{M}(S^1_k) + \mathbf{M}_k = 0, \\ &{}^{-}M^{\Lambda}(S^1_k) = {}^{+}M^{\Lambda}(S^1_k) + M^{\Lambda}_k \quad (\Lambda = 1, 2, 3). \end{aligned} \tag{8.309}$$

The moment boundary condition at the boundary point $\mathscr{P}_r$ is

$$\mathbf{M}(S^1_r) + \mathbf{M}_r = 0 \;\text{ or }\; M^{\Lambda}(S^1_r) + M^{\Lambda}_r = 0 \quad (\Lambda = 1, 2, 3). \tag{8.310}$$

The displacement continuity and boundary conditions are similar to Eq. (8.207), i.e., $u^{\Lambda}_{k-} = u^{\Lambda}_{k+}$ and $u^{\Lambda}_r = B^{\Lambda}_r$ ($\Lambda = 1, 2, 3$).

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