NONLINEAR PHYSICAL SCIENCE

Albert C.J. Luo

Nonlinear Deformable-body Dynamics

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Preface

Deformable-body dynamics is a subject to investigate the states of strains and internal relative motions in deformable solids subject to the action of external forces. This is an old and interesting topic, and many problems still are unsolved or solved incompletely. Rethinking such problems in this topic may bring new vitality to the modern science and technology. The first consideration of the nature of the resistance of deformable-bodies to rupture was given by Galileo in 1638. The theory of deformable-bodies, started from Galileo's problem, is based on the discovery of Hooke's Law in 1660 and the general differential equations of elasticity by Navier in 1821. The Hooke's law is an experimental discovery about the stress and strain relation. This law provides the basis to develop the mathematical theory of deformable bodies. In 1821, Navier was the first to investigate the general equations of equilibrium and vibration of elastic solids. In 1850, Kirchhoff proposed two assumptions: (i) that linear filaments of the plate initially normal to the middle-surface remain straight and normal to the middle-surface after deformed, and (ii) that all fibers in middle surface remain unstretched. Based on the Kirchhoff assumptions, the approximate theories for beams, rods, plates and shells have been developed for recent 150 years. From the theory of 3-dimensional deformable body, with certain assumptions, this book will present a mathematical treatise of such approximate theories for thin deformable-bodies including cables, beams, rods, webs, membranes, plates and shells. The nonlinear theory for deformable body based on the Kirchhoff assumptions is a special case to be discussed. This book consists of eight chapters. Chapter 1 discusses the history of the deformable body dynamics. Chapter 2 presents the mathematical tool for the deformation and kinematics of deformable-bodies. Chapter 3 addresses the deformation geometry, kinematics and dynamics of deformable body. Chapter 4 discusses constitutive laws and damage theory for deformable-bodies. In Chapter 5, nonlinear dynamics of cables is addressed. Chapter 6 discusses nonlinear plates and waves, and the nonlinear theories for webs, membranes and shells are presented in Chapter 7. Finally, Chapter 8 presents the nonlinear theory for thin beams and rods.

The purpose to write this book is to answer a question from Professor Huancun Sun (my thesis advisor) during my master thesis defense in 1990. In my master thesis, I considered the higher order terms to correct the strains in the von Karman plate theory. However, such a correction did not consider curvature effects on the balance equations. Professor Sun asked me what is the error compared to the exact theory of the 3-dimensional deformable bodies. After about 20 years, I believe that I can give an appropriate answer to his question. In fact, after my thesis defense, I almost place such a question away. However, in 1996, I worked with Professor C.D. Mote, Jr. at UC Berkeley on nonlinear dynamical behaviors of high speed rotating disks in disk drives. Such a problem drove me to rethink about the accurate plate theory. To express my indebtedness to both of them for their guidance and advice, this book is my gift for the 80^{th} birthday of Professor Sun and the 70^{th} birthday of Professor Mote, Jr.. This book is also dedicated to my friend and colleague, Professor Zhongheng Guo. His book on *Nonlinear Elasticity* stimulated my research interest in nonlinear deformable solids 25 years ago. His book was an excellent book for graduate students. Some inspirations of this book originated from the book of *Nonlinear Elasticity*. In addition, I would like to thank Professor Youjin Che for lending his books "*Tensor Analysis*" and "*Variational Principles*" to me during my sophomore year in 1981. After almost 30 years, I cannot find both of books to return to him. I sincerely hope this book can bring my apology and appreciation to him.

Herein, I would like to thank my wife (Sherry X. Huang) and my children (Yanyi Luo, Robin Ruo-Bing Luo, and Robert Zong-Yuan Luo) for their tolerance, patience, understanding and support.

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Contents

Chapter 1
Introduction

To investigate deformable-body dynamics, it is very important to learn a development history of the mathematical theory of deformable solids. From such a development history, one can find how the deformable-body dynamics to stimulate the development of modern physical science, which will give people a kind of indication for new discoveries. In this chapter, a brief history for establishing the approximate theories of deformable solids will be given. Especially, the cable dynamics will be discussed first, and a mathematical treatise of nonlinear beams and rods will be presented. In addition, the past and current status of plates and shell theory will be discussed, and the current status of soft web theory and applications will be presented. Finally, the book layout will be presented, and a brief summarization for each chapter will be given.

1.1. Deformable-body dynamics

Deformable-body dynamics is a subject to investigate the states of strain and internal relative motions in deformable solids subject to the action of the external forces. The first consideration of the nature of the resistance of solids to rupture was given by Galileo (1638). He treated the deformable body as inelastic without any laws and hypotheses between the displacement and forces. Galileo studied the resistance of a beam clamped at one end into the wall under its own weight or applied weight. He concluded that the beam rotates about the axis perpendicular to its length and in the plane of the wall. The determination of this axis is known as the Galileo's problem. The theory of deformable-bodies started from the Galileo's problem is based on the discovery of Hooke's Law in 1660 and the general differential equations of elasticity by Navier in 1821. The Hooke's law (Hooke, 1678) is an experimental discovery about the stress and strain relation. This law provides the basis to develop the mathematical theory of deformable-bodies. In 1821, Navier was the first to investigate the general equations of equilibrium and vibration of elastic solids, as presented in Love (1944). Although equilibrium and vibrations of plates and shells were treated before the general theory of elasticity was developed, one was interested in reduction of the general theory of elasticity to the theory of plates and shells by the power series of the distance from the middle surface. The problem is that the resultant forces and moments at the edge must be equal to the internal forces and moments generated by the strain. However, too many unknowns cannot be solved. Kirchhoff (1850a,b) proposed two assumptions: (i) that linear filaments of the plate initially normal to the middle-surface remain straight and normal to the middle-surface after deformed, and (ii) that all fibers in middle surface remain unstretched. Independent of the general equation of elasticity, the theory of the bending and twisting of thin rods and wires was developed by methods akin to those employed by Euler. One thought how to connect the general theory of elasticity to the theory of thin rods. Kirchhoff (1859) pointed out that the general equations of elasticity are strictly applicable to any small portion of a thin rod if all the linear dimensions of the portion are of the same order of magnitude as the diameters of the cross section. The equation of motion for such a portion of the rod could be simplified from the first approximation of deformation and kinematics. Based on the Kirchhoff assumptions, the approximate theories for beams, rods, plates and shells have been developed for recent 150 years. From 3-dimensional deformable body theory, with certain assumptions, this book will present a mathematic frame to develop such approximate theories for thin deformable-bodies including cables, beams, rods, webs, membranes, plates and shells. The theory for deformable body based on the Kirchhoff assumptions is a special case to be discussed. In this chapter, the development history for the theory of nonlinear cable dynamics will be discussed first.

1.1.1. Cable dynamics

Cables are used as one of the simplest structures for human being at least thousands of years. The cable configuration attracted scientists to investigate since it was used for the suspension bridge in the early human-being history. Based on the historical record, the sophisticated suspension bridges in China appeared before the start of the Christian era. The iron chain suspension bridge was built in Yunnan, China in A.D. 65 in Needham (1954). In 1586, Stevin established the triangle forces experimentally with loaded string to understand the catenary and the collapse mechanism in a voussoir arch, as reported in Hopkins (1970). From Truesdell (1960), Beeckman, in 1615, for the first time solved the suspension bridge problem that the configuration of hanged cables with the in-plane, uniformly distributed loading is a parabolic arc. Galileo mused on the shape of a hanging chain and concluded that it is parabolic primarily by analogy to the flight of a projectile, which was published in *Discourse on Two New Sciences* in 1638. However, it was proved that this view was incorrect as Bernoullis (James and John), Leibnitz and Huygens jointly discovered the catenary in Truesdell (1960). To solve the catenary, Huygens relied on the geometrical principle, and Leibnitz and Bernoullis used the calculus and Hooke's law to develop the general differential equations of equilibrium of a chain element under various loading. In addition, Bernoullis provided the basic fundamental of the calculus of variations to keep the center of gravity of the chain as low as possible. Furthermore, the principle of virtual work was developed. In the early 18th century, the vibration of taut string was extensively investigated to get the nature of the solution of partial differential equations. In 1738, Daniel Bernoulli (son of John Bernoulli) published a solution for natural frequencies of a chain that hangs from one end, and the solution was in the form of an infinite series (Watson, 1966). In 1764, Euler obtained the equation of motion for the vibrating taut membrane and obtained the infinite series solution through the variable separation. The partial solution was given by Poisson in 1829 and Clebsch in 1862. In addition, Lagrange in 1760 used the discretized string of beads model of the taut string as an illustration of the application of his equations of motion in Whittacker (1970). This work was done for the first time on the solution of vibration problems by the difference equations.

The equilibrium configuration, tension and displacement of elastic cables under arbitrary loading are needed in the design of cable structures. Rohrs (1851) first modeled the vibration of a uniform, inextensible suspended chain hanging freely under its own weight and obtained the approximate natural frequencies and responses of the cable. Routh (1884) considered the symmetric transverse vibration of a heterogeneous chain hanging in the form of a cycloid, and application of this chain model to the uniform chain yielded the Rohrs' model when the sag ratio is small. The chain was still modeled as inextensible. Pugsley (1949) developed a semi-empirical theory for the in-plane natural frequencies of the first three modes of a uniform, inextensible suspended chain. Saxon and Cahn (1953) developed an asymptotic method for the natural frequencies of the chain for large sag to span ratios. Simpson (1966) investigated the in-plane vibration of a stretched cable through its equilibrium and also determined the natural frequencies of multispan, sagged transmission lines using the transfer matrix method. Irvine and Caughey (1974) used a similar approach to investigate the free vibrations of a sagged, stretched cable hanging under its own weight. Hagedorn and Schafer (1980) showed that geometrical nonlinearity is significant in the computation of natural frequencies of in-plane vibration of an elastic cable. Luongo et al. (1984) analyzed the planar, non-linear, free vibrations of sagged cables through a perturbation method. Perkins (1992) considered the nonlinear vibrations of 3-dimensional, elastic, sagged cables analytically and experimentally, and gave a brief review of recent developments in cable dynamics. For translating cables, Simpson (1972) investigated planar oscillations by the linearized equations of motion around the equilibrium. Triantafyllou (1985) used an alternative approach to derive the linearized equations of motion at the equilibrium. Perkins and Mote (1987, 1989) developed a 3-dimensional cable theory for traveling elastic cables. The natural modes for the vibration and stability of translating cable at equilibria were obtained from the eigensolutions of discretized continuum models, and also some experimental results were reported. Luo et al (1996) presented the analytical solution and resonant motion for a stretched, spinning, nonlinear tether.

The non-straight equilibria of the cable have been determined by approximate means. For the stationary cables or strings, Dickey (1969) investigated the nonlinear string under a vertical force and gave tensile and compressive equilibrium solutions. Antman (1979) extended the Dickey investigation and investigated comprehensively the existence, multiplicity and qualitative behaviors of equilibrium for nonlinear elastic strings under different loads. The translating, sagged string possesses two non-trivial equilibrium states because of centrifugal loading. O'Reilly and Varadi (1995) investigated the equilibria of translating elastic cables. O'Reilly (1996) showed that if one used an observation due to Routh (1884) for inextensible strings, then the work of Antman (1979) and Dickey (1969) on static equilibria for strings can be extended to examine the steady motions of these strings. Healey and Papadopoulos (1990) extended the inextensible cable results to all the elastic strings. O'Reilly (1996) obtained the steady motion and stability of elastic and inextensible strings, and it was also shown that multiple steady motions were possible. In the quantitative investigation of elastic cables, Irvine (1981) used the method of Dickey (1969) to determine the exact equilibrium configuration and the approximate displacements of 2-dimensional cables under positive tension. For a single concentrated vertical load, the predicted displacement is constrained by the assumption that the equilibrium configuration is parabolic and that the ends of the cable are fixed. For multiple concentrated masses, the solutions given by Irvine (1981) require specificity of the initial configuration. To overcome these limitations, Yu et al. (1995) followed Irvine's procedure and computed the tension and equilibrium configuration of a 3-dimensional cable under uniform and concentrated transverse loading. The aforementioned exact solutions describe the equilibrium but not the deformation displacement because the initial configuration is not known. Luo and Mote (2000a) developed a nonlinear theory for traveling, arbitrarily sagged, elastic cables, and the closed-form equilibrium solution and existence were developed analytically. To investigate dynamics of nonlinear sagged, elastic cable, the dynamics of the inextensible cables should be investigated. Luo and Wang (2002) gave a series solution for the oscillation of the traveling, inextensible cable. Wang and Luo (2004) presented an alternative analytic solution for the motion of the in-plane, traveling inextensible cable. This analytical solution is also valid for the traveling speed over the critical speed. Based on dynamics of the inextensible cables, the dynamics of the sagged cables can be determined.

1.1.2. Beams and rods

Galileo (1638) studied the resistance of a beam clamped at one end into the wall under its own weight or applied weight, which caused modern science to develop. Through waves and vibrations in deformable-bodies, one understood the light and sound propagations. Before the theory of elasticity based on the Hooke's law and Navier's general differential equations for deformable-bodies, one investigated the theory of the bending and twisting of thin rods and wires. To obtain the solutions and extension of the Galileo's problems, the related, approximate theories for the vibration of bars and plates and the stability of columns were developed between 1638 and 1821. The 1-dimensional rod theory from 3-dimensional models by averaging the stress on the cross section was introduced by Leibniz in 1684. Since then, the first investigation of the elastic line or elastica was presented by James Bernoulli in 1705. In that research, the resistance of the bent rod is assumed to arise from the extension and contraction of its longitudinal fibers in the elastica, and the equation of the curve assumed by the axis is given, in which the resistance to bending is a bending moment proportional to curvature of the rod as bent. Once the concept about the bending moment perpendicular to curvature was established, the work done in bending a rod is proportional to the square of its curvature. Danial Bernoulli suggested to Euler that the differential equation of the bent rod can be obtained by minimizing the integral of the square of the curvature along the rod. From that suggestion, in 1744, Euler obtained the differential equation of the bent rod and classified the various form for such a problem. From this problem, Euler worked on what is the least length of elastica to bend under its own weight or applied weight (distributed force). Following the Euler theory, Lagrange determined the strongest form of column. Such an idea is a base for the variational principle, and such research is the earliest research on elastic stability. In the Euler's investigation, the rod was assumed as a line of particles to resist bending. In 1776, Coulomb considered the cross section of rod to present the flexure theory of beams and investigated the torsion of the thin rods. The theory improved the rod theory presented by Daniel Bernoulli and Euler. The concept of shear was proposed for the first time. From variation of energy function, the differential equations for the transverse vibration of bars were obtained by Euler and Daniel Bernoulli, and the vibration of rods with different boundary conditions was discussed. In 1802, Chldni presented an investigation of those modes of vibrations, and discussed the longitudinal and torsional vibrations of the bar. Based on the Hooke's law, in 1821, Navier developed the general differential equation for the theory of elasticity. Since the theory of rods was independently developed, one thought how to connect the general theory of elasticity to the theory of thin rods. Kirchhoff (1859) pointed out that the general equations of elasticity are strictly applicable to any small portion of a thin rod if all the linear dimensions of the portion are of the same order of magnitude as the diameters of the cross section. The equation of motions for such a portion of the rod could be simplified from the first approximation of deformation and kinematics. The earlier beam theories were developed by Kirchhoff (1859) and Clebsch (1862), as also presented in Love (1944). The comprehensive history of elasticity can be found in Todhunter and Pearson (1960) and Truesdell (1960).

Since 1940's, one has been interested in the systematic development of rod theories from 3-dimensional continuum mechanics. Hay (1942) obtained the strain from the power series in a thickness parameter. Novozhilov (1948) developed nonlinear theory for a rod with a large deformation. The other approximate theories for 1-dimensional rods or bars were presented by Midlin and Herrmann

(1952), Volterra (1955, 1956, 1961), Midlin and McNiven (1960) and Medick (1966). The theories were used to investigate the wave propagation and vibrations. On the other hand, to develop a theory of rod based on a 1-dimensional continuum model, E. and F. Cosserat (1909) introduced a concept of four vector fields to describe deformable vectors (directors) at a point of the directed and oriented curve. Ericksen and Truesdell (1958) used the Cosserat approach to develop a nonlinear theory of stress and strain in rods and shells through the oriented bodies. Cohn (1966) developed a static, isothermal theory of elastic curves. Whitman and DeSilva (1969) followed the Cohn's work to obtain the dynamical case and gave an explicit expression for the director inertia terms, and DeSilva and Whitman (1971) presented a thermo-dynamical theory for the directed curves with constitutive equations of materials. Such a theory can reduce to classic elastica and the linear theory of the Timoshenko beam when the assumptions were introduced to the corresponding theories, and an exact solution for such a nonlinear theory rods was presented (e.g., Whitman and DeSilva, 1970, 1972, 1974). On the other hand, Green (1959) presented the exact equilibrium equations for resultant force and moments by integration of the 3-dimensional equations over the cross section. Green and Laws (1966) extended this concept and developed a general theory of rods through two directors at each point in rods which requires specification of three vector fields. Antman and Warner (1966) used the polynomials in transverse coordinates to express the location of particle in rods and obtained the equation of motion with powers of the transverse coordinates for hyperelastic rods. Green, Laws and Naghdi (1967) used the idea of Green and Laws (1966) to present a linear theory of straight elastic rods, and Green, Knops and Laws (1968) used the same treatment for small deformation superimposed on finite deformation of elastic rods. A more detailed discussion of rod theories with directors can be referred to Antman (1972). Reissner (1972, 1973) developed a 1-dimensional finite-strain, static beam theory but how to treat the moment was not given. Wempner (1973) presented mechanics of curved rods, but the strain is the Almansi-Hamel strain. The strain energy of nonlinear rods was presented in Berdichevsky (1982). Maewal (1983) gave strain-displacement relations in nonlinear rods and shells. Danielson and Hodges (1987) discussed nonlinear beam kinematics through the deposition of the rotation tensor, and a mixed variational formulation for dynamics of moving beams was presented in Hodges (1990). Simo and Vu-Quoc (1987, 1991) used the exact strain to develop a theory for geometrically-nonlinear, planar rods, and several higher-order approximate theories were also given. Recently, this approach was used for development of the 3-D composite beam theory and numerical approaches were developed for prediction of dynamic responses in Vu-Quoc and Ebcioglu (1995, 1996) and Vu-Quoc and Deng (1995, 1997). The other derivation of equations of motion for geometrically-nonlinear rods can be referred to Crespo da Silva and Glynn (1978a), Crespo da Silva (1991), Pai and Nayfeh (1990, 1992, 1994).

The vibration of nonlinear, planar rods based on an accurate beam theory was investigated through a perturbation approach in Verma (1972). The free, nonlinear transverse vibration of beams was investigated in Nayfeh (1973) when the beam properties varied along with length. Ho et al. (1975, 1976) discussed the nonlinear vibration of rods through a single mode model and a perturbation approach. The forced vibration of nonlinear, torsional, inextensional beams was investigated in Crespo da Silva and Glynn (1978b). The planar, forced oscillations of shear in deformable beams were investigated through a specific, single-mode response and perturbation method in Luongo et al. (1986) and the planar motion of an elastic rod under a compressive force was analyzed in Atanackovic and Cveticanin (1996). Holmes and Marsden (1981) used the Melnikov method to investigate the chaotic oscillation of a forced beam. Maewal (1986) investigated chaotic motion in a harmonically excited elastic beam through the perturbation approach and Lyapunov exponent method. The dynamical potential for the nonlinear vibration of cantilevered beams was discussed in Berdichevsky et al. (1995), and the numerical simulations of chaotic motions in non-dampened nonlinear rods were also presented. Luo and Han (1999) presented the nonlinear equations of an in-plane rod to investigate its chaos. In practical applications, one often used the approximate theories to discuss the deformations and vibration of nonlinear rods and beams. In recent decades, in order to more accurately describe DNA structures and micro-electromechanical-systems (MEMS), one tried to revisit the theory of rods. The nonlinear theory of rods in Kirchhoff (1859) was revisited. Tsuru (1987) discussed equilibrium shape and vibrations of thin elastic rods. Coleman and Dill (1992) discussed the flexure waves in elastic rods (also see, Coleman et al., 1993). Tobias and Olson (1993) used a homogeneous inextensible elastic rod with a uniform cross section to describe a segment of DNA (also see, Coleman et al., 1995, 1996; Swigon et al., 1998). Lembo (2001) discussed the free shapes of elastic rods, and Colemen and Swigon (2004) presented the theory of self-contact in Kirchhoff rods with applications in supercoiling of knotted and unknotted DNA plasmids. Recently, the Cosserat theory of elastic rods was used to model MEMS (e.g., Cao et al., 2005; 2006), and the systematic description of elastic rod based on the Cosserat theory was presented in Cao and Tucker (2008). From the aforementioned survey, it is very important to develop an accurate theory for beams and rods. This book will present a theoretic frame for one to develop accurate theories for beams and rods.

1.1.3. Plates and shells

In the 17th century, based on special hypotheses, the theories of thin rods were developed. In the same fashion, the theory for plates and shells could be developed. Euler was the first to consider the plate consisting of annuli bars. In fact, the linear bending theory of plates was really developed by Kirchhoff (1850a,b) from his assumptions for the theory of thin rods. Love (1888) developed the linear theory of shells from the 3-dimensional equation of linear elasticity, as also presented in Love (1944). The nonlinear strains were determined by the first-order approximation of the extension. Such a theory originated from the small free vibration of a thin elastic shell in Love (1888). Such a work drew the criticism from Rayleigh (1888) because such an extensional deformation theory of shells is against the inextensional deformation theory. Lamb (1890) used the alternative way to derive the same equations as in Love (1888) and Basset (1890) considered the higher-order terms of the extension for thin cylindrical and spherical shells. To solve this argument, around 1940, with the framework of the Kirchhoff-Love assumption, Chien (1944 a,b) presented an intrinsic theory of plates and shells. Gol'denveizer (1944) discussed the applicability of the general theorems of the theory of elasticity to the thin shells. Reissner (1944) introduced the deformation caused by shear strain into the bending of elastic plates through an assumed displacement field. Discussion on the developments of the linear theory can be referred to Naghdi (1972), and other books.

The 3-dimensional thin continuous medium can be described by a 2-dimensional surface with a director. Such a concept of the continuous and oriented media was initialed by Duhem (1906). E. and F. Cosserat (1909) extended such concepts to develop the theory for shells and rods. Such a concept provides a base for development of the field theory for plates and shells. In addition, the existing approximate nonlinear theories for plates and shells have been derived from the 3-dimensional equations. In the early stage, it was assumed that the strain is very small but the rotation is large or moderately large, and the linear constitutive equations are assumed to be valid. von Karman (1910) extended the Love's strain based on the first order approximation of extension and developed an approximate theory for plates, and von Karman and Tsien (1939, 1941) used such approximate theory to investigate the buckling of thin spherical and cylindrical shells by external pressure. However, Galerkin (1915) discussed series solutions of some problems of elastic equilibrium of rods and plates. Novozhilov (1941) presented a general theory for stability of thin shells, and followed Galerkin's idea systematically presented the nonlinear theory for elasticity in Nolvozhilov (1948) or Nolvozhilov (1953) (English version). Following the von Karman theory, Reissner (1957) presented his nonlinear plate theory including shear deformation. Herrmann (1955) derived a plate theory governing dynamic motion with small elongation and shear deformation but moderately large rotation. Wang (1990) developed the 2-dimensional theory reduced from the 3-dimensional theory for transversely isotropic plates. Hodges et al. (1993) developed the geometrically nonlinear plate theory through the warping displacement. Since von Karman (1910) developed a nonlinear theory for thin plates with large deflection, ones used that nonlinear theory to investigate the buckling stability (e.g., Levy, 1942) and the nonlinear vibration of a spinning disk (e.g., Nowinski, 1964, 1981).

Based on the concept of continuous and oriented media, Ericksen and Truesdell (1958) presented a general development of the kinematics of the oriented media through n-stretchable directors in the n-dimensional space. The concept of directors was introduced. Truesdell and Toupin (1960) gave an exposition of the kinematics of the theory of oriented bodies. The 3-dimensional theory of an oriented medium with a single deformable director at all points of the body was developed in Green, Naghdi and Rivlin (1965). Cohen and DeSilva (1966) used the kinematics of Ericksen and Truesdell (1958) to introduce a triad of deformable directors to every point on the Cosserat surface. Toupin (1964) remarked only one single director should be enough to develop the nonlinear theory for plates and shells. Based on such development of the kinematics, the linearized kinematical measures were obtained by Green, Naghdi and Wainwright (1965). In fact, such kinematics of the plates and shells were completed by computation of the Lagrangian strains based on the Cosserat surface with directors. The general formulas presented in Naghdi (1972) are the first order approximation of extensions and shear deformation angles from the 3-dimensional deformable bodies. The equation of motions for plates and shells are based on the Cosserat surface. In fact, the deformation of the cross section of the deformed plates and shells may not be in the director. Luo (2000) developed a general frame for the approximate theories of plates. From the Kichhoff-assumptions, an approximate nonlinear theory of plates was presented, which can easily reduce to the existing theories. Luo and Mote (2000b) used the more accurate approximate plate theory in Luo (2000) to investigate the nonlinear vibration of rotating disks and presented an analytical solution for the nonlinear vibration of thin rotating disks. Luo and Tan (2001) investigated the resonant and stationary waves in rotating disks. Luo and Hamidzadeh (2004) used such an approximate theory to investigate the steady-state motion and buckling stability of the axially moving plates, and Luo (2003, 2005) investigated the resonant and stationary waves and chaos in axially traveling plates. In this book, the author will present a mathematical treatise for the approximate theories of plates and shells from a different aspect.

1.1.4. Soft webs

Generally speaking, the deformable soft web structures are extensible thin surfaces which can resist the tension only. Such deformable webs exist extensively in civil, textile and space engineering and bio-tissues, such as textile and paper materials (e.g., flag, clothes, balloons and papers) and bio-membranes (e.g., cell membranes). So far, one did not find an appropriate way to exactly describe the deformable webs in textile materials and bio-membranes. The problem on the inextensible soft webs rather than the elastic soft webs was investigated in the 19^{th} century, which was induced by the inextensible chains and nets. The difficulty for elastic webs is where the initial configurations of the elastic webs are. Even for the inextensible cables and nets, one has a difficulty to find the exact equilibrium configuration. To obtain exact equilibrium configuration, the knowledge of differential geometry is required and the wrinkling instability of the web structure blocks one to further think about such a problem. For simplicity, a pre-tensioned state for cables and nets was considered to obtain the corresponding equilibrium. To avoid such a difficulty, one adopted the membrane or plate and shell theory to apply such problems because the membrane or plate and shell theory assumed the corresponding initial configuration exists. In this book, from the differential geometry of deformable body, the theory for the deformable webs will be presented, which include non-continuous, deformable network webs, non-continuous fabric, deformable webs and continuous deformable webs.

In practical engineering, one is used to adopt the existing membrane or plate and shell theory to investigate the web dynamics. Lin and Mote (1995) investigated the axially moving rectangular webs to determine equilibrium solutions through the von Karman theory. The eigenvalue solutions predicting the wrinkling of rectangular webs under nonlinearly distributed edge loading were presented in Lin and Mote (1996). The elastic wrinkling of a tensioned circular plate was investigated using von Karman plate theory in Adams (1993). For an further extension of such an investigation, Luo and Hamidzadeh (2004) used the new plate theory of Luo (2000) to investigate the stability of traveling webs (exactly speaking, traveling thin plates), and Luo (2003, 2005) investigated resonant and chaotic waves motion in the traveling plates. In addition, Marynowksi (2004) investigated bifurcations and chaos of the axially moving viscoelastic webs through the beam model (also see, Marynowski and Kapitaniak, 2007). Recently, Marynowski (2009) summarized the recent results of the axially moving orthotropic webs in a lecture note. However, such results based on the traditional plate and beam theory cannot be applied to the nonlinear dynamics of deformable soft webs. The deformable soft webs cannot support any negative forces and bending moments.

The early work is about the net webs presented by Tchebychew (1878), and such a net web consists of inextensible fibers. The further development for the inextensible cable-net webs was presented by Pipkin (1981,1984). To avoid non-tension in the nets, Pipkin (1986) considered the relaxed energy density rather than the tension-field theory for isotropic elastic membranes. Steigmann and Pipkin (1989) discussed the wrinkling of pressurized membranes through the relaxed energy density. Steigemann and Pipkin (1991) presented the theory for equilibrium of elastic nets through the theory of relaxed energy density and discussed the wrinkling problem of the elastic nets. Based on such relaxed energy density, the shear on the webs or surfaces of elastic networks can be discussed. In 1996, the small oscillations of discrete elastic networks near the equilibrium were discussed in Wang and Steigmann (1996). The necessary and sufficient conditions for minimum-energy configuration were developed for numerical computation of the soft elastic networks in Atai and Steigman (1997). Recently, Nadler et al. (2006) discussed the convexity of the strain energy-function in the two-scale model of ideal fabrics. Nadler (2008) considered the relative stiffness of elastic nets with Cartesian and polar underlying structure. The aforementioned developments of the theory of the elastic network (or webs) were based on the finite deformation and relaxed strain-energy. All the discussion are about the initial pre-stressed nets. In this book, the direct derivation will be presented to obtain the equation of motion rather than the relaxed-strain energy with the deformed configuration of webs. This idea originated from the traveling cable dynamics in Luo and Mote (2000). In that paper, the exact equations of motion for traveling cables were developed and the closed-form solution for steady-state equilibrium was obtained. Based on such equilibrium, the vibration of the traveling inextensible cables were presented (e.g., Luo and Wang, 2001;Wang and Luo, 2004). For the cable structure, the wrinkling and swaying phenomena were observed. The author believes that the soft web structures will possess the similar phenomena. Soft deformable webs do not have any "snap-through" phenomena, but both the wrinkling behavior and swaying phenomenon exist. However, membrane structures can resist the negative membrane forces, thus the "snap-through" can be observed.

1.2. Book layout

This book consists of eight chapters. Chapter 1 discusses the history of the deformable body dynamics. In Chapter 2, the mathematical tool for the deformation and kinematics of the deformable bodies will be presented. Chapter 3 will address the deformation geometry, kinematics and dynamics of deformable bodies. Chapter 4 will present constitutive laws and damage theory for deformable bodies. In Chapter 5, nonlinear cable dynamics will be presented. Chapter 6 will discuss the nonlinear theory and vibration waves of plates. In Chapter 7, the nonlinear theory for webs, membranes and shells will be presented. Finally, Chapter 8 will present the nonlinear theory for beams and rods. The main contents in this book are summarized as follows.

Chapter 2 will review the basic vector algebra first. The base vectors and metric tensors will be introduced, and the local base vectors in curvilinear coordinates and tensor algebra will be presented. The second-order tensors will be discussed in detail. The differentiation and derivatives of tensor fields will be presented, and the gradient, invariant differential operators and integral theorems for tensors are presented. The Riemann-Christoffel curvature tensor will also be discussed. Finally, two-point tensor fields will be presented.

Chapter 3 will present the deformation geometry, kinematics and dynamics of continuous media. To discuss deformation geometry, the deformation gradients will be introduced in the local curvilinear coordinate systems, and the Green and Cauchy strain tensors will be presented. The stretch and angle changes for line elements will be discussed through Green and Cauchy strain tensors. The velocity gradient will be introduced for kinematics, and the material derivatives of deformation gradient, infinitesimal line element, area and volume in the deformed configuration will be presented. The Cauchy stress and couple stress tensors will be defined to discuss the dynamics of continuous media, and the local balances for the Cauchy momentum and angular momentum will be discussed. The Piola-Kirchhoff stress tensors will be introduced and the Boussinesq and Kirchhoff local balance of momentum will be discussed. The local principles of the energy conservation will be discussed by the virtual work principle.

Chapter 4 will discuss the constitutive laws and basic invariant requirements in continuous media. To develop a continuum damage theory, the concepts of damage variables will be introduced. The equivalent principles in continuum damage mechanics will be presented to obtain effective material properties, which include the *strain equivalence principle*, the *complementary energy equivalence principle* and the *incremental complementary energy equivalence principle*. A large damage theory for anisotropic damaged materials will be discussed from the incremental complementary energy equivalence principle, and three examples will be illustrated for application.

Chapter 5 will discuss the nonlinear dynamics of traveling and rotating cables. A general nonlinear theory of cables will be presented. The basic equations of motion for rotating and longitudinally traveling cables will be derived. The closed-form solutions for equilibriums of elastic cables will be developed. To investigate the cable dynamics, the rigid body dynamics of cables will be discussed. The equation of motion for the deformation displacements of deformable cables will be addressed.

Chapter 6 will present a nonlinear plate theory from the 3-dimensional theory of deformable-bodies, and the approximate theories of thin plates will be discussed. From such a theory, approximate solutions for nonlinear waves in axially traveling plates and rotating disks will be presented. Stationary and resonant waves in the traveling plates and rotating disks will be discussed. Finally, chaotic waves in axially traveling plates under a periodic excitation will be presented.

Chapter 7 will present the nonlinear theories for webs, membranes and shells. The theory for network, fabric non-continuum and continuum webs will be presented first from the nonlinear theory of cables, and the theory for the continuous web will be discussed as well. Further, the nonlinear theory of membranes will be developed in an analogy way. The nonlinear theory of shells will be developed from the general theory of the 3-dimensional deformable body, and such a theory of shells can easily reduce to the existing linear and nonlinear theories.

Chapter 8 will present the nonlinear theories for rods and beams in the Cartesian coordinate frame and the curvilinear frame of the initial configuration. Without torsion, the nonlinear theory for in-plane beams will be developed first under certain assumptions, and the nonlinear theory of rods will be presented systematically from the general theory of the 3-dimensional deformable body.

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Chapter 2
Tensor Analysis

This chapter will prepare basic knowledge about the tensor analysis in $\mathbb{R}^3$. The base vectors and metric tensors will be introduced, and the local base vectors in curvilinear coordinates and tensor algebra will be presented. The second-order tensors will be discussed in detail. The differentiation and derivatives of tensor fields will be presented, and the gradient, invariant differential operators and integral theorems for tensors will be presented. The Riemann-Christoffel curvature tensor will also be discussed. Finally, two-point tensor fields will be presented. This chapter will provide mathematical tools for nonlinear continuous media. In this chapter, the notations will be adopted from Guo (1980) and Marsden and Hughes (1983).

2.1. Vectors and tensors

In this section, vectors and vector analysis will be introduced first. From vector analysis, the base vectors in curvilinear coordinate systems will be presented, and the metric tensors will be introduced. The transformation between the base vectors of two curvilinear coordinate systems will be presented. The concept of tensors is introduced. The tensor algebra will be discussed through the dyadic expression.

2.1.1. Vector algebra

In a vector space $\mathscr{E}$, there are three vectors (i.e., $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$). These vectors satisfy the following rules:

(i) *Vector addition*

$$\begin{aligned} &\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}, \\ &\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{v}+\mathbf{u})+\mathbf{w}, \end{aligned} \tag{2.1}$$

$$\mathbf{u} + \mathbf{0} = \mathbf{u},$$
$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}. \tag{2.2}$$

(ii) *Vector scalar multiplication*

$$\begin{aligned} k(\mathbf{u}) &= k\mathbf{u}, \\ a(b\mathbf{u}) &= (ab)\mathbf{u}, \\ (a+b)\mathbf{u} &= a\mathbf{u} + b\mathbf{u}, \\ a(\mathbf{u}+\mathbf{v}) &= a\mathbf{u} + a\mathbf{v}. \end{aligned} \tag{2.3}$$

Definition 2.1. A set of *n*-vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N$ is *linearly dependent* if

$$\sum_{i=1}^{N} k_i \mathbf{v}_i = 0 \tag{2.4}$$

with at least one of $k_i \neq 0$ for $i = 1, 2, \ldots, N$. Otherwise, such a set of *N*-vectors is *linearly independent*, i.e.,

$$\sum_{i=1}^{N} k_i \mathbf{v}_i \neq 0. \tag{2.5}$$

Definition 2.2. A vector space $\mathscr{E}_N$ is *N-dimensional* if a set of *N*-vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N$ is *linearly independent*, an $(N+1)^{\text{th}}$-vector $\mathbf{v}_{N+1}$ with $k_{N+1} \neq 0$ satisfies

$$\sum_{i=1}^{N+1} k_i \mathbf{v}_i = 0. \tag{2.6}$$

For Eq.(2.6), consider $k_{N+1} = -1$ and $\mathbf{v}_{N+1} = \mathbf{v}$, one obtains a new vector as

$$\mathbf{v} = \sum_{i=1}^{N} k_i \mathbf{v}_i. \tag{2.7}$$

Therefore, a new vector $\mathbf{v}$ can be expressed by the a set of *N*-linearly independent vectors (i.e., $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N$). In other words, such a set of *N*-linearly independent vectors is said a basis of the *N*-dimensional vector space.

If the angle between vectors $\mathbf{u}$ and $\mathbf{v}$ is θ, the *vector dot product* (or *vector scalar product*) of the vectors $\mathbf{u}$ and $\mathbf{v}$ are computed by

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta \tag{2.8}$$

where $|\cdot|$ is the magnitude of the vector. If $\mathbf{u} = \mathbf{v}$, $\theta = 0$, then we have $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$. Therefore, the *magnitude of a vector* $\mathbf{v}$ is computed as

$$|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}. \tag{2.9}$$

The vector $\mathbf{v}$ is a *unit vector* if

$$|\mathbf{v}| = 1. \tag{2.10}$$

Definition 2.3. In a Euclidean vector space, the vectors $\mathbf{u}$ and $\mathbf{v}$ are *perpendicular* if

$$\mathbf{u} \cdot \mathbf{v} = 0. \tag{2.11}$$

Definition 2.4. A set of *N*-linearly independent vectors $\mathbf{v}_i$ ($i = 1, 2, \ldots, N$) is *orthogonal* if

$$\begin{aligned} \mathbf{v}_i \cdot \mathbf{v}_j &= 0 \quad \text{if } i \neq j, \\ \mathbf{v}_i \cdot \mathbf{v}_j &\neq 0 \quad \text{if } i = j. \end{aligned} \tag{2.12}$$

Furthermore, such a set of *N*-linear independent vectors is termed an orthogonal base.

If the angle between vectors $\mathbf{u}$ and $\mathbf{v}$ is θ, the *vector cross product* (or *vector product*) of the vectors $\mathbf{u}$ and $\mathbf{v}$ defines a new vector $\mathbf{w}$, computed by

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} \text{ and } |\mathbf{w}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta. \tag{2.13}$$

Such a new vector is perpendicular to both vectors $\mathbf{u}$ and $\mathbf{v}$; and the corresponding *unit vector* is defined as

$$\mathbf{e}_w = \frac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|}. \tag{2.14}$$

The direction of the vector $\mathbf{w}$ is based on the right-handed system of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. In other words,

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}. \tag{2.15}$$

2.1.2. Base vectors and metric tensors

Definition 2.5. Consider three Cartesian coordinates z^i ($i = 1, 2, 3$) of a point $P(\mathbf{z})$ with $\mathbf{z} = \Sigma_{i=1}^{3} z^i \mathbf{i}_i$ in $\mathbb{R}^3$. Three functions in a neighborhood of $\mathbf{z}$

$$x^i = x^i(z^1, z^2, z^3) \quad (i = 1, 2, 3) \tag{2.16}$$

are called *curvilinear coordinates* of point $P(\mathbf{z})$ if there is a unique one-to-one inverse

$$z^i = z^i(x^1, x^2, x^3) \quad (i = 1, 2, 3). \tag{2.17}$$

From differential geometry, the condition for the unique one-to-one inverse maps of $x^i = x^i(z^1, z^2, z^3)$ $(i = 1, 2, 3)$ requires that the first-order derivatives of z^i are continuous and the determinant of the corresponding *Jacobian matrix* is non-zero, i.e.,

$$J = \left|\frac{\partial z^i}{\partial x^j}\right| = \begin{vmatrix} \partial z^1/\partial x^1 & \partial z^1/\partial x^2 & \partial z^1/\partial x^3 \\ \partial z^2/\partial x^1 & \partial z^2/\partial x^2 & \partial z^2/\partial x^3 \\ \partial z^3/\partial x^1 & \partial z^3/\partial x^2 & \partial z^3/\partial x^3 \end{vmatrix} \neq 0. \tag{2.18}$$

From the above definition, the curvilinear coordinates are shown in Fig.2.1. The Cartesian coordinate system is defined by $\{z^i\}$ $(i = 1, 2, 3)$ and the corresponding unit vectors are $\{\mathbf{i}_i\}$ $(i = 1, 2, 3)$. The point P is described by

$$\begin{aligned} \mathbf{p} &= \sum\nolimits_{j=1}^{3} z^j \mathbf{i}_j = \sum\nolimits_{j=1}^{3} z^j(x^1, x^2, x^3)\mathbf{i}_j \\ &= z^j \mathbf{i}_j = z^j(x^1, x^2, x^3)\mathbf{i}_j. \end{aligned} \tag{2.19}$$

The Einstein summation convention is adopted. The summation convention states that summation $(i = 1, 2, 3)$ is only implied by repeated indices when one of the repeated indices appears as a superscript and one as a subscript. Such repeated indices are termed the *blind index*, which can be replaced arbitrarily. For instance, $\mathbf{p} = z^i \mathbf{i}_i = z^j \mathbf{i}_j$.

The base vectors $\mathbf{g}_i(x^1, x^2, x^3)$ tangential to the curves x^i $(i = 1, 2, 3)$ are defined as

$$\mathbf{g}_i = \frac{\partial \mathbf{p}}{\partial x^i} = \frac{\partial z^j}{\partial x^i}\mathbf{i}_j. \tag{2.20}$$

On the other hand, the point P can be expressed by

$$\mathbf{p} = x^i \mathbf{g}_i = x^i(z^1, z^2, z^3)\mathbf{g}_i. \tag{2.21}$$

The corresponding base vector $\mathbf{i}_j$ $(j = 1, 2, 3)$ is determined by

$$\mathbf{i}_j = \frac{\partial \mathbf{p}}{\partial z^j} = \frac{\partial x^i}{\partial z^j}\mathbf{g}_i. \tag{2.22}$$

From any three linearly independent vectors $\mathbf{g}_i$ $(i = 1,2,3)$ in 3-dimensional vector space, any vector can be expressed by a combination of the three vectors

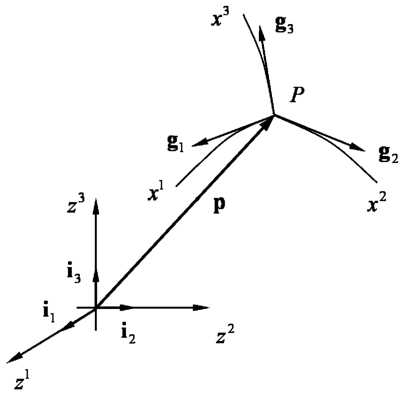


Fig. 2.1 Curvilinear coordinates.

$$\mathbf{v} = v^i \mathbf{g}_i , \tag{2.23}$$

where v^i is the component of vector $\mathbf{v}$ in the direction of $\mathbf{g}_i$.

Definition 2.6. The base vectors $\mathbf{g}_i$ ($i = 1, 2, 3$) of a vector $\mathbf{v} = v^i \mathbf{g}_i$ are called the *covariant base vectors* of the vector $\mathbf{v}$; and the corresponding components v^i ($i = 1, 2, 3$) are the *contravariant* components of the vector $\mathbf{v}$.

Since the base vectors $\mathbf{g}_i$ ($i = 1, 2, 3$) are not necessary to be orthogonal and unit vectors, a quantity to measure the dot product of two non-orthogonal $\mathbf{g}_i$ and $\mathbf{g}_j$ is introduced as

$$g_{ij} \equiv \mathbf{g}_i \cdot \mathbf{g}_j , \tag{2.24}$$

from which one obtains $g_{ij} = g_{ji}$. Once the vector $\mathbf{v}$ and three base vectors $\mathbf{g}_i$ ($i = 1, 2, 3$) are given, the component v^i of the vector $\mathbf{v}$ in the direction of $\mathbf{g}_i$ is determined by

$$g_{ij} v^j = \mathbf{v} \cdot \mathbf{g}_i \quad (i = 1, 2, 3) \tag{2.25}$$

because of $\mathbf{v} \cdot \mathbf{g}_i = v^j \mathbf{g}_j \cdot \mathbf{g}_i = g_{ij} v^j$. The foregoing equation gives a unique solution of v^i ($i = 1, 2, 3$) because the determinant of coefficients of linear algebraic equations is non-zero:

$$g \equiv | g_{ij} | = | \mathbf{g}_i \cdot \mathbf{g}_j | = [\mathbf{g}_1 \cdot (\mathbf{g}_2 \times \mathbf{g}_3)]^2 \equiv [\mathbf{g}_1 \mathbf{g}_2 \mathbf{g}_3]^2 \neq 0, \tag{2.26}$$

where $| \cdot |$ is matrix determinant. To prove the foregoing equation, three base vectors $\mathbf{g}_i$ ($i = 1, 2, 3$) expressed by three orthogonal unit vectors ($\mathbf{i}_r$, $r = 1, 2, 3$) on three axes in $\mathbb{R}^3$ are considered for an example, i.e.,

$$\mathbf{g}_i = g_i^r \mathbf{i}_r \quad (i = 1, 2, 3), \tag{2.27}$$

where $g_i^r = \partial z^r / \partial x^i$ and $| g_i^r | = | \partial z^r / \partial x^i |$. The cross product of the two base vectors $\mathbf{g}_i$ and $\mathbf{g}_j$ are

$$\mathbf{g}_i \times \mathbf{g}_j = \begin{vmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ g_i^1 & g_i^2 & g_i^3 \\ g_j^1 & g_j^2 & g_j^3 \end{vmatrix} \tag{2.28}$$

and

$$[\mathbf{g}_1\mathbf{g}_2\mathbf{g}_3] = \mathbf{g}_1 \cdot (\mathbf{g}_2 \times \mathbf{g}_3) = \begin{vmatrix} g_1^1 & g_1^2 & g_1^3 \\ g_2^1 & g_2^2 & g_2^3 \\ g_3^1 & g_3^2 & g_3^3 \end{vmatrix} = \begin{vmatrix} g_1^1 & g_2^1 & g_3^1 \\ g_1^2 & g_2^2 & g_3^2 \\ g_1^3 & g_2^3 & g_3^3 \end{vmatrix} = | g_r^i |. \tag{2.28}$$

On the other hand,

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = g_i^r g_j^r, \tag{2.29}$$

$$| g_{ij} | = | g_i^r g_j^r | = | g_i^r | \cdot | g_i^r | = | g_i^r |^2 = [\mathbf{g}_1\mathbf{g}_2\mathbf{g}_3]^2. \tag{2.30}$$

Therefore, equation (2.26) can be obtained.

For simplicity, another set of base vectors is also introduced to measure the vector $\mathbf{v}$.

Definition 2.7. A set of three base vectors $\mathbf{g}^i$ ($i = 1, 2, 3$) is *contravariant* if the following equation holds for the covariant base vectors of $\mathbf{g}_j$ ($j = 1, 2, 3$),

$$\mathbf{g}^i \cdot \mathbf{g}_j = \delta_j^i, \tag{2.31}$$

where the *Kronecker delta* δ_i^j is defined as

$$\delta_i^j = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases} \tag{2.32}$$

From the foregoing definition, the contravariant base vector $\mathbf{g}^i$ is perpendicular to the covariant base vector $\mathbf{g}_j$ ($j \neq i$). The two sets of base vectors $\mathbf{g}^i$ ($i = 1, 2, 3$) and $\mathbf{g}_j$ ($j = 1, 2, 3$) are of dual basis (or reciprocal basis) in a Euclidean vector space. As in Eq.(2.24), a new quantity of the base vector $\mathbf{g}^i$ ($i = 1, 2, 3$) is introduced as

$$g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j \text{ and } g^{ij} = g^{ji}. \tag{2.33}$$

With Eq.(2.31), equation (2.33) gives the base vector $\mathbf{g}^i$ as

$$\mathbf{g}^i = g^{ij}\mathbf{g}_j. \tag{2.34}$$

If the matrix $(g^{ij})_{3\times 3}$ is uniquely given, then the base vector $\mathbf{g}^i$ is determined uniquely. Multiplying $\mathbf{g}_k$ at both sides of Eq.(2.34) and using Eq.(2.31) gives

$$\delta^i_k = g^{ij}\mathbf{g}_j \cdot \mathbf{g}_k = g^{ij} g_{jk} \quad (i, j, k = 1,\ 2,\ 3\ ). \tag{2.35}$$

The determinant of the foregoing equation gives $1 = | g^{ij} g_{jk} |$. With Eq.(2.26), one obtains

$$| g^{ij} | = \frac{1}{g} \neq 0. \tag{2.36}$$

Thus the three base vectors $\mathbf{g}^i$ ($i = 1, 2, 3$) are not in the same plane. For $k = i$, (g^{ij}) is the inverse matrix of (g_{ji}), i.e.,

$$(g^{ij}) = (g_{ji})^{-1} = \frac{1}{g}\text{cofactor}(g_{ji}) \tag{2.37}$$

due to $g = \det(g_{ji}) \neq 0$. Thus, the *contravariant base vector* is given by

$$\mathbf{g}^i = (g_{ji})^{-1}\mathbf{g}_j = \frac{1}{g}\text{cofactor}(g_{ji})\mathbf{g}_j. \tag{2.38}$$

On the other hand, because $\mathbf{g}^i$ is normal to the basic vectors $\mathbf{g}_j$ and $\mathbf{g}_k$ ($i \neq j \neq k$). If $i \to j \to k \to i$ rotates clockwise, then the two vectors $\mathbf{g}^i$ and $\mathbf{g}_j \times \mathbf{g}_k$ are collinear, but their magnitudes are different. So we have

$$\alpha\mathbf{g}^i = \mathbf{g}_j \times \mathbf{g}_k. \tag{2.39}$$

Left multiplication of $\mathbf{g}_i$ in Eq.(2.39) and using Eq.(2.31) gives

$$\alpha\mathbf{g}_i \cdot \mathbf{g}^i = \mathbf{g}_i \cdot (\mathbf{g}_j \times \mathbf{g}_k) \ \Rightarrow \alpha = [\mathbf{g}_i\mathbf{g}_j\mathbf{g}_k]. \tag{2.40}$$

Thus,

$$\mathbf{g}^i = \frac{\mathbf{g}_j \times \mathbf{g}_k}{[\mathbf{g}_i\mathbf{g}_j\mathbf{g}_k]} \quad (i \to j \to k \to i). \tag{2.41}$$

Similarly, the covariant base vectors $\mathbf{g}_i$ can be determined by the contravariant base vectors, i.e.,

$$\mathbf{g}_i = \frac{\mathbf{g}^j \times \mathbf{g}^k}{[\mathbf{g}^i \mathbf{g}^j \mathbf{g}^k]} \quad (i \to j \to k \to i). \tag{2.42}$$

Once the covariant base vectors are given, the contravariant component of v^i in the vector $\mathbf{v} = v^i \mathbf{g}_i$ in Eq.(2.23) is determined by

$$v^j = \mathbf{v} \cdot \mathbf{g}^j \tag{2.43}$$

because of $\mathbf{v} \cdot \mathbf{g}^j = v^i \mathbf{g}_i \cdot \mathbf{g}^j = v^i \delta_i^j = v^j$. In addition, the vector $\mathbf{v}$ can be expressed by the contravariant base vector $\mathbf{g}^j$, i.e.,

$$\mathbf{v} = v_i \mathbf{g}^i \text{ and } v_i = \mathbf{v} \cdot \mathbf{g}_i . \tag{2.44}$$

Definition 2.8. The base vector $\mathbf{g}^i$ ($i = 1, 2, 3$) of a vector $\mathbf{v} = v_i \mathbf{g}^i$ is called the *contravariant base vector* of the vector $\mathbf{v}$; and the corresponding component $v_i = \mathbf{v} \cdot \mathbf{g}_i$ ($i = 1, 2, 3$) is the *covariant* component of the vector $\mathbf{v}$.

Definition 2.9. Two quantities $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ and $g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j$ are termed the *covariant and contravariant metric tensors*, respectively.

From the previous discussion, once the covariant base vectors $\mathbf{g}_i$ are given, the two metric tensors g_{ij} and g^{ij}, contravariant base vector $\mathbf{g}^i$ and vector $\mathbf{v}$ can be determined. Once the vector can be expressed through two covariant and contravariant base vectors, the dot and cross products of two vectors can be discussed. For instance, from Eq.(2.34), the dot product of the base vectors is

$$\mathbf{g}^i \cdot \mathbf{g}^j = g^{ik} \mathbf{g}_k \cdot \mathbf{g}^j = g^{ik} \delta_k^j = g^{ij}, \tag{2.45}$$

but from Eq.(2.35),

$$g_{ki} \mathbf{g}^i = g_{ki} g^{ij} \mathbf{g}_j = \delta_k^j \mathbf{g}_j = \mathbf{g}_k . \tag{2.46}$$

From Eqs.(2.43) and (2.44),

$$\begin{aligned} v_i &= \mathbf{g}_i \cdot \mathbf{v} = \mathbf{g}_i \cdot \mathbf{g}_k v^k = g_{ik} v^k, \\ v^i &= \mathbf{g}^i \cdot \mathbf{v} = \mathbf{g}^i \cdot \mathbf{g}^k v_k = g^{ik} v_k . \end{aligned} \tag{2.47}$$

Consider any two vectors $\mathbf{u} = u^i \mathbf{g}_i = u_i \mathbf{g}^i$ and $\mathbf{v} = v^i \mathbf{g}_i = v_i \mathbf{g}^i$. The corresponding dot product is given by

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= \mathbf{g}_i \cdot \mathbf{g}_j u^i v^j = g_{ij} u^i v^j \\ &= \mathbf{g}_i \cdot \mathbf{g}^j u^i v_j = \delta_i^j u^i v_j = u^i v_i \end{aligned}$$

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{g}^i\cdot\mathbf{g}^j u_i v_j = g^{ij}u_i v_j$$
$$= \mathbf{g}^i\cdot\mathbf{g}_j u_i v^j = \delta^i_j u_i v^j = u_i v^i. \tag{2.48}$$

If $\mathbf{u} = \mathbf{v}$, then

$$|\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v} = g_{ij}v^i v^j = v^i v_i = g^{ij}v_i v_j = v_i v^i. \tag{2.49}$$

In conclusion, g_{ij} and g^{ij} are key quantities to determine the magnitude of the vector $\mathbf{v}$, which are often called *the metric tensors*.

Definition 2.10. *Eddington tensor* is defined as

$$\varepsilon_{ijk} = [\mathbf{g}_i\mathbf{g}_j\mathbf{g}_k] \text{ and } \varepsilon^{ijk} = [\mathbf{g}^i\mathbf{g}^j\mathbf{g}^k]. \tag{2.50}$$

The covariant and contravariant components of the cross product of two vectors ($\mathbf{w} = \mathbf{u}\times\mathbf{v}$) are computed by

$$w_i = \mathbf{w}\cdot\mathbf{g}_i = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{g}_i = (u^j\mathbf{g}_j\times v^k\mathbf{g}_k)\cdot\mathbf{g}_i$$
$$= u^j v^k(\mathbf{g}_j\times\mathbf{g}_k)\cdot\mathbf{g}_i = u^j v^k[\mathbf{g}_i\mathbf{g}_j\mathbf{g}_k] = \varepsilon_{ijk}u^j v^k, \tag{2.51}$$
$$w^i = \mathbf{w}\cdot\mathbf{g}^i = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{g}^i = (u_j\mathbf{g}^j\times v_k\mathbf{g}^k)\cdot\mathbf{g}^i$$
$$= u_j v_k(\mathbf{g}^j\times\mathbf{g}^k)\cdot\mathbf{g}^i = u_j v_k[\mathbf{g}^i\mathbf{g}^j\mathbf{g}^k] = \varepsilon^{ijk}u_j v_k. \tag{2.52}$$

From the foregoing equations, the *Eddington tensor* is a kind of the metric tensor as the metric tensors g^{ij} and g_{ij} in the dot product of vectors.

The cosine of the angle between two vectors $\mathbf{u}$ and $\mathbf{v}$ is computed by

$$\cos\theta_{(\mathbf{u},\mathbf{v})} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\cdot|\mathbf{v}|} = \frac{u^i v_i}{\sqrt{u^j u_j}\sqrt{v^k v_k}} \tag{2.53}$$

and the volume of three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is determined by

$$[\mathbf{uvw}] = [\mathbf{g}_i\mathbf{g}_j\mathbf{g}_k]u^i v^j w^k = \varepsilon_{ijk}u^i v^j w^k$$
$$= [\mathbf{g}^i\mathbf{g}^j\mathbf{g}^k]u_i v_j w_k = \varepsilon^{ijk}u_i v_j w_k. \tag{2.54}$$

2.1.3. Local base vector transformation

As in the previous section, in a coordinate system of $\{x^i\}$ ($i = 1, 2, 3$), any vector to a point p can be expressed by $\mathbf{p} = x^i\mathbf{g}_i$. However, in practical computation, a convenient coordinate system is needed. For example, there is a coordinate system of $x^{i'}$ ($i=1, 2, 3$) with the corresponding base vector $\mathbf{g}_{i'} = (\partial z^j / \partial x^{i'})\mathbf{i}_j$. Such a system is sketched through the dashed curves in Fig.2.2. Under a certain condition, the two coordinate systems can be transformed.

If the three functions $\{x^{i'}\}$ ($i=1, 2, 3$) in $\Omega \subset \mathbb{R}^3$,

$$x^{i'} = x^{i'}(x^1, x^2, x^3) \tag{2.55}$$

are unique, continuous and differentiable with respect to $\{x^i\}$ ($i=1, 2, 3$), then, in the domain Ω of $\{x^{i'}\}$, there are three corresponding unique, continuous and differentiable, inverse functions, i.e.,

$$x^i = x^i(x^{1'}, x^{2'}, x^{3'}). \tag{2.56}$$

Therefore, the two coordinate systems $\{x^{i'}\}$ and $\{x^i\}$ can be transformed each other.

From the aforementioned conditions, the Jacobian matrices of the two transformations are non-zero, and the continuity requires that the determinants of the two Jacobian matrices do not change sign. Because

$$1 = |\delta_{i'}^{i'}| = \left|\frac{\partial x^{i'}}{\partial x^r} \cdot \frac{\partial x^r}{\partial x^{i'}}\right| = \left|\frac{\partial x^{i'}}{\partial x^r}\right| \cdot \left|\frac{\partial x^r}{\partial x^{i'}}\right|, \tag{2.57}$$

and if

$$\left|\frac{\partial x^{i'}}{\partial x^i}\right| > 0 \ \left(\text{or } \left|\frac{\partial x^{i'}}{\partial x^i}\right| < 0\right), \tag{2.58}$$

equation (2.57) gives

$$\left|\frac{\partial x^i}{\partial x^{i'}}\right| > 0 \ \left(\text{or } \left|\frac{\partial x^i}{\partial x^{i'}}\right| < 0\right) \tag{2.59}$$

and vice versa. Therefore, the vector $\mathbf{p}$ for a point P in Fig. 2.2 can be expressed through two coordinate systems $\{x^{i'}\}$ and $\{x^i\}$. The corresponding geometry properties can be discussed as follows.

To determine the tangential vector of $\{x^{i'}\}$ ($i=1, 2, 3$), consider the vector $\mathbf{p} = x^i(x^1, x^2, x^3)\mathbf{g}_i$. The definition of the base vector gives

$$\mathbf{g}_{i'} \equiv \frac{\partial \mathbf{p}}{\partial x^{i'}} = \frac{\partial}{\partial x^{i'}}(x^i \mathbf{g}_i) = \frac{\partial x^i}{\partial x^{i'}}\mathbf{g}_i. \tag{2.60}$$

The base vectors for $\{x^{i'}\}$ are local, and at the different point, they are different. From the properties of $\{x^i\}$, the base vector $\mathbf{g}_{i'}$ is continuous and differentiable. Due to the linear independence of $\mathbf{g}_i$ $\{i=1, 2, 3\}$, $[\mathbf{g}_1\mathbf{g}_2\mathbf{g}_3] > 0$ for the right-handed coordinate system. So

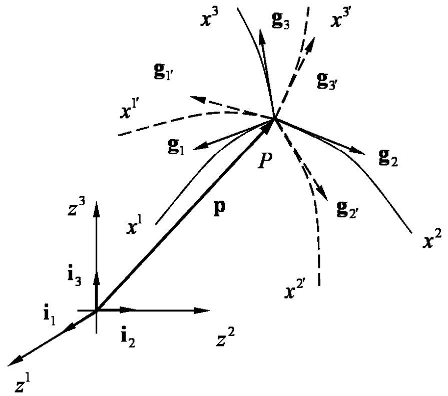


Fig. 2.2 Curvilinear coordinates.

$$[\mathbf{g}_{1'}\mathbf{g}_{2'}\mathbf{g}_{3'}] = \frac{\partial x^i}{\partial x^{1'}}\frac{\partial x^j}{\partial x^{2'}}\frac{\partial x^k}{\partial x^{3'}}[\mathbf{g}_i\mathbf{g}_j\mathbf{g}_k] = \left|\frac{\partial x^i}{\partial x^{i'}}\right|[\mathbf{g}_1\mathbf{g}_2\mathbf{g}_3] > 0, \tag{2.61}$$

which implies that three base vectors of $\mathbf{g}_{i'}$ $(i = 1, 2, 3)$ are linearly independent. Similarly, if there is another curvilinear coordinate system of $\{x^{i''}\}$ $(i = 1, 2, 3)$ which is transformed from the curvilinear coordinate system of $\{x^{i'}\}$ $(i = 1, 2, 3)$, the corresponding local base vector is defined by

$$\mathbf{g}_{i''} \equiv \frac{\partial \mathbf{p}}{\partial x^{i''}} = \frac{\partial}{\partial x^{i''}}(x^{i'}\mathbf{g}_{i'}) = \frac{\partial x^{i'}}{\partial x^{i''}}\mathbf{g}_{i'}. \tag{2.62}$$

At any point in $\mathbb{R}^3$, the matrices $(\partial x^i / \partial x^{i'})$ is the inverse of $(\partial x^{i'} / \partial x^i)$, vice versa. The elements of the two transformation matrices are defined as

$$A^i_{i'} = \frac{\partial x^i}{\partial x^{i'}} \text{ and } A^{i'}_i = \frac{\partial x^{i'}}{\partial x^i}. \tag{2.63}$$

For any vector $\mathbf{v}$, one can use different points to describe the vector. For a point p, suppose the vector $\mathbf{v}$ is expressed by

$$\mathbf{v} = v^i(p)\mathbf{g}_i(p). \tag{2.64}$$

Similarly, at another point q, the vector $\mathbf{v}$ is expressed as

$$\mathbf{v} = v^i(q)\mathbf{g}_i(q). \tag{2.65}$$

If two points p and q are different, then $v^i(p) \neq v^i(q)$ and $\mathbf{g}_i(p) \neq \mathbf{g}_i(q)$. But both of them describe the same vector. Similarly, such a concept will be used in tensor analysis. Consider the two vectors to describe by the local base vectors of different points. The two vector addition is

$$\mathbf{u}+\mathbf{v}=u^i(p)\mathbf{g}_i(p)+v^i(q)\mathbf{g}_i(q). \tag{2.66}$$

From Eqs.(2.60) and (2.63), a new set of covariant base vectors $\mathbf{g}_{i'}$ is expressed by

$$\mathbf{g}_{i'}=A^i_{i'}\mathbf{g}_i. \tag{2.67}$$

The new contravariant base vector is defined by

$$\mathbf{g}^{i'}=g^{i'j'}\mathbf{g}_{j'}. \tag{2.68}$$

Similarly,

$$\mathbf{g}^{j'}=A^{j'}_j\mathbf{g}^j. \tag{2.69}$$

From Eqs.(2.67) and (2.69),

$$\delta^{i'}_{j'}=\mathbf{g}^{i'}\cdot\mathbf{g}_{j'}=A^{i'}_i\mathbf{g}^i\cdot A^j_{j'}\mathbf{g}_j=A^{i'}_iA^j_{j'}\delta^i_j=A^{i'}_kA^k_{j'}. \tag{2.70}$$

Thus the matrix $(A^{i'}_i)_{3\times3}$ is full, which is an inverse of the matrix $(A^i_{i'})_{3\times3}$.

$$\begin{aligned}
g_{i'j'}&=\mathbf{g}_{i'}\cdot\mathbf{g}_{j'}=A^i_{i'}\mathbf{g}_i\cdot A^j_{j'}\mathbf{g}_j=A^i_{i'}A^j_{j'}g_{ij},\\
g^{i'j'}&=\mathbf{g}^{i'}\cdot\mathbf{g}^{j'}=A^{i'}_i\mathbf{g}^i\cdot A^{j'}_j\mathbf{g}^j=A^{i'}_iA^{j'}_jg^{ij};\\
\varepsilon_{i'j'k'}&=[\mathbf{g}_{i'}\mathbf{g}_{j'}\mathbf{g}_{k'}]=A^i_{i'}A^j_{j'}A^k_{k'}[\mathbf{g}_i\mathbf{g}_j\mathbf{g}_k]=A^i_{i'}A^j_{j'}A^k_{k'}\varepsilon_{ijk},\\
\varepsilon^{i'j'k'}&=[\mathbf{g}^{i'}\mathbf{g}^{j'}\mathbf{g}^{k'}]=A^{i'}_iA^{j'}_jA^{k'}_k[\mathbf{g}^i\mathbf{g}^j\mathbf{g}^k]=A^{i'}_iA^{j'}_jA^{k'}_k\varepsilon^{ijk},\\
\sqrt{g'}&=|A^i_{i'}|\sqrt{g}.
\end{aligned} \tag{2.71}$$

Although the components of a vector $\mathbf{v}$ vary with different coordinates, the vector $\mathbf{v}$ is independent of such coordinates. For different coordinate systems, the vector $\mathbf{v}$ is expressed by

$$\begin{aligned}
\mathbf{v}&=v^{i'}\mathbf{g}_{i'}=v^{i'}A^i_{i'}\mathbf{g}_i=v^i\mathbf{g}_i=v^iA^{i'}_i\mathbf{g}_{i'}\\
&=v_{i'}\mathbf{g}^{i'}=v_{i'}A^{i'}_i\mathbf{g}^i=v_i\mathbf{g}^i=v_iA^i_{i'}\mathbf{g}^{i'}.
\end{aligned} \tag{2.72}$$

The foregoing equation yields

$$\begin{aligned}
v^{i'}&=v^iA^{i'}_i,\ v^i=v^{i'}A^i_{i'},\\
v_{i'}&=v_iA^i_{i'},\ v_i=v_{i'}A^{i'}_i.
\end{aligned} \tag{2.73}$$

For a given coordinate system $\{x^i\}$, the corresponding covariant vector bases can be determined (i.e., $\mathbf{g}_i=\partial\mathbf{p}/\partial x^i$). Such a coordinate system is called a complete coordinate frame. Consider any three linearly-independent base vectors $\mathbf{g}_{(i)}=A^i_{(i)}\mathbf{g}_i$ ($A^i_{(i)}$ is differentiable). If one can find a coordinate system $\{x^{(i)}\}$ to make

$\mathbf{g}_{(i)} = \partial \mathbf{p} / \partial x^{(i)}$ exist, then this coordinate system is complete. Otherwise, this coordinate system is incomplete. From $\mathbf{g}_{(i)} = A^i_{(i)} \mathbf{g}_i$, tensor components in a complete coordinate system can be converted in the incomplete coordinate system.

2.1.4. Tensor algebra

As in vectors, any mixed tensor can be expressed in a dyadic form of

$$\boldsymbol{\psi} = \psi^{i_1 \cdots i_m}{}_{j_1 \cdots j_n} \sqrt{g}^{-W} \mathbf{g}_{i_1} \cdots \mathbf{g}_{i_m} \mathbf{g}^{j_1} \cdots \mathbf{g}^{j_n}, \tag{2.74}$$

where $\psi^{i_1 \cdots i_m}{}_{j_1 \cdots j_n}$ is the component of the tensor. The number of total indices is *the order of tensor*. W is the weight of the tensor. If $W = 0$, the tensor $\boldsymbol{\psi}$ is called *the absolute tensor*. Otherwise, the tensor $\boldsymbol{\psi}$ is called the relative tensor or the tensor density. If $W = 1$, the tensor possesses *the zero-order density*. If $\psi^{i_1 \cdots i_m}{}_{j_1 \cdots j_n} = 0$, the tensor is called the *zero tensor*. If two tensors have the same weight and the same order, the two tensors are of the same type. Any vector is a tensor of the first-order. Any scalar is a tensor of the zero-order. Because a tensor is the same as a vector in different coordinate systems, any tensor is independent of coordinate systems. Any tensor can be expressed through the base vectors of different coordinate systems. Thus, the tensor in Eq.(2.74) is expressed as

$$\begin{aligned}
\boldsymbol{\psi} &= \psi^{i'_1 \cdots i'_m}{}_{j'_1 \cdots j'_n} \sqrt{g'}^{-W} \mathbf{g}_{i'_1} \cdots \mathbf{g}_{i'_m} \mathbf{g}^{j'_1} \cdots \mathbf{g}^{j'_n} \\
&= (| A^{p'}_p |^{-W} A^{i_1}_{i'_1} \cdots A^{i_m}_{i'_m} A^{j'_1}_{j_1} \cdots A^{j'_n}_{j_n} \psi^{i'_1 \cdots i'_m}{}_{j'_1 \cdots j'_n}) \sqrt{g}^{-W} \mathbf{g}_{i_1} \cdots \mathbf{g}_{i_m} \mathbf{g}^{j_1} \cdots \mathbf{g}^{j_n} \\
&= (| A^{p}_{p'} |^{W} A^{i_1}_{i'_1} \cdots A^{i_m}_{i'_m} A^{j'_1}_{j_1} \cdots A^{j'_n}_{j_n} \psi^{i'_1 \cdots i'_m}{}_{j'_1 \cdots j'_n}) \sqrt{g}^{-W} \mathbf{g}_{i_1} \cdots \mathbf{g}_{i_m} \mathbf{g}^{j_1} \cdots \mathbf{g}^{j_n} \\
&= \psi^{i_1 \cdots i_m}{}_{j_1 \cdots j_n} \sqrt{g}^{-W} \mathbf{g}_{i_1} \cdots \mathbf{g}_{i_m} \mathbf{g}^{j_1} \cdots \mathbf{g}^{j_n} = \cdots \\
&= \psi_{i_1 \cdots i_m}{}^{j_1 \cdots j_n} \sqrt{g}^{-W} \mathbf{g}^{i_1} \cdots \mathbf{g}^{i_m} \mathbf{g}_{j_1} \cdots \mathbf{g}_{j_n}.
\end{aligned} \tag{2.75}$$

The dyadic expressions of the *metric tensors* and *Eddington tensors* are given as

$$\begin{aligned}
\mathbf{I} &= g_{ij} \mathbf{g}^i \mathbf{g}^j = \mathbf{g}_i \mathbf{g}^i = \delta^i_j \mathbf{g}_i \mathbf{g}^j = g^{ij} \mathbf{g}_i \mathbf{g}_j, \\
\boldsymbol{\varepsilon} &= \varepsilon_{ijk} \mathbf{g}^i \mathbf{g}^j \mathbf{g}^k = \varepsilon^i_{\cdot jk} \mathbf{g}_i \mathbf{g}^j \mathbf{g}^k = \cdots = \varepsilon^{ijk} \mathbf{g}_i \mathbf{g}_j \mathbf{g}_k.
\end{aligned} \tag{2.76}$$

From the foregoing equation, the metric tensor and Kronecker-delta are the same geometric characteristic quantities.

Definition 2.11. For a transformation of two coordinates $x^i = x^i(x^{1'}, x^{2'}, \ldots, x^{N'})$ ($i = 1, 2, \ldots, N$),

(i) a function $\phi(x^1, x^2, \ldots, x^N)$ is termed *an absolute scalar* if it does not change its original values, i.e.,

$$\phi(x^1, x^2, \ldots, x^N) = \phi'(x^{1'}, x^{2'}, \ldots, x^{N'}). \tag{2.77}$$

(ii) quantities v^i and v_i ($i = 1, 2, \ldots, N$) are called the *absolute contravariant, and covariant components* of a vector (or simply called, *contravariant and covariant vectors*), respectively if

$$\begin{aligned} v^{i'} &= A^{i'}_j v^j && \text{(contravariant vector)},\\ v_{i'} &= A^j_{i'} v_j && \text{(covariant vector)}. \end{aligned} \tag{2.78}$$

(iii) quantities $v^{i_1\cdots i_m j_1\cdots j_n}$, $v_{i_1\cdots i_m j_1\cdots j_n}$ and $v^{i_1\cdots i_m}{}_{j_1\cdots j_n}$ ($i_1, \ldots, i_m, j_1, \ldots, j_n = 1, 2, \ldots, N$) are respectively called the *absolute* contravariant, covariant and mixed components of a tensor (or simply called, *contravariant, covariant, and mixed tensors*), respectively if

$$\begin{aligned} v^{i'_1\cdots i'_m j'_1\cdots j'_n} &= A^{i'_1}_{i_1}\cdots A^{i'_m}_{i_m} A^{j'_1}_{j_1}\cdots A^{j'_n}_{j_n} v^{i_1\cdots i_m j_1\cdots j_n} \text{ (contravariant tensor)},\\ v_{i'_1\cdots i'_m j'_1\cdots j'_n} &= A^{i_1}_{i'_1}\cdots A^{i_m}_{i'_m} A^{j_1}_{j'_1}\cdots A^{j_n}_{j'_n} v_{i_1\cdots i_m j_1\cdots j_n} \text{ (covariant tensor)},\\ v^{i'_1\cdots j'_m}{}_{j'_1\cdots j'_n} &= A^{i'_1}_{i_1}\cdots A^{i'_m}_{i_m} A^{j_1}_{j'_1}\cdots A^{j_n}_{j'_n} v^{i_1\cdots i_m}{}_{j_1\cdots j_n} \text{ (mixed tensor)}. \end{aligned} \tag{2.79}$$

Definition 2.12. For a transformation of two coordinates $x^i = x^i(x^{1'}, x^{2'}, \ldots, x^{N'})$ ($i = 1, 2, \ldots, N$),

(i) a function $\phi(x^1, x^2, \ldots, x^N)$ is termed *a relative scalar of weight W* if it does not change its original values, i.e.,

$$\phi(x^1, x^2, \ldots, x^N) = |A^p_{p'}|^W \phi'(x^{1'}, x^{2'}, \ldots, x^{N'}), \tag{2.80}$$

(ii) quantities v^i and v_i ($i = 1, 2, \ldots, N$) are called the relative *contravariant and covariant components of a vector of weight W* (or simply called, *relative contravariant* and *covariant vectors of weight W*) , respectively if

$$\begin{aligned} v^{i'} &= |A^{p'}_p|^W A^{i'}_j v^j && \text{(relative contravariant vector)},\\ v_{i'} &= |A^p_{p'}|^W A^j_{i'} v_j && \text{(relative covariant vector)}. \end{aligned} \tag{2.81}$$

(iii) quantities $v^{i_1\cdots i_m j_1\cdots j_n}$, $v_{i_1\cdots i_m j_1\cdots j_n}$ and $v^{i_1\cdots i_m}{}_{j_1\cdots j_n}$ ($i_1, \ldots, i_m, j_1, \ldots, j_n = 1, 2, \ldots, N$) are respectively called the *relative contravariant, covariant and mixed components* of a tensor *of weight W* (or simply called, *contravariant*, *covariant* and *mixed tensors of weight W*) , respectively if

$$
\begin{aligned}
v^{i'_1\cdots i'_m j'_1\cdots j'_n} &= |A^{p}_{p'}|^{W}\ A^{i'_1}_{i_1}\cdots A^{i'_m}_{i_m} A^{j'_1}_{j_1}\cdots A^{j'_n}_{j_n} v^{i_1\cdots i_m j_1\cdots j_n} \ \text{(contravariant tensor)},\\
v_{i'_1\cdots i'_m j'_1\cdots j'_n} &= |A^{p}_{p'}|^{W}\ A^{i_1}_{i'_1}\cdots A^{i_m}_{i'_m} A^{j_1}_{j'_1}\cdots A^{j_n}_{j'_n} v_{i_1\cdots i_m j_1\cdots j_n} \ \text{(covariant tensor)},\\
v^{i'_1\cdots i'_m}{}_{j'_1\cdots j'_n} &= |A^{p}_{p'}|^{W}\ A^{i'_1}_{i_1}\cdots A^{i'_m}_{i_m} A^{j_1}_{j'_1}\cdots A^{j_n}_{j'_n} v^{i_1\cdots i_m}{}_{j_1\cdots j_n} \ \text{(mixed tensor)}.
\end{aligned}
\tag{2.82}
$$

In a tensor space $\mathscr{E}$, there are three tensors of the same type (i.e., $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$ and $\boldsymbol{\eta}$). These tensors satisfy the following rules.

(i) *Tensor addition*

$$
\begin{aligned}
\boldsymbol{\varphi}+\boldsymbol{\psi} &= \boldsymbol{\psi}+\boldsymbol{\varphi},\\
\boldsymbol{\varphi}+(\boldsymbol{\psi}+\boldsymbol{\eta}) &= (\boldsymbol{\psi}+\boldsymbol{\varphi})+\boldsymbol{\eta},\\
\boldsymbol{\varphi}+\mathbf{0} &= \boldsymbol{\varphi},\\
\boldsymbol{\varphi}+(-\boldsymbol{\varphi}) &= \mathbf{0}.
\end{aligned}
\tag{2.83}
$$

(ii) *Tensor scalar multiplication*

$$
\begin{aligned}
k(\boldsymbol{\varphi}) &= k\boldsymbol{\varphi},\\
a(b\boldsymbol{\varphi}) &= (ab)\boldsymbol{\varphi},\\
(a+b)\boldsymbol{\varphi} &= a\boldsymbol{\varphi}+b\boldsymbol{\varphi},\\
a(\boldsymbol{\varphi}+\boldsymbol{\psi}) &= a\boldsymbol{\varphi}+a\boldsymbol{\psi}.
\end{aligned}
\tag{2.84}
$$

(A) *Tensor addition* Two tensors of the *same* order and type can be added to form a new tensor.

$$
\begin{aligned}
\boldsymbol{\psi}+\boldsymbol{\varphi} &= \psi^{i_1\cdots i_m}{}_{j_1\cdots j_n}\sqrt{g}^{-W}\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}\\
&\quad+\varphi^{i_1\cdots i_m}{}_{j_1\cdots j_n}\sqrt{g}^{-W}\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}\\
&= (\psi^{i_1\cdots i_m}{}_{j_1\cdots j_n}+\varphi^{i_1\cdots i_m}{}_{j_1\cdots j_n})\sqrt{g}^{-W}\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}\\
&\equiv \eta^{i_1\cdots i_m}{}_{j_1\cdots j_n}\sqrt{g}^{-W}\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}\\
&= \boldsymbol{\eta},
\end{aligned}
\tag{2.85}
$$

$$
\eta^{i_1\cdots i_m}{}_{j_1\cdots j_n} = \psi^{i_1\cdots i_m}{}_{j_1\cdots j_n} + \varphi^{i_1\cdots i_m}{}_{j_1\cdots j_n}. \tag{2.86}
$$

When the components of two tensors of the same type are different expressions, before addition, the component expressions should be changed to be same via the coordinate transformation tensor and the metric tensor. For a tensor, a new tensor obtained by lowering and/or raising indices is called the *associated tensor* of the given tensor.

(B) *Tensor multiplication* The *outer product of two tensors* is carried out by simply multiplying their components. This operation gives a new tensor and the tensor order is the sum of those of multipliers. The base vectors will follow the dyadic rules to express a new tensor.

$$
\begin{aligned}
\boldsymbol{\psi}\boldsymbol{\varphi} &= (\psi^{i_1\cdots i_m}{}_{j_1\cdots j_n}\sqrt{g}^{-W_1}\,\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}) \\
&\quad\cdot(\varphi^{k_1\cdots k_r}{}_{l_1\cdots l_s}\sqrt{g}^{-W_2}\,\mathbf{g}_{k_1}\cdots\mathbf{g}_{k_r}\mathbf{g}^{l_1}\cdots\mathbf{g}^{l_s}) \\
&= (\psi^{i_1\cdots i_m}{}_{j_1\cdots j_n}\varphi^{k_1\cdots k_r}{}_{l_1\cdots l_s})\sqrt{g}^{-(W_1+W_2)} \\
&\quad\cdot\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}\mathbf{g}_{k_1}\cdots\mathbf{g}_{k_r}\mathbf{g}^{l_1}\cdots\mathbf{g}^{l_s} \\
&\equiv \eta^{i_1\cdots i_m}{}_{j_1\cdots j_n}{}^{k_1\cdots k_r}{}_{l_1\cdots l_s}\sqrt{g}^{-W}\,\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}\mathbf{g}_{k_1}\cdots\mathbf{g}_{k_r}\mathbf{g}^{l_1}\cdots\mathbf{g}^{l_s} \\
&= \boldsymbol{\eta},
\end{aligned}
\tag{2.87}
$$

$$
\eta^{i_1\cdots i_m}{}_{j_1\cdots j_n}{}^{k_1\cdots k_r}{}_{l_1\cdots l_s} = \psi^{i_1\cdots i_m}{}_{j_1\cdots j_n}\varphi^{k_1\cdots k_r}{}_{l_1\cdots l_s}\,. \tag{2.88}
$$

(C) *Tensor contraction* Contraction is the dot product of any two base vectors in the tensor. In a mixed tensor the operation of equating a superscript index to a subscript index. *A contraction operation decreases the order a tensor by two.*

$$
\begin{aligned}
\overbrace{\boldsymbol{\psi}}^{\bullet} &= \psi^{i_1\cdots i_k\cdots i_l}{}_{j_1\cdots j_m\cdots j_n}\sqrt{g}^{-W}\,\mathbf{g}_{i_1}\cdots\overbrace{\mathbf{g}_{i_k}\cdots\mathbf{g}_{i_l}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_m}}^{\bullet}\cdots\mathbf{g}^{j_n} \\
&= \psi^{i_1\cdots i_k\cdots i_l}{}_{j_1\cdots j_m\cdots j_n}\delta^{j_m}_{i_k}\sqrt{g}^{-W}\,\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_{k-1}}\mathbf{g}_{i_{k+1}}\cdots\mathbf{g}_{i_l}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_{m-1}}\mathbf{g}^{j_{m+1}}\cdots\mathbf{g}^{j_n} \\
&= \psi^{i_1\cdots i_{k-1}i_k i_{k+1}\cdots i_l}{}_{j_1\cdots j_{m-1}i_k j_{m+1}\cdots j_n}\sqrt{g}^{-W}\,\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_{k-1}}\mathbf{g}_{i_{k+1}}\cdots\mathbf{g}_{i_l}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_{m-1}}\mathbf{g}^{j_{m+1}}\cdots\mathbf{g}^{j_n} \\
&\equiv \varphi^{i_1\cdots i_{k-1}i_{k+1}\cdots i_l}{}_{j_1\cdots j_{m-1}j_{m+1}\cdots j_n}\sqrt{g}^{-W}\,\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_{k-1}}\mathbf{g}_{i_{k+1}}\cdots\mathbf{g}_{i_l}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_{m-1}}\mathbf{g}^{j_{m+1}}\cdots\mathbf{g}^{j_n} \\
&= \boldsymbol{\varphi}.
\end{aligned}
\tag{2.89}
$$

After contraction, a new quantity still is a tensor, and the tensor character can be verified by the *quotient rule*. That is,

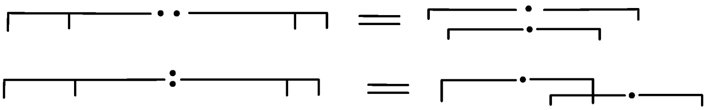
$$
\begin{aligned}
\boldsymbol{\varphi} &= \varphi^{i'_1\cdots i'_{k-1}i'_{k+1}\cdots i'_l}{}_{j'_1\cdots j'_{m-1}j'_{m+1}\cdots j'_n}\sqrt{g'}^{-W} \\
&\quad\cdot\mathbf{g}_{i'_1}\cdots\mathbf{g}_{i'_{k-1}}\mathbf{g}_{i'_{k+1}}\cdots\mathbf{g}_{i'_l}\mathbf{g}^{j'_1}\cdots\mathbf{g}^{j'_{m-1}}\mathbf{g}^{j'_{m+1}}\cdots\mathbf{g}^{j'_n} \\
&= \varphi^{i'_1\cdots i'_{k-1}i'_{k-1}i'_{k+1}\cdots i'_l}{}_{j'_1\cdots j'_{m-1}j'_m j'_{m+1}\cdots j'_n}\delta^{i'_k}_{j'_m}\sqrt{g'}^{-W} \\
&\quad\cdot\mathbf{g}_{i'_1}\cdots\mathbf{g}_{i'_{k-1}}\mathbf{g}_{i'_k}\mathbf{g}_{i'_{k+1}}\cdots\mathbf{g}_{i'_l}\mathbf{g}^{j'_1}\cdots\mathbf{g}^{j'_{m-1}}\mathbf{g}^{j'_m}\mathbf{g}^{j'_{m+1}}\cdots\mathbf{g}^{j'_n}, \\
&= \varphi^{i'_1\cdots i'_{k-1}i'_k i'_{k+1}\cdots i'_l}{}_{j'_1\cdots j'_{m-1}j'_m j'_{m+1}\cdots j'_n}\delta^{i'_k}_{j'_m}\sqrt{g'}^{-W}A^{i_1}_{i'_1}\cdots A^{i_k}_{i'_k}\cdots A^{i_l}_{i'_l}A^{j'_1}_{j_1}\cdots A^{j'_m}_{j_m}\cdots A^{j'_n}_{j_n} \\
&\quad\cdot\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_{k-1}}\overbrace{\mathbf{g}_{i_k}\mathbf{g}_{i_{k+1}}\cdots\mathbf{g}_{i_l}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_{m-1}}}^{\bullet}\mathbf{g}^{j_m}\mathbf{g}^{j_{m+1}}\cdots\mathbf{g}^{j_n} \\
&= \psi^{i_1\cdots i_{k-1}i_{k+1}\cdots i_l}{}_{j_1\cdots j_{m-1}j_{m+1}\cdots j_n}\sqrt{g}^{-W} \\
&\quad\cdot\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_{k-1}}\mathbf{g}_{i_{k+1}}\cdots\mathbf{g}_{i_l}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_{m-1}}\mathbf{g}^{j_{m+1}}\cdots\mathbf{g}^{j_n}.
\end{aligned}
\tag{2.90}
$$

The contraction can be operated continuously. For instance, two continuous contractions are given as follows.

$$\psi^{i_1\cdots i_{k-1}i_k i_{k+1}\cdots i_l}{}_{j_1\cdots j_{m-1}j_m j_{m+1}\cdots j_n}\sqrt{g}^{-W}\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_{k-1}}\mathbf{g}_{i_k}\mathbf{g}_{i_{k+1}}\cdots\mathbf{g}_{i_l}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_{m-1}}\mathbf{g}^{j_m}\mathbf{g}^{j_{m+1}}\cdots\mathbf{g}^{j_n}$$
$$=\psi^{i_1\cdots i_{k-1}i_k i_{k+1}\cdots i_l}{}_{j_1\cdots j_{m-1}j_m j_{m+1}\cdots j_n}\delta^{j_m}_{i_{k-1}}\delta^{j_{m+1}}_{i_{k+1}}\sqrt{g}^{-W}\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_{k-2}}\mathbf{g}_{i_k}\mathbf{g}_{i_{k+2}}\cdots\mathbf{g}_{i_l}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_{m-1}}\mathbf{g}^{j_{m+2}}$$
$$=\varphi^{i_1\cdots i_{k-2}i_k i_{k+2}\cdots i_l}{}_{j_1\cdots j_{m-1}j_{m+2}\cdots j_n}\sqrt{g}^{-W}\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_{k-2}}\mathbf{g}_{i_k}\mathbf{g}_{i_{k+2}}\cdots\mathbf{g}_{i_l}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_{m-1}}\mathbf{g}^{j_{m+2}}\cdots\mathbf{g}^{j_n}. \tag{2.91}$$

$$\psi^{i_1\cdots i_{k-1}i_k i_{k+1}\cdots i_l}{}_{j_1\cdots j_{m-1}j_m j_{m+1}\cdots j_n}\sqrt{g}^{-W}\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_{k-1}}\mathbf{g}_{i_k}\mathbf{g}_{i_{k+1}}\cdots\mathbf{g}_{i_l}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_{m-1}}\mathbf{g}^{j_m}\mathbf{g}^{j_{m+1}}\cdots\mathbf{g}^{j_n}$$
$$=\psi^{i_1\cdots i_{k-1}i_k i_{k+1}\cdots i_l}{}_{j_1\cdots j_{m-1}j_m j_{m+1}\cdots j_n}\delta^{j_{m+1}}_{i_{k-1}}\delta^{j_m}_{i_{k+1}}\sqrt{g}^{-W}$$
$$\cdot\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_{k-2}}\mathbf{g}_{i_k}\mathbf{g}_{i_{k+2}}\cdots\mathbf{g}_{i_l}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_{m-1}}\mathbf{g}^{j_{m+2}}\cdots\mathbf{g}^{j_n}$$
$$\equiv\eta^{i_1\cdots i_{k-2}i_k i_{k+2}\cdots i_l}{}_{j_1\cdots j_{m-1}j_{m+2}\cdots j_n}\sqrt{g}^{-W}\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_{k-2}}\mathbf{g}_{i_k}\mathbf{g}_{i_{k+2}}\cdots\mathbf{g}_{i_l}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_{m-1}}\mathbf{g}^{j_{m+2}}\cdots\mathbf{g}^{j_n} \tag{2.92}$$

Note that the contraction symbols represent the following meaning.



(D) *Tensor dot product* The tensor dot product is a simple multiplication of tensors with the prescribed base vector contraction.

$$\overset{\bullet}{\boldsymbol{\psi}\,\boldsymbol{\varphi}}=(\psi^{ij}{}_{kl}\sqrt{g}^{-W_1}\mathbf{g}_i\mathbf{g}_j\mathbf{g}^k\mathbf{g}^l)(\varphi^{pq}{}_{st}\sqrt{g}^{-W_2}\mathbf{g}_p\mathbf{g}_q\mathbf{g}^s\mathbf{g}^t)$$
$$=\psi^{ij}{}_{kl}\varphi^{pk}{}_{st}\sqrt{g}^{-(W_1+W_2)}\mathbf{g}_i\mathbf{g}_j\mathbf{g}^l\mathbf{g}_p\mathbf{g}^s\mathbf{g}^t$$
$$\equiv\eta^{ij\cdot p}_{l\cdot st}\,\mathbf{g}_i\mathbf{g}_j\mathbf{g}^l\mathbf{g}_p\mathbf{g}^s\mathbf{g}^t. \tag{2.93}$$

If the tensor dot product is based on the two adjacent base vector contraction, then the above expression becomes

$$\boldsymbol{\psi}\cdot\boldsymbol{\varphi}=(\psi^{ij}{}_{kl}\sqrt{g}^{-W_1}\mathbf{g}_i\mathbf{g}_j\mathbf{g}^k\mathbf{g}^l)\cdot(\varphi^{pq}{}_{st}\sqrt{g}^{-W_2}\mathbf{g}_p\mathbf{g}_q\mathbf{g}^s\mathbf{g}^t)$$
$$=(\psi^{ij}{}_{kl}\sqrt{g}^{-W_1}\mathbf{g}_i\mathbf{g}_j\mathbf{g}^k\mathbf{g}^l)(\varphi^{pq}{}_{st}\sqrt{g}^{-W_2}\mathbf{g}_p\mathbf{g}_q\mathbf{g}^s\mathbf{g}^t)$$
$$=\psi^{ij}{}_{kl}\varphi^{lq}{}_{st}\sqrt{g}^{-(W_1+W_2)}\mathbf{g}_i\mathbf{g}_j\mathbf{g}^k\mathbf{g}_q\mathbf{g}^s\mathbf{g}^t$$
$$\equiv\eta^{ij\cdot q}_{k\cdot st}\,\mathbf{g}_i\mathbf{g}_j\mathbf{g}^l\mathbf{g}_p\mathbf{g}^s\mathbf{g}^t. \tag{2.94}$$

If the tensor dot product is based on the contraction of two pairs of the adjacent base vectors, then

$$
\begin{aligned}
\boldsymbol{\psi} : \boldsymbol{\varphi} &= (\psi^{ij}{}_{kl}\sqrt{g}^{-W_1}\mathbf{g}_i\mathbf{g}_j\mathbf{g}^k\mathbf{g}^l) : (\varphi^{pq}{}_{st}\sqrt{g}^{-W_2}\mathbf{g}_p\mathbf{g}_q\mathbf{g}^s\mathbf{g}^t) \\
&= (\psi^{ij}{}_{kl}\sqrt{g}^{-W_1}\mathbf{g}_i\mathbf{g}_j\mathbf{g}^k\mathbf{g}^l)(\varphi^{pq}{}_{st}\sqrt{g}^{-W_2}\mathbf{g}_p\mathbf{g}_q\mathbf{g}^s\mathbf{g}^t) \\
&= \psi^{ij}{}_{kl}\varphi^{pq}{}_{st}\delta^k_p\delta^l_q\sqrt{g}^{-(W_1+W_2)}\mathbf{g}_i\mathbf{g}_j\mathbf{g}^s\mathbf{g}^t \\
&= \psi^{ij}{}_{kl}\varphi^{kl}{}_{st}\sqrt{g}^{-(W_1+W_2)}\mathbf{g}_i\mathbf{g}_j\mathbf{g}^s\mathbf{g}^t \\
&\equiv \eta^{ij}{}_{st}\,\mathbf{g}_i\mathbf{g}_j\mathbf{g}^s\mathbf{g}^t.
\end{aligned}
\tag{2.95}
$$

$$
\begin{aligned}
\boldsymbol{\psi} \cdot\cdot\, \boldsymbol{\varphi} &= (\psi^{ij}{}_{kl}\sqrt{g}^{-W_1}\mathbf{g}_i\mathbf{g}_j\mathbf{g}^k\mathbf{g}^l) \cdot\cdot\, (\varphi^{pq}{}_{st}\sqrt{g}^{-W_2}\mathbf{g}_p\mathbf{g}_q\mathbf{g}^s\mathbf{g}^t) \\
&= (\psi^{ij}{}_{kl}\sqrt{g}^{-W_1}\mathbf{g}_i\mathbf{g}_j\mathbf{g}^k\mathbf{g}^l)(\varphi^{pq}{}_{st}\sqrt{g}^{-W_2}\mathbf{g}_p\mathbf{g}_q\mathbf{g}^s\mathbf{g}^t) \\
&= \psi^{ij}{}_{kl}\varphi^{pq}{}_{st}\delta^k_q\delta^l_p\sqrt{g}^{-(W_1+W_2)}\mathbf{g}_i\mathbf{g}_j\mathbf{g}^s\mathbf{g}^t \\
&= \psi^{ij}{}_{kl}\varphi^{lk}{}_{st}\sqrt{g}^{-(W_1+W_2)}\mathbf{g}_i\mathbf{g}_j\mathbf{g}^s\mathbf{g}^t \equiv \xi^{ij}{}_{st}\mathbf{g}_i\mathbf{g}_j\mathbf{g}^s\mathbf{g}^t.
\end{aligned}
\tag{2.96}
$$

(E) *Tensor cross product* The cross product is based on the cross product of the adjacent base vectors of the same type.

$$
\begin{aligned}
\boldsymbol{\psi} \times \boldsymbol{\varphi} &= (\psi^{ij}{}_{kl}\sqrt{g}^{-W_1}\mathbf{g}_i\mathbf{g}_j\mathbf{g}^k\mathbf{g}^l) \times (\varphi_{pq}{}^{st}\sqrt{g}^{-W_2}\mathbf{g}^p\mathbf{g}^q\mathbf{g}_s\mathbf{g}_t) \\
&= (\psi^{ij}{}_{kl}\varphi_{pq}{}^{st}\sqrt{g}^{-(W_1+W_2)}\mathbf{g}_i\mathbf{g}_j\mathbf{g}^k(\mathbf{g}^l\times\mathbf{g}^p)\mathbf{g}^q\mathbf{g}_s\mathbf{g}_t) \\
&= \psi^{ij}{}_{kl}\varphi_{pq}{}^{st}\varepsilon^{lpm}\sqrt{g}^{-W}\mathbf{g}_i\mathbf{g}_j\mathbf{g}^k\mathbf{g}_m\mathbf{g}^q\mathbf{g}_s\mathbf{g}_t \\
&\equiv \eta^{ij\cdot m\cdot st}_{k\cdot q}\mathbf{g}_i\mathbf{g}_j\mathbf{g}^k\mathbf{g}_m\mathbf{g}^q\mathbf{g}_s\mathbf{g}_t.
\end{aligned}
\tag{2.97}
$$

$$
\begin{aligned}
\boldsymbol{\psi} {}^{\times}_{\times} \boldsymbol{\varphi} &= (\psi^{ij}{}_{kl}\sqrt{g}^{-W_1}\mathbf{g}_i\mathbf{g}_j\mathbf{g}^k\mathbf{g}^l)(\varphi_{pq}{}^{rs}\sqrt{g}^{-W_2}\mathbf{g}^p\mathbf{g}^q\mathbf{g}_r\mathbf{g}_s) \\
&= (\psi^{ij}{}_{kl}\varphi_{pq}{}^{rs}\sqrt{g}^{-(W_1+W_2)}\mathbf{g}_i\mathbf{g}_j(\mathbf{g}^k\times\mathbf{g}^p)(\mathbf{g}^l\times\mathbf{g}^q)\mathbf{g}_r\mathbf{g}_s \\
&= \psi^{ij}{}_{kl}\varphi_{pq}{}^{rs}\varepsilon^{kpm}\varepsilon^{lqn}\sqrt{g}^{-W}\mathbf{g}_i\mathbf{g}_j\mathbf{g}_m\mathbf{g}_n\mathbf{g}_r\mathbf{g}_s \\
&\equiv \xi^{ijmnrs}\mathbf{g}_i\mathbf{g}_j\mathbf{g}_m\mathbf{g}_n\mathbf{g}_r\mathbf{g}_s.
\end{aligned}
\tag{2.98}
$$

(F) *Tensor index permutation* Permutation of superscript indices $i, \ldots, j, \ldots, k$ (or subscript indices $r, \ldots, s, \ldots, t$) of a tensor gives a new tensor with the same type. For instance,

$$
\begin{aligned}
\boldsymbol{\psi} &= \psi^{k\cdots i\cdots j}{}_{r\cdots s\cdots t}\sqrt{g}^{-W}\mathbf{g}_i\cdots\mathbf{g}_j\cdots\mathbf{g}_k\mathbf{g}^r\cdots\mathbf{g}^s\cdots\mathbf{g}^t \\
&= \varphi^{i\cdots j\cdots k}{}_{r\cdots s\cdots t}\sqrt{g}^{-W}\mathbf{g}_i\cdots\mathbf{g}_j\cdots\mathbf{g}_k\mathbf{g}^r\cdots\mathbf{g}^s\cdots\mathbf{g}^t = \boldsymbol{\varphi}.
\end{aligned}
\tag{2.99}
$$

From the foregoing equation, the locations of subscript (or superscript) indices are very important.

(G) *Tensor symmetry* If two-index permutation of superscript indices $i, \ldots, j, \ldots, k$ (or subscript indices $r, \ldots, s, \ldots, t$) of a tensor generates a new tensor that is the same as the old tensor, such a tensor is symmetric for two indices.

$$\begin{aligned}\boldsymbol{\psi} &= \psi^{k\cdots j\cdots i}{}_{r\cdots s\cdots t}\sqrt{g}^{-W}\mathbf{g}_i\cdots\mathbf{g}_j\cdots\mathbf{g}_k\mathbf{g}^r\cdots\mathbf{g}^s\cdots\mathbf{g}^t \\ &= \psi^{i\cdots j\cdots k}{}_{r\cdots s\cdots t}\sqrt{g}^{-W}\mathbf{g}_i\cdots\mathbf{g}_j\cdots\mathbf{g}_k\mathbf{g}^r\cdots\mathbf{g}^s\cdots\mathbf{g}^t.\end{aligned} \tag{2.100}$$

(H) *Tensor asymmetry* If two-index permutation of superscript indices $i, \ldots, j, \ldots, k$ (or subscript indices $r, \ldots, s, \ldots, t$) of a tensor gives a new tensor, its sign with the old tensor is opposite. Such a tensor is asymmetric for such two indices.

$$\begin{aligned}\boldsymbol{\psi} &= \psi^{k\cdots j\cdots i}{}_{r\cdots s\cdots t}\sqrt{g}^{-W}\mathbf{g}_i\cdots\mathbf{g}_j\cdots\mathbf{g}_k\mathbf{g}^r\cdots\mathbf{g}^s\cdots\mathbf{g}^t \\ &= -\psi^{i\cdots j\cdots k}{}_{r\cdots s\cdots t}\sqrt{g}^{-W}\mathbf{g}_i\cdots\mathbf{g}_j\cdots\mathbf{g}_k\mathbf{g}^r\cdots\mathbf{g}^s\cdots\mathbf{g}^t.\end{aligned} \tag{2.101}$$

For a given tensor with *M*-indices on the same level (subscript or superscript), $M!$ times index exchange form $M!$ new tensors, and the average of $M!$ tensors is to form *a symmetric tensor*. Such a symmetric tensor of the original tensor is expressed by the subscript or subscript with parenthesis $(\cdot)$. For instance,

$$\begin{aligned}\varphi_{(ij)} &= \frac{1}{2!}(\varphi_{ij} + \varphi_{ji}), \\ \varphi_{(ijk)} &= \frac{1}{3!}(\varphi_{ijk} + \varphi_{jki} + \varphi_{kij} + \varphi_{kji} + \varphi_{ikj} + \varphi_{jik}).\end{aligned} \tag{2.102}$$

For a given tensor with *N*-indices on the same level (subscript or superscript), $N!$ times index-permutation forms $N!$ new tensors. Among the $N!$ tensors, after the $N!/2$ new tensors with odd permutation of index multiply negative one (-1), with the other new tensors, the average of the new tensors is to form an asymmetric tensor. Such an asymmetric tensor is expressed by the subscript or subscript with bracket $[\cdot]$. For instance,

$$\begin{aligned}\varphi_{[ij]} &= \frac{1}{2!}(\varphi_{ij} - \varphi_{ji}), \\ \varphi_{[ijk]} &= \frac{1}{3!}(\varphi_{ijk} + \varphi_{jki} + \varphi_{kij} - \varphi_{kji} - \varphi_{ikj} - \varphi_{jik}).\end{aligned} \tag{2.103}$$

(I) *Tensor quotient rule*

Theorem 2.1. *If A_i is an arbitrary covariant vector and A_iX^i is invariant, i.e.,*

$$A_iX^i = A_{i'}X^{i'}, \tag{2.104}$$

then X^i *is a contravariant vector.*

Proof: Because of

$$A_{i'} X^{i'} = A_i X^i = A_i^{i'} A_{i'} X^i$$

one obtains

$$A_{i'}(X^{i'} - A_i^{i'} X^i) = 0.$$

Because $A_{i'}$ is selected arbitrarily, the foregoing equation gives

$$X^{i'} = A_i^{i'} X^i.$$

It implies that X^i is a contravariant vector. This theorem is proved. ■

Theorem 2.2. If $A^{i_1 \cdots i_m}{}_{j_1 \cdots j_n}$ *is an arbitrary n-order covariant and m-order contravariant tensor and* $B^{j_{n+1} \cdots j_p}{}_{i_{m+1} \cdots i_q} = X^{j_1 \cdots j_n j_{n+1} \cdots j_p}{}_{i_1 \cdots i_m i_{m+1} \cdots i_q} A^{i_1 \cdots i_m}{}_{j_1 \cdots j_n}$ *is an* $(q-m)$*-order covariant and* $(p-n)$*-contravariant tensor, then* $X^{j_1 \cdots j_n j_{n+1} \cdots j_p}{}_{i_1 \cdots i_m i_{m+1} \cdots i_q}$ *is a q-order covariant and p-order contravariant tensor.*

Proof: Because $B^{j_{n+1} \cdots j_p}{}_{i_{m+1} \cdots i_q}$ is a tensor,

$$\begin{aligned}
B^{j'_{n+1} \cdots j'_p}{}_{i'_{m+1} \cdots i'_q} &= A^{j'_{n+1}}_{j_{n+1}} \cdots A^{j'_p}_{j_p} A^{i_{m+1}}_{i'_{m+1}} \cdots A^{i_q}_{i'_q} B^{j_{n+1} \cdots j_p}{}_{i_{m+1} \cdots i_q} \\
&= A^{j'_{n+1}}_{j_{n+1}} \cdots A^{j'_p}_{j_p} A^{i_{m+1}}_{i'_{m+1}} \cdots A^{i_q}_{i'_q} X^{j_1 \cdots j_n j_{n+1} \cdots j_p}{}_{i_1 \cdots i_m \cdots i_q} A^{i_1 \cdots i_m}{}_{j_1 \cdots j_n} \\
&= A^{j'_{n+1}}_{j_{n+1}} \cdots A^{j'_p}_{j_p} A^{i_{m+1}}_{i'_{m+1}} \cdots A^{i_q}_{i'_q} X^{j_1 \cdots j_n j_{n+1} \cdots j_p}{}_{i_1 \cdots i_m \cdots i_q} A^{j'_1}_{j_1} \cdots A^{j'_n}_{j_n} A^{i_1}_{i'_1} \cdots A^{i_m}_{i'_m} A^{i'_1 \cdots i'_m}{}_{j'_1 \cdots j'_n} \\
&= A^{j'_1}_{j_1} \cdots A^{j'_n}_{j_n} A^{j'_{m+1}}_{j_{n+1}} \cdots A^{j'_p}_{j_p} A^{i_1}_{i'_1} \cdots A^{i_m}_{i'_m} A^{i_{m+1}}_{i'_{m+1}} \cdots A^{i_q}_{i'_q} X^{j_1 \cdots j_n j_{n+1} \cdots j_p}{}_{i_1 \cdots i_m i_{m+1} \cdots i_q} A^{i'_1 \cdots i'_m}{}_{j'_1 \cdots j'_n}
\end{aligned}$$

For a new coordinate, $B^{j'_{n+1} \cdots j'_p}{}_{i'_{m+1} \cdots i'_q} = X^{j'_1 \cdots j'_n \cdots j'_p}{}_{i'_1 \cdots i'_m \cdots i'_q} A^{i'_1 \cdots i'_m}{}_{j'_1 \cdots j'_n}$. The foregoing two equations yield

$$\begin{aligned}
&(A^{j'_1}_{j_1} \cdots A^{j'_n}_{j_n} A^{j'_{n+1}}_{j_{n+1}} \cdots A^{j'_p}_{j_p} A^{i_1}_{i'_1} \cdots A^{i_m}_{i'_m} A^{i_{m+1}}_{i'_{m+1}} \cdots A^{i_q}_{i'_q} X^{j_1 \cdots j_n j_{n+1} \cdots j_p}{}_{i_1 \cdots i_m i_{m+1} \cdots i_q} \\
&- X^{j'_1 \cdots j'_n \cdots j'_p}{}_{i'_1 \cdots i'_m \cdots i'_q}) A^{i'_1 \cdots i'_m}{}_{j'_1 \cdots j'_n} = 0.
\end{aligned}$$

Because $A^{i'_1 \cdots i'_m}{}_{j'_1 \cdots j'_n}$ is selected arbitrarily, the following equation is obtained.

$$\begin{aligned}
&X^{j'_1 \cdots j'_n \cdots j'_p}{}_{i'_1 \cdots i'_m \cdots i'_q} \\
&= A^{j'_1}_{j_1} \cdots A^{j'_n}_{j_n} A^{j'_{n+1}}_{j_{n+1}} \cdots A^{j'_p}_{j_p} A^{i_1}_{i'_1} \cdots A^{i_m}_{i'_m} A^{i_{m+1}}_{i'_{m+1}} \cdots A^{i_q}_{i'_q} X^{j_1 \cdots j_n j_{n+1} \cdots j_p}{}_{i_1 \cdots i_m i_{m+1} \cdots i_q}.
\end{aligned}$$

Therefore, $X^{j_1 \cdots j_n j_{n+1} \cdots j_p}{}_{i_1 \cdots i_m i_{m+1} \cdots i_q}$ *is a q-order covariant and p-order contravariant* *tensor. This theorem is proved.* ■

(J) *Ricci symbol* For simplicity, the Ricci symbol (or permutation symbol) $e^{i\cdots j\cdots k}$ or $e_{i\cdots j\cdots k}$ are introduced as

$$e^{i_1 i_2\cdots i_N}\ (\text{or } e_{i_1 i_2\cdots i_N}) = \begin{cases} 0 & \text{when any two indices are equal,} \\ +1 & \text{when } (i_1, i_2, \ldots, i_N) \text{ are } (1, 2, \ldots, N) \text{ or} \\ & \quad \text{an even permutation of } (1, 2, \ldots, N), \\ -1 & \text{when } (i_1, i_2, \ldots, i_N) \text{ are an odd} \\ & \text{permutation of } (1, 2, \ldots, N). \end{cases} \tag{2.105}$$

Generalized Kronecker-delta:

$$\delta^{i_1 i_2\cdots i_N}_{j_1 j_2\cdots j_N} = \begin{cases} 0 & \text{when any two or more upper} \\ & \text{(or lower) indices are equal,} \\ +1 & \text{for even permutation between} \\ & \text{the upper index and lower index,} \\ -1 & \text{for odd permutation between} \\ & \text{the upper index and lower index,} \end{cases} \tag{2.106}$$

where

$$\delta^{i_1 i_2\cdots i_N}_{j_1 j_2\cdots j_N} = e^{i_1 i_2\cdots i_N} e_{j_1 j_2\cdots j_N}, \tag{2.107}$$

$$\delta^{i_1 i_2\cdots i_{N-1}}_{j_1 j_2\cdots j_{N-1}} = \delta^{i_1 i_2\cdots i_{N-1} i_N}_{j_1 j_2\cdots j_{N-1} i_N}. \tag{2.108}$$

From the Ricci symbol, the determinant of a matrix $(a^{m}_{\cdot n})_{q\times q}$ can be expressed by

$$|a^{m}_{\cdot n}| = e_{i_1 i_2\cdots i_q} a^{i_1}_{\cdot 1} a^{i_2}_{\cdot 2} \cdots a^{i_q}_{\cdot q} = e^{i_1 i_2\cdots i_q} a^{1}_{\cdot i_1} a^{2}_{\cdot i_2} \cdots a^{q}_{\cdot i_q}. \tag{2.109}$$

Theorem 2.3. *For a matrix* $(a^{m}_{\cdot n})_{3\times 3}$, *the following relations hold*

$$\begin{aligned} &e_{ijk} a^{i}_{\cdot r} a^{j}_{\cdot s} a^{k}_{\cdot t} = |a^{m}_{\cdot n}|\, e_{rst} \text{ and } e^{ijk} a^{r}_{\cdot i} a^{s}_{\cdot j} a^{t}_{\cdot k} = |a^{m}_{\cdot n}|\, e^{rst}, \\ &e^{ijk} a_{ir} a_{js} a_{kt} = e^{ijk} a_{ri} a_{sj} a_{tk} = |a_{mn}|\, e^{rst}, \\ &e_{ijk} a^{ir} a^{js} a^{kt} = e_{ijk} a^{ri} a^{sj} a^{tk} = |a^{mn}|\, e_{rst}. \end{aligned} \tag{2.110}$$

Proof: The asymmetry of Ricci symbol gives

$$e_{ijk} a^{i}_{\cdot r} a^{j}_{\cdot s} a^{k}_{\cdot t} = e_{kji} a^{k}_{\cdot r} a^{j}_{\cdot s} a^{i}_{\cdot t} = e_{kji} a^{i}_{\cdot t} a^{j}_{\cdot s} a^{k}_{\cdot r} = -e_{ijk} a^{i}_{\cdot t} a^{j}_{\cdot s} a^{k}_{\cdot r}.$$

For index (i, j, k), $e_{ijk} a^{i}_{\cdot r} a^{j}_{\cdot s} a^{k}_{\cdot t}$ are asymmetric. In addition, if (r, s, t) is $(1, 2, 3)$, then $e_{ijk} a^{i}_{\cdot r} a^{j}_{\cdot s} a^{k}_{\cdot t} = |a^{m}_{\cdot n}|$. Thus,

$$e_{ijk}a^i_{\cdot r}a^j_{\cdot s}a^k_{\cdot t} = |a^m_{\cdot n}|\, e_{rst}.$$

Similarly, the following formula can be proved.

$$e^{ijk}a^r_{\cdot i}a^s_{\cdot j}a^t_{\cdot k} = |a^m_{\cdot n}|\, e^{rst},$$

$$e^{ijk}a_{ir}a_{js}a_{kt} = e^{ijk}a_{ri}a_{sj}a_{tk} = |a_{mn}|\, e^{rst},$$
$$e_{ijk}a^{ir}a^{js}a^{kt} = e_{ijk}a^{ri}a^{sj}a^{tk} = |a^{mn}|\, e_{rst}.$$

Therefore, this theorem is proved. ■

Theorem 2.4. *Two Ricci symbols (* e_{ijk} *and* e^{ijk} *) are two tensors.*

Proof: Because

$$e_{ijk}A^i_{i'}A^j_{j'}A^k_{k'} = e_{i'j'k'}\,|A^p_{p'}| \;\Rightarrow\; e_{i'j'k'} = |A^p_{p'}|^{-1}\, A^i_{i'}A^j_{j'}A^k_{k'}e_{ijk}.$$
$$e^{ijk}A^{i'}_iA^{j'}_jA^{k'}_k = e^{i'j'k'}\,|A^{p'}_p| \;\Rightarrow\; e^{i'j'k'} = |A^p_{p'}|\, A^i_{i'}A^j_{j'}A^k_{k'}e^{ijk}.$$

It implies that two Ricci symbols (e_{ijk} and e^{ijk}) are two tensors. This theorem is proved. ■

Because $e_{ijk}g^{ir}g^{js}g^{kt} \neq e^{rst}$, e_{ijk} and e^{ijk} are two different tensors. However, Eddington tensors (ε_{ijk} and ε^{ijk}) are asymmetric because

$$\begin{aligned} \varepsilon_{i_1\cdots i_q} &= e_{i_1\cdots i_q}[\mathbf{g}_1\cdots\mathbf{g}_q] = e_{i_1\cdots i_q}\sqrt{g}, \\ \varepsilon^{i_1\cdots i_q} &= e^{i_1\cdots i_q}[\mathbf{g}^1\cdots\mathbf{g}^q] = e^{i_1\cdots i_q}\big/\sqrt{g}; \\ \varepsilon_{ijk} &= e_{ijk}\left[\mathbf{g}_1\mathbf{g}_2\mathbf{g}_3\right] = e_{ijk}\sqrt{g}, \\ \varepsilon^{ijk} &= e^{ijk}\left[\mathbf{g}^1\mathbf{g}^2\mathbf{g}^3\right] = e^{ijk}\big/\sqrt{g}; \end{aligned} \tag{2.111}$$

$$\begin{aligned} \delta^{ijk}_{rst} &= e^{ijk}e_{rst} = \varepsilon^{ijk}\varepsilon_{rst} = [\mathbf{g}^i\mathbf{g}^j\mathbf{g}^k][\mathbf{g}_r\mathbf{g}_s\mathbf{g}_t] \\ &= \begin{vmatrix} \mathbf{g}^i\cdot\mathbf{g}_r & \mathbf{g}^i\cdot\mathbf{g}_s & \mathbf{g}^i\cdot\mathbf{g}_t \\ \mathbf{g}^j\cdot\mathbf{g}_r & \mathbf{g}^j\cdot\mathbf{g}_s & \mathbf{g}^j\cdot\mathbf{g}_t \\ \mathbf{g}^k\cdot\mathbf{g}_r & \mathbf{g}^k\cdot\mathbf{g}_s & \mathbf{g}^k\cdot\mathbf{g}_t \end{vmatrix} = \begin{vmatrix} \delta^i_r & \delta^i_s & \delta^i_t \\ \delta^j_r & \delta^j_s & \delta^j_t \\ \delta^k_r & \delta^k_s & \delta^k_t \end{vmatrix}, \\ \delta^{ijt}_{rst} &= \delta^i_r\delta^j_s - \delta^i_s\delta^j_r, \\ \delta^{ist}_{rst} &= \delta^i_r\delta^s_s - \delta^i_s\delta^s_r = 3\delta^i_r - \delta^i_r = 2\delta^i_r, \\ \delta^{rst}_{rst} &= 2\delta^r_r = 2\times 3 = 3!. \end{aligned} \tag{2.112}$$

The generalized Kronecker-delta can be determined by

$$\delta^{i_1\cdots i_q}_{j_1\cdots j_q} = e^{i_1\cdots i_q} e_{j_1\cdots j_q} = \varepsilon^{i_1\cdots i_q} \varepsilon_{j_1\cdots j_q} = [\mathbf{g}^{i_1} \cdots \mathbf{g}^{i_q}][\mathbf{g}_{j_1} \cdots \mathbf{g}_{j_q}]$$
$$= \begin{vmatrix} \mathbf{g}^{i_1} \cdot \mathbf{g}_{j_1} & \cdots & \mathbf{g}^{i_1} \cdot \mathbf{g}_{j_q} \\ \vdots & & \vdots \\ \mathbf{g}^{i_q} \cdot \mathbf{g}_{j_1} & \cdots & \mathbf{g}^{i_q} \cdot \mathbf{g}_{j_q} \end{vmatrix} = \begin{vmatrix} \delta^{i_1}_{j_1} & \cdots & \delta^{i_1}_{j_q} \\ \vdots & & \vdots \\ \delta^{i_q}_{j_1} & \cdots & \delta^{i_q}_{j_q} \end{vmatrix}, \tag{2.113}$$
$$\delta^{i_1\cdots i_p j_{p+1}\cdots j_q}_{j_1\cdots j_p j_{p+1}\cdots j_q} = (q-p)!\,\delta^{i_1\cdots i_p}_{j_1\cdots j_p},$$
$$\delta^{j_1\cdots j_q}_{j_1\cdots j_q} = q!.$$

From Eq.(2.110), the determinant of a matrix $(a^m_{\cdot n})$ or (a_{mn}) is given by

$$|\,a^m_{\cdot n}\,| = \frac{1}{3!}\delta^{rst}_{ijk} a^i_{\cdot r} a^j_{\cdot s} a^k_{\cdot t}, \text{ and } |\,a_{mn}\,| = \frac{1}{3!} e^{ijk} e^{rst} a_{ir} a_{js} a_{kt}. \tag{2.114}$$

This is because $e^{rst} e_{ijk} a^i_{\cdot r} a^j_{\cdot s} a^k_{\cdot t} = |\,a^m_{\cdot n}\,|\, e_{rst} e^{rst}$ and $\delta^{rst}_{ijk} a^i_{\cdot r} a^j_{\cdot s} a^k_{\cdot t} = |\,a^m_{\cdot n}\,|\,\delta^{rst}_{rst}$ for the first equation. Similarly, the second equation can be obtained. From the foregoing equation, the cofactors of element $a^p_{\cdot q}$ in determinant $|\,a^m_{\cdot n}\,|$ and element a_{pq} in determinant $|\,a_{mn}\,|$ are determined by

$$\frac{\partial |\,a^m_{\cdot n}\,|}{\partial a^p_{\cdot q}} = \frac{1}{3!}\delta^{rst}_{ijk}(\delta^i_p \delta^q_r a^j_{\cdot s} a^k_{\cdot t} + \delta^j_p \delta^q_s a^i_{\cdot r} a^k_{\cdot t} + \delta^k_p \delta^q_t a^i_{\cdot r} a^j_{\cdot s})$$
$$= \frac{1}{3!}(\delta^{qst}_{pjk} a^j_{\cdot s} a^k_{\cdot t} + \delta^{rqt}_{ipk} a^i_{\cdot r} a^k_{\cdot t} + \delta^{rsq}_{ijp} a^i_{\cdot r} a^j_{\cdot s})$$
$$= \frac{1}{3!}(\delta^{qst}_{pjk} a^j_{\cdot s} a^k_{\cdot t} + \delta^{sqt}_{jpk} a^j_{\cdot s} a^k_{\cdot t} + \delta^{ksq}_{tjp} a^k_{\cdot t} a^j_{\cdot s})$$
$$= \frac{1}{2!}\delta^{qst}_{pjk} a^j_{\cdot s} a^k_{\cdot t}. \tag{2.115}$$

Similarly,

$$\frac{\partial |\,a_{mn}\,|}{\partial a_{pq}} = \frac{1}{2!} e^{pjk} e^{qst} a_{js} a_{kt}. \tag{2.116}$$

The generalized discussion on the property of *generalized Kronecker-delta* can be found in Schouten (1951) and Eringen (1971).

2.2. Second-order tensors

In this section, the second-order tensor will be discussed. The second-order tensor algebra will be presented, and the basic properties and principal direction of tensors will be discussed, and the tensor decompositions and functions will be presented.

2.2.1. Second-order tensor algebra

The second-order tensor is extensively used, which is also called an *affine tensor*. The second-order tensor is expressed by

$$\mathbf{B} = B^{i}_{\cdot j}\mathbf{g}_i\mathbf{g}^j. \tag{2.117}$$

A dot product of a second-order tensor with a vector is a vector, i.e.,

$$\mathbf{B}\cdot\mathbf{v} = B^{i}_{\cdot j}\mathbf{g}_i\mathbf{g}^j \cdot v^k\mathbf{g}_k = B^{i}_{\cdot j}v^k\delta^j_k\mathbf{g}_i = B^{i}_{\cdot j}v^j\mathbf{g}_i \equiv u^i\mathbf{g}_i = \mathbf{u}. \tag{2.118}$$

where $\mathbf{v} = v^k\mathbf{g}_k$ is adopted in the foregoing equation.

For two second-order tensors $\mathbf{B} = B^{i}_{\cdot j}\mathbf{g}_i\mathbf{g}^j$ and $\mathbf{D} = D^{i}_{\cdot j}\mathbf{g}_i\mathbf{g}^j$. Their summation and dot products are the second-order tensors.

$$\begin{aligned}
\mathbf{B}+\mathbf{D} &= B^{i}_{\cdot j}\mathbf{g}_i\mathbf{g}^j + D^{i}_{\cdot j}\mathbf{g}_i\mathbf{g}^j = (B^{i}_{\cdot j} + D^{i}_{\cdot j})\mathbf{g}_i\mathbf{g}^j,\\
\mathbf{B}\cdot\mathbf{D} &= B^{i}_{\cdot j}\mathbf{g}_i\mathbf{g}^j \cdot D^{r}_{\cdot s}\mathbf{g}_r\mathbf{g}^s = B^{i}_{\cdot j}D^{r}_{\cdot s}\mathbf{g}_i\mathbf{g}^j\cdot\mathbf{g}_r\mathbf{g}^s\\
&= B^{i}_{\cdot j}D^{r}_{\cdot s}\delta^j_r\mathbf{g}_i\mathbf{g}^s = B^{i}_{\cdot r}D^{r}_{\cdot s}\mathbf{g}_i\mathbf{g}^s = B^{i}_{\cdot r}D^{r}_{\cdot j}\mathbf{g}_i\mathbf{g}^j.
\end{aligned} \tag{2.119}$$

If $\mathbf{D} = \mathbf{B}$, the following relations exist.

$$\begin{aligned}
\mathbf{B}^2 &\equiv \mathbf{B}\cdot\mathbf{B} = B^{i}_{\cdot r}B^{r}_{\cdot j}\mathbf{g}_i\mathbf{g}^j,\\
\mathbf{B}^N &\equiv \underbrace{\mathbf{B}\cdot\cdots\cdot\mathbf{B}}_{N} = B^{i}_{\cdot r_1}B^{r_1}_{\cdot r_2}\cdots B^{r_{N-2}}_{\cdot r_{N-1}}B^{r_{N-1}}_{\cdot j}\mathbf{g}_i\mathbf{g}^j.
\end{aligned} \tag{2.120}$$

Definition 2.13. For a second-order tensor of $\mathbf{B} = B^{i}_{\cdot j}\mathbf{g}_i\mathbf{g}^j$, a tensor $\mathbf{B}^{\mathrm{T}}$ is termed the *transpose of the second-order tensor* $\mathbf{B}$ if

$$\mathbf{B}^{\mathrm{T}} = (B^{i}_{\cdot j})^{\mathrm{T}}\mathbf{g}_i\mathbf{g}^j = B^{\cdot i}_{j}\mathbf{g}_i\mathbf{g}^j. \tag{2.121}$$

Theorem 2.5. *Consider two second-order tensors* $\mathbf{B}$ *and* $\mathbf{D}$. *If* $\mathbf{B}^{\mathrm{T}}$ *is the transpose of* $\mathbf{B}$, *then the following relation holds*

$$(\mathbf{B}\cdot\mathbf{D})^{\mathrm{T}} = \mathbf{D}^{\mathrm{T}}\cdot\mathbf{B}^{\mathrm{T}}. \tag{2.122}$$

Proof: For any two vectors $\mathbf{a} = a_i\mathbf{g}^i$ and $\mathbf{b} = b^j\mathbf{g}_j$, we have

$$\mathbf{a}\cdot\mathbf{B}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{B}^{\mathrm{T}}\cdot\mathbf{a},$$

If $\mathbf{b} = \mathbf{D} \cdot \mathbf{c}$ is selected, the foregoing equation becomes

$$\mathbf{a} \cdot \mathbf{B} \cdot \mathbf{D} \cdot \mathbf{c} = \mathbf{D} \cdot \mathbf{c} \cdot \mathbf{B}^{\mathrm{T}} \cdot \mathbf{a} = \mathbf{c} \cdot \mathbf{D}^{\mathrm{T}} \cdot \mathbf{B}^{\mathrm{T}} \cdot \mathbf{a}.$$

On the other hand, the following equation should hold:

$$\mathbf{a} \cdot (\mathbf{B} \cdot \mathbf{D}) \cdot \mathbf{c} = \mathbf{c} \cdot (\mathbf{B} \cdot \mathbf{D})^{\mathrm{T}} \cdot \mathbf{a}.$$

Because $\mathbf{a}$ and $\mathbf{c}$ are selected arbitrarily, the two equations give

$$(\mathbf{B} \cdot \mathbf{D})^{\mathrm{T}} = \mathbf{D}^{\mathrm{T}} \cdot \mathbf{B}^{\mathrm{T}}.$$

This theorem is proved. ■

(A) *Non-degenerate and degenerate tensors*

Definition 2.14. A second-order tensor $\mathbf{B}$ is *non-degenerate* if three quantities of $\mathbf{B} \cdot \mathbf{a}$, $\mathbf{B} \cdot \mathbf{b}$ and $\mathbf{B} \cdot \mathbf{c}$ are linearly independent for any three independent vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Otherwise, the second-order tensor $\mathbf{B}$ is *degenerate.*

Theorem 2.6. *For any three linearly-independent vectors of* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$, *there is an invariant quantity of a second-order tensor* $\mathbf{B}$, *defined by*

$$\mathrm{III} = \frac{(\mathbf{B} \cdot \mathbf{a}) \times (\mathbf{B} \cdot \mathbf{b}) \cdot (\mathbf{B} \cdot \mathbf{c})}{[\mathbf{abc}]}. \tag{2.123}$$

Proof : Consider three transformed vectors $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$ of the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Thus we have

$$\mathbf{a}' = A_{\mathbf{a}}^{\mathbf{a}'} \mathbf{a}, \quad \mathbf{b}' = A_{\mathbf{b}}^{\mathbf{b}'} \mathbf{b}, \quad \mathbf{c}' = A_{\mathbf{c}}^{\mathbf{c}'} \mathbf{c}$$

where

$$A_{\mathbf{a}}^{\mathbf{a}'} = \frac{\partial \mathbf{a}'}{\partial \mathbf{a}}, \quad A_{\mathbf{b}}^{\mathbf{b}'} = \frac{\partial \mathbf{b}'}{\partial \mathbf{b}}, \quad A_{\mathbf{c}}^{\mathbf{c}'} = \frac{\partial \mathbf{c}'}{\partial \mathbf{c}}.$$

So we have

$$\frac{(\mathbf{B} \cdot \mathbf{a}') \times (\mathbf{B} \cdot \mathbf{b}') \cdot (\mathbf{B} \cdot \mathbf{c}')}{[\mathbf{a}'\mathbf{b}'\mathbf{c}']} = \frac{(\mathbf{B} \cdot \mathbf{a}) \times (\mathbf{B} \cdot \mathbf{b}) \cdot (\mathbf{B} \cdot \mathbf{c})}{[\mathbf{abc}]} = \mathrm{III}.$$

So the quantity III is invariant. This theorem is proved. ■

Definition 2.15. A second-order tensor $\mathbf{B}$ is *degenerate* if there is a vector $\mathbf{v}$ to make the following relation hold:

$$\mathbf{B} \cdot \mathbf{v} = \mathbf{0}, \tag{2.124}$$

and the direction of $\mathbf{v}$ is called the *zero-direction of the tensor* $\mathbf{B}$.

Definition 2.16. A second-order tensor $\mathbf{B}$ is called a *zero tensor* if the directions of any three independent vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the zero-directions of $\mathbf{B}\cdot\mathbf{a}$, $\mathbf{B}\cdot\mathbf{b}$ and $\mathbf{B}\cdot\mathbf{c}$, i.e.,

$$\mathbf{B}\cdot\mathbf{a}=\mathbf{0},\ \mathbf{B}\cdot\mathbf{b}=\mathbf{0},\ \mathbf{B}\cdot\mathbf{c}=\mathbf{0}. \tag{2.125}$$

Consider any three independent vectors to be three base vectors $\mathbf{g}_i$ ($i=1,2,3$) of the second-order tensor $\mathbf{B}$. The invariant quantity III is independent of the base vectors, so this invariant quantity is called the *third invariant* quantity of the second-order tensor $\mathbf{B}$. Such an invariant is determined by

$$\begin{aligned}
\mathrm{III}&=\frac{(\mathbf{B}\cdot\mathbf{g}_1)\times(\mathbf{B}\cdot\mathbf{g}_2)\cdot(\mathbf{B}\cdot\mathbf{g}_3)}{[\mathbf{g}_1\mathbf{g}_2\mathbf{g}_3]}\\
&=\frac{(B^{i_1}_{\cdot j_1}\mathbf{g}_{i_1}\mathbf{g}^{j_1}\cdot\mathbf{g}_1)\times(B^{i_2}_{\cdot j_2}\mathbf{g}_{i_2}\mathbf{g}^{j_2}\cdot\mathbf{g}_2)\cdot(B^{i_3}_{\cdot j_3}\mathbf{g}_{i_3}\mathbf{g}^{j_3}\cdot\mathbf{g}_3)}{\sqrt{g}}\\
&=\frac{(B^{i_1}_{\cdot 1}\mathbf{g}_{i_1})\times(B^{i_2}_{\cdot 2}\mathbf{g}_{i_2})\cdot(B^{i_3}_{\cdot 3}\mathbf{g}_{i_3})}{\sqrt{g}}=e_{i_1i_2i_3}B^{i_1}_{\cdot 1}B^{i_2}_{\cdot 2}B^{i_3}_{\cdot 3}=\left|B^{i}_{\cdot j}\right|.
\end{aligned} \tag{2.126}$$

For a second-order non-degenerate tensor $\mathbf{B}$, the third invariant quantity is non-zero (i.e., $\mathrm{III}=\left|B^{i}_{\cdot j}\right|\neq 0$). Thus, the inverse of the tensor $\mathbf{B}$ (i.e., $\mathbf{B}^{-1}$) will exist. Further, the following relations hold

$$\mathbf{B}^{-1}\cdot\mathbf{B}=\mathbf{I}\ \text{ and }\ \mathbf{B}\cdot\mathbf{B}^{-1}=\mathbf{I}. \tag{2.127}$$

where $\mathbf{I}$ is the *identity tensor* or *metric tensor*. Similarly, the transpose of the tensor $\mathbf{B}$ is non-degenerate because the third invariant quantity of the transpose of $\mathbf{B}$ (i.e., $\mathbf{B}^{\mathrm{T}}$) is non-zero.

$$\mathrm{III}(B_{j}^{\cdot i})=|B_{j}^{\cdot i}|=|g_{jr}B^{r}_{\cdot s}g^{si}|=|g_{jr}|\cdot|B^{r}_{\cdot s}|\cdot|g^{si}|=|B^{i}_{\cdot j}|\neq 0. \tag{2.128}$$

So the third invariant quantity of the tensor $\mathbf{B}$ is its determinant, written by $\det\mathbf{B}=|B^{i}_{\cdot j}|=|B_{j}^{\cdot i}|$. If $g\neq 1$, $\det\mathbf{B}\neq|B_{ij}|\neq|B^{ij}|$. Further, $(\mathbf{B}^{-1})^{\mathrm{T}}=(\mathbf{B}^{\mathrm{T}})^{-1}$ because $(\mathbf{B}^{-1})^{\mathrm{T}}\cdot\mathbf{B}^{\mathrm{T}}=(\mathbf{B}\cdot\mathbf{B}^{-1})^{\mathrm{T}}=\mathbf{I}^{\mathrm{T}}=\mathbf{I}$ and $\mathbf{I}=(\mathbf{B}^{\mathrm{T}})^{-1}\cdot\mathbf{B}^{\mathrm{T}}$.

Exercise: Prove $(\mathbf{B}\cdot\mathbf{a})\times(\mathbf{B}\cdot\mathbf{b})=\mathrm{III}(\mathbf{B}^{-1})^{\mathrm{T}}\cdot(\mathbf{a}\times\mathbf{b})$.

(B) *Characteristic direction and invariance of tensor*

Definition 2.17. For a second-order tensor $\mathbf{B}$, if the direction of a vector $\mathbf{v}$ is the direction of $(\mathbf{B}\cdot\mathbf{v})$, i.e.,

$$\mathbf{B}\cdot\mathbf{v}=B\mathbf{v}\quad\text{or}\quad(\mathbf{B}-B\mathbf{I})\cdot\mathbf{v}=\mathbf{0}, \tag{2.129}$$

then the direction is called the *characteristic (or eigenvector) direction of the ten-*

sor $\mathbf{B}$.

From the foregoing equation, the direction of $\mathbf{v}$ is the zero direction of $\mathbf{B}-B\mathbf{I}$, which indicates that the third invariant quantity of $\mathbf{B}-B\mathbf{I}$ is zero, i.e.,

$$\mathrm{III}(\mathbf{B}-B\mathbf{I})=\frac{[(\mathbf{B}-B\mathbf{I})\cdot\mathbf{a}]\times[(\mathbf{B}-B\mathbf{I})\cdot\mathbf{b}]\cdot[(\mathbf{B}-B\mathbf{I})\cdot\mathbf{c}]}{[\mathbf{abc}]}=0, \quad \text{or} \quad \det(\mathbf{B}-B\mathbf{I})=0. \tag{2.130}$$

The expanded characteristic equation is

$$B^3-\mathrm{I}B^2+\mathrm{II}B-\mathrm{III}=0 \tag{2.131}$$

where

$$\mathrm{I}=\frac{\mathbf{a}\times\mathbf{b}\cdot(\mathbf{B}\cdot\mathbf{c})+\mathbf{b}\times\mathbf{c}\cdot(\mathbf{B}\cdot\mathbf{a})+\mathbf{c}\times\mathbf{a}\cdot(\mathbf{B}\cdot\mathbf{b})}{[\mathbf{abc}]},$$
$$\mathrm{II}=\frac{(\mathbf{B}\cdot\mathbf{a})\times(\mathbf{B}\cdot\mathbf{b})\cdot\mathbf{c}+(\mathbf{B}\cdot\mathbf{b})\times(\mathbf{B}\cdot\mathbf{c})\cdot\mathbf{a}+(\mathbf{B}\cdot\mathbf{c})\times(\mathbf{B}\cdot\mathbf{a})\cdot\mathbf{b}}{[\mathbf{abc}]}. \tag{2.132}$$

The quantities I and II of the tensor $\mathbf{B}$ are independent of three independent vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Therefore, both of them are called the *first and second invariant quantities of the tensor* $\mathbf{B}$. Consider three independent vectors to be three base vectors $\mathbf{g}_i$ ($i=1,2,3$) of the second-order tensor $\mathbf{B}$. Such an invariance is determined by

$$\mathrm{I}=\frac{\mathbf{g}_1\times\mathbf{g}_2\cdot(B^i_{\cdot j}\mathbf{g}_i\mathbf{g}^j\cdot\mathbf{g}_3)+\mathbf{g}_2\times\mathbf{g}_3\cdot(B^i_{\cdot j}\mathbf{g}_i\mathbf{g}^j\cdot\mathbf{g}_1)+\mathbf{g}_3\times\mathbf{g}_1\cdot(B^i_{\cdot j}\mathbf{g}_i\mathbf{g}^j\cdot\mathbf{g}_2)}{[\mathbf{g}_1\mathbf{g}_2\mathbf{g}_3]}$$
$$=\mathbf{g}^3\cdot(B^i_{\cdot 3}\mathbf{g}_i)+\mathbf{g}^1\cdot(B^i_{\cdot 1}\mathbf{g}_i)+\mathbf{g}^2\cdot(B^i_{\cdot 2}\mathbf{g}_i)=B^i_{\cdot i}=\frac{1}{1!}\delta^r_i B^i_{\cdot r}. \tag{2.133}$$

$$\mathrm{II}=\frac{B^i_{\cdot 1}\mathbf{g}_i\times B^j_{\cdot 2}\mathbf{g}_j\cdot\mathbf{g}_3+B^p_{\cdot 2}\mathbf{g}_p\times B^q_{\cdot 3}\mathbf{g}_q\cdot\mathbf{g}_1+B^r_{\cdot 3}\mathbf{g}_r\times B^s_{\cdot 1}\mathbf{g}_s\cdot\mathbf{g}_2}{[\mathbf{g}_1\mathbf{g}_2\mathbf{g}_3]}$$
$$=B^i_{\cdot 1}B^j_{\cdot 2}e_{ij3}+B^p_{\cdot 2}B^q_{\cdot 3}e_{pq1}+B^r_{\cdot 3}B^s_{\cdot 1}e_{rs2}=\frac{1}{2}B^i_{\cdot r}B^j_{\cdot s}e_{ijk}e^{rsk}=\frac{1}{2!}B^i_{\cdot r}B^j_{\cdot s}\delta^{rs}_{ij}. \tag{2.134}$$

If the direction of $\mathbf{v}$ is the characteristic direction of the tensor $\mathbf{B}$, then

$$\mathbf{B}^2\cdot\mathbf{v}=B^2\mathbf{v} \tag{2.135}$$

because of $\mathbf{B}^2\cdot\mathbf{v}=\mathbf{B}\cdot\mathbf{B}\cdot\mathbf{v}=\mathbf{B}\cdot B\cdot\mathbf{v}=B\mathbf{B}\cdot\mathbf{v}=B^2\mathbf{v}$. Similarly,

$$\mathbf{B}^N\cdot\mathbf{v}=B^N\mathbf{v}. \tag{2.136}$$

Exercise: Prove the first and second invariant quantity of the tensor $\mathbf{B}$ is independent of three independent vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

2.2.2. Basic properties

(A) *Symmetric and asymmetric tensors*

Definition 2.18. The tensor $\mathbf{S}$ is *symmetric* if

$$\mathbf{S} = \mathbf{S}^{\mathrm{T}} \text{ i.e., } S^{i}_{\cdot j} = S^{\cdot i}_{j} \text{ or } S_{ij} = S_{ji}. \tag{2.137}$$

Definition 2.19. The tensor $\mathbf{A}$ is *asymmetric* if

$$\mathbf{A} = -\mathbf{A}^{\mathrm{T}} \text{ i.e., } A^{i}_{\cdot j} = -A^{\cdot i}_{j} \text{ or } A_{ij} = -A_{ji}. \tag{2.138}$$

Definition 2.20. For two unit vectors $\mathbf{t}$ and $\mathbf{n}$, the scalar quantity

$$\mathbf{t} \cdot \mathbf{B} \cdot \mathbf{n} = B_{ij} n^i t^j \tag{2.139}$$

is called the *shear component of the tensor* $\mathbf{B}$ in the direction of $\mathbf{t}$ on the normal surface of vector $\mathbf{n}$. If $\mathbf{B} = \mathbf{S}$, the following quantity exists,

$$\mathbf{t} \cdot \mathbf{S} \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{S} \cdot \mathbf{t} = B_{ij} n^i t^j. \tag{2.140}$$

If $\mathbf{t} \cdot \mathbf{n} = 0$, such a component is called the *orthogonal shear component* of $\mathbf{B}$ in the direction of $\mathbf{t}$ on the normal surface of vector $\mathbf{n}$. For $\mathbf{t} = \mathbf{n}$, the above quantity becomes

$$\mathbf{n} \cdot \mathbf{B} \cdot \mathbf{n} = B_{ij} n^i n^j \tag{2.141}$$

which is called the *normal component* of tensor $\mathbf{B}$ on the surface with its normal vector $\mathbf{n}$.

(B) *Orthogonal affine tensors*

Definition 2.21. A tensor $\mathbf{R}$ is called an *orthogonal affine tensor* if the magnitudes of both $\mathbf{R} \cdot \mathbf{v}$ and $\mathbf{v}$ are same, i.e.,

$$(\mathbf{R} \cdot \mathbf{v})^2 = \mathbf{v}^2 \text{ or } \mathbf{v} \cdot \mathbf{R}^{\mathrm{T}} \cdot \mathbf{R} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{v}. \tag{2.142}$$

From the definition,

$$\mathbf{R}^{\mathrm{T}} \cdot \mathbf{R} = \mathbf{I} \quad \Rightarrow \quad \mathbf{R}^{\mathrm{T}} = \mathbf{R}^{-1}. \tag{2.143}$$

The orthogonal tensor does not change the dot product of two vectors and volume of three vectors. In other words,

$$\begin{gathered} (\mathbf{R} \cdot \mathbf{u}) \cdot (\mathbf{R} \cdot \mathbf{v}) = \mathbf{u} \cdot \mathbf{v}, \\ [(\mathbf{R} \cdot \mathbf{u})(\mathbf{R} \cdot \mathbf{v})(\mathbf{R} \cdot \mathbf{w})] = [\mathbf{uvw}]. \end{gathered} \tag{2.144}$$

(C) *Characteristic directions of symmetric tensors*

If a unit vector $\mathbf{i}$ is a vector for the characteristic direction of symmetric tensor $\mathbf{S}$ (i.e., $\mathbf{S}\cdot\mathbf{i} = S\mathbf{i}$), the corresponding characteristic equation is

$$S^3 - \mathrm{I}S^2 + \mathrm{II}S - \mathrm{III} = 0. \tag{2.145}$$

If the tensor $\mathbf{S}$ is degenerate, one of the characteristic root is zero (i.e., $S_1 = 0$). The characteristic vector is a zero vector in the direction of the unit vector $\mathbf{i}$. For any non-zero vector $\mathbf{v}$ normal to the vector $\mathbf{i}$,

$$\mathbf{i}\cdot\mathbf{S}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{S}\cdot\mathbf{i} = S_1\mathbf{v}\cdot\mathbf{i} = 0 \tag{2.146}$$

from which the vector $\mathbf{S}\cdot\mathbf{v}$ is normal to the unit vector $\mathbf{i}$. In physics, the tensor $\mathbf{S}$ has a zero component in the direction of the unit vector $\mathbf{i}$ on the normal surface of the vector $\mathbf{v}$. The symmetric tensor $\mathbf{S}$ has three characteristic directions of ($\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$) with the corresponding values S_i ($i = 1, 2, 3$). The three characteristic directions are perpendicular each other.

(D) *Principal directions of tensors*

Theorem 2.7. *For a second-order tensor* $\mathbf{B}$, *there are three principal directions* (***i***, ***j*** *and* ***k***) *to be perpendicular each other, and three vectors* $\mathbf{B}\cdot\mathbf{i}$, $\mathbf{B}\cdot\mathbf{j}$ *and* $\mathbf{B}\cdot\mathbf{k}$ *are perpendicular each other.*

Proof : Because $(\mathbf{B}^{\mathrm{T}}\cdot\mathbf{B})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\cdot\mathbf{B}$, it means that $\mathbf{B}^{\mathrm{T}}\cdot\mathbf{B}$ are symmetric. Thus, there are three orthogonal unit vectors. Suppose (**i**, **j** and **k**) are a set of three orthogonal unit vectors for $\mathbf{B}^{\mathrm{T}}\cdot\mathbf{B}$.

$$\mathbf{B}^{\mathrm{T}}\cdot\mathbf{B}\cdot\mathbf{i} = b_1\mathbf{i},\ \mathbf{B}^{\mathrm{T}}\cdot\mathbf{B}\cdot\mathbf{j} = b_2\mathbf{j} \text{ and } \mathbf{B}^{\mathrm{T}}\cdot\mathbf{B}\cdot\mathbf{k} = b_3\mathbf{k}.$$

Furthermore,

$$(\mathbf{B}\cdot\mathbf{i})\cdot(\mathbf{B}\cdot\mathbf{j}) = \mathbf{i}\cdot(\mathbf{B}^{\mathrm{T}}\cdot\mathbf{B}\cdot\mathbf{j}) = b_2\mathbf{i}\cdot\mathbf{j} = 0.$$

Similarly, $(\mathbf{B}\cdot\mathbf{k})\cdot(\mathbf{B}\cdot\mathbf{i}) = (\mathbf{B}\cdot\mathbf{j})\cdot(\mathbf{B}\cdot\mathbf{k}) = 0$. Three vectors $\mathbf{B}\cdot\mathbf{i}$, $\mathbf{B}\cdot\mathbf{j}$ and $\mathbf{B}\cdot\mathbf{k}$ are perpendicular each other. Therefore, (**i**, **j** and **k**) are the principal directions of a second-order tensor $\mathbf{B}$. This theorem is proved. ■

The symmetric tensor is expressed by

$$\begin{aligned}\mathbf{S} = {} & S_{11}\mathbf{ii} + S_{12}\mathbf{ij} + S_{13}\mathbf{ik} + S_{21}\mathbf{ji} + S_{22}\mathbf{jj} + S_{23}\mathbf{jk} \\ & + S_{31}\mathbf{ki} + S_{32}\mathbf{kj} + S_{33}\mathbf{kk}.\end{aligned} \tag{2.147}$$

The projections on the three principal directions are

$$\begin{aligned} S_1\mathbf{i} &= \mathbf{S}\cdot\mathbf{i} = S_{11}\mathbf{i}+S_{21}\mathbf{j}+S_{31}\mathbf{k},\\ S_2\mathbf{j} &= \mathbf{S}\cdot\mathbf{j} = S_{12}\mathbf{i}+S_{22}\mathbf{j}+S_{32}\mathbf{k},\\ S_3\mathbf{k} &= \mathbf{S}\cdot\mathbf{k} = S_{13}\mathbf{i}+S_{23}\mathbf{j}+S_{33}\mathbf{k}. \end{aligned} \tag{2.148}$$

So $S_{ij}=0$ $(i\neq j)$ and $S_{ij}=S_i$ $(i=j)$. Finally,

$$\mathbf{S} = S_1\mathbf{ii}+S_2\mathbf{jj}+S_3\mathbf{kk}. \tag{2.149}$$

Definition 2.22. For any symmetric tensor $\mathbf{S}$, the tensor functions are defined as

$$\mathbf{S}^n = S_1^n\mathbf{ii}+S_2^n\mathbf{jj}+S_3^n\mathbf{kk}. \tag{2.150}$$

Definition 2.23. For any symmetric and positive-definite tensor $\mathbf{S}$, the tensor functions are defined as

$$\begin{aligned} \mathbf{S}^{\frac{1}{n}} &= S_1^{\frac{1}{n}}\mathbf{ii}+S_2^{\frac{1}{n}}\mathbf{jj}+S_3^{\frac{1}{n}}\mathbf{kk},\\ \log\mathbf{S} &= \log S_1\mathbf{ii}+\log S_2\mathbf{jj}+\log S_3\mathbf{kk}. \end{aligned} \tag{2.151}$$

2.2.3. Tensor decompositions

(A) *Addition decomposition* The *addition* decomposition of a tensor $\mathbf{B}$ is summation of its symmetric and asymmetric tensors, i.e.,

$$\begin{aligned} \mathbf{B} &= \frac{1}{2}(\mathbf{B}+\mathbf{B}^{\mathrm{T}})+\frac{1}{2}(\mathbf{B}-\mathbf{B}^{\mathrm{T}}) \equiv \mathbf{S}+\mathbf{A},\\ B_{ij} &= B_{(ij)}+B_{[ij]}. \end{aligned} \tag{2.152}$$

(B) *Polar decomposition*

Theorem 2.8. *For a non-degenerate tensor* $\mathbf{B}$, *there is a symmetric positive definite tensor* ($\mathbf{V}$ *or* $\mathbf{U}$) *and an orthogonal* (*or rotation*) *tensor* $\mathbf{R}$. *The left and right polar decompositions of* $\mathbf{B}$,

$$\mathbf{B} = \mathbf{V}\cdot\mathbf{R} = \mathbf{R}\cdot\mathbf{U} \tag{2.153}$$

is unique with $\mathbf{V}=(\mathbf{B}\cdot\mathbf{B}^{\mathrm{T}})^{\frac{1}{2}}$ and $\mathbf{U}=(\mathbf{B}^{\mathrm{T}}\cdot\mathbf{B})^{\frac{1}{2}}$.

Proof: Because a tensor $\mathbf{B}$ is non-degenerate, its transpose $\mathbf{B}^{\mathrm{T}}$ is non-degenerate. For any vector $\mathbf{v}$,

$$\mathbf{B}\cdot\mathbf{v}\neq 0 \text{ and } \mathbf{B}^{\mathrm{T}}\cdot\mathbf{v}\neq 0,$$
$$(\mathbf{B}^{\mathrm{T}}\cdot\mathbf{v})^2 = \mathbf{v}\cdot(\mathbf{B}\cdot\mathbf{B}^{\mathrm{T}})\cdot\mathbf{v} > 0 \text{ and}$$

$$(\mathbf{B}\cdot\mathbf{v})^2 = \mathbf{v}\cdot(\mathbf{B}^{\mathrm{T}}\cdot\mathbf{B})\cdot\mathbf{v} > 0.$$

In addition, $(\mathbf{B}\cdot\mathbf{B}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{B}\cdot\mathbf{B}^{\mathrm{T}}$ and $(\mathbf{B}^{\mathrm{T}}\cdot\mathbf{B})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\cdot\mathbf{B}$. Thus $\mathbf{B}\cdot\mathbf{B}^{\mathrm{T}}$ and $\mathbf{B}^{\mathrm{T}}\cdot\mathbf{B}$ are symmetric and positive-definite. Define two symmetric, positive-definite tensors as

$$\mathbf{V} = (\mathbf{B}\cdot\mathbf{B}^{\mathrm{T}})^{\frac{1}{2}} \text{ and } \mathbf{U} = (\mathbf{B}^{\mathrm{T}}\cdot\mathbf{B})^{\frac{1}{2}}.$$

If $\mathbf{R} = \mathbf{V}^{-1}\cdot\mathbf{B}$, then,

$$\mathbf{R}\cdot\mathbf{R}^{\mathrm{T}} = (\mathbf{V}^{-1}\cdot\mathbf{B})\cdot\mathbf{B}^{\mathrm{T}}\cdot(\mathbf{V}^{-1})^{\mathrm{T}} = \mathbf{V}^{-1}\cdot(\mathbf{V}\cdot\mathbf{V})\cdot\mathbf{V}^{-1} = \mathbf{I}.$$

Consider another left decomposition as

$$\overline{\mathbf{V}}\cdot\overline{\mathbf{R}} = \mathbf{B} = \mathbf{V}\cdot\mathbf{R},$$

from which

$$\mathbf{R} = \mathbf{V}^{-1}\cdot\overline{\mathbf{V}}\cdot\overline{\mathbf{R}}.$$

However, $\mathbf{R}\cdot\mathbf{R}^{\mathrm{T}} = \mathbf{I}$ gives $\mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}} = \overline{\mathbf{R}}^{\mathrm{T}}\cdot\overline{\mathbf{V}}^{\mathrm{T}}\cdot(\mathbf{V}^{-1})^{\mathrm{T}} = \overline{\mathbf{R}}^{\mathrm{T}}\cdot\overline{\mathbf{V}}\cdot\mathbf{V}^{-1}$ with symmetry of $\mathbf{V}$ (i.e., $\mathbf{V} = \mathbf{V}^{\mathrm{T}}$ and $(\mathbf{V}^{-1})^{\mathrm{T}} = \mathbf{V}^{-1}$). Taking inverse of tensor $\mathbf{R}^{-1}$ gives

$$\mathbf{R} = \mathbf{V}\cdot(\overline{\mathbf{V}})^{-1}\cdot(\overline{\mathbf{R}}^{\mathrm{T}})^{-1} = \mathbf{V}\cdot\overline{\mathbf{V}}^{-1}\cdot\overline{\mathbf{R}}.$$

Comparison of two expressions of $\mathbf{R}$ gives

$$(\mathbf{V}^{-1}\cdot\overline{\mathbf{V}} - \mathbf{V}\cdot\overline{\mathbf{V}}^{-1})\cdot\overline{\mathbf{R}} = 0.$$

Further,

$$(\mathbf{V}^{-1}\cdot\overline{\mathbf{V}} - \mathbf{V}\cdot\overline{\mathbf{V}}^{-1})\cdot\overline{\mathbf{R}}\cdot\overline{\mathbf{R}}^{\mathrm{T}} = 0 \Rightarrow \mathbf{V}^{-1}\cdot\overline{\mathbf{V}} = \mathbf{V}\cdot\overline{\mathbf{V}}^{-1}.$$

So $\overline{\mathbf{V}}\cdot\overline{\mathbf{V}} = \mathbf{V}\cdot\mathbf{V} \Rightarrow \overline{\mathbf{V}} = \mathbf{V}$. From $\mathbf{R} = \mathbf{V}^{-1}\cdot\overline{\mathbf{V}}\cdot\overline{\mathbf{R}}$, one obtains $\mathbf{R} = \overline{\mathbf{R}}$. Therefore, the left decomposition is unique. Similarly, the uniqueness for the right decomposition can be proved. If the orthogonal tensor $\mathbf{R}$ in the left and right decompositions are different (i.e., $\mathbf{R} \neq \tilde{\mathbf{R}}$),

$$\mathbf{B} = \tilde{\mathbf{R}}\cdot\mathbf{U} = \mathbf{V}\cdot\mathbf{R} = (\mathbf{R}\cdot\mathbf{R}^{-1})\cdot\mathbf{V}\cdot\mathbf{R} = \mathbf{R}\cdot(\mathbf{R}^{-1}\cdot\mathbf{V}\cdot\mathbf{R}).$$

From the uniqueness of the right decomposition,

$$\mathbf{U} = (\mathbf{R}^{-1}\cdot\mathbf{V}\cdot\mathbf{R}) \text{ and } \tilde{\mathbf{R}} = \mathbf{R},$$

also because $(\mathbf{R}^{-1}\cdot\mathbf{V}\cdot\mathbf{R})^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}}\cdot\mathbf{V}^{\mathrm{T}}\cdot(\mathbf{R}^{-1})^{\mathrm{T}} = \mathbf{R}^{-1}\cdot\mathbf{V}\cdot\mathbf{R}$ is symmetric. This theorem is proved. ■

2.2.4. Tensor functions

(A) *Tensor function* For a tensor $\mathbf{B}$, the *tensor function* is defined as

$$\varphi = \varphi(\mathbf{B}) \text{ and } \mathbf{C} = \mathbf{f}(\mathbf{B}). \tag{2.154}$$

For instance, if a tensor $\mathbf{B}$ is the second-order tensor, the tensor functions can be expressed by 9-components as

$$\varphi = \varphi(B^{i}_{\cdot j}) \text{ and } C_{ij} = f_{ij}(B^{p}_{\cdot q}). \tag{2.155}$$

(B) *Tensor function gradient*

$$\frac{d\varphi}{d\mathbf{B}} = \frac{d\varphi}{dB^{i}_{\cdot j}}\mathbf{g}^{i}\mathbf{g}_{j} \text{ and } \frac{df_{ij}}{d\mathbf{B}} = \frac{df_{ij}}{dB^{p}_{\cdot q}}\mathbf{g}^{p}\mathbf{g}_{q} \tag{2.156}$$

where $d\mathbf{B} = dB^{i}_{\cdot j}\mathbf{g}_{i}\mathbf{g}^{j}$.

$$d\varphi = \frac{d\varphi}{d\mathbf{B}} : d\mathbf{B} \text{ and } d\mathbf{f} = \frac{d\mathbf{f}}{d\mathbf{B}} : d\mathbf{B}. \tag{2.157}$$

The foregoing equation is obtained from

$$\begin{aligned} d\varphi &= \frac{d\varphi}{dB^{i}_{\cdot j}} dB^{i}_{\cdot j} = \frac{d\varphi}{dB^{i}_{\cdot j}} \delta^{i}_{r}\delta^{s}_{j} dB^{r}_{\cdot s} = \frac{d\varphi}{dB^{i}_{\cdot j}} \mathbf{g}^{i}\cdot\mathbf{g}_{r}\mathbf{g}_{j}\cdot\mathbf{g}^{s} dB^{r}_{\cdot s} \\ &= \left(\frac{d\varphi}{dB^{i}_{\cdot j}}\mathbf{g}^{i}\mathbf{g}_{j}\right) : dB^{r}_{\cdot s}\mathbf{g}_{r}\mathbf{g}^{s} = \frac{d\varphi}{d\mathbf{B}} : d\mathbf{B}. \end{aligned} \tag{2.158}$$

For $\mathbf{B} = B^{i_1\cdots i_m}{}_{j_1\cdots j_n}\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}$, the tensor function gradient is

$$\begin{aligned} \frac{d\varphi}{d\mathbf{B}} &= \frac{d\varphi}{dB^{i_1\cdots i_m}{}_{j_1\cdots j_n}} \mathbf{g}^{i_1}\cdots\mathbf{g}^{i_m}\mathbf{g}_{j_1}\cdots\mathbf{g}_{j_n}, \\ \frac{df^{r_1\cdots r_k}{}_{s_1\cdots s_l}}{d\mathbf{B}} &= \frac{df^{r_1\cdots r_k}{}_{s_1\cdots s_l}}{dB^{i_1\cdots i_m}{}_{j_1\cdots j_n}} \mathbf{g}^{i_1}\cdots\mathbf{g}^{i_m}\mathbf{g}_{j_1}\cdots\mathbf{g}_{j_n} \end{aligned} \tag{2.159}$$

and $d\mathbf{B} = dB^{i_1\cdots i_m}{}_{j_1\cdots j_n}\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}$.

It is observed that the tensor function gradient is an associated tensor of the tensor $\mathbf{B}$. The tensor characters of the gradient can be proved by

$$\begin{aligned} \frac{\partial\varphi}{\partial B^{i'_1\cdots i'_m}{}_{j'_1\cdots j'_n}} &= \frac{\partial\varphi}{\partial B^{i_1\cdots i_m}{}_{j_1\cdots j_n}} \frac{\partial(A^{i_1}_{i'_1}\cdots A^{i_m}_{i'_m}A^{j'_1}_{j_1}\cdots A^{j'_n}_{j_n}B^{i'_1\cdots i'_m}{}_{j'_1\cdots j'_n})}{\partial B^{i'_1\cdots i'_m}{}_{j'_1\cdots j'_n}} \\ &= A^{i_1}_{i'_1}\cdots A^{i_m}_{i'_m}A^{j'_1}_{j_1}\cdots A^{j'_n}_{j_n} \frac{\partial\varphi}{\partial B^{i_1\cdots i_m}{}_{j_1\cdots j_n}}. \end{aligned} \tag{2.160}$$

2.3. Tensor calculus

In this section, the Christoffel symbol will be introduced to discuss the differentiation of tensors. Further, the invariant differential operators and the integral theorems will be presented. The Riemann-Christoffel curvature tensors will be discussed.

2.3.1. Differentiation

Consider a tensor field as

$$\boldsymbol{\psi} = \psi^{i_1\cdots i_m}{}_{j_1\cdots j_n}\sqrt{g}^{-W}\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}. \tag{2.161}$$

The components and bases of the tensor $\boldsymbol{\psi}$ are functions of curvilinear coordinates x^k and x_l. For the differentiation and integration of a tensor, the base vectors cannot be treated as constants. Without loss of generality, consider a vector $\mathbf{p}$, and its differentiation is

$$d\mathbf{p} = \frac{\partial\mathbf{p}}{\partial x^i}dx^i = dx^i\mathbf{g}_i \text{ and } d\mathbf{p} = \frac{\partial\mathbf{p}}{\partial x_i}dx_i = dx_i\mathbf{g}^i. \tag{2.162}$$

Note that dx^i (or dx_i) with x_i (or x^i) does not have any direct differentiation, but the local bases $\mathbf{g}^i$ (or $\mathbf{g}_i$) are relative to coordinates x_k (or x^k).

$$\frac{\partial\mathbf{g}_j}{\partial x^i} = \frac{\partial}{\partial x^i}(\frac{\partial\mathbf{p}}{\partial x^j}) = \mathbf{p}_{,ji}. \tag{2.163}$$

If $\mathbf{p} = z^k(x^1, x^2, x^3)\mathbf{i}_k$, $\mathbf{g}_i = (\partial z^k/\partial x^i)\mathbf{i}_k$ and $\mathbf{i}_k = (\partial x^i/\partial z^k)\mathbf{g}_i$, the foregoing equation becomes

$$\begin{aligned}\frac{\partial\mathbf{g}_j}{\partial x^i} &= \frac{\partial}{\partial x^i}(\frac{\partial\mathbf{p}}{\partial x^j}) = \frac{\partial^2 z^k}{\partial x^j\partial x^i}\mathbf{i}_k = \frac{\partial^2 z^k}{\partial x^j\partial x^i}\frac{\partial x^l}{\partial z^k}\mathbf{g}_l \\ &= \frac{\partial^2 z^k}{\partial x^j\partial x^i}\frac{\partial x^l}{\partial z^k}\mathbf{g}_l\cdot\mathbf{g}_m\cdot\mathbf{g}^m = \frac{\partial^2 z^k}{\partial x^j\partial x^i}\frac{\partial x^l}{\partial z^k}g_{lm}\mathbf{g}^m.\end{aligned} \tag{2.164}$$

Introduce the *Christoffel symbols of the first and second kinds* as

$$\Gamma_{ijl} = \frac{\partial^2 z^k}{\partial x^j\partial x^i}\frac{\partial x^m}{\partial z^k}g_{ml} \text{ and } \Gamma_{ij}^l = \frac{\partial^2 z^k}{\partial x^j\partial x^i}\frac{\partial x^l}{\partial z^k} \tag{2.165}$$

or

$$\Gamma_{ij}^l = \Gamma_{ijm}g^{ml} \text{ and } \Gamma_{ijl} = \Gamma_{ij}^m g_{ml}. \tag{2.166}$$

Equation (2.164) becomes

$$\frac{\partial \mathbf{g}_j}{\partial x^i} = \Gamma^l_{ij}\mathbf{g}_l \text{ and } \frac{\partial \mathbf{g}_j}{\partial x^i} = \Gamma_{ijl}\mathbf{g}^l. \tag{2.167}$$

Thus,

$$\frac{\partial \mathbf{g}_j}{\partial x^i}\cdot\mathbf{g}^l = \Gamma^l_{ij} \text{ and } \frac{\partial \mathbf{g}_j}{\partial x^i}\cdot\mathbf{g}_l = \Gamma_{ijl}. \tag{2.168}$$

Because

$$\begin{aligned} 0 = \frac{\partial \delta^k_j}{\partial x^i} &= \frac{\partial}{\partial x^i}\left(\mathbf{g}_j\cdot\mathbf{g}^k\right) = \frac{\partial \mathbf{g}_j}{\partial x^i}\cdot\mathbf{g}^k + \frac{\partial \mathbf{g}^k}{\partial x^i}\cdot\mathbf{g}_j \\ &= \Gamma^k_{ij} + \frac{\partial \mathbf{g}^k}{\partial x^i}\cdot\mathbf{g}_j, \end{aligned} \tag{2.169}$$

the following relations are obtained.

$$\frac{\partial \mathbf{g}^k}{\partial x^i}\cdot\mathbf{g}_j = -\Gamma^k_{ij} \text{ and } \frac{\partial \mathbf{g}^k}{\partial x^i} = -\Gamma^k_{ij}\mathbf{g}^j. \tag{2.170}$$

Using Eq.(2.164), the derivative of the fundamental metric tensor g_{ij} with respect to coordinate x^k ($k = 1, 2, 3$) gives

$$\begin{aligned} g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k} &= \frac{\partial}{\partial x^k}(\mathbf{g}_i\cdot\mathbf{g}_j) = \frac{\partial \mathbf{g}_i}{\partial x^k}\cdot\mathbf{g}_j + \frac{\partial \mathbf{g}_j}{\partial x^k}\cdot\mathbf{g}_i \\ &= \Gamma_{kij} + \Gamma_{kji}. \end{aligned} \tag{2.171}$$

Similarly,

$$g_{jk,i} = \Gamma_{ijk} + \Gamma_{ikj} \text{ and } g_{ki,j} = \Gamma_{jki} + \Gamma_{jik}. \tag{2.172}$$

Because of $\Gamma_{ijk} = \Gamma_{jik}$, from Eq.(2.165), equation (2.172) becomes

$$g_{jk,i} = \Gamma_{ijk} + \Gamma_{kij} \text{ and } g_{ki,j} = \Gamma_{kji} + \Gamma_{ijk}. \tag{2.173}$$

Adding both of equations in Eq.(2.173) and subtracting Eq.(2.171) yields

$$\Gamma_{ijk} = \frac{1}{2}(g_{jk,i} + g_{ki,j} - g_{ij,k}). \tag{2.174}$$

The derivative of volume $\sqrt{g} = [\mathbf{g}_1\mathbf{g}_2\mathbf{g}_3] = \mathbf{g}_1\cdot(\mathbf{g}_2\times\mathbf{g}_3)$ with respect to coordinate x^i ($i = 1, 2, 3$) is

$$\frac{\partial}{\partial x^i}\sqrt{g} = \frac{\partial}{\partial x^i}\mathbf{g}_1\cdot(\mathbf{g}_2\times\mathbf{g}_3) + \mathbf{g}_1\cdot(\frac{\partial}{\partial x^i}\mathbf{g}_2\times\mathbf{g}_3) + \mathbf{g}_1\cdot(\mathbf{g}_2\times\frac{\partial}{\partial x^i}\mathbf{g}_3)$$

$$= \Gamma^r_{i1}[\mathbf{g}_r\mathbf{g}_2\mathbf{g}_3] + \Gamma^r_{i2}[\mathbf{g}_1\mathbf{g}_r\mathbf{g}_3] + \Gamma^r_{i3}[\mathbf{g}_1\mathbf{g}_2\mathbf{g}_r]$$
$$= \Gamma^r_{ir}[\mathbf{g}_1\mathbf{g}_2\mathbf{g}_3] = \Gamma^r_{ir}\sqrt{g}. \tag{2.175}$$

Notice that $[\mathbf{g}_i\mathbf{g}_j\mathbf{g}_k] \neq 0$ for $i \neq j \neq k \neq i$ and $[\mathbf{g}_i\mathbf{g}_j\mathbf{g}_k] = 0$ for the other are adopted.

Introduce the *Hamilton operator* as

$$\nabla = \mathbf{g}^i \frac{\partial}{\partial x^i} = \mathbf{g}_i \frac{\partial}{\partial x_i} \text{ for } i = 1, 2, \ldots, n. \tag{2.176}$$

(A) *Scalar function gradient* For a zero-order tensor (or scalar) $\varphi(x^1, x^2, \ldots, x^N)$, the gradient of such a quantity is given by

$$\nabla\varphi = \mathbf{g}^i \frac{\partial\varphi}{\partial x^i} \text{ for } i = 1, 2, \ldots, N; \tag{2.177}$$

from which the differentiation of $\varphi(x^1, x^2, \ldots, x^N)$ is expressed by

$$d\varphi = \frac{\partial\varphi}{\partial x^i}dx^i = dx^j\mathbf{g}_j \cdot \mathbf{g}^i \frac{\partial\varphi}{\partial x^i} = d\mathbf{p}\cdot\nabla\varphi. \tag{2.178}$$

(B) *Vector gradient* For a first-order tensor (or vector) $\mathbf{v} = v^i\mathbf{g}_i = v_i\mathbf{g}^i$,

$$d\mathbf{v} = \frac{\partial\mathbf{v}}{\partial x^i}dx^i = dx^j\mathbf{g}_j \cdot \mathbf{g}^i \frac{\partial\mathbf{v}}{\partial x^i} = d\mathbf{p}\cdot\mathbf{g}^i(\frac{\partial v^j}{\partial x^i}\mathbf{g}_j + v^j\frac{\partial\mathbf{g}_j}{\partial x^i})$$
$$= d\mathbf{p}\cdot\mathbf{g}^i\mathbf{g}_j(\frac{\partial v^j}{\partial x^i} + \Gamma^j_{ir}v^r) = d\mathbf{p}\cdot\mathbf{g}^i\mathbf{g}_j(v^j_{,i} + \Gamma^j_{ir}v^r) = d\mathbf{p}\cdot\nabla\mathbf{v}. \tag{2.179}$$

Therefore,

$$\nabla\mathbf{v} = \mathbf{g}^i\mathbf{g}_j(v^j{}_{,i} + \Gamma^j_{ir}v^r) \text{ and } \nabla_i v^j \equiv v^j{}_{;i} = v^j{}_{,i} + \Gamma^j_{ir}v^r. \tag{2.180}$$

$$\frac{\partial\mathbf{v}}{\partial x^i} = v^j{}_{;i}\mathbf{g}_j = \mathbf{g}_j(v^j{}_{,i} + \Gamma^j_{ir}v^r) \text{ and } \nabla\mathbf{v} = v^j{}_{;i}\mathbf{g}^i\mathbf{g}_j. \tag{2.181}$$

Thus the covariant partial derivative of a contravariant vector with respect to x^i is

$$v^j{}_{;i} = v^j{}_{,i} + \Gamma^j_{ir}v^r. \tag{2.182}$$

Similarly, for $\mathbf{v} = v_i\mathbf{g}^i$, the following equations are achieved.

$$\begin{gathered}\nabla\mathbf{v} = \mathbf{g}^i\mathbf{g}^j(v_{j,i} - \Gamma^r_{ij}v_r) \text{ and } \nabla_i v_j \equiv v_{j;i} = v_{j,i} - \Gamma^r_{ij}v_r \\ \frac{\partial\mathbf{v}}{\partial x^i} = v_{j;i}\mathbf{g}^j = \mathbf{g}^j(v_{j,i} - \Gamma^r_{ij}v_r) \text{ and } \nabla\mathbf{v} = v_{j;i}\mathbf{g}^i\mathbf{g}^j.\end{gathered} \tag{2.183}$$

The covariant partial derivative of a covariant vector with respect to x^i is

$$v_{j;i} = v_{j,i} - \Gamma^r_{ij} v_r. \tag{2.184}$$

As in Eq.(2.179),

$$\begin{aligned} d\mathbf{v} &= \frac{\partial \mathbf{v}}{\partial x^i} dx^i = \frac{\partial \mathbf{v}}{\partial x^i} \mathbf{g}^i \cdot \mathbf{g}_j dx^j = \left[\frac{\partial v^j}{\partial x^i} \mathbf{g}_j + v^j \frac{\partial \mathbf{g}_j}{\partial x^i} \right] \mathbf{g}^i \cdot d\mathbf{p} \\ &= \left[\frac{\partial v^j}{\partial x^i} + \Gamma^j_{ir} v^r \right] \mathbf{g}_j \mathbf{g}^i \cdot d\mathbf{p} = (v^j_{,i} + \Gamma^j_{ir} v^r) \mathbf{g}_j \mathbf{g}^i \cdot d\mathbf{p} \\ &\equiv v^j{}_{;i} \mathbf{g}_j \mathbf{g}^i \cdot d\mathbf{p} = \mathbf{v}\nabla \cdot d\mathbf{p}. \end{aligned} \tag{2.185}$$

Thus,

$$\mathbf{v}\nabla = v^j{}_{;i} \mathbf{g}_j \mathbf{g}^i \quad \text{and} \quad v^j{}_{;i} = v^j_{,i} + \Gamma^j_{ir} v^r. \tag{2.186}$$

In a similar fashion, for a contravariant vector,

$$\mathbf{v}\nabla = v_{i;j} \mathbf{g}^j \mathbf{g}^i \quad \text{and} \quad v_{i;j} = v_{i,j} + \Gamma^i_{jr} v^r. \tag{2.187}$$

Therefore,

$$\nabla \mathbf{v} \neq \mathbf{v}\nabla \text{ and } \nabla \mathbf{v} = (\mathbf{v}\nabla)^{\mathrm{T}} \tag{2.188}$$

(C) *Tensor gradient*

To extend the concept of the vector gradient to the tensor, the gradient of a tensor $\boldsymbol{\varphi}$ can be discussed via its total differentiation, i.e.,

$$d\boldsymbol{\psi} = \boldsymbol{\psi}_{,r} dx^r = dx^s \mathbf{g}_s \cdot \mathbf{g}^r \frac{\partial}{\partial x^r} \boldsymbol{\psi} \equiv d\mathbf{p} \cdot \nabla \boldsymbol{\psi}. \tag{2.189}$$

$$d\boldsymbol{\psi} = \boldsymbol{\psi}_{,r} dx^r = \boldsymbol{\psi} \frac{\partial}{\partial x^r} \mathbf{g}^r \cdot dx^s \mathbf{g}_s \equiv \boldsymbol{\psi}\nabla \cdot d\mathbf{p}. \tag{2.190}$$

Thus,

$$\begin{aligned} \nabla \boldsymbol{\psi} &= \mathbf{g}^r \frac{\partial}{\partial x^r} (\psi^{i_1 \cdots i_m}{}_{j_1 \cdots j_n} \sqrt{g}^{-W} \mathbf{g}_{i_1} \cdots \mathbf{g}_{i_m} \mathbf{g}^{j_1} \cdots \mathbf{g}^{j_n}) \\ &= \mathbf{g}^r \left[\frac{\partial}{\partial x^r} (\psi^{i_1 \cdots i_m}{}_{j_1 \cdots j_n}) \sqrt{g}^{-W} \mathbf{g}_{i_1} \cdots \mathbf{g}_{i_m} \mathbf{g}^{j_1} \cdots \mathbf{g}^{j_n} \right. \\ &\quad + \psi^{i_1 \cdots i_m}{}_{j_1 \cdots j_n} \frac{\partial}{\partial x^r} (\sqrt{g}^{-W}) \mathbf{g}_{i_1} \cdots \mathbf{g}_{i_m} \mathbf{g}^{j_1} \cdots \mathbf{g}^{j_n} \\ &\quad + \psi^{i_1 \cdots i_m}{}_{j_1 \cdots j_n} \sqrt{g}^{-W} \frac{\partial}{\partial x^r} (\mathbf{g}_{i_1}) \cdots \mathbf{g}_{i_m} \mathbf{g}^{j_1} \cdots \mathbf{g}^{j_n} + \cdots \\ &\quad \left. + \psi^{i_1 \cdots i_m}{}_{j_1 \cdots j_n} \sqrt{g}^{-W} \mathbf{g}_{i_1} \cdots \mathbf{g}_{i_m} \mathbf{g}^{j_1} \cdots \frac{\partial}{\partial x^r} (\mathbf{g}^{j_n}) \right] \end{aligned}$$

$$=\left[\frac{\partial}{\partial x^r}(\psi^{i_1\cdots i_m}{}_{j_1\cdots j_n})-W\Gamma^s_{rs}\psi^{i_1\cdots i_m}{}_{j_1\cdots j_n}+\Gamma^{i_1}_{rs}\psi^{si_2\cdots i_m}{}_{j_1\cdots j_n}+\cdots\right.$$
$$\left.+\Gamma^{i_m}_{rs}\psi^{i_1\cdots i_{m-1}s}{}_{j_1\cdots j_n}-\Gamma^s_{rj_1}\psi^{i_1\cdots i_m}{}_{sj_2\cdots j_n}-\cdots-\Gamma^s_{rj_n}\psi^{i_1\cdots i_m}{}_{j_1\cdots j_{n-1}s}\right]$$
$$\times\sqrt{g}^{-W}\mathbf{g}^r\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}. \tag{2.191}$$

Introduce the component of $\nabla\boldsymbol{\psi}$ (i.e., the covariant partial derivative of a tensor component)

$$\begin{aligned}\nabla_r\psi^{i_1\cdots i_m}{}_{j_1\cdots j_n}&=\frac{\partial}{\partial x^r}(\psi^{i_1\cdots i_m}{}_{j_1\cdots j_n})-W\Gamma^s_{rs}\psi^{i_1\cdots i_m}{}_{j_1\cdots j_n}\\&\quad+\Gamma^{i_1}_{rs}\psi^{si_2\cdots i_m}{}_{j_1\cdots j_n}+\cdots+\Gamma^{i_m}_{rs}\psi^{i_1\cdots i_{n-1}s}{}_{j_1\cdots j_n}\\&\quad-\Gamma^s_{rj_1}\psi^{i_1\cdots i_m}{}_{sj_2\cdots j_n}-\cdots-\Gamma^s_{rj_n}\psi^{i_1\cdots i_m}{}_{j_1\cdots j_{n-1}s}\\&\equiv\psi^{i_1\cdots i_m}{}_{j_1\cdots j_n;r}.\end{aligned} \tag{2.192}$$

Thus

$$\begin{aligned}\nabla\boldsymbol{\psi}&=\nabla_r\psi^{i_1\cdots i_m}{}_{j_1\cdots j_n}\sqrt{g}^{-W}\mathbf{g}^r\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}\\&=\psi^{i_1\cdots i_m}{}_{j_1\cdots j_n;r}\sqrt{g}^{-W}\mathbf{g}^r\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}.\end{aligned} \tag{2.193}$$

Similarly,

$$\begin{aligned}\boldsymbol{\psi}\nabla&=\nabla_r\psi^{i_1\cdots i_m}{}_{j_1\cdots j_n}\sqrt{g}^{-W}\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}\mathbf{g}^r\\&=\psi^{i_1\cdots i_m}{}_{j_1\cdots j_n;r}\sqrt{g}^{-W}\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}\mathbf{g}^r.\end{aligned} \tag{2.194}$$

Note that $\nabla_r\neq\partial/\partial x^r$ and ∇_r cannot apply to the base vector and g. In the contravariant coordinate system, the Hamilton operator can be defined in a similar fashion.

Theorem 2.9. (Ricci's theorem). *The covariant partial differentiation of metric tensors is a zero tensor, i.e.,*

$$\nabla_i g_{jk}=g_{jk;i}=0,\ \nabla_i g^{jk}=g^{jk}{}_{;i}=0,\ \nabla_i\delta^j_k=\delta^j_{k;i}=0. \tag{2.195}$$

Proof : Because $\mathbf{I}=g_{jk}\mathbf{g}^j\mathbf{g}^k=\delta^j_k\mathbf{g}_j\mathbf{g}^k=g^{jk}\mathbf{g}_j\mathbf{g}_k$ is a constant tensor, $d\mathbf{I}=0$ and $\nabla\mathbf{I}=0$. So all components should be zero, i.e.,

$$\nabla_i g_{jk}=g_{jk;i}=0,\ \nabla_i g^{jk}=g^{jk}{}_{;i}=0,\ \nabla_i\delta^j_k=\delta^j_{k;i}=0.$$

On the other hand,

$$\begin{aligned}\nabla_i g_{jk} &= g_{jk;i} = \frac{\partial}{\partial x^i} g_{jk} - \Gamma_{ij}^r g_{rk} - \Gamma_{ik}^r g_{jr} \\ &= \Gamma_{ijk} + \Gamma_{ikj} - \Gamma_{ijk} - \Gamma_{ikj} \\ &= 0, \\ \nabla_i g^{jk} &= g^{jk}{}_{;i} = \frac{\partial}{\partial x^i} g^{jk} + \Gamma_{ir}^j g^{rk} + \Gamma_{ir}^k g^{jr} \\ &= -\Gamma_{ir}^j g^{rk} - \Gamma_{ir}^k g^{jr} + \Gamma_{ir}^j g^{rk} + \Gamma_{ir}^k g^{jr} \\ &= 0, \\ \nabla_i \delta_k^j &= \delta_{k;i}^j = \frac{\partial}{\partial x^i} \delta_k^j + \Gamma_{ir}^j \delta_k^r - \Gamma_{ik}^r \delta_r^j \\ &= 0 + \Gamma_{ik}^j - \Gamma_{ik}^j \\ &= 0.\end{aligned}$$

This theorem is proved. ■

From the foregoing theorem,

$$v^j{}_{;i} = (g^{jr} v_r)_{;i} = g^{jr} v_{r;i} \text{ and } v_{j;i} = (g_{jr} v^r)_{;i} = g_{jr} v^r{}_{;i}. \tag{2.196}$$

Theorem 2.10. *The covariant partial differentiation of the Eddington tensor is a zero tensor, i.e.,*

$$\nabla_r \varepsilon^{ijk} = \varepsilon^{ijk}{}_{;r} = 0 \text{ and } \nabla_r \varepsilon_{ijk} = \varepsilon_{ijk;r} = 0. \tag{2.197}$$

Proof: Because $\boldsymbol{\varepsilon} = \varepsilon^{ijk} \mathbf{g}_i \mathbf{g}_j \mathbf{g}_k$ and $\varepsilon^{ijk} = [\mathbf{g}^i \mathbf{g}^j \mathbf{g}^k]$, its differentiation is

$$\begin{aligned}\nabla\boldsymbol{\varepsilon} &= \nabla_r \varepsilon^{ijk} \mathbf{g}^r \mathbf{g}_i \mathbf{g}_j \mathbf{g}_k = \varepsilon^{ijk}{}_{;r} \mathbf{g}^r \mathbf{g}_i \mathbf{g}_j \mathbf{g}_k = \mathbf{g}^r \frac{\partial}{\partial x^r} (\varepsilon^{ijk} \mathbf{g}_i \mathbf{g}_j \mathbf{g}_k) \\ &= \mathbf{g}^r \{([\frac{\partial}{\partial x^r}(\mathbf{g}^i)\mathbf{g}^j \mathbf{g}^k] + [\mathbf{g}^i \frac{\partial}{\partial x^r}(\mathbf{g}^j)\mathbf{g}^k] + [\mathbf{g}^i \mathbf{g}^j \frac{\partial}{\partial x^r}(\mathbf{g}^k)])\mathbf{g}_i \mathbf{g}_j \mathbf{g}_k \\ &\quad + [\mathbf{g}^i \mathbf{g}^j \mathbf{g}^k][\frac{\partial}{\partial x^r}(\mathbf{g}_i)\mathbf{g}_j \mathbf{g}_k + \mathbf{g}_i \frac{\partial}{\partial x^r}(\mathbf{g}_j)\mathbf{g}_k + \mathbf{g}_i \mathbf{g}_j \frac{\partial}{\partial x^r}(\mathbf{g}_k)]\} \\ &= \mathbf{g}^r \{(-\Gamma_{rs}^i [\mathbf{g}^s \mathbf{g}^j \mathbf{g}^k] - \Gamma_{rs}^j [\mathbf{g}^i \mathbf{g}^s \mathbf{g}^k] - \Gamma_{rs}^k [\mathbf{g}^i \mathbf{g}^j \mathbf{g}^s])\mathbf{g}_i \mathbf{g}_j \mathbf{g}_k \\ &\quad + [\mathbf{g}^i \mathbf{g}^j \mathbf{g}^k] \Gamma_{ri}^s \mathbf{g}_s \mathbf{g}_j \mathbf{g}_k + \Gamma_{rj}^s \mathbf{g}_i \mathbf{g}_s \mathbf{g}_k + \Gamma_{rk}^s \mathbf{g}_i \mathbf{g}_j \mathbf{g}_s\} \\ &= 0.\end{aligned}$$

Therefore,

$$\nabla_r \varepsilon^{ijk} = \varepsilon^{ijk}{}_{;r} = 0.$$

In an alike fashion, the following formulas hold:

$$\nabla_r \varepsilon_{ijk} = \varepsilon_{ijk;r} = 0.$$

This theorem is proved. ■

(D) *Derivative distribution*

$$\begin{aligned}&\nabla_r(\alpha\psi^{i_1\cdots i_m}{}_{j_1\cdots j_n} + \beta\varphi^{i_1\cdots i_m}{}_{j_1\cdots j_n})\\ &= \alpha\nabla_r\psi^{i_1\cdots i_m}{}_{j_1\cdots j_n} + \beta\nabla_r\varphi^{i_1\cdots i_m}{}_{j_1\cdots j_n}.\end{aligned} \tag{2.198}$$

(E) *Leibniz rule*

$$\begin{aligned}&\nabla_r(\psi^{i_1\cdots i_m}{}_{j_1\cdots j_n}\varphi^{k_1\cdots k_r}{}_{l_1\cdots l_s})\\ &= \nabla_r\psi^{i_1\cdots i_m}{}_{j_1\cdots j_n}\varphi^{k_1\cdots k_r}{}_{l_1\cdots l_s} + \psi^{i_1\cdots i_m}{}_{j_1\cdots j_n}\nabla_r\varphi^{k_1\cdots k_r}{}_{l_1\cdots l_s}.\end{aligned} \tag{2.199}$$

From the gradient operator in the covariant coordinate bases, the gradient operator in the contravariant bases is defined as

$$\begin{aligned}\nabla^r\psi^{i_1\cdots i_m}{}_{j_1\cdots j_n} &= \frac{\partial}{\partial x^r}\psi^{i_1\cdots i_m}{}_{j_1\cdots j_n} = \frac{\partial}{g_{rs}\partial x^s}\psi^{i_1\cdots i_m}{}_{j_1\cdots j_n}\\ &= g^{rs}\frac{\partial}{\partial x^s}\psi^{i_1\cdots i_m}{}_{j_1\cdots j_n} = g^{rs}\nabla_r\psi^{i_1\cdots i_m}{}_{j_1\cdots j_n}.\end{aligned} \tag{2.200}$$

The relation $\partial x_r = \partial x^s g_{rs}$ is used because of $d\mathbf{p} = dx^s\mathbf{g}_s = dx_r\mathbf{g}^r$.

2.3.2. Invariant differential operators and integral theorems

After discussed the gradient of a tensor, as in the vector analysis, there are four types of differential operators. Such a concept can be extended to the tensor. For a tensor $\boldsymbol{\psi}$, the following differentiations are

$$\begin{aligned}\operatorname{grad}\boldsymbol{\psi} \equiv \nabla\boldsymbol{\psi} &= \nabla_r\psi^{i_1\cdots i_m}{}_{j_1\cdots j_n}\sqrt{g}^{-W}\mathbf{g}^r\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}\\ &= \psi^{i_1\cdots i_m}{}_{j_1\cdots j_n;r}\sqrt{g}^{-W}\mathbf{g}^r\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}.\end{aligned} \tag{2.201}$$

$$\begin{aligned}\operatorname{div}\boldsymbol{\psi} \equiv \nabla\cdot\boldsymbol{\psi} &= \nabla_r\psi^{ri_2\cdots i_m}{}_{j_1\cdots j_n}\sqrt{g}^{-W}\mathbf{g}_{i_2}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}\\ &= \psi^{ri_2\cdots i_m}{}_{j_1\cdots j_n;r}\sqrt{g}^{-W}\mathbf{g}_{i_2}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}.\end{aligned} \tag{2.202}$$

$$\begin{aligned}\operatorname{rot}\boldsymbol{\psi} \equiv \nabla\times\boldsymbol{\psi} &= \mathbf{g}^r\times\nabla_r\psi_{i_1\cdots i_m}{}^{j_1\cdots j_n}\sqrt{g}^{-W}\mathbf{g}^{i_1}\cdots\mathbf{g}^{i_m}\mathbf{g}_{j_1}\cdots\mathbf{g}_{j_n}\\ &= \varepsilon^{sri_1}\psi_{i_1\cdots i_m}{}^{j_1\cdots j_n}{}_{;r}\sqrt{g}^{-W}\mathbf{g}_s\mathbf{g}^{i_2}\cdots\mathbf{g}^{i_m}\mathbf{g}_{j_1}\cdots\mathbf{g}_{j_n}.\end{aligned} \tag{2.203}$$

$$\begin{aligned}\nabla^2\boldsymbol{\psi} &\equiv \operatorname{div}\operatorname{grad}\boldsymbol{\psi} = \nabla\cdot\nabla\boldsymbol{\psi}\\ &= \nabla^r\nabla_r\psi^{i_1\cdots i_m}{}_{j_1\cdots j_n}\sqrt{g}^{-W}\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}.\end{aligned} \tag{2.204}$$

From Schouten (1951), the integration on a manifold can be expressed by

$$\int_{\mathfrak{V}_{m+1}} d\mathfrak{V}^{ri_1\cdots i_m}\frac{\partial}{\partial x^r}\odot\psi^{j_1\cdots j_n} = \oint_{\mathfrak{V}_m} d\mathfrak{V}^{i_1\cdots i_m}\odot\psi^{j_1\cdots j_n} \tag{2.205}$$

where $d\mathfrak{V}^{ri_1\cdots i_m}$ and $d\mathfrak{V}^{i_1\cdots i_m}$ are $(m+1)$ and m-dimensional Grassmann volume elements. The volume $\mathfrak{V}_m$ is the boundary of the volume $\mathfrak{V}_{m+1}$. The operator "$\odot$" represents any operation between the tensor ψ and the volume elements $d\mathfrak{A}$.

$$\begin{aligned}&\int_{\mathfrak{V}_{m+1}}\mathbf{g}_r\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}d\mathfrak{V}^{ri_1\cdots i_m}\cdot\mathbf{g}^r\frac{\partial}{\partial x^r}\odot\psi^{j_1\cdots j_n}\mathbf{g}_{j_1}\cdots\mathbf{g}_{j_n}\\ &= \oint_{\mathfrak{V}_m}\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}d\mathfrak{V}^{i_1\cdots i_m}\odot\psi^{j_1\cdots j_n}\mathbf{g}_{j_1}\cdots\mathbf{g}_{j_n},\end{aligned} \tag{2.206}$$

$$\int_{\mathfrak{V}_{m+1}} d\boldsymbol{\mathfrak{V}}^{(m+1)}\cdot\nabla\odot\boldsymbol{\psi} = \oint_{\mathfrak{V}_m} d\boldsymbol{\mathfrak{V}}^{(m)}\odot\boldsymbol{\psi} \tag{2.207}$$

where

$$\begin{aligned}d\boldsymbol{\mathfrak{V}}^{(m+1)} &= d\mathfrak{V}^{ri_1\cdots i_m}\mathbf{g}_r\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m},\\ d\boldsymbol{\mathfrak{V}}^{(m)} &= d\mathfrak{V}^{i_1\cdots i_m}\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m},\\ \boldsymbol{\psi} &= \psi^{j_1\cdots j_n}\mathbf{g}_{j_1}\cdots\mathbf{g}_{j_n}.\end{aligned} \tag{2.208}$$

In the 3-D Euclidean space, $m\in\{1,2\}$. The volume integration changes to the area integration by the Green transformation. The area integration is converted into the linear integration by the Kelvin transformation.

$$\begin{aligned}d\mathfrak{V}^{i_1i_2i_3} &= 3!\,\underset{1}{dx}^{i_1}\,\underset{2}{dx}^{i_2}\,\underset{3}{dx}^{i_3}, &\mathrm{d}\boldsymbol{\mathfrak{V}}^{(3)} &= d\mathfrak{V}^{i_1i_2i_3}\mathbf{g}_{i_1}\mathbf{g}_{i_2}\mathbf{g}_{i_3},\\ d\mathfrak{V}^{i_2i_3} &= 2!\,\underset{4}{dx}^{i_2}\,\underset{5}{dx}^{i_3}, &\mathrm{d}\boldsymbol{\mathfrak{V}}^{(2)} &= d\mathfrak{V}^{i_2i_3}\mathbf{g}_{i_2}\mathbf{g}_{i_3},\\ d\mathfrak{V}^{i_3} &= 1!\,\underset{6}{dx}^{i_3}, &\mathrm{d}\boldsymbol{\mathfrak{V}}^{(1)} &= d\mathfrak{V}^{i_3}\mathbf{g}_{i_3}.\end{aligned} \tag{2.209}$$

Note that

$$\begin{aligned}dv &= \frac{1}{3!}\boldsymbol{\varepsilon}\,\vdots\,d\boldsymbol{\mathfrak{V}}^{(3)} = \frac{1}{3!}\varepsilon_{i_1i_2i_3}d\mathfrak{V}^{i_1i_2i_3}\\ &= \varepsilon_{i_1i_2i_3}\,\underset{1}{dx}^{i_1}\,\underset{2}{dx}^{i_2}\,\underset{3}{dx}^{i_3} = [\underset{1}{d\mathbf{p}}\,\underset{2}{d\mathbf{p}}\,\underset{3}{d\mathbf{p}}],\\ d\mathbf{a} &= \frac{1}{2!}\boldsymbol{\varepsilon}:d\boldsymbol{\mathfrak{V}}^{(2)} = \frac{1}{3!}\varepsilon_{ijk}\mathbf{g}^i\mathbf{g}^j\mathbf{g}^k:d\mathfrak{V}^{i_2i_3}\mathbf{g}_{i_2}\mathbf{g}_{i_3}\\ &= \varepsilon_{ii_2i_3}\,\underset{4}{dx}^{i_2}\,\underset{5}{dx}^{i_3}\,\mathbf{g}^i = \underset{4}{d\mathbf{p}}\times\underset{5}{d\mathbf{p}},\end{aligned} \tag{2.210}$$

$$d\mathfrak{V}^{(3)} = \frac{1}{0!}\boldsymbol{\varepsilon} dv, \quad d\mathfrak{V}^{(2)} = \frac{1}{1!}\boldsymbol{\varepsilon} \cdot d\mathbf{a}. \tag{2.211}$$

For $m = 2$,

$$\int_{\mathfrak{V}_3} d\mathfrak{V}^{(3)} \cdot \nabla \odot \boldsymbol{\psi} = \oint_{\mathfrak{V}_2} d\mathfrak{V}^{(2)} \odot \boldsymbol{\psi}. \tag{2.212}$$

The left double dot product of $\frac{1}{2!}\boldsymbol{\varepsilon}$ to both sides of Eq.(2.212) gives

$$\int_{\mathfrak{V}_3} \frac{1}{2!}\boldsymbol{\varepsilon} : d\mathfrak{V}^{(3)} \cdot \nabla \odot \boldsymbol{\psi} = \oint_{\mathfrak{V}_2} \frac{1}{2!}\boldsymbol{\varepsilon} : d\mathfrak{V}^{(2)} \odot \boldsymbol{\psi}, \tag{2.213}$$

$$\int_v \frac{1}{2!}\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} dv \cdot \nabla \odot \boldsymbol{\psi} = \oint_a d\mathbf{a} \odot \boldsymbol{\psi}, \tag{2.214}$$

$$\int_v dv \mathbf{I} \cdot \nabla \odot \boldsymbol{\psi} = \oint_a d\mathbf{a} \odot \boldsymbol{\psi}. \tag{2.215}$$

Finally,

$$\int_v dv \nabla \odot \boldsymbol{\psi} = \oint_a d\mathbf{a} \odot \boldsymbol{\psi}. \tag{2.216}$$

The above equation can be applied to any curvilinear coordinate systems. Setting "$\odot$" be contraction, dot product and cross product, the foregoing equation becomes

$$\begin{aligned} \int_v dv \ \mathrm{grad}\, \boldsymbol{\psi} &= \oint_a d\mathbf{a} \boldsymbol{\psi}, \\ \int_v dv \ \mathrm{div}\, \boldsymbol{\psi} &= \oint_a d\mathbf{a} \cdot \boldsymbol{\psi}, \\ \int_v dv \ \mathrm{rot}\, \boldsymbol{\psi} &= \oint_a d\mathbf{a} \times \boldsymbol{\psi}. \end{aligned} \tag{2.217}$$

For $m = 1$,

$$-\int_{\mathfrak{V}_2} d\mathfrak{V}^{(2)} \cdot \nabla \odot \boldsymbol{\psi} = \oint_{\mathfrak{V}_1} d\mathfrak{V}^{(1)} \odot \boldsymbol{\psi}, \tag{2.218}$$

$$-\int_a (\boldsymbol{\varepsilon} \cdot d\mathbf{a}) \cdot \nabla \odot \boldsymbol{\psi} = \oint_c d\mathbf{p} \odot \boldsymbol{\psi}, \tag{2.219}$$

$$\int_a d\mathbf{a} \times \nabla \odot \boldsymbol{\psi} = \oint_c d\mathbf{p} \odot \boldsymbol{\psi}. \tag{2.220}$$

Setting "$\odot$" to be a dot product,

$$\int_a d\mathbf{a} \times \nabla \cdot \boldsymbol{\psi} = \int_a d\mathbf{a} \cdot \nabla \times \boldsymbol{\psi} = \int_a d\mathbf{a} \cdot \mathrm{rot}\, \boldsymbol{\psi} = \oint_c d\mathbf{p} \cdot \boldsymbol{\psi}. \tag{2.221}$$

If $\boldsymbol{\psi}$ is a vector, the foregoing equation is the *Stokes formula*.

2.3.3. Riemann-Christoffel curvature tensors

In calculus, the order of mixed partial derivative of a scalar is not important, i.e.,

$$\frac{\partial^2\phi}{\partial x^1\partial x^2}=\frac{\partial^2\phi}{\partial x^2\partial x^1}. \tag{2.222}$$

For a vector $\mathbf{u}=u_i\mathbf{g}^i$, one would like to know whether the second-order covariant derivatives of components of a vector can commute or not. Thus, consider

$$\begin{aligned}u_{i;jk}&=(u_{i;j})_{,k}-\Gamma^r_{ki}u_{r;j}-\Gamma^r_{kj}u_{i;r}\\&=u_{i,jk}-\Gamma^r_{ji,k}u_r-\Gamma^s_{ji}u_{s,k}-\Gamma^r_{ki}(u_{r,j}-\Gamma^s_{jr}u_s)-\Gamma^r_{kj}(u_{i,r}-\Gamma^s_{ri}u_s)\\&=u_{i,jk}-\Gamma^r_{kj}u_{i,r}-\Gamma^r_{ki}u_{r,j}-\Gamma^r_{ji}u_{r,k}-u_s(\Gamma^s_{ji,k}-\Gamma^r_{ik}\Gamma^s_{jr}-\Gamma^r_{kj}\Gamma^s_{ri}).\end{aligned} \tag{2.223}$$

Similarly,

$$u_{i;kj}=u_{i,kj}-\Gamma^r_{jk}u_{i,r}-\Gamma^r_{ji}u_{r,k}-\Gamma^r_{ki}u_{r,j}-u_s(\Gamma^s_{ki,j}-\Gamma^r_{ij}\Gamma^s_{kr}-\Gamma^r_{jk}\Gamma^s_{ri}). \tag{2.224}$$

So Eq.(2.223) minus Eq.(2.224) yields

$$u_{i;jk}-u_{i;kj}=u_s(\Gamma^s_{ki,j}-\Gamma^s_{ji,k}+\Gamma^r_{ik}\Gamma^s_{jr}-\Gamma^r_{ij}\Gamma^s_{kr}). \tag{2.225}$$

Since u_s is an arbitrary vector, and the left-hand side of Eq.(2.225) is a covariant tensor of third order. From the quotient law, the coefficient of u_s on the right-hand side should be a fourth-order tensor, and this tensor possesses once contravariant and three times covariant. Such a tensor is called the *Riemann-Christoffel* tensor. Therefore, the *Riemann-Christoffel tensor* is defined as

$$R^s_{\cdot ijk}=\Gamma^s_{ki,j}-\Gamma^s_{ji,k}+\Gamma^r_{ik}\Gamma^s_{jr}-\Gamma^r_{ij}\Gamma^s_{kr}. \tag{2.226}$$

Further, equation (2.224) becomes

$$2u_{i;[jk]}=u_{i;jk}-u_{i;kj}=u_sR^s_{\cdot ijk}, \tag{2.227}$$

with

$$\begin{aligned}(u_{i;jk}-u_{i;kj})\mathbf{g}\,\mathbf{g}^j\mathbf{g}^k&=u_sR^s_{\cdot ijk}\mathbf{g}^i\mathbf{g}^j\mathbf{g}^k=u_s\mathbf{g}^s\cdot R^r_{\cdot ijk}\mathbf{g}_r\mathbf{g}^i\mathbf{g}^j\mathbf{g}^k\\&=\mathbf{u}\cdot\mathbf{R}.\end{aligned} \tag{2.228}$$

Theorem 2.11. *Cross-covariant derivatives of any vector are equal* ($u_{i;jk}=u_{i;kj}$) *if and only if the Riemann-Christoffel tensor vanishes* ($R^s_{\cdot ijk}=0$).

Proof: Since

$$u_{i;jk}-u_{i;kj}=u_sR^s_{\cdot ijk},$$

if $u_{i;jk} = u_{i;kj}$, because of any selection of u_s, $R^{s}_{\cdot ijk} = 0$; vice versa. This theorem is proved. ■

Definition 2.24. The *curvature tensor* is defined by

$$R_{ijkl} = g_{ir} R^{r}_{\cdot jkl}. \tag{2.229}$$

Theorem 2.12. *The curvature tensor is expressed by*

$$R_{ijkl} = \frac{1}{2}(g_{il,jk} + g_{jk,il} - g_{ik,jl} - g_{jl,ik}) + g^{rs}(\Gamma_{ilr}\Gamma_{jks} - \Gamma_{ikr}\Gamma_{jls}). \tag{2.230}$$

Proof: Since

$$\Gamma_{jli,k} = (g_{ir}\Gamma^{r}_{jl})_{,k} = g_{ir,k}\Gamma^{r}_{jl} + g_{ir}\Gamma^{r}_{jl,k} = (\Gamma_{ikr} + \Gamma_{rki})\Gamma^{r}_{jl} + g_{ir}\Gamma^{r}_{jl,k},$$

The deformation of the foregoing equation gives

$$g_{ir}\Gamma^{r}_{jl,k} = \Gamma_{jli,k} - (\Gamma_{ikr} + \Gamma_{rki})\Gamma^{r}_{jl}.$$

Similarly,

$$g_{ir}\Gamma^{r}_{jk,l} = \Gamma_{jki,l} - (\Gamma_{ilr} + \Gamma_{rli})\Gamma^{r}_{jk}.$$

Using the foregoing equations, the definition of the curvature gives

$$\begin{aligned} R_{ijkl} &= g_{ir} R^{r}_{\cdot jkl} = g_{ir}(\Gamma^{r}_{lj,k} - \Gamma^{r}_{kj,l} + \Gamma^{s}_{jl}\Gamma^{r}_{ks} - \Gamma^{s}_{jk}\Gamma^{r}_{ls}) \\ &= \Gamma_{jli,k} - (\Gamma_{ikr} + \Gamma_{rki})\Gamma^{r}_{jl} - \Gamma_{jki,l} + (\Gamma_{ilr} + \Gamma_{rli})\Gamma^{r}_{jk} + \Gamma^{r}_{jl}\Gamma_{ikr} - \Gamma^{r}_{jk}\Gamma_{ilr} \\ &= \Gamma_{jli,k} - \Gamma_{rki}\Gamma^{r}_{jl} - \Gamma_{jki,l} + \Gamma_{rli}\Gamma^{r}_{jk}. \end{aligned}$$

Because

$$\begin{aligned} \Gamma_{jli,k} &= \frac{1}{2}(g_{li,j} + g_{ij,l} - g_{jl,i})_{,k} = \frac{1}{2}(g_{li,jk} + g_{ij,lk} - g_{jl,ik}), \\ \Gamma_{jki,l} &= \frac{1}{2}(g_{ki,j} + g_{ij,k} - g_{jk,i})_{,l} = \frac{1}{2}(g_{ki,jl} + g_{ij,kl} - g_{jk,il}), \end{aligned}$$

with $\Gamma_{ijk} = g_{kr}\Gamma^{r}_{ij}$, the curvature tensor is given by

$$\begin{aligned} R_{ijkl} &= \Gamma_{jli,k} - \Gamma_{rki}\Gamma^{r}_{jl} - \Gamma_{jki,l} + \Gamma_{rli}\Gamma^{r}_{jk} \\ &= \frac{1}{2}(g_{li,jk} + g_{jk,il} - g_{ij,lk} - g_{ki,jl}) + g^{rs}(\Gamma_{rli}\Gamma_{jks} - \Gamma_{rki}\Gamma_{jls}). \end{aligned}$$

This theorem is proved. ■

The following symmetry of the curvature tensor is very easily proved as

$$R_{ijkl} = R_{iklj} = -R_{jikl} = -R_{ijlk}. \tag{2.231}$$

Consider a second-order tensor $\mathbf{A} = A_{ij}\mathbf{g}^i\mathbf{g}^j = A^{ij}\mathbf{g}_i\mathbf{g}_j$. The cross-covariant derivative of the second-order tensor is

$$\begin{aligned} A_{ij;kl} - A_{ij;kl} &= A_{rj}R^r_{\cdot ikl} + A_{ir}R^r_{\cdot jkl}, \\ A^{ij}{}_{;kl} - A^{ij}{}_{;kl} &= -A^{rj}R^i_{\cdot rkl} - A^{ir}R^j_{\cdot rkl}. \end{aligned} \tag{2.232}$$

Bianchi's identities:

$$\begin{aligned} R^p_{\cdot qij;k} + R^p_{\cdot qjk;i} + R^p_{\cdot qki;j} &= 0, \\ R_{pqij;k} + R_{pqjk;i} + R_{pqki;j} &= 0. \end{aligned} \tag{2.233}$$

Consider a tensor $\boldsymbol{\psi} = \psi^{i_1\cdots i_m}{}_{j_1\cdots j_n}\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}$. The cross-covariant derivative of the second-order tensor is

$$\begin{aligned} \psi^{i_1\cdots i_m}{}_{j_1\cdots j_n;kl} - \psi^{i_1\cdots i_m}{}_{j_1\cdots j_n;lk} &= -\psi^{r\cdots i_m}{}_{j_1\cdots j_n}R^{i_1}_{\cdot rkl} - \cdots - \psi^{i_1\cdots r}{}_{j_1\cdots j_n}R^{i_m}_{\cdot rkl} \\ &\quad + \psi^{i_1\cdots i_m}{}_{rj_2\cdots j_n}R^r_{\cdot j_1kl} + \cdots + \psi^{i_1\cdots i_m}{}_{j_1\cdots j_{n-1}r}R^r_{\cdot j_nkl}. \end{aligned} \tag{2.234}$$

Since the Riemann-Christoffel tensor is a measure of the curvature of the space, a space of vanishing such a tensor is called *a flat space*. In the Euclidean space, the Riemann-Christoffel tensor vanishes, thus Euclidean space is a flat space.

In the curvilinear coordinate, the covariant partial derivatives of the tensor components in the incomplete coordinate system can be carried out via $\partial/\partial x^{(i)} \equiv A^i_{(i)}\partial/\partial x^{(i)}$. Because of $(\partial/\partial x^k)A^{(k)}_j \neq (\partial/\partial x^j)A^{(k)}_k$, the covariant mixed derivative order cannot be exchanged. So the Christoffel symbol is not symmetric. The other operations of tensors in the incomplete coordinate system are similar in the complete coordinate system. In orthogonal curvilinear coordinates ($g_{ij} = g^{ij} = 0$, $i \neq j$, $g^{ii} = 1/g_{ii}$) $\Gamma_{ijk} = 0$ ($i \neq j \neq k \neq i$). The components of a tensor do not have the same units in general. Consider a vector $\mathbf{u} = u^i\mathbf{g}_i$ and introduce a unit vector $\mathbf{e}_i = \mathbf{g}_i/\sqrt{g_{ii}}$. So $\mathbf{u} = u^i\mathbf{g}_i = u^{(i)}\mathbf{e}_i$ from which $u^{(i)} = u^i\sqrt{g_{ii}}$ or $u^i = u^{(i)}/\sqrt{g_{ii}}$. The component $u^{(i)}$ is called the physical component. Such orthogonal coordinates system with the unit vector is called the physical coordinates.

2.4. Two-point tensor fields

In this section, the two-point tensor fields will be discussed. The two point tensors and differentiations for two *independent* and *correlated* reference frames are discussed. Finally the properties of shifter tensors will be presented.

2.4.1. Two-point tensors

Consider two curvilinear coordinates in two domains $\mathfrak{B}$ and $\mathfrak{b}$ to be $\{X^I\}$ and $\{x^i\}$. In the domain $\mathfrak{B}$, the base vectors and metric tensors can be expressed by $\mathbf{G}_I$ and $\mathbf{G}^I$, G_{IJ} and G^{IJ} with $G = |G_{IJ}|$. In this book, the Guo's representation of two-point tensor fields will be adopted in Guo (1980). That is, the symbol "$<$" over a tensor indicates the tensor described in domain $\mathfrak{B}$. The symbol "$>$" over the tensor indicates the tensor described in domain $\mathfrak{b}$. The number of "$<$" or "$>$" should be the same as the tensor order. For any order tensor, one can simply express by "$<\cdot<$" and "$>\cdot>$".

Definition 2.25. A *two-point tensor field* is defined in the two coordinates $\{X^I\}$ and $\{x^i\}$ by

$$\begin{aligned} \overset{<\cdot>}{\boldsymbol{\psi}}(\mathbf{P},\mathbf{p}) &= \psi^{I_1\cdots I_M}{}_{J_1\cdots J_N}{}^{i_1\cdots i_m}{}_{j_1\cdots j_n}\sqrt{G}^{-W}\sqrt{g}^{-w} \\ &\quad\cdot\mathbf{G}_{I_1}\cdots\mathbf{G}_{I_M}\mathbf{G}^{J_1}\cdots\mathbf{G}^{J_N}\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}, \\ \overset{>\cdot<}{\boldsymbol{\psi}}(\mathbf{p},\mathbf{P}) &= \psi^{i_1\cdots i_m}{}_{j_1\cdots j_n}{}^{I_1\ldots I_M}{}_{J_1\cdots J_N}\sqrt{g}^{-w}\sqrt{G}^{-W} \\ &\quad\cdot\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}\mathbf{G}_{I_1}\cdots\mathbf{G}_{I_M}\mathbf{G}^{J_1}\cdots\mathbf{G}^{J_N}. \end{aligned} \tag{2.235}$$

For a simple first-order tensor, the above expression is given by

$$\begin{aligned} \overset{<}{\mathbf{P}} &= X^I\mathbf{G}_I = X_I\mathbf{G}^I \text{ and } \overset{<}{\mathbf{V}} = V^I\mathbf{G}_I = V_I\mathbf{G}^I, \\ \overset{>}{\mathbf{p}} &= x^i\mathbf{g}_i = x_i\mathbf{g}^i \text{ and } \overset{>}{\mathbf{v}} = v^i\mathbf{g}_i = v_i\mathbf{g}^i. \end{aligned} \tag{2.236}$$

It is found that the vector can be expressed by only one coordinate. Thus, the hat of the vector can be dropped.

Any second-order one-point tensor is described by only one coordinate, i.e.,

$$\begin{aligned} \overset{<<}{\mathbf{A}} &= A^{IJ}\mathbf{G}_I\mathbf{G}_J = A_{IJ}\mathbf{G}^I\mathbf{G}^J = A^I{}_J\mathbf{G}_I\mathbf{G}^J, \\ \overset{>>}{\mathbf{B}} &= B^{ij}\mathbf{g}_i\mathbf{g}_j = B_{ij}\mathbf{g}^i\mathbf{g}^j = B^i{}_j\mathbf{g}_i\mathbf{g}^j. \end{aligned} \tag{2.237}$$

The above two tensors are expressed by the coordinates $\{X^I\}$ and $\{x^i\}$, respectively. However, two second-order two-point tensors are

$$\begin{aligned} \overset{<>}{\mathbf{A}} &= A^I{}_i\mathbf{G}_I\mathbf{g}^i = A^{Ii}\mathbf{G}_I\mathbf{g}_i = A_{Ii}\mathbf{G}^I\mathbf{g}^i = A_I{}^i\mathbf{G}^I\mathbf{g}_i, \\ \overset{><}{\mathbf{B}} &= B^i{}_I\mathbf{g}_i\mathbf{G}^I = B_i{}^I\mathbf{g}^i\mathbf{G}_I = B^{iI}\mathbf{g}_i\mathbf{G}_I = B_{iI}\mathbf{g}^i\mathbf{G}^I. \end{aligned} \tag{2.238}$$

In Eq.(2.235), the tensor components have the following transformation as

$$\psi^{I'_1\cdots I'_M}{}_{J'_1\cdots J'_N}{}^{i'_1\cdots i'_m}{}_{j'_1\cdots j'_n} = |A^P_{P'}|^W |A^p_{p'}|^w A^{I'_1}_{I_1}\cdots A^{I'_M}_{I_M} A^{J_1}_{J'_1}\cdots A^{J_N}_{J'_N} \cdot A^{i'_1}_{i_1}\cdots A^{i'_m}_{i_m} A^{j_1}_{j'_1}\cdots A^{j_n}_{j'_n} \psi^{I_1\cdots I_M}{}_{J_1\cdots J_N}{}^{i_1\cdots i_m}{}_{j_1\cdots j_n}, \tag{2.239}$$

where $A^{I'}_I = \partial X^{I'}/\partial X^I$ and $A^J_{J'} = \partial X^J/\partial X^{J'}$. Other tensor algebraic operations of two-point tensors are the same as one-point tensors. The base vectors at points **P** and **p** can be either independent or relevant. Thus, two-point tensors can be discussed herein.

2.4.2. Independent coordinates

Consider the coordinate systems of the two points **P** and **p** to be independent. The covariant differentiations of two points **P** and **p** are

$$d\mathbf{P} = dX^I \mathbf{G}_I \text{ and } d\mathbf{p} = dx^i \mathbf{g}_i. \tag{2.240}$$

Definition 2.26. The total covariant differentiation of a two-point tensor field $\overset{<\cdot>}{\psi}(\mathbf{P},\mathbf{p})$ for the independent coordinates of two points **P** and **p** is defined as

$$\begin{aligned} d\overset{<\cdot>}{\psi} &= d\mathbf{P}\cdot\overset{<}{\nabla}\overset{<\cdot>}{\psi} + d\mathbf{p}\cdot\overset{>}{\nabla}\overset{<\cdot>}{\psi} \\ &= \overset{<\cdot>}{\psi}\overset{<}{\nabla}\cdot d\mathbf{P} + \overset{<\cdot>}{\psi}\overset{>}{\nabla}\cdot d\mathbf{p}, \end{aligned} \tag{2.241}$$

with

$$\begin{aligned} \overset{<}{\nabla}\overset{<\cdot>}{\psi} &= \mathbf{G}^R \frac{\partial}{\partial X^R}\overset{<\cdot>}{\psi} = \psi^{I_1\cdots I_M}{}_{J_1\cdots J_N}{}^{i_1\cdots i_m}{}_{j_1\cdots j_n;R}\sqrt{G}^{-W}\sqrt{g}^{-w} \\ &\quad \cdot \mathbf{G}^R\mathbf{G}_{I_1}\cdots\mathbf{G}_{I_M}\mathbf{G}^{J_1}\cdots\mathbf{G}^{J_N}\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}, \\ \overset{>}{\nabla}\overset{<\cdot>}{\psi} &= \mathbf{g}^r \frac{\partial}{\partial x^r}\overset{<\cdot>}{\psi} = \psi^{I_1\cdots I_M}{}_{J_1\cdots J_N}{}^{i_1\cdots i_m}{}_{j_1\cdots j_n;r}\sqrt{G}^{-W}\sqrt{g}^{-w} \\ &\quad \cdot \mathbf{g}^r\mathbf{G}_{I_1}\cdots\mathbf{G}_{I_M}\mathbf{G}^{J_1}\cdots\mathbf{G}^{J_N}\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}, \\ \overset{<\cdot>}{\psi}\overset{<}{\nabla} &= \frac{\partial}{\partial X^R}\overset{<\cdot>}{\psi}\mathbf{G}^R = \psi^{I_1\cdots I_M}{}_{J_1\cdots J_N}{}^{i_1\cdots i_m}{}_{j_1\cdots j_n;R}\sqrt{G}^{-W}\sqrt{g}^{-w} \\ &\quad \cdot \mathbf{G}_{I_1}\cdots\mathbf{G}_{I_M}\mathbf{G}^{J_1}\cdots\mathbf{G}^{J_N}\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}\mathbf{G}^R, \\ \overset{<\cdot>}{\psi}\overset{>}{\nabla} &= \frac{\partial}{\partial x^r}\overset{<\cdot>}{\psi}\mathbf{g}^r = \psi^{I_1\cdots I_M}{}_{J_1\cdots J_N}{}^{i_1\cdots i_m}{}_{j_1\cdots j_n;r}\sqrt{G}^{-W}\sqrt{g}^{-w} \\ &\quad \cdot \mathbf{G}_{I_1}\cdots\mathbf{G}_{I_M}\mathbf{G}^{J_1}\cdots\mathbf{G}^{J_N}\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_m}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_n}\mathbf{g}^r, \end{aligned} \tag{2.242}$$

where

$$\begin{aligned}
&\psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q;R} \equiv \nabla_R \psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q} \\
&= \psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q,R} + \Gamma^{I_1}_{RS}\psi^{SI_2\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q} + \cdots \\
&\quad + \Gamma^{I_P}_{RS}\psi^{I_1\cdots I_{P-1}S}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q} - \Gamma^{S}_{RJ_1}\psi^{I_1\cdots I_P}{}_{SJ_2\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q} - \cdots \\
&\quad - \Gamma^{S}_{RJ_Q}\psi^{I_1\cdots I_P}{}_{J_1\cdots J_{Q-1}S}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q} - W\Gamma^{S}_{RS}\psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q}; \\
&\psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q;r} \equiv \nabla_r \psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q} \\
&= \psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q,r} + \Gamma^{i_1}_{rs}\psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{ri_2\cdots i_p}{}_{j_1\cdots j_q} + \cdots \\
&\quad + \Gamma^{i_p}_{rs}\psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_{p-1}r}{}_{j_1\cdots j_q} - \Gamma^{s}_{rj_1}\psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{sj_2\cdots j_q} - \cdots \\
&\quad - \Gamma^{s}_{rj_q}\psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_{q-1}s} - w\Gamma^{s}_{rs}\psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q}.
\end{aligned} \tag{2.243}$$

The above definition is given because of

$$\begin{aligned}
d\overset{<\cdot>}{\boldsymbol{\psi}} &= \overset{<\cdot>}{\boldsymbol{\psi}}_{,R}\, dX^R + \overset{<\cdot>}{\boldsymbol{\psi}}_{,r}\, dx^r \\
&= dX^S \mathbf{G}_S \cdot \mathbf{G}^R \Big(\frac{\partial}{\partial X^R}\overset{<\cdot>}{\boldsymbol{\psi}}\Big) + dx^s \mathbf{g}_s \cdot \mathbf{g}^r \Big(\frac{\partial}{\partial x^r}\overset{<\cdot>}{\boldsymbol{\psi}}\Big) \\
&= d\mathbf{P}\cdot\overset{<}{\nabla}\overset{<\cdot>}{\boldsymbol{\psi}} + d\mathbf{p}\cdot\overset{>}{\nabla}\overset{<\cdot>}{\boldsymbol{\psi}}; \\
d\overset{<\cdot>}{\boldsymbol{\psi}} &= \overset{<\cdot>}{\boldsymbol{\psi}}_{,R}\, dX^R + \overset{<\cdot>}{\boldsymbol{\psi}}_{,r}\, dx^r \\
&= \Big(\frac{\partial}{\partial X^R}\overset{<\cdot>}{\boldsymbol{\psi}}\Big)\mathbf{G}^R \cdot \mathbf{G}_S dX^S + \Big(\frac{\partial}{\partial x^r}\overset{<\cdot>}{\boldsymbol{\psi}}\Big)\mathbf{g}^r \cdot \mathbf{g}_s dx^s \\
&= \overset{<\cdot>}{\boldsymbol{\psi}}\overset{<}{\nabla}\cdot d\mathbf{P} + \overset{<\cdot>}{\boldsymbol{\psi}}\overset{>}{\nabla}\cdot d\mathbf{p}.
\end{aligned} \tag{2.244}$$

For a zero-order tensor (scalar) $\varphi(\mathbf{P},\mathbf{p})$,

$$d\varphi = d\mathbf{P}\cdot\overset{<}{\nabla}\varphi + d\mathbf{p}\cdot\overset{>}{\nabla}\varphi = \varphi\overset{<}{\nabla}\cdot d\mathbf{P} + \varphi\overset{>}{\nabla}\cdot d\mathbf{p}, \tag{2.245}$$

where

$$\begin{aligned}
&\overset{<}{\nabla}\varphi = \frac{\partial\varphi}{\partial X^I}\mathbf{G}^I = \varphi\overset{<}{\nabla} \quad \text{and} \quad \overset{>}{\nabla}\varphi = \frac{\partial\varphi}{\partial x^i}\mathbf{g}^i = \varphi\overset{>}{\nabla}, \\
&\varphi_{;I} = \varphi_{,I} = \frac{\partial\varphi}{\partial X^I},\ \ \varphi_{;i} = \varphi_{,i} = \frac{\partial\varphi}{\partial x^i}.
\end{aligned} \tag{2.246}$$

2.4.3. Correlated coordinates

Consider the coordinates of the two points $\mathbf{P}$ and $\mathbf{p}$ to be dependent, and the two coordinates have one-to-one relation as

$$\begin{aligned} &\mathbf{P} = \mathbf{P}(\mathbf{p}) \;\Rightarrow\; X^I = X^I(x^1, x^2, x^3) \;\text{ for } I = 1,2,3; \\ &\mathbf{p} = \mathbf{p}(\mathbf{P}) \;\Rightarrow\; x^i = x^i(X^1, X^2, X^3) \;\text{ for } i = 1,2,3. \end{aligned} \tag{2.247}$$

The covariant differentiations of two points **P** and **p** are

$$\begin{aligned} d\mathbf{P} &= \frac{\partial}{\partial x^i}\mathbf{P}dx^i = (dx^r\mathbf{g}_r)\cdot(\mathbf{g}^s\frac{\partial}{\partial x^s}\mathbf{P}) = d\mathbf{p}\cdot\overset{>}{\nabla}\mathbf{P} \\ &= (\frac{\partial}{\partial x^s}\mathbf{P})\,\mathbf{g}^s\cdot(\mathbf{g}_r dx^r) = \mathbf{P}\overset{>}{\nabla}\cdot d\mathbf{p}. \\ d\mathbf{p} &= \frac{\partial}{\partial X^I}\mathbf{p}dX^I = (dX^R\mathbf{G}_R)\cdot(\mathbf{G}^S\frac{\partial}{\partial X^S})\,\mathbf{p} = d\mathbf{P}\cdot\overset{<}{\nabla}\mathbf{p} \\ &= (\frac{\partial}{\partial X^S}\mathbf{p})\,\mathbf{G}^S\cdot(\mathbf{G}_R dX^R) = \mathbf{p}\overset{<}{\nabla}\cdot d\mathbf{P}. \end{aligned} \tag{2.248}$$

Furthermore,

$$\begin{aligned} \overset{>}{\nabla}\mathbf{P} &= \mathbf{g}^i\frac{\partial\mathbf{P}}{\partial x^i} = \mathbf{g}^i\frac{\partial X^I}{\partial x^i}\frac{\partial\mathbf{P}}{\partial X^I} = X^I_{,i}\mathbf{g}^i\mathbf{G}_I \equiv X^I_{;i}\mathbf{g}^i\mathbf{G}_I, \\ \mathbf{P}\overset{>}{\nabla} &= \frac{\partial\mathbf{P}}{\partial x^i}\mathbf{g}^i = \frac{\partial X^I}{\partial x^i}\frac{\partial\mathbf{P}}{\partial X^I}\mathbf{g}^i = X^I_{,i}\mathbf{G}_I\mathbf{g}^i \equiv X^I_{;i}\mathbf{G}_I\mathbf{g}^i; \\ \overset{<}{\nabla}\mathbf{p} &= \mathbf{G}^I\frac{\partial\mathbf{p}}{\partial X^I} = \mathbf{G}^I\frac{\partial x^i}{\partial X^I}\frac{\partial\mathbf{p}}{\partial x^i} = x^i_{,I}\mathbf{G}^I\mathbf{g}_i \equiv x^i_{;I}\mathbf{G}^I\mathbf{g}_i, \\ \mathbf{p}\overset{<}{\nabla} &= \frac{\partial\mathbf{p}}{\partial X^I}\mathbf{G}^I = \frac{\partial x^i}{\partial X^I}\frac{\partial\mathbf{p}}{\partial x^i}\mathbf{G}^I = x^i_{,I}\mathbf{g}_i\mathbf{G}^I \equiv x^i_{;I}\mathbf{g}_i\mathbf{G}^I. \end{aligned} \tag{2.249}$$

From the above expressions, $\overset{>}{\nabla}\mathbf{P}$, $\mathbf{P}\overset{>}{\nabla}$, $\overset{<}{\nabla}\mathbf{p}$ and $\mathbf{p}\overset{<}{\nabla}$ are two-point tensor fields because

$$\begin{aligned} X^{I'}_{;i'} &= \frac{\partial X^{I'}}{\partial X^I}\frac{\partial X^I}{\partial x^i}\frac{\partial x^i}{\partial x^{i'}} = A^{I'}_I A^i_{i'} X^I_{;i}, \\ x^{i'}_{;I'} &= \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^i}{\partial X^I}\frac{\partial X^I}{\partial X^{I'}} = A^{i'}_i A^I_{I'} x^i_{;I}. \end{aligned} \tag{2.250}$$

Definition 2.27. The *total covariant differentiations* of the two-point tensor fields $\overset{<\cdot>}{\boldsymbol{\psi}}\left(\mathbf{P},\mathbf{p}(\mathbf{P})\right)$ and $\overset{<\cdot>}{\boldsymbol{\psi}}\left(\mathbf{P}(\mathbf{p}),\mathbf{p}\right)$, respectively, are defined by

$$\begin{aligned} d\overset{<\cdot>}{\boldsymbol{\psi}}\left(\mathbf{P},\mathbf{p}(\mathbf{P})\right) &= \overset{<\cdot>}{\boldsymbol{\psi}}_{,I}\,dX^I + \overset{<\cdot>}{\boldsymbol{\psi}}_{,i}\,x^i_{;I}dX^I = d\mathbf{P}\cdot\overset{<}{\square}\overset{<\cdot>}{\boldsymbol{\psi}} = (\overset{<\cdot>}{\boldsymbol{\psi}}\overset{<}{\square})\cdot d\mathbf{P}; \\ d\overset{<\cdot>}{\boldsymbol{\psi}}\left(\mathbf{P}(\mathbf{p}),\mathbf{p}\right) &= \overset{<\cdot>}{\boldsymbol{\psi}}_{,I}\,X^I_{;i}dx^i + \overset{<\cdot>}{\boldsymbol{\psi}}_{,i}\,dx^i = d\mathbf{p}\cdot\overset{>}{\square}\overset{<\cdot>}{\boldsymbol{\psi}} = (\overset{<\cdot>}{\boldsymbol{\psi}}\overset{>}{\square})\cdot d\mathbf{p}, \end{aligned} \tag{2.251}$$

where the total covariant derivatives of the two-point tensors $\overset{<\cdot>}{\boldsymbol{\psi}}$ are

$$
\begin{aligned}
&\overset{<}{\square}\overset{<\cdot>}{\boldsymbol{\psi}} \equiv \overset{<}{\nabla}\overset{<\cdot>}{\boldsymbol{\psi}} + (\overset{<}{\nabla}\mathbf{p})\cdot(\overset{>}{\nabla}\overset{<\cdot>}{\boldsymbol{\psi}}) = \mathbf{G}^I\overset{<\cdot>}{\boldsymbol{\psi}}_{,I} + (\mathbf{G}^I x^i_{;I}\mathbf{g}_i)\cdot(\mathbf{g}^j\overset{<\cdot>}{\boldsymbol{\psi}}_{,j}),\\
&\overset{<\cdot>}{\boldsymbol{\psi}}\overset{<}{\square} \equiv \overset{<\cdot>}{\boldsymbol{\psi}}\overset{<}{\nabla} + (\overset{<\cdot>}{\boldsymbol{\psi}}\overset{>}{\nabla})\cdot(\mathbf{p}\overset{<}{\nabla}) = \overset{<\cdot>}{\boldsymbol{\psi}}_{,I}\mathbf{G}^I + (\overset{<\cdot>}{\boldsymbol{\psi}}_{,j}\mathbf{g}^j)\cdot(\mathbf{g}_i x^i_{;I}\mathbf{G}^I);\\
&\overset{>}{\square}\overset{<\cdot>}{\boldsymbol{\psi}} \equiv \overset{>}{\nabla}\overset{<\cdot>}{\boldsymbol{\psi}} + (\overset{>}{\nabla}\mathbf{P})\cdot(\overset{<}{\nabla}\overset{<\cdot>}{\boldsymbol{\psi}}) = \mathbf{g}^i\overset{<\cdot>}{\boldsymbol{\psi}}_{,i} + (\mathbf{g}^i X^I_{;i}\mathbf{G}_I)\cdot(\mathbf{G}^J\overset{<\cdot>}{\boldsymbol{\psi}}_{,J}),\\
&\overset{<\cdot>}{\boldsymbol{\psi}}\overset{<}{\square} \equiv \overset{<\cdot>}{\boldsymbol{\psi}}\overset{>}{\nabla} + (\overset{<\cdot>}{\boldsymbol{\psi}}\overset{<}{\nabla})\cdot(\mathbf{P}\overset{>}{\nabla}) = \overset{<\cdot>}{\boldsymbol{\psi}}_{,i}\mathbf{g}^i + (\overset{<\cdot>}{\boldsymbol{\psi}}_{,J}\mathbf{G}^J)\cdot(\mathbf{G}_I X^I_{;i}\mathbf{g}^i);
\end{aligned}
\tag{2.252}
$$

and the corresponding components are

$$
\begin{aligned}
&\square_I\psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q} = \psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q:I}\\
&= \psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q;I} + \psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q;i}x^i_{;I};\\
&\square_i\psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q} = \psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q:i}\\
&= \psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q;i} + \psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q;I}X^I_{;i}.
\end{aligned}
\tag{2.253}
$$

Because the following relations exist, i.e.,

$$
x^i_{;I}X^I_{;j} = \delta^i_j\ ,\ X^I_{;i}x^i_{;J} = \delta^I_J, \tag{2.254}
$$

the total derivatives of the components of the two-point tensors are

$$
\begin{aligned}
&\psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q:I}X^I_{;i}\\
&= \psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q;I}X^I_{;i} + \psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q;i}\\
&= \psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q:i};
\end{aligned}
\tag{2.255}
$$

$$
\begin{aligned}
&\psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q:i}x^i_{;I}\\
&= \psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q;i}x^i_{;I} + \psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q;I}\\
&= \psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}{}^{i_1\cdots i_p}{}_{j_1\cdots j_q:I}.
\end{aligned}
\tag{2.256}
$$

For a zero-order tensor (scalar) $\varphi(\mathbf{P},\mathbf{p}(\mathbf{P}))$ and $\varphi(\mathbf{P}(\mathbf{p}),\mathbf{p})$,

$$
d\varphi = d\mathbf{P}\cdot(\overset{<}{\nabla}\varphi + \overset{<}{\nabla}\mathbf{p}\cdot\overset{>}{\nabla}\varphi) = (\varphi\overset{<}{\nabla} + \varphi\overset{>}{\nabla}\cdot\overset{<}{\nabla}\mathbf{p})\cdot d\mathbf{P} \tag{2.257}
$$

where

$$
\begin{aligned}
&\overset{<}{\nabla}\varphi = \frac{\partial\varphi}{\partial X^I}\mathbf{G}^I = \varphi\overset{<}{\nabla}\quad\text{and}\quad \overset{>}{\nabla}\varphi = \frac{\partial\varphi}{\partial x^i}\mathbf{g}^i = \varphi\overset{>}{\nabla};\\
&\frac{d\varphi}{dX^I} \equiv \varphi_{:I} = \varphi_{;I} + x^r_{;I}\varphi_{;r} = \frac{\partial\varphi}{\partial X^I} + \frac{\partial\varphi}{\partial x^r}\frac{\partial x^r}{\partial X^I},\\
&\frac{d\varphi}{dx^i} \equiv \varphi_{:i} = \varphi_{;i} + x^r_{;i}\varphi_{;r} = \frac{\partial\varphi}{\partial x^i} + \frac{\partial\varphi}{\partial X^r}\frac{\partial X^r}{\partial x^i}.
\end{aligned}
\tag{2.258}
$$

For a two-point tensor of $\mathbf{B} = B^i{}_I \mathbf{g}_i \mathbf{G}^I$,

$$d\mathbf{B} = d\mathbf{P} \cdot (\overset{<}{\nabla}\mathbf{B} + \overset{<}{\nabla}\mathbf{p} \cdot \overset{>}{\nabla}\mathbf{B}) = (\mathbf{B}\overset{<}{\nabla} + \mathbf{B}\overset{>}{\nabla} \cdot \overset{<}{\nabla}\mathbf{p}) \cdot d\mathbf{P} \tag{2.259}$$

where

$$\begin{aligned} &\overset{<}{\nabla}\mathbf{B} = \frac{\partial \mathbf{B}}{\partial X^J}\mathbf{G}^J = B^i{}_{I;J}\mathbf{g}_i\mathbf{G}^I\mathbf{G}^J, \quad \mathbf{B}\overset{<}{\nabla} = \mathbf{G}^J\frac{\partial \mathbf{B}}{\partial X^J} = B^i{}_{I;J}\mathbf{G}^J\mathbf{g}_i\mathbf{G}^I, \\ &\overset{>}{\nabla}\mathbf{B} = \frac{\partial \mathbf{B}}{\partial x^j}\mathbf{g}^j = B^i{}_{I;j}\mathbf{g}_i\mathbf{G}^I\mathbf{g}^j, \quad \mathbf{B}\overset{>}{\nabla} = \mathbf{g}^j\frac{\partial \mathbf{B}}{\partial x^j} = B^i{}_{I;j}\mathbf{g}^j\mathbf{g}_i\mathbf{G}^I. \end{aligned} \tag{2.260}$$

$$\begin{aligned} &\frac{dB^i{}_I}{dX^J} \equiv B^i{}_{I:J} = B^i{}_{I;J} + x^r_{;J}B^i{}_{I;r}, \quad \frac{dB^i{}_I}{dx^j} \equiv B^i{}_{I:j} = B^i{}_{I;j} + X^R_{;j}B^i{}_{I;R}; \\ &B^i{}_{I;J} = \frac{\partial B^i{}_I}{\partial X^J} - \Gamma^L_{IJ}B^i{}_L, \quad B^i{}_{I;j} = \frac{\partial B^i{}_I}{\partial x^j} + \Gamma^i_{jl}B^l{}_I. \end{aligned} \tag{2.261}$$

The total covariant derivatives of components to the two-point tensors are

$$\begin{aligned} B^i{}_{I:J} &= \frac{\partial B^i{}_I}{\partial X^J} - \Gamma^L_{IJ}B^i{}_L + x^r_{;J}\left(\frac{\partial B^i{}_I}{\partial x^r} + \Gamma^i_{rl}B^l{}_I\right), \\ B^i{}_{I:j} &= \frac{\partial B^i{}_I}{\partial x^j} + \Gamma^L_{jl}B^l{}_I + X^R_{;j}\left(\frac{\partial B^i{}_I}{\partial X^R} - \Gamma^L_{IR}B^i{}_L\right). \end{aligned} \tag{2.262}$$

2.4.4. Shifter tensor fields

Definition 2.28. The *shifter tensor fields* are defined by

$$\begin{aligned} \overset{><}{\mathbf{I}} &= g_{iI}\mathbf{g}^i\mathbf{G}^I = g^i_I\mathbf{g}_i\mathbf{G}^I = g^I_i\mathbf{g}^i\mathbf{G}_I = g^{iI}\mathbf{g}_i\mathbf{G}_I, \\ \overset{<>}{\mathbf{I}} &= g_{Ii}\mathbf{G}^I\mathbf{g}^i = g^I_i\mathbf{G}_I\mathbf{g}^i = g^i_I\mathbf{g}_i\mathbf{G}^I = g^{iI}\mathbf{g}_i\mathbf{G}_I. \end{aligned} \tag{2.263}$$

To explain the shifter tensor, two vectors $\overset{<}{\mathbf{v}} = v^I\mathbf{G}_I$ and $\overset{>}{\mathbf{v}} = v^i\mathbf{g}_i$ in two coordinates at points P and p are sketched in Fig.2.3. Using the shifter tensor, the two vectors can be translated each other. For instance,

$$\overset{><}{\mathbf{I}} \cdot \overset{<}{\mathbf{v}} = \overset{>}{\mathbf{v}} \quad \text{and} \quad \overset{<>}{\mathbf{I}} \cdot \overset{>}{\mathbf{v}} = \overset{<}{\mathbf{v}}. \tag{2.264}$$

From the previous relation, the two vectors are of the same. They just are observed in the different reference frames.

$$\overset{><}{\mathbf{I}} \cdot \overset{<>}{\mathbf{I}} = \overset{>>}{\mathbf{I}} \quad \text{and} \quad \overset{<>}{\mathbf{I}} \cdot \overset{><}{\mathbf{I}} = \overset{<<}{\mathbf{I}}, \tag{2.265}$$

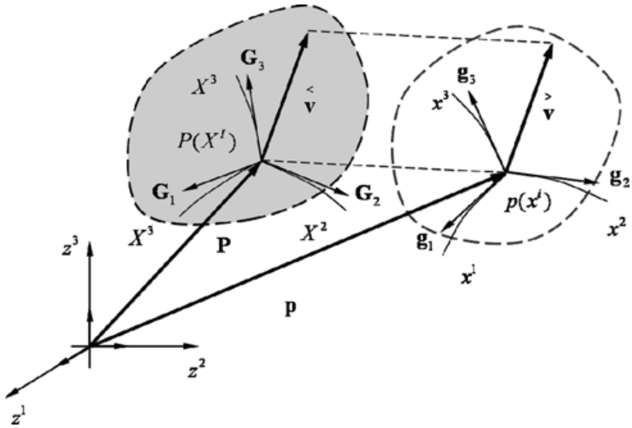


Fig. 2.3 Two parallel vectors in two coordinates.

$$g^i{}_I g^I{}_j = \delta^i_j \quad \text{and} \quad g^I{}_i g^i{}_J = \delta^I_J, \tag{2.266}$$

where $\overset{\ll}{\mathbf{I}}$ and $\overset{\gg}{\mathbf{I}}$ are the two second-order unit tensors in two domains $\mathfrak{B}$ and $\mathfrak{b}$, respectively. The components of the shifter tensor is given by

$$\begin{aligned}
\overset{><}{\mathbf{I}} \cdot \mathbf{G}_I &= g_{iI} \mathbf{g}^i \mathbf{G}^I \cdot \mathbf{G}_I = g_{iI} \mathbf{g}^i = g^i{}_I \mathbf{g}_i, \\
\overset{><}{\mathbf{I}} \cdot \mathbf{G}^I &= g_i{}^I \mathbf{g}^i \mathbf{G}_I \cdot \mathbf{G}^I = g_i{}^I \mathbf{g}^i = g^{iI} \mathbf{g}_i; \\
\overset{<>}{\mathbf{I}} \cdot \mathbf{g}_i &= g_{Ii} \mathbf{G}^I \mathbf{g}^i \cdot \mathbf{g}_i = g_{iI} \mathbf{G}^I = g^I{}_i \mathbf{G}_I, \\
\overset{<>}{\mathbf{I}} \cdot \mathbf{g}^i &= g_I{}^i \mathbf{G}^I \mathbf{g}^i \cdot \mathbf{g}_i = g_I{}^i \mathbf{G}^I = g^{Ii} \mathbf{G}_I.
\end{aligned} \tag{2.267}$$

The foregoing equations can be rewritten as

$$\begin{aligned}
\overset{><}{\mathbf{I}} \cdot \mathbf{G}_I &= (\mathbf{g}_i \cdot \mathbf{G}_I)\mathbf{g}^i = (\mathbf{g}^i \cdot \mathbf{G}_I)\mathbf{g}_i, \\
\overset{><}{\mathbf{I}} \cdot \mathbf{G}^I &= (\mathbf{g}_i \cdot \mathbf{G}^I)\mathbf{g}^i = (\mathbf{g}^i \cdot \mathbf{G}^I)\mathbf{g}_i; \\
\overset{<>}{\mathbf{I}} \cdot \mathbf{g}_i &= (\mathbf{G}_I \cdot \mathbf{g}_i)\mathbf{G}^I = (\mathbf{G}^I \cdot \mathbf{g}_i)\mathbf{G}_I, \\
\overset{<>}{\mathbf{I}} \cdot \mathbf{g}^i &= (\mathbf{G}_I \cdot \mathbf{g}^i)\mathbf{G}^I = (\mathbf{G}^I \cdot \mathbf{g}^i)\mathbf{G}_I.
\end{aligned} \tag{2.268}$$

In addition,

$$\begin{aligned}
g_{iI} &= \mathbf{g}_i \cdot \mathbf{G}_I = \mathbf{G}_I \cdot \mathbf{g}_i = g_{Ii}, \\
g^{Ii} &= \mathbf{G}^I \cdot \mathbf{g}^i = \mathbf{g}^i \cdot \mathbf{G}^I = g^{iI}; \\
g_i{}^I &= \mathbf{g}_i \cdot \mathbf{G}^I = \mathbf{G}^I \cdot \mathbf{g}_i = g^I{}_i \equiv g_i^I,
\end{aligned} \tag{2.269a}$$

$$g^{i}{}_{I} = \mathbf{g}^{i} \cdot \mathbf{G}_{I} = \mathbf{G}_{I} \cdot \mathbf{g}^{i} = g_{I}{}^{i} = g_{I}^{i}. \tag{2.269b}$$

Theorem 2.13. *The following relation holds*:

$$\sqrt{G} = |\, g_{I}^{i} \,| \sqrt{g} \;\; \textit{and} \;\; \sqrt{\mathrm{g}} = |\, g_{i}^{I} \,| \sqrt{G}. \tag{2.270}$$

Proof: From the definition of $\sqrt{G}$,

$$\begin{aligned} \sqrt{G} &= [\mathbf{G}_1\mathbf{G}_2\mathbf{G}_3] = [g_1^i \mathbf{g}_i g_2^j \mathbf{g}_j g_3^k \mathbf{g}_k] = g_1^i g_2^j g_3^k [\mathbf{g}_i \mathbf{g}_j \mathbf{g}_k] \\ &= |\, g_I^i \,| [\mathbf{g}_1\mathbf{g}_2\mathbf{g}_3] = |\, g_I^i \,| \sqrt{g}. \end{aligned}$$

Thus,

$$\sqrt{G} = |\, g_I^i \,| \sqrt{g} \;\; \text{or} \;\; \sqrt{g} = |\, g_i^I \,| \sqrt{G}.$$

This theorem is proved. ■

Theorem 2.14. *For two vectors* $\overset{<}{\mathbf{v}} = v^I \mathbf{G}_I$ *and* $\overset{>}{\mathbf{v}} = v^i \mathbf{g}_i$, *the following relations hold*:

$$\begin{aligned} v^I &= g_i^I v^i \;\; \text{and} \;\; v_I = g_I^i v_i; \\ v^i &= g_I^i v^I \;\; \text{and} \;\; v_i = g_i^I v_I. \end{aligned} \tag{2.271}$$

Proof: The vector translation can be given by

$$\begin{aligned} \overset{>}{\mathbf{v}} &= v^i \mathbf{g}_i = \overset{><}{\mathbf{I}} \cdot \overset{<}{\mathbf{v}} = (g_I^i \mathbf{g}_i \mathbf{G}^I) \cdot (v^R \mathbf{G}_R) = g_R^i v^R \mathbf{g}_i, \\ \overset{<}{\mathbf{v}} &= v^I \mathbf{G}_I = \overset{<>}{\mathbf{I}} \cdot \overset{>}{\mathbf{v}} = (g_i^I \mathbf{G}_I \mathbf{g}^i) \cdot (v^r \mathbf{g}_r) = g_r^I v^r \mathbf{G}_I. \end{aligned}$$

So Eq.(2.271) is obtained. ■

Similarly, the tensor can be translated partially and completely translated.

Theorem 2.15. *Consider a tensor* $\overset{<<}{\mathbf{B}}$, $\overset{><}{\mathbf{B}}$, $\overset{<>}{\mathbf{B}}$ *and* $\overset{>>}{\mathbf{B}}$, *the following relations exist*:

$$\begin{aligned} B_{iI} &= g_i^J B_{JI}, \;\; B_{Ii} = g_i^J B_{IJ}, \;\; B_{ij} = g_i^I g_j^J B_{IJ}; \\ B_{Ii} &= g_I^r B_{ri}, \;\; B_{iI} = g_I^r B_{ir}, \;\; B_{IJ} = g_I^r g_J^s B_{rs}. \end{aligned} \tag{2.272}$$

Proof: Because

$$\begin{aligned} \overset{><}{\mathbf{B}} &= B_{iI} \mathbf{g}^i \mathbf{G}^I = \overset{><}{\mathbf{I}} \cdot \overset{<<}{\mathbf{B}} = (g_i^R \mathbf{g}^i \mathbf{G}_R) \cdot (B_{IJ} \mathbf{G}^I \mathbf{G}^J) = g_i^R B_{RI} \mathbf{g}^i \mathbf{G}^I, \\ \overset{<>}{\mathbf{B}} &= B_{Ii} \mathbf{G}^I \mathbf{g}^i = \overset{<<}{\mathbf{B}} \cdot \overset{<>}{\mathbf{I}} = (B_{IJ} \mathbf{G}^I \mathbf{G}^J) \cdot (g_i^R \mathbf{G}_R \mathbf{g}^i) = B_{IR} g_i^R \mathbf{G}^I \mathbf{g}^i, \end{aligned}$$

$$\overset{>>}{\mathbf{B}} = B_{ij}\mathbf{g}^i\mathbf{g}^j = \overset{><}{\mathbf{I}}\cdot\overset{<<}{\mathbf{B}}\cdot\overset{<>}{\mathbf{I}} = (g_i^R\mathbf{g}^i\mathbf{G}_R)\cdot(B_{IJ}\mathbf{G}^I\mathbf{G}^J)\cdot(g_j^S\mathbf{G}_S\mathbf{g}^j)$$
$$= g_i^I B_{IJ} g_j^J \mathbf{g}^i\mathbf{g}^j,$$

one obtains

$$\overset{><}{\mathbf{B}} = \overset{><}{\mathbf{I}}\cdot\overset{<<}{\mathbf{B}} \quad\Rightarrow\quad B_{iI} = g_i^R B_{RI};$$
$$\overset{<>}{\mathbf{B}} = \overset{<<}{\mathbf{B}}\cdot\overset{<>}{\mathbf{I}} \quad\Rightarrow\quad B_{Ii} = g_i^R B_{IR};$$
$$\overset{>>}{\mathbf{B}} = \overset{><}{\mathbf{I}}\cdot\overset{<<}{\mathbf{B}}\cdot\overset{<>}{\mathbf{I}} \quad\Rightarrow\quad B_{ij} = g_i^I g_j^J B_{IJ}.$$

Similarly,

$$\overset{<>}{\mathbf{B}} = \overset{<>}{\mathbf{I}}\cdot\overset{>>}{\mathbf{B}} \quad\Rightarrow\quad B_{Ii} = g_R^r B_{ri};$$
$$\overset{><}{\mathbf{B}} = \overset{>>}{\mathbf{B}}\cdot\overset{><}{\mathbf{I}} \quad\Rightarrow\quad B_{iI} = g_I^r B_{ir};$$
$$\overset{<<}{\mathbf{B}} = \overset{<>}{\mathbf{I}}\cdot\overset{>>}{\mathbf{B}}\cdot\overset{><}{\mathbf{I}} \quad\Rightarrow\quad B_{IJ} = g_I^i g_J^j B_{ij}.$$

This theorem is proved. ■

Such an idea can be extended to higher-order tensors. For instance, the following theorem can be stated as an example.

Theorem 2.16. *For the following two-point tensors,*

$$\overset{<\cdot<}{\boldsymbol{\psi}} = \psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}\sqrt{G}^{-W}\mathbf{G}_{I_1}\cdots\mathbf{G}_{I_P}\mathbf{G}^{J_1}\cdots\mathbf{G}^{J_Q},$$
$$\overset{>\cdot>}{\boldsymbol{\psi}} = \psi^{i_1\cdots i_P}{}_{j_1\cdots j_Q}\sqrt{g}^{-W}\mathbf{g}_{i_1}\cdots\mathbf{g}_{i_P}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_Q}, \tag{2.273}$$

the following relations hold

$$\overset{>\cdot>}{\boldsymbol{\psi}} = (|\,g_M^m\,|\sqrt{\frac{G}{g}})^W \underbrace{\overset{><}{\mathbf{I}}\cdots\overset{><}{\mathbf{I}}}_{P}\underbrace{\overset{><}{\mathbf{I}}\cdots\overset{><}{\mathbf{I}}}_{Q}\vdots\,\overset{<\cdot<}{\boldsymbol{\psi}},$$
$$\overset{<\cdot<}{\boldsymbol{\psi}} = (|\,g_M^m\,|\sqrt{\frac{g}{G}})^W \underbrace{\overset{<>}{\mathbf{I}}\cdots\overset{<>}{\mathbf{I}}}_{P}\underbrace{\overset{<>}{\mathbf{I}}\cdots\overset{<>}{\mathbf{I}}}_{Q}\vdots\,\overset{>\cdot>}{\boldsymbol{\psi}}. \tag{2.274}$$

$$\psi^{i_1\cdots i_P}{}_{j_1\cdots j_Q} = |\,g_M^m\,|^W\; g_{I_1}^{i_1}\cdots g_{I_P}^{i_P} g_{j_1}^{J_1}\cdots g_{j_Q}^{J_Q}\psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q},$$
$$\psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q} = |\,g_M^m\,|^W\; g_{i_1}^{I_1}\cdots g_{i_P}^{I_P} g_{J_1}^{j_1}\cdots g_{J_Q}^{j_Q}\psi^{i_1\cdots i_P}{}_{j_1\cdots j_Q}. \tag{2.275}$$

Proof: Consider

$$\begin{aligned}\overset{>\cdot>}{\boldsymbol{\psi}} &= \psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}[\overset{><}{\mathbf{I}}\cdot\mathbf{G}_1\ \overset{><}{\mathbf{I}}\cdot\mathbf{G}_2\ \overset{><}{\mathbf{I}}\cdot\mathbf{G}_3]^{-W}\ \overset{><}{\mathbf{I}}\cdot\mathbf{G}_{I_1}\cdots\overset{><}{\mathbf{I}}\cdot\mathbf{G}_{I_P}\ \overset{><}{\mathbf{I}}\cdot\mathbf{G}^{J_1}\cdots\overset{><}{\mathbf{I}}\cdot\mathbf{G}^{J_Q}\\ &= |\,g^m_M\,|^w\ g^{i_1}_{I_1}\cdots g^{i_P}_{I_P}g^{J_1}_{j_1}\cdots g^{J_Q}_{j_Q}\psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}\sqrt{g}^{-W}\ \mathbf{g}_{i_1}\cdots\mathbf{g}_{i_P}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_Q}\\ &= \psi^{i_1\cdots i_P}{}_{j_1\cdots j_Q}\sqrt{g}^{-W}\ \mathbf{g}_{i_1}\cdots\mathbf{g}_{i_P}\mathbf{g}^{j_1}\cdots\mathbf{g}^{j_Q}.\end{aligned}$$

So,

$$\psi^{i_1\cdots i_P}{}_{j_1\cdots j_Q} = |\,g^m_M\,|^w\ g^{i_1}_{I_1}\cdots g^{i_P}_{I_P}g^{J_1}_{j_1}\cdots g^{J_Q}_{j_Q}\psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q}.$$

Similarly,

$$\psi^{I_1\cdots I_P}{}_{J_1\cdots J_Q} = |\,g^m_M\,|^W\ g^{I_1}_{i_1}\cdots g^{I_P}_{i_P}g^{j_1}_{J_1}\cdots g^{j_Q}_{J_Q}\psi^{i_1\cdots i_P}{}_{j_1\cdots j_Q}.$$

This theorem is proved. ■

Theorem 2.17. *The following relations exist for the shifter tensor*

$$\begin{aligned}&\overset{<>\,>}{\mathbf{I}\,\nabla} = \overset{<>\,<}{\mathbf{I}\,\nabla} = \overset{><\,>}{\mathbf{I}\,\nabla} = \overset{><\,<}{\mathbf{I}\,\nabla} = 0,\\ &g^i_{I;j} = g^i_{I;J} = g^I_{i;j} = g^I_{i;J} = 0;\end{aligned}\tag{2.276}$$

and

$$\begin{aligned}&\overset{<>\,>}{\mathbf{I}\,\square} = \overset{<>\,<}{\mathbf{I}\,\square} = \overset{><\,>}{\mathbf{I}\,\square} = \overset{><\,<}{\mathbf{I}\,\square} = 0,\\ &g^i_{I:j} = g^i_{I:J} = g^I_{i:j} = g^I_{i:J} = 0.\end{aligned}\tag{2.277}$$

Proof: From the derivative of the tensor,

$$\begin{aligned}g^i_{I;j} &= g^i_{I,j} + \Gamma^i_{jr}g^r_I = (\mathbf{g}^i\cdot\mathbf{G}_I)_{,j} + \Gamma^i_{jr}g^r_I\\ &= \mathbf{g}^i{}_{,j}\cdot\mathbf{G}_I + \Gamma^i_{jr}g^r_I = -\Gamma^i_{jr}\mathbf{g}^r\cdot\mathbf{G}_I + \Gamma^i_{jr}g^r_I = 0,\\ g^i_{I;J} &= g^i_{I,J} - \Gamma^R_{JI}g^i_R = (\mathbf{g}^i\cdot\mathbf{G}_I)_{,J} - \Gamma^R_{JI}g^i_R\\ &= \mathbf{g}^i\cdot\mathbf{G}_{I,J} - \Gamma^R_{JI}g^i_R = \Gamma^R_{JI}\mathbf{g}^i\cdot\mathbf{G}_R - \Gamma^R_{JI}g^i_R = 0.\end{aligned}$$

Similarly, $g^I_{i;j} = 0$ and $g^I_{i;J} = 0$. So

$$\overset{<>\,>}{\mathbf{I}\,\nabla} = g^i_{I;j} = 0,\quad \overset{<>\,<}{\mathbf{I}\,\nabla} = g^i_{I;J} = 0,$$

$$\overset{><\,>}{\mathbf{I}\,\nabla} = g^I_{i;j} = 0,\quad \overset{><\,<}{\mathbf{I}\,\nabla} = g^I_{i;J} = 0.$$

$$\overset{<>\,>}{\mathbf{I}\,\square} = g^i_{I:j} = g^i_{I;j} + g^i_{I;R}X^R_{;j} = 0,$$

$$\overset{<>\,<}{\mathbf{I}\,\square} = g^i_{I:J} = g^i_{I;J} + g^i_{I;r}x^r_{;J} = 0,$$

$$\overset{><>}{\mathbf{I}}\square = g^{I}_{i:j} = g^{I}_{i;j} + g^{I}_{i;R} X^{R}_{;j} = 0,$$

$$\overset{><<}{\mathbf{I}}\square = g^{I}_{i:J} = g^{I}_{i;J} + g^{I}_{i;r} x^{r}_{;J} = 0.$$

This theorem is proved. ■

If one is interested in more materials on tensor analysis in space $\mathbb{R}^n$, the further reading materials can be found in Schouten (1951), Eringen (1962, 1971) and Marsden and Hudges (1983).

References

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Chapter 3
Deformation, Kinematics and Dynamics

This chapter will systematically discuss the differential geometry, kinematics and dynamics of deformation in continuous media. To discuss deformation geometry, the deformation gradients will be introduced in the local curvilinear coordinate system, and the Green and Cauchy strain tensors will be presented. The length and angle changes will be discussed through Green and Cauchy strain tensors. The velocity gradient will be introduced for discussion of the kinematics, and the material derivatives of deformation gradient, infinitesimal line element, area and volume in the deformed configuration will be presented. The Cauchy stress and couple stress tensors will be defined to discuss the dynamics of continuous media, and the local balances for the Cauchy momentum and angular momentum will be discussed. Piola-Kirchhoff stress tensors will be presented and the Boussinesq and Kirchhoff local balance of momentum will be discussed. The local principles of the energy conservation will be discussed by the virtual work principle. This chapter will present an important foundation of continuum mechanics. From such a foundation, one can further understand other approximate existing theories in deformable body and fluids.

3.1. Deformation geometry

In this section, the deformation gradients on the curvilinear coordinates will be defined. Based on the deformation gradient, the Green and Cauchy strain tensors will be discussed. The stretches and angle changes of two line elements before and after deformation will be presented through Green and Cauchy strain tensors. The principal values and directions of the strain tensors will be discussed. The fundamental deformation theorem in continuous media will be presented.

3.1.1. Curvilinear coordinates

Consider a domain $\mathfrak{B}$ occupied by all material points of a continuous medium at time t_0, which possesses a material volume V and its surface S. Any material point P in such a domain is described by a curvilinear coordinate system X^I ($I = 1, 2, 3$). A vector from the origin of the coordinate system to point P is expressed by $\mathbf{P}$. Such a material domain is called *an initial configuration of the continuous body*. For time t, after the material domain is deformed, the material domain moves a new domain $\mathfrak{b}$ with a material volume v and its surface s. In the deformed domain, the material point p is described by a curvilinear coordinate system x^i ($i = 1, 2, 3$), and the position vector from the origin of the new coordinate system is given by $\mathbf{p}$. The deformed domain of the continuous material body is called *a deformed configuration*. Such a motion of material domain is sketched in Fig.3.1. The curvilinear coordinates $\{X^I\}$ and $\{x^i\}$ in the initial and deformed configurations are depicted, and the corresponding base vectors $\{\mathbf{G}_I\}$ and $\{\mathbf{g}_i\}$ are presented. The displacement between two points P and p is expressed by a vector $\mathbf{u}$, and such a vector is called a *displacement vector*. This motion of the deformed continuous body is described by

$$\mathbf{p} = \mathbf{P} + \mathbf{u}. \tag{3.1}$$

From a time t, a position vector $\mathbf{p}$ of the material point in an instantaneous configuration $\mathfrak{b}$ can be expressed by the position vector $\mathbf{P}$ of material points in the initial configuration $\mathfrak{B}$, i.e.,

$$\mathbf{p} = \mathbf{p}(\mathbf{P}, t) \quad \text{or} \quad x^i = x^i(X^I, t). \tag{3.2}$$

If the motion is single-valued, continuous and differentiable, in the neighborhood of point $\mathbf{P}$, there is a one-to-one inverse transformation which is single-valued, continuous and differentiable, i.e.,

$$\mathbf{P} = \mathbf{P}(\mathbf{p}, t) \quad \text{or} \quad X^I = X^I(x^i, t). \tag{3.3}$$

In other words, any particle described by the coordinate system $\{X^I\}$ for time t can be described through the coordinate system $\{x^i\}$. For Eq.(3.2), its inverse in Eq.(3.3) exists if and only if the Jacobian matrix is not singular, vice versa. That is,

$$|x^i_{;I}| \neq 0 \quad \text{and} \quad |X^I_{;i}| \neq 0. \tag{3.4}$$

The above continuity condition expresses the *indestructibility* of continuous media. From Eq.(3.4), it implies that the positive and finite volume of a material configuration cannot become zero and infinite. The matter in the continuous body cannot be impenetrable.

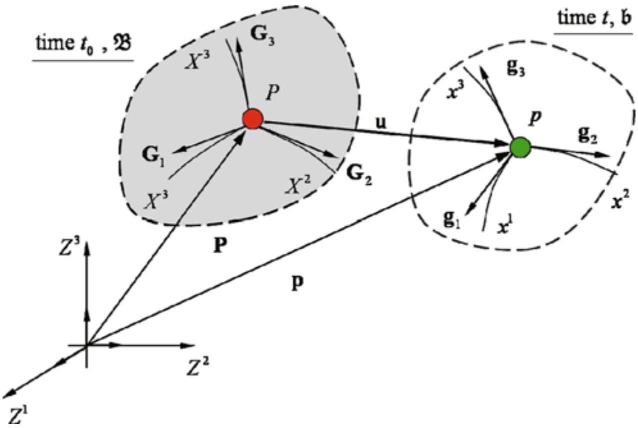

Fig. 3.1 Two configurations in two local coordinates.

Definition 3.1. The coordinates X^I ($I = 1, 2, 3$) for time t_0 are called the *Lagrange coordinates* (or *material coordinates*) if the coordinate X^I does not vary with time *t*. The corresponding reference frame { X^I } is called the *Lagrangian* (*material*) *coordinate system.*

Definition 3.2. The coordinates x^i ($i = 1, 2, 3$) for time *t* are called the *Eulerian coordinates* (or *spatial coordinates*) if the coordinate x^i varies with time *t*. The corresponding frame is called the Eulerian (spatial) coordinate system.

To further explain the movement of a continuous body, two independent, curvilinear systems are sketched in Fig.3.2. In the initial configuration $\mathfrak{B}$, the curvilinear coordinate system is { X^I }, and the curvilinear system after movement and deformed is given by $\{x^i, t_0\}$. For any time t, the continuous body are deformed and moved with displacement $\mathbf{u}$ to the instantaneous configuration $\mathfrak{b}$. In this configuration, the new curvilinear coordinate system is { x^i }. However, the coordinate system in the initial configuration becomes $\{X^I, t\}$. Such a relation can be described through Eqs.(3.2) and (3.3). Further, the position vectors $\mathbf{P}$ and $\mathbf{p}$ with the displacement vector $\mathbf{u}$ can be expressed through two coordinates. Based on the coordinates in $\mathfrak{B}$, three vectors $\mathbf{P}$, $\mathbf{p}$ and $\mathbf{u}$ are denoted by $\overset{<}{\mathbf{P}}$, $\overset{<}{\mathbf{p}}$ and $\overset{<}{\mathbf{u}}$, and from the coordinates in $\mathfrak{b}$, three vectors $\mathbf{P}$, $\mathbf{p}$ and $\mathbf{u}$ are denoted by $\overset{>}{\mathbf{P}}$, $\overset{>}{\mathbf{p}}$ and $\overset{>}{\mathbf{u}}$. Notice that $\mathbf{P} \equiv \overset{<}{\mathbf{P}}$ and $\mathbf{p} \equiv \overset{>}{\mathbf{p}}$. In other words, we have $\overset{<}{\mathbf{p}} \equiv \mathbf{p}(\mathbf{P})$, $\overset{<}{\mathbf{u}} = \mathbf{u}(\mathbf{P})$ and $\overset{>}{\mathbf{P}} \equiv \mathbf{P}(\mathbf{p})$, $\overset{>}{\mathbf{u}} = \mathbf{u}(\mathbf{p})$.

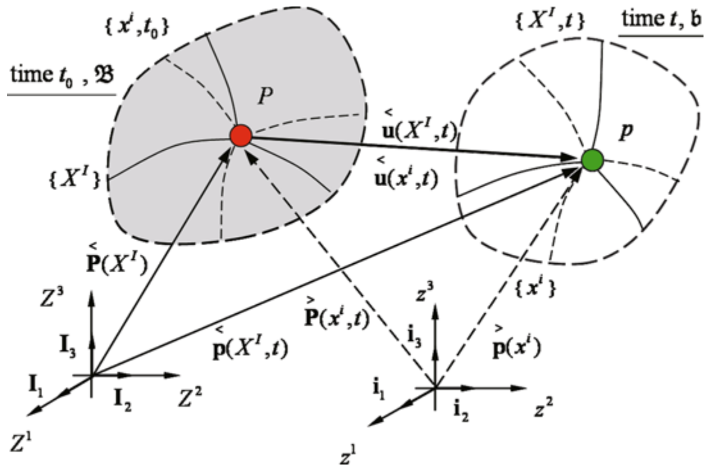

Fig. 3.2 Two local curvilinear coordinates and movements.

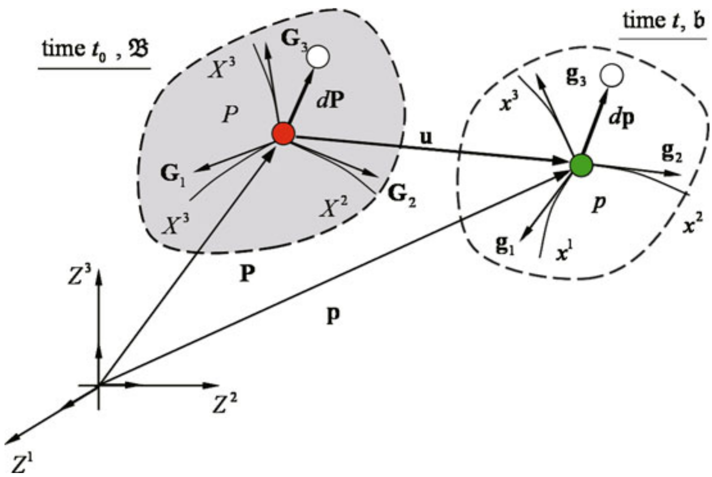

Fig. 3.3 Infinitesimal position vectors $d\mathbf{P}$ and $d\mathbf{p}$ in two configurations.

As in Chapter 2, the infinitesimal vectors $d\mathbf{P}$ in 𝔅 and $d\mathbf{p}$ in 𝔟 can be defined as in Fig.3.3. For simplicity, the position vector can be through the Cartesian coordinates Z^I ($I = 1, 2, 3$) and z^i ($i = 1, 2, 3$). The definition is given as follows.

Definition 3.3. For two positions $\mathbf{P}$ in 𝔅 and $\mathbf{p}$ in 𝔟,

$$\mathbf{P} = Z^I(X^1, X^2, X^3)\mathbf{I}_I \quad \text{and} \quad \mathbf{p} = z^i(x^1, x^2, x^3)\mathbf{i}_i, \tag{3.5}$$

the infinitesimal vectors $d\mathbf{P}$ and $d\mathbf{p}$ are defined as

$$d\mathbf{P} = \frac{\partial \mathbf{P}}{\partial X^I} dX^I = \mathbf{G}_I dX^I \text{ and } d\mathbf{p} = \frac{\partial \mathbf{p}}{\partial x^i} dx^i = \mathbf{g}_i dx^i, \tag{3.6}$$

where *the base vectors* for the initial and deformed configurations are

$$\mathbf{G}_I = \frac{\partial \mathbf{P}}{\partial X^I} = \frac{\partial Z^J}{\partial X^I} \mathbf{I}_J \text{ and } \mathbf{g}_i = \frac{\partial \mathbf{p}}{\partial x^i} = \frac{\partial z^j}{\partial x^i} \mathbf{i}_j; \tag{3.7}$$

and the squares of the lengths of the two infinitesimal vectors are defined as

$$dS^2 = d\mathbf{P} \cdot d\mathbf{P} = G_{IJ} dX^I \mathrm{d}X^J \text{ and } ds^2 = d\mathbf{p} \cdot d\mathbf{p} = g_{ij} dx^i dx^j, \tag{3.8}$$

where *the metric tensors* for the initial and deformed configurations are

$$G_{IJ} = \mathbf{G}_I \cdot \mathbf{G}_J = \delta_{MN} \frac{\partial Z^M}{\partial X^I} \frac{\partial Z^N}{\partial X^J} \text{ and } g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = \delta_{mn} \frac{\partial z^m}{\partial x^i} \frac{\partial z^n}{\partial x^j}. \tag{3.9}$$

Definition 3.4. For two positions $\overset{>}{\mathbf{P}} \equiv \mathbf{P}(\mathbf{p})$ in $\mathfrak{B}$ and $\overset{<}{\mathbf{p}} \equiv \mathbf{p}(\mathbf{P})$ in $\mathfrak{b}$,

$$\begin{aligned} \mathbf{P} &= Z^I(X^1, X^2, X^3)\mathbf{I}_I \text{ and } \mathbf{p} = z^i(x^1, x^2, x^3)\mathbf{i}_i, \\ X^K &= X^K(x^1, x^2, x^3) \text{ and } x^k = x^k(X^1, X^2, X^3). \end{aligned} \tag{3.10}$$

The infinitesimal vectors $d\mathbf{P}$ and $d\mathbf{p}$ are defined as

$$d\mathbf{P} = \frac{\partial \mathbf{P}}{\partial x^i} dx^i = \mathbf{c}_i dx^i \text{ and } d\mathbf{p} = \frac{\partial \mathbf{p}}{\partial X^I} dX^I = \mathbf{C}_I dX^I, \tag{3.11}$$

where *the covariant base vectors* are

$$\begin{aligned} \mathbf{c}_i &= \frac{\partial \mathbf{P}}{\partial x^i} = \frac{\partial Z^J}{\partial X^M} \frac{\partial X^M}{\partial x^i} \mathbf{I}_J = \mathbf{G}_M \frac{\partial X^M}{\partial x^i} = \mathbf{G}_M X^M_{;i} \text{ and} \\ \mathbf{C}_I &= \frac{\partial \mathbf{p}}{\partial X^I} = \frac{\partial z^j}{\partial x^m} \frac{\partial x^m}{\partial X^I} \mathbf{i}_j = \mathbf{g}_m \frac{\partial x^m}{\partial X^I} = \mathbf{g}_m x^m_{;I}. \end{aligned} \tag{3.12}$$

The squares of the lengths are defined as

$$dS^2 = d\mathbf{P} \cdot d\mathbf{P} = c_{ij} dx^i dx^j \text{ and } ds^2 = d\mathbf{p} \cdot d\mathbf{p} = C_{IJ} dX^I dX^J, \tag{3.13}$$

where *the covariant tensors* are

$$\begin{aligned} c_{ij} &= \mathbf{c}_i \cdot \mathbf{c}_j = G_{MN} \frac{\partial X^M}{\partial x^i} \frac{\partial X^N}{\partial x^j} = G_{MN} X^M_{;i} X^N_{;j}; \\ C_{IJ} &= \mathbf{C}_I \cdot \mathbf{C}_J = g_{mn} \frac{\partial x^m}{\partial X^I} \frac{\partial x^n}{\partial X^J} = g_{mn} x^m_{;I} x^n_{;J}, \end{aligned} \tag{3.14}$$

which are also called *Cauchy's deformation tensor* and *Green's deformation tensor,* respectively.

From the definitions of Cauchy's and Green's deformation tensors, both of the two tensors are *symmetric* (i.e., $c_{ij} = c_{ji}$ and $C_{IJ} = C_{JI}$) and both are positive-definite.

Definition 3.5. For two position vectors **P** in $\mathfrak{B}$ and **p** in $\mathfrak{b}$, *the contravariant base vectors* are defined as

$$\mathbf{c}^i = \mathbf{G}^M \frac{\partial x^i}{\partial X^M} = \mathbf{G}^M x^i_{;M} \text{ and } \mathbf{C}^I = \mathbf{g}^m \frac{\partial X^I}{\partial x^m} = \mathbf{g}^m X^I_{;m}, \tag{3.15}$$

where the contravariant base vectors with covariant base vectors satisfy

$$\mathbf{c}_i \cdot \mathbf{c}^j = \delta_i^j \text{ and } \mathbf{C}_I \cdot \mathbf{C}^J = \delta_I^J. \tag{3.16}$$

Definition 3.6. For two position vectors **P** in $\mathfrak{B}$ and **p** in $\mathfrak{b}$, the *contravariant metric tensors* are defined as

$$\begin{aligned} {}^{-1}c^{ij} &= \mathbf{c}^i \cdot \mathbf{c}^j = \mathbf{G}^M \cdot \mathbf{G}^N \frac{\partial x^i}{\partial X^M}\frac{\partial x^j}{\partial X^N} = G^{MN} x^i_{;M} x^j_{;N}, \\ {}^{-1}C^{IJ} &= \mathbf{C}^I \cdot \mathbf{C}^J = \mathbf{g}^m \cdot \mathbf{g}^n \frac{\partial X^I}{\partial x^m}\frac{\partial X^J}{\partial x^n} = g^{mn} X^I_{;m} X^J_{;n}. \end{aligned} \tag{3.17}$$

Remarks:

$$C^{IJ} = C_{MN} G^{MI} G^{NJ} \neq {}^{-1}C^{IJ} \text{ and } c^{ij} = c_{mn} g^{mi} g^{nj} \neq {}^{-1}c^{ij}. \tag{3.18}$$

Definition 3.7. For two position vectors **P** in $\mathfrak{B}$ and **p** in $\mathfrak{b}$, the *displacement vectors* are defined as

$$\overset{<}{\mathbf{u}} = U^I \mathbf{G}_I = \overset{<>}{\mathbf{I}} \cdot \overset{>}{\mathbf{u}} = g_i^I u^i \mathbf{G}_I \text{ and } \overset{>}{\mathbf{u}} = u^i \mathbf{g}_i = \overset{><}{\mathbf{I}} \cdot \overset{<}{\mathbf{u}} = g_I^i U^I \mathbf{g}_i. \tag{3.19}$$

where

$$g_i^I = \mathbf{G}^I \cdot \mathbf{g}_i = \mathbf{g}_i \cdot \mathbf{G}^I \text{ and } g_I^i = \mathbf{g}^i \cdot \mathbf{G}_I = \mathbf{G}_I \cdot \mathbf{g}^i. \tag{3.20}$$

3.1.2. Deformation gradient and tensors

Before the deformation gradient is discussed, the Hamilton operators will be introduced first, and the corresponding metric tensors will be presented. Further, the deformation gradient will be presented.

Definition 3.8. The *Hamilton operators* to the coordinates $\{X^I\}$ and $\{x^i\}$ are defined as

$$\overset{<}{\nabla} = \frac{\partial}{\partial X^I}\mathbf{G}^I \text{ and } \overset{>}{\nabla} = \frac{\partial}{\partial x^i}\mathbf{g}^i. \tag{3.21}$$

From the previous definition, the gradient of position vectors can be expressed by the two-point tensors in Chapter 2. The definition is given as follows.

Definition 3.9. For two position vectors $\mathbf{P}$ in $\mathfrak{B}$ and $\mathbf{p}$ in $\mathfrak{b}$, the gradients of the two position vectors are defined as

$$\begin{aligned}
&\overset{>}{\nabla}\mathbf{P} = \frac{\partial X^I}{\partial x^i}\mathbf{g}^i\mathbf{G}_I = X^I_{;i}\mathbf{g}^i\mathbf{G}_I,\ \mathbf{P}\overset{>}{\nabla} = \frac{\partial X^I}{\partial x^i}\mathbf{G}_I\mathbf{g}^i = X^I_{;i}\mathbf{G}_I\mathbf{g}^i; \\
&\overset{<}{\nabla}\mathbf{p} = \frac{\partial x^i}{\partial X^I}\mathbf{G}^I\mathbf{g}_i = x^i_{;I}\mathbf{G}^I\mathbf{g}_i,\ \mathbf{p}\overset{<}{\nabla} = \frac{\partial x^i}{\partial X^I}\mathbf{g}_i\mathbf{G}^I = x^i_{;I}\mathbf{g}_i\mathbf{G}^I,
\end{aligned} \tag{3.22}$$

which are also called *the deformation gradient tensors.*

$$\begin{aligned}
&\overset{<<}{\mathbf{C}} = C_{IJ}\mathbf{G}^I\mathbf{G}^J = C^I_{\cdot J}\mathbf{G}_I\mathbf{G}^J \ \text{ and } \ \overset{>>}{\mathbf{c}} = c_{ij}\mathbf{g}^i\mathbf{g}^j = c^i_{\cdot j}\mathbf{g}_i\mathbf{g}^j; \\
&\overset{-1}{\overset{<<}{\mathbf{C}}} = {}^{-1}C^{IJ}\mathbf{G}_I\mathbf{G}_J = {}^{-1}C^I_{\cdot J}\mathbf{G}_I\mathbf{G}^J \text{and } \overset{-1}{\overset{>>}{\mathbf{c}}} = {}^{-1}c^{ij}\mathbf{g}_i\mathbf{g}_j = {}^{-1}c^i_{\cdot j}\mathbf{g}_i\mathbf{g}^j
\end{aligned} \tag{3.23}$$

are the *covariant and contravariant metric tensors*, which are also called the covariant and contravariant *Cauchy's deformation tensors* and *Green's deformation tensors*, respectively.

Theorem 3.1. *For two position vectors* $\mathbf{P}$ *in* $\mathfrak{B}$ *and* $\mathbf{p}$ *in* $\mathfrak{b}$*, the covariant metric tensors are determined by*

$$\begin{aligned}
&\overset{<<}{\mathbf{C}} = (\overset{<}{\nabla}\overset{<}{\mathbf{p}})\cdot(\overset{<}{\mathbf{p}}\overset{<}{\nabla}) \ \text{ and } \ \overset{>>}{\mathbf{c}} = (\overset{>}{\nabla}\overset{>}{\mathbf{P}})\cdot(\overset{>}{\mathbf{P}}\overset{>}{\nabla}); \\
&\overset{-1}{\overset{<<}{\mathbf{C}}} = (\overset{>}{\mathbf{P}}\overset{>}{\nabla})\cdot(\overset{>}{\nabla}\overset{>}{\mathbf{P}}) \ \text{ and } \ \overset{-1}{\overset{>>}{\mathbf{c}}} = (\overset{<}{\mathbf{p}}\overset{<}{\nabla})\cdot(\overset{<}{\nabla}\overset{<}{\mathbf{p}}).
\end{aligned} \tag{3.24}$$

Proof: The two metric tensors are

$$\begin{aligned}
\overset{<<}{\mathbf{C}} &= C_{IJ}\mathbf{G}^I\mathbf{G}^J = g_{ij}x^i_{;I}x^j_{;J}\mathbf{G}^I\mathbf{G}^J = (x^i_{;I}\mathbf{G}^I\mathbf{g}_i)\cdot(x^j_{;J}\mathbf{g}_j\mathbf{G}^J) \\
&= (\overset{<}{\nabla}\mathbf{p})\cdot(\mathbf{p}\overset{<}{\nabla}) = (\overset{<}{\nabla}\overset{<}{\mathbf{p}})\cdot(\overset{<}{\mathbf{p}}\overset{<}{\nabla}); \\
\overset{>>}{\mathbf{c}} &= c_{ij}\mathbf{g}^i\mathbf{g}^j = G_{MN}X^M_{;i}X^N_{;j}\mathbf{g}^i\mathbf{g}^j = (X^M_{;i}\mathbf{g}^i\mathbf{G}_M)\cdot(X^N_{;j}\mathbf{G}_N\mathbf{g}^j) \\
&= (\overset{>}{\nabla}\mathbf{P})\cdot(\mathbf{P}\overset{>}{\nabla}) = (\overset{>}{\nabla}\overset{>}{\mathbf{P}})\cdot(\overset{>}{\mathbf{P}}\overset{>}{\nabla}); \\
\overset{-1}{\overset{<<}{\mathbf{C}}} &= {}^{-1}C^{IJ}\mathbf{G}_I\mathbf{G}_J = g^{mn}X^I_{;m}X^J_{;n}\mathbf{G}_I\mathbf{G}_J = (X^I_{;m}\mathbf{G}_I\mathbf{g}^m)\cdot(X^J_{;n}\mathbf{g}^n\mathbf{G}_J) \\
&= (\mathbf{P}\overset{>}{\nabla})\cdot(\overset{>}{\nabla}\mathbf{P}) = (\overset{>}{\mathbf{P}}\overset{>}{\nabla})\cdot(\overset{>}{\nabla}\overset{>}{\mathbf{P}}); \\
\overset{-1}{\overset{>>}{\mathbf{c}}} &= {}^{-1}c^{ij}\mathbf{g}_i\mathbf{g}_j = G^{MN}x^i_{;M}x^j_{;N}\mathbf{g}_i\mathbf{g}_j = (x^i_{;M}\mathbf{g}_i\mathbf{G}^M)\cdot(x^j_{;N}\mathbf{G}^N\mathbf{g}_j) \\
&= (\mathbf{p}\overset{<}{\nabla})\cdot(\overset{<}{\nabla}\mathbf{p}) = (\overset{<}{\mathbf{p}}\overset{<}{\nabla})\cdot(\overset{<}{\nabla}\overset{<}{\mathbf{p}}).
\end{aligned}$$

This theorem is proved. ■

Theorem 3.2. *For two position vectors* $\mathbf{P}$ *in* $\mathfrak{B}$ *and* $\mathbf{p}$ *in* $\mathfrak{b}$, *the invariants of covariant metric tensors are determined by*

$$\begin{aligned}
\det(\overset{<<}{\mathbf{C}}) &= |C^I_{\cdot J}| = |G^{IR} g_{ij} x^i_{;R} x^j_{;J}| = \mathscr{J}^2,\\
\det(\overset{>>}{\mathbf{c}}) &= |c^i_{\cdot j}| = |g^{ir} G_{IJ} X^I_{;r} X^J_{;j}| = \mathscr{j}^2,\\
\det(\overset{\overset{-1}{<<}}{\mathbf{C}}) &= |{}^{-1}C^I_{\cdot J}| = |g^{mn} G_{IR} X^R_{;m} X^J_{;n}| = \mathscr{j}^2,\\
\det(\overset{\overset{-1}{>>}}{\mathbf{c}}) &= |{}^{-1}c^i_{\cdot j}| = |G^{IJ} g_{ir} x^r_{;I} x^j_{;J}| = \mathscr{J}^2,
\end{aligned} \tag{3.25}$$

where

$$\mathscr{J} = \sqrt{\frac{g}{G}}\,|x^i_{;I}|, \quad \mathscr{j} = \sqrt{\frac{G}{g}}\,|X^I_{;i}|. \tag{3.26}$$

Proof: Because

$$\det(\overset{<<}{\mathbf{C}}) = |C^I_{\cdot J}| = |G^{IR} C_{RJ}| = |G^{IR} g_{ij} x^i_{;R} x^j_{;J}|$$

and

$$|G_{IR}| = \frac{1}{|G^{IR}|} = G \text{ and } |g_{ij}| = \frac{1}{|g^{ij}|} = g$$

then,

$$\det(\overset{<<}{\mathbf{C}}) = |C^I_{\cdot J}| = |G^{IR} g_{ij} x^i_{;R} x^j_{;J}| = \mathscr{J}^2.$$

Similarly,

$$\begin{aligned}
\det(\overset{>>}{\mathbf{c}}) &= |c^i_{\cdot j}| = |g^{ir} c_{rj}| = |g^{ir} G_{IJ} X^I_{;r} X^J_{;j}| = \mathscr{j}^2,\\
\det(\overset{\overset{-1}{<<}}{\mathbf{C}}) &= |{}^{-1}C^I_{\cdot J}| = |G_{IR}\,{}^{-1}C^{RJ}| = |G_{IR} g^{mn} X^R_{;m} X^J_{;n}| = \mathscr{j}^2,\\
\det(\overset{\overset{-1}{>>}}{\mathbf{c}}) &= |{}^{-1}c^i_{\cdot j}| = |g_{ir}\,{}^{-1}c^{rj}| = |g_{ir} G^{IJ} x^r_{;I} x^j_{;J}| = \mathscr{J}^2.
\end{aligned}$$

This theorem is proved. ■

If x^i and X^I are independent, $X^I_{;i} \equiv X^I_{,i}$ and $x^i_{;I} \equiv x^i_{,I}$. Because of the movement and deformation of the continuous body, the displacement between the initial configuration $\mathfrak{B}$ and instantaneous configuration $\mathfrak{b}$ is $\mathbf{u}$. For the infinitesimal position vectors $d\mathbf{P}$ and $d\mathbf{p}$ in the two configurations, the infinitesimal displacement vector is expressed by $d\mathbf{u}$, as shown in Fig.3.4. The infinitesimal line elements $d\mathbf{P}$ and $d\mathbf{p}$ are defined as follows.

Definition 3.10. For two position vectors $\mathbf{P}$ in $\mathfrak{B}$ and $\mathbf{p}$ in $\mathfrak{b}$, the gradients of two vectors are defined as

$$\begin{aligned} d\mathbf{p} &= dx^i\mathbf{g}_i = (x^i_{;I}\mathbf{g}_i\mathbf{G}^I)\cdot(dX^J\mathbf{G}_J) = (\mathbf{p}\overset{<}{\nabla})\cdot d\mathbf{P} = \overset{><}{\mathbf{F}}\cdot d\mathbf{P},\\ d\mathbf{P} &= dX^I\mathbf{G}_I = (x^I_{;i}\mathbf{G}_I\mathbf{g}^i)\cdot(dx^j\mathbf{g}_j) = (\mathbf{P}\overset{>}{\nabla})\cdot d\mathbf{p} = \overset{-1}{\overset{<>}{\mathbf{F}}}\cdot d\mathbf{p}, \end{aligned} \tag{3.27}$$

where

$$\overset{><}{\mathbf{F}} \equiv \mathbf{p}\overset{<}{\nabla} = X^{\;i}_{;\;I}\mathbf{g}_i\mathbf{G}^I \text{ and } \overset{-1}{\overset{<>}{\mathbf{F}}} \equiv \mathbf{P}\overset{>}{\nabla} = X^{\;I}_{;\;i}\mathbf{G}_I\mathbf{g}^i, \tag{3.28}$$

which are called the *deformation gradient tensors.*

From the foregoing definition,

$$\begin{aligned} \overset{><}{\mathbf{F}}\cdot\overset{-1}{\overset{<>}{\mathbf{F}}} &= (x^i_{;I}\mathbf{g}_i\mathbf{G}^I)\cdot(X^J_{;j}\mathbf{G}_J\mathbf{g}^j) = \delta^i_j\mathbf{g}_i\mathbf{g}^j = \overset{>>}{\mathbf{I}},\\ \overset{-1}{\overset{<>}{\mathbf{F}}}\cdot\overset{><}{\mathbf{F}} &= (X^I_{;i}\mathbf{G}_I\mathbf{g}^i)\cdot(x^j_{;J}\mathbf{g}_j\mathbf{G}^J) = \delta^I_J\mathbf{G}_I\mathbf{G}^J = \overset{<<}{\mathbf{I}}. \end{aligned} \tag{3.29}$$

From Eq.(3.29), the two-point second-order tensors of $\overset{><}{\mathbf{F}}$ and $\overset{-1}{\overset{<>}{\mathbf{F}}}$ are inverse each other. The deformation gradient with "-1" is on the deformed configuration.

Theorem 3.3. *For two position vectors* $\mathbf{P}$ *in* $\mathfrak{B}$ *and* $\mathbf{p}$ *in* $\mathfrak{b}$*, the transposes of deformation gradient tensors are determined by*

$$(\overset{<>}{\mathbf{F}})^{\mathrm{T}} = \overset{<}{\nabla}\mathbf{p} = x^{\;i}_{;\;I}\mathbf{G}^I\mathbf{g}_i \text{ and } (\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}} = \overset{>}{\nabla}\mathbf{P} = X^{\;I}_{;\;i}\mathbf{g}^i\mathbf{G}_I. \tag{3.30}$$

Proof: Consider two vectors $\overset{>}{\mathbf{u}} = u_i\mathbf{g}^i$ and $\overset{<}{\mathbf{v}} = v^I\mathbf{G}_I$. From Eqs.(3.22) and (3.28),

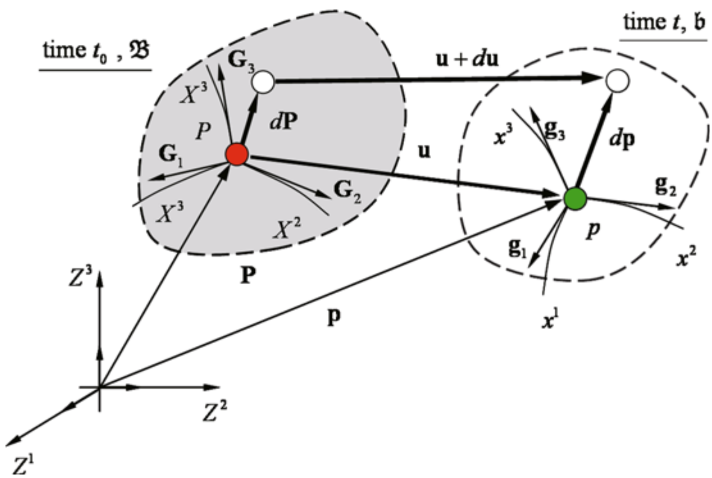

Fig. 3.4 Infinitesimal displacement vector $d\mathbf{u}$ between two configurations.

$$\overset{>}{\mathbf{u}}\cdot\overset{><}{\mathbf{F}}\cdot\overset{<}{\mathbf{v}} = (u_j\mathbf{g}^j)\cdot(X^i_{;I}\mathbf{g}_i\mathbf{G}^I)\cdot(v^J\mathbf{G}_J) = u_i X^i_{;I} v^I = v^I X^i_{;I} u_i$$

$$= (v^J\mathbf{G}_J)\cdot(X^i_{;I}\mathbf{G}^I\mathbf{g}_i)\cdot(u_j\mathbf{g}^j) = \overset{<}{\mathbf{v}}\cdot(\overset{<}{\nabla}\mathbf{p})\cdot\overset{>}{\mathbf{u}} \equiv \overset{<}{\mathbf{v}}\cdot(\overset{<>}{\mathbf{F}})^{\mathrm{T}}\cdot\overset{>}{\mathbf{u}}.$$

Similarly, consider two vectors $\overset{<}{\mathbf{u}} = u_I\mathbf{G}^I$ and $\overset{>}{\mathbf{v}} = v^i\mathbf{g}_i$, and the following equation holds:

$$\overset{<}{\mathbf{u}}\cdot\overset{-1}{\overset{<>}{\mathbf{F}}}\cdot\overset{>}{\mathbf{v}} = (u_J\mathbf{G}^J)\cdot(x^I_{;i}\mathbf{G}_I\mathbf{g}^i)\cdot(v^j\mathbf{g}_j) = u_I x^I_{;i} v^i = v^i x^I_{;i} u_I$$

$$= (v^j\mathbf{g}_j)\cdot(x^I_{;i}\mathbf{g}^i\mathbf{G}_I)\cdot(u_J\mathbf{G}^J) = \overset{>}{\mathbf{v}}\cdot(\overset{>}{\nabla}\mathbf{P})\ \cdot\overset{<}{\mathbf{u}}. = \overset{>}{\mathbf{v}}\cdot(\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}}\cdot\overset{<}{\mathbf{u}}.$$

This theorem is proved. ■

Definition 3.11. For two position vectors $\mathbf{P}$ in $\mathfrak{B}$ and $\mathbf{p}$ in $\mathfrak{b}$, from the shifter tensors, *the other deformation gradients* are defined as

$$\begin{aligned}&\overset{><}{\mathbf{F}} = \mathbf{p}\overset{<}{\nabla} = x^i_{;I}\mathbf{g}_i\mathbf{G}^I,\quad \overset{<>}{\mathbf{F}} = \overset{<>}{\mathbf{I}}\cdot\overset{><}{\mathbf{F}}\cdot\overset{<>}{\mathbf{I}} = g^I_r x^r_{;R} g^R_i\mathbf{G}_I\mathbf{g}^i,\\ &\overset{<<}{\mathbf{F}} = \overset{<>}{\mathbf{I}}\cdot\overset{><}{\mathbf{F}} = g^I_r x^r_{;J}\mathbf{G}_I\mathbf{G}^J,\quad \overset{>>}{\mathbf{F}} = \overset{><}{\mathbf{F}}\cdot\overset{<>}{\mathbf{I}} = x^i_{;R} g^R_j\mathbf{g}_i\mathbf{g}^j.\end{aligned} \tag{3.31}$$

$$\begin{aligned}&(\overset{<>}{\mathbf{F}})^{\mathrm{T}} = \overset{<}{\nabla}\mathbf{p} = x^i_{;I}\mathbf{G}^I\mathbf{g}_i,\quad (\overset{><}{\mathbf{F}})^{\mathrm{T}} = \overset{><}{\mathbf{I}}\cdot(\overset{<>}{\mathbf{F}})^{\mathrm{T}}\cdot\overset{><}{\mathbf{I}} = g^R_i x^r_{;R} g^I_r\mathbf{g}^i\mathbf{G}_I,\\ &(\overset{>>}{\mathbf{F}})^{\mathrm{T}} = \overset{><}{\mathbf{I}}\cdot(\overset{<>}{\mathbf{F}})^{\mathrm{T}} = x^i_{;R} g^R_j\mathbf{g}^j\mathbf{g}_i,\quad (\overset{<<}{\mathbf{F}})^{\mathrm{T}} = (\overset{<>}{\mathbf{F}})^{\mathrm{T}}\cdot\overset{><}{\mathbf{I}} = x^r_{;I} g^J_r\mathbf{G}^I\mathbf{G}_J.\end{aligned} \tag{3.32}$$

$$\begin{aligned}&\overset{-1}{\overset{<>}{\mathbf{F}}} = \mathbf{P}\overset{>}{\nabla} = X^I_{;i}\mathbf{G}_I\mathbf{g}^i,\quad \overset{-1}{\overset{><}{\mathbf{F}}} = \overset{><}{\mathbf{I}}\cdot\overset{-1}{\overset{<>}{\mathbf{F}}}\cdot\overset{><}{\mathbf{I}} = g^i_R X^R_{;r} g^r_I\mathbf{g}_i\mathbf{G}^I,\\ &\overset{-1}{\overset{>>}{\mathbf{F}}} = \overset{><}{\mathbf{I}}\cdot\overset{-1}{\overset{<>}{\mathbf{F}}} = g^i_R X^R_{;j}\mathbf{g}_i\mathbf{g}^j,\quad \overset{-1}{\overset{<<}{\mathbf{F}}} = \overset{-1}{\overset{<>}{\mathbf{F}}}\cdot\overset{><}{\mathbf{I}} = X^I_{;r} g^r_J\mathbf{G}_I\mathbf{G}^J.\end{aligned} \tag{3.33}$$

$$\begin{aligned}&(\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}} = \overset{>}{\nabla}\mathbf{P} = X^I_{;i}\mathbf{g}^i\mathbf{G}_I,\quad (\overset{-1}{\overset{<>}{\mathbf{F}}})^{\mathrm{T}} = \overset{<>}{\mathbf{I}}\cdot(\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}}\cdot\overset{<>}{\mathbf{I}} = g^r_I X^J_{;r} g^i_J\mathbf{G}^I\mathbf{g}_i,\\ &(\overset{-1}{\overset{>>}{\mathbf{F}}})^{\mathrm{T}} = (\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}}\cdot\overset{<>}{\mathbf{I}} = X^R_{;i} g^j_R\mathbf{g}^i\mathbf{g}_j,\quad (\overset{-1}{\overset{<<}{\mathbf{F}}})^{\mathrm{T}} = \overset{<>}{\mathbf{I}}\cdot(\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}} = g^r_J X^I_{;r}\mathbf{G}^J\mathbf{G}_I.\end{aligned} \tag{3.34}$$

From the deformation gradient, the *covariant and contravariant metric tensors* (or *Cauchy's deformation tensor* and *Green's deformation tensor*) are expressed by

$$\begin{aligned}&\overset{<<}{\mathbf{C}} = (\overset{<>}{\mathbf{F}})^{\mathrm{T}}\cdot(\overset{><}{\mathbf{F}}) = (\overset{<<}{\mathbf{F}})^{\mathrm{T}}\cdot(\overset{<<}{\mathbf{F}}),\quad \overset{-1}{\overset{<<}{\mathbf{C}}} = \overset{-1}{\overset{<>}{\mathbf{F}}}\cdot(\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}} = (\overset{-1}{\overset{<<}{\mathbf{F}}})\cdot(\overset{-1}{\overset{<<}{\mathbf{F}}})^{\mathrm{T}},\\ &\overset{>>}{\mathbf{c}} = (\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}}\cdot(\overset{-1}{\overset{<>}{\mathbf{F}}}) = (\overset{-1}{\overset{>>}{\mathbf{F}}})^{\mathrm{T}}\cdot(\overset{-1}{\overset{>>}{\mathbf{F}}}),\quad (\overset{-1}{\overset{>>}{\mathbf{c}}}) = \overset{-1}{\overset{><}{\mathbf{F}}}\cdot(\overset{<>}{\mathbf{F}})^{\mathrm{T}} = \overset{>>}{\mathbf{F}}\cdot(\overset{>>}{\mathbf{F}})^{\mathrm{T}}.\end{aligned} \tag{3.35}$$

Definition 3.12. For two position vectors $\mathbf{P}$ in $\mathfrak{B}$ and $\mathbf{p}$ in $\mathfrak{b}$, the *infinitesimal areas and volumes* are defined as

$$d\overset{>}{\mathbf{a}} = d\underset{4}{\mathbf{p}}\times d\underset{5}{\mathbf{p}} \quad \text{and} \quad d\overset{<}{\mathbf{a}} = d\overset{<}{\underset{4}{\mathbf{p}}}\times d\overset{<}{\underset{5}{\mathbf{p}}},$$
$$d\overset{<}{\mathbf{A}} = d\underset{4}{\mathbf{P}}\times d\underset{5}{\mathbf{P}} \quad \text{and} \quad d\overset{>}{\mathbf{A}} = d\overset{>}{\underset{4}{\mathbf{P}}}\times d\overset{>}{\underset{5}{\mathbf{P}}}. \tag{3.36}$$

$$dv = [d\underset{1}{\mathbf{p}}\, d\underset{2}{\mathbf{p}}\, d\underset{3}{\mathbf{p}}] \quad \text{and} \quad dV = [d\underset{1}{\mathbf{P}}\, d\underset{2}{\mathbf{P}}\, d\underset{3}{\mathbf{P}}]. \tag{3.37}$$

Theorem 3.4. For two position vectors $\mathbf{P}$ in $\mathfrak{B}$ and $\mathbf{p}$ in $\mathfrak{b}$, the infinitesimal volumes and areas are

$$dv = [d\overset{<}{\underset{1}{\mathbf{p}}}\, d\overset{<}{\underset{2}{\mathbf{p}}}\, d\overset{<}{\underset{3}{\mathbf{p}}}] = \sqrt{\frac{g}{G}}\,|\,x^i_{;I}\,|\,dV = \mathscr{J} dV,$$
$$dV = [d\overset{>}{\underset{1}{\mathbf{P}}}\, d\overset{>}{\underset{2}{\mathbf{P}}}\, d\overset{>}{\underset{3}{\mathbf{P}}}] = \sqrt{\frac{G}{g}}\,|\,X^I_{;i}\,|\,dv = \mathscr{j} dv, \tag{3.38}$$

$$d\overset{>}{\mathbf{a}} = d\underset{4}{\mathbf{p}}\times d\underset{5}{\mathbf{p}} = \mathscr{J}(\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}}\cdot d\overset{<}{\mathbf{A}},$$
$$d\overset{<}{\mathbf{A}} = d\underset{4}{\mathbf{P}}\times d\underset{5}{\mathbf{P}} = \mathscr{j}(\overset{<>}{\mathbf{F}})^{\mathrm{T}}\cdot d\overset{>}{\mathbf{a}}. \tag{3.39}$$

Proof: From the definition of infinitesimal volume,

$$dv = [d\overset{<}{\underset{1}{\mathbf{p}}}\, d\overset{<}{\underset{2}{\mathbf{p}}}\, d\overset{<}{\underset{3}{\mathbf{p}}}] = (\overset{<>}{\mathbf{I}}\cdot d\underset{1}{\mathbf{p}})\times(\overset{<>}{\mathbf{I}}\cdot d\underset{2}{\mathbf{p}})\cdot(\overset{<>}{\mathbf{I}}\cdot d\underset{3}{\mathbf{p}})$$
$$= \frac{[(\overset{<<}{\mathbf{F}}\cdot d\underset{1}{\mathbf{P}})\times(\overset{<<}{\mathbf{F}}\cdot d\underset{2}{\mathbf{P}})]\cdot(\overset{<<}{\mathbf{F}}\cdot d\underset{3}{\mathbf{P}})}{(d\underset{1}{\mathbf{P}}\times d\underset{2}{\mathbf{P}})\cdot d\underset{3}{\mathbf{P}}}\,dV = \mathrm{III}(\overset{<<}{\mathbf{F}})dV$$
$$= |\,g^I_r x^r_{;J}\,|\,dV = |\,g^I_r\,||\,x^r_{;J}\,|\,dV = \sqrt{\frac{g}{G}}\,|\,x^i_{;I}\,|\,dV = \mathscr{J} dV.$$

Similarly, the infinitesimal volume for the initial configuration is

$$dV = [d\overset{>}{\underset{1}{\mathbf{P}}}\, d\overset{>}{\underset{2}{\mathbf{P}}}\, d\overset{>}{\underset{3}{\mathbf{P}}}] = (\overset{><}{\mathbf{I}}\cdot d\underset{1}{\mathbf{P}})\times(\overset{><}{\mathbf{I}}\cdot d\underset{2}{\mathbf{P}})\cdot(\overset{><}{\mathbf{I}}\cdot d\underset{3}{\mathbf{P}})$$
$$= \frac{[(\overset{-1}{\overset{>>}{\mathbf{F}}}\cdot d\underset{1}{\mathbf{p}})\times(\overset{-1}{\overset{>>}{\mathbf{F}}}\cdot d\underset{2}{\mathbf{p}})]\cdot(\overset{-1}{\overset{>>}{\mathbf{F}}}\cdot d\underset{3}{\mathbf{p}})}{(d\underset{1}{\mathbf{p}}\times d\underset{2}{\mathbf{p}})\cdot d\underset{3}{\mathbf{p}}}\,dv = \mathbf{III}(\overset{-1}{\overset{>>}{\mathbf{F}}})dv$$
$$= |\,g^i_R X^R_{;i}\,|\,dv = |\,g^i_I\,||\,X^I_{;i}\,|\,dv = \sqrt{\frac{G}{g}}\,|\,X^I_{;i}\,|\,dV = \mathscr{j} dv.$$

From the definition of infinitesimal area,

$$d\overset{>}{\mathbf{a}} = d\underset{4}{\mathbf{p}}\times d\underset{5}{\mathbf{p}} = \overset{><}{\mathbf{I}}\cdot d\overset{<}{\mathbf{a}} = \overset{><}{\mathbf{I}}\cdot(d\overset{<}{\underset{4}{\mathbf{p}}}\times d\overset{<}{\underset{5}{\mathbf{p}}}) = \overset{><}{\mathbf{I}}\cdot[(\overset{<<}{\mathbf{F}}\cdot d\overset{<}{\underset{4}{\mathbf{P}}})\times(\overset{<<}{\mathbf{F}}\cdot d\overset{<}{\underset{5}{\mathbf{P}}})].$$

Because

$$\mathrm{III}(\overset{<<}{\mathbf{F}})\,\underset{3}{d\mathbf{P}}\cdot(\underset{4}{d\mathbf{P}}\times\underset{5}{d\mathbf{P}})=(\overset{<<}{\mathbf{F}}\cdot\underset{3}{d\mathbf{P}})\cdot[(\overset{<<}{\mathbf{F}}\cdot\underset{4}{d\mathbf{P}})\times(\overset{<<}{\mathbf{F}}\cdot\underset{5}{d\mathbf{P}})]$$

$$=[(\overset{<<}{\mathbf{F}}\cdot\underset{4}{d\mathbf{P}})\times(\overset{<<}{\mathbf{F}}\cdot\underset{5}{d\mathbf{P}})]\cdot(\overset{<<}{\mathbf{F}}\cdot\underset{3}{d\mathbf{P}})=\underset{3}{d\mathbf{P}}\cdot(\overset{<<}{\mathbf{F}})^{\mathrm{T}}\cdot[(\overset{<<}{\mathbf{F}}\cdot\underset{4}{d\mathbf{P}})\times(\overset{<<}{\mathbf{F}}\cdot\underset{5}{d\mathbf{P}})],$$

$$(\overset{<<}{\mathbf{F}})^{\mathrm{T}}\cdot[(\overset{<<}{\mathbf{F}}\cdot\underset{4}{d\mathbf{P}})\times(\overset{<<}{\mathbf{F}}\cdot\underset{5}{d\mathbf{P}})]=\mathrm{III}(\overset{<<}{\mathbf{F}})\cdot(\underset{4}{d\mathbf{P}}\times\underset{5}{d\mathbf{P}}),$$

$$(\overset{<<}{\mathbf{F}}\cdot\underset{4}{d\mathbf{P}})\times(\overset{<<}{\mathbf{F}}\cdot\underset{5}{d\mathbf{P}})=\mathrm{III}(\overset{<<}{\mathbf{F}})[(\overset{<<}{\mathbf{F}})^{-1}]^{\mathrm{T}}\cdot(\underset{4}{d\mathbf{P}}\times\underset{5}{d\mathbf{P}})=\mathrm{III}(\overset{<<}{\mathbf{F}})(\overset{\overset{-1}{<<}}{\mathbf{F}})^{\mathrm{T}}\cdot(\underset{4}{d\mathbf{P}}\times\underset{5}{d\mathbf{P}}),$$

thus

$$d\overset{>}{\mathbf{a}}=\mathrm{III}(\overset{<<}{\mathbf{F}})\,\overset{><}{\mathbf{I}}\cdot(\overset{\overset{-1}{<<}}{\mathbf{F}})^{\mathrm{T}}\cdot(\underset{4}{d\mathbf{P}}\times\underset{5}{d\mathbf{P}})=\mathscr{J}(\overset{\overset{-1}{><}}{\mathbf{F}})^{\mathrm{T}}\cdot d\overset{<}{\mathbf{A}}.$$

Similarly,

$$(\overset{\overset{-1}{>>}}{\mathbf{F}}\cdot\underset{4}{d\mathbf{p}})\times(\overset{\overset{-1}{>>}}{\mathbf{F}}\cdot\underset{5}{d\mathbf{p}})=\mathrm{III}(\overset{\overset{-1}{>>}}{\mathbf{F}})[(\overset{\overset{-1}{>>}}{\mathbf{F}})^{-1}]^{\mathrm{T}}\cdot(\underset{4}{d\mathbf{p}}\times\underset{5}{d\mathbf{p}})=\mathrm{III}(\overset{\overset{-1}{>>}}{\mathbf{F}})(\overset{>>}{\mathbf{F}})^{\mathrm{T}}\cdot(\underset{4}{d\mathbf{p}}\times\underset{5}{d\mathbf{p}}),$$

and the infinitesimal area in the initial configuration is determined by

$$d\overset{<}{\mathbf{A}}=\overset{<>}{\mathbf{I}}\cdot d\overset{>}{\mathbf{A}}=\overset{<>}{\mathbf{I}}\cdot(d\underset{4}{\overset{>}{\mathbf{P}}}\times d\underset{5}{\overset{>}{\mathbf{P}}})=\overset{<>}{\mathbf{I}}\cdot[(\overset{\overset{-1}{>>}}{\mathbf{F}}\cdot d\underset{4}{\overset{>}{\mathbf{p}}})\times(\overset{\overset{-1}{>>}}{\mathbf{F}}\cdot d\underset{5}{\overset{>}{\mathbf{p}}})]$$

$$=\mathrm{III}(\overset{\overset{-1}{>>}}{\mathbf{F}})\cdot\overset{<>}{\mathbf{I}}\cdot(\overset{>>}{\mathbf{F}})^{\mathrm{T}}\cdot d\overset{>}{\mathbf{a}}=\mathscr{j}(\overset{<>}{\mathbf{F}})^{\mathrm{T}}\cdot d\overset{>}{\mathbf{a}}.$$

This theorem is proved. ■

The component expressions of the area are

$$da_i=\mathscr{J}X^I_{;i}dA_I \ \text{ and } \ dA_I=\mathscr{j}x^i_{;I}da_i. \tag{3.40}$$

Theorem 3.5. *For two position vectors* $\mathbf{P}$ *in* $\mathfrak{B}$ *and* $\mathbf{p}$ *in* $\mathfrak{b}$, *the following equations holds*

$$\overset{<}{\square}\cdot(\mathscr{J}\overset{\overset{-1}{<>}}{\mathbf{F}})=0 \ \text{ and } \ \overset{>}{\square}\cdot(\mathscr{j}\overset{><}{\mathbf{F}})=0 \tag{3.41}$$

with the component expression as

$$(\mathscr{J}X^I_{;i})_{:I}=0 \ \text{ and } \ (\mathscr{j}x^i_{;I})_{:i}=0. \tag{3.42}$$

Proof: Because $\operatorname{div}\overset{>>}{\mathbf{I}}=0$, from Eq.(2.217),

$$0=\int_v dv\,\operatorname{div}\overset{>>}{\mathbf{I}}=\oint_a d\overset{>}{\mathbf{a}}\cdot\overset{>>}{\mathbf{I}}=\oint_a d\overset{>}{\mathbf{a}}.$$

With Eq.(3.39), the integration domain is converted from a to $\mathscr{A}$ via $\overset{<>}{\mathbf{I}}$

$$0=\oint_{\mathscr{A}}\mathscr{J}\,\overset{<>}{\mathbf{I}}\cdot(\overset{\overset{-1}{><}}{\mathbf{F}})^{\mathrm{T}}\cdot d\overset{<}{\mathbf{A}}=\oint_{\mathscr{A}}\mathscr{J}(\overset{\overset{-1}{<<}}{\mathbf{F}})^{\mathrm{T}}\cdot d\overset{<}{\mathbf{A}}=\oint_{\mathscr{A}}d\overset{<}{\mathbf{A}}\cdot(\mathscr{J}\overset{\overset{-1}{<<}}{\mathbf{F}})=\oint_{\mathscr{V}}dV\,\overset{<}{\square}\cdot(\mathscr{J}\overset{\overset{-1}{<<}}{\mathbf{F}})$$

where $\overset{<}{\square}\cdot(\mathscr{J}\overset{-1}{\overset{\ll}{\mathbf{F}}}) \equiv \mathrm{div}(\mathscr{J}\overset{-1}{\overset{\ll}{\mathbf{F}}})$ with the total differentiation to point $\mathbf{P}$. Because of any arbitrarily selection of the volume, the foregoing equation gives

$$\overset{<}{\square}\cdot(\mathscr{J}\overset{-1}{\overset{\ll}{\mathbf{F}}}) = 0.$$

Also the foregoing equation gives

$$\overset{<}{\square}\cdot(\mathscr{J}\overset{-1}{\overset{\ll}{\mathbf{F}}}) = \overset{<}{\square}\cdot(\mathscr{J}\overset{-1}{\overset{<>}{\mathbf{F}}}\cdot\overset{><}{\mathbf{I}}) = \overset{<}{\square}\cdot(\mathscr{J}\overset{-1}{\overset{<>}{\mathbf{F}}})\cdot\overset{><}{\mathbf{I}} + \mathscr{J}\overset{-1}{\overset{<>}{\mathbf{F}}}\cdot(\overset{<}{\square}\cdot\overset{><}{\mathbf{I}}) = 0.$$

Because of $\overset{<}{\square}\cdot\overset{><}{\mathbf{I}} = 0$,

$$\overset{<}{\square}\cdot(\mathscr{J}\overset{-1}{\overset{<>}{\mathbf{F}}})\cdot\overset{><}{\mathbf{I}} = 0.$$

Also because $\overset{><}{\mathbf{I}}$ is non-zero, the following equation is achieved,

$$\overset{<}{\square}\cdot(\mathscr{J}\overset{-1}{\overset{<>}{\mathbf{F}}}) = 0$$

from which the component expression is

$$(\mathscr{J}X^{I}_{;i})_{:I} = 0.$$

Similarly,

$$0 = \int_{\mathscr{V}} dV \,\mathrm{div}\,\overset{\ll}{\mathbf{I}} = \oint_{\mathscr{A}} d\overset{<}{\mathbf{A}}\cdot\overset{\ll}{\mathbf{I}} = \oint_{\mathscr{A}} d\overset{<}{\mathbf{A}},$$

With Eq.(3.39), the integration domain is converted from $\mathscr{A}$ to a via $\overset{><}{\mathbf{I}}$

$$0 = \oint_{a} \jmath\,\overset{><}{\mathbf{I}}\cdot(\overset{<>}{\mathbf{F}})^{\mathrm{T}}\cdot d\overset{>}{\mathbf{a}} = \oint_{a} \jmath(\overset{\gg}{\mathbf{F}})^{\mathrm{T}}\cdot d\overset{>}{\mathbf{A}} = \oint_{a} d\overset{>}{\mathbf{A}}\cdot(\jmath\overset{\gg}{\mathbf{F}}) = \oint_{v} dv\overset{>}{\square}\cdot(\jmath\overset{-1}{\overset{\gg}{\mathbf{F}}}).$$

any arbitrarily selection of the volume S

$$\overset{>}{\square}\cdot(\jmath\overset{\gg}{\mathbf{F}}) = 0.$$

Further, the following relation holds,

$$\overset{>}{\square}\cdot(\jmath\overset{\gg}{\mathbf{F}}) = \overset{>}{\square}\cdot(\jmath\overset{><}{\mathbf{F}}\cdot\overset{<>}{\mathbf{I}}) = \overset{>}{\square}\cdot(\jmath\overset{><}{\mathbf{F}})\cdot\overset{<>}{\mathbf{I}} + \jmath\overset{><}{\mathbf{F}}\cdot(\overset{>}{\square}\cdot\overset{<>}{\mathbf{I}}) = 0$$

with $\overset{>}{\square}\cdot\overset{<>}{\mathbf{I}} = 0$, the foregoing equation gives

$$\overset{>}{\square}\cdot(\jmath\overset{><}{\mathbf{F}})\cdot\overset{<>}{\mathbf{I}} = 0.$$

Also because $\overset{<>}{\mathbf{I}}$ is non-zero,

$$\overset{>}{\square}\cdot(\jmath\overset{><}{\mathbf{F}}) = 0.$$

Therefore, the component expression is given by

$$(\mathscr{J} x^i_{;I})_{:i} = 0.$$

This theorem is proved. ■

3.1.3. Green-Cauchy strain tensors and engineering strain

From Definition 3.10, for the two vectors $\mathbf{P}$ in $\mathfrak{B}$ and $\mathbf{p}$ in $\mathfrak{b}$,

$$d\mathbf{p} = \overset{><}{\mathbf{F}}\cdot d\mathbf{P} \text{ and } d\mathbf{P} = \overset{<>}{\mathbf{F}}{}^{-1}\cdot d\mathbf{p} \tag{3.43}$$

and from Definition 3.7, the *displacement vectors* are

$$\overset{<}{\mathbf{u}} = U^I\mathbf{G}_I = g_i^I u^i \mathbf{G}_I \;\text{ and }\; \overset{>}{\mathbf{u}} = u^i\mathbf{g}_i = g_I^i U^I \mathbf{g}_i. \tag{3.44}$$

If the positions are described by

$$\overset{<}{\mathbf{p}} = \overset{<}{\mathbf{P}} + \overset{<}{\mathbf{u}} = (X^I + U^I)\mathbf{G}_I \;\text{ and }\; \overset{>}{\mathbf{P}} = \overset{>}{\mathbf{p}} - \overset{>}{\mathbf{u}} = (x^i - u^i)\mathbf{g}_i, \tag{3.45}$$

then the covariant *Green and Cauchy* base vectors are

$$\mathbf{C}_I = \frac{\partial \overset{<}{\mathbf{p}}}{\partial X^I} = \mathbf{G}_I + U^M_{;I}\mathbf{G}_M \;\text{ and }\; \mathbf{c}_i = \frac{\partial \overset{>}{\mathbf{P}}}{\partial x^i} = \mathbf{g}_i - u^m{}_{;i}\mathbf{g}_m. \tag{3.46}$$

$$\begin{aligned} C_{IJ} &= \mathbf{C}_I\cdot\mathbf{C}_J = G_{IJ} + G_{MJ}U^M_{;I} + G_{IN}U^N_{;J} + G_{MN}U^M_{;I}U^N_{;J} \\ &= G_{IJ} + U_{J;I} + U_{I;J} + U_{M;I}U^M_{;J}, \\ c_{ij} &= \mathbf{c}_i\cdot\mathbf{c}_j = g_{ij} - g_{mj}u^m_{;i} - g_{in}u^n_{;j} + g_{mn}u^m_{;i}u^n_{;j} \\ &= g_{ij} - u_{j;i} - u_{i;j} + u_{m;i}u^m_{;j}. \end{aligned} \tag{3.47}$$

$$\begin{aligned} \overset{<<}{\mathbf{C}} &= C_{IJ}\mathbf{G}^I\mathbf{G}^J = (G_{IJ} + U_{J;I} + U_{I;J} + U_{M;I}U^M_{;J})\mathbf{G}^I\mathbf{G}^J, \\ \overset{>>}{\mathbf{c}} &= c_{ij}\mathbf{g}^i\mathbf{g}^j = (g_{ij} - u_{j;i} - u_{i;j} + u_{m;i}u^m_{;j})\mathbf{g}^i\mathbf{g}^j. \end{aligned} \tag{3.48}$$

From Eqs.(3.8) and (3.13), the squares of the infinitesimal lengths are

$$\begin{aligned} dS^2 &= G_{IJ}dX^I\mathrm{d}X^J = c_{ij}dx^idx^j, \\ ds^2 &= C_{IJ}dX^IdX^J = g_{ij}dx^idx^j. \end{aligned} \tag{3.49}$$

Definition 3.13. For two position vectors $\mathbf{P}$ in $\mathfrak{B}$ and $\mathbf{p}$ in $\mathfrak{b}$, the difference between the square of the initial and deformed infinitesimal lengths is

$$ds^2 - dS^2 = 2E_{IJ}dX^IdX^J = 2e_{ij}dx^idx^j, \tag{3.50}$$

where two new quantities

$$E_{IJ}=\frac{1}{2}(C_{IJ}-G_{IJ})\text{ and }e_{ij}=\frac{1}{2}(g_{ij}-c_{ij}),$$
$$\overset{<<}{\mathbf{E}}=\frac{1}{2}(\overset{<<}{\mathbf{C}}-\overset{<<}{\mathbf{I}})\text{ and }\overset{>>}{\mathbf{e}}=\frac{1}{2}(\overset{>>}{\mathbf{I}}-\overset{>>}{\mathbf{c}}),\tag{3.51}$$

are called the *Lagrangian and Eulerian strain tensors* accordingly.

Theorem 3.6. *For two position vectors* $\mathbf{P}$ *in* $\mathfrak{B}$ *and* $\mathbf{p}$ *in* $\mathfrak{b}$*, there is a displacement* $\mathbf{u}$*. The Lagrangian and Eulerian strain tensors are computed by*

$$E_{IJ}=\frac{1}{2}(U_{J;I}+U_{I;J}+U_{M;I}U^{M}_{;J})\text{ and }e_{ij}=\frac{1}{2}(u_{j;i}+u_{i;j}-u_{m;i}u^{m}_{;j}).\tag{3.52}$$

Proof: With Eq.(3.47), definitions in Eq.(3.51) gives

$$E_{IJ}=\frac{1}{2}(C_{IJ}-G_{IJ})=\frac{1}{2}(U_{J;I}+U_{I;J}+U_{M;I}U^{M}_{;J}),$$
$$e_{ij}=\frac{1}{2}(g_{ij}-c_{ij})=\frac{1}{2}(u_{j;i}+u_{i;j}-u_{m;i}u^{m}_{;j}).$$

This theorem is proved. ■

From the definitions of the Lagrangian and Eulerian strain tensors, both strain tensors are not strain tensors in the physical sense. Both of the two strain tensors are two geometric quantities similar to the Cauchy and Green deformation tensors. In the physical sense, the strain is measured based on the changes of the length and angle. The normal strain is relative to the elongation and stretch or extension of an element length. The shear strain is based on the angle change between the two line elements. Therefore, such concepts should be discussed.

Definition 3.14. For two position vectors $\mathbf{P}$ in $\mathfrak{B}$ and $\mathbf{p}$ in $\mathfrak{b}$, the unit vectors of the infinitesimal vectors $d\mathbf{P}$ and $d\mathbf{p}$ are defined as

$$\mathbf{N}=\frac{d\mathbf{P}}{|d\mathbf{P}|}=N^{I}\mathbf{G}_{I}\text{ and }\mathbf{n}=\frac{d\mathbf{p}}{|d\mathbf{p}|}=n^{i}\mathbf{g}_{i},\tag{3.53}$$

where

$$N^{I}=\frac{dX^{I}}{|d\mathbf{P}|}=\frac{dX^{I}}{dS}=\frac{dX^{I}}{\sqrt{G_{MN}dX^{M}dX^{N}}},$$
$$n^{i}=\frac{dx^{i}}{|d\mathbf{p}|}=\frac{dx^{i}}{ds}=\frac{dx^{i}}{\sqrt{g_{mn}dx^{m}dx^{n}}}.\tag{3.54}$$

In the Cartesian coordinate system, both of the two quantities in the foregoing definition are the *directional cosines*. For the curvilinear coordinate systems, the directional cosine will be discussed later.

Definition 3.15. For two position vectors **P** in $\mathfrak{B}$ and **p** in $\mathfrak{b}$, the *elongation along the direction* **N** is defined as

$$\varepsilon_{\mathbf{N}} = \frac{|d\mathbf{p}|\cos\theta_{(\mathbf{N},\mathbf{n})} - |d\mathbf{P}|}{|d\mathbf{P}|} = \frac{d\mathbf{p}\cdot d\mathbf{P}}{|d\mathbf{P}|^2} - 1, \tag{3.55}$$

with

$$\varepsilon_{\mathbf{N}} = x^m_{;I} g^I_m N^I N^J - 1 = (G_{IJ} + U^I_{;J}) N^I N^J - 1. \tag{3.56}$$

Definition 3.16. For two position vectors **P** in $\mathfrak{B}$ and **p** in $\mathfrak{b}$, the *elongation along the direction* **n** is defined as

$$\varepsilon_{\mathbf{n}} = \frac{|d\mathbf{p}| - |d\mathbf{P}|\cos\theta_{(\mathbf{N},\mathbf{n})}}{|d\mathbf{p}|} = 1 - \frac{d\mathbf{p}\cdot d\mathbf{P}}{|d\mathbf{p}|^2}, \tag{3.57}$$

with

$$\varepsilon_{\mathbf{n}} = 1 - X^M_{;i} g^i_M n^i n^j = 1 - (g_{ij} - u^i_{ij}) n^i n^j. \tag{3.58}$$

Definition 3.17. For two position vectors **P** in $\mathfrak{B}$ and **p** in $\mathfrak{b}$, the ratio of the lengths of vectors $d\mathbf{P}$ and $d\mathbf{p}$ is called *the stretch* ($\Lambda_{\mathbf{N}} = \lambda_{\mathbf{n}}$), defined by

$$\begin{aligned}\Lambda_{\mathbf{N}} &= \frac{|d\mathbf{p}|}{|d\mathbf{P}|} = \sqrt{\frac{d\mathbf{P}}{|d\mathbf{P}|}\cdot \overset{\ll}{\mathbf{C}} \cdot \frac{d\mathbf{P}}{|d\mathbf{P}|}} = \sqrt{\mathbf{N}\cdot \overset{\ll}{\mathbf{C}}\cdot \mathbf{N}} \\ &= \sqrt{C_{IJ} N^I N^J},\end{aligned} \tag{3.59}$$

$$\begin{aligned}\lambda_{\mathbf{n}} &= \frac{|d\mathbf{p}|}{|d\mathbf{P}|} = \frac{1}{\sqrt{\frac{d\mathbf{p}}{|d\mathbf{p}|}\cdot \overset{\gg}{\mathbf{c}} \cdot \frac{d\mathbf{p}}{|d\mathbf{p}|}}} = \frac{1}{\sqrt{\mathbf{n}\cdot \overset{\gg}{\mathbf{c}}\cdot \mathbf{n}}} \\ &= \frac{1}{\sqrt{c_{ij} n^i n^j}}.\end{aligned} \tag{3.60}$$

Definition 3.18. For two position vectors **P** in $\mathfrak{B}$ and **p** in $\mathfrak{b}$, the *extension* ($E_{\mathbf{N}} = e_{\mathbf{n}}$) between two vectors $d\mathbf{P}$ and $d\mathbf{p}$ is defined as

$$E_{\mathbf{N}} = \frac{|d\mathbf{p}| - |d\mathbf{P}|}{|d\mathbf{P}|} = \Lambda_{\mathbf{N}} - 1 = \sqrt{C_{IJ} N^I N^J} - 1, \tag{3.61}$$

$$e_{\mathbf{n}} = \frac{|d\mathbf{p}| - |d\mathbf{P}|}{|d\mathbf{P}|} = \lambda_{\mathbf{n}} - 1 = \frac{1}{\sqrt{c_{ij} n^i n^j}} - 1. \tag{3.62}$$

Definition 3.19. For two unit vectors in $\mathfrak{B}$ of

$$\underset{1}{\mathbf{N}} = \frac{d\underset{1}{\mathbf{P}}}{|d\underset{1}{\mathbf{P}}|} \text{ and } \underset{2}{\mathbf{N}} = \frac{d\underset{2}{\mathbf{P}}}{|d\underset{2}{\mathbf{P}}|}, \tag{3.63}$$

the angle between the two vectors ($\Theta_{(\underset{1}{\mathbf{N}},\underset{2}{\mathbf{N}})}$) is defined as

$$\cos\Theta_{(\underset{1}{\mathbf{N}},\underset{2}{\mathbf{N}})} = \frac{d\underset{1}{\mathbf{P}}\cdot d\underset{2}{\mathbf{P}}}{|d\underset{1}{\mathbf{P}}|\cdot|d\underset{2}{\mathbf{P}}|}. \tag{3.64}$$

For two unit vectors in $\mathfrak{b}$ of

$$\underset{1}{\mathbf{n}} = \frac{d\underset{1}{\mathbf{p}}}{|d\underset{1}{\mathbf{p}}|} \text{ and } \underset{2}{\mathbf{n}} = \frac{d\underset{2}{\mathbf{p}}}{|d\underset{2}{\mathbf{p}}|}, \tag{3.65}$$

the *angle* between the two vectors ($\theta_{(\underset{1}{\mathbf{n}},\underset{2}{\mathbf{n}})}$) is defined as

$$\cos\theta_{(\underset{1}{\mathbf{n}},\underset{2}{\mathbf{n}})} = \frac{d\underset{1}{\mathbf{p}}\cdot d\underset{2}{\mathbf{p}}}{|d\underset{1}{\mathbf{p}}|\cdot|d\underset{2}{\mathbf{p}}|}. \tag{3.66}$$

The *shear strain in the plane of* $\underset{1}{\mathbf{N}}$ and $\underset{2}{\mathbf{N}}$ is defined as

$$\Gamma_{(\underset{1}{\mathbf{N}},\underset{2}{\mathbf{N}})} \equiv \gamma_{(\underset{1}{\mathbf{n}},\underset{2}{\mathbf{n}})} = \Theta_{(\underset{1}{\mathbf{N}},\underset{2}{\mathbf{N}})} - \theta_{(\underset{1}{\mathbf{n}},\underset{2}{\mathbf{n}})}. \tag{3.67}$$

Theorem 3.7. *For two unit vectors in* $\mathfrak{B}$, *the angle between the two vectors* ($\Theta_{(\underset{1}{\mathbf{N}},\underset{2}{\mathbf{N}})}$) *is determined by*

$$\cos\Theta_{(\underset{1}{\mathbf{N}},\underset{2}{\mathbf{N}})} = \frac{G_{IJ}\, d\underset{1}{X^I}\, d\underset{2}{X^J}}{|d\underset{1}{\mathbf{P}}|\cdot|d\underset{2}{\mathbf{P}}|} = G_{IJ}\, \underset{1}{N^I}\, \underset{2}{N^J}. \tag{3.68}$$

For two unit vectors in $\mathfrak{b}$, *the angle between the two vectors* ($\theta_{(\underset{1}{\mathbf{n}},\underset{2}{\mathbf{n}})}$) *is determined by*

$$\cos\theta_{(\underset{1}{\mathbf{n}},\underset{2}{\mathbf{n}})} = \frac{1}{\Lambda_{\underset{1}{\mathbf{N}}}\Lambda_{\underset{2}{\mathbf{N}}}} \underset{1}{\mathbf{N}}\cdot \overset{\ll}{\mathbf{C}}\cdot \underset{2}{\mathbf{N}} = \frac{1}{\Lambda_{\underset{1}{\mathbf{N}}}\Lambda_{\underset{2}{\mathbf{N}}}} C_{IJ}\, \underset{1}{N^I}\, \underset{2}{N^J}. \tag{3.69}$$

Proof : From definition, the angle between the two vectors ($\Theta_{(\underset{1}{\mathbf{N}},\underset{2}{\mathbf{N}})}$) is computed by

$$\cos\Theta_{(\underset{1}{\mathbf{N}},\underset{2}{\mathbf{N}})} = \frac{d\underset{1}{\mathbf{P}}\cdot d\underset{2}{\mathbf{P}}}{|d\underset{1}{\mathbf{P}}|\cdot|d\underset{2}{\mathbf{P}}|} = \frac{G_{IJ}\, d\underset{1}{X^I}\, d\underset{2}{X^J}}{|d\underset{1}{\mathbf{P}}|\cdot|d\underset{2}{\mathbf{P}}|} = G_{IJ}\, \underset{1}{N^I}\, \underset{2}{N^J}.$$

The angle between the two vectors ($\theta_{(\underset{1}{\mathbf{n}},\underset{2}{\mathbf{n}})}$) is computed by

$$\begin{aligned}
\cos\theta_{(\underset{1}{\mathbf{n}},\underset{2}{\mathbf{n}})} &= \frac{\underset{1}{d\mathbf{p}}\cdot\underset{2}{d\mathbf{p}}}{|\underset{1}{d\mathbf{p}}|\cdot|\underset{2}{d\mathbf{p}}|} = \frac{(\overset{\ll}{\mathbf{F}}\cdot\underset{1}{d\mathbf{P}})\cdot(\overset{\ll}{\mathbf{F}}\cdot\underset{2}{d\mathbf{P}})}{\Lambda_{\underset{1}{\mathbf{N}}}\Lambda_{\underset{2}{\mathbf{N}}}\,|\underset{1}{d\mathbf{P}}|\cdot|\underset{2}{d\mathbf{P}}|}\\
&= \frac{1}{\Lambda_{\underset{1}{\mathbf{N}}}\Lambda_{\underset{2}{\mathbf{N}}}}\underset{1}{\mathbf{N}}\cdot(\overset{\ll}{\mathbf{F}})^{\mathrm{T}}\cdot\overset{\ll}{\mathbf{F}}\cdot\underset{2}{\mathbf{N}} = \frac{1}{\Lambda_{\underset{1}{\mathbf{N}}}\Lambda_{\underset{2}{\mathbf{N}}}}\underset{1}{\mathbf{N}}\cdot\overset{\ll}{\mathbf{C}}\cdot\underset{2}{\mathbf{N}}\\
&= \frac{1}{\Lambda_{\underset{1}{\mathbf{N}}}\Lambda_{\underset{2}{\mathbf{N}}}}C_{IJ}\,\underset{1}{N^{I}}\,\underset{2}{N^{J}}\,.
\end{aligned}$$

This theorem is proved. ■

Notice that the afore-defined extension and shear are called *the engineering strain*.

Definition 3.20. For the infinitesimal area $d\overset{<}{\mathbf{A}}$ in $\mathfrak{B}$, the unit normal vector is ${}_{A}\mathbf{N} = d\overset{<}{\mathbf{A}}/|d\overset{<}{\mathbf{A}}|$. For the infinitesimal area $d\overset{>}{\mathbf{a}}$ in $\mathfrak{b}$, *the ratio of areas* $d\overset{>}{\mathbf{a}}$ and $d\overset{<}{\mathbf{A}}$ is defined by

$$\sigma_{{}_{A}\mathbf{N}} = \frac{|d\overset{>}{\mathbf{a}}|}{|d\overset{<}{\mathbf{A}}|}. \tag{3.70}$$

Theorem 3.8. *For two infinitesimal areas* $d\overset{<}{\mathbf{A}}$ *in* $\mathfrak{B}$ *and* $d\overset{>}{\mathbf{a}}$ *in* $\mathfrak{b}$ *with the unit normal vector* ${}_{A}\mathbf{N} = d\overset{<}{\mathbf{A}}/|d\overset{<}{\mathbf{A}}|$, *the ratio of two areas* $d\overset{>}{\mathbf{a}}$ *and* $d\overset{<}{\mathbf{A}}$ *is*

$$\sigma_{{}_{A}\mathbf{N}} = \mathscr{J}\sqrt{{}_{A}\mathbf{N}\cdot\overset{-1}{\overset{\ll}{\mathbf{C}}}\cdot{}_{A}\mathbf{N}} = \mathscr{J}\sqrt{{}^{-1}C^{IJ}\;{}_{A}N_{I}\;{}_{A}N_{J}}\,. \tag{3.71}$$

Proof: From definition, the ratio of two areas is

$$\begin{aligned}
\sigma_{{}_{A}\mathbf{N}} &= \frac{|d\overset{>}{\mathbf{a}}|}{|d\overset{<}{\mathbf{A}}|} = \frac{\sqrt{d\overset{>}{\mathbf{a}}\cdot d\overset{>}{\mathbf{a}}}}{|d\overset{<}{\mathbf{A}}|}\\
&= \frac{\sqrt{(\mathscr{J}\,(\overset{-1}{\overset{\ll}{\mathbf{F}}})^{\mathrm{T}}\cdot d\overset{<}{\mathbf{A}})\cdot(\mathscr{J}\,(\overset{-1}{\overset{\ll}{\mathbf{F}}})^{\mathrm{T}}\cdot d\overset{<}{\mathbf{A}})}}{|d\overset{<}{\mathbf{A}}|}\\
&= \mathscr{J}\sqrt{{}_{A}\mathbf{N}\cdot\overset{-1}{\overset{\ll}{\mathbf{F}}}\cdot(\overset{-1}{\overset{\ll}{\mathbf{F}}})^{\mathrm{T}}\cdot{}_{A}\mathbf{N}} = \mathscr{J}\sqrt{{}_{A}\mathbf{N}\cdot\overset{-1}{\overset{\ll}{\mathbf{C}}}\cdot{}_{A}\mathbf{N}}\\
&= \mathscr{J}\sqrt{{}^{-1}C^{IJ}\;{}_{A}N_{I}\;{}_{A}N_{J}}\,.
\end{aligned}$$

This theorem is proved. ■

Definition 3.21. For the infinitesimal area $d\overset{>}{\mathbf{a}}$ in $\mathfrak{b}$, the unit normal vector is ${}_a\mathbf{n} = d\overset{>}{\mathbf{a}} / |d\overset{>}{\mathbf{a}}|$. For the infinitesimal area $d\overset{<}{\mathbf{A}}$ in $\mathfrak{B}$, *the ratio of areas* $d\overset{>}{\mathbf{a}}$ and $d\overset{<}{\mathbf{A}}$ is defined by

$$\sigma_{{}_a\mathbf{n}} = \frac{|d\overset{>}{\mathbf{a}}|}{|d\overset{<}{\mathbf{A}}|}. \tag{3.72}$$

Theorem 3.9. *For two infinitesimal areas $d\overset{<}{\mathbf{A}}$ in $\mathfrak{B}$ and $d\overset{>}{\mathbf{a}}$ in $\mathfrak{b}$ with the unit normal vector ${}_a\mathbf{n} = d\overset{>}{\mathbf{a}} / |d\overset{>}{\mathbf{a}}|$, the ratio of areas $d\overset{>}{\mathbf{a}}$ and $d\overset{<}{\mathbf{A}}$ is*

$$\sigma_{{}_a\mathbf{n}} = \frac{1}{\mathscr{J}\sqrt{{}_a\mathbf{n}\cdot \overset{-1}{\overset{<<}{\mathbf{c}}} \cdot {}_a\mathbf{n}}} = \frac{1}{\mathscr{J}\sqrt{{}^{-1}c^{ij}\, {}_a n_i\, {}_a n_j}} \tag{3.73}$$

Proof: From definition, the ratio of two areas is

$$\sigma_{{}_a\mathbf{n}} = \frac{|d\overset{>}{\mathbf{a}}|}{|d\overset{<}{\mathbf{A}}|} = \frac{|d\overset{>}{\mathbf{a}}|}{\sqrt{d\overset{<}{\mathbf{A}}\cdot d\overset{<}{\mathbf{A}}}} = \frac{|d\overset{>}{\mathbf{a}}|}{\sqrt{(\mathscr{J}(\overset{>>}{\mathbf{F}})^{\mathrm{T}}\cdot d\overset{>}{\mathbf{a}})\cdot(\mathscr{J}(\overset{>>}{\mathbf{F}})^{\mathrm{T}}\cdot d\overset{>}{\mathbf{a}})}}$$

$$= \left[\mathscr{J}\sqrt{((\overset{>>}{\mathbf{F}})^{\mathrm{T}}\cdot {}_a\mathbf{n})\cdot((\overset{>>}{\mathbf{F}})^{\mathrm{T}}\cdot {}_a\mathbf{n})}\right]^{-1} = \left[\mathscr{J}\sqrt{{}_a\mathbf{n}\cdot \overset{>>}{\mathbf{F}}\cdot(\overset{>>}{\mathbf{F}})^{\mathrm{T}}\cdot {}_a\mathbf{n}}\right]^{-1}$$

$$= \left[\mathscr{J}\sqrt{{}_a\mathbf{n}\cdot \overset{-1}{\overset{>>}{\mathbf{c}}}\cdot {}_a\mathbf{n}}\right]^{-1} = (\mathscr{J}\sqrt{{}^{-1}c^{ij}\, {}_a n_i\, {}_a n_j})^{-1}.$$

This theorem is proved. ■

Definition 3.22. For the infinitesimal volume dV in $\mathfrak{B}$ and dv in $\mathfrak{b}$, *the ratio of volume* is defined by

$$\sigma_V = \frac{dv}{dV} = \frac{[d\underset{1}{\mathbf{p}}\, d\underset{2}{\mathbf{p}}\, d\underset{3}{\mathbf{p}}]}{[d\underset{1}{\mathbf{P}}\, d\underset{2}{\mathbf{P}}\, d\underset{3}{\mathbf{P}}]}. \tag{3.74}$$

Theorem 3.10. *For the infinitesimal volume dV in $\mathfrak{B}$ and dv in $\mathfrak{b}$, the ratio of volume is determined by*

$$\sigma_V = \frac{dv}{dV} = \mathrm{III}(\overset{<<}{\mathbf{F}}) = \sqrt{\frac{g}{G}}\,|x^i_{;I}| = \mathscr{J}. \tag{3.75}$$

Proof: From definition, the ratio of the volume is

$$\sigma_V = \frac{dv}{dV} = \frac{[\underset{1}{d\mathbf{p}}\,\underset{2}{d\mathbf{p}}\,\underset{3}{d\mathbf{p}}]}{[\underset{1}{d\mathbf{P}}\,\underset{2}{d\mathbf{P}}\,\underset{3}{d\mathbf{P}}]} = \frac{(\overset{\ll}{\mathbf{F}}\cdot \underset{1}{d\mathbf{P}})\times(\overset{\ll}{\mathbf{F}}\cdot \underset{2}{d\mathbf{P}})\cdot(\overset{\ll}{\mathbf{F}}\cdot \underset{3}{d\mathbf{P}})}{\underset{1}{d\mathbf{P}}\times \underset{2}{d\mathbf{P}}\cdot \underset{3}{d\mathbf{P}}} = \mathrm{III}(\overset{\ll}{\mathbf{F}})$$

$$= |\, g_r^I x_{,J}^r \,| = |\, g_r^I \,||\, x_{,J}^r \,| = \sqrt{\frac{g}{G}}\,|\, x_{,I}^i \,| = \mathscr{J}.$$

This theorem is proved. ■

3.1.4. Principal strains and directions

At any point in $\mathfrak{B}$, the strain tensors can be determined by the Green-Cauchy strain tensors. In different directions, the strain of line element is different. To determine the extremum values of the strain, consider the minimization of the extension $E_{\mathbf{N}}$ by the following equation:

$$\Lambda_{\mathbf{N}}^2 = C_{IJ} N^I N^J \tag{3.76}$$

with the initial extension condition of $E_{(\mathbf{N})} = 0$ (or $\Lambda_{(\mathbf{N})} = 1$), i.e.,

$$G_{IJ} N^I N^J - 1 = 0. \tag{3.77}$$

It is found that the extremum extension is to determine the extremum stretch. Such a problem is a conditional extremum problem. Langrange's method of multipliers can be adopted. That is, one has a function

$$\mathfrak{F} = \Lambda_{(\mathbf{N})}^2 - \Lambda(G_{IJ} N^I N^J - 1) = C_{IJ} N^I N^J - \Lambda(G_{IJ} N^I N^J - 1). \tag{3.78}$$

Thus,

$$\frac{\partial}{\partial N^K}\left[C_{IJ} N^I N^J - \Lambda(G_{IJ} N^I N^J - 1)\right] = 0, \tag{3.79}$$

where Λ is an unknown Lagrange multiplier.

$$(C_{IJ} - \Lambda G_{IJ}) N^J = 0, \tag{3.80}$$

which can be expressed by

$$(C_{\cdot J}^I - \Lambda \delta_J^I) N^J = 0 \tag{3.81}$$

or

$$(\overset{\ll}{\mathbf{C}} - \Lambda \overset{\ll}{\mathbf{I}})\cdot \mathbf{N} = 0. \tag{3.82}$$

The non-trivial solution requires

$$|\, C_{\cdot J}^I - \Lambda \delta_J^I \,| = 0 \ \text{ or } \ \det(\overset{\ll}{\mathbf{C}} - \Lambda \overset{\ll}{\mathbf{I}}) = 0. \tag{3.83}$$

The foregoing equation gives

$$-\Lambda^3 + \mathrm{I}(\overset{<<}{\mathbf{C}})\Lambda^2 + \mathrm{II}(\overset{<<}{\mathbf{C}})\Lambda + \mathrm{III}(\overset{<<}{\mathbf{C}}) = 0, \tag{3.84}$$

where

$$\begin{aligned} &\mathrm{I}(\overset{<<}{\mathbf{C}}) = \frac{1}{1!}\delta_I^J C_{\cdot J}^I = C_{\cdot I}^I, \quad \mathrm{II}(\overset{<<}{\mathbf{C}}) = \frac{1}{2!}\delta_{IK}^{JL} C_{\cdot J}^I C_{\cdot L}^K, \\ &\mathrm{III}(\overset{<<}{\mathbf{C}}) = \frac{1}{3!}\delta_{IKM}^{JLN} C_{\cdot J}^I C_{\cdot L}^K C_{\cdot N}^M = |\, C_{\cdot J}^I \,|. \end{aligned} \tag{3.85}$$

From Eq.(3.84), three principal values Λ_Γ ($\Gamma = 1, 2, 3$) with $\Lambda_1 \geq \Lambda_2 \geq \Lambda_3$ are obtained. Because $\overset{<<}{\mathbf{C}}$ is positive-definite, $\Lambda_\Gamma > 0$ ($\Gamma = 1, 2, 3$), and the corresponding principal directions $\underset{\Gamma}{\mathbf{N}}$ are orthogonal, which are the principal directions of the Green-strain tensor. Such orthogonality of the principal directions can be proved as follows.

$$\overset{<<}{\mathbf{C}}\cdot\underset{1}{\mathbf{N}} = \Lambda_1 \underset{1}{\mathbf{N}} \quad\text{and}\quad \overset{<<}{\mathbf{C}}\cdot\underset{2}{\mathbf{N}} = \Lambda_2 \underset{2}{\mathbf{N}}. \tag{3.86}$$

Left multiplication of $\underset{2}{\mathbf{N}}$ in the first equation of Eq.(3.86) gives

$$\underset{2}{\mathbf{N}}\cdot\overset{<<}{\mathbf{C}}\cdot\underset{1}{\mathbf{N}} = \Lambda_1 \underset{2}{\mathbf{N}}\cdot\underset{1}{\mathbf{N}} \quad\Rightarrow\quad \underset{1}{\mathbf{N}}\cdot(\overset{<<}{\mathbf{C}})^{\mathrm{T}}\cdot\underset{2}{\mathbf{N}} = \Lambda_1 \underset{1}{\mathbf{N}}\cdot\underset{2}{\mathbf{N}}\ . \tag{3.87}$$

Because $\overset{<<}{\mathbf{C}}$ is symmetric,

$$\underset{1}{\mathbf{N}}\cdot\overset{<<}{\mathbf{C}}\cdot\underset{2}{\mathbf{N}} = \Lambda_1 \underset{1}{\mathbf{N}}\cdot\underset{2}{\mathbf{N}} \quad\text{but}\quad \underset{1}{\mathbf{N}}\cdot\overset{<<}{\mathbf{C}}\cdot\underset{2}{\mathbf{N}} = \Lambda_2 \underset{1}{\mathbf{N}}\cdot\underset{2}{\mathbf{N}}. \tag{3.88}$$

The difference between the two equations gives

$$0 = (\Lambda_1 - \Lambda_2)\underset{1}{\mathbf{N}}\cdot\underset{2}{\mathbf{N}}. \tag{3.89}$$

If $\Lambda_1 \neq \Lambda_2$, the foregoing equation gives

$$0 = \underset{1}{\mathbf{N}}\cdot\underset{2}{\mathbf{N}}. \tag{3.90}$$

The two principal directional vectors are perpendicular. The principal directions are also called the *spatial strain directions*. Based on the three principal directions, the three principal stretches are computed by

$$\Lambda_{\underset{\Gamma}{\mathbf{N}}} = \sqrt{\underset{\Gamma}{\mathbf{N}}\cdot\overset{<<}{\mathbf{C}}\cdot\underset{\Gamma}{\mathbf{N}}} = \sqrt{\Lambda_\Gamma \underset{\Gamma}{\mathbf{N}}\cdot\underset{\Gamma}{\mathbf{N}}} = \sqrt{\Lambda_\Gamma}. \tag{3.91}$$

From Chapter 2,

$$\overset{1/2}{\overset{<<}{\mathbf{C}}}\cdot\underset{\Gamma}{\mathbf{N}} = \Lambda_{\underset{\Gamma}{\mathbf{N}}}\underset{\Gamma}{\mathbf{N}} \quad\text{or}\quad (\overset{1/2}{\overset{<<}{\mathbf{C}}} - \overset{<<}{\mathbf{I}})\cdot\underset{\Gamma}{\mathbf{N}} = E_{\underset{\Gamma}{\mathbf{N}}}\underset{\Gamma}{\mathbf{N}}, \tag{3.92}$$

where

$$E_{\underset{\Gamma}{\mathbf{N}}} = \Lambda_{\underset{\Gamma}{\mathbf{N}}} - 1 = \sqrt{\Lambda_\Gamma} - 1. \tag{3.93}$$

In a similar fashion, consider the minimization of the extension $e_{\mathbf{n}}$ by the following equation.

$$\lambda_{\mathbf{n}}^{-2} = c_{ij} n^i n^j \tag{3.94}$$

with the initial extension condition of $e_{\mathbf{n}} = 0$ (or $\lambda_{\mathbf{n}} = 1$), i.e.,

$$g_{ij} n^i n^j - 1 = 0. \tag{3.95}$$

The three principal values of the Cauchy strain tensor are λ_Γ ($\Gamma = 1, 2, 3$) with $\lambda_1 \geq \lambda_2 \geq \lambda_3$, and the corresponding, orthogonal principal directions are $\underset{\Gamma}{\mathbf{n}}$. The principal directions are called *the material strain directions*. The principal stretch is computed by

$$\lambda_{\underset{\Gamma}{\mathbf{n}}} = \frac{1}{\sqrt{\underset{\Gamma}{\mathbf{n}} \cdot \overset{>>}{\mathbf{c}} \cdot \underset{\Gamma}{\mathbf{n}}}} = \frac{1}{\sqrt{\lambda_\Gamma \underset{\Gamma}{\mathbf{n}} \cdot \underset{\Gamma}{\mathbf{n}}}} = \frac{1}{\sqrt{\lambda_\Gamma}}. \tag{3.96}$$

Further,

$$\overset{\overset{-1/2}{>>}}{\mathbf{c}} \cdot \underset{\Gamma}{\mathbf{n}} = \lambda_{\underset{\Gamma}{\mathbf{n}}} \underset{\Gamma}{\mathbf{n}} \quad \text{or} \quad (\overset{\overset{-1/2}{>>}}{\mathbf{c}} - \overset{>>}{\mathbf{I}}) \cdot \underset{\Gamma}{\mathbf{n}} = e_{\underset{\Gamma}{\mathbf{n}}} \underset{\Gamma}{\mathbf{n}}, \tag{3.97}$$

where

$$e_{\underset{\Gamma}{\mathbf{n}}} = \lambda_{\underset{\Gamma}{\mathbf{n}}} - 1 = \frac{1}{\sqrt{\lambda_\Gamma}} - 1. \tag{3.98}$$

Definition 3.23. For the infinitesimal vector $d\mathbf{p}$ in $\mathfrak{b}$, the following relation at point $\mathbf{p}$ holds:

$$c_{ij} dx^i dx^j = K^2, \tag{3.99}$$

which is called the *material strain ellipsoid* if for the infinitesimal vector $d\mathbf{P}$ in $\mathfrak{B}$,

$$G_{IJ} dX^I dX^J = K^2 \text{ at point } \mathbf{P}. \tag{3.100}$$

For the infinitesimal vector $d\mathbf{P}$, there is a relation at point $\mathbf{P}$ as

$$C_{IJ} dX^I dX^J = k^2, \tag{3.101}$$

which is called the *spatial strain ellipsoid* if for the infinitesimal vector $d\mathbf{p}$ in $\mathfrak{b}$

$$g_{ij} dx^i dx^j = k^2 \text{ at point } \mathbf{p}. \tag{3.102}$$

Theorem 3.11. *If two infinitesimal vectors* $d\underset{1}{\mathbf{P}}$ *and* $d\underset{2}{\mathbf{P}}$ *in* $\mathfrak{B}$ *are perpendicular at point* $\mathbf{P}$, *i.e.,*

$$d\underset{1}{\mathbf{P}} \cdot d\underset{2}{\mathbf{P}} = 0 \text{ or } G_{IJ}\, d\underset{1}{X^I}\, d\underset{2}{X^J} = 0, \tag{3.103}$$

then after deformation at point $\mathbf{p}$ *in* $\mathfrak{b}$*, the following equations exist*:

$$\underset{1}{d\mathbf{p}}\cdot(\mathbf{c}\cdot\underset{2}{d\mathbf{p}})=0 \text{ or } \underset{1}{dx^i}(\mathrm{c}_{ij}\underset{2}{dx^j})=0;$$
$$(\mathbf{c}\cdot\underset{1}{d\mathbf{p}})\cdot\underset{2}{d\mathbf{p}}=0 \text{ or } (\mathrm{c}_{ij}\underset{1}{dx^i})\underset{2}{dx^j}=0. \tag{3.104}$$

Proof: This theorem can be easily proved by the definition, i.e.,

$$dS^2=\underset{1}{d\mathbf{P}}\cdot\underset{2}{d\mathbf{P}}=G_{IJ}\underset{1}{dX^I}\underset{2}{dX^J}=G_{IJ}X^I_{;i}X^J_{;j}\underset{1}{dx^i}\underset{2}{dx^j}=\mathrm{c}_{ij}\underset{1}{dx^i}\underset{2}{dx^j}.$$

If $\underset{1}{d\mathbf{P}}\cdot\underset{2}{d\mathbf{P}}=0$ or $G_{IJ}\underset{1}{dX^I}\underset{2}{dX^J}=0$,

$$(\mathrm{c}_{ij}\underset{1}{dx^i})\underset{2}{dx^j}=\underset{1}{dx^i}(\mathrm{c}_{ij}\underset{2}{dx^j})=0.$$

In other words,

$$(\overset{>>}{\mathbf{c}}\cdot\underset{1}{d\mathbf{p}})\cdot\underset{2}{d\mathbf{p}}=(\underset{1}{d\mathbf{p}})\cdot(\overset{>>}{\mathbf{c}}\cdot\underset{2}{d\mathbf{p}})=0.$$

This theorem is proved. ■

Theorem 3.12. *If the deformed infinitesimal vector of* $d\mathbf{P}$ *in* $\mathfrak{B}$ *is an infinitesimal vector* $d\mathbf{p}$ *in* $\mathfrak{b}$*, then the two infinitesimal vectors can be decomposed by three motions of the translation, rotation and deformations*:

$$d\mathbf{p}=\overset{><}{\mathbf{I}}\cdot\overset{<<}{\mathbf{R}}\cdot\overset{1/2}{\overset{<<}{\mathbf{C}}}\cdot d\mathbf{P}=\overset{>>}{\mathbf{R}}\cdot\overset{><}{\mathbf{I}}\cdot\overset{1/2}{\overset{<<}{\mathbf{C}}}\cdot d\mathbf{P}=\overset{>>}{\mathbf{R}}\cdot\overset{1/2}{\overset{>>}{\mathbf{C}}}\cdot\overset{><}{\mathbf{I}}\cdot d\mathbf{P}$$
$$=\overset{><}{\mathbf{I}}\cdot\overset{-1/2}{\overset{<<}{\mathbf{c}}}\cdot\overset{<<}{\mathbf{R}}\cdot d\mathbf{P}=\overset{-1/2}{\overset{>>}{\mathbf{c}}}\cdot\overset{><}{\mathbf{I}}\cdot\overset{<<}{\mathbf{R}}\cdot d\mathbf{P}=\overset{-1/2}{\overset{>>}{\mathbf{c}}}\cdot\overset{>>}{\mathbf{R}}\cdot\overset{><}{\mathbf{I}}\cdot d\mathbf{P}; \tag{3.105}$$

and

$$d\mathbf{P}=\overset{-1/2}{\overset{<<}{\mathbf{C}}}\cdot\overset{-1}{\overset{<<}{\mathbf{R}}}\cdot\overset{<>}{\mathbf{I}}\cdot d\mathbf{p}=\overset{-1/2}{\overset{<<}{\mathbf{C}}}\cdot\overset{<>}{\mathbf{I}}\cdot\overset{-1}{\overset{>>}{\mathbf{R}}}\cdot d\mathbf{p}=\overset{<>}{\mathbf{I}}\cdot\overset{-1/2}{\overset{>>}{\mathbf{C}}}\cdot\overset{-1}{\overset{>>}{\mathbf{R}}}\cdot d\mathbf{p}$$
$$=\overset{-1}{\overset{<<}{\mathbf{R}}}\cdot\overset{1/2}{\overset{<<}{\mathbf{c}}}\cdot\overset{<>}{\mathbf{I}}\cdot d\mathbf{p}=\overset{-1}{\overset{<<}{\mathbf{R}}}\cdot\overset{<>}{\mathbf{I}}\cdot\overset{1/2}{\overset{>>}{\mathbf{c}}}\cdot d\mathbf{p}=\overset{<>}{\mathbf{I}}\cdot\overset{-1}{\overset{>>}{\mathbf{R}}}\cdot\overset{1/2}{\overset{>>}{\mathbf{c}}}\cdot d\mathbf{p}. \tag{3.106}$$

where $\mathbf{I}$, $\mathbf{R}$ *and* $\mathbf{C}$ (*or* $\mathbf{c}$) *are the translation, rigid rotation and deformation, respectively.*

Proof: From definition, the deformed infinitesimal vector is

$$d\mathbf{p}=\overset{><}{\mathbf{F}}\cdot d\mathbf{P}=\overset{><}{\mathbf{I}}\cdot\overset{<<}{\mathbf{F}}\cdot d\mathbf{P}=\overset{>>}{\mathbf{F}}\cdot\overset{><}{\mathbf{I}}\cdot d\mathbf{P}.$$

Because of $\det(\overset{<<}{\mathbf{F}})\neq 0$, the deformation gradient $\overset{<<}{\mathbf{F}}$ is a non-degenerte, second-order tensor, decomposed by the left and right polar decompositions:

$$\overset{<<}{\mathbf{F}}=(\overset{<<}{\mathbf{F}}\cdot(\overset{<<}{\mathbf{F}})^{\mathrm{T}})^{\frac{1}{2}}\cdot\overset{<<}{\mathbf{R}}=\overset{-1/2}{\overset{<<}{\mathbf{c}}}\cdot\overset{<<}{\mathbf{R}} \text{ and } \overset{<<}{\mathbf{F}}=\overset{<<}{\mathbf{R}}\cdot((\overset{<<}{\mathbf{F}})^{\mathrm{T}}\cdot\overset{<<}{\mathbf{F}})^{\frac{1}{2}}=\overset{<<}{\mathbf{R}}\cdot\overset{1/2}{\overset{<<}{\mathbf{C}}}.$$

Similarly,

$$\overset{\gg}{\mathbf{F}} = \overset{\gg}{\mathbf{c}}^{-1/2} \cdot \overset{\gg}{\mathbf{R}} \text{ and } \overset{\gg}{\mathbf{F}} = \overset{\gg}{\mathbf{R}} \cdot \overset{\gg}{\mathbf{C}}^{1/2}.$$

Substitution of the polar decompositions of the deformation gradient tensors into the definition of $d\mathbf{p}$ gives Eq.(3.105).

In a similar fashion,

$$d\mathbf{P} = \overset{\lessgtr}{\mathbf{F}}^{-1} \cdot d\mathbf{P} = \overset{\lessgtr}{\mathbf{I}} \cdot \overset{\gg}{\mathbf{F}}^{-1} \cdot d\mathbf{P} = \overset{\ll}{\mathbf{F}}^{-1} \cdot \overset{\lessgtr}{\mathbf{I}} \cdot d\mathbf{p}.$$

The tensor is decomposed by the left and right polar decompositions:

$$\overset{\gg}{\mathbf{F}}^{-1} = (\overset{\gg}{\mathbf{F}}^{-1} \cdot (\overset{\gg}{\mathbf{F}}^{-1})^{\mathrm{T}})^{\frac{1}{2}} \cdot (\overset{\gg}{\mathbf{R}})^{\mathrm{T}} = \overset{\gg}{\mathbf{C}}^{-1/2} \cdot (\overset{\gg}{\mathbf{R}})^{\mathrm{T}} \text{and } \overset{\ll}{\mathbf{F}} = (\overset{\gg}{\mathbf{R}})^{\mathrm{T}} \cdot ((\overset{\gg}{\mathbf{F}}^{-1})^{\mathrm{T}} \cdot \overset{\gg}{\mathbf{F}}^{-1})^{\frac{1}{2}} = (\overset{\gg}{\mathbf{R}})^{\mathrm{T}} \cdot \overset{\gg}{\mathbf{c}}^{1/2}$$

and

$$\overset{\ll}{\mathbf{F}}^{-1} = \overset{\ll}{\mathbf{C}}^{-1/2} \cdot (\overset{\gg}{\mathbf{R}})^{\mathrm{T}} \text{ and } \overset{\gg}{\mathbf{F}} = (\overset{\ll}{\mathbf{R}})^{\mathrm{T}} \cdot \overset{\ll}{\mathbf{c}}^{1/2}.$$

Because

$$\overset{\ll}{\mathbf{I}} = \overset{\ll}{\mathbf{F}}^{-1} \cdot \overset{\ll}{\mathbf{F}} = (\overset{\gg}{\mathbf{R}})^{\mathrm{T}} \cdot \overset{\ll}{\mathbf{c}}^{1/2} \cdot \overset{\ll}{\mathbf{c}}^{-1/2} \cdot \overset{\ll}{\mathbf{R}} = (\overset{\gg}{\mathbf{R}})^{\mathrm{T}} \cdot \overset{\ll}{\mathbf{R}},$$

one obtains

$$(\overset{\gg}{\mathbf{R}})^{\mathrm{T}} = \overset{\ll}{\mathbf{R}}^{-1}.$$

Finally,

$$\overset{\gg}{\mathbf{F}}^{-1} = \overset{\gg}{\mathbf{C}}^{-1/2} \cdot \overset{\gg}{\mathbf{R}}^{-1} = \overset{\gg}{\mathbf{R}}^{-1} \cdot \overset{\gg}{\mathbf{c}}^{1/2} \text{ and } \overset{\ll}{\mathbf{F}}^{-1} = \overset{\ll}{\mathbf{C}}^{-1/2} \cdot \overset{\gg}{\mathbf{R}}^{-1} = \overset{\ll}{\mathbf{R}}^{-1} \cdot \overset{\ll}{\mathbf{c}}^{1/2}.$$

Substitution of the foregoing equations into the definition of $d\mathbf{P}$ gives Eq.(3.106). This theorem is proved. ■

From $\overset{\ll}{\mathbf{F}} = \overset{\ll}{\mathbf{c}}^{-1/2} \cdot \overset{\ll}{\mathbf{R}}$ and $\overset{\ll}{\mathbf{F}} = \overset{\ll}{\mathbf{R}} \cdot \overset{\ll}{\mathbf{C}}^{1/2}$, the rotation tensor can be determined by

$$\overset{\ll}{\mathbf{R}} = \overset{\ll}{\mathbf{F}} \cdot \overset{\ll}{\mathbf{C}}^{-1/2} \text{ and } \overset{\gg}{\mathbf{R}} = \overset{\gg}{\mathbf{c}}^{1/2} \cdot \overset{\gg}{\mathbf{F}}. \tag{3.107}$$

Theorem 3.13. *For the infinitesimal vectors* $d\mathbf{P}$ *in* $\mathfrak{B}$ *and* $d\mathbf{p}$ *in* $\mathfrak{b}$, *if the Green and Cauchy tensors* C_{IJ} *and* c_{ij} *are nonsingular, positive-definite tensors, then their Riemann-Christoffel tensors are zero*:

$$R_{IJKL} = 0 \text{ and } R_{ijkl} = 0. \tag{3.108}$$

Proof: From the definition of the Riemann-Christoffel tensor,

$$R_{IJKL} = \frac{1}{2}(C_{IL;JK} + C_{JK;IL} - C_{IK,JL} - C_{JL;IK}) \\ + {}^{-1}C^{MN}(\Gamma_{ILM}\Gamma_{JKN} - \Gamma_{IKM}\Gamma_{JLN}) = 0,$$

where

$$\Gamma_{IJK} = \frac{1}{2}(C_{JK,I} + C_{KI,J} - C_{IJ,K}).$$

Similarly,

$$R_{ijkl} = \frac{1}{2}(c_{il;jk} + c_{jk;il} - c_{ik,jl} - c_{jl;ik}) \\ + {}^{-1}c^{rs}(\Gamma_{ilr}\Gamma_{jks} - \Gamma_{ikr}\Gamma_{jls}) = 0$$

where

$$\Gamma_{ijk} = \frac{1}{2}(c_{jk,i} + c_{ki,j} - c_{ij,k})$$

This theorem is proved. ■

The conditions in Eq.(3.108) is called *the compatibility condition*, and also is called the *integrability condition* of the following equations

$$g_{ij}x^i_{;I}x^j_{;J} = C_{IJ} \text{ and } g_{IJ}X^I_{;i}X^I_{;j} = c_{ij}. \tag{3.109}$$

3.2. Kinematics

In this section, the vectors and tensors varying with time will be discussed in the deformed configuration. The velocity and acceleration of such position vectors and deformation tensors in the deformed configuration will be discussed.

3.2.1. Material derivatives

Before discussion on material derivatives, the velocity of particles in the deformed configuration should be defined. The material description is a material particle moving to all possible spatial points, as shown in Fig.3.5. The special material point with (X^I , $I = 1, 2, 3$) is invariant. With varying time, the spatial locations of the material point changes.

Definition 3.24. For two position vectors $\mathbf{P}$ in $\mathfrak{B}$ and $\mathbf{p}$ in $\mathfrak{b}$ for time t, there is a relation

$$\mathbf{p} = \mathbf{p}(\mathbf{P}, t) \quad \text{or}\, x^i = x^i(X^I, t). \tag{3.110}$$

The time rate of change of the position of a given material particle $\mathbf{P}$ going through the spatial points $\mathbf{p}$ is called the *velocity of the particle* $\mathbf{P}$ at the location point $\mathbf{p}$, defined by

$$\begin{aligned}\mathbf{v} &\equiv \lim_{\Delta t \to 0} \frac{\mathbf{p}(\mathbf{P}, t+\Delta t) - \mathbf{p}(\mathbf{P}, t)}{\Delta t} = \left.\frac{\partial \mathbf{p}}{\partial t}\right|_{\mathbf{P}} \\ &= \mathbf{p}_{,i} \left.\frac{\partial x^i}{\partial t}\right|_{\mathbf{P}} = \left.\frac{\partial x^i}{\partial t}\right|_{\mathbf{P}} \mathbf{g}_i .\end{aligned} \tag{3.111}$$

The covariant component of the vector $\mathbf{v}(\mathbf{p}, t) = v^i \mathbf{g}_i$ at point $\mathbf{p}(\mathbf{P}, t)$ is

$$v^i = \mathbf{v} \cdot \mathbf{g}^i = \left.\frac{\partial x^i}{\partial t}\right|_{\mathbf{P}} . \tag{3.112}$$

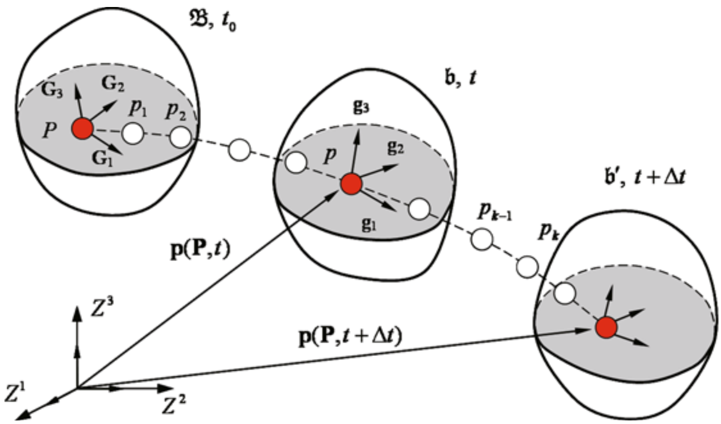

Fig. 3.5 Material description

However, the spatial description is all possible material points going through the fixed spatial point. The spatial point with (x^i, $i = 1, 2, 3$) is invariant. The material points at the fixed spatial point are switched. Such a spatial description is shown in Fig.3.6.

Definition 3.25. For two position vectors $\mathbf{P}$ in $\mathfrak{B}$ and $\mathbf{p}$ in $\mathfrak{b}$ for time t,

$$\mathbf{P} = \mathbf{P}(\mathbf{p}, t) \quad \text{or}\, X^I = X^I(x^i, t), \tag{3.113}$$

the time rate of material points P switching at the fixed location point p is called *the switching velocity of the material point* $\mathbf{P}$ at the location point $\mathbf{p}$, defined by

$$\begin{aligned}\mathbf{V} &\equiv \lim_{\Delta t\to 0}\frac{\mathbf{P}(\mathbf{p},t+\Delta t)-\mathbf{P}(\mathbf{p},t)}{\Delta t}=\left.\frac{\partial \mathbf{P}}{\partial t}\right|_{\mathbf{p}}\\ &=\mathbf{P}_{,I}\left.\frac{\partial X^I}{\partial t}\right|_{\mathbf{p}}=\left.\frac{\partial X^I}{\partial t}\right|_{\mathbf{p}}\mathbf{G}_I\equiv V^I\mathbf{G}_I.\end{aligned} \tag{3.144}$$

Notice that $\partial(\cdot)/\partial t\,|_{\mathbf{p}}$ and $\partial(\cdot)/\partial t\,|_{\mathbf{P}}$ are the partial derivatives relative to point $\mathbf{P}(X^I)$ and $\mathbf{p}(x^i)$. Based on the *two fixed variables* $(\mathbf{p}, t)$ and $(\mathbf{P}, t)$ at time *t*, the total derivatives of a vector and tensor are from the Leibniz rule, i.e.,

$$\begin{aligned}\left.\frac{\partial(\cdot)}{\partial t}\right|_{\mathbf{P}}&=\left.\frac{\partial(\cdot)}{\partial t}\right|_{\mathbf{p}}+\left.\frac{\partial x^i}{\partial t}\right|_{\mathbf{P}}\frac{\partial(\cdot)}{\partial x^i}=\left.\frac{\partial(\cdot)}{\partial t}\right|_{\mathbf{p}}+v^i\frac{\partial(\cdot)}{\partial x^i};\\ \left.\frac{\partial(\cdot)}{\partial t}\right|_{\mathbf{p}}&=\left.\frac{\partial(\cdot)}{\partial t}\right|_{\mathbf{P}}+\left.\frac{\partial X^I}{\partial t}\right|_{\mathbf{p}}\frac{\partial(\cdot)}{\partial X^I}=\left.\frac{\partial(\cdot)}{\partial t}\right|_{\mathbf{P}}+V^I\frac{\partial(\cdot)}{\partial X^I}.\end{aligned} \tag{3.115}$$

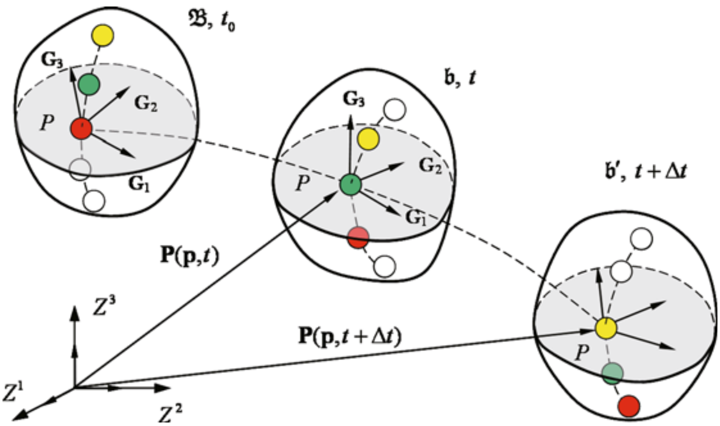

Fig. 3.6 Spatial description.

If the *variables* $(\mathbf{P}, t)$ and $(\mathbf{p}, t)$ at time *t* are used as the fixed material points, the foregoing total derivatives are called *the material derivatives*. In many textbooks, the *point* $(\mathbf{P}, t)$ at time *t* is as a fixed material point $\mathbf{P}$ to the spatial place $(\mathbf{p}, t)$ at time *t*.

Consider the fixed base vectors of the coordinates $\{X^I\}$ and $\{x^i\}$. The following relations hold:

$$\left.\frac{\partial \mathbf{G}_I}{\partial t}\right|_{\mathbf{P}} = \left.\frac{\partial \mathbf{G}^I}{\partial t}\right|_{\mathbf{P}} = \left.\frac{\partial \sqrt{G}}{\partial t}\right|_{\mathbf{P}} = 0,$$
$$\left.\frac{\partial \mathbf{g}_i}{\partial t}\right|_{\mathbf{p}} = \left.\frac{\partial \mathbf{g}^i}{\partial t}\right|_{\mathbf{p}} = \left.\frac{\partial \sqrt{g}}{\partial t}\right|_{\mathbf{p}} = 0. \tag{3.116}$$

$$\left.\frac{\partial \mathbf{G}_I}{\partial t}\right|_{\mathbf{p}} = \left.\frac{\partial \mathbf{G}_I}{\partial t}\right|_{\mathbf{P}} + V^J \frac{\partial \mathbf{G}_I}{\partial X^J} = V^J \Gamma^K_{JI} \mathbf{G}_K,$$
$$\left.\frac{\partial \mathbf{G}^I}{\partial t}\right|_{\mathbf{p}} = \left.\frac{\partial \mathbf{G}^I}{\partial t}\right|_{\mathbf{P}} + V^J \frac{\partial \mathbf{G}^I}{\partial X^J} = -V^J \Gamma^I_{JK} \mathbf{G}^K,$$
$$\left.\frac{\partial G_{IJ}}{\partial t}\right|_{\mathbf{p}} = \left.\frac{\partial (\mathbf{G}_I \cdot \mathbf{G}_J)}{\partial t}\right|_{\mathbf{p}} = V^K (\Gamma_{KIJ} + \Gamma_{KJI}),$$
$$\left.\frac{\partial \sqrt{G}}{\partial t}\right|_{\mathbf{p}} = \frac{\partial [\mathbf{G}_1 \mathbf{G}_2 \mathbf{G}_3]}{\partial t} = V^J \Gamma^I_{JI} \sqrt{G}. \tag{3.117}$$

$$\left.\frac{\partial \mathbf{g}_i}{\partial t}\right|_{\mathbf{P}} = \left.\frac{\partial \mathbf{g}_i}{\partial t}\right|_{\mathbf{p}} + v^j \frac{\partial \mathbf{g}_i}{\partial x^j} = v^j \Gamma^k_{ji} \mathbf{g}_k,$$
$$\left.\frac{\partial \mathbf{g}^i}{\partial t}\right|_{\mathbf{P}} = \left.\frac{\partial \mathbf{g}^i}{\partial t}\right|_{\mathbf{p}} + v^j \frac{\partial \mathbf{g}^i}{\partial x^j} = -v^j \Gamma^i_{jk} \mathbf{g}^k,$$
$$\left.\frac{\partial g_{ij}}{\partial t}\right|_{\mathbf{P}} = \left.\frac{\partial (\mathbf{g}_i \cdot \mathbf{g}_j)}{\partial t}\right|_{\mathbf{P}} = v^k (\Gamma_{kij} + \Gamma_{kji}),$$
$$\left.\frac{\partial \sqrt{g}}{\partial t}\right|_{\mathbf{P}} = \left.\frac{\partial [\mathbf{g}_1 \mathbf{g}_2 \mathbf{g}_3]}{\partial t}\right|_{\mathbf{P}} = v^j \Gamma^i_{ji} \sqrt{g}. \tag{3.118}$$

$$\left.\frac{\partial g^i_I}{\partial t}\right|_{\mathbf{p}} = \left.\frac{\partial (\mathbf{g}^i \cdot \mathbf{G}_I)}{\partial t}\right|_{\mathbf{p}} = \mathbf{g}^i \cdot \left.\frac{\partial \mathbf{G}_I}{\partial t}\right|_{\mathbf{p}} = V^J \Gamma^K_{JI} g^i_K,$$
$$\left.\frac{\partial g^I_i}{\partial t}\right|_{\mathbf{p}} = \left.\frac{\partial (\mathbf{g}_i \cdot \mathbf{G}^I)}{\partial t}\right|_{\mathbf{p}} = \mathbf{g}_i \cdot \left.\frac{\partial \mathbf{G}^I}{\partial t}\right|_{\mathbf{p}} = -V^J \Gamma^I_{JK} g^K_i,$$
$$\left.\frac{\partial |g^i_I|}{\partial t}\right|_{\mathbf{p}} = \left.\frac{1}{\sqrt{g}} \frac{\partial \sqrt{G}}{\partial t}\right|_{\mathbf{p}} = V^J \Gamma^K_{JK} \, |g^i_I|,$$
$$\left.\frac{\partial |g^I_i|}{\partial t}\right|_{\mathbf{p}} = \left.\sqrt{g} \frac{\partial}{\partial t} \left(\frac{1}{\sqrt{G}}\right)\right|_{\mathbf{p}} = -V^J \Gamma^K_{JK} \, |g^I_i|. \tag{3.119}$$

$$\left.\frac{\partial g_i^I}{\partial t}\right|_{\mathbf{P}} = \left.\frac{\partial(\mathbf{g}_i\cdot\mathbf{G}^I)}{\partial t}\right|_{\mathbf{P}} = \mathbf{G}^I\cdot\left.\frac{\partial\mathbf{g}_i}{\partial t}\right|_{\mathbf{P}} = v^j\Gamma^k_{ji}g_k^I,$$
$$\left.\frac{\partial g_I^i}{\partial t}\right|_{\mathbf{P}} = \left.\frac{\partial(\mathbf{g}^i\cdot\mathbf{G}_I)}{\partial t}\right|_{\mathbf{P}} = \mathbf{G}_I\cdot\left.\frac{\partial\mathbf{g}^i}{\partial t}\right|_{\mathbf{P}} = -v^j\Gamma^i_{jk}g_I^k,$$
$$\left.\frac{\partial|g_i^I|}{\partial t}\right|_{\mathbf{P}} = \frac{1}{\sqrt{G}}\left.\frac{\partial\sqrt{G}}{\partial t}\right|_{\mathbf{P}} = v^j\Gamma^k_{jk}\,|g_i^I|, \tag{3.120}$$
$$\left.\frac{\partial|g_I^i|}{\partial t}\right|_{\mathbf{P}} = \sqrt{G}\left.\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{g}}\right)\right|_{\mathbf{P}} = -v^j\Gamma^k_{jk}\,|g_I^i|.$$

If the velocity $\mathbf{v}$ at point $\mathbf{p}$ is based on material and spatial coordinates $\mathbf{P}$ and $\mathbf{p}$, the corresponding descriptions are called the *material* and *spatial* descriptions, respectively.

Definition 3.26. For two position vectors $\mathbf{P}$ in $\mathfrak{B}$ and $\mathbf{p}$ in $\mathfrak{b}$ for time t, the velocity of the material point $\mathbf{P}$ at the location point $\mathbf{p}$ is

$$\mathbf{v} = \mathbf{v}(\mathbf{P},t) = v^i(\mathbf{P},t)\mathbf{g}_i \quad \text{or } v^i = v^i(\mathbf{P},t). \tag{3.121}$$

The time rate of change of the velocity $\mathbf{v}(\mathbf{P},t)$ of the material point at the point $\mathbf{p}$ is called the *acceleration of the material point* $\mathbf{P}$ at the location point $\mathbf{p}$ in the deformed configuration $\mathfrak{b}$, defined by

$$\mathbf{a}(\mathbf{P},t) \equiv \lim_{\Delta t\to 0}\frac{\mathbf{v}(\mathbf{P},t+\Delta t)-\mathbf{v}(\mathbf{P},t)}{\Delta t} = \left.\frac{\partial\mathbf{v}}{\partial t}\right|_{\mathbf{P}} = \left.\frac{\partial(v^i\mathbf{g}_i)}{\partial t}\right|_{\mathbf{P}}. \tag{3.122}$$

Theorem 3.14. *For two position vectors* $\mathbf{P}$ *in* $\mathfrak{B}$ *and* $\mathbf{p}$ *in* $\mathfrak{b}$ *for time t, under Eq.*(3.121)*, the acceleration of the material point* $\mathbf{P}$ *at the location point* $\mathbf{p}$ *is determined by*

$$\mathbf{a}(\mathbf{P},t) = \left(\left.\frac{\partial v^i}{\partial t}\right|_{\mathbf{P}} + v^jv^k\Gamma^i_{jk}\right)\mathbf{g}_i = \left(\left.\frac{\partial v^i}{\partial t}\right|_{\mathbf{p}} + v^jv^i_{;j}\right)\mathbf{g}_i. \tag{3.123}$$

Proof: From definition of the acceleration of the material point $\mathbf{P}$ at the location point $\mathbf{p}$, under Eqs. (3.118) and (3.121),

$$\mathbf{a}(\mathbf{P},t)=\left.\frac{\partial(v^i\mathbf{g}_i)}{\partial t}\right|_{\mathbf{P}}=\left.\frac{\partial v^i}{\partial t}\mathbf{g}_i\right|_{\mathbf{P}}+v^i\left.\frac{\partial \mathbf{g}_i}{\partial t}\right|_{\mathbf{P}}=(\left.\frac{\partial v^i}{\partial t}\right|_{\mathbf{P}}+v^jv^k\Gamma^i_{jk})\mathbf{g}_i$$
$$=(\left.\frac{\partial v^i}{\partial t}\right|_{\mathbf{p}}+v^j\frac{\partial v^i}{\partial x^j})\mathbf{g}_i+v^jv^k\Gamma^i_{jk}\mathbf{g}_i=(\left.\frac{\partial v^i}{\partial t}\right|_{\mathbf{p}}+v^j\frac{\partial v^i}{\partial x^j}+v^jv^k\Gamma^i_{jk})\mathbf{g}_i$$
$$=(\left.\frac{\partial v^i}{\partial t}\right|_{\mathbf{p}}+v^jv^i_{;j})\mathbf{g}_i.$$

This theorem is proved. ■

Definition 3.27. For two position vectors $\mathbf{P}$ in $\mathfrak{B}$ and $\mathbf{p}$ in $\mathfrak{b}$ for time t, the velocity of the material point $\mathbf{P}$ passing through the local point $\mathbf{p}$ is

$$\mathbf{v}=\mathbf{v}(\mathbf{p},t)=v^i(\mathbf{p},t)\mathbf{g}_{\mathrm{i}}\quad\text{or } v^i=v^i(\mathbf{p},t). \tag{3.124}$$

The time rate of change of the passing velocity $\mathbf{v}(\mathbf{p},t)$ of the material point $\mathbf{P}$ is called *the switching acceleration of the material points* $\mathbf{P}$ at the fixed location point $\mathbf{p}$, defined by

$$\mathbf{a}(\mathbf{p},t)\equiv\lim_{\Delta t\to 0}\frac{\mathbf{v}(\mathbf{p},t+\Delta t)-\mathbf{v}(\mathbf{p},t)}{\Delta t}=\left.\frac{\partial \mathbf{v}}{\partial t}\right|_{\mathbf{P}}. \tag{3.125}$$

Theorem 3.15. *For two position vectors* $\mathbf{P}$ *in* $\mathfrak{B}$ *and* $\mathbf{p}$ *in* $\mathfrak{b}$ *for time* t, *under Eq.*(3.124), *the passing acceleration of the material point* $\mathbf{P}$ *at the location point* $\mathbf{p}$ *in the deformed configuration* $\mathfrak{b}$ *is determined by*

$$\mathbf{a}(\mathbf{p},t)=\left.\frac{\partial \mathbf{v}}{\partial t}\right|_{\mathbf{P}}=\left.\frac{\partial \mathbf{v}}{\partial t}\right|_{\mathbf{p}}+\mathbf{v}\cdot\overset{>}{\nabla}\mathbf{v}=(\left.\frac{\partial v^i}{\partial t}\right|_{\mathbf{p}}+v^j\nabla_j v^i)\mathbf{g}_i=a^i\mathbf{g}_i. \tag{3.126}$$

Proof: From the definition of acceleration, with Eq.(3.124),

$$\mathbf{a}(\mathbf{p},t)=\left.\frac{\partial \mathbf{v}}{\partial t}\right|_{\mathbf{P}}=\left.\frac{\partial \mathbf{v}}{\partial t}\right|_{\mathbf{p}}+v^j\frac{\partial \mathbf{v}}{\partial x^j}=\left.\frac{\partial \mathbf{v}}{\partial t}\right|_{\mathbf{p}}+\mathbf{v}\cdot\overset{>}{\nabla}\mathbf{v}$$
$$=(\left.\frac{\partial v^i}{\partial t}\right|_{\mathbf{p}}+v^j\nabla_j v^i)\mathbf{g}_i=a^i\mathbf{g}_i.$$

This theorem is proved. ■

No matter how the material or spatial description is used, the same acceleration at the same point in the deformed configuration is described. Therefore, the two descriptions should give the same acceleration. The acceleration based on the deformed configuration can be translated into the initial configuration by the shifter tensor, which can be done through the velocity expression in the initial configuration in $\mathfrak{B}$.

Theorem 3.16. *For two position vectors* $\mathbf{P}$ *in* $\mathfrak{B}$ *and* $\mathbf{p}$ *in* $\mathfrak{b}$ *for time* t, *the velocity of the material point* $\mathbf{P}$ *at the location point* $\mathbf{p}$ *is*

$$\mathbf{v} = \mathbf{v}(\mathbf{P}, t) = v^I(\mathbf{P}, t)\mathbf{G}_I \quad \text{or } v^i = v^i(\mathbf{P}, t). \tag{3.127}$$

The acceleration of the material point $\mathbf{P}$ *at the location point* $\mathbf{p}$, *based on the coordinates in the initial configuration* $\mathfrak{B}$, *is*

$$\overset{<}{\mathbf{a}}(\mathbf{P}, t) = \overset{<>}{\mathbf{I}} \cdot \mathbf{a}(\mathbf{P}, t). \tag{3.128}$$

Proof: From the definition of acceleration, with Eq.(3.127),

$$\begin{aligned}
\overset{<}{\mathbf{a}}(\mathbf{P}, t) &= \left.\frac{\partial \mathbf{v}(\mathbf{P}, t)}{\partial t}\right|_{\mathbf{P}} = \left.\frac{\partial (v^I(\mathbf{P}, t)\mathbf{G}_I)}{\partial t}\right|_{\mathbf{P}} = \left.\frac{\partial v^I(\mathbf{P}, t)}{\partial t}\right|_{\mathbf{P}} \mathbf{G}_I \\
&= \left.\frac{\partial (v^i(\mathbf{P}, t) g_i^I)}{\partial t}\right|_{\mathbf{P}} \mathbf{G}_I = \left(\left.\frac{\partial v^i(\mathbf{P}, t)}{\partial t}\right|_{\mathbf{P}} g_i^I + v^i(\mathbf{P}, t)\left.\frac{\partial g_i^I}{\partial t}\right|_{\mathbf{P}}\right)\mathbf{G}_I \\
&= \left(\left.\frac{\partial v^i(\mathbf{P}, t)}{\partial t}\right|_{\mathbf{P}} + v^j v^k \Gamma^i_{jk}\Big|_{\mathbf{P}}\right) g_i^I \mathbf{G}_I = \overset{<>}{\mathbf{I}} \cdot \mathbf{a}(\mathbf{P}, t).
\end{aligned}$$

This theorem is proved. ■

Theorem 3.17. *For two position vectors* $\mathbf{P}$ *in* $\mathfrak{B}$ *and* $\mathbf{p}$ *in* $\mathfrak{b}$ *for time* t, *the passing velocity of the material point* $\mathbf{P}$ *at the location point* $\mathbf{p}$ *is*

$$\mathbf{v} = \mathbf{v}(\mathbf{p}, t) = v^I(\mathbf{p}, t)\mathbf{G}_I \quad \text{or } v^I = v^I(\mathbf{p}, t). \tag{3.129}$$

The passing acceleration of the material point $\mathbf{P}$ *at the location point* $\mathbf{p}$, *based on the coordinates in the initial configuration* $\mathfrak{B}$, *is*

$$\overset{<}{\mathbf{a}}(\mathbf{p}, t) = \overset{<>}{\mathbf{I}} \cdot \mathbf{a}(\mathbf{p}, t). \tag{3.130}$$

Proof: From the definition of acceleration, with Eq.(3.129),

$$\begin{aligned}
\overset{<}{\mathbf{a}}(\mathbf{p}, t) &= \left.\frac{\partial \mathbf{v}(\mathbf{p}, t)}{\partial t}\right|_{\mathbf{P}} = \left.\frac{\partial v^I(\mathbf{p}, t)}{\partial t}\right|_{\mathbf{P}} \mathbf{G}_I = \left.\frac{\partial (v^i(\mathbf{p}, t) g_i^I)}{\partial t}\right|_{\mathbf{P}} \mathbf{G}_I \\
&= \left(\left.\frac{\partial v^i(\mathbf{p}, t)}{\partial t}\right|_{\mathbf{P}} + v^j v^k \Gamma^i_{jk}\right) g_i^I \mathbf{G}_I = \left(\left.\frac{\partial v^i(\mathbf{p}, t)}{\partial t}\right|_{\mathbf{p}} + v^j \nabla_j v^i\right) g_i^I \mathbf{G}_I \\
&= \overset{<>}{\mathbf{I}} \cdot \mathbf{a}(\mathbf{p}, t).
\end{aligned}$$

This theorem is proved. ■

Definition 3.28. For two position vectors $\mathbf{P}$ in $\mathfrak{B}$ and $\mathbf{p}$ in $\mathfrak{b}$ for time t, there is a two-point tensor

$$\overset{<\cdot>}{\boldsymbol{\psi}} = \psi^{I_1 \cdots I_P}{}_{J_1 \cdots J_Q}{}^{i_1 \cdots i_m}{}_{j_1 \cdots j_n} \sqrt{G}^{-W} \sqrt{g}^{-w} \mathbf{G}_{I_1} \cdots \mathbf{G}_{I_P} \cdot \mathbf{G}^{J_1} \cdots \mathbf{G}^{J_Q} \mathbf{g}_{i_1} \cdots \mathbf{g}_{i_m} \mathbf{g}^{j_1} \cdots \mathbf{g}^{j_n}. \tag{3.131}$$

The *time rate of change of the tensor* $\overset{<\cdot>}{\boldsymbol{\psi}}(\mathbf{P},t)$ for the material point $\mathbf{P}$ at the location points $\mathbf{p}$ is defined by the material description, i.e.,

$$\begin{aligned} \left.\frac{\partial \overset{<\cdot>}{\boldsymbol{\psi}}(\mathbf{P},t)}{\partial t}\right|_{\mathbf{P}} = \Bigg[& \left.\frac{\partial \psi^{I_1 \cdots I_P}{}_{J_1 \cdots J_Q}{}^{i_1 \cdots i_m}{}_{j_1 \cdots j_n}}{\partial t}\right|_{\mathbf{P}} + v^r \Gamma^{i_1}_{rs} \psi^{I_1 \cdots I_P}{}_{J_1 \cdots J_Q}{}^{s i_2 \cdots i_m}{}_{j_1 \cdots j_n} \\ & + \cdots + \Gamma^{i_m}_{rs} \psi^{I_1 \cdots I_P}{}_{J_1 \cdots J_Q}{}^{i_1 \cdots i_{m-1} s}{}_{j_1 \cdots j_n} - \Gamma^{s}_{r j_1} \psi^{I_1 \cdots I_P}{}_{J_1 \cdots J_Q}{}^{i_1 \cdots i_m}{}_{s j_2 \cdots j_n} \\ & - \cdots - \Gamma^{s}_{r j_n} \psi^{I_1 \cdots I_P}{}_{J_1 \cdots J_Q}{}^{i_1 \cdots i_m}{}_{j_1 \cdots j_{n-1} s} - w \Gamma^{s}_{rs} \psi^{I_1 \cdots I_P}{}_{J_1 \cdots J_Q}{}^{i_1 \cdots i_m}{}_{j_1 \cdots j_n} \Bigg] \\ & \cdot \sqrt{G}^{-W} \sqrt{g}^{-w} \mathbf{G}_{I_1} \cdots \mathbf{G}_{I_P} \mathbf{G}^{J_1} \cdots \mathbf{G}^{J_Q} \mathbf{g}_{i_1} \cdots \mathbf{g}_{i_m} \mathbf{g}^{j_1} \cdots \mathbf{g}^{j_n}. \end{aligned} \tag{3.132}$$

The *time rate of change of the tensor* $\overset{<\cdot>}{\boldsymbol{\psi}}(\mathbf{p},t)$ for the material point $\mathbf{P}$ at the location point $\mathbf{p}$ is defined by the spatial description, i.e.,

$$\begin{aligned} \left.\frac{\partial \overset{<\cdot>}{\boldsymbol{\psi}}(\mathbf{P},t)}{\partial t}\right|_{\mathbf{P}} = & \Bigg(\left.\frac{\partial \psi^{I_1 \cdots I_P}{}_{J_1 \cdots J_Q}{}^{i_1 \cdots i_m}{}_{j_1 \cdots j_n}}{\partial t}\right|_{\mathbf{p}} + \psi^{I_1 \cdots I_P}{}_{J_1 \cdots J_Q}{}^{s i_2 \cdots i_m}{}_{j_1 \cdots j_n ; r} v^r \Bigg) \\ & \cdot \sqrt{G}^{-W} \sqrt{g}^{-w} \mathbf{G}_{I_1} \cdots \mathbf{G}_{I_P} \mathbf{G}^{J_1} \cdots \mathbf{G}^{J_Q} \mathbf{g}_{i_1} \cdots \mathbf{g}_{i_m} \mathbf{g}^{j_1} \cdots \mathbf{g}^{j_n}. \end{aligned} \tag{3.133}$$

The foregoing operation is expressed by

$$\begin{aligned} \frac{D}{Dt} \overset{<\cdot>}{\boldsymbol{\psi}}(\mathbf{P},t) = & \frac{D}{Dt} \psi^{I_1 \cdots I_P}{}_{J_1 \cdots J_Q}{}^{i_1 \cdots i_m}{}_{j_1 \cdots j_n} \\ & \cdot \sqrt{G}^{-W} \sqrt{g}^{-w} \mathbf{G}_{I_1} \cdots \mathbf{G}_{I_P} \mathbf{G}^{J_1} \cdots \mathbf{G}^{J_Q} \mathbf{g}_{i_1} \cdots \mathbf{g}_{i_m} \mathbf{g}^{j_1} \cdots \mathbf{g}^{j_n} \end{aligned} \tag{3.134}$$

where for the material derivative

$$\frac{D(\cdot)}{Dt} = \left.\frac{\partial(\cdot)}{\partial t}\right|_{\mathbf{P}} \tag{3.135}$$

for components

$$\frac{D(\cdot)}{Dt} = \left.\frac{\partial(\cdot)}{\partial t}\right|_{\mathbf{p}} + v^r \nabla_r = \left.\frac{\partial(\cdot)}{\partial t}\right|_{\mathbf{P}} + v^r \left(\nabla_r - \frac{\partial}{\partial x^r}\right)(\cdot). \tag{3.136}$$

For a tensor $\overset{>\bullet>}{\boldsymbol{\psi}}$, the corresponding material derivative is

$$\frac{D\overset{>\cdot>}{\psi}}{Dt}=\left.\frac{\partial\overset{>\cdot>}{\psi}}{\partial \boldsymbol{t}}\right|_{\mathbf{p}}+v^r\frac{\partial\overset{>\cdot>}{\psi}}{\partial x^r}=\left.\frac{\partial\overset{>\cdot>}{\psi}}{\partial \boldsymbol{t}}\right|_{\mathbf{p}}+\mathbf{v}\cdot\overset{>}{\nabla}\overset{>\cdot>}{\psi}. \tag{3.137}$$

As in Chapter 2, the material derivatives of the identity tensors, Eddington's tensors and shifter tensors are zero.

Definition 3.29. For two position vectors $\mathbf{P}$ in $\mathfrak{B}$ and $\mathbf{p}$ in $\mathfrak{b}$ for time t, the *gradient of the velocity vector field* $\mathbf{v}(\mathbf{p},t)$ is defined by

$$\overset{>>}{\mathbf{D}}\equiv\mathbf{v}\overset{>}{\nabla}=v^i_{;j}\mathbf{g}_i\mathbf{g}^j \tag{3.138}$$

and the relative velocity of the point $\mathbf{P}+d\mathbf{P}$ to point $\mathbf{P}$ is given by the differentiation of the velocity vector field $\mathbf{v}(\mathbf{p},t)$ at point $\mathbf{p}$, i.e.,

$$d\mathbf{v}=\mathbf{v}_{,i}dx^i=(\mathbf{v}_{,i}\mathbf{g}^i)\cdot(dx^j\mathbf{g}_j)=\mathbf{v}\overset{>}{\nabla}\cdot d\mathbf{p}=\overset{>>}{\mathbf{D}}\cdot d\mathbf{p}. \tag{3.139}$$

Definition 3.30. For two position vectors $\mathbf{P}$ in $\mathfrak{B}$ and $\mathbf{p}$ in $\mathfrak{b}$ for time t, the *material derivative of deformation gradient tensor* is defined by

$$\frac{D}{Dt}\overset{><}{\mathbf{F}}\equiv\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\frac{\partial\mathbf{p}(\mathbf{P},t+\Delta t)}{\partial X^I}-\frac{\partial\mathbf{p}(\mathbf{P},t)}{\partial X^I}\right]=\mathbf{v}_{,I}\mathbf{G}^I. \tag{3.140}$$

Theorem 3.18. *For two position vectors* $\mathbf{P}$ *in* $\mathfrak{B}$ *and* $\mathbf{p}$ *in* $\mathfrak{b}$ *for time* t*, the material derivatives of deformation gradient tensors* $\overset{><}{\mathbf{F}}$ and $\overset{-1}{\overset{<>}{\mathbf{F}}}$ *are determined by*

$$\frac{D}{Dt}\overset{><}{\mathbf{F}}=\overset{>>}{\mathbf{D}}\cdot\overset{><}{\mathbf{F}}\quad\text{and}\quad\frac{D}{Dt}\overset{-1}{\overset{<>}{\mathbf{F}}}=-\overset{-1}{\overset{<>}{\mathbf{F}}}\cdot\overset{>>}{\mathbf{D}}. \tag{3.141}$$

Proof: From the definition of the total derivative of tensors,

$$\begin{aligned}\frac{D}{Dt}\overset{><}{\mathbf{F}}&=\mathbf{v}_{,I}\mathbf{G}^I=\frac{\partial\mathbf{v}}{\partial x^i}\frac{\partial x^i}{\partial X^I}\mathbf{G}^I=\frac{\partial\mathbf{v}}{\partial x^i}\mathbf{g}^i\cdot\mathbf{g}_j\frac{\partial x^j}{\partial X^I}\mathbf{G}^I\\&=\frac{\partial\mathbf{v}}{\partial x^r}\mathbf{g}^r\cdot(\frac{\partial x^j}{\partial X^I}\mathbf{g}_j\mathbf{G}^I)=\frac{\partial v^i}{\partial x^r}\mathbf{g}_i\mathbf{g}^r\cdot(\frac{\partial x^j}{\partial X^I}\mathbf{g}_j\mathbf{G}^I)\\&=(v^i_{;r}\mathbf{g}_i\mathbf{g}^r)\cdot(\frac{\partial x^j}{\partial X^I}\mathbf{g}_j\mathbf{G}^I)=\overset{>>}{\mathbf{D}}\cdot\overset{><}{\mathbf{F}}.\end{aligned}$$

Because of $D\overset{>>}{\mathbf{I}}/Dt=0$,

$$0=\frac{D}{Dt}\overset{>>}{\mathbf{I}}=\frac{D}{Dt}(\overset{><}{\mathbf{F}}\cdot\overset{-1}{\overset{<>}{\mathbf{F}}})=(\frac{D}{Dt}\overset{><}{\mathbf{F}})\cdot\overset{-1}{\overset{<>}{\mathbf{F}}}+\overset{><}{\mathbf{F}}\cdot(\frac{D}{Dt}\overset{-1}{\overset{<>}{\mathbf{F}}}).$$

Thus,

$$\frac{D}{Dt}\overset{-1}{\overset{\lt\gt}{\mathbf{F}}} = -\overset{-1}{\overset{\lt\gt}{\mathbf{F}}}\cdot\Big(\frac{D}{Dt}\overset{\gt\lt}{\mathbf{F}}\Big)\cdot\overset{-1}{\overset{\lt\gt}{\mathbf{F}}} = -\overset{-1}{\overset{\lt\gt}{\mathbf{F}}}\cdot\overset{\gt\gt}{\mathbf{D}}\cdot\overset{\gt\lt}{\mathbf{F}}\cdot\overset{-1}{\overset{\lt\gt}{\mathbf{F}}} = -\overset{-1}{\overset{\lt\gt}{\mathbf{F}}}\cdot\overset{\gt\gt}{\mathbf{D}}\cdot\overset{\gt\gt}{\mathbf{I}} = -\overset{-1}{\overset{\lt\gt}{\mathbf{F}}}\cdot\overset{\gt\gt}{\mathbf{D}}.$$

This theorem is proved. ■

Theorem 3.19. *For two position vectors* $\mathbf{P}$ *in* $\mathfrak{B}$ *and* $\mathbf{p}$ *in* $\mathfrak{b}$ *for time* t, *the material derivatives of the infinitesimal line, area and volume elements in* $\mathfrak{B}$ *are zero, i.e.,*

$$\frac{D}{Dt}d\mathbf{P} = \frac{D}{Dt}d\overset{\lt}{\mathbf{A}} = \frac{D}{Dt}dV = 0; \tag{3.142}$$

the material derivatives of the infinitesimal line, area and volume elements in $\mathfrak{b}$ *are*

$$\frac{D}{Dt}d\mathbf{p} = \overset{\gt\gt}{\mathbf{D}}\cdot d\mathbf{p} = \overset{\gt\gt}{\mathbf{D}}\cdot\overset{\gt\lt}{\mathbf{F}}\cdot d\mathbf{P},$$

$$\begin{aligned}&\frac{D}{Dt}d\overset{\gt}{\mathbf{a}} = [\mathrm{I}(\overset{\gt\gt}{\mathbf{D}})\overset{\gt\gt}{\mathbf{I}} - (\overset{\gt\gt}{\mathbf{D}})^{\mathrm{T}}]\cdot d\overset{\gt}{\mathbf{a}} = [\mathrm{I}(\overset{\gt\gt}{\mathbf{D}})\overset{\gt\gt}{\mathbf{I}} - (\overset{\gt\gt}{\mathbf{D}})^{\mathrm{T}}]\cdot\mathscr{J}\overset{-1}{\overset{\gt\lt}{\mathbf{F}}}\cdot d\overset{\gt}{\mathbf{A}},\\ &\frac{D}{Dt}dv = \mathrm{I}(\overset{\gt\gt}{\mathbf{D}})dv = \mathrm{I}(\overset{\gt\gt}{\mathbf{D}})\mathrm{III}(\overset{\lt\lt}{\mathbf{F}})\cdot dV \text{ and } \frac{D}{Dt}\mathscr{J} = \mathrm{I}(\overset{\gt\gt}{\mathbf{D}})\mathscr{J}.\end{aligned} \tag{3.143}$$

Proof: Because $d\mathbf{P}$ is independent of time,

$$\frac{D}{Dt}d\mathbf{P} = \frac{D}{Dt}d\overset{\lt}{\mathbf{A}} = \frac{D}{Dt}dV = 0.$$

However,

$$\frac{D}{Dt}d\mathbf{p} = \frac{D}{Dt}(\overset{\gt\lt}{\mathbf{F}}\cdot d\mathbf{P}) = \frac{D}{Dt}(\overset{\gt\lt}{\mathbf{F}})\cdot d\mathbf{P} = \overset{\gt\gt}{\mathbf{D}}\cdot\overset{\gt\lt}{\mathbf{F}}\cdot d\mathbf{P} = \overset{\gt\gt}{\mathbf{D}}\cdot d\mathbf{p} = d\mathbf{v} = d\Big(\frac{D\mathbf{p}}{Dt}\Big).$$

$$\begin{aligned}\frac{D}{Dt}dv &= \frac{D}{Dt}[d\underset{1}{\mathbf{p}}\,d\underset{2}{\mathbf{p}}\,d\underset{3}{\mathbf{p}}]\\ &= \frac{(\overset{\gt\gt}{\mathbf{D}}\cdot d\underset{1}{\mathbf{p}})\times d\underset{2}{\mathbf{p}}\cdot d\underset{3}{\mathbf{p}} + d\underset{1}{\mathbf{p}}\times(\overset{\gt\gt}{\mathbf{D}}\cdot d\underset{2}{\mathbf{p}})\cdot d\underset{3}{\mathbf{p}} + d\underset{1}{\mathbf{p}}\times d\underset{2}{\mathbf{p}}\cdot(\overset{\gt\gt}{\mathbf{D}}\cdot d\underset{3}{\mathbf{p}})}{[d\underset{1}{\mathbf{p}}\,d\underset{2}{\mathbf{p}}\,d\underset{3}{\mathbf{p}}]}dv\\ &= \mathrm{I}(\overset{\gt\gt}{\mathbf{D}})dv = \mathrm{I}(\overset{\gt\gt}{\mathbf{D}})\mathrm{III}(\overset{\lt\lt}{\mathbf{F}})\cdot dV.\end{aligned}$$

Because

$$\frac{D}{Dt}dv = \frac{D}{Dt}(\mathscr{J}dV) = \frac{D}{Dt}(\mathscr{J})dV = \mathbf{I}(\overset{\gt\gt}{\mathbf{D}})dv,$$

then,

$$\frac{D}{Dt}\mathscr{J} = \mathrm{I}(\overset{\gt\gt}{\mathbf{D}})\frac{dv}{dV} = \mathrm{I}(\overset{\gt\gt}{\mathbf{D}})\mathscr{J}.$$

Thus,

$$\begin{aligned}
\frac{D}{Dt}d\overset{>}{\mathbf{a}} &= \frac{D}{Dt}[\mathscr{J}(\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}}\cdot d\overset{<}{\mathbf{A}}] = \frac{D}{Dt}(\mathscr{J})[(\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}}\cdot d\overset{<}{\mathbf{A}}] + \mathscr{J}\frac{D}{Dt}(\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}}\cdot d\overset{<}{\mathbf{A}} \\
&= \mathscr{J}\mathrm{I}(\overset{>>}{\mathbf{D}})(\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}}\cdot d\overset{<}{\mathbf{A}} - \mathscr{J}(\overset{>>}{\mathbf{D}})^{\mathrm{T}}\cdot(\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}}\cdot d\overset{<}{\mathbf{A}} \\
&= [\mathrm{I}(\overset{>>}{\mathbf{D}})\overset{>>}{\mathbf{I}} - (\overset{>>}{\mathbf{D}})^{\mathrm{T}}]\cdot\mathscr{J}\overset{-1}{\overset{><}{\mathbf{F}}}\cdot d\overset{>}{\mathbf{A}} \\
&= [\mathrm{I}(\overset{>>}{\mathbf{D}})\overset{>>}{\mathbf{I}} - (\overset{>>}{\mathbf{D}})^{\mathrm{T}}]\cdot d\overset{>}{\mathbf{a}}.
\end{aligned}$$

This theorem is proved. ■

Theorem 3.20. *For two position vectors* $\mathbf{P}$ *in* $\mathfrak{B}$ *and* $\mathbf{p}$ *in* $\mathfrak{b}$ *for time* t, *for* $\mathbf{n} = d\mathbf{p}/|d\mathbf{p}|$ *and* ${}_a\mathbf{n} = d\overset{>}{\mathbf{a}}/|d\overset{>}{\mathbf{a}}|$, *the time rates of changes of the length, volume and area elements are*

$$\begin{aligned}
d_{\mathbf{n}} &\equiv \frac{1}{|d\mathbf{p}|}(\frac{D}{Dt}|d\mathbf{p}|) = \mathbf{n}\cdot\overset{>>}{\mathbf{d}}\cdot\mathbf{n}, \\
&\frac{1}{dv}\frac{D}{Dt}(dv) = I(\overset{>>}{\mathbf{d}}), \\
&\frac{1}{|d\overset{>}{\mathbf{a}}|}\frac{D}{Dt}|d\overset{>}{\mathbf{a}}| = I(\overset{>>}{\mathbf{d}}) - d_{{}_a\mathbf{n}}
\end{aligned} \tag{3.144}$$

where *the deformation rate* is

$$\overset{>>}{\mathbf{d}} = \frac{1}{2}[\overset{>>}{\mathbf{D}} + (\overset{>>}{\mathbf{D}})^{\mathrm{T}}] \text{ and } d_{{}_a\mathbf{n}} \equiv {}_a\mathbf{n}\cdot\overset{>>}{\mathbf{d}}\cdot{}_a\mathbf{n}. \tag{3.145}$$

Proof: Because the time rate of change of the arc length is

$$\begin{aligned}
\frac{D}{Dt}|d\mathbf{p}| &= \frac{D}{Dt}\sqrt{d\mathbf{p}\cdot d\mathbf{p}} = \frac{1}{2\sqrt{d\mathbf{p}\cdot d\mathbf{p}}}(\frac{D}{Dt}d\mathbf{p}\cdot d\mathbf{p} + d\mathbf{p}\cdot\frac{D}{Dt}d\mathbf{p}) \\
&= \frac{1}{2|d\mathbf{p}|^2}(\frac{D}{Dt}d\mathbf{p}\cdot d\mathbf{p} + d\mathbf{p}\cdot\frac{D}{Dt}d\mathbf{p})|d\mathbf{p}|,
\end{aligned}$$

from Theorem 3.19, $D(d\mathbf{p})/Dt = \overset{>>}{\mathbf{D}}\cdot d\mathbf{p}$, the time rate of change of the arc length is

$$\begin{aligned}
\frac{D}{Dt}|d\mathbf{p}| &= \frac{1}{2}\frac{\overset{>>}{\mathbf{D}}\cdot d\mathbf{p}\cdot d\mathbf{p} + d\mathbf{p}\cdot\overset{>>}{\mathbf{D}}\cdot d\mathbf{p}}{|d\mathbf{p}|^2}|d\mathbf{p}| \\
&= (\mathbf{n}\cdot\frac{1}{2}[(\overset{>>}{\mathbf{D}})^{\mathrm{T}} + \overset{>>}{\mathbf{D}}]\cdot\mathbf{n})|d\mathbf{p}| \\
&= (\mathbf{n}\cdot\overset{>>}{\mathbf{d}}\cdot\mathbf{n})|d\mathbf{p}|.
\end{aligned}$$

Thus, the time rate of change of the volume is

$$d_{\mathbf{n}} \equiv \frac{1}{|d\mathbf{p}|}\frac{D}{Dt}(|d\mathbf{p}|) = \mathbf{n}\cdot\overset{\gg}{\mathbf{d}}\cdot\mathbf{n}.$$

For the rate of the volume, one obtains

$$\begin{aligned}\frac{1}{dv}\frac{D}{Dt}(dv) &= \mathrm{I}(\overset{\gg}{\mathbf{D}}) = Tr[\frac{1}{2}(\overset{\gg}{\mathbf{D}}+(\overset{\gg}{\mathbf{D}})^{\mathrm{T}})+\frac{1}{2}(\overset{\gg}{\mathbf{D}}-(\overset{\gg}{\mathbf{D}})^{\mathrm{T}})]\\ &= Tr[\frac{1}{2}(\overset{\gg}{\mathbf{D}}+(\overset{\gg}{\mathbf{D}})^{\mathrm{T}})]+Tr[\frac{1}{2}(\overset{\gg}{\mathbf{D}}-(\overset{\gg}{\mathbf{D}})^{\mathrm{T}})] = Tr(\overset{\gg}{\mathbf{d}}) = \mathrm{I}(\overset{\gg}{\mathbf{d}}).\end{aligned}$$

The time rate of change of the area is

$$\begin{aligned}\frac{D}{Dt}|d\overset{>}{\mathbf{a}}| &= \frac{D}{Dt}\sqrt{d\overset{>}{\mathbf{a}}\cdot d\overset{>}{\mathbf{a}}} = \frac{1}{2\sqrt{d\overset{>}{\mathbf{a}}\cdot d\overset{>}{\mathbf{a}}}}(\frac{D}{Dt}d\overset{>}{\mathbf{a}}\cdot d\overset{>}{\mathbf{a}}+d\overset{>}{\mathbf{a}}\cdot\frac{D}{Dt}d\overset{>}{\mathbf{a}})\\ &= \frac{1}{2|d\overset{>}{\mathbf{a}}|^2}(\frac{D}{Dt}d\overset{>}{\mathbf{a}}\cdot d\overset{>}{\mathbf{a}}+d\overset{>}{\mathbf{a}}\cdot\frac{D}{Dt}d\overset{>}{\mathbf{a}})|d\overset{>}{\mathbf{a}}|.\end{aligned}$$

From Theorem 3.19, the following relation holds:

$$\frac{D}{Dt}d\overset{>}{\mathbf{a}} = (\mathrm{I}(\overset{\gg}{\mathbf{D}})\,\overset{\gg}{\mathbf{I}} - (\overset{\gg}{\mathbf{D}})^{\mathrm{T}})\cdot d\overset{>}{\mathbf{a}}.$$

Substitution of the foregoing equation into the time rate of change of the area gives

$$\begin{aligned}\frac{D}{Dt}|d\overset{>}{\mathbf{a}}| &= \frac{[\mathrm{I}(\overset{\gg}{\mathbf{D}})\,\overset{\gg}{\mathbf{I}}-(\overset{\gg}{\mathbf{D}})^{\mathrm{T}}]\cdot d\overset{>}{\mathbf{a}}\cdot d\overset{>}{\mathbf{a}}+d\overset{>}{\mathbf{a}}\cdot[\mathrm{I}(\overset{\gg}{\mathbf{D}})\,\overset{\gg}{\mathbf{I}}-(\overset{\gg}{\mathbf{D}})^{\mathrm{T}}]\cdot d\overset{>}{\mathbf{a}}}{2|d\overset{>}{\mathbf{a}}|^2}|d\overset{>}{\mathbf{a}}|\\ &= \frac{1}{2}\frac{d\overset{>}{\mathbf{a}}\cdot[\mathrm{I}(\overset{\gg}{\mathbf{D}})\,\overset{\gg}{\mathbf{I}}-\overset{\gg}{\mathbf{D}}]\cdot d\overset{>}{\mathbf{a}}+d\overset{>}{\mathbf{a}}\cdot[\mathrm{I}(\overset{\gg}{\mathbf{D}})\,\overset{\gg}{\mathbf{I}}-(\overset{\gg}{\mathbf{D}})^{\mathrm{T}}]\cdot d\overset{>}{\mathbf{a}}}{|d\overset{>}{\mathbf{a}}|^2}|d\overset{>}{\mathbf{a}}|\\ &= \{\mathrm{I}(\overset{\gg}{\mathbf{D}}) - {}_a\mathbf{n}\cdot\frac{1}{2}[(\overset{\gg}{\mathbf{D}})^{\mathrm{T}}+\overset{\gg}{\mathbf{D}}]\cdot{}_a\mathbf{n}\}|d\overset{>}{\mathbf{a}}|\\ &= [\mathrm{I}(\overset{\gg}{\mathbf{D}}) - {}_a\mathbf{n}\cdot\overset{\gg}{\mathbf{d}}\cdot{}_a\mathbf{n}]|d\overset{>}{\mathbf{a}}|.\end{aligned}$$

So the time rate of change of the deformed area is

$$\frac{1}{|d\overset{>}{\mathbf{a}}|}\frac{D}{Dt}|d\overset{>}{\mathbf{a}}| = \mathrm{I}(\overset{\gg}{\mathbf{D}}) - {}_a\mathbf{n}\cdot\overset{\gg}{\mathbf{d}}\cdot{}_a\mathbf{n}.$$

This theorem is proved. ■

Theorem 3.21. *For two position vectors* $\mathbf{P}$ *in* $\mathfrak{B}$ *and* $\mathbf{p}$ *in* $\mathfrak{b}$ *for time* t*, for* $\mathbf{n} = d\mathbf{p}/|d\mathbf{p}|$*, the following relations exist*:

$$\frac{D\mathbf{n}}{Dt} = \overset{\gg}{\mathbf{D}}\cdot\mathbf{n} - (\mathbf{n}\cdot\overset{\gg}{\mathbf{d}}\cdot\mathbf{n})\mathbf{n} = (\overset{\gg}{\mathbf{D}} - d_{\mathbf{n}}\,\overset{\gg}{\mathbf{I}})\cdot\mathbf{n}, \tag{3.146}$$

$$\frac{D}{Dt}\gamma_{(\underset{1}{\mathbf{n}},\underset{2}{\mathbf{n}})}=\frac{1}{\sin\theta_{(\underset{1}{\mathbf{n}},\underset{2}{\mathbf{n}})}}\{[\underset{1}{\mathbf{n}}-\cos\theta_{(\underset{1}{\mathbf{n}},\underset{2}{\mathbf{n}})}\underset{2}{\mathbf{n}}]\cdot\overset{\gg}{\mathbf{d}}\cdot\underset{2}{\mathbf{n}}+[\underset{2}{\mathbf{n}}-\cos\theta_{(\underset{1}{\mathbf{n}},\underset{2}{\mathbf{n}})}\underset{1}{\mathbf{n}}]\cdot\overset{\gg}{\mathbf{d}}\cdot\underset{1}{\mathbf{n}}\}. \tag{3.147}$$

Proof: The material derivative of the unit vector is

$$\frac{D\mathbf{n}}{Dt}=\frac{D}{Dt}\left(\frac{d\mathbf{p}}{|d\mathbf{p}|}\right)=\frac{1}{|d\mathbf{p}|^2}\left[\frac{D}{Dt}(d\mathbf{p})\,|d\mathbf{p}|-d\mathbf{p}\frac{D}{Dt}(|d\mathbf{p}|)\right].$$

Because

$$\frac{D}{Dt}(d\mathbf{p})=\overset{\gg}{\mathbf{D}}\cdot d\mathbf{p} \text{ and } \frac{D}{Dt}(|d\mathbf{p}|)=(\mathbf{n}\cdot\overset{\gg}{\mathbf{d}}\cdot\mathbf{n})\,|d\mathbf{p}|,$$

then,

$$\frac{D\mathbf{n}}{Dt}=\overset{\gg}{\mathbf{D}}\cdot\mathbf{n}-(\mathbf{n}\cdot\overset{\gg}{\mathbf{d}}\cdot\mathbf{n})\mathbf{n}=(\overset{\gg}{\mathbf{D}}-d_{\mathbf{n}}\overset{\gg}{\mathbf{I}})\cdot\mathbf{n}.$$

$$\begin{aligned}\frac{D}{Dt}(\cos\theta_{(\underset{1}{\mathbf{n}},\underset{2}{\mathbf{n}})})&=\frac{D}{Dt}(\underset{1}{\mathbf{n}}\cdot\underset{2}{\mathbf{n}})=\frac{D}{Dt}(\underset{1}{\mathbf{n}})\cdot\underset{2}{\mathbf{n}}+\underset{1}{\mathbf{n}}\cdot\frac{D}{Dt}(\underset{2}{\mathbf{n}})\\
&=\overset{\gg}{\mathbf{D}}\cdot\underset{1}{\mathbf{n}}\cdot\underset{2}{\mathbf{n}}-(\underset{1}{\mathbf{n}}\cdot\overset{\gg}{\mathbf{d}}\cdot\underset{1}{\mathbf{n}})\underset{1}{\mathbf{n}}\cdot\underset{2}{\mathbf{n}}+\underset{1}{\mathbf{n}}\cdot[\overset{\gg}{\mathbf{D}}\cdot\underset{2}{\mathbf{n}}-(\underset{2}{\mathbf{n}}\cdot\overset{\gg}{\mathbf{d}}\cdot\underset{2}{\mathbf{n}})]\underset{2}{\mathbf{n}}\\
&=2\underset{1}{\mathbf{n}}\cdot\overset{\gg}{\mathbf{d}}\cdot\underset{2}{\mathbf{n}}-[(\underset{1}{\mathbf{n}}\cdot\overset{\gg}{\mathbf{d}}\cdot\underset{1}{\mathbf{n}})+(\underset{2}{\mathbf{n}}\cdot\overset{\gg}{\mathbf{d}}\cdot\underset{2}{\mathbf{n}})]\underset{1}{\mathbf{n}}\cdot\underset{2}{\mathbf{n}}\\
&=[\underset{1}{\mathbf{n}}-(\underset{1}{\mathbf{n}}\cdot\underset{2}{\mathbf{n}})\underset{2}{\mathbf{n}}]\cdot\overset{\gg}{\mathbf{d}}\cdot\underset{2}{\mathbf{n}}+[\underset{2}{\mathbf{n}}-(\underset{2}{\mathbf{n}}\cdot\underset{1}{\mathbf{n}})\underset{1}{\mathbf{n}}]\cdot\overset{\gg}{\mathbf{d}}\cdot\underset{1}{\mathbf{n}}\\
&=[\underset{1}{\mathbf{n}}-\cos\theta_{(\underset{1}{\mathbf{n}},\underset{2}{\mathbf{n}})}\underset{2}{\mathbf{n}}]\cdot\overset{\gg}{\mathbf{d}}\cdot\underset{2}{\mathbf{n}}+[\underset{2}{\mathbf{n}}-\cos\theta_{(\underset{1}{\mathbf{n}},\underset{2}{\mathbf{n}})}\underset{1}{\mathbf{n}}]\cdot\overset{\gg}{\mathbf{d}}\cdot\underset{1}{\mathbf{n}},\end{aligned}$$

and

$$\begin{aligned}\frac{D}{Dt}(\cos\theta_{(\underset{1}{\mathbf{n}},\underset{2}{\mathbf{n}})})&=\frac{D}{Dt}[\cos(\Theta_{(\underset{1}{\mathbf{N}},\underset{2}{\mathbf{N}})}-\gamma_{(\underset{1}{\mathbf{n}},\underset{2}{\mathbf{n}})})]\\
&=-\sin\theta_{(\underset{1}{\mathbf{n}},\underset{2}{\mathbf{n}})}\frac{D}{Dt}(\Theta_{(\underset{1}{\mathbf{N}},\underset{2}{\mathbf{N}})}-\gamma_{(\underset{1}{\mathbf{n}},\underset{2}{\mathbf{n}})})=\sin\theta_{(\underset{1}{\mathbf{n}},\underset{2}{\mathbf{n}})}\frac{D}{Dt}\gamma_{(\underset{1}{\mathbf{n}},\underset{2}{\mathbf{n}})}.\end{aligned}$$

From the foregoing equations,

$$\frac{D}{Dt}\gamma_{(\underset{1}{\mathbf{n}},\underset{2}{\mathbf{n}})}=\frac{1}{\sin\theta_{(\underset{1}{\mathbf{n}},\underset{2}{\mathbf{n}})}}\{[\underset{1}{\mathbf{n}}-\cos\theta_{(\underset{1}{\mathbf{n}},\underset{2}{\mathbf{n}})}\underset{2}{\mathbf{n}}]\cdot\overset{\gg}{\mathbf{d}}\cdot\underset{2}{\mathbf{n}}+[\underset{2}{\mathbf{n}}-\cos\theta_{(\underset{1}{\mathbf{n}},\underset{2}{\mathbf{n}})}\underset{1}{\mathbf{n}}]\cdot\overset{\gg}{\mathbf{d}}\cdot\underset{1}{\mathbf{n}}\}.$$

This theorem is proved. ■

3.2.2. Strain rates

From the previous discussion of the rate of arc length, area and volume, the strain rate with respect to time is very important in nonlinear continuum mechanics.

Therefore, the strain rates will be discussed as follows.

Theorem 3.22. *For two position vectors* $\mathbf{P}$ *in* $\mathfrak{B}$ *and* $\mathbf{p}$ *in* $\mathfrak{b}$ *for time* t, *the material derivatives of strain tensors (or strain rates) are given by*

$$
\begin{aligned}
&\frac{D}{Dt}\overset{\ll}{\mathbf{C}} = 2(\overset{<>}{\mathbf{F}})^{\mathrm{T}}\cdot\overset{\gg}{\mathbf{d}}\cdot\overset{><}{\mathbf{F}},\quad \frac{D}{Dt}\overset{-1}{\overset{\ll}{\mathbf{C}}} = -2\overset{-1}{\overset{<>}{\mathbf{F}}}\cdot\overset{\gg}{\mathbf{d}}\cdot(\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}};\\
&\frac{D}{Dt}\overset{-1}{\overset{\gg}{\mathbf{c}}} = \overset{\gg}{\mathbf{D}}\cdot\overset{-1}{\overset{\gg}{\mathbf{c}}}+\overset{-1}{\overset{\gg}{\mathbf{c}}}\cdot(\overset{\gg}{\mathbf{D}})^{\mathrm{T}},\quad \frac{D}{Dt}\overset{\gg}{\mathbf{c}} = -(\overset{\gg}{\mathbf{c}}\cdot\overset{\gg}{\mathbf{D}}+(\overset{\gg}{\mathbf{D}})^{\mathrm{T}}\cdot\overset{\gg}{\mathbf{c}});\\
&\frac{D}{Dt}\overset{\ll}{\mathbf{E}} = (\overset{<>}{\mathbf{F}})^{\mathrm{T}}\cdot\overset{\gg}{\mathbf{d}}\cdot\overset{><}{\mathbf{F}},\quad \frac{D}{Dt}\overset{\gg}{\mathbf{e}} = \overset{\gg}{\mathbf{d}}-[\overset{\gg}{\mathbf{e}}\cdot\overset{\gg}{\mathbf{D}}+(\overset{\gg}{\mathbf{D}})^{\mathrm{T}}\cdot\overset{\gg}{\mathbf{e}}].
\end{aligned}
\tag{3.148}
$$

Proof: From the definitions of the deformation gradients,

$$
\begin{aligned}
\frac{D}{Dt}\overset{\ll}{\mathbf{C}} &= \frac{D}{Dt}[(\overset{<>}{\mathbf{F}})^{\mathrm{T}}\cdot\overset{><}{\mathbf{F}}] = \frac{D}{Dt}(\overset{<>}{\mathbf{F}})^{\mathrm{T}}\cdot\overset{><}{\mathbf{F}}+(\overset{<>}{\mathbf{F}})^{\mathrm{T}}\cdot\frac{D}{Dt}\overset{><}{\mathbf{F}}\\
&= (\overset{<>}{\mathbf{F}})^{\mathrm{T}}\cdot(\overset{\gg}{\mathbf{D}})^{\mathrm{T}}\cdot\overset{><}{\mathbf{F}}+(\overset{<>}{\mathbf{F}})^{\mathrm{T}}\cdot\overset{\gg}{\mathbf{D}}\cdot\overset{><}{\mathbf{F}}\\
&= 2(\overset{<>}{\mathbf{F}})^{\mathrm{T}}\cdot\overset{\gg}{\mathbf{d}}\cdot\overset{><}{\mathbf{F}},\\
\frac{D}{Dt}\overset{-1}{\overset{\ll}{\mathbf{C}}} &= \frac{D}{Dt}[\overset{-1}{\overset{<>}{\mathbf{F}}}\cdot(\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}}] = \frac{D}{Dt}\overset{-1}{\overset{<>}{\mathbf{F}}}\cdot(\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}}+\overset{-1}{\overset{<>}{\mathbf{F}}}\cdot\frac{D}{Dt}(\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}}\\
&= -\overset{-1}{\overset{<>}{\mathbf{F}}}\cdot\overset{\gg}{\mathbf{D}}\cdot(\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}}-\overset{-1}{\overset{<>}{\mathbf{F}}}\cdot(\overset{\gg}{\mathbf{D}})^{\mathrm{T}}\cdot(\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}} = -2\overset{-1}{\overset{<>}{\mathbf{F}}}\cdot\overset{\gg}{\mathbf{d}}\cdot(\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}},\\
\frac{D}{Dt}\overset{-1}{\overset{\gg}{\mathbf{c}}} &= \frac{D}{Dt}[\overset{><}{\mathbf{F}}\cdot(\overset{<>}{\mathbf{F}})^{\mathrm{T}}] = \frac{D}{Dt}\overset{><}{\mathbf{F}}\cdot(\overset{<>}{\mathbf{F}})^{\mathrm{T}}+\overset{><}{\mathbf{F}}\cdot\frac{D}{Dt}(\overset{<>}{\mathbf{F}})^{\mathrm{T}}\\
&= \overset{\gg}{\mathbf{D}}\cdot\overset{><}{\mathbf{F}}\cdot(\overset{<>}{\mathbf{F}})^{\mathrm{T}}+\overset{><}{\mathbf{F}}\cdot(\overset{<>}{\mathbf{F}})^{\mathrm{T}}\cdot(\overset{\gg}{\mathbf{D}})^{\mathrm{T}} = \overset{\gg}{\mathbf{D}}\cdot\overset{-1}{\overset{\gg}{\mathbf{c}}}+\overset{-1}{\overset{\gg}{\mathbf{c}}}\cdot(\overset{\gg}{\mathbf{D}})^{\mathrm{T}},\\
\frac{D}{Dt}\overset{\gg}{\mathbf{c}} &= \frac{D}{Dt}[(\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}}\cdot\overset{-1}{\overset{<>}{\mathbf{F}}}] = \frac{D}{Dt}(\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}}\cdot\overset{-1}{\overset{<>}{\mathbf{F}}}+(\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}}\cdot\frac{D}{Dt}\overset{-1}{\overset{<>}{\mathbf{F}}}\\
&= -(\overset{\gg}{\mathbf{D}})^{\mathrm{T}}\cdot(\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}}\cdot\overset{-1}{\overset{<>}{\mathbf{F}}}-(\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}}\cdot\overset{-1}{\overset{<>}{\mathbf{F}}}\cdot\overset{\gg}{\mathbf{D}} = -(\overset{\gg}{\mathbf{D}})^{\mathrm{T}}\cdot\overset{\gg}{\mathbf{c}}-\overset{\gg}{\mathbf{c}}\cdot\overset{\gg}{\mathbf{D}},\\
\frac{D}{Dt}\overset{\ll}{\mathbf{E}} &= \frac{D}{Dt}[\frac{1}{2}(\overset{\ll}{\mathbf{C}}-\overset{\ll}{\mathbf{I}})] = \frac{1}{2}\frac{D}{Dt}\overset{\ll}{\mathbf{C}} = (\overset{<>}{\mathbf{F}})^{\mathrm{T}}\cdot\overset{\gg}{\mathbf{d}}\cdot\overset{><}{\mathbf{F}},\\
\frac{D}{Dt}\overset{\gg}{\mathbf{e}} &= \frac{D}{Dt}[\frac{1}{2}(\overset{\gg}{\mathbf{I}}-\overset{\gg}{\mathbf{c}})] = -\frac{1}{2}\frac{D}{Dt}\overset{\gg}{\mathbf{c}} = \frac{1}{2}[(\overset{\gg}{\mathbf{D}})^{\mathrm{T}}\cdot\overset{\gg}{\mathbf{c}}+\overset{\gg}{\mathbf{c}}\cdot\overset{\gg}{\mathbf{D}}]\\
&= \frac{1}{2}[(\overset{\gg}{\mathbf{D}})^{\mathrm{T}}\cdot(\overset{\gg}{\mathbf{I}}-2\overset{\gg}{\mathbf{e}})+(\overset{\gg}{\mathbf{I}}-2\overset{\gg}{\mathbf{e}})\cdot\overset{\gg}{\mathbf{D}}]\\
&= \overset{\gg}{\mathbf{d}}-[(\overset{\gg}{\mathbf{D}})^{\mathrm{T}}\cdot\overset{\gg}{\mathbf{e}}+\overset{\gg}{\mathbf{e}}\cdot\overset{\gg}{\mathbf{D}}].
\end{aligned}
$$

This theorem is proved. ■

Theorem 3.23. *For two position vectors* $\mathbf{P}$ *in* $\mathfrak{B}$ *and* $\mathbf{p}$ *in* $\mathfrak{b}$ *for time* t *, the material derivatives of stretch and area ratios are given by*

$$\begin{aligned}\frac{D}{Dt}\Lambda_{\mathbf N} &= \frac{D}{Dt}\lambda_{\mathbf n} = \lambda_{\mathbf n} d_{\mathbf n} = \lambda_{\mathbf n}\mathbf n\cdot\overset{>>}{\mathbf d}\cdot\mathbf n,\\ \frac{D}{Dt}\sigma_{{}_A\mathbf N} &= \frac{D}{Dt}\sigma_{{}_a\mathbf n} = \sigma_{{}_a\mathbf n}[\mathrm I(\overset{>>}{\mathbf d}) - d_{{}_a\mathbf n}]\\ &= \sigma_{{}_a\mathbf n}[\mathbf I(\overset{>>}{\mathbf d}) - {}_a\mathbf n\cdot\overset{>>}{\mathbf d}\cdot{}_a\mathbf n].\end{aligned} \tag{3.149}$$

Proof: Because $\Lambda_{\mathbf N} = \sqrt{\mathbf N\cdot\overset{>>}{\mathbf C}\cdot\mathbf N}$,

$$\begin{aligned}\frac{D}{Dt}\Lambda_{\mathbf N} &= \frac{1}{2\Lambda_{\mathbf N}}(\mathbf N\cdot\frac{D}{Dt}\overset{>>}{\mathbf C}\cdot\mathbf N) = \frac{1}{\Lambda_{\mathbf N}}\mathbf N\cdot(\overset{>>}{\mathbf F})^{\mathrm T}\cdot\overset{>>}{\mathbf d}\cdot\overset{>>}{\mathbf F}\cdot\mathbf N\\ &= (\frac{|d\mathbf p|}{|d\mathbf P|})^{-1}\overset{>>}{\mathbf F}\cdot\mathbf N\cdot\overset{>>}{\mathbf d}\cdot\overset{>>}{\mathbf F}\cdot\mathbf N = (\frac{|d\mathbf p|}{|d\mathbf P|})^{-1}\overset{>>}{\mathbf F}\cdot\frac{d\mathbf P}{|d\mathbf P|}\cdot\overset{>>}{\mathbf d}\cdot\overset{>>}{\mathbf F}\cdot\frac{d\mathbf P}{|d\mathbf P|}\\ &= \frac{(\overset{>>}{\mathbf F}\cdot d\mathbf P)\cdot\overset{>>}{\mathbf d}\cdot(\overset{>>}{\mathbf F}\cdot d\mathbf P)}{|d\mathbf p||d\mathbf P|} = \frac{d\mathbf p\cdot\overset{>>}{\mathbf d}\cdot d\mathbf p}{|d\mathbf p|\cdot|d\mathbf P|}\\ &= \frac{|d\mathbf p|}{|d\mathbf P|}\frac{d\mathbf p\cdot\overset{>>}{\mathbf d}\cdot d\mathbf p}{|d\mathbf p|\cdot|d\mathbf p|} = \Lambda_{\mathbf N}\mathbf n\cdot\overset{>>}{\mathbf d}\cdot\mathbf n,\end{aligned}$$

and also because $\Lambda_{\mathbf N} = \lambda_{\mathbf n}$,

$$\frac{D}{Dt}\lambda_{\mathbf n} = \lambda_{\mathbf n}\mathbf n\cdot\overset{>>}{\mathbf d}\cdot\mathbf n = \lambda_{\mathbf n} d_{\mathbf n}.$$

For $\sigma_{{}_A\mathbf N} = \mathscr J({}_A\mathbf N\cdot\overset{-1}{\overset{<<}{\mathbf C}}\cdot{}_A\mathbf N)^{\frac{1}{2}} = \sigma_{{}_a\mathbf n}$ and $D\mathscr J/Dt = \mathscr J\mathbf I(\overset{>>}{\mathbf d})$ in Eq.(3.143),

$$\begin{aligned}\frac{D}{Dt}\sigma_{{}_A\mathbf N} &= \frac{1}{\mathscr J}\frac{D\mathscr J}{Dt}\sigma_{{}_A\mathbf N} + \mathscr J^2\frac{1}{2\sigma_{{}_A\mathbf N}}({}_A\mathbf N\cdot\frac{D}{Dt}\overset{-1}{\overset{<<}{\mathbf C}}\cdot{}_A\mathbf N)\\ &= \mathrm I(\overset{>>}{\mathbf d})\sigma_{{}_A\mathbf N} - \frac{1}{\sigma_{{}_A\mathbf N}}\{[\mathscr J(\overset{-1}{\overset{<>}{\mathbf F}})^T\cdot{}_A\mathbf N]\cdot\overset{>>}{\mathbf d}\cdot[\mathscr J(\overset{-1}{\overset{><}{\mathbf F}})^T\cdot{}_A\mathbf N]\}\\ &= \mathrm I(\overset{>>}{\mathbf d})\sigma_{{}_A\mathbf N} - \frac{d\overset{>}{\mathbf a}\cdot\overset{>>}{\mathbf d}\cdot d\overset{>}{\mathbf a}}{|d\overset{>}{\mathbf a}|\cdot|d\overset{<}{\mathbf A}|},\\ &= \mathrm I(\overset{>>}{\mathbf d})\sigma_{{}_A\mathbf N} - \frac{d\overset{>}{\mathbf a}\cdot\overset{>>}{\mathbf d}\cdot d\overset{>}{\mathbf a}}{|d\overset{>}{\mathbf a}|\cdot|d\overset{>}{\mathbf a}|}\frac{|d\overset{>}{\mathbf a}|}{|d\overset{<}{\mathbf A}|}\\ &= \mathrm I(\overset{>>}{\mathbf d})\sigma_{{}_A\mathbf N} - {}_a\mathbf n\cdot\overset{>>}{\mathbf d}\cdot{}_a\mathbf n\sigma_{{}_A\mathbf N},\end{aligned}$$

$$\frac{D}{Dt}\sigma_{_A\mathbf{N}} = \frac{D}{Dt}\sigma_{_a\mathbf{n}} = (\mathrm{I}(\overset{\gg}{\mathbf{d}}) - {_a\mathbf{n}}\cdot\overset{\gg}{\mathbf{d}}\cdot{_a\mathbf{n}})\sigma_{_a\mathbf{n}} = (\mathrm{I}(\overset{\gg}{\mathbf{d}}) - d_{_a\mathbf{n}})\sigma_{_a\mathbf{n}}.$$

This theorem is proved. ■

3.3. Dynamics

In this section, forces and stresses in the continuous body will be presented. Further the relation between the force and acceleration in the continuous media will be addressed.

3.3.1. Forces and stresses

Dynamics is to determine the response of continuous media under external forces. The external forces include body force and surface force. For two configurations $\mathfrak{B}$ and $\mathfrak{b}$ for time t, the body force in $\mathfrak{b}$ per unit volume is denoted by $\mathbf{f}(\mathbf{P},t)$, and the body moment in $\mathfrak{b}$ per unit volume is represented by $\mathfrak{M}(\mathbf{P},t)$. The total force and moment to the origin are given by

$$\boldsymbol{\mathcal{F}} = \int_v \mathbf{f}(\mathbf{P},t)dv \text{ and } \boldsymbol{\mathcal{M}} = \int_v [\mathbf{p}\times\mathbf{f}(\mathbf{P},t) + \mathfrak{M}(\mathbf{P},t)]dv. \tag{3.150}$$

The surface force includes the distributed and concentrated forces on the surface with an exterior unit normal $\mathbf{n}$. The concentrated forces and moments act on the location of point $\mathbf{p}_\alpha$, are expressed by $\boldsymbol{\mathcal{F}}_\alpha$ and $\boldsymbol{\mathcal{M}}_\alpha$. The distributed force and moment on the surface are denoted by $\mathbf{t}_{(\mathbf{n})}$ and $\mathbf{m}_{(\mathbf{n})}$, respectively. The total external force on the continuous body $\mathfrak{b}$ is

$$\boldsymbol{\mathcal{F}} = \oint_a \mathbf{t}_{(\mathbf{n})}da + \int_v \mathbf{f}(\mathbf{P},t)dv + \sum\nolimits_\alpha \boldsymbol{\mathcal{F}}_\alpha \tag{3.151}$$

and the total external moment on the continuous body $\mathfrak{b}$ to point O is

$$\begin{aligned}\boldsymbol{\mathcal{M}} = &\oint_a \left[\mathbf{m}_{(\mathbf{n})} + \mathbf{p}\times\mathbf{t}_{(\mathbf{n})}\right]da + \int_v [\mathfrak{M} + \mathbf{p}\times\mathbf{f}]dv \\ &+ \sum\nolimits_\alpha [\boldsymbol{\mathcal{M}}_\alpha + \mathbf{p}_\alpha \times \boldsymbol{\mathcal{F}}_\alpha].\end{aligned} \tag{3.152}$$

In addition to the external forces, the internal force is an important concept in continuous body, which is used to describe the interaction between two adjacent material points. Such the internal force is a contact force.

Consider a deformed continuous body $\mathfrak{b}$ under the external force ($\boldsymbol{\mathcal{F}}$) and the external moment ($\boldsymbol{\mathcal{M}}$). To investigate the internal force between two adjacent material points, the cross-section method is adopted. The deformed continuous body $\mathfrak{b}$ is cut into two parts through a cross-section Σ, as shown in Fig.3.7. A normal vector of the cross-section at point $\mathbf{p}(\mathbf{P},t)$ is a positive direction $\mathbf{n}$ for one of two parts but as a negative direction ($-\mathbf{n}$) for the other part. The corresponding cross-sections are represented by $\Sigma_{\mathbf{n}}$ and $\Sigma_{-\mathbf{n}}$. Suppose that the external force and moment on the part with the positive normal vector are $\mathbf{F}_1$ and $\mathbf{M}_1$, respectively. The external force and moment on the rest part are $\mathbf{F}_2$ and $\mathbf{M}_2$ accordingly. The cross-section is a new surface. Thus, the distributed force and moment on the new surface are the contact force and moment, which can be expressed by $\mathbf{t}_{(\mathbf{n})}$ and $\mathbf{m}_{(\mathbf{n})}$. Such distributed force and moments can be defined by the density of the force and moment on the new surface. On the cross section with the exterior unit normal $\mathbf{n}$, consider an area Δa of the cross section at point $\mathbf{p}(\mathbf{P},t)$, on which the total force and moment are $\Delta\mathbf{t}$ and $\Delta\mathbf{m}$. Further, the *stress vectors* and *couple stress vectors* can be defined.

Definition 3.31. At point $\mathbf{p}$ in $\mathfrak{b}$ for time t, the *stress vector* and *couple stress vector* on the exterior unit normal direction ($\mathbf{n}$) of the cross section Σ are defined by

$$\mathbf{t}_{(\mathbf{n})} \equiv \lim_{\Delta a\to 0}\frac{\Delta\mathbf{t}}{\Delta a} \quad\text{and}\quad \mathbf{m}_{(\mathbf{n})} \equiv \lim_{\Delta a\to 0}\frac{\Delta\mathbf{m}}{\Delta a}. \tag{3.153}$$

Note that $\mathbf{t}_{(\mathbf{n})}$ and $\mathbf{m}_{(\mathbf{n})}$ are the *densities of force and moment* on the exterior normal direction of the cross-section, respectively. For the cross section, the total force and moments to the point O are computed by

$$\mathbf{F}_{(\mathbf{n})} = \int_{a'}\mathbf{t}_{(\mathbf{n})}da \quad\text{and}\quad \mathbf{M}_O = \int_{a'}(\mathbf{p}\times\mathbf{t}_{(\mathbf{n})} + \mathbf{m}_{(\mathbf{n})})da. \tag{3.154}$$

On the cross sections $\Sigma_{\mathbf{n}}$ and $\Sigma_{-\mathbf{n}}$, the stress vectors and couple stress vectors are

$$\mathbf{t}_{-\mathbf{n}} = -\mathbf{t}_{\mathbf{n}} \quad\text{and}\quad \mathbf{m}_{-\mathbf{n}} = -\mathbf{m}_{\mathbf{n}}. \tag{3.155}$$

3.3.2. Transport theorem

Under the condition in Eq.(3.4), the integral of a two-point tensor $\overset{<\cdot>}{\boldsymbol{\psi}}(\mathbf{p},t)$ based on the volume $\mathscr{V}$ of the configuration $\mathfrak{B}$ is expressed by a time-related quantity

$$\mathfrak{I}(t) = \int_{\mathscr{V}}\overset{<\cdot>}{\boldsymbol{\psi}}(\mathbf{p},t)dv. \tag{3.156}$$

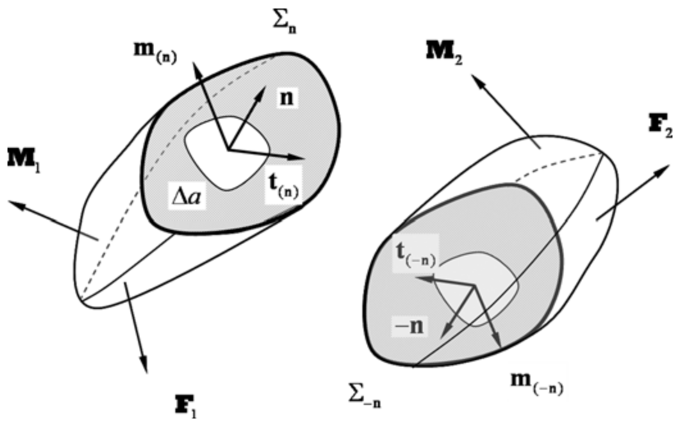

Fig. 3.7 Stress and couple stress vectors on the cross section $\Sigma_{\mathbf{n}}$ and $\Sigma_{-\mathbf{n}}$.

Theorem 3.24. *For two position vectors* $\mathbf{P}$ *in* $\mathfrak{B}$ *and* $\mathbf{p}$ *in* $\mathfrak{b}$ *for time* t *, there is a two-point tensor* $\overset{<\cdot>}{\boldsymbol{\psi}}(\mathbf{p},t)$ *. The time rate of change of the integral of the two-point tensor is*

$$\frac{D}{Dt}\int_{\mathscr{V}}\overset{<\cdot>}{\boldsymbol{\psi}}(\mathbf{p},t)dv=\int_{\mathscr{V}}[\frac{D}{Dt}\overset{<\cdot>}{\boldsymbol{\psi}}(\mathbf{p},t)+\overset{<\cdot>}{\boldsymbol{\psi}}(\mathbf{p},t)\mathrm{div}(\mathbf{v})]dv, \tag{3.157}$$

where $\mathrm{div}(\mathbf{v})=v^i_{;i}$.

Proof : By changes of variable, the material differentiation under the integral sign gives

$$\begin{aligned}\frac{D}{Dt}\int_{\mathscr{V}}\overset{<\cdot>}{\boldsymbol{\psi}}(\mathbf{p},t)dv&=\frac{D}{Dt}\int_{\mathscr{V}}\overset{<\cdot>}{\boldsymbol{\psi}}(\mathbf{P},t)\mathscr{J}(\mathbf{P},t)dV\\&=\int_{\mathscr{V}}[\frac{D}{Dt}\overset{<\cdot>}{\boldsymbol{\psi}}(\mathbf{P},t)\mathscr{J}(\mathbf{P},t)+\overset{<\cdot>}{\boldsymbol{\psi}}(\mathbf{P},t)\frac{D}{Dt}\mathscr{J}(\mathbf{P},t)]dV\\&=\int_{\mathscr{V}}[\frac{D}{Dt}\overset{<\cdot>}{\boldsymbol{\psi}}(\mathbf{P},t)+\overset{<\cdot>}{\boldsymbol{\psi}}(\mathbf{P},t)\mathrm{div}(\mathbf{v})]\mathscr{J}(\mathbf{P},t)dV\\&=\int_{\mathscr{V}}[\frac{D}{Dt}\overset{<\cdot>}{\boldsymbol{\psi}}(\mathbf{p},t)+\overset{<\cdot>}{\boldsymbol{\psi}}(\mathbf{p},t)\mathrm{div}(\mathbf{v})]dv.\end{aligned}$$

This theorem is proved. ■

If the two-point tensor $\overset{<\cdot>}{\boldsymbol{\psi}}(\mathbf{p},t)$ based on the configuration $\mathfrak{B}$ is translated into one-point tensor $\overset{>\cdot>}{\boldsymbol{\psi}}(\mathbf{p},t)$, the time change ratio of the integral is

$$\begin{aligned}\frac{D}{Dt}\int_{\mathscr{V}} \overset{>\cdot>}{\boldsymbol{\psi}}\, dv &= \int_{\mathscr{V}} [\frac{D}{Dt}\overset{>\cdot>}{\boldsymbol{\psi}} + \overset{>\cdot>}{\boldsymbol{\psi}}\,\mathrm{div}(\mathbf{v})]dv \\ &= \int_{\mathscr{V}} [(\frac{\partial \overset{>\cdot>}{\boldsymbol{\psi}}}{\partial t})_{\mathbf{P}} + \mathbf{v}\cdot\overset{>}{\nabla}\overset{>\cdot>}{\boldsymbol{\psi}} + (\overset{>}{\nabla}\cdot\mathbf{v})\overset{>\cdot>}{\boldsymbol{\psi}}]dv \\ &= \int_{\mathscr{V}} [(\frac{\partial \overset{>\cdot>}{\boldsymbol{\psi}}}{\partial t})_{\mathbf{P}} + \mathrm{div}(\mathbf{v}\overset{>\cdot>}{\boldsymbol{\psi}})]dv \\ &= \int_{v} (\frac{\partial \overset{>\cdot>}{\boldsymbol{\psi}}}{\partial t})_{\mathbf{P}}\, dv + \oint_{s} d\overset{>}{\mathbf{a}}\cdot\mathbf{v}\overset{>\cdot>}{\boldsymbol{\psi}}, \end{aligned} \tag{3.158}$$

where v is the volume of material volume V for time t, and s is the corresponding boundary surface of volume v.

Consider a mass density of the continuous body $\mathfrak{B}$ to be $\rho(\mathbf{P},t)$ as a one-point tensor. Suppose that the mass distribution in $\mathfrak{B}$ is continuous (i.e., $0 \le \rho < \infty$). From the conservation law of mass, the mass cannot increase and decrease during deformation. From the transport theorem,

$$0 = \frac{D}{Dt}\int_{\mathscr{V}} \rho dv = \int_{\mathscr{V}} [\frac{D\rho}{Dt} + \rho \mathrm{div}(\mathbf{v})]dv. \tag{3.159}$$

Because the mass volume is arbitrarily selected, the Euler continuous equation is given by

$$\begin{aligned} &0 = \frac{D\rho}{Dt} + \rho\mathrm{div}(\mathbf{v}) \text{ or } 0 = \frac{D\rho}{Dt} + \mathbf{v}\cdot\nabla\rho + (\mathbf{v}\cdot\nabla)\rho, \\ &0 = \frac{D\rho}{Dt} + (\rho v^i)_{,i}. \end{aligned} \tag{3.160}$$

Similarly,

$$0 = \frac{D}{Dt}\int_{\mathscr{V}} \rho dv = \frac{D}{Dt}\int_{\mathscr{V}} \rho \mathscr{J} dV = \int_{\mathscr{V}} \frac{D}{Dt}(\rho \mathscr{J})dV. \tag{3.161}$$

Thus,

$$0 = \frac{D}{Dt}(\rho\mathscr{J}) \;\Rightarrow\; \rho\mathscr{J} = \rho_0, \tag{3.162}$$

where $\rho(\mathbf{P},t_0) = \rho_0$. Equation (3.162) gives Lagrange's continuous equation. Further,

$$\int_{\mathscr{V}} \rho dv = \int_{\mathscr{V}} \rho \mathscr{J} dV = \int_{\mathscr{V}} \rho_0 dV. \tag{3.163}$$

In the transport theorem, the time rate of change of the integral of the two-point tensor can be determined by

$$\frac{D}{Dt}\int_{\mathscr{V}} \overset{<\cdot>}{\boldsymbol{\psi}}(\mathbf{p},t)\rho dv = \int_{\mathscr{V}} [\frac{D}{Dt}(\overset{<\cdot>}{\boldsymbol{\psi}}(\mathbf{p}\mathscr{J}))\rho + \overset{<\cdot>}{\boldsymbol{\psi}}(\mathbf{p}\mathscr{J})\frac{D\rho}{Dt}]dv$$
$$= \int_{\mathscr{V}} \frac{D}{Dt}(\overset{<\cdot>}{\boldsymbol{\psi}}(\mathbf{p},t))\rho dv. \tag{3.164}$$

3.3.3. Cauchy stress and couple-stress tensors

The strain state is determined by all the stretches ($\Lambda_{\mathbf{N}}$) or extensions ($E_{\mathbf{N}}$). The Green strain tensor $\overset{<<}{\mathbf{C}}$ can be used to determine the strain state because of $\Lambda_{\mathbf{N}} = (\mathbf{N}\cdot\overset{<<}{\mathbf{C}}\cdot\mathbf{N})^{\frac{1}{2}}$ (or $E_{\mathbf{N}} = (\mathbf{N}\cdot\overset{<<}{\mathbf{C}}\cdot\mathbf{N})^{\frac{1}{2}} - 1$). However, the stress state is formed by the stress tensor $\{\mathbf{t}_{_a\mathbf{n}}\}$ and the couple stress tensor $\{\mathbf{m}_{_a\mathbf{n}}\}$. Consider a mass body $\mathfrak{b}'$ with volume v' and boundary area a', which is corresponding to the material configuration $\mathfrak{B}'$ with volume $\mathscr{V}'$ and boundary area $\mathscr{A}'$. From the principle of momentum,

$$\frac{D}{Dt}\int_{\mathscr{V}'} \mathbf{v}\rho dv = \int_{\mathscr{V}'} \mathbf{f} dv + \oint_{\mathscr{A}'} \mathbf{t}_{_a\mathbf{n}} da. \tag{3.165}$$

The principle of angular momentum gives

$$\frac{D}{Dt}\int_{\mathscr{V}'} \mathbf{p}\times\mathbf{v}\rho dv = \int_{\mathscr{V}'} (\mathfrak{M} + \mathbf{p}\times\mathbf{f}) dv + \oint_{\mathscr{A}'} (\mathbf{m}_{_a\mathbf{n}} + \mathbf{p}\times\mathbf{t}_{_a\mathbf{n}}) da. \tag{3.166}$$

Because

$$\begin{gathered}\frac{D(\rho\mathscr{J})}{Dt} = 0 \quad \text{and} \quad \frac{D\mathbf{v}}{Dt} = \mathbf{a}, \\ \frac{D}{Dt}(\mathbf{p}\times\mathbf{v}) = \frac{D\mathbf{p}}{Dt}\times\mathbf{v} + \mathbf{p}\times\frac{D\mathbf{v}}{Dt} = \mathbf{v}\times\mathbf{v} + \mathbf{p}\times\mathbf{a} = \mathbf{p}\times\mathbf{a}\end{gathered} \tag{3.167}$$

and also because the external forces are invariant on the initial and deformed volume, equations (3.165) and (3.166) become

$$\int_{v'} \mathbf{a}\rho dv = \int_{v'} \mathbf{f} dv + \oint_{a'} \mathbf{t}_{_a\mathbf{n}} da, \tag{3.168}$$

$$\int_{v'} \mathbf{p}\times\mathbf{a}\rho dv = \int_{v'} (\mathfrak{M} + \mathbf{p}\times\mathbf{f}) dv + \oint_{a'} (\mathbf{m}_{_a\mathbf{n}} + \mathbf{p}\times\mathbf{t}_{_a\mathbf{n}}) da. \tag{3.169}$$

Definition 3.32. At point $\mathbf{p}$ in configuration $\mathfrak{b}$ for time t, the *stress tensor* and *couple stress tensor* are, respectively, defined as

$$\overset{>>}{\mathbf{t}} \equiv \mathbf{t}_i \mathbf{g}^i = t^i_{\cdot j} \mathbf{g}_i \mathbf{g}^j \quad \text{and} \quad \overset{>>}{\mathbf{m}} \equiv \mathbf{m}_i \mathbf{g}^i = m^i_{\cdot j} \mathbf{g}_i \mathbf{g}^j \tag{3.170}$$

which are also called the Cauchy stress tensor and Cauchy couple stress tensor accordingly.

Theorem 3.25. *For two position vectors* $\mathbf{P}$ *in* $\mathfrak{B}$ *and* $\mathbf{p}$ *in* $\mathfrak{b}$ *for time* t *, the stress vector* $\mathbf{t}_{{}_a\mathbf{n}}$ *and couple stress vector* $\mathbf{m}_{{}_a\mathbf{n}}$ *at point* $\mathbf{p}(\mathbf{P},t)$ *can be determined by the stress tensor* ($\overset{\gg}{\mathbf{t}}$) *and couple stress tensor* ($\overset{\gg}{\mathbf{m}}$) *on the other surfaces of the infinitesimal volume, i.e.,*

$$\mathbf{t}_{\mathbf{n}} = \overset{\gg}{\mathbf{t}} \cdot {}_a\mathbf{n} \quad \text{and}\ \mathbf{m}_{\mathbf{n}} = \overset{\gg}{\mathbf{m}} \cdot {}_a\mathbf{n}. \tag{3.171}$$

Proof : For the stress tetrahedron in Fig.3.8, using the mean-value theorem, the principles for the momentum and angular momentum give

$$(\mathbf{a}\rho)\Delta v = \mathbf{f}\Delta v + \mathbf{t}_i \Delta a^i + \mathbf{t}_{{}_a\mathbf{n}}\Delta a,$$

$$(\mathbf{p}\times\mathbf{a})\rho\Delta v = (\mathfrak{M} + \mathbf{p}\times\mathbf{f})\Delta v + (\mathbf{m}_{{}_a\mathbf{n}} + \mathbf{p}\times\mathbf{t}_{{}_a\mathbf{n}})\Delta a$$
$$-(\mathbf{m}_i + \mathbf{p}\times\mathbf{t}_i)\Delta a^i,$$

Since $\Delta a^i = \Delta a \mathbf{g}^i \cdot {}_a\mathbf{n}$ and $\lim_{\Delta a \to 0} \Delta v / \Delta a \to 0$, one obtains

$$\mathbf{t}_{{}_a\mathbf{n}} = \mathbf{t}_i \mathbf{g}^i \cdot {}_a\mathbf{n} \ \text{ and }\ \mathbf{m}_{{}_a\mathbf{n}} = \mathbf{m}_i \mathbf{g}^i \cdot {}_a\mathbf{n},$$

Because $\mathbf{g}^j \times \mathbf{g}^k / [\mathbf{g}^1\mathbf{g}^2\mathbf{g}^3] = \mathbf{g}_i$ ($i, j, k \in \{1, 2, 3\}$), $\mathbf{g}_i$ will be normal to the surface of $\mathbf{g}^j \times \mathbf{g}^k$. The normal vector ${}_a\mathbf{n}_i$ is also normal to the surface of $\mathbf{g}^j \times \mathbf{g}^k$. So the two vectors ${}_a\mathbf{n}_i$ and $\mathbf{g}_i$ are on the same direction and let ${}_a\mathbf{n}_i = \mathbf{g}_i$. So the Cauchy stress tensor and couple stress tensors can be determined by

$$\overset{\gg}{\mathbf{t}} \equiv \mathbf{t}_i \mathbf{g}^i = t^i_{.j} \mathbf{g}_i \mathbf{g}^j \ \text{ and }\ \overset{\gg}{\mathbf{m}} \equiv \mathbf{m}_i \mathbf{g}^i = m^i_{.j} \mathbf{g}_i \mathbf{g}^j,$$

So

$$\mathbf{t}_{\mathbf{n}} = \overset{\gg}{\mathbf{t}} \cdot {}_a\mathbf{n} \quad \text{and}\ \mathbf{m}_{\mathbf{n}} = \overset{\gg}{\mathbf{m}} \cdot {}_a\mathbf{n}.$$

This theorem is proved. ■

From Eq.(3.170),

$$\mathbf{t}_{{}_a\mathbf{n}} da = \overset{\gg}{\mathbf{t}} \cdot {}_a\mathbf{n} da = \overset{\gg}{\mathbf{t}} \cdot d\mathbf{a} \quad \text{and}\ \mathbf{m}_{{}_a\mathbf{n}} da = \overset{\gg}{\mathbf{m}} \cdot {}_a\mathbf{n} da = \overset{\gg}{\mathbf{m}} \cdot d\mathbf{a}. \tag{3.172}$$

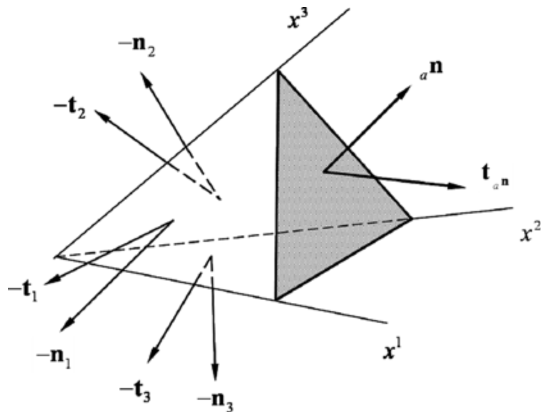

Fig. 3.8 Stress tetrahedron.

The normal stress on the direction of **n** can be determined by

$$t_{{}_a\mathbf{n}\,{}_a\mathbf{n}} = {}_a\mathbf{n}\cdot\mathbf{t}_{{}_a\mathbf{n}} = {}_a\mathbf{n}\cdot\overset{\gg}{\mathbf{t}}\cdot{}_a\mathbf{n}$$
$$= {}_a\mathbf{n}\cdot(\overset{\gg}{\mathbf{t}})^{\mathrm{T}}\cdot{}_a\mathbf{n} = {}_a\mathbf{n}\cdot\frac{1}{2}[\overset{\gg}{\mathbf{t}}+(\overset{\gg}{\mathbf{t}})^{\mathrm{T}}]\cdot{}_a\mathbf{n}. \quad (3.173)$$

In Fig.3.9, the stress tensors in the Cartesian coordinate system are depicted through two directions of x^1 and x^2 in order to make view clear. In a similar fashion, the stress tensor components and stress vectors can be sketched in the direction of x^3. The components of the stress tensor are expressed by t_{ij} ($i, j = 1, 2, 3$) which "i" and "j" are the surface direction along the axis of x^i and the stress component direction in the axis of x^j, respectively. $\mathbf{t}_i$ is the stress vector on the direction of the surface along the axis of x^i. In Fig.3.9, the relation between the stress vector and stress tensor component is clearly shown. Similarly, the couple stress tensor and couple stress vectors can be expressed.

Theorem 3.26. *For two position vectors* **P** *in* $\mathfrak{B}$ *and* **p** *in* $\mathfrak{b}$ *for time* t*, the necessary and sufficient conditions for the local balance of momentum are*

$$\overset{\gg}{\mathbf{t}}\cdot\overset{>}{\nabla}+\mathbf{f} = \rho\mathbf{a} \quad \text{and} \quad t^{ij}_{;j}+f^i = \rho a^i; \quad (3.174)$$

and the necessary and sufficient conditions for the local balance of angular momentum are

$$\overset{\gg}{\mathbf{m}}\cdot\overset{>}{\nabla}+\mathfrak{M} = \overset{\ggg}{\boldsymbol{\varepsilon}} : \overset{\gg}{\mathbf{t}} \quad \text{and} \quad \mathrm{m}^{ij}_{;j}+\mathfrak{M}^i = \varepsilon^{ijk}t_{jk}. \quad (3.175)$$

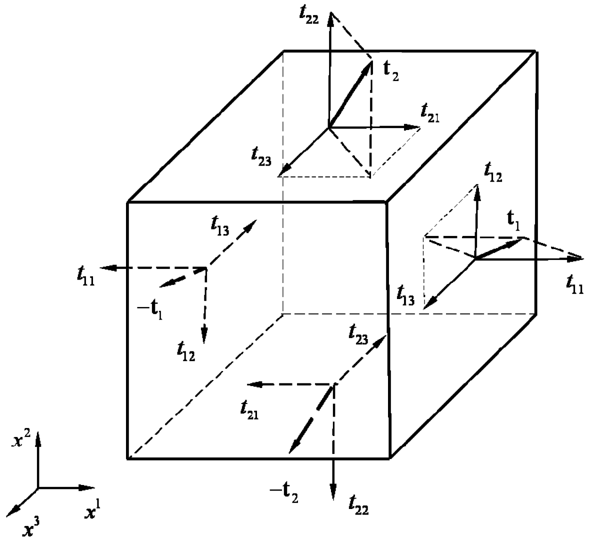

Fig. 3.9 Stress tensor in two directions of the rectangular coordinates.

Proof: Consider the Cauchy stress tensor $\overset{>>}{\mathbf{t}}$. From the divergence theorem,

$$\oint_a \overset{>>}{\mathbf{t}}\cdot d\overset{>}{\mathbf{a}} = \oint_a d\overset{>}{\mathbf{a}}\cdot(\overset{>>}{\mathbf{t}})^{\mathrm{T}} = \int_v dv\,\overset{>}{\nabla}\cdot(\overset{>>}{\mathbf{t}})^{\mathrm{T}} = \int_v dv\,\overset{>>}{\mathbf{t}}\cdot\overset{>}{\nabla},$$

$$\oint_a \mathbf{p}\times\overset{>>}{\mathbf{t}}\cdot d\overset{>}{\mathbf{a}} = \oint_a d\overset{>}{\mathbf{a}}\cdot(\mathbf{p}\times\overset{>>}{\mathbf{t}})^{\mathrm{T}} = \int_v dv(\mathbf{p}\times\overset{>>}{\mathbf{t}})\cdot\overset{>}{\nabla},$$

Because

$$\mathbf{p}\times\overset{>>}{\mathbf{t}} = x_i\mathbf{g}^i\times t_{jk}\mathbf{g}^j\mathbf{g}^k = (\mathbf{g}^i\times\mathbf{g}^j)\cdot(\mathbf{g}^l\cdot\mathbf{g}_l)x_it_{jk}\mathbf{g}^k$$

$$= \varepsilon^{ijl}\mathbf{g}_l\mathbf{g}_i\mathbf{g}_j : x_r\mathbf{g}^r t_{sk}\mathbf{g}^s\mathbf{g}^k = \overset{>>>}{\boldsymbol{\varepsilon}} : \mathbf{p}\overset{>>}{\mathbf{t}},$$

with $\overset{>>>}{\boldsymbol{\varepsilon}}\cdot\overset{>}{\nabla} = 0$, $\overset{>>>}{\boldsymbol{\varepsilon}} : (\overset{>>>}{\mathbf{t}})^{\mathrm{T}} = -\overset{>>>}{\boldsymbol{\varepsilon}} : \overset{>>}{\mathbf{t}}$ and $(\mathbf{p}\overset{>}{\nabla}) = \mathbf{I}$, then,

$$\oint_a \mathbf{p}\times\overset{>>}{\mathbf{t}}\cdot d\overset{>}{\mathbf{a}} = \int_v dv(\overset{>>>}{\boldsymbol{\varepsilon}} : \mathbf{p}\overset{>>}{\mathbf{t}})\cdot\overset{>}{\nabla} = \int_v dv\{\overset{>>>}{\boldsymbol{\varepsilon}} : [\mathbf{p}(\overset{>>}{\mathbf{t}}\cdot\overset{>}{\nabla}) + (\mathbf{p}\overset{>}{\nabla})\cdot(\overset{>>}{\mathbf{t}})^{\mathrm{T}}]\}$$

$$= \int_v dv[(\mathbf{p}\times\overset{>>}{\mathbf{t}}\cdot\overset{>}{\nabla}) - \overset{>>>}{\boldsymbol{\varepsilon}} : \overset{>>}{\mathbf{t}}].$$

Thus,

$$\oint_a \mathbf{p}\times\overset{>>}{\mathbf{t}}\cdot d\overset{>}{\mathbf{a}} = \int_v dv[\mathbf{p}\times(\overset{>>}{\mathbf{t}}\cdot\overset{>}{\nabla}) - \overset{>>>}{\boldsymbol{\varepsilon}} : \overset{>>}{\mathbf{t}}],$$

From Eq.(3.168),

$$\int_v \mathbf{a}\rho dv = \int_v \mathbf{f}dv + \oint_a (\mathbf{t}_{{}_a\mathbf{n}})da = \int_v \mathbf{f}dv + \oint_a \overset{>>}{\mathbf{t}}\cdot({}_a\mathbf{n}da)$$
$$= \int_v \mathbf{f}dv + \oint_a \overset{>>}{\mathbf{t}}\cdot d\mathbf{a}.$$

Further,

$$\int_v (\mathbf{a}\rho - \mathbf{f} - \overset{>>}{\mathbf{t}}\cdot\overset{>}{\nabla})dv = 0.$$

Since the volume is selected arbitrarily, the local balance of momentum is obtained, i.e.,

$$\overset{>>}{\mathbf{t}}\cdot\overset{>}{\nabla} + \mathbf{f} = \rho\mathbf{a},$$

or

$$t^{ij}_{;j} + f^i = \rho a^i.$$

From Eq.(3.168),

$$\int_v \mathbf{p}\times\mathbf{a}\rho dv = \int_v (\mathfrak{M} + \mathbf{p}\times\mathbf{f})dv + \oint_a (\mathbf{m}_n + \mathbf{p}\times\mathbf{t}_n)da$$
$$= \int_v (\mathfrak{M} + \mathbf{p}\times\mathbf{f})dv + \oint_a (\overset{>>}{\mathbf{m}} + \mathbf{p}\times\overset{>>}{\mathbf{t}})\cdot\mathbf{n}da$$
$$= \int_v (\mathfrak{M} + \mathbf{p}\times\mathbf{f})dv + \oint_a (\overset{>>}{\mathbf{m}} + \mathbf{p}\times\overset{>>}{\mathbf{t}})\cdot d\mathbf{a}$$
$$= \int_v (\mathfrak{M} + \mathbf{p}\times\mathbf{f})dv + \oint_a \overset{>>}{\mathbf{m}}\cdot d\mathbf{a} + \oint_a (\mathbf{p}\times\overset{>>}{\mathbf{t}})\cdot d\mathbf{a}$$
$$= \int_v (\mathfrak{M} + \mathbf{p}\times\mathbf{f})dv + \oint_a \overset{>>}{\mathbf{m}}\cdot d\mathbf{a} + \int_v dv[\mathbf{p}\times(\overset{>>}{\mathbf{t}}\cdot\overset{>}{\nabla}) - \overset{>>>}{\boldsymbol{\varepsilon}} : \overset{>>}{\mathbf{t}}].$$

The re-arrangement of the foregoing equation gives

$$\int_v [\overset{>>}{\mathbf{m}}\cdot\overset{>}{\nabla} + \mathfrak{M} - \overset{>>>}{\boldsymbol{\varepsilon}} : \overset{>>}{\mathbf{t}} + \mathbf{p}\times(\rho\mathbf{a} - \mathbf{f} - \overset{>>}{\mathbf{t}}\cdot\overset{>}{\nabla})]dv$$
$$= \int_v (\overset{>>}{\mathbf{m}}\cdot\overset{>}{\nabla} + \mathfrak{M} - \overset{>>>}{\boldsymbol{\varepsilon}} : \overset{>>}{\mathbf{t}})dv = 0.$$

Because the volume is chosen arbitrarily,

$$\overset{>>}{\mathbf{m}}\cdot\overset{>}{\nabla} + \mathfrak{M} - \overset{>>>}{\boldsymbol{\varepsilon}} : \overset{>>}{\mathbf{t}} = 0,$$

or

$$m^{ij}_{;j} + \mathfrak{M}^i = \varepsilon^{ijk} t_{jk}.$$

The local balance of angular momentum is obtained. This theorem is proved. ■

Left multiplication of $\overset{>>>}{\boldsymbol{\varepsilon}}$ in Eq.(3.175) gives

$$\overset{>>>}{\boldsymbol{\varepsilon}}\cdot\overset{>>}{\mathbf{m}}\cdot\overset{>}{\nabla} + \overset{>>>}{\boldsymbol{\varepsilon}}\cdot\mathfrak{M} = \overset{>>>}{\boldsymbol{\varepsilon}}\cdot\overset{>>>}{\boldsymbol{\varepsilon}} : \overset{>>}{\mathbf{t}}. \tag{3.176}$$

Because

$$\overset{>>>}{\boldsymbol{\varepsilon}} \cdot \overset{>>>}{\boldsymbol{\varepsilon}} : \overset{>>}{\mathbf{t}} = \varepsilon_{ijk}\mathbf{g}^i\mathbf{g}^j\mathbf{g}^k \cdot \varepsilon^{rst} t_{rs}\mathbf{g}_t = \varepsilon_{ijk}\varepsilon^{rsk}t_{rs}\mathbf{g}^i\mathbf{g}^j$$

$$= \delta^{rs}_{ij} t_{rs}\mathbf{g}^i\mathbf{g}^j = (t_{ij}-t_{ji})\mathbf{g}^i\mathbf{g}^j = \overset{>>}{\mathbf{t}} - (\overset{>>}{\mathbf{t}})^{\mathrm{T}}. \tag{3.177}$$

Equation (3.176) gives

$$\overset{>>>}{\mathbf{m}} \cdot \overset{>}{\nabla} + \overset{>>}{\mathfrak{M}} = \frac{1}{2}[\overset{>>}{\mathbf{t}} - (\overset{>>}{\mathbf{t}})^{\mathrm{T}}], \tag{3.178}$$

where

$$\overset{>>}{\mathfrak{M}} \equiv \frac{1}{2}\overset{>>>}{\boldsymbol{\varepsilon}} : \mathfrak{M} \text{ and } \overset{>>>}{\mathbf{m}} = \frac{1}{2}\overset{>>>}{\boldsymbol{\varepsilon}} : \overset{>>}{\mathbf{m}}. \tag{3.179}$$

If $\overset{>>}{\mathfrak{M}} = 0$ and $\overset{>>>}{\mathbf{m}} = 0$, equation (3.178) gives

$$\overset{>>}{\mathbf{t}} = (\overset{>>}{\mathbf{t}})^{\mathrm{T}}. \tag{3.180}$$

The Cauchy stress tensor belongs to the Euler description. Sometimes, Lagrange's description will be very convenient in application. The Piola stress tensor is defined as follows.

Definition 3.33. A two-point stress tensor in $\mathfrak{B}$ and $\mathfrak{b}$,

$$\overset{><}{\boldsymbol{\tau}} = \tau^{iM}\mathbf{g}_i\mathbf{G}_M \equiv \mathscr{J}\,\overset{>>}{\mathbf{t}} \cdot (\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}} = \mathscr{J} t^{ir} X^M_{,r}\mathbf{g}_i\mathbf{G}_M \tag{3.181}$$

is called the *Piola stress tensor*.

The Piola stress tensor $\overset{><}{\boldsymbol{\tau}}$ is *the stress tensor on the initial configuration* $\mathfrak{B}$ in the direction of $\mathbf{N}$ to express the Cauchy stress tensor $\overset{>>}{\mathbf{t}}$ in the deformed configuration $\mathfrak{b}$, which can be presented through the following theorem.

Theorem 3.27. *For two position vectors* $\mathbf{P}$ *in* $\mathfrak{B}$ *and* $\mathbf{p}$ *in* $\mathfrak{b}$ *for time* t*, the stress vectors on the corresponding surface at point* $\mathbf{P}$ *and* $\mathbf{p}$ *satisfy*

$$\overset{><}{\boldsymbol{\tau}}_{{}_A\mathrm{N}} = \overset{>>}{\mathbf{t}}_{{}_a\mathbf{n}}\,\sigma_{{}_a\mathbf{n}}, \tag{3.182}$$

where

$$\boldsymbol{\tau}_{{}_A\mathbf{N}} = \overset{><}{\boldsymbol{\tau}} \cdot {}_A\mathbf{N} \text{ and } \mathbf{t}_{{}_a\mathbf{n}} = \overset{>>}{\mathbf{t}} \cdot {}_a\mathbf{n}. \tag{3.183}$$

Proof: The force in the initial configuration $\mathfrak{B}$ and in the deformed configuration $\mathfrak{b}$ are equal, i.e.,

$$\overset{><}{\boldsymbol{\tau}} \cdot d\overset{<}{\mathbf{A}} = \overset{>>}{\mathbf{t}} \cdot d\overset{>}{\mathbf{a}}.$$

Because of $d\overset{<}{\mathbf{A}} = {}_A\mathbf{N}dA$ and $d\overset{>}{\mathbf{a}} = {}_a\mathbf{n}da$,

$$\overset{><}{\boldsymbol{\tau}} \cdot {}_A\mathbf{N} = \overset{>>}{\mathbf{t}} \cdot {}_a\mathbf{n}\frac{da}{dA} = \overset{>>}{\mathbf{t}} \cdot {}_a\mathbf{n}\sigma_{\mathbf{n}} \;\Rightarrow\; \overset{><}{\boldsymbol{\tau}}_{{}_A\mathbf{N}} = \overset{>>}{\mathbf{t}}_{{}_a\mathbf{n}} \cdot \sigma_{\mathbf{n}}.$$

This theorem is proved. ■

From the definition of the Piola stress tensor, the corresponding local balance of momentum can be given as follows.

Theorem 3.28. *For two position vectors* $\mathbf{P}$ *in* $\mathfrak{B}$ *and* $\mathbf{p}$ *in* $\mathfrak{b}$ *for time* t, *the local balance of momentum* (*Boussinesq equation*) *are given by*

$$\begin{aligned}
&\overset{<}{\Box} \cdot (\overset{><}{\boldsymbol{\tau}})^{\mathrm{T}} + \mathscr{J}\mathbf{f} = \rho_0 \mathbf{a} \quad \text{or} \quad \overset{><}{\boldsymbol{\tau}} \cdot \overset{<}{\Box} + \mathscr{J}\mathbf{f} = \rho_0 \mathbf{a},\\
&\tau^{iJ}_{;J} + \mathscr{J} f^i = \rho_0 a^i \quad \text{or} \quad (\mathscr{J} t^{ir} X^M_{,r})_{:M} + \mathscr{J} f^i = \rho_0 a^i.
\end{aligned} \tag{3.184}$$

Proof: The momentum integral equation is

$$\int_v \mathbf{a}\rho dv = \int_v \mathbf{f} dv + \oint_a \overset{>>}{\mathbf{t}} \cdot d\mathbf{a}.$$

Because

$$\rho_0 = \mathscr{J}\rho \text{ and } dv = \mathscr{J} dV,$$

with the force invariance before and after deformation, i.e.,

$$\overset{>>}{\mathbf{t}} \cdot d\overset{>}{\mathbf{a}} = \overset{><}{\boldsymbol{\tau}} \cdot d\overset{<}{\mathbf{A}},$$

the momentum integral equation is

$$\int_{\mathscr{V}} \mathbf{a}\rho_0 dV = \int_{\mathscr{V}} \mathscr{J}\mathbf{f} dV + \oint_{\mathscr{A}} \overset{><}{\boldsymbol{\tau}} \cdot d\overset{<}{\mathbf{A}},$$

$$\begin{aligned}
\oint_{\mathscr{A}} \overset{><}{\boldsymbol{\tau}} \cdot d\overset{<}{\mathbf{A}} &= \oint_{\mathscr{A}} d\overset{<}{\mathbf{A}} \cdot \overset{><}{\boldsymbol{\tau}} = \int_{\mathscr{V}} dV \overset{<}{\Box} \cdot (\overset{<>}{\boldsymbol{\tau}})^{\mathrm{T}} = \int_{\mathscr{V}} dV \overset{<}{\Box} \cdot (\mathscr{J} \overset{-1}{\overset{<>}{\mathbf{F}}} \cdot \overset{>>}{\mathbf{t}})\\
&= \int_{\mathscr{V}} dV \overset{<>}{\boldsymbol{\tau}} \cdot \overset{<}{\Box} = \int_{\mathscr{V}} dV (\mathscr{J} \overset{>>}{\mathbf{t}} \cdot (\overset{-1}{\overset{><}{\mathbf{F}}})^{\mathrm{T}}) \cdot \overset{<}{\Box}
\end{aligned}$$

Further,

$$\begin{aligned}
\int_{\mathscr{V}} \mathbf{a}\rho_0 dV &= \int_{\mathscr{V}} \mathscr{J}\mathbf{f} dV + \int_{\mathscr{V}} dV \overset{<}{\Box} \cdot (\overset{<>}{\boldsymbol{\tau}})^{\mathrm{T}}\\
&= \int_{\mathscr{V}} \mathscr{J}\mathbf{f} dV + \int_{\mathscr{V}} dV \overset{<>}{\boldsymbol{\tau}} \cdot \overset{<}{\Box}.
\end{aligned}$$

With an arbitrary selection of volume, the foregoing equation gives

$$\overset{<}{\Box} \cdot (\overset{><}{\boldsymbol{\tau}})^{\mathrm{T}} + \mathscr{J}\mathbf{f} = \rho_0 \mathbf{a} \quad \text{or} \quad \overset{><}{\boldsymbol{\tau}} \cdot \overset{<}{\Box} + \mathscr{J}\mathbf{f} = \rho_0 \mathbf{a}.$$

This theorem is proved. ■

To define the stress on the initial configuration $\mathfrak{B}$, the Kirchhoff stress tensor will be introduced.

Definition 3.34. A one-point stress tensor in $\mathfrak{B}$,

$$\overset{<<}{\mathbf{T}} = T^{IJ}\mathbf{G}_I\mathbf{G}_J \equiv \overset{<>}{\overset{-1}{\mathbf{F}}}\cdot\overset{><}{\boldsymbol{\tau}} = \mathscr{J}\,\overset{<>}{\overset{-1}{\mathbf{F}}}\cdot\overset{>>}{\mathbf{t}}\cdot(\overset{><}{\overset{-1}{\mathbf{F}}})^{\mathrm{T}}$$
$$= \mathscr{J} X^M_{,r}\, t^{sr} X^N_{,s}\,\mathbf{G}_M\mathbf{G}_N, \tag{3.185}$$

is called *the Kirchhoff stress tensor*.

The expression in Eq.(3.185) is a stress transformation from the final configuration $\mathfrak{b}$ to the initial configuration $\mathfrak{B}$.

Theorem 3.29. *For two position vectors* $\mathbf{P}$ *in* $\mathfrak{B}$ *and* $\mathbf{p}$ *in* $\mathfrak{b}$ *for time* t, *the following relation exists*:

$$\sigma_{{}_a\mathbf{n}}\,\mathbf{t}_{{}_a\mathbf{n}} = \overset{><}{\mathbf{I}}\cdot\mathbf{T}_{{}_A\mathbf{N}} \tag{3.186}$$

with

$$\mathbf{t}_{{}_a\mathbf{n}} = \overset{>>}{\mathbf{t}}\cdot{}_a\mathbf{n} \;\text{ and }\; \overset{><}{\mathbf{I}}\cdot\mathbf{T}_{{}_A\mathbf{N}} = \overset{><}{\mathbf{F}}\cdot(\overset{<<}{\mathbf{T}}\cdot{}_A\mathbf{N}). \tag{3.187}$$

Proof : The forces in the initial configuration $\mathfrak{B}$ and in the deformed configuration $\mathfrak{b}$ are equal, i.e.,

$$\overset{>>}{\mathbf{t}}\cdot d\overset{>}{\mathbf{a}} = \overset{><}{\boldsymbol{\tau}}\cdot d\overset{<}{\mathbf{A}} = \mathscr{J}\,\overset{>>}{\mathbf{t}}\cdot(\overset{><}{\overset{-1}{\mathbf{F}}})^{\mathrm{T}}\cdot d\overset{<}{\mathbf{A}} = \overset{><}{\mathbf{F}}\cdot\mathscr{J}\,\overset{<>}{\overset{-1}{\mathbf{F}}}\cdot\overset{>>}{\mathbf{t}}\cdot(\overset{><}{\overset{-1}{\mathbf{F}}})^{\mathrm{T}}\cdot d\overset{<}{\mathbf{A}}.$$

Using the definition of the Kirchhoff stress tensor, the foregoing equation becomes

$$\overset{>>}{\mathbf{t}}\cdot d\overset{>}{\mathbf{a}} = \overset{><}{\mathbf{F}}\cdot\overset{<<}{\mathbf{T}}\cdot d\overset{<}{\mathbf{A}}.$$

The above equation is deformed as

$$\overset{>>}{\mathbf{t}}\cdot{}_a\mathbf{n}da = \overset{><}{\mathbf{F}}\cdot(\overset{<<}{\mathbf{T}}\cdot d\overset{<}{\mathbf{A}}) = \overset{><}{\mathbf{F}}\cdot(\overset{<<}{\mathbf{T}}\cdot{}_A\mathbf{N}dA).$$

With Eq.(3.187),

$$\mathbf{t}_{{}_a\mathbf{n}}\frac{da}{dA} = \mathbf{T}_{{}_A\mathbf{N}} \;\Rightarrow \sigma_{{}_a\mathbf{n}}\,\mathbf{t}_{{}_a\mathbf{n}} = \overset{><}{\mathbf{I}}\cdot\mathbf{T}_{{}_A\mathbf{N}}.$$

This theorem is proved. ■

The Piola and Kirchhoff stress tensors act on the surface $d\overset{<}{\mathbf{A}}$ on the initial configuration of $\mathfrak{B}$ in the direction of $\mathbf{N}$. If $\overset{>>}{\mathfrak{M}} = 0$ and $\overset{>>>}{\mathbf{m}} = 0$, the Cauchy stress tensor is symmetric (i.e., $\overset{>>}{\mathbf{t}} = (\overset{>>}{\mathbf{t}})^{\mathrm{T}}$). So the corresponding Kirchhoff stress tensor is symmetric because of

$$\begin{aligned}\overset{<<}{(\mathbf{T})}{}^{\mathrm{T}} &= (\mathscr{J}\,\overset{\overset{-1}{<>}}{\mathbf{F}}\cdot\overset{>>}{\mathbf{t}}\cdot(\overset{\overset{-1}{><}}{\mathbf{F}})^{\mathrm{T}})^{\mathrm{T}} = \mathscr{J}((\overset{\overset{-1}{><}}{\mathbf{F}})^{\mathrm{T}})^{\mathrm{T}}\cdot(\overset{>>}{\mathbf{t}})^{\mathrm{T}}\cdot(\overset{\overset{-1}{<>}}{\mathbf{F}})^{\mathrm{T}} \\ &= \mathscr{J}\,\overset{\overset{-1}{<>}}{\mathbf{F}}\cdot\overset{>>}{\mathbf{t}}\cdot(\overset{\overset{-1}{><}}{\mathbf{F}})^{\mathrm{T}} = \overset{<<}{\mathbf{T}}.\end{aligned} \tag{3.188}$$

From the stress vector definition, the principal stress and direction can be determined by the extremum of the stress vector. Consider the principal stress t_σ in the principal stress ${}_a\mathbf{n}_\sigma$. The corresponding equation is

$$\overset{>>}{\mathbf{t}}\cdot{}_a\mathbf{n}_\sigma = t_\sigma\,{}_a\mathbf{n}_\sigma \text{ and } (\overset{>>}{\mathbf{t}} - t_\sigma\overset{>>}{\mathbf{I}})\cdot{}_a\mathbf{n}_\sigma = 0. \tag{3.189}$$

From the definition of the Kirchhoff stress tensor,

$$\overset{>>}{\mathbf{t}} = \mathscr{J}\,\overset{><}{\mathbf{F}}\cdot\overset{<<}{\mathbf{T}}\cdot(\overset{<>}{\mathbf{F}})^{\mathrm{T}} = \mathscr{J}\,\overset{><}{\mathbf{R}}\cdot\overset{\overset{-1/2}{<<}}{\mathbf{C}}\cdot\overset{<<}{\mathbf{T}}\cdot\overset{\overset{1/2}{<<}}{\mathbf{C}}\cdot\overset{\overset{-1}{<>}}{\mathbf{R}}. \tag{3.190}$$

Deformation of the foregoing equation gives

$$\mathscr{J}\,\overset{\overset{-1}{<>}}{\mathbf{R}}\cdot\overset{>>}{\mathbf{t}} = \overset{\overset{-1/2}{<<}}{\mathbf{C}}\cdot\overset{<<}{\mathbf{T}}\cdot\overset{\overset{1/2}{<<}}{\mathbf{C}}\cdot\overset{\overset{-1}{<>}}{\mathbf{R}} \;\Rightarrow\; \overset{\overset{-1/2}{<<}}{\mathbf{C}}\cdot\overset{<<}{\mathbf{T}}\cdot\overset{\overset{1/2}{<<}}{\mathbf{C}}\cdot\overset{\overset{-1}{<>}}{\mathbf{R}}\cdot{}_a\mathbf{n}_\sigma = \mathscr{J}\,\overset{\overset{-1}{<>}}{\mathbf{R}}\cdot\overset{>>}{\mathbf{t}}\cdot{}_a\mathbf{n}_\sigma. \tag{3.191}$$

Thus,

$$\overset{\overset{-1/2}{<<}}{\mathbf{C}}\cdot\overset{<<}{\mathbf{T}}\cdot\overset{\overset{1/2}{<<}}{\mathbf{C}}\cdot(\overset{\overset{-1}{<>}}{\mathbf{R}}\cdot{}_a\mathbf{n}_\sigma) = t_\sigma\mathscr{J}(\overset{\overset{-1}{<>}}{\mathbf{R}}\cdot{}_a\mathbf{n}_\sigma). \tag{3.192}$$

From which it is observed that $t_\sigma\mathscr{J}$ is the principal value for $\overset{\overset{-1/2}{<<}}{\mathbf{C}}\cdot\overset{<<}{\mathbf{T}}\cdot\overset{\overset{1/2}{<<}}{\mathbf{C}}$ on the direction of $\mathbf{N}_\sigma = (\overset{\overset{-1}{<>}}{\mathbf{R}}\cdot{}_a\mathbf{n}_\sigma)$.

Theorem 3.30. *For two position vectors* $\mathbf{P}$ *in* $\mathfrak{B}$ *and* $\mathbf{p}$ *in* $\mathfrak{b}$ *for time* t, *the local balance of momentum* (*Kirchhoff equation*) *is given by the Kirchhoff stress*

$$\begin{gathered}\overset{<}{\square}\cdot(\overset{<<}{\mathbf{T}}\cdot(\overset{<>}{\mathbf{F}})^{\mathrm{T}}) + \mathscr{J}\mathbf{f} = \rho_0\mathbf{a} \quad\text{or}\quad (\overset{><}{\mathbf{F}}\cdot\overset{<<}{\mathbf{T}})\cdot\overset{<}{\square} + \mathscr{J}\mathbf{f} = \rho_0\mathbf{a}, \\ (T^{IJ}x^i_{;I})_{:J} + \mathscr{J}f^i = \rho_0 a^i.\end{gathered} \tag{3.193}$$

Proof: From the definition of the Kirchhoff stress tensor,

$$\overset{><}{\boldsymbol{\tau}} = \overset{><}{\mathbf{F}}\cdot\overset{<<}{\mathbf{T}} \text{ and } (\overset{><}{\boldsymbol{\tau}})^{\mathrm{T}} = \overset{<<}{\mathbf{T}}\cdot(\overset{><}{\mathbf{F}})^{\mathrm{T}},$$

and

$$\tau^{iJ} = x^i_{;I}T^{IJ} \text{ and } \tau^{Ij} = T^{IJ}x^j_{;J}.$$

Equation (3.184) gives

$$\overset{<}{\square}\cdot(\overset{<<}{\mathbf{T}}\cdot(\overset{<>}{\mathbf{F}})^{\mathrm{T}}) + \mathscr{J}\mathbf{f} = \rho_0\mathbf{a} \quad\text{or}\quad (\overset{><}{\mathbf{F}}\cdot\overset{<<}{\mathbf{T}})\cdot\overset{<}{\square} + \mathscr{J}\mathbf{f} = \rho_0\mathbf{a}.$$

The component expression is given as

$$(T^{IJ}x^i_{;I})_{:J} + \mathscr{J}f^i = \rho_0 a^i.$$

This theorem is proved. ■

Using $\overset{><}{\mathbf{F}} = \overset{><}{\mathbf{I}} \cdot \overset{<<}{\mathbf{F}}$ and the total differentiation operator for two-point tensor, $(\cdot)\cdot\square$ will become $(\cdot)\cdot\nabla$, the Kirchhoff equation becomes

$$(\overset{<<}{\mathbf{F}}\cdot\overset{<<}{\mathbf{T}})\cdot\overset{<}{\nabla} + \mathscr{J}\overset{<}{\mathbf{f}} = \rho_0\overset{<}{\mathbf{a}}. \tag{3.194}$$

Using $\overset{<<}{\boldsymbol{\tau}} = \overset{<<}{\mathbf{F}}\cdot\overset{<<}{\mathbf{T}}$, the foregoing equation gives

$$\overset{<<}{\boldsymbol{\tau}}\cdot\overset{<}{\nabla} + \mathscr{J}\overset{<}{\mathbf{f}} = \rho_0\overset{<}{\mathbf{a}}. \tag{3.195}$$

Consider

$$\overset{<<}{\mathbf{F}} = \overset{<}{\mathbf{p}}\overset{<}{\nabla} = (\mathbf{P} + \overset{<}{\mathbf{u}})\overset{<}{\nabla} = \overset{<<}{\mathbf{I}} + \overset{<}{\mathbf{u}}\overset{<}{\nabla}. \tag{3.196}$$

Equation (3.194) gives

$$[(\overset{<<}{\mathbf{I}} + \overset{<}{\mathbf{u}}\overset{<}{\nabla})\cdot\overset{<<}{\mathbf{T}}]\cdot\overset{<}{\nabla} + \mathscr{J}\overset{<}{\mathbf{f}} = \rho_0\overset{<}{\mathbf{a}}, \tag{3.197}$$

$$[(\delta_I^K + u^K_{;I})T^{IJ}]_{;J} + \mathscr{J} f^K = \rho_0 a^K. \tag{3.198}$$

3.4. Energy conservation

The energy principle of the conservation of energy requires

$$\frac{D}{Dt}(\mathscr{K} + \mathscr{U}) = \mathscr{W} + \sum\nolimits_{\alpha} \frac{D\mathscr{U}_{\alpha}}{Dt}, \tag{3.199}$$

where the kinetic energy, internal energy, external work are

$$\begin{aligned}
\mathscr{K} &= \int_v \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\rho dv = \int_{\mathscr{V}} \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\rho_0 dV, \\
\mathscr{U} &= \int_v u dv = \int_{\mathscr{V}} U dV, \\
\mathscr{W} &= \int_v \mathbf{v}\cdot\mathbf{f} dv + \oint_a \mathbf{v}\cdot\mathbf{t}_{a\mathbf{n}} da = \int_{\mathscr{V}} \mathbf{v}\cdot\mathbf{f}\mathscr{J} dV + \oint_{\mathscr{A}} \mathbf{v}\cdot\mathbf{T}_{A\mathbf{N}} dA,
\end{aligned} \tag{3.200}$$

and $D\mathscr{U}_{\alpha}/Dt$ ($\alpha = 1, 2, \ldots$) is the rate of energies entering or leaving the body per unit time. In other words, the time rate of change of the kinetic plus internal energy is equal to the summation of the rate of work of the external forces and all other energies entering or leaving the body per unit time. Equation (3.199) gives

$$\begin{aligned}
&\frac{D}{Dt}\Big(\int_{\mathscr{V}}\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\rho_0 dV+\int_{\mathscr{V}}U dV\Big)\\
&=\int_{\mathscr{V}}\mathbf{v}\cdot\mathbf{f}\mathscr{J}dV+\oint_{\mathscr{A}}\mathbf{v}\cdot\mathbf{T}_{{}_A\mathbf{N}}dA+\sum\nolimits_{\alpha}\frac{D\mathscr{U}_\alpha}{Dt},\\
&\frac{D}{Dt}\Big(\int_{v}\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\rho dv+\int_{v}u dv\Big)\\
&=\int_{v}\mathbf{v}\cdot\mathbf{f}dv+\oint_{a}\mathbf{v}\cdot\mathbf{t}_{{}_a\mathbf{n}}da+\sum\nolimits_{\alpha}\frac{D\mathscr{U}_\alpha}{Dt}.
\end{aligned}\tag{3.201}$$

Theorem 3.31. *For two position vectors* $\mathbf{P}$ *in* $\mathfrak{B}$ *and* $\mathbf{p}$ *in* $\mathfrak{b}$ *for time* t, *under* $D\mathscr{U}_\alpha/Dt=0$ $(\alpha=1,2,\ldots)$, *the local energy conservation is*

$$\begin{aligned}
&\frac{DU}{Dt}=\mathscr{J}\,\overset{\gg}{\mathbf{t}}:\overset{\gg}{\mathbf{d}}=\overset{\times}{\boldsymbol{\tau}}:\frac{D\overset{\times}{\mathbf{F}}}{Dt}=\overset{\ll}{\mathbf{T}}:\frac{D\overset{\ll}{\mathbf{E}}}{Dt}\quad \textit{in }\mathfrak{B},\\
&\frac{Du}{Dt}=\overset{\gg}{\mathbf{t}}:\overset{\gg}{\mathbf{d}}={}_{\mathscr{J}}\overset{\times}{\boldsymbol{\tau}}:\frac{D\overset{\times}{\mathbf{F}}}{Dt}={}_{\mathscr{J}}\overset{\ll}{\mathbf{T}}:\frac{D\overset{\ll}{\mathbf{E}}}{Dt}\quad \textit{in }\mathfrak{b}.
\end{aligned}\tag{3.202}$$

Proof: Consider

$$\begin{aligned}
\int_v\mathbf{v}\cdot\mathbf{f}dv+\oint_a\mathbf{v}\cdot\mathbf{t}_{\mathbf{n}}da&=\int_v\mathbf{v}\cdot\mathbf{f}dv+\oint_a\mathbf{v}\cdot\overset{\gg}{\mathbf{t}}\cdot d\overset{>}{\mathbf{a}}\\
&=\int_v\mathbf{v}\cdot\mathbf{f}dv+\int_v\overset{>}{\nabla}\cdot(\overset{\gg}{\mathbf{t}}\cdot\mathbf{v})dv
\end{aligned}$$

$$\begin{aligned}
\int_v\mathbf{v}\cdot\mathbf{f}dv+\oint_a\mathbf{v}\cdot\mathbf{t}_{\mathbf{n}}da&=\int_v\mathbf{v}\cdot\mathbf{f}dv+\int_v\overset{>}{\nabla}\,\overset{\gg}{\mathbf{t}}\cdot\mathbf{v}dv+\int_v\overset{\gg}{\mathbf{t}}:(\overset{>}{\nabla}\,\mathbf{v})dv\\
&=\int_v\mathbf{v}\cdot(\mathbf{f}+\overset{>}{\nabla}\,\overset{\gg}{\mathbf{t}})dv+\int_v\overset{\gg}{\mathbf{t}}:(\overset{>}{\nabla}\,\mathbf{v})dv.
\end{aligned}$$

Using

$$\overset{\gg}{\mathbf{t}}:(\overset{>}{\nabla}\,\mathbf{v})=(\overset{\gg}{\mathbf{t}})^{\mathrm{T}}:(\mathbf{v}\overset{>}{\nabla})=\overset{\gg}{\mathbf{t}}:(\mathbf{v}\overset{>}{\nabla})\ \text{ and}$$

$$\mathbf{f}+\overset{>}{\nabla}\,\overset{\gg}{\mathbf{t}}=\rho\mathbf{a},$$

the proceeding integration equation becomes

$$\begin{aligned}
\int_v\mathbf{v}\cdot\mathbf{f}dv+\oint_a\mathbf{v}\cdot\mathbf{t}_{\mathbf{n}}da&=\int_v\mathbf{v}\cdot\rho\mathbf{a}dv+\int_v\overset{\gg}{\mathbf{t}}:\frac{1}{2}(\overset{>}{\nabla}\,\mathbf{v}+\mathbf{v}\overset{>}{\nabla})dv\\
&=\int_v\mathbf{v}\cdot\frac{D\mathbf{v}}{Dt}\rho dv+\int_v\overset{\gg}{\mathbf{t}}:\overset{\gg}{\mathbf{d}}\,dv\\
&=\int_v\mathbf{v}\cdot\frac{D\mathbf{v}}{Dt}\rho dv+\int_{\mathscr{V}}\overset{\gg}{\mathbf{t}}:\overset{\gg}{\mathbf{d}}\,\mathscr{J}dV.
\end{aligned}$$

Further,

$$\int_v \mathbf{v}\cdot\mathbf{f}dv + \oint_a \mathbf{v}\cdot\mathbf{t_n}da - \int_{\mathscr{V}} \overset{\gg}{\mathbf{t}} : \overset{\gg}{\mathbf{d}} \mathscr{J} dV$$
$$= \frac{D}{Dt}\left(\int_v \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\rho dv\right).$$

From the energy conservation in Eq.(3.201) and $D\mathcal{U}_\alpha / Dt = 0$,

$$\frac{DU}{Dt} = \mathscr{J}\, \overset{\gg}{\mathbf{t}} : \overset{\gg}{\mathbf{d}}.$$

Because

$$\rho dv = \rho_0 dV \text{ and } \overset{\gg}{\mathbf{t}}\cdot d\overset{>}{\mathbf{a}} = \overset{\gg}{\boldsymbol{\tau}}\cdot d\overset{<}{\mathbf{A}},$$

then

$$\int_v \mathbf{v}\cdot\mathbf{f}dv + \oint_a \mathbf{v}\cdot\mathbf{t_n}da = \int_v \mathbf{v}\cdot\mathbf{f}dv + \oint_a \mathbf{v}\cdot\overset{\gg}{\mathbf{t}}\cdot d\overset{>}{\mathbf{a}}$$
$$= \int_{\mathscr{V}} \mathbf{v}\cdot\mathbf{f}\mathscr{J} dV + \oint_{\mathscr{A}} (\mathbf{v}\cdot\overset{><}{\boldsymbol{\tau}})\cdot d\overset{>}{\mathbf{A}}$$
$$= \int_{\mathscr{V}} \mathbf{v}\cdot\mathbf{f}\mathscr{J} dV + \int_{\mathscr{V}} (\mathbf{v}\cdot\overset{><}{\boldsymbol{\tau}})\cdot\overset{<}{\square} dV;$$

and also because

$$(\mathbf{v}\cdot\overset{><}{\boldsymbol{\tau}})\cdot\overset{<}{\square} = (\mathbf{v}\overset{<}{\square}) : \overset{><}{\boldsymbol{\tau}} + \mathbf{v}\cdot(\overset{><}{\boldsymbol{\tau}}\cdot\overset{<}{\square}),$$
$$\int_v \mathbf{v}\cdot\mathbf{f}dv + \oint_a \mathbf{v}\cdot\mathbf{t_n}da = \int_{\mathscr{V}} \mathbf{v}\cdot(\mathscr{J}\mathbf{f} + \overset{><}{\boldsymbol{\tau}}\cdot\overset{<}{\square})dV + \int_{\mathscr{V}} (\mathbf{v}\overset{<}{\square}) : \overset{><}{\boldsymbol{\tau}}\, dV$$
$$= \int_{\mathscr{V}} \mathbf{v}\cdot\rho_0\mathbf{a}dV + \int_{\mathscr{V}} \overset{><}{\boldsymbol{\tau}} : (\mathbf{v}\overset{<}{\square})dV.$$

Using $\mathbf{v}\overset{<}{\square} = \overset{\gg}{\mathbf{D}}\cdot\overset{><}{\mathbf{F}}$ and $\overset{><}{\boldsymbol{\tau}} = \overset{><}{\mathbf{F}}\cdot\overset{\ll}{\mathbf{T}}$, one obtains

$$\overset{><}{\boldsymbol{\tau}} : (\mathbf{v}\overset{<}{\square}) = (\overset{><}{\mathbf{F}}\cdot\overset{\ll}{\mathbf{T}}) : (\overset{\gg}{\mathbf{D}}\cdot\overset{><}{\mathbf{F}}) = tr(\overset{\ll}{\mathbf{T}}\cdot(\overset{<>}{\mathbf{F}})^{\mathrm{T}}\cdot\overset{\gg}{\mathbf{D}}\cdot\overset{><}{\mathbf{F}})$$
$$= tr((\overset{<>}{\mathbf{F}})^{\mathrm{T}}\cdot(\overset{\gg}{\mathbf{D}})^{\mathrm{T}}\cdot((\overset{><}{\mathbf{F}})^{\mathrm{T}})^{\mathrm{T}}\cdot(\overset{\ll}{\mathbf{T}})^{\mathrm{T}}) = tr(\overset{\ll}{\mathbf{T}}\cdot(\overset{<>}{\mathbf{F}})^{\mathrm{T}}\cdot\overset{\gg}{\mathbf{D}}\cdot\overset{><}{\mathbf{F}})$$
$$= tr(\overset{\ll}{\mathbf{T}}\cdot(\overset{<>}{\mathbf{F}})^{\mathrm{T}}\cdot\frac{1}{2}(\overset{\gg}{\mathbf{D}} + \overset{\gg}{\mathbf{D}})\cdot\overset{><}{\mathbf{F}}) = tr(\overset{\ll}{\mathbf{T}}\cdot(\overset{<>}{\mathbf{F}})^{\mathrm{T}}\cdot\overset{\gg}{\mathbf{d}}\cdot\overset{><}{\mathbf{F}}) = tr(\overset{\ll}{\mathbf{T}} : \frac{D\overset{\ll}{\mathbf{E}}}{Dt}),$$

where

$$\frac{D\overset{\ll}{\mathbf{E}}}{Dt} = (\overset{<>}{\mathbf{F}})^T\cdot\overset{\gg}{\mathbf{d}}\cdot\overset{><}{\mathbf{F}},\quad \frac{D}{Dt}\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) = \mathbf{v}\cdot\frac{D\mathbf{v}}{Dt} \text{ and } \mathbf{a} = \frac{D\mathbf{v}}{Dt}.$$

So

$$\int_v \mathbf{v}\cdot\mathbf{f}dv+\oint_a \mathbf{v}\cdot\mathbf{t}_{\mathbf{n}}da=\frac{D}{Dt}\Big(\int_{\mathscr{V}}\frac{1}{2}\mathbf{v}\cdot\mathbf{v}dV\Big)+\int_{\mathscr{V}}\overset{><}{\boldsymbol{\tau}}:\frac{D\overset{><}{\mathbf{F}}}{Dt}dV$$

$$=\frac{D}{Dt}\Big(\int_{\mathscr{V}}\frac{1}{2}\mathbf{v}\cdot\mathbf{v}dV\Big)+\int_{\mathscr{V}}\overset{<<}{\mathbf{T}}:\frac{D\overset{><}{\mathbf{E}}}{Dt}dV.$$

Again, from the energy conservation in Eq.(3.201) and $D\mathscr{U}_\alpha / Dt=0$,

$$\frac{DU}{Dt}=\overset{>>}{\boldsymbol{\tau}}:\frac{D\overset{><}{\mathbf{F}}}{Dt}=\overset{<<}{\mathbf{T}}:\frac{D\overset{<<}{\mathbf{E}}}{Dt}.$$

The proof of this theorem is completed. ■

This foregoing theorem gives the rate of the internal energy density in the initial and deformed configurations. In the previous proof,

$$\int_v \mathbf{v}\cdot(\mathbf{f}-\rho\frac{D\mathbf{v}}{Dt})dv+\oint_a \mathbf{v}\cdot\mathbf{t}_{{}_a\mathbf{n}}da$$

$$=\int_v \overset{>>}{\mathbf{t}}:\overset{>>}{\mathbf{d}}\,dv=\int_{\mathscr{V}}\mathscr{J}\,\overset{>>}{\mathbf{t}}:\overset{>>}{\mathbf{d}}\,dV. \tag{3.203}$$

The left-hand side of the foregoing equation gives the rate of energy under external force and inertial force. If the internal energy is zero, equation (3.203) gives the *D'Alembert principle* for rigid body. With $\delta\mathbf{p}=\mathbf{v}\delta t$, equation (3.203) gives

$$\int_v \delta\mathbf{p}\cdot(\mathbf{f}-\rho\frac{D\mathbf{v}}{Dt})dv+\oint_a \delta\mathbf{p}\cdot\mathbf{t}_{{}_a\mathbf{n}}da=\delta t\int_v \overset{>>}{\mathbf{t}}:\overset{>>}{\mathbf{d}}\,dv$$

$$=\delta t\int_{\mathscr{V}}\mathscr{J}\,\overset{>>}{\mathbf{t}}:\overset{>>}{\mathbf{d}}\,dV. \tag{3.204}$$

The foregoing equation gives the *virtual work principle*. In other words, the effective force work on the virtual displacement is equal to the internal force work. If the internal force work is zero, the virtual work principle for the rigid body is obtained.

In the previous proof of theorem,

$$\int_v \mathbf{v}\cdot\mathbf{f}dv+\oint_a \mathbf{v}\cdot\mathbf{t}_{\mathbf{n}}da=\int_v \mathbf{v}\cdot\frac{D\mathbf{v}}{Dt}\rho dv+\int_v \overset{>>}{\mathbf{t}}:\overset{>>}{\mathbf{d}}\,dv$$

$$=\int_v \mathbf{v}\cdot\frac{D\mathbf{v}}{Dt}\rho dv+\int_{\mathscr{V}}\overset{>>}{\mathbf{t}}:\overset{>>}{\mathbf{d}}\,\mathscr{J}dV, \tag{3.205}$$

$$\int_v \mathbf{v}\cdot\mathbf{f}dv+\oint_a \mathbf{v}\cdot\mathbf{t}_{{}_a\mathbf{n}}da=\frac{D}{Dt}\Big(\int_{\mathscr{V}}\frac{1}{2}\mathbf{v}\cdot\mathbf{v}dV\Big)+\int_{\mathscr{V}}\overset{><}{\boldsymbol{\tau}}:\frac{D\overset{><}{\mathbf{F}}}{Dt}dV$$

$$=\frac{D}{Dt}\Big(\int_{\mathscr{V}}\frac{1}{2}\mathbf{v}\cdot\mathbf{v}dV\Big)+\int_{\mathscr{V}}\overset{<<}{\mathbf{T}}:\frac{D\overset{><}{\mathbf{E}}}{Dt}dV. \tag{3.206}$$

The foregoing two equations give the *kinetic energy theorem*. Namely, the time rate of change of kinetic energy and internal energy is equal to the time rate of change of the work done by external forces. The other discussion on thermodynamical laws can be found in Eringen (1962, 1971). The generalized variational principle for nonlinear elasticity can be referred to Guo (1980).

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Chapter 4

Constitutive Laws and Damage Theory

This chapter will discuss constitutive laws and basic invariant requirements in continuous media. To develop a continuous damage theory, the concepts of damage variables will be briefly introduced. The equivalent principles on continuum damage mechanics will be presented to obtain effective material properties. A large damage theory for anisotropic damaged materials will be discussed from the incremental complementary energy equivalence principle, and a few simple applications will be presented.

4.1. Constitutive equations

Constitutive laws of materials are developed from invariance requirements. The four basic invariance requirements are:

(i) *The principle of determinism* states "the behavior of materials at time t is determined by all the past history of the motion of all material points in the body until time t".
(ii) *The principle of neighborhood* states "the behavior of a material point P at time t is determined by the behavior of an arbitrary small neighborhood".
(iii) *The principle of coordinate invariance* states "the constitutive laws of materials are independent of coordinates".
(iv) *The principle* of *material objectivity* states "the constitutive laws of materials are independent of the rigid motion of the spatial coordinates".

The detailed discussion of the objectivity of stress and strain tensors can be refered to Eringen (1962), Guo (1980) and Marsden and Hughes (1983).

To develop the constitutive laws, the following two assumptions are extensively adopted.

(A1) The natural states lie in the zero stress in the initial configuration $\mathfrak{B}$.

(A2) The behaviors of materials are dependent on the current deformation state $\mathfrak{b}$ to the natural state.

From the foregoing hypothesis, there is a relation of the stress and strain, i.e.,

$$\overset{\gg}{\mathbf{t}} = \mathbf{f}(\overset{-1}{\overset{\gg}{\mathbf{c}}}, \mathbf{P}). \tag{4.1}$$

Because $\overset{-1}{\overset{\gg}{\mathbf{c}}} = \overset{><}{\mathbf{F}} \cdot (\overset{<>}{\mathbf{F}})^{\mathrm{T}} = \overset{\gg}{\mathbf{F}} \cdot (\overset{\gg}{\mathbf{F}})^{\mathrm{T}}$, the fundamental deformation theorem gives

$$\overset{\gg}{\mathbf{F}} = \overset{-1/2}{\overset{\gg}{\mathbf{c}}} \cdot \overset{\gg}{\mathbf{R}} \text{ and } \overset{\ll}{\mathbf{F}} = \overset{\ll}{\mathbf{R}} \cdot \overset{1/2}{\overset{\ll}{\mathbf{C}}}; \tag{4.2}$$

and also

$$\overset{\gg}{\mathbf{F}} = \overset{-1/2}{\overset{\gg}{\mathbf{c}}} \cdot \overset{\gg}{\mathbf{R}} \text{ and } \overset{\ll}{\mathbf{F}} = \overset{\ll}{\mathbf{R}} \cdot \overset{1/2}{\overset{\ll}{\mathbf{C}}}, \tag{4.3}$$

then,

$$\mathrm{III}(\overset{\ll}{\mathbf{F}}) = \mathrm{III}(\overset{\ll}{\mathbf{R}} \cdot \overset{1/2}{\overset{\ll}{\mathbf{C}}}) = \mathrm{III}(\overset{\ll}{\mathbf{R}}) \cdot \mathrm{III}(\overset{1/2}{\overset{\ll}{\mathbf{C}}}). \tag{4.4}$$

Thus,

$$\mathrm{III}(\overset{\ll}{\mathbf{R}}) = \frac{\mathrm{III}(\overset{\ll}{\mathbf{F}})}{\mathrm{III}(\overset{\ll}{\mathbf{C}})^{\frac{1}{2}}} = \frac{\mathrm{III}(\overset{\ll}{\mathbf{F}})}{\sqrt{\mathrm{III}(\overset{\ll}{\mathbf{C}})}} = \frac{\mathrm{III}(\overset{\ll}{\mathbf{F}})}{\sqrt{\mathrm{III}(\overset{\ll}{\mathbf{F}} \cdot (\overset{\ll}{\mathbf{F}})^{\mathrm{T}})}} = 1. \tag{4.5}$$

The orthogonal tensor $\overset{\ll}{\mathbf{R}}$ represents the rotation only, which will be independent of deformation from the initial configuration $\mathfrak{B}$ to the deformed configuration $\mathfrak{b}$.

From $\overset{><}{\mathbf{F}} = \overset{-1/2}{\overset{\ll}{\mathbf{c}}} \cdot \overset{\ll}{\mathbf{R}} \cdot \overset{><}{\mathbf{I}}$ and $\overset{><}{\mathbf{F}} = \overset{><}{\mathbf{I}} \cdot \overset{\ll}{\mathbf{R}} \cdot \overset{1/2}{\overset{\ll}{\mathbf{C}}}$,

$$\overset{><}{\mathbf{R}} \cdot \overset{1/2}{\overset{\ll}{\mathbf{C}}} = \overset{-1/2}{\overset{\ll}{\mathbf{c}}} \cdot \overset{><}{\mathbf{R}}. \tag{4.6}$$

With $\overset{1/2}{\overset{\ll}{\mathbf{C}}} \cdot \underset{\Gamma}{\mathbf{N}} = \Lambda_{\Gamma} \underset{\Gamma}{\mathbf{N}}$, left multiplication of $\underset{\Gamma}{\mathbf{N}}$ of the foregoing equation yields

$$\overset{-1/2}{\overset{\ll}{\mathbf{c}}} \cdot \overset{><}{\mathbf{R}} \cdot \underset{\Gamma}{\mathbf{N}} = \overset{><}{\mathbf{R}} \cdot \overset{1/2}{\overset{\ll}{\mathbf{C}}} \cdot \underset{\Gamma}{\mathbf{N}} = \Lambda_{\Gamma} \overset{><}{\mathbf{R}} \cdot \underset{\Gamma}{\mathbf{N}}. \tag{4.7}$$

Using $\overset{-1/2}{\overset{\ll}{\mathbf{c}}} \cdot \underset{\Gamma}{\mathbf{n}} = \lambda_{\Gamma} \underset{\Gamma}{\mathbf{n}}$ and $\lambda_{\Gamma} = \Lambda_{\Gamma}$,

$$\underset{\Gamma}{\mathbf{n}} = \overset{><}{\mathbf{R}} \cdot \underset{\Gamma}{\mathbf{N}}. \tag{4.8}$$

From Eq.(3.107),

$$\overset{\ll}{\mathbf{R}} = \overset{\ll}{\mathbf{F}} \cdot \overset{-1/2}{\overset{\ll}{\mathbf{C}}} \text{ and } \overset{\ll}{\mathbf{R}} = \overset{1/2}{\overset{\gg}{\mathbf{c}}} \cdot \overset{\ll}{\mathbf{F}} \;\; \Rightarrow \;\; \overset{><}{\mathbf{R}} = \overset{><}{\mathbf{F}} \cdot \overset{-1/2}{\overset{\ll}{\mathbf{C}}} \text{ and } \overset{><}{\mathbf{R}} = \overset{1/2}{\overset{\gg}{\mathbf{c}}} \cdot \overset{><}{\mathbf{F}}. \tag{4.9}$$

From the previous discussion, the stress tensor in Eq.(4.1) is expressed by

$$\overset{\gg}{\mathbf{t}} = \mathbf{f}(\overset{-1}{\overset{\gg}{\mathbf{c}}}, \mathbf{P}) = \mathbf{f}(\overset{-1/2}{\overset{\gg}{\mathbf{c}}}, \overset{\ll}{\mathbf{F}}, \mathbf{P}) = \mathbf{g}(\lambda_{\underset{i}{\mathbf{n}}}, \gamma_{(\underset{i_1}{\mathbf{n}}, \underset{i_2}{\mathbf{n}})}, \mathbf{P}), \tag{4.10}$$

where $i, i_1, i_2 = 1, 2, 3$.

Introduce a new variable

$$\varepsilon_{ii} = e_{\underset{i}{\mathbf{n}}} = \lambda_{\underset{i}{\mathbf{n}}\underset{i}{\mathbf{n}}} - 1 \equiv \lambda_{\underset{i}{\mathbf{n}}} - 1 \text{ and } \varepsilon_{ij} = \gamma_{(\underset{i}{\mathbf{n}}, \underset{j}{\mathbf{n}})},$$
$$\overset{\gg}{\boldsymbol{\mathcal{E}}} \equiv \boldsymbol{\mathcal{E}} = \varepsilon_{ij} \underset{i}{\mathbf{n}} \underset{j}{\mathbf{n}}. \tag{4.11}$$

Further, the material coefficients are defined as

$$\overset{\gg}{\mathbf{t}} = \sum_n \mathscr{C}_n (\overset{-1}{\overset{\gg}{\mathbf{c}}})^n = \sum_n \mathscr{D}_n (\overset{-1/2}{\overset{\gg}{\mathbf{c}}})^n \tag{4.12}$$

and

$$\overset{\gg}{\mathbf{t}} = \mathbf{g}(\overset{\gg}{\boldsymbol{\mathcal{E}}}, \mathbf{P}) = \sum_n \mathscr{E}_n \boldsymbol{\mathcal{E}}^n \text{ or } \overset{\gg}{\boldsymbol{\mathcal{E}}} = \mathscr{F}(\overset{\gg}{\mathbf{t}}, t). \tag{4.13}$$

For linear elasticity, the foregoing equation is expressed by

$$\overset{\gg}{\mathbf{t}} = \boldsymbol{E} : \overset{\gg}{\boldsymbol{\mathcal{E}}}, \tag{4.14}$$

where

$$\boldsymbol{E} = E_{ijkl} \underset{i}{\mathbf{n}} \underset{j}{\mathbf{n}} \underset{k}{\mathbf{n}} \underset{l}{\mathbf{n}} \text{ and } \overset{\gg}{\boldsymbol{\mathcal{E}}} = \mathcal{E}_{kl} \underset{k}{\mathbf{n}} \underset{l}{\mathbf{n}}. \tag{4.15}$$

From energy points of view, the internal energy function for a specific material point $\mathbf{P}$ in $\mathfrak{B}$ is determined by the movement history of the all material points $\mathbf{P}' \in \mathfrak{B}$, i.e.,

$$U = U\left(\mathbf{p}(\mathbf{P}', t'); \mathbf{P}, t\right) \text{ for } t' \le t. \tag{4.16}$$

From *the material objectivity*,

$$U\left(\tilde{\mathbf{p}}(\mathbf{P}', \tilde{t}'), \mathbf{P}, \tilde{t}\right) = U\left(\mathbf{p}(\mathbf{P}', t'); \mathbf{P}, t\right). \tag{4.17}$$

For the two coordinates, consider

$$\tilde{\mathbf{p}}(\mathbf{P}', \tilde{t}') = \mathbf{Q}(t')\mathbf{p} + \mathbf{b}(t'),$$
$$\tilde{t}' = t' + a. \tag{4.18}$$

The material objectivity requires $\mathbf{Q}(t') = const$ and $\mathbf{b}(t') = const$. Let $\mathbf{Q}(t') = \overset{\gg}{\mathbf{I}}$ and $\mathbf{b}(t') = -\mathbf{p}(\mathbf{P}, t')$ with $a = 0$. Consider the translation of coordinate from any material point $\mathbf{P}' \in \mathfrak{B}$ to the specific material point, i.e.,

$$\tilde{\mathbf{p}}(\mathbf{P}', \tilde{t}') = \mathbf{p}(\mathbf{P}', t') - \mathbf{p}(\mathbf{P}, t'). \tag{4.19}$$

Further,

$$U = U\big(\mathbf{p}(\mathbf{P}', t') - \mathbf{p}(\mathbf{P}, t'); \mathbf{P}, t'\big). \tag{4.20}$$

Setting $\mathbf{Q}(t') = \overset{\gg}{\mathbf{I}}$ and $\mathbf{b}(t') = 0$ with $a = -t$, $\tilde{t}' = t' - t$ and $\tilde{t} = 0$.

$$\tilde{\mathbf{p}}(\mathbf{P}', \tilde{t}') = \mathbf{p}(\mathbf{P}', \tilde{t}' + t). \tag{4.21}$$

The internal energy function is expressed by

$$U = U\big(\mathbf{p}(\mathbf{P}', \tilde{t}' + t) - \mathbf{p}(\mathbf{P}, \tilde{t}' + t); \mathbf{P}\big). \tag{4.22}$$

For a new time variable of $\tau' = t - t' \geq 0$,

$$U = U\big(\mathbf{p}(\mathbf{P}', t - \tau') - \mathbf{p}(\mathbf{P}, t - \tau'); \mathbf{P}\big). \tag{4.23}$$

The Taylor series expansion of displacement in the deformed configuration gives

$$\mathbf{p}(\mathbf{P}', t') - \mathbf{p}(\mathbf{P}, t') = [\mathbf{p}(\mathbf{P}, t')\nabla] \cdot (\mathbf{P}' - \mathbf{P}) + \cdots. \tag{4.24}$$

From the principle of neighborhood,

$$U = U([\mathbf{p}(\mathbf{P}, t')\overset{<}{\nabla}]; \mathbf{P}) = U(\overset{><}{\mathbf{F}}(\mathbf{P}, t'); \mathbf{P}). \tag{4.25}$$

From $\overset{><}{\mathbf{F}} = \overset{\gg}{\mathbf{c}}^{-1/2} \cdot \overset{\gg}{\mathbf{R}} \cdot \overset{><}{\mathbf{I}}$ and $\overset{><}{\mathbf{F}} = \overset{><}{\mathbf{I}} \cdot \overset{\ll}{\mathbf{R}} \cdot \overset{\ll}{\mathbf{C}}^{1/2}$,

$$U = U(\overset{\ll}{\mathbf{C}}^{1/2}; \mathbf{P}) = U(\overset{\gg}{\mathbf{c}}^{-1/2}; \mathbf{P}) = U(\lambda_{\underset{i}{\mathbf{n}}}, \gamma_{(\underset{i_1}{\mathbf{n}}, \underset{i_2}{\mathbf{n}})};\ \mathbf{P}) = U(\mathcal{E};\ \mathbf{P}). \tag{4.26}$$

The internal energy function (or internal energy density) at the specific material point $\mathbf{P}$ is determined by the strain state. Such an internal energy function is also called *the strain energy*.

Theorem 4.1. *For two position vectors* $\mathbf{P}$ *in* $\mathfrak{B}$ *and* $\mathbf{p}$ *in* $\mathfrak{b}$ *for time* t, *under* $D\mathcal{U}_\alpha / Dt = 0$ *(*$\alpha = 1, 2, \ldots$*), based on the strain energy U, the following relations hold*:

(i)

$$\overset{\ll}{\mathbf{T}} = \frac{dU}{d\overset{\ll}{\mathbf{E}}} = 2\frac{dU}{d\overset{\ll}{\mathbf{C}}},\ T^{IJ} = \frac{\partial U}{\partial E_{IJ}} = 2\frac{\partial U}{\partial C_{IJ}}, \tag{4.27}$$

which is called the Cosserat stress and strain *relation.*

(ii)

$$\overset{><}{\boldsymbol{\tau}} = \frac{dU}{d\overset{><}{\mathbf{F}}},\ \tau_i^{\cdot I} = \frac{\partial U}{\partial x_{;I}^i}, \tag{4.28}$$

which is called the Kirchhoff form of *stress and strain*

(iii)

$$\begin{aligned} &\overset{>>}{\mathbf{t}} = \frac{1}{\mathscr{J}}\frac{dU}{d\overset{>>}{\boldsymbol{\mathcal{E}}}};\ t_{\underset{i\ i}{\mathbf{n}\mathbf{n}}} = \frac{1}{\mathscr{J}}\frac{\partial U}{\partial e_{\underset{i}{\mathbf{n}}}}(e_{\underset{i}{\mathbf{n}}}+1), \\ &t_{\underset{i_1 i_2}{\mathbf{n}\mathbf{n}}} = \frac{1}{\mathscr{J}}\frac{\partial U}{\partial \gamma_{(\underset{i_1 i_2}{\mathbf{n}\mathbf{n}})}}\sin\theta_{(\underset{i_1 i_2}{\mathbf{n}\mathbf{n}})} = \frac{1}{\mathscr{J}}\frac{\partial U}{\partial \gamma_{(\underset{i_1 i_2}{\mathbf{n}\mathbf{n}})}}\sin(\Theta_{(\underset{i_1\ i_2}{\mathbf{N}\mathbf{N}})} - \gamma_{(\underset{i_1 i_2}{\mathbf{n}\mathbf{n}})}), \end{aligned} \tag{4.29}$$

which gives the stress and strain relation in $\mathfrak{b}$.

Proof: For different strains, the time-change rate of strain energy gives

$$\frac{DU}{Dt} = \frac{dU}{d\overset{<<}{\mathbf{E}}} : \frac{D\overset{<<}{\mathbf{E}}}{Dt} = 2\frac{dU}{d\overset{<<}{\mathbf{C}}} : \frac{1}{2}\frac{D\overset{<<}{\mathbf{C}}}{Dt} = \frac{dU}{d\overset{><}{\mathbf{F}}} : \frac{D\overset{<<}{\mathbf{F}}}{Dt} = \frac{dU}{d\overset{>>}{\boldsymbol{\mathcal{E}}}} : \frac{D\overset{>>}{\boldsymbol{\mathcal{E}}}}{Dt}\ \text{ in } \mathfrak{B}.$$

From Eq.(3.200),

$$\Big(\frac{dU}{d\overset{<<}{\mathbf{E}}} - \overset{<<}{\mathbf{T}}\Big) : \frac{D\overset{<<}{\mathbf{E}}}{Dt} = 0 \text{ or } \Big(2\frac{dU}{d\overset{<<}{\mathbf{C}}} - \overset{<<}{\mathbf{T}}\Big) : \frac{1}{2}\frac{D\overset{<<}{\mathbf{C}}}{Dt} = 0;\ \ \Big(\frac{dU}{d\overset{><}{\mathbf{F}}} - \overset{><}{\boldsymbol{\tau}}\Big) : \frac{D\overset{<<}{\mathbf{F}}}{Dt} = 0.$$

Thus,

$$\overset{<<}{\mathbf{T}} = \frac{dU}{d\overset{<<}{\mathbf{E}}} = 2\frac{dU}{d\overset{<<}{\mathbf{C}}},\ T^{IJ} = \frac{\partial U}{\partial E_{IJ}} = 2\frac{\partial U}{\partial C_{IJ}},$$

and

$$\overset{><}{\boldsymbol{\tau}} = \frac{dU}{d\overset{><}{\mathbf{F}}},\ \tau_i^{\cdot I} = \frac{\partial U}{\partial x_{;I}^i}.$$

Therefore, Equations (4.27) and (4.28) are obtained.

Consider

$$\frac{DU}{Dt} = \frac{\partial U}{\partial e_{\underset{i}{\mathbf{n}}}} \cdot \frac{D}{Dt}(\lambda_{\underset{i}{\mathbf{n}}} - 1) + \frac{\partial U}{\partial \gamma_{(\underset{i_1 i_2}{\mathbf{n}\mathbf{n}})}} \cdot \frac{D\gamma_{(\underset{i_1 i_2}{\mathbf{n}\mathbf{n}})}}{Dt}.$$

From Eq.(4.11),

$$\Big(\frac{dU}{d\overset{>>}{\boldsymbol{\varepsilon}}}-\mathscr{J}\,\overset{>>}{\mathbf{t}}\Big):\frac{1}{\mathscr{J}}\frac{D\overset{>>}{\boldsymbol{\varepsilon}}}{Dt}=\Big(\frac{dU}{d\overset{>>}{\boldsymbol{\varepsilon}}}-\mathscr{J}\,\overset{>>}{\mathbf{t}}\Big):\overset{>>}{\mathbf{d}}=0.$$

Because

$$\frac{\partial U}{\partial e_{\underset{i}{\mathbf{n}}}}\cdot\frac{D\lambda_{\underset{i}{\mathbf{n}}}}{Dt}=\frac{\partial U}{\partial e_{\underset{i}{\mathbf{n}}}}\lambda_{\underset{i}{\mathbf{n}}}\,\underset{i}{\mathbf{n}}\cdot\overset{>>}{\mathbf{d}}\cdot\underset{i}{\mathbf{n}}=\frac{\partial U}{\partial e_{\underset{i}{\mathbf{n}}}}\lambda_{\underset{i}{\mathbf{n}}}\,\underset{i}{\mathbf{n}}\underset{i}{\mathbf{n}}:\overset{>>}{\mathbf{d}}=\mathscr{J}\,t_{\underset{i}{\mathbf{n}}\underset{i}{\mathbf{n}}}\,\underset{i}{\mathbf{n}}\underset{i}{\mathbf{n}}:\overset{>>}{\mathbf{d}},$$

the stress component in $\underset{i}{\mathbf{n}}$ is given by

$$t_{\underset{i}{\mathbf{n}}\underset{i}{\mathbf{n}}}=\frac{1}{\mathscr{J}}\frac{\partial U}{\partial e_{\underset{i}{\mathbf{n}}}}\lambda_{\underset{i}{\mathbf{n}}}=\frac{1}{\mathscr{J}}\frac{\partial U}{\partial e_{\underset{i}{\mathbf{n}}}}(e_{\underset{i}{\mathbf{n}}}+1),$$

and also because

$$\begin{aligned}\frac{\partial U}{\partial\gamma_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}}\cdot\frac{D\gamma_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}}{Dt}&=\mathscr{J}\,t_{\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}}}\,\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}}:\overset{>>}{\mathbf{d}}\\&=\frac{\partial U}{\partial\gamma_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}}\frac{1}{\sin\theta_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}}\Big[(\underset{i_1}{\mathbf{n}}-\cos\theta_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}\underset{i_2}{\mathbf{n}})\underset{i_2}{\mathbf{n}}+(\underset{i_2}{\mathbf{n}}-\cos\theta_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}\underset{i_1}{\mathbf{n}})\underset{i_1}{\mathbf{n}}\Big]:\overset{>>}{\mathbf{d}},\end{aligned}$$

with $\cos\theta_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}=\underset{i_1}{\mathbf{n}}\cdot\underset{i_2}{\mathbf{n}}=\underset{i_2}{\mathbf{n}}\cdot\underset{i_1}{\mathbf{n}}$ and $\underset{i_1}{\mathbf{n}}\cdot\underset{i_1}{\mathbf{n}}=\underset{i_2}{\mathbf{n}}\cdot\underset{i_2}{\mathbf{n}}=1$, one obtains

$$\begin{aligned}t_{\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}}}&=\frac{1}{\mathscr{J}}\frac{\partial U}{\partial\gamma_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}}\frac{1}{\sin\theta_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}}\Big[(\underset{i_1}{\mathbf{n}}-\cos\theta_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}\underset{i_2}{\mathbf{n}})\underset{i_2}{\mathbf{n}}+(\underset{i_2}{\mathbf{n}}-\cos\theta_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}\underset{i_1}{\mathbf{n}})\underset{i_1}{\mathbf{n}}\Big]:\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}}\\&=\frac{1}{\mathscr{J}}\frac{\partial U}{\partial\gamma_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}}\frac{1}{\sin\theta_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}}\Big[(\underset{i_1}{\mathbf{n}}-\cos\theta_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}\underset{i_2}{\mathbf{n}})\cdot\underset{i_1}{\mathbf{n}}+(\underset{i_2}{\mathbf{n}}-\cos\theta_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}\underset{i_1}{\mathbf{n}})\cdot\underset{i_1}{\mathbf{n}}\cos\theta_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}\Big]\\&=\frac{1}{\mathscr{J}}\frac{\partial U}{\partial\gamma_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}}\frac{1}{\sin\theta_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}}\Big[(1-\cos^2\theta_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})})+(\cos\theta_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}-\cos\theta_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})})\cos\theta_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}\Big]\\&=\frac{1}{\mathscr{J}}\frac{\partial U}{\partial\gamma_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}}\sin\theta_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}=\frac{1}{\mathscr{J}}\frac{\partial U}{\partial\gamma_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}}\sin(\Theta_{(\underset{i_1}{\mathbf{N}}\underset{i_2}{\mathbf{N}})}-\gamma_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}).\end{aligned}$$

Therefore, Equation (4.29) is obtained, i.e.,

$$\overset{>>}{\mathbf{t}}=\frac{1}{\mathscr{J}}\frac{dU}{d\overset{>>}{\boldsymbol{\varepsilon}}},\quad t_{\underset{i}{\mathbf{n}}\underset{i}{\mathbf{n}}}=\frac{1}{\mathscr{J}}\frac{\partial U}{\partial e_{\underset{i}{\mathbf{n}}}}\lambda_{\underset{i}{\mathbf{n}}}=\frac{1}{\mathscr{J}}\frac{\partial U}{\partial e_{\underset{i}{\mathbf{n}}}}(e_{\underset{i}{\mathbf{n}}}+1),$$

$$t_{\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}}}=\frac{1}{\mathscr{J}}\frac{\partial U}{\partial\gamma_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}}\sin\theta_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}=\frac{1}{\mathscr{J}}\frac{\partial U}{\partial\gamma_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}}\sin(\Theta_{(\underset{i_1}{\mathbf{N}}\underset{i_2}{\mathbf{N}})}-\gamma_{(\underset{i_1}{\mathbf{n}}\underset{i_2}{\mathbf{n}})}).$$

This theorem is proved. ■

On the other hand, the stress $\overset{\gg}{\mathbf{t}}$ is obtained by

$$
\begin{aligned}
&\overset{\gg}{\mathbf{t}} = \frac{1}{\mathscr{J}} \overset{>\!<}{\mathbf{F}} \cdot \overset{\ll}{\mathbf{T}} \cdot (\overset{<\!>}{\mathbf{F}})^{\mathrm{T}} = \frac{1}{\mathscr{J}} \overset{>\!<}{\mathbf{F}} \cdot \frac{dU}{d\overset{\ll}{\mathbf{E}}} \cdot (\overset{<\!>}{\mathbf{F}})^{\mathrm{T}} = \frac{2}{\mathscr{J}} \overset{>\!<}{\mathbf{F}} \cdot \frac{dU}{d\overset{\ll}{\mathbf{C}}} \cdot (\overset{<\!>}{\mathbf{F}})^{\mathrm{T}};\\
&t^{ij} = \frac{1}{\mathscr{J}} \frac{dU}{dE_{IJ}} x^i_{;I} x^j_{;J} = \frac{2}{\mathscr{J}} \frac{dU}{dC_{IJ}} x^i_{;I} x^j_{;J}.
\end{aligned}
\tag{4.30}
$$

Consider

$$
\begin{aligned}
\frac{dU}{dx^i_{;I}} &= \frac{\partial U}{\partial C_{MN}} \frac{\partial C_{MN}}{\partial x^i_{;I}} = \frac{\partial U}{\partial C_{MN}} \frac{\partial}{\partial x^i_{;I}} (g_{rs} x^r_{;M} x^s_{;N})\\
&= \frac{\partial U}{\partial C_{MN}} (g_{is} \delta^I_M x^s_{;N} + g_{ri} \delta^I_N x^r_{;M}) = 2 g_{is} \frac{\partial U}{\partial C_{IN}} x^s_{;N}\\
&= g_{is} x^s_{;M} T^{MI} = \tau_i^{\cdot I}.
\end{aligned}
\tag{4.31}
$$

Using the Kirchhoff stress gives the Cauchy stress, i.e.,

$$
\begin{aligned}
&\overset{\gg}{\mathbf{t}} = \frac{1}{\mathscr{J}} \overset{>\!<}{\boldsymbol{\tau}} \cdot (\overset{<\!>}{\mathbf{F}})^{\mathrm{T}} = \frac{1}{\mathscr{J}} \frac{dU}{d\overset{>\!<}{\mathbf{F}}} \cdot (\overset{<\!>}{\mathbf{F}})^{\mathrm{T}};\\
&t^{ij} = \frac{1}{\mathscr{J}} \frac{dU}{dx^r_{;I}} g^{ri} x^j_{;I}.
\end{aligned}
\tag{4.32}
$$

Rewriting Eq.(4.26) gives

$$
U = U(\overset{\ll}{\mathbf{E}}; \mathbf{P}) = U(\overset{>\!<}{\mathbf{F}}; \mathbf{P}) = U(\overset{\gg}{\boldsymbol{\mathcal{E}}}; \mathbf{P}). \tag{4.33}
$$

Further,

$$
\delta U = \overset{\ll}{\mathbf{T}} : \delta \overset{\ll}{\mathbf{E}} = \overset{>\!<}{\boldsymbol{\tau}} : \delta \overset{>\!<}{\mathbf{F}} = \mathscr{J}\, \overset{\gg}{\mathbf{t}} : \delta \overset{\gg}{\boldsymbol{\mathcal{E}}}. \tag{4.34}
$$

Definition 4.1. For two position vectors $\mathbf{P}$ in $\mathfrak{B}$ and $\mathbf{p}$ in $\mathfrak{b}$ for time t, based on the strain energy density U or u, the quantity U^C or u^c is termed a *complementary energy density* if

$$
\left.
\begin{aligned}
&U^C(\overset{\ll}{\mathbf{T}}) + U(\overset{\ll}{\mathbf{E}}) = \overset{\ll}{\mathbf{T}} : \overset{\ll}{\mathbf{E}};\\
&U^C(\overset{>\!<}{\boldsymbol{\tau}}) + U(\overset{>\!<}{\mathbf{F}}) = \overset{>\!<}{\boldsymbol{\tau}} : \overset{>\!<}{\mathbf{F}};\\
&U^C(\overset{\gg}{\mathbf{t}}) + U(\overset{\gg}{\boldsymbol{\mathcal{E}}}) = \mathscr{J}\, \overset{\gg}{\mathbf{t}} : \overset{\gg}{\boldsymbol{\mathcal{E}}}
\end{aligned}
\right\} \text{ in } \mathfrak{B}
\tag{4.35}
$$

or

$$\left.\begin{aligned} &u^c(\overset{\ll}{\mathbf{T}})+u(\overset{\ll}{\mathbf{E}})=\mathscr{J}\overset{\ll}{\mathbf{T}}:\overset{\ll}{\mathbf{E}};\\ &u^c(\overset{><}{\boldsymbol{\tau}})+u(\overset{><}{\mathbf{F}})=\mathscr{J}\overset{><}{\boldsymbol{\tau}}:\overset{><}{\mathbf{F}};\\ &u^c(\overset{\gg}{\mathbf{t}})+u(\overset{\gg}{\boldsymbol{\mathcal{E}}})=\overset{\gg}{\mathbf{t}}:\overset{\gg}{\boldsymbol{\mathcal{E}}} \end{aligned}\right\}\ \text{in } \mathfrak{b}. \tag{4.36}$$

In Fig.4.1, the strain energy density U and complementary energy density U^C are presented based on three pair of stress and strain, i.e., $(\overset{\ll}{\mathbf{T}},\overset{\ll}{\mathbf{E}})$, $(\overset{><}{\boldsymbol{\tau}},\overset{><}{\mathbf{F}})$ and $(\mathscr{J}\overset{\gg}{\mathbf{t}},\overset{\gg}{\boldsymbol{\mathcal{E}}})$ in the initial configuration $\mathfrak{B}$. For a given state of stress and strain, the total energy density is determined by the dot product of the stress and strain tensors (i.e., $\overset{\ll}{\mathbf{T}}:\overset{\ll}{\mathbf{E}}$, $\overset{><}{\boldsymbol{\tau}}:\overset{><}{\mathbf{F}}$ and $\mathscr{J}\overset{\gg}{\mathbf{t}}:\overset{\gg}{\boldsymbol{\mathcal{E}}}$). The hatched area is the strain energy density U and the rest part is the complementary energy U^C. From Eq.(4.35),

$$\left.\begin{aligned} &\delta U^C(\overset{\ll}{\mathbf{T}})+\delta U(\overset{\ll}{\mathbf{E}})=\delta\overset{\ll}{\mathbf{T}}:\overset{\ll}{\mathbf{E}}+\overset{\ll}{\mathbf{T}}:\delta\overset{\ll}{\mathbf{E}};\\ &\delta U^C(\overset{><}{\boldsymbol{\tau}})+\delta U(\overset{><}{\mathbf{F}})=\delta\overset{><}{\boldsymbol{\tau}}:\overset{><}{\mathbf{F}}+\overset{><}{\boldsymbol{\tau}}:\delta\overset{><}{\mathbf{F}};\\ &\delta U^C(\overset{\gg}{\mathbf{t}})+\delta U(\overset{\gg}{\boldsymbol{\mathcal{E}}})=\mathscr{J}\delta\overset{\gg}{\mathbf{t}}:\overset{\gg}{\boldsymbol{\mathcal{E}}}+\mathscr{J}\overset{\gg}{\mathbf{t}}:\delta\overset{\gg}{\boldsymbol{\mathcal{E}}} \end{aligned}\right\}\ \text{in } \mathfrak{B}. \tag{4.37}$$

With Eq.(4.36), the foregoing equations give

$$\delta U^C(\overset{\ll}{\mathbf{T}})=\delta\overset{\ll}{\mathbf{T}}:\overset{\ll}{\mathbf{E}}=\delta\overset{><}{\boldsymbol{\tau}}:\overset{><}{\mathbf{F}}=\mathscr{J}\delta\overset{\gg}{\mathbf{t}}:\overset{\gg}{\boldsymbol{\mathcal{E}}}. \tag{4.38}$$

In other words,

$$\begin{aligned} &\overset{\ll}{\mathbf{E}}=\frac{dU^C(\overset{\ll}{\mathbf{T}})}{d\overset{\ll}{\mathbf{T}}},\ E_{IJ}=\frac{\partial U^C}{\partial T^{IJ}};\\ &\overset{><}{\mathbf{F}}=\frac{dU^C(\overset{><}{\boldsymbol{\tau}})}{d\overset{><}{\boldsymbol{\tau}}},\ x^{\cdot i}_{;I}=\frac{\partial U^C}{\partial \tau^{\cdot I}_{i}};\\ &\overset{\gg}{\boldsymbol{\mathcal{E}}}=\frac{1}{\mathscr{J}}\frac{dU^C(\overset{\gg}{\mathbf{t}})}{d\overset{\gg}{\mathbf{t}}},\ \varepsilon_{ij}=\frac{1}{\mathscr{J}}\frac{\partial U^C}{\partial t^{ij}}. \end{aligned} \tag{4.39}$$

Further, the time-rate of change of the total energy density is

$$\left.\begin{aligned} &\frac{D}{Dt}U^C(\overset{\ll}{\mathbf{T}})+\frac{D}{Dt}U(\overset{\ll}{\mathbf{E}})=\frac{D}{Dt}\overset{\ll}{\mathbf{T}}:\overset{\ll}{\mathbf{E}}+\overset{\ll}{\mathbf{T}}:\frac{D}{Dt}\overset{\ll}{\mathbf{E}};\\ &\frac{D}{Dt}U^C(\overset{><}{\boldsymbol{\tau}})+\frac{D}{Dt}U(\overset{><}{\mathbf{F}})=\frac{D}{Dt}\overset{><}{\boldsymbol{\tau}}:\overset{><}{\mathbf{F}}+\overset{><}{\boldsymbol{\tau}}:\frac{D}{Dt}\overset{><}{\mathbf{F}}; \end{aligned}\right\}\ \text{in } \mathfrak{B};$$

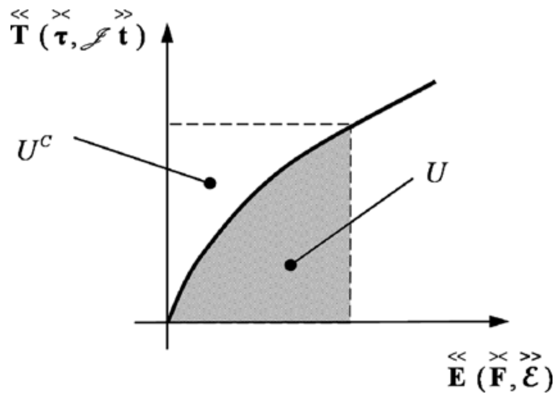

Fig. 4.1 The strain energy density and complementary energy density based on three pairs of stress and strain.

$$\frac{D}{Dt}U^C(\overset{\gg}{\mathbf{t}})+\frac{D}{Dt}U(\overset{\gg}{\boldsymbol{\mathcal{E}}}) = \mathscr{J}\frac{D}{Dt}\overset{\gg}{\mathbf{t}}:\overset{\gg}{\boldsymbol{\mathcal{E}}}+\mathscr{J}\overset{\gg}{\mathbf{t}}:\frac{D}{Dt}\overset{\gg}{\boldsymbol{\mathcal{E}}} \ \text{ in } \mathfrak{B}. \tag{4.40}$$

For the Cauchy stress tensor in Eq.(4.14), equation (4.34) gives

$$\begin{aligned}\delta U &= \mathscr{J}\overset{\gg}{\mathbf{t}}:\delta\overset{\gg}{\boldsymbol{\mathcal{E}}} = \mathscr{J}\mathbf{g}(\overset{\gg}{\boldsymbol{\mathcal{E}}},\mathbf{P}):\delta\overset{\gg}{\boldsymbol{\mathcal{E}}} = \mathscr{J}\sum\nolimits_n \mathscr{E}_n(\overset{\gg}{\boldsymbol{\mathcal{E}}})^n:\delta\overset{\gg}{\boldsymbol{\mathcal{E}}};\\ \delta U &= \mathscr{J}t^{ij}\delta\varepsilon_{ij} = \mathscr{J}g^{ij}(\varepsilon_{kl},\mathbf{P})\,\delta\varepsilon_{ij} = \mathscr{J}\sum\nolimits_n \mathscr{E}_n(\varepsilon_{ij})^n\,\delta\varepsilon_{ij}.\end{aligned} \tag{4.41}$$

From Eq.(4.34), the strain energy density is computed by

$$\begin{aligned}U &= \mathscr{J}\int_0^{\overset{\gg}{\boldsymbol{\mathcal{E}}}}\overset{\gg}{\mathbf{t}}:\delta\overset{\gg}{\boldsymbol{\mathcal{E}}} = \mathscr{J}\int_0^{\overset{\gg}{\boldsymbol{\mathcal{E}}}}\mathbf{g}(\overset{\gg}{\boldsymbol{\mathcal{E}}},\mathbf{P}):\delta\overset{\gg}{\boldsymbol{\mathcal{E}}}\\ &= \int_0^{\overset{\ll}{\mathbf{E}}}\overset{\ll}{\mathbf{T}}:\delta\overset{\ll}{\mathbf{E}} = \int_0^{\overset{\asymp}{\mathbf{F}}}\overset{\asymp}{\boldsymbol{\tau}}:\delta\overset{\asymp}{\mathbf{F}};\end{aligned} \tag{4.42}$$

and the strain energy is computed by

$$\begin{aligned}\mathcal{U} &= \int_{\mathscr{V}}U\,dV = \int_{\mathscr{V}}\Big(\mathscr{J}\int_0^{\overset{\gg}{\boldsymbol{\mathcal{E}}}}\overset{\gg}{\mathbf{t}}:\delta\overset{\gg}{\boldsymbol{\mathcal{E}}}\Big)dV = \int_v\Big(\int_0^{\overset{\gg}{\boldsymbol{\mathcal{E}}}}\overset{\gg}{\mathbf{t}}:\delta\overset{\gg}{\boldsymbol{\mathcal{E}}}\Big)dv\\ &= \int_{\mathscr{V}}\Big(\int_0^{\overset{\gg}{\mathbf{E}}}\overset{\ll}{\mathbf{T}}:\delta\overset{\ll}{\mathbf{E}}\Big)dV = \int_{\mathscr{V}}\Big(\int_0^{\overset{\asymp}{\mathbf{F}}}\overset{\asymp}{\boldsymbol{\tau}}:\delta\overset{\asymp}{\mathbf{F}}\Big)dV.\end{aligned} \tag{4.43}$$

From Eq.(4.38), the complementary energy density is computed by

$$U^C = \mathscr{J}\int_0^{\overset{\gg}{\mathbf{t}}}\overset{\gg}{\boldsymbol{\mathcal{E}}}:\delta\overset{\gg}{\mathbf{t}} = \int_0^{\overset{\ll}{\mathbf{T}}}\overset{\ll}{\mathbf{E}}:\delta\overset{\ll}{\mathbf{T}} = \int_0^{\overset{\asymp}{\boldsymbol{\tau}}}\overset{\asymp}{\mathbf{F}}:\delta\overset{\asymp}{\boldsymbol{\tau}}; \tag{4.44}$$

and the complementary energy is computed by

$$\mathcal{U}^C = \int_{\mathscr{V}} U^c dV = \int_{\mathscr{V}} (\mathscr{J} \int_0^{\overset{>>}{\mathbf{t}}} \overset{>>}{\boldsymbol{\mathcal{E}}} : \delta \overset{>>}{\mathbf{t}}) dV = \int_v (\int_0^{\overset{>>}{\mathbf{t}}} \overset{>>}{\boldsymbol{\mathcal{E}}} : \delta \overset{>>}{\mathbf{t}}) dv$$
$$= \int_{\mathscr{V}} (\int_0^{\overset{<<}{\mathbf{T}}} \overset{<<}{\mathbf{E}} : \delta \overset{<<}{\mathbf{T}}) dV = \int_{\mathscr{V}} (\int_0^{\overset{><}{\boldsymbol{\tau}}} \overset{><}{\mathbf{F}} : \delta \overset{><}{\boldsymbol{\tau}}) dV. \tag{4.45}$$

4.2. Material damage and effective stress

Consider a deformed body to be a damaged material at time t, and the corresponding stress on the cross section $\Sigma_{\mathbf{n}}$ for the *undamaged* and *damaged* bodies are shown in Fig.4.2. The undamaged and damage bodies are on the left and right sides, respectively. The effective area for the damaged body is $\Delta\tilde{a}$ to the area of Δa for the undamaged body. The *effective* stress vector and the *effective* couple stress vector on the cross section $\Sigma_{\mathbf{n}}$ are sketched by $\tilde{\mathbf{t}}_{(\mathbf{n})}$ and $\tilde{\mathbf{m}}_{(\mathbf{n})}$. From the idea of Kachanov (1958), a damage variable is introduced for development of continuum damage mechanics. The detailed description of continuum damage mechanics can be found in Kachanov (1986) and Krajcinovic and Lemaitre (1987).

Definition 4.2. At point $\mathbf{p}$ in $\mathfrak{b}$ for time t in a damaged material, suppose the effective cross section of the damaged material is $\Delta\tilde{a}$ to the cross section Δa of the undamaged material with the external normal direction ($\mathbf{n}$). The *damage variable* in the external normal direction ($\mathbf{n}$) is defined as

$$D_{(\mathbf{n})}\mathbf{n} \equiv \lim_{\Delta\mathbf{a}\to\mathbf{0}} \frac{\Delta\mathbf{a} - \Delta\tilde{\mathbf{a}}}{|\Delta\mathbf{a}|} = \lim_{\Delta a\to 0} \frac{\Delta a - \Delta\tilde{a}}{\Delta a}\mathbf{n}. \tag{4.46}$$

The *effective* stress vector and *effective* couple stress vector on the exterior unit normal direction of the cross section Σ are defined as

$$\tilde{\mathbf{t}}_{(\mathbf{n})} \equiv \lim_{\Delta a\to 0} \frac{\Delta\mathbf{t}}{\Delta\tilde{a}} \quad \text{and} \quad \tilde{\mathbf{m}}_{(\mathbf{n})} \equiv \lim_{\Delta a\to 0} \frac{\Delta\mathbf{m}}{\Delta\tilde{a}}. \tag{4.47}$$

For the damage variable, $D_n = 0$ is for the undamaged state; $D_n = 1$ is for a breaking state of the damaged body along a surface with an external normal direction $\mathbf{n}$; $0 < D_n < 1$ is for the damaged state. From the previous definition, the *effective* and *conventional* stress vectors and couple stress vectors are

$$\tilde{\mathbf{t}}_{(\mathbf{n})} \equiv \frac{1}{1 - D_{(\mathbf{n})}}\mathbf{t}_{(\mathbf{n})} \quad \text{and} \quad \tilde{\mathbf{m}}_{(\mathbf{n})} \equiv \frac{1}{1 - D_{(\mathbf{n})}}\mathbf{m}_{(\mathbf{n})}. \tag{4.48}$$

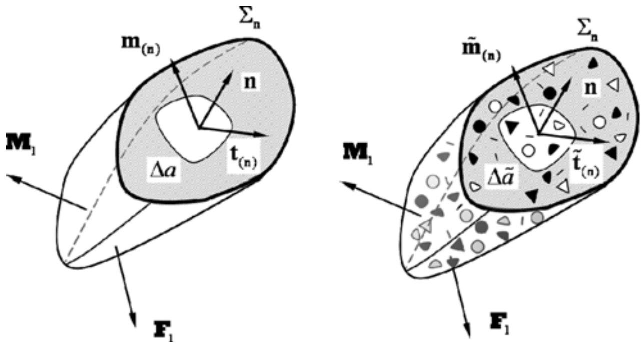

Fig. 4.2 Stress vectors on a cross section Σ_n of undamaged (left) and damaged (right) materials.

Definition 4.3. The damage of a deformable body to its undamaged state is called *an isotropic damage* if the micro-crack and cavities in all directions of the damaged body are uniformly distributed.

Definition 4.4. The damage of a deformable body to its undamaged state is called *an anisotropic damage* if the micro-cracks and cavities of the damaged body in all directions are non-uniformly distributed.

For the isotropic damaged body, the damage variable is independent of the direction. That is, $D_n = D$ for all directions. The damage characteristics of the material can be described through a damage variable. However, for the anisotropic damaged body, the damage variables are different in the different directions. The *damage variable vector* is introduced by

$$\mathbf{D}_{(\mathbf{n})} = D_{(\mathbf{n})}\mathbf{n}. \tag{4.49}$$

From the effective and conventional stress vector definitions, the *effective and conventional stress tensors* satisfy

$$\tilde{\mathbf{t}} = \mathbf{M}(\mathbf{D}) : \mathbf{t}, \tag{4.50}$$

where the fourth-order tensor for the conversion is

$$\mathbf{M}(\mathbf{D}) = M_{ijkl}(\mathbf{D})\underset{i}{\mathbf{n}}\underset{j}{\mathbf{n}}\underset{k}{\mathbf{n}}\underset{l}{\mathbf{n}} = M_{ijkl}((1-D_{(\mathbf{n})})^{-1})\underset{i}{\mathbf{n}}\underset{j}{\mathbf{n}}\underset{k}{\mathbf{n}}\underset{l}{\mathbf{n}}. \tag{4.51}$$

The concept of the damage variable can be extended to strengthened materials. If the damage variable becomes negative (i.e., $D_n < 0$), the effective stress will become small to the undamaged state in the direction of $\mathbf{n}$. The damage variable should become the strengthening variable. Without loss of generality, the damage variables can represent both damaged and strengthened states of the deformable body for $D_n \in (-\infty, 1]$ as a generalized concept. For an anisotropic damaged body, the deformable body can be damaged in some directions and can be strengthened in some other directions. Therefore, the damage variable is a new variable to describe the material properties in the entire deformation rather than the linear relation. Because such a concept is introduced, one can easily describe the material properties from the elastic state to fracture state. To describe the anisotropic damage for materials, the *second-order damage tensor* is defined as

$$\mathbf{D} = D_{ij}\,\mathbf{n}_i\mathbf{n}_j. \tag{4.52}$$

4.3. Equivalence principles

As aforementioned in the previous section, the damage variable is used to measure the material characteristics from the elastic state to fracture. To investigate the damaged deformable body, it is assumed that the material properties of the damaged deformable body are in the *undamaged* state, and the *effective* stress will be adopted to measure the change caused by the damage of materials. However, it is assumed that the damaged deformable body is in the conventional stress (or real stress) state, and the material characteristics of the damaged deformable body is determined by the *effective* material properties. Lemaitre and Chaboche (1978) used the strain equivalence to determine an effective stress in the constitutive laws rather than the conventional stress (also see, Chaboche, 1978). The principle of strain equivalence can be stated as follows.

(A) *The principle of strain equivalence* For a damaged body, the effective stress with the conventional material characteristics possesses the same strain as the real stress with the effective material properties, i.e.,

$$\boldsymbol{\mathcal{E}} = \mathscr{F}(\tilde{\mathbf{t}}, \boldsymbol{E}^{-1}, t) = \mathscr{F}(\mathbf{t}, \tilde{\boldsymbol{E}}^{-1}, t). \tag{4.53}$$

For the uniaxial linear elastic law of a damaged material,

$$\varepsilon^e = \frac{\tilde{\sigma}}{E} = \frac{\sigma}{\tilde{E}}, \tag{4.54}$$

where $t_{11} = \sigma$ and $t_{ii} = 0$ for $i = 2, 3$, in addition, $t_{ij} = 0$ for $i, j = 1, 2, 3$ but $i \neq j$. With Eq.(4.47), the foregoing equation gives

$$\frac{\tilde{\sigma}}{E} = \frac{\sigma}{(1-D)E} = \frac{\sigma}{\tilde{E}}. \tag{4.55}$$

Therefore,

$$\tilde{E} = (1 - D)E \;\Rightarrow\; D = 1 - \frac{\tilde{E}}{E}. \tag{4.56}$$

From the foregoing equation, the damage variable is used to measure the properties of the damaged materials, which is independent of stress and strain. The damage variable can be measured experimentally. This hypothesis works well for isotropic damage materials but yields asymmetric stiffness matrix for anisotropic materials. Gordebois and Sidoroff (1979, 1982) suggested a damage model based on the hypothesis of the total complementary elastic strain energy equivalence.

(B) *The principle of complementary strain energy equivalence* For a damaged deformable body, the total complementary elastic energy of *a damage state* with the conventional stress is equivalent to the total complementary elastic energy of the *conventional* state with the effective stress.

For undamaged, linear elastic materials, the stress tensor is given in Eq.(4.14). For anisotropic damaged, linear elastic materials, the corresponding effective stress tensor is expressed by

$$\overset{\gg}{\tilde{\mathbf{t}}} = \tilde{\boldsymbol{E}} : \overset{\gg}{\tilde{\boldsymbol{\mathcal{E}}}} \;\text{ or }\; \overset{\gg}{\tilde{\boldsymbol{\mathcal{E}}}} = \tilde{\boldsymbol{E}}^{-1} : \overset{\gg}{\tilde{\mathbf{t}}}, \tag{4.57}$$

where $\overset{\gg}{\tilde{\mathbf{t}}}$ and $\tilde{\boldsymbol{E}}$ are effective stress tensor and effective elasticity tensor, respectively. From Eqs.(4.14) and (4.57), the complementary strain energy hypothesis can be expressed by

$$\mathcal{U}^C(\overset{\gg}{\mathbf{t}}, \tilde{\boldsymbol{E}}^{-1}, \mathbf{D}) = \mathcal{U}^C(\overset{\gg}{\tilde{\mathbf{t}}}, \boldsymbol{E}^{-1}, 0). \tag{4.58}$$

If the volume for the conventional and damage states of damaged materials is invariant, the foregoing equation can be expressed by the density of the complementary energy, i.e.,

$$U^C(\overset{\gg}{\mathbf{t}}, \tilde{\boldsymbol{E}}^{-1}, \mathbf{D}) = U^C(\overset{\gg}{\tilde{\mathbf{t}}}, \boldsymbol{E}^{-1}, 0). \tag{4.59}$$

For damaged materials with linear elasticity, $U^C(\overset{\gg}{\mathbf{t}}, \tilde{\boldsymbol{E}}^{-1}, \mathbf{D}) = \frac{1}{2}\overset{\gg}{\mathbf{t}} : \tilde{\boldsymbol{E}}^{-1} : \overset{\gg}{\mathbf{t}}$ is obtained. For undamaged materials, $\mathbf{D} = 0$. For damaged materials with nonlinear elasticity,

$$\begin{aligned} U^C(\overset{\gg}{\mathbf{t}}, \tilde{\mathscr{E}}^{-1}, \mathbf{D}) &= \mathscr{J}\int_0^{\overset{\gg}{\mathbf{t}}} \overset{\gg}{\boldsymbol{\mathcal{E}}}(\overset{\gg}{\mathbf{t}}, \tilde{\mathscr{E}}^{-1}, \mathbf{D}) : \delta\overset{\gg}{\mathbf{t}}; \\ U^C(\overset{\gg}{\tilde{\mathbf{t}}}, \mathscr{E}^{-1}, 0) &= \mathscr{J}\int_0^{\overset{\gg}{\tilde{\mathbf{t}}}} \overset{\gg}{\tilde{\boldsymbol{\mathcal{E}}}}(\overset{\gg}{\tilde{\mathbf{t}}}, \mathscr{E}^{-1}, 0) : \delta\overset{\gg}{\tilde{\mathbf{t}}}. \end{aligned} \tag{4.60}$$

Note that $\tilde{\mathscr{E}}^{-1}$ and $\mathscr{E}^{-1}$ are nonlinear coefficients from which the strain can be expressed by the stress. For linear case, the coefficient tensors $\tilde{\mathscr{E}}^{-1}$ and $\mathscr{E}^{-1}$ become the tensors $\tilde{\boldsymbol{E}}^{-1}$ and $\boldsymbol{E}^{-1}$. The corresponding complementary energy is computed by

$$\begin{aligned}\mathcal{U}^C(\overset{>>}{\mathbf{t}},\tilde{\mathscr{E}}^{-1},\mathbf{D}) &= \int_{\mathscr{V}} U^c(\overset{>>}{\mathbf{t}},\tilde{\mathscr{E}}^{-1},\mathbf{D})dV \\ &= \int_{\mathscr{V}}[\mathscr{J}\int_0^{\overset{>>}{\mathbf{t}}} \boldsymbol{\mathcal{E}}(\overset{>>}{\mathbf{t}},\tilde{\mathscr{E}}^{-1},\mathbf{D}) : \delta\overset{>>}{\mathbf{t}}]\,dV \\ &= \int_v[\int_0^{\overset{>>}{\mathbf{t}}} \boldsymbol{\mathcal{E}}(\overset{>>}{\mathbf{t}},\tilde{\mathscr{E}}^{-1},\mathbf{D}) : \delta\overset{>>}{\mathbf{t}}]\,dv,\end{aligned} \tag{4.61}$$

$$\begin{aligned}\mathcal{U}^C(\overset{>>}{\tilde{\mathbf{t}}},\mathscr{E}^{-1},0) &= \int_{\mathscr{V}} U^c(\overset{>>}{\tilde{\mathbf{t}}},\mathscr{E}^{-1},0)dV \\ &= \int_{\mathscr{V}}[\mathscr{J}\int_0^{\overset{>>}{\tilde{\mathbf{t}}}} \tilde{\boldsymbol{\mathcal{E}}}(\overset{>>}{\tilde{\mathbf{t}}},\mathscr{E}^{-1},0) : \delta\overset{>>}{\tilde{\mathbf{t}}}]dV \\ &= \int_v[\int_0^{\overset{>>}{\tilde{\mathbf{t}}}} \tilde{\boldsymbol{\mathcal{E}}}(\overset{>>}{\tilde{\mathbf{t}}},\mathscr{E}^{-1},0) : \delta\overset{>>}{\tilde{\mathbf{t}}}]dv.\end{aligned} \tag{4.62}$$

The complementary energy equivalence hypothesis can be illustrated through the density of complementary energy with the volume invariance between the damaged and conventional states, as shown in Fig.4.3. The dashed-line hatched area is the complementary energy density for the effective stress with undamaged materials. The solid-line hatched area is the complementary energy density for the conventional stress with damaged materials. Under the same volume, the integrations of the densities should be same to make such equivalence of the complementary energy for the damage and conventional states.

Similarly, such a complementary strain energy hypothesis can be expressed by the *Cosserat and Kirchhoff stresses*:

$$\begin{aligned}\mathcal{U}^C(\overset{><}{\boldsymbol{\tau}},\tilde{\mathscr{E}}^{-1},\mathbf{D}) &= \mathcal{U}^C(\overset{><}{\tilde{\boldsymbol{\tau}}},\mathscr{E}^{-1},0); \\ \mathcal{U}^C(\overset{<<}{\mathbf{T}},\tilde{\mathscr{E}}^{-1},\mathbf{D}) &= \mathcal{U}^C(\overset{<<}{\tilde{\mathbf{T}}},\mathscr{E}^{-1},0),\end{aligned} \tag{4.63}$$

and the corresponding complementary energies are

$$\begin{aligned}\mathcal{U}^C(\overset{><}{\boldsymbol{\tau}},\tilde{\mathscr{E}}^{-1},\mathbf{D}) &= \int_{\mathscr{V}} U^C(\overset{><}{\boldsymbol{\tau}},\tilde{\mathscr{E}}^{-1},\mathbf{D})dV \\ &= \int_{\mathscr{V}}(\int_0^{\overset{><}{\boldsymbol{\tau}}} \mathbf{F}(\overset{><}{\boldsymbol{\tau}},\tilde{\mathscr{E}}^{-1},\mathbf{D}) : \delta\overset{><}{\boldsymbol{\tau}})dV, \\ \mathcal{U}^C(\overset{<<}{\mathbf{T}},\tilde{\mathscr{E}}^{-1},\mathbf{D}) &= \int_{\mathscr{V}} U^C(\overset{<<}{\mathbf{T}},\tilde{\mathscr{E}}^{-1},\mathbf{D})dV \\ &= \int_{\mathscr{V}}(\int_0^{\overset{<<}{\mathbf{T}}} \mathbf{E}(\overset{<<}{\mathbf{T}},\tilde{\mathscr{E}}^{-1},\mathbf{D}) : \delta\overset{<<}{\mathbf{T}})dV.\end{aligned} \tag{4.64}$$

and

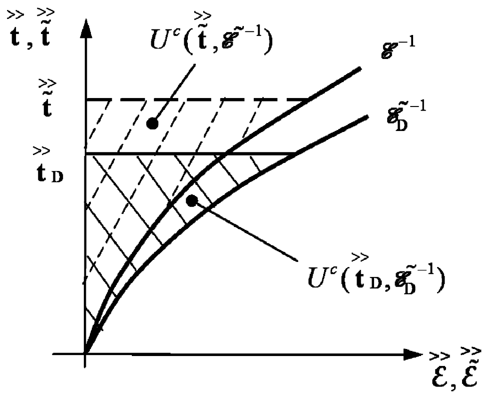

Fig. 4.3 Complementary energy equivalence at a conventional stress state of $(\overset{\gg}{\mathbf{t}},\mathbf{D})$ with damaged material and at the effective stress state of $(\overset{\gg}{\tilde{\mathbf{t}}},0)$ with undamaged material.

$$
\begin{aligned}
\mathcal{U}^C(\overset{><}{\tilde{\boldsymbol{\tau}}},\mathscr{E}^{-1},0) &= \int_{\mathscr{V}} U^C(\overset{><}{\tilde{\boldsymbol{\tau}}},\mathscr{E}^{-1},0)dV \\
&= \int_{\mathscr{V}} \Big(\int_0^{\overset{><}{\tilde{\boldsymbol{\tau}}}} \overset{><}{\tilde{\mathbf{F}}}(\overset{><}{\tilde{\boldsymbol{\tau}}},\mathscr{E}^{-1},0):\delta\overset{><}{\tilde{\boldsymbol{\tau}}}\Big)dV, \\
\mathcal{U}^C(\overset{\ll}{\mathbf{T}},\mathscr{E}^{-1},0) &= \int_{\mathscr{V}} U^C(\overset{\ll}{\tilde{\mathbf{T}}},\mathscr{E}^{-1},0)dV \\
&= \int_{\mathscr{V}} \Big(\int_0^{\overset{\ll}{\tilde{\mathbf{T}}}} \overset{\ll}{\tilde{\mathbf{E}}}(\overset{\ll}{\tilde{\mathbf{T}}},\mathscr{E}^{-1},0):\delta\overset{\ll}{\tilde{\mathbf{T}}}\Big)dV.
\end{aligned}
\tag{4.65}
$$

Again, if the volume for the conventional and damage states of damaged materials is invariant, the complementary energies can be compared by

$$
\begin{aligned}
U^C(\overset{\gg}{\boldsymbol{\tau}},\tilde{\mathscr{E}}^{-1},\mathbf{D}) &= U^C(\overset{\gg}{\tilde{\boldsymbol{\tau}}},\mathscr{E}^{-1},0); \\
U^C(\overset{\gg}{\mathbf{T}},\tilde{\mathscr{E}}^{-1},\mathbf{D}) &= U^C(\overset{\gg}{\tilde{\mathbf{T}}},\mathscr{E}^{-1},0),
\end{aligned}
\tag{4.66}
$$

where

$$
\begin{aligned}
U^C(\overset{><}{\boldsymbol{\tau}},\tilde{\mathscr{E}}^{-1},\mathbf{D}) &= \int_0^{\overset{><}{\boldsymbol{\tau}}} \overset{><}{\mathbf{F}}(\overset{><}{\boldsymbol{\tau}},\tilde{\mathscr{E}}^{-1},\mathbf{D}):\delta\overset{><}{\boldsymbol{\tau}}, \\
U^C(\overset{><}{\tilde{\boldsymbol{\tau}}},\mathscr{E}^{-1},0) &= \int_0^{\overset{><}{\tilde{\boldsymbol{\tau}}}} \overset{><}{\tilde{\mathbf{F}}}(\overset{><}{\tilde{\boldsymbol{\tau}}},\mathscr{E}^{-1},0):\delta\overset{><}{\tilde{\boldsymbol{\tau}}};
\end{aligned}
\tag{4.67}
$$

$$
\begin{aligned}
U^C(\overset{\ll}{\mathbf{T}},\tilde{\mathscr{E}}^{-1},\mathbf{D}) &= \int_0^{\overset{\ll}{\mathbf{T}}} \overset{\ll}{\mathbf{E}}(\overset{\ll}{\mathbf{T}},\tilde{\mathscr{E}}^{-1},\mathbf{D}):\delta\overset{\ll}{\mathbf{T}}, \\
U^C(\overset{\ll}{\tilde{\mathbf{T}}},\mathscr{E}^{-1},0) &= \int_0^{\overset{\ll}{\tilde{\mathbf{T}}}} \overset{\ll}{\tilde{\mathbf{E}}}(\overset{\ll}{\tilde{\mathbf{T}}},\mathscr{E}^{-1},0):\delta\overset{\ll}{\tilde{\mathbf{T}}}.
\end{aligned}
\tag{4.68}
$$

The foregoing hypothesis asserts that the complementary strain energies for the conventional and damage states are equivalent. Herein, such an idea was extended (Luo, 1991), i.e., the complementary strain energy for the conventional stress with $\mathbf{D}+d\mathbf{D}$ is equivalent to the complementary strain energy for the effective stress with $\mathbf{D}$. The detailed description was given in Luo et al. (1995). Further, the incremental complementary energy hypothesis was proposed.

(C) *The principle of incremental complementary strain energy equivalence* For a damaged body with $\mathbf{D}+d\mathbf{D}$, the incremental complementary strain energy of a damage state with the instantaneous conventional stress relative to $\mathbf{D}+d\mathbf{D}$ is equivalent to the incremental complementary strain energy of the damage state of $\mathbf{D}$ with the instantaneous effective stress.

The foregoing hypothesis is expressed mathematically by

$$\delta \mathcal{U}^C(\overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}}, \tilde{\mathscr{E}}^{-1}_{\mathbf{D}+d\mathbf{D}}, \mathbf{D}+d\mathbf{D}) = \delta\mathcal{U}^C(\overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}}, \tilde{\mathscr{E}}^{-1}_{\mathbf{D}}, \mathbf{D}), \tag{4.69}$$

where

$$\begin{aligned}\delta \mathcal{U}^C(\overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}}, \tilde{\mathscr{E}}^{-1}_{\mathbf{D}+d\mathbf{D}}, \mathbf{D}+d\mathbf{D}) &= \mathcal{U}^C(\overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}} + \delta\overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}}, \tilde{\mathscr{E}}^{-1}_{\mathbf{D}+d\mathbf{D}}, \mathbf{D}+d\mathbf{D}) \\ &\quad - \mathcal{U}^C(\overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}}, \tilde{\mathscr{E}}^{-1}_{\mathbf{D}+d\mathbf{D}}, \mathbf{D}+d\mathbf{D}),\end{aligned} \tag{4.70}$$

$$\delta \mathcal{U}^C(\overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}}, \tilde{\mathscr{E}}^{-1}_{\mathbf{D}}, \mathbf{D}) = \mathcal{U}^C(\overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}} + \delta\overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}}, \tilde{\mathscr{E}}^{-1}_{\mathbf{D}}, \mathbf{D}) - \mathcal{U}^C(\overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}}, \tilde{\mathscr{E}}^{-1}_{\mathbf{D}}, \mathbf{D}). \tag{4.71}$$

If the volume for the conventional and damage states of damaged materials is invariant, the foregoing hypothesis can be expressed by the density of the complementary energy, i.e.,

$$\delta U^C(\overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}}, \tilde{\mathscr{E}}^{-1}_{\mathbf{D}+d\mathbf{D}}, \mathbf{D}+d\mathbf{D}) = \delta U^C(\overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}}, \tilde{\mathscr{E}}^{-1}_{\mathbf{D}}, \mathbf{D}), \tag{4.72}$$

where

$$\begin{aligned}\delta U^C(\overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}}, \tilde{\mathscr{E}}^{-1}_{\mathbf{D}+d\mathbf{D}}, \mathbf{D}+d\mathbf{D}) &= U^C(\overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}} + \delta\overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}}, \tilde{\mathscr{E}}^{-1}_{\mathbf{D}+d\mathbf{D}}, \mathbf{D}+d\mathbf{D}) \\ &\quad - U^C(\overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}}, \tilde{\mathscr{E}}^{-1}_{\mathbf{D}+d\mathbf{D}}, \mathbf{D}+d\mathbf{D}),\end{aligned} \tag{4.73}$$

$$\delta U^C(\overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}}, \tilde{\mathscr{E}}^{-1}_{\mathbf{D}}, \mathbf{D}) = U^C(\overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}} + \delta\overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}}, \tilde{\mathscr{E}}^{-1}_{\mathbf{D}}, \mathbf{D}) - U^C(\overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}}, \tilde{\mathscr{E}}^{-1}_{\mathbf{D}}, \mathbf{D}). \tag{4.74}$$

The hypothesis of incremental complementary strain energy equivalence can be presented through the incremental density of complementary energy because of the invariance of volumes for conventional and damage states, as sketched in Fig.4.4.

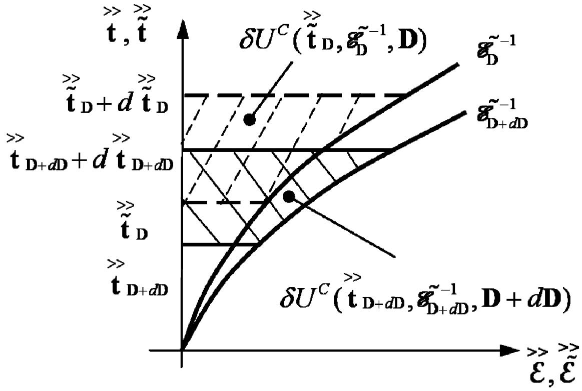

Fig. 4.4 Incremental complementary energy equivalence at a damage state of $(\overset{\gg}{\mathbf{t}}, \mathbf{D})$ with incremental stress and damage $(d\overset{\gg}{\mathbf{t}}, d\mathbf{D})$.

Based on the damage state of $\mathbf{D}$, the incremental complementary energy can be computed from the effective stress state $\overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}}$ to $\overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}} + d\overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}}$, which is presented by the dashed-line hatched area ($\delta U^C(\overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}}, \tilde{\mathscr{E}}^{-1}_{\mathbf{D}}, \mathbf{D})$). However, based on the damage state of $\mathbf{D} + d\mathbf{D}$, the incremental complementary energy can be computed from the conventional stress state $\overset{\gg}{\mathbf{t}}_{\mathbf{D}}$ to $\overset{\gg}{\mathbf{t}}_{\mathbf{D}} + d\overset{\gg}{\mathbf{t}}_{\mathbf{D}}$, which is presented by the solid-line hatched area ($\delta U^C(\overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}}, \tilde{\mathscr{E}}^{-1}_{\mathbf{D}+d\mathbf{D}}, \mathbf{D}+d\mathbf{D})$). The incremental complementary energy equivalence requires both of the incremental complementary energy be equal.

From the complementary energy, the constitutive equation for incremental continuous damage is

$$\overset{\gg}{\boldsymbol{\varepsilon}}_{\mathbf{D}+d\mathbf{D}} = \frac{1}{\mathscr{J}} \frac{\partial[\delta U^C(\overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}}\, \tilde{E}^{-1}_{\mathbf{D}+d\mathbf{D}}, \mathbf{D}+d\mathbf{D})]}{\partial(\delta \overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}})}. \tag{4.75}$$

For linear elastic, damaged materials, equation (4.72) gives

$$\delta U^C(\overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}}, \tilde{E}^{-1}_{\mathbf{D}+d\mathbf{D}}, \mathbf{D}+d\mathbf{D}) = \delta U^C(\overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}}, \tilde{E}^{-1}_{\mathbf{D}}, \mathbf{D}) \tag{4.76}$$

where

$$\delta U^C(\overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}}, \tilde{E}^{-1}_{\mathbf{D}+d\mathbf{D}}, \mathbf{D}+d\mathbf{D}) = \delta\overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}} : \tilde{E}^{-1}_{\mathbf{D}+d\mathbf{D}} : \overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}} \tag{4.77}$$

$$\delta U^C(\overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}}, \tilde{E}^{-1}_{\mathbf{D}}, \mathbf{D}) = \delta\overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}} : \tilde{E}^{-1}_{\mathbf{D}} : \overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}}. \tag{4.78}$$

From the incremental density of complementary energy, the hypothesis for the incremental complementary energy is expressed for the linear anisotropic damaged materials via

$$\delta\mathcal{U}^{C}(\overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}}\, \tilde{\boldsymbol{E}}^{-1}_{\mathbf{D}+d\mathbf{D}}, \mathbf{D}+d\mathbf{D}) = \delta\mathcal{U}^{C}(\overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}}, \tilde{\boldsymbol{E}}^{-1}_{\mathbf{D}}, \mathbf{D}), \tag{4.79}$$

where

$$\begin{aligned}\delta\mathcal{U}^{C}(\overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}}, \tilde{\boldsymbol{E}}^{-1}_{\mathbf{D}+d\mathbf{D}}, \mathbf{D}+d\mathbf{D}) &= \int_{\mathscr{V}} \delta U^{c}(\overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}+d\mathbf{D}}, \tilde{\boldsymbol{E}}^{-1}_{\mathbf{D}+d\mathbf{D}}, \mathbf{D}+d\mathbf{D})dV \\ &= \int_{\mathscr{V}} (\mathscr{J}\delta \overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}} : \tilde{\boldsymbol{E}}^{-1}_{\mathbf{D}+d\mathbf{D}} : \overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}+d\mathbf{D}})dV \\ &= \int_{v} (\delta \overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}} : \tilde{\boldsymbol{E}}^{-1}_{\mathbf{D}+d\mathbf{D}} : \overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}+d\mathbf{D}})dv,\end{aligned} \tag{4.80}$$

$$\begin{aligned}\delta\mathcal{U}^{C}(\overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}}, \tilde{\boldsymbol{E}}^{-1}_{\mathbf{D}}, \mathbf{D}) &= \int_{\mathscr{V}} \delta U^{c}(\overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}}, \tilde{\boldsymbol{E}}^{-1}_{\mathbf{D}}, \mathbf{D})dV \\ &= \int_{\mathscr{V}} (\mathscr{J}\delta \overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}} : \tilde{\boldsymbol{E}}^{-1}_{\mathbf{D}} : \overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}})dV \\ &= \int_{v} (\delta \overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}} : \tilde{\boldsymbol{E}}^{-1}_{\mathbf{D}} : \overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}})dv.\end{aligned} \tag{4.81}$$

For an anisotropic damage, the relation between the conventional stress $\overset{\gg}{\mathbf{t}}$ and effective stress $\overset{\gg}{\mathbf{t}}_{\mathbf{D}}$ is determined by Eq.(4.50). For a small, incremental damage, such a relation holds, i.e.,

$$\begin{aligned}\overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}} &= \mathbf{M}(d\mathbf{D}) : \overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}}, \\ \delta \overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}} &= \mathbf{M}(d\mathbf{D}) : \delta \overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}}.\end{aligned} \tag{4.82}$$

Substitution of Eq.(4.82) into Eq.(4.81) yields

$$\begin{aligned}\delta\mathcal{U}^{C}(\overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}}, \tilde{\boldsymbol{E}}^{-1}_{\mathbf{D}}, \mathbf{D}) &= \int_{V} \delta U^{c}(\overset{\gg}{\tilde{\mathbf{t}}}_{\mathbf{D}}, \tilde{\boldsymbol{E}}^{-1}_{\mathbf{D}}, \mathbf{D})dV \\ &= \int_{V} (\mathscr{J}\delta \overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}} : \mathbf{M}(d\mathbf{D}) : \tilde{\boldsymbol{E}}^{-1}_{\mathbf{D}} : \mathbf{M}(d\mathbf{D}) : \overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}})dV \\ &= \int_{v} (\delta \overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}} : \mathbf{M}(d\mathbf{D}) : \tilde{\boldsymbol{E}}^{-1}_{\mathbf{D}} : \mathbf{M}(d\mathbf{D}) : \overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}})dv.\end{aligned} \tag{4.83}$$

With Eqs.(4.80) and (4.83), the incremental complementary energy for linear anisotropic materials gives

$$\tilde{\boldsymbol{E}}^{-1}_{\mathbf{D}+d\mathbf{D}} = \mathbf{M}(d\mathbf{D}) : \tilde{\boldsymbol{E}}^{-1}_{\mathbf{D}} : \mathbf{M}(d\mathbf{D}). \tag{4.84}$$

Such a relation can be developed in the deformed configuration of $\mathfrak{b}$ under the invariance of volumes for conventional and damage states.

For linear elastic damaged materials, equation (4.75) gives

$$\overset{\gg}{\boldsymbol{\varepsilon}}_{\mathbf{D}+d\mathbf{D}} = \tilde{\boldsymbol{E}}_{\mathbf{D}+d\mathbf{D}}^{-1} : \overset{\gg}{\mathbf{t}}_{\mathbf{D}+d\mathbf{D}} \text{ or } \overset{\gg}{\boldsymbol{\varepsilon}}_{\mathbf{D}} = \tilde{\boldsymbol{E}}_{\mathbf{D}}^{-1} : \overset{\gg}{\mathbf{t}}_{\mathbf{D}} \tag{4.85}$$

4.4. An anisotropic damage theory

To demonstrate how to use the equivalence principle of the incremental complementary energy to determine effective stiffness matrix, the notations are

$$\begin{aligned} \boldsymbol{\sigma} &= (t_{11}, t_{22}, t_{33}, t_{12}, t_{13}, t_{23})^{\mathrm{T}}, \\ \boldsymbol{\varepsilon} &= (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23})^{\mathrm{T}}. \end{aligned} \tag{4.86}$$

The corresponding damage effective tensor is

$$\begin{aligned} \mathbf{M}(\mathbf{D}) &= diag(M_{11}, M_{22}, M_{33}, M_{12}, M_{13}, M_{23})_{6\times 6}, \\ \text{with } M_{ij}(\mathbf{D}) &= (1-D_i)^{-1/2}(1-D_j)^{-1/2} \text{ for } i, j = 1, 2, 3, \end{aligned} \tag{4.87}$$

where $D_i (i = 1,2,3)$ are the damage variables in their principal axes. The incremental damage effective tensor is

$$\begin{aligned} \mathbf{M}(d\mathbf{D}) &= diag\left(M_{11}, M_{22}, M_{33}, M_{12}, M_{13}, M_{23}\right)_{6\times 6}, \\ \text{with } M_{ij}(d\mathbf{D}) &= (1-dD_i)^{-1/2}(1-dD_j)^{-1/2} \text{ for } i, j = 1, 2, 3. \end{aligned} \tag{4.88}$$

Consider a total damage of **D** to be discretized n-piecewise linear damage quantity, as shown in Fig.4.5. Between two piecewise linear damages, the incremental quantity of damage variable in the ith-direction is given by $dD_i^{(k)}$ ($k =$ $1, 2, \cdots, n$). Therefore,

$$D_i = \sum\nolimits_{k=1}^{n} dD_i^{(k)} \text{ for } i = 1, 2, 3 \tag{4.89}$$

From Eq.(4.83),

$$\begin{aligned} \tilde{\boldsymbol{E}}_{\mathbf{D}^{(1)}}^{-1} &= \mathbf{M}(d\mathbf{D}^{(1)}) : \boldsymbol{E}^{-1} : \mathbf{M}(d\mathbf{D}^{(1)}), \\ \tilde{\boldsymbol{E}}_{\mathbf{D}^{(k)}}^{-1} &= \mathbf{M}(d\mathbf{D}^{(k)}) : \tilde{\boldsymbol{E}}_{\mathbf{D}^{(k-1)}}^{-1} : \mathbf{M}(d\mathbf{D}^{(k)}) \text{ for } k = 2, \ldots, n, \end{aligned} \tag{4.90}$$

where $\tilde{\boldsymbol{E}}_{\mathbf{D}^{(k)}}^{-1}$ and $\tilde{\boldsymbol{E}}^{-1}$ are the second order tensor of 6×6 owing to definitions in Eqs.(4.86)–(4.88). With iteration, Equation (4.90) yields

$$\begin{aligned} \tilde{\boldsymbol{E}}_{\mathbf{D}}^{-1} = & \underbrace{\mathbf{M}(d\mathbf{D}^{(n)}) : \cdots : \mathbf{M}(d\mathbf{D}^{(k)}) : \cdots : \mathbf{M}(d\mathbf{D}^{(1)})}_{n} : \boldsymbol{E}^{-1} \\ & : \underbrace{\mathbf{M}(d\mathbf{D}^{(1)}) : \cdots : \mathbf{M}(d\mathbf{D}^{(k)}) : \cdots : \mathbf{M}(d\mathbf{D}^{(n)})}_{n}, \end{aligned} \tag{4.91}$$

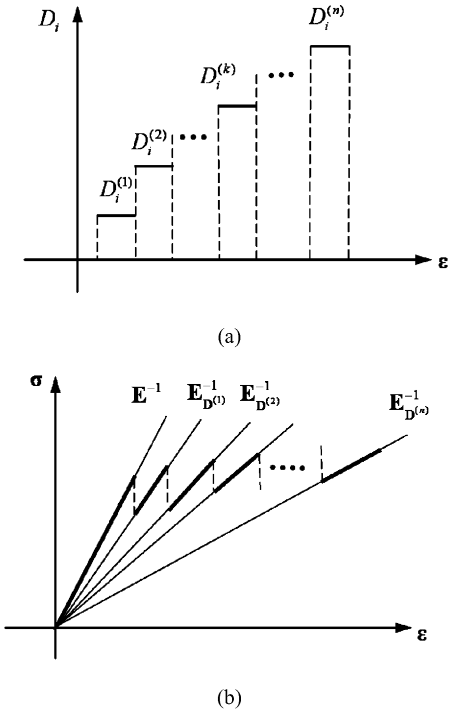

Fig. 4.5 Piecewise linear damage of materials: (a) piecewise damage and (b) stress and strain relation for each incremental damage.

$$\begin{aligned}\tilde{\boldsymbol{E}}_{\mathbf{D}}^{-1} &= diag\left(\Pi_{k=1}^{n} M_{ij}(d\mathbf{D}^{(k)})\right)_{6\times 6} : \boldsymbol{E}^{-1} : diag\left(\Pi_{k=1}^{n} M_{ij}(d\mathbf{D}^{(k)})\right)_{6\times 6} \\ &= diag\left(\Pi_{k=1}^{n}(1-dD_i^{(k)})^{-1/2}(1-dD_j^{(k)})^{-1/2}\right)_{6\times 6} : \boldsymbol{E}^{-1} \\ &\quad : diag\left(\Pi_{k=1}^{n}(1-dD_i^{(k)})^{-1/2}(1-dD_j^{(k)})^{-1/2}\right)_{6\times 6}. \end{aligned} \tag{4.92}$$

Suppose that the incremental damage variable is equal (i.e., $dD_i^{(k)} = dD_i = D_i / n$). Equation (4.92) becomes, for $i, j = 1,2,3$,

$$\begin{aligned}\tilde{\boldsymbol{E}}_{\mathbf{D}}^{-1} &= diag\left((1-dD_i)^{-n/2}(1-dD_j)^{-n/2}\right)_{6\times 6} : \boldsymbol{E}^{-1} \\ &\quad : diag\left((1-dD_i)^{-n/2}(1-dD_j)^{-n/2}\right)_{6\times 6} \\ &= diag\left((1-\frac{D_i}{n})^{-n/2}(1-\frac{D_j}{n})^{-n/2}\right)_{6\times 6} : \boldsymbol{E}^{-1} \\ &\quad : diag\left((1-\frac{D_i}{n})^{-n/2}(1-\frac{D_j}{n})^{-n/2}\right)_{6\times 6}. \end{aligned} \tag{4.93}$$

For $n \to \infty$, the foregoing equation gives

$$\tilde{\boldsymbol{E}}_{\mathbf{D}}^{-1} = diag\left(e^{(D_i+D_j)/2}\right)_{6\times6} : \boldsymbol{E}^{-1} : diag\left(e^{(D_i+D_j)/2}\right)_{6\times6}. \tag{4.94}$$

In addition, a new damage effect tensor is obtained, i.e.,

$$\begin{aligned} \mathbf{M}(\mathbf{D}) &= diag(M_{11}, M_{22}, M_{33}, M_{12}, M_{13}, M_{23})_{6\times6}, \\ M_{ij}(\mathbf{D}) &= e^{(D_i+D_j)/2} \text{ for } i, j = 1,2,3; \end{aligned} \tag{4.95}$$

Finally,

$$\tilde{\boldsymbol{E}}_{\mathbf{D}}^{-1} = \mathbf{M}(\mathbf{D}) : \boldsymbol{E}^{-1} : \mathbf{M}(\mathbf{D}). \tag{4.96}$$

Consider an isotropic material with the elastic tensor as

$$\mathbf{E}^{-1} = \frac{1}{E}(a_{ij})_{6\times6}, \tag{4.97}$$

where

$$\begin{aligned} &a_{ii} = 1,\ a_{ij} = -\nu \text{ for } i, j = 1,2,3; \\ &a_{ii} = 2(1+\nu) \text{ for } i = 4, 5, 6; \\ &a_{ij} = 0; \text{ for } i = 1, 2, \ldots, 6 \text{ and } j = 4, 5, 6; \\ &a_{ij} = 0; \text{ for } i = 4, 5, 6 \text{ and } j = 1, 2, \ldots, 6; \\ &(i \neq j). \end{aligned} \tag{4.98}$$

The damage elastic tensor is given by

$$\tilde{\mathbf{E}}_D^{-1} = \frac{1}{E}(a_{ij}^D)_{6\times6}, \tag{4.99}$$

where

$$\begin{aligned} &a_{ii}^D = e^{2D_i},\ a_{ij}^D = -\nu e^{(D_i+D_j)} \text{ for } i, j = 1, 2, 3; \\ &a_{44}^D = 2(1+\nu)e^{(D_1+D_2)},\ a_{55}^D = 2(1+\nu)e^{(D_1+D_3)}, \\ &a_{66}^D = 2(1+\nu)e^{(D_2+D_3)}; \\ &a_{ij}^D = 0; \text{ for } i = 1, 2, \ldots, 6 \text{ and } j = 4, 5, 6; \\ &a_{ij}^D = 0; \text{ for } i = 4, 5, 6 \text{ and } j = 1, 2, \ldots, 6; \\ &(i \neq j). \end{aligned} \tag{4.100}$$

If the damage variable is very small (i.e., $D_i << 1$), each component in $\mathbf{M}(\mathbf{D})$ can be expanded by the Taylor series. Keeping the first order and neglecting the higher order terms of such Taylor series yields $e^{-D_i} \approx 1 - D_i$. The components in Eq.(4.99) becomes

$$
\begin{aligned}
& a_{ii}^{D} = \frac{1}{(1-D_i)^2}, \ a_{ij}^{D} = \frac{-\nu}{(1-D_i)(1-D_j)} \ \text{ for } i, \ j = 1, 2, 3 \\
& a_{44}^{D} = \frac{2(1+\nu)}{(1-D_1)(1-D_2)}, \ a_{55}^{D} = \frac{2(1+\nu)}{(1-D_1)(1-D_3)}, \\
& a_{66}^{D} = \frac{2(1+\nu)}{(1-D_2)(1-D_3)}; \\
& \left. \begin{aligned} & a_{ij}^{D} = 0; \ \text{for } i = 1, 2, \ldots, 6 \text{ and } j = 4, 5, 6 \\ & a_{ij}^{D} = 0; \ \text{for } i = 4, 5, 6 \text{ and } j = 1, 2, \ldots, 6 \end{aligned} \right\} \text{with } i \neq j.
\end{aligned} \tag{4.101}
$$

With Eq.(4.101), equation (4.99) gives the effective elastic tensor, identical to that in Chow and Wang (1987).

4.5. Applications

This section will demonstrate applications of the large damage theory and the uniaxial tension, pure torsion and perfect-plastic materials will be presented.

4.5.1. Uniaxial tensional models

From the large damage theory with anisotropic damage, the strains under a tension stress are

$$
\begin{aligned}
\varepsilon_1 &= \frac{1}{E} e^{2D_1} \sigma_1 = \frac{1}{\tilde{E}} \sigma_1, \\
\varepsilon_2 &= -\frac{\nu_{12}}{E} e^{D_1 + D_2} \sigma_1 = -\frac{\tilde{\nu}_{12}}{\tilde{E}} \sigma_1, \\
\varepsilon_3 &= -\frac{\nu_{13}}{E} e^{D_1 + D_3} \sigma_1 = -\frac{\tilde{\nu}_{13}}{\tilde{E}} \sigma_1,
\end{aligned} \tag{4.102}
$$

where the effective Young's modulus and Poisson's ratios are

$$
\tilde{E} = e^{-2D_1} E, \ \tilde{\nu}_{12} = e^{D_2 - D_1} \nu_{12} \text{ and } \tilde{\nu}_{13} = e^{D_3 - D_1} \nu_{13}. \tag{4.103}
$$

From the foregoing equation, the damage variables are

$$
D_1 = -\frac{1}{2} \ln \frac{\tilde{E}}{E},
$$

$$
\begin{aligned}
D_2 &= D_1 - \ln\frac{\nu_{12}}{\tilde{\nu}_{12}} = -\ln\Big(\frac{\nu_{12}}{\tilde{\nu}_{12}}\sqrt{\frac{\tilde{E}}{E}}\Big),\\
D_3 &= D_1 - \ln\frac{\nu_{13}}{\tilde{\nu}_{13}} = -\ln\Big(\frac{\nu_{13}}{\tilde{\nu}_{13}}\sqrt{\frac{\tilde{E}}{E}}\Big).
\end{aligned}
\tag{4.104}
$$

From the small damage theory with anisotropic damage, the strains under a tension stress are given by

$$
\begin{aligned}
\varepsilon_1 &= \frac{1}{E(1-D_1)^2}\sigma_1 = \frac{1}{\tilde{E}}\sigma_1,\\
\varepsilon_2 &= -\frac{\nu_{12}}{E(1-D_1)(1-D_2)}\sigma_1 = -\frac{\tilde{\nu}_{12}}{\tilde{E}}\sigma_1,\\
\varepsilon_3 &= -\frac{\nu_{13}}{E(1-D_1)(1-D_3)}\sigma_1 = -\frac{\tilde{\nu}_{13}}{\tilde{E}}\sigma_1.
\end{aligned}
\tag{4.105}
$$

where the effective Young's modulus and Poisson's ratios are

$$
\tilde{E} = (1-D_1)^2 E,\ \tilde{\nu}_{12} = \frac{1-D_1}{1-D_2}\nu_{12} \text{ and } \tilde{\nu}_{13} = \frac{1-D_1}{1-D_3}\nu_{13}.
\tag{4.106}
$$

From the foregoing equation, the damage variables are computed by

$$
\begin{aligned}
D_1 &= 1 - \sqrt{\frac{\tilde{E}}{E}},\\
D_2 &= 1 - \frac{\nu_{12}}{\tilde{\nu}_{12}}(1-D_1) = 1 - \frac{\nu_{12}}{\tilde{\nu}_{12}}\sqrt{\frac{\tilde{E}}{E}},\\
D_3 &= 1 - \frac{\nu_{13}}{\tilde{\nu}_{13}}(1-D_1) = 1 - \frac{\nu_{13}}{\tilde{\nu}_{13}}\sqrt{\frac{\tilde{E}}{E}}.
\end{aligned}
\tag{4.107}
$$

It is observed that Eq.(4.107) can be recovered from Eq.(4.104) through the Taylor series expansion and neglecting the higher order terms when the damage is very small.

4.5.2. Pure torsion

Consider an anisotropic damage of a shaft under a pure torsion. Suppose that the maximum shear stress is τ, the corresponding principal stresses are $\sigma_1 = \tau$, $\sigma_1 = -\tau$ and $\sigma_3 = 0$. From the large damage theory,

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} e^{2D_1} & -\nu e^{(D_1+D_2)} & 0 \\ -\nu e^{-(D_1+D_2)} & e^{2D_2} & 0 \\ 0 & 0 & 2(1+\nu)e^{(D_1+D_2)} \end{bmatrix} \begin{Bmatrix} \tau \\ -\tau \\ 0 \end{Bmatrix}. \tag{4.108}$$

Solving the foregoing equation gives

$$D_1 = \frac{1}{2}\ln\frac{Y^2 X}{Y^2-1} \text{ and } D_2 = \frac{1}{2}\ln\frac{X}{Y^2-1}, \tag{4.109}$$

where

$$\begin{aligned} X &= \frac{\tau}{E}(\varepsilon_1+\varepsilon_2), \\ Y &= \frac{1}{2}\left[-\nu(1+\frac{\varepsilon_1}{\varepsilon_2}) + \sqrt{\nu^2(1+\frac{\varepsilon_1}{\varepsilon_2})^2 - 4\frac{\varepsilon_1}{\varepsilon_2}}\right]. \end{aligned} \tag{4.110}$$

The effective Young's moduli and Poison's ratios are

$$\tilde{E}_1 = Ee^{-2D_1} = E\frac{Y^2 X}{Y^2-1} \text{ and } \tilde{E}_2 = Ee^{-2D_2} = E\frac{Y^2-1}{X}, \tag{4.111}$$

$$\tilde{\nu}_{12} = \nu e^{D_2-D_1} = \frac{\nu}{Y} \text{ and } \tilde{\nu}_{21} = \nu e^{D_1-D_2} = \nu Y. \tag{4.112}$$

The effective shear modulus $\tilde{G}$ can be evaluated by the transformation of the damage tensor $\mathbf{D}$ and damage effective tensor $\mathbf{M}(\mathbf{D})$ from the principal stress axes to the maximum shear stress axes. For this case, the maximum shear stress state is a pure shear state given by the pure torsion. The direction cosines between the two coordinate systems are

$$\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.113}$$

Thus, the effective shear modulus $\tilde{G}$ for the anisotropic damage is

$$\tilde{G} = \frac{2(1+\nu)G}{e^{2D_1} + e^{2D_2} + 2\nu e^{D_1+D_2}}. \tag{4.114}$$

With Taylor series expansion, the foregoing equation gives an expression for small damage as in Chow and Wang (1987). That is,

$$\tilde{G} = \frac{2(1+\nu)G}{(1-D_1)^{-2} + (1-D_2)^{-2} + 2\nu(1-D_1)^{-1}(1-D_2)^{-1}}. \tag{4.115}$$

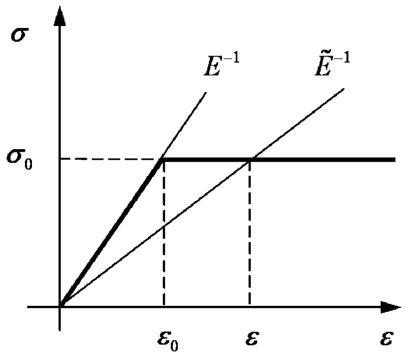

Fig. 4.6 An elastic perfectly-plastic material.

4.5.3. Elastic perfectly-plastic materials

In this section, the large damage theory will be applied to a large damage model of a material with an elastic perfectly-plastic behavior, as shown in Fig.4.6. The un-damage constitutive equation for uniaxial tension is

$$\sigma = \begin{cases} E\varepsilon & \text{if } \varepsilon \le \varepsilon_0, \\ E\varepsilon_0 & \text{if } \varepsilon \ge \varepsilon_0, \end{cases} \tag{4.116}$$

where ε_0 is the strain at yield. For isotropic materials with anisotropic damage, the effective elastic modulus is computed from Eq.(4.99), i.e.,

$$\tilde{E} = e^{-2D} E \quad \text{or} \quad D = -\frac{1}{2}\ln\frac{\tilde{E}}{E}. \tag{4.117}$$

In terms of strains, the damage variable is evaluated by

$$D = \begin{cases} -\dfrac{1}{2}\ln\dfrac{\varepsilon_0}{\varepsilon} & \text{for } \varepsilon \ge \varepsilon_0, \\ 0 & \text{for } \varepsilon \le \varepsilon_0. \end{cases} \tag{4.118}$$

If the damage is very small, equation (4.118) becomes

$$D = \begin{cases} 1 - \sqrt{\dfrac{\varepsilon_0}{\varepsilon}} & \text{for } \varepsilon \ge \varepsilon_0, \\ 0 & \text{for } \varepsilon \le \varepsilon_0. \end{cases} \tag{4.119}$$

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Chapter 5
Nonlinear Cables

This chapter will discuss nonlinear cables as the simplest soft deformable element. A general nonlinear theory of cables will be presented. Equations of motion for traveling and rotating cables will be discussed. The closed-formed solution for equilibrium of elastic cables will be developed. To investigate the cable dynamics, the rigid body dynamics of cables will be discussed. Further, the elastic cable dynamics can be investigated.

5.1. A nonlinear theory of cables

In this section, the theory for nonlinear cables will be discussed. Before the nonlinear cable theory is discussed, the following concepts of deformable and inextensible cables are introduced first.

Definition 5.1. If a 1-dimensional deformable body only resists the tensile forces, the deformable body is called a *deformable cable*.

Definition 5.2. If a 1-dimensional, non-deformable body only resists the tensile forces, the 1-dimensional, non-deformable body is called an *inextensible cable*.

From the two definitions, if the internal tensile force of a cable becomes compressive, the current configurations of the deformable and inextensible cables cannot exist. In other words, the compressive forces cannot exist in the entire deformable and inextensible cables. If all the internal tensile forces on the cable become zero, the cable configuration will keep the configuration of the inextensible cable. If the internal tensile force on the cable becomes zero in a local segment, the local configuration of the local segment of the cable will keep the inextensible cable configuration. On such a local segment, the corresponding inextensible configuration may be changed with any small perturbation force to form a new configuration with tensile forces or to be in any knotted state without any configuration. Such a local new configuration is discontinuous to the non-knotted, global configuration. Such a phenomenon of the cable is called *the local knotting of cable*. After the cable is locally knotted, the knotted segment cannot form any configuration. In other words, only if the tensile force on the cable exists, the cable configuration can be formed.

Consider a nonlinear cable with an initial configuration in coordinates (X^I, $I=1,2,3$) with unit vectors ($\mathbf{I}_I$, $I=1,2,3$), as shown in Fig.5.1. A point P on the initial configuration is given by (X^I, $I=1,2,3$) which is the function of a curvilinear coordinate ($S^1 \equiv S$) with the base vector ($\mathbf{G}_1$). On the cable cross section with the normal direction collinear to $\mathbf{G}_1$, two coordinates (S^Λ, $\Lambda=2,3$) with the corresponding directional vectors ($\mathbf{G}_\Lambda$, $\Lambda=2,3$) can be selected arbitrarily. Therefore, one assumes that $X^I = X^I(S)$ and the point P on the initial configuration is

$$\mathbf{R} = X^I(S)\mathbf{I}_I. \tag{5.1}$$

Under external forces, the nonlinear cable will form a new configuration in a coordinate (x^I, $I=1,2,3$) with unit vectors ($\mathbf{I}_I$, $I=1,2,3$), as shown in Fig.5.2. A point p on the new configuration is given by (x^I, $I=1,2,3$). For the new configuration, the corresponding curvilinear coordinate ($s^1 \equiv s$) with the base vector ($\mathbf{g}_1$) exists. Such a configuration is also called a final configuration under such external forces. Thus, a point p (x^I, $I=1,2,3$) can also be described through a curvilinear coordinate (S) with the base vector ($\mathbf{G}_1$). With $x^I = x^I(S)$, the vector $\mathbf{r}$ is

$$\mathbf{r} = x^I(S)\mathbf{I}_I, \tag{5.2}$$

and the displacement between point P and point p is

$$\mathbf{u} = u^I(S)\mathbf{I}_I. \tag{5.3}$$

Thus,

$$\mathbf{r} = \mathbf{R} + \mathbf{u},\ x^I(S) = X^I(S) + u^I(S). \tag{5.4}$$

The base vectors and the corresponding unit vectors are

$$\mathbf{G}_1 \equiv \frac{d\mathbf{R}}{dS} = X^I_{,1}\mathbf{I}_I \text{ and } \mathbf{g}_1 \equiv \frac{d\mathbf{r}}{dS} = (X^I_{,1} + u^I_{,1})\mathbf{I}_I; \tag{5.5}$$

$$\mathbf{N}_1 \equiv \frac{\mathbf{G}_1}{|\mathbf{G}_1|} = \frac{1}{\sqrt{G_{11}}} X^I_{,1}\mathbf{I}_I \text{ and } \mathbf{n}_1 \equiv \frac{\mathbf{g}_1}{|\mathbf{g}_1|} = \frac{X^I_{,1} + u^I_{,1}}{\sqrt{G_{11} + 2E_{11}}}\mathbf{I}_I, \tag{5.6}$$

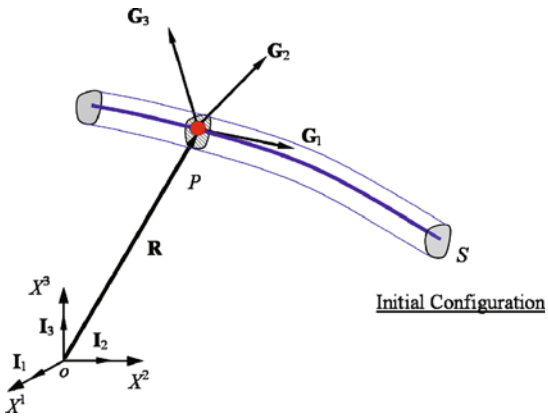


Fig. 5.1 A nonlinear cable with an initial configuration.

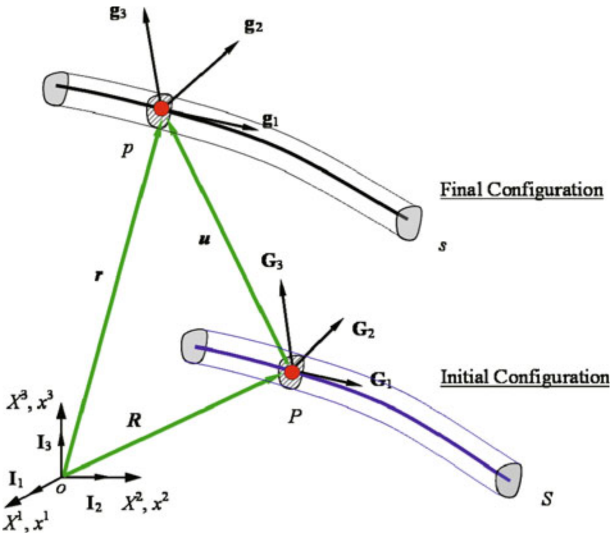


Fig. 5.2 Displacement on the cables with the initial and final configurations.

where $(\cdot)_{,1} = d(\cdot)/dS$ and summation on I should be completed, and

$$G_{11} = X^I_{,1} X^I_{,1} \text{ and } E_{11} = X^I_{,1} u^I_{,1} + X^I_{,1} u^I_{,1} + u^I_{,1} u^I_{,1}. \tag{5.7}$$

The strain on the direction of $\mathbf{G}_1$ is

$$\mathscr{E}(S) = \frac{|d\mathbf{r}| - |d\mathbf{R}|}{|d\mathbf{R}|} = \frac{1}{\sqrt{G_{11}}} \sqrt{(X^I_{,1} + u^I_{,1})(X^I_{,1} + u^I_{,1})} - 1. \tag{5.8}$$

From the material law, the tension is expressed by

$$T = \int_A f(\mathscr{E})dA, \tag{5.9}$$

where $f(\mathscr{E}) = 0$ for $\mathscr{E} = 0$. With $E = \partial f / \partial \mathscr{E}|_{\mathscr{E}=0}$, the tension for a linear elastic material is

$$T = \int_A E\mathscr{E}dA = \int_A E\,[\frac{1}{\sqrt{G_{11}}}\sqrt{(X^I_{,1}+u^I_{,1})(X^I_{,1}+u^I_{,1})}-1]\,dA, \tag{5.10}$$

where Young's modulus and cross-section area in the $\mathbf{N}_1$-direction are E and A, respectively. If the initial configuration is in the deformed state with initial tension T^0 in the S^1-direction, the corresponding strain is

$$\varepsilon = \mathscr{E}^0(S) + \frac{1}{\sqrt{G_{11}}}\sqrt{(X^I_{,1}+u^I_{,1})(X^I_{,1}+u^I_{,1})}-1, \tag{5.11}$$

where the initial tension $T^0(S) = \int_A f(\mathscr{E}^0)dA,$ and the tension on the deformed configuration of the cable is determined by

$$T = \int_A f(\varepsilon)dA = \int_A f(\mathscr{E}^0 + \mathscr{E})dA. \tag{5.12}$$

For the linear elasticity $(E = \text{const})$,

$$T = T^0(S) + EA[\frac{1}{\sqrt{G_{11}}}\sqrt{(X^I_{,1}+u^I_{,1})(X^I_{,1}+u^I_{,1})}-1]. \tag{5.13}$$

If the displacement $u^I = 0$ ($I = 1, 2, 3$), then, equation (5.11) becomes

$$\varepsilon = \mathscr{E}^0(S) + (\frac{1}{\sqrt{G_{11}}}\sqrt{X^I_{,1}X^I_{,1}}-1). \tag{5.14}$$

The geometric relation gives

$$\frac{1}{\sqrt{G_{11}}}\sqrt{X^I_{,1}X^I_{,1}} = 1. \tag{5.15}$$

The initial strain and tension can be recovered (i.e., $\varepsilon = \mathscr{E}^0(S)$). However, the initial tension is very difficult to obtain. On the other hand, the inextensible cable possesses the fact of $EA \to \infty$. Equation (5.9) gives

$$\frac{T}{EA} = \frac{1}{EA}\int_A f(\mathscr{E})dA = \frac{1}{EA}\int_A \left[E\mathscr{E} + o(E\mathscr{E})\right]dA. \tag{5.16}$$

As $EA \to \infty$, the foregoing equation leads to $\mathscr{E}=0$, i.e.,

$$\frac{1}{\sqrt{G_{11}}}\sqrt{(X_{,1}^I+u_{,1}^I)(X_{,1}^I+u_{,1}^I)}=1 \text{ (summation on } I). \tag{5.17}$$

However, with Eq.(5.15), equation (5.17) yields

$$u^I=0 \ \ (I=1,2,3). \tag{5.18}$$

It means that the inextensible cable can support any tension without stretch. Therefore, the inextensible cable configuration should be considered as an initial configuration. To apply the external forces on the cable and to investigate the corresponding nonlinear dynamics, the point and segment sets on the initial configuration of the deformable cable are defined by

$$\mathscr{P}=\bigcup_{k=0}^m \mathscr{P}_k \text{ and } \mathscr{P}_k\equiv\{S\,|\,S=A_k\}, \tag{5.19}$$

$$\mathcal{S}=\bigcup_{k=1}^m \mathcal{S}_k, \text{ and } \mathcal{S}_k\equiv\{S\,|\,S\in[A_{k-1},A_k]\}. \tag{5.20}$$

The point and segment sets on the deformed configuration are defined as

$$\boldsymbol{p}\equiv\bigcup_{k=0}^m \boldsymbol{p}_k \text{ and } \boldsymbol{p}_k\equiv\{s\,|\,s=a_k\}, \tag{5.21}$$

$$\boldsymbol{s}=\bigcup_{k=1}^m \boldsymbol{s}_k, \text{ and } \boldsymbol{s}_k\equiv\{s\,|\,s\in[a_{k-1},a_k]\}. \tag{5.22}$$

Consider the force distributed force on the segment $\mathcal{S}_k$ and the concentrated force on a point $\mathscr{P}_k$ for ($k=1,2,\ldots,m$), i.e.,

$$\begin{aligned}\mathbf{q}_k&\equiv q_k^I(S)\mathbf{I}_I \ \text{ on } \mathcal{S}_k,\\ \mathbf{F}_k&\equiv F_k^I(S)\mathbf{I}_I \ \text{ at } \mathscr{P}_k.\end{aligned} \tag{5.23}$$

The corresponding forces on the segment $\boldsymbol{s}_k$ and concentrated forces on point $\boldsymbol{p}_k$ are

$$\begin{aligned}\mathbf{p}_k&\equiv p_k^I(s)\mathbf{I}_I \ \text{ on } \boldsymbol{s}_k,\\ \mathbf{f}_k&\equiv f_k^I(s)\mathbf{I}_I \ \text{ at } \boldsymbol{p}_k.\end{aligned} \tag{5.24}$$

From Eqs.(5.23) and (5.24),

$$\begin{aligned}&\mathbf{F}_k=\mathbf{f}_k, \text{ or } F_k^I(S_k)=f_k^I(s_k);\\ &\mathbf{p}_k\equiv\frac{1}{\sqrt{G_{11}}(1+\mathscr{E})}\mathbf{q}_k, \text{ or } p_k^I(s)=\frac{1}{\sqrt{G_{11}}(1+\mathscr{E})}q_k^I(S),\end{aligned} \tag{5.25}$$

where

$$\sqrt{G_{11}}(1+\mathscr{E})\equiv\sqrt{(X_{,1}^I+u_{,1}^I)(X_{,1}^I+u_{,1}^I)}. \tag{5.26}$$

Therefore, equation of motion for segments $S \in \mathcal{S}_k$ is

$$\rho_0 A(\mathbf{X}_{,tt} + \mathbf{u}_{,tt}) = \mathbf{q} + \mathbf{T}_{,1}, \text{ or}$$

$$\rho_0 A(X^I_{,tt} + u^I_{,tt}) = q^I + \left[\frac{T(X^I_{,1} + u^I_{,1})}{\sqrt{G_{11}}(1+\mathscr{E})} \right]_{,1}, \tag{5.27}$$

where $\rho_0 A$ is based on the initial configuration. In Eq. (5.27), the *tension of cable* is computed by

$$T = \begin{cases} \int_A f(\varepsilon)\, dA & \text{for any materials,} \\ T^0(S) + \int_A E\mathscr{E}\, dA & \text{for linear elasticity,} \end{cases} \tag{5.28}$$

where $T^0(S) = \int_A E\mathscr{E}^0 dA$ and $\varepsilon = \mathscr{E}^0 + \mathscr{E}$.

The tension vector is defined as

$$\mathbf{T}(S) = T(S)\, \mathbf{n}_1 = T^I(S)\mathbf{I}_I, \tag{5.29}$$

where $\mathbf{n}_1$ is the unit normal direction of the cable.

$$T^I(S) = T(S)\, \mathbf{n}_1 \cdot \mathbf{I}_I = T(S)\cos\theta_{(\mathbf{n}_1,\mathbf{I}_I)} \tag{5.30}$$

and

$$\mathbf{n}_1 = \frac{(X^I_{,1} + u^I_{,1})}{\sqrt{G_{11} + 2E_{11}}}\mathbf{I}_I \text{ and } \cos\theta_{(\mathbf{n}_1,\mathbf{I}_I)} = \frac{X^I_{,1} + u^I_{,1}}{\sqrt{G_{11}}(1+\mathscr{E})}. \tag{5.31}$$

Consider the sign convention of internal forces. On the positive (or negative) cross section, the internal forces in the positive (or negative) direction are positive, otherwise negative. The force condition at the node $\mathscr{P}_k$ with $S = A_k$ is

$$\begin{aligned} &{}^-\mathbf{T}_k(S)\big|_{S=A_k} + {}^+\mathbf{T}_k(S)\big|_{S=A_k} + \mathbf{F}_k = 0, \text{ or} \\ &{}^-T_k(S)\cos\theta_{(\mathbf{n}_1,\mathbf{I}_I)}\big|_{S=A_k} = {}^+T_k(S)\cos\theta_{(\mathbf{n}_1,\mathbf{I}_I)}\big|_{S=A_k} + F^I_k, \end{aligned} \tag{5.32}$$

where the tension vector on such a normal direction of $\mathbf{n}_1$ is

$${}^{\pm}\mathbf{T}_k(S) = {}^{\pm}T_k(S)\mathbf{n}_1 \tag{5.33}$$

and

$${}^{\pm}T^I(S)\big|_{S=A_k} = {}^{\pm}T_k(S)\, \mathbf{n}_1 \cdot \mathbf{I}_I\big|_{S=A_k} = {}^{\pm}T_k(S)\cos\theta_{(\mathbf{n}_1,\mathbf{I}_I)}\big|_{S=A_k}. \tag{5.34}$$

The displacement boundary condition at a node point $\mathscr{P}_r$ is

$$u_r^I = b_r^I \text{ and } X_r^I = B_r^I \text{ for } r = 0, m. \tag{5.35}$$

For the force boundary, the corresponding conditions are given as in Eq.(5.31), i.e.,

$$\left.\begin{array}{l} \mathbf{T}_r(S)\big|_{S=A_r} + \mathbf{F}_r^I = 0, \\ T_r(S)\cos\theta_{(\mathbf{n}_1,\mathbf{I}_I)}\big|_{S=A_r} + F_r^I = 0 \end{array}\right\} \text{for } r = 0, m. \tag{5.36}$$

Definition 5.3. A deformable cable is called *a locally knotted cable* if

$$T_k(S) < 0 \;\text{ for } S \in [A_{k-1}, A_k]. \tag{5.37}$$

A deformable cable is called a *globally knotted cable* if all segments $\mathcal{S}_k \subset \mathcal{S}$ ($k = 1, 2, \ldots, m$) in such a cable satisfy Eq.(5.37). The two knotted boundaries on the segment $\mathcal{S}_k$ are determined by

$$T_k(S) = 0 \;\text{ for } S \in [A_{k-1}, A_k]. \tag{5.38}$$

Definition 5.4. If a 1-dimensional straight deformable body only resists the tensile forces, the deformable body is called a *deformable* string.

For the deformable string, there is an initial configuration. To investigate the dynamics of the string, it is not necessary to determine the initial configuration because the inextensible state is the same as the initial straight state of the string. However, for the deformable cable, the initial configuration is very difficult to be determined, which should be from the inextensible cable. For the deformable string, one can simply use $X^1(S) = S$ and $X^I(S) = 0$ ($I = 2, 3$). The afore-developed theory can be directly applied to the string. From the definition of cables, the cable cannot support any compressive forces. In addition, one is interested in the 1-dimensional deformable body that supports the tensile and compressive forces. Thus, the corresponding definitions are given as follows.

Definition 5.5. If a 1-dimensional deformable body only resists the tensile and compressive forces, the deformable body is called a *deformable* arch.

Definition 5.6. If a 1-dimensional straight deformable body only resists the tensile and compressive forces, the deformable body is called a *deformable* truss.

Compared to the deformed cable, the deformable arch must possess a specific, initial configuration to support such compressive internal forces. The deformed arch can be described as a deformed cable. Herein, such a description will not be repeated. The normal force on the arch is determined by letting $N \equiv T$ for all equations presented in this section. In addition, $\ddot{X}^I = \dot{X}^I = 0$ is admitted because the initial configuration is fixed. However, *the continuum cable possesses a dynamical initial configuration.* In other words, the non-deformable configuration of an arch is invariant with time, but the non-deformable configuration of a cable will be changed with time. The truss is a straight arch. The initial configuration is a straight line. Thus, one can use $X^1(S)=S$ and $X^I(S)=0$ ($I=2,3$) to describe like a string. In addition, the theory for the 1-dimensional deformable beams and rods will be discussed in Chapter 8.

5.2. Traveling and rotating cables

Consider a traveling and rotating, sagged, elastic cable passing through two eyelets, as sketched in Fig.5.3. The horizontal and vertical separations of the eyelets are L and H, respectively, and the length of cable is $\mathscr{L}$. This cable travels at a translation speed $\bar{c}$ along the longitudinal direction and rotates with a rotation speed $\mathbf{\Omega}$ at origin o. $\bar{x}$, $\bar{y}$ and $\bar{z}$ are Cartesian coordinates rotating with cable together; $\bar{y}$ is co-linear with the gravitational acceleration. The fixed end points B_1 and B_2 are positioned arbitrarily. The initial and final configurations with displacement are

$$\bar{\mathbf{R}}=\bar{X}^I\mathbf{I}_I,\ \bar{\mathbf{r}}=\bar{x}^I\mathbf{I}_I \text{ and } \bar{\mathbf{u}}=\bar{\mathbf{r}}-\bar{\mathbf{R}}=\bar{u}^I\mathbf{I}_I, \tag{5.39}$$

where $\sqrt{\bar{X}^I_{,\bar{S}}\bar{X}^I_{,\bar{S}}}=1$ ($\mathbf{I}_1=\mathbf{i},\mathbf{I}_2=\mathbf{j},\mathbf{I}_3=\mathbf{k}$) and $\bar{S}$ is the arc length of the cable. For a straight cable, $\bar{S}=\bar{X}^1(\bar{S})$ and $\bar{X}^2=\bar{X}^3=0$. The initial configuration $\bar{\mathbf{R}}$ can be or cannot be independent of time. The cable is subject to the distributed forces as $\bar{\mathbf{q}}=\bar{q}^I\mathbf{I}_I$. The rotation speed to the origin o is

$$\bar{\mathbf{\Omega}}=\bar{\Omega}^I\mathbf{I}_I. \tag{5.40}$$

For the constant rotation speed, the velocity and acceleration are

$$\begin{aligned}\frac{D\bar{\mathbf{r}}}{Dt}&=\frac{\partial\bar{\mathbf{r}}}{\partial t}+(\bar{\mathbf{\Omega}}\times\bar{\mathbf{r}})=(\bar{x}^I_{,t}+\bar{\Omega}^J\bar{x}^K e_{IJK})\mathbf{I}_I,\\ \frac{D^2\bar{\mathbf{r}}}{Dt^2}&=\frac{\partial^2\bar{\mathbf{r}}}{\partial t^2}+2\bar{\mathbf{\Omega}}\times\frac{\partial\bar{\mathbf{r}}}{\partial t}+\bar{\mathbf{\Omega}}\times(\bar{\mathbf{\Omega}}\times\bar{\mathbf{r}})\\ &=(\bar{x}^I_{,t}+2\bar{\Omega}^J\bar{x}^K_{,t}e_{IJK}+\bar{\Omega}^J\bar{\Omega}^K\bar{x}^L e_{IJM}e_{MKL})\mathbf{I}_I,\end{aligned} \tag{5.41}$$

where e_{IJK} is the Ricci function, and $(\cdot)_{,t}=\partial(\cdot)/\partial t$ and $(\cdot)_{,tt}=\partial^2(\cdot)/\partial t^2$. If the rotation speed changes with time, the velocity and acceleration are expressed by

$$\frac{D\bar{\mathbf{r}}}{Dt}=\frac{\partial\bar{\mathbf{r}}}{\partial t}+(\bar{\mathbf{\Omega}}\times\bar{\mathbf{r}})=(\bar{x}^I_{,t}+\bar{\Omega}^J\bar{x}^K e_{IJK})\mathbf{I}_I,$$

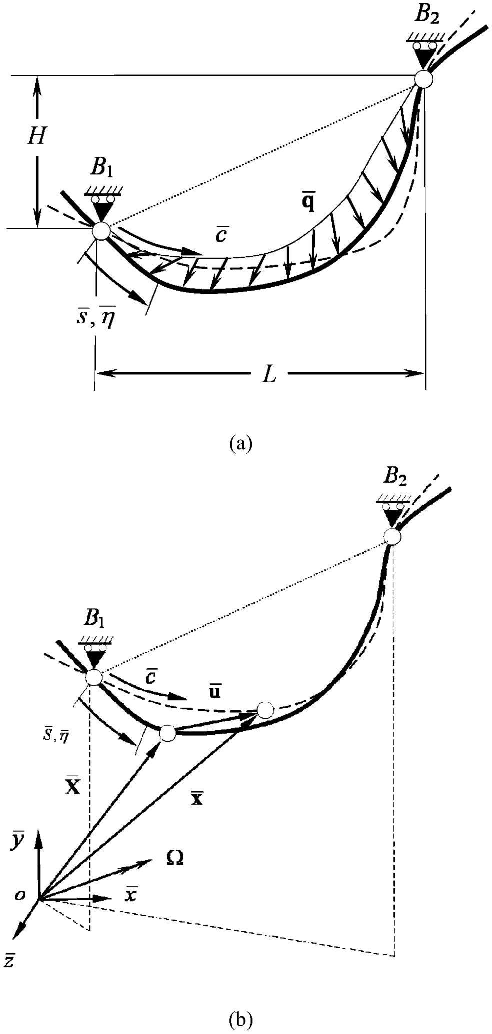


Fig. 5.3 Equilibrium and deformation of a traveling, sagged cable under a distributed loads (a) loading and (b) displacement.

$$
\begin{aligned}
\frac{D^2\bar{\mathbf{r}}}{Dt^2} &= \frac{\partial^2\bar{\mathbf{r}}}{\partial t^2} + \frac{\partial\bar{\mathbf{\Omega}}}{\partial t}\times\bar{\mathbf{r}} + 2\mathbf{\Omega}\times\frac{\partial\mathbf{r}}{\partial t} + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) \\
&= (\bar{x}^I_{,t} + \bar{\Omega}^J_{,t}\bar{x}^K e_{IJK} + 2\bar{\Omega}^J\bar{x}^K_{,t} e_{IJK} + \bar{\Omega}^J\bar{\Omega}^K\bar{x}^L e_{IJM}e_{MKL})\mathbf{I}_I .
\end{aligned}
\tag{5.42}
$$

The strain of equilibrium under the initial tension $\bar{T}^0$ is

$$
\varepsilon = \mathscr{E}^0(\bar{S}) + \left[\sqrt{(\bar{X}^I_{,\bar{S}} + \bar{u}^I_{,\bar{S}})(\bar{X}^I_{,\bar{S}} + \bar{u}^I_{,\bar{S}})} - 1\right] \text{(summation on } I\text{)} \tag{5.43}
$$

where $(\cdot)_{,\bar{S}} = \partial(\cdot)/\partial\bar{S}$. The tension in the deformable cable at equilibrium is

$$\begin{aligned}
&\bar{T}(\bar{S}) = \int_A \bar{f}(\varepsilon)dA \qquad \text{for any materials,}\\
&\bar{T}(\bar{S}) = \bar{T}^0(\bar{S}) + EA\left[\sqrt{(\bar{X}^I_{,\bar{S}} + \bar{u}^I_{,\bar{S}})(\bar{X}^I_{,\bar{S}} + \bar{u}^I_{,\bar{S}})} - 1\right]\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for elastic material,}
\end{aligned} \tag{5.44}$$

where $\mathscr{E}^0 = \bar{T}^0 / EA$, and E, A and $\bar{S}$ are Young's modulus, the cross-sectional area and the arc length of the cable in the initial configuration. For the inextensible cable of $(EA \to \infty)$, $\varepsilon = \mathscr{E}^0$ because of $\sqrt{\bar{X}^I_{,\bar{S}}\bar{X}^I_{,\bar{S}}} = 1$ and $u^I = 0$, $\{\bar{X}^I\}$ depends on the external loading except for the initial tension. The non-dimensional variables

$$\begin{aligned}
&S = \frac{\bar{S}}{\mathscr{L}},\ x^I = \frac{\bar{x}^I}{\mathscr{L}},\ X^I = \frac{\bar{X}^I}{\mathscr{L}},\ u^I = \frac{\bar{u}^I}{\mathscr{L}},\ q^I = \frac{\bar{q}^I}{\rho_0 Ag},\\
&t = \frac{c_q \bar{t}}{\mathscr{L}},\ c_p = \frac{1}{c_q}\sqrt{\frac{E}{\rho_0}},\ c^0(S) = \frac{1}{c_q}\sqrt{\frac{\bar{T}^0}{\rho_0 A}},\ c = \frac{\bar{c}}{c_q},\\
&c_q = \sqrt{g\mathscr{L}},\ T = \frac{\bar{T}}{\rho_0 Ag\mathscr{L}},\ T^0 = \frac{\bar{T}^0}{\rho_0 Ag\mathscr{L}},\ \Omega^I = \frac{\bar{\Omega}^I\mathscr{L}}{c_q},\\
&\mathbf{R} = \frac{\bar{\mathbf{R}}}{\mathscr{L}},\ \mathbf{r} = \frac{\bar{\mathbf{r}}}{\mathscr{L}},\ \mathbf{q} = \frac{\bar{\mathbf{q}}}{\rho_0 Ag},\ \mathbf{\Omega} = \frac{\bar{\mathbf{\Omega}}\mathscr{L}}{c_q}
\end{aligned} \tag{5.45}$$

are introduced, where $\mathscr{L}$ is the total length of the cable, g is the gravitational acceleration, and ρ_0 is the density. Suppose the density is invariant. The equation of motion for the rotating cable is

$$\frac{D^2\mathbf{r}}{Dt^2} = \mathbf{q} + \mathbf{T}_{,S}, \tag{5.46}$$

where

$$\mathbf{T} = T(S)\mathbf{n}_1 = T^I(S)\mathbf{I}_I, \tag{5.47}$$

and

$$\begin{aligned}
&T^I(S') = T(S)\cos\theta_{(\mathbf{n}_1,\mathbf{I}_I)} = \frac{T(S)(X^I_{,S} + u^I_{,S})}{(1+\mathscr{E})},\\
&\mathscr{E} = \sqrt{(X^I_{,S} + u^I_{,S})(X^I_{,S} + u^I_{,S})} - 1.
\end{aligned} \tag{5.48}$$

As in Luo et al. (1996),

$$(X^I_{,tt}+u^I_{,tt})+\Omega^J_{,t}(X^K+u^K)e_{IJK}+2\Omega^J(X^K_{,t}+u^K_{,t})e_{IJK}$$
$$+\Omega^J\Omega^K(X^L+u^L)e_{IJM}e_{MKL}=q^I+\left[\frac{T(S)(X^I_{,S}+u^I_{,S})}{(1+\mathscr{E})}\right]_{,S}. \tag{5.49}$$

For a traveling cable with speed $c(t)$, transformation of s to η:

$$\eta = S+\int_0^t c d\xi, \tag{5.50}$$

results in equations of motion for the segment $[\eta_{B_1},\eta_{B_2}]$ mapped from $[s_{B_1},s_{B_2}]$,

$$(X^I_{,tt}+u^I_{,tt})+2c(X^I_{,\eta t}+u^I_{,\eta t})+c_{,t}(X^I_{,\eta}+u^I_{,\eta})+c^2(X^I_{,\eta\eta}+u^I_{,\eta\eta})$$
$$+\Omega^J_{,t}(X^K+u^K)e_{IJK}+2\Omega^J[(X^K_{,t}+u^K_{,t})+c(X^K_{,\eta}+u^K_{,\eta})]e_{IJK}$$
$$+\Omega^J\Omega^K(X^L+u^L)e_{IJM}e_{MKL}=q^I+\left[\frac{T(\eta)(X^I_{,\eta}+u^I_{,\eta})}{(1+\mathscr{E})}\right]_{,\eta}. \tag{5.51}$$

The boundary conditions for the equilibrium displacement and initial configuration are

$$u^I|_{\eta=0}=a^I,\ \ u^I|_{\eta=1}=b^I \ \ \text{and}\ \ X^I|_{\eta=0}=A^I,\ \ X^I|_{\eta=1}=B^I. \tag{5.52}$$

From Eq.(5.52), the ratio of the chord to the arc length of the cable at the initial configuration is a non-dimensionalized variable $x_{\mathrm{B}}=\sqrt{(A^I-B^I)(A^I-B^I)}$.

For non-eyelet supports, the boundary conditions can be given by forces or displacements. The displacement conditions are the same as in Eq.(5.52). However, the force boundary conditions are

$$T^I|_{\eta=0}=c^I \ \ \text{and}\ \ T^I|_{\eta=1}=d^I. \tag{5.53}$$

Consider a cable rotating around the x-axis with the two supports A and B in the same height. The rotation speed vector becomes $\mathbf{\Omega}=\Omega\mathbf{I}_1$. Equation (5.51) reduces to

$$(X^I_{,tt}+u^I_{,tt})+2c(X^I_{,\eta t}+u^I_{,\eta t})+c_{,t}(X^I_{,\eta}+u^I_{,\eta})+c^2(X^I_{,\eta\eta}+u^I_{,\eta\eta})$$
$$+\Omega_{,t}(X^K+u^K)e_{I1K}+2\Omega[(X^K_{,t}+u^K_{,t})+c(X^K_{,\eta}+u^K_{,\eta})]e_{I1K}$$
$$-\Omega^2(X^I+u^I)(1-\delta^I_1)=q^I+\left[\frac{T(\eta)(X^I_{,\eta}+u^I_{,\eta})}{(1+\mathscr{E})}\right]_{,\eta}. \tag{5.54}$$

For a traveling and rotating, non-sagged cable with constant rotation and traveling speeds (i.e., $c_{,t}=0$ and $\Omega_{,t}=0$), we have $X_2=X_3=0$ and $X_1=S$ with $\eta=X_1+ct$. Equation (5.54) reduces to

$$u^I_{,tt} + 2cu^I_{,\eta t} + c^2 u^I_{,\eta\eta} + 2\Omega[u^K_{,t} + c(\delta^K_1 + u^K_{,\eta})]e_{I1K}$$
$$-\Omega^2 u^I(1-\delta^I_1) = q^I + \left[\frac{T(S)(X^I_{,\eta} + u^I_{,\eta})}{(1+\mathscr{E})}\right]_{,\eta}. \tag{5.55}$$

Without rotation, the foregoing equation becomes

$$u^I_{,tt} + 2cu^I_{,\eta t} + c^2 u^I_{,\eta\eta} = q^I + \left[\frac{T(\eta)(X^I_{,\eta} + u^I_{,\eta})}{(1+\mathscr{E})}\right]_{,\eta}. \tag{5.56}$$

For elastic materials, the tension is given by

$$T(\eta) = [c_0(\eta)]^2 + c_p^2(\sqrt{(X^K_{,\eta} + u^K_{,\eta})(X^K_{,\eta} + u^K_{,\eta})} - 1). \tag{5.57}$$

With Eq.(5.57), equation (5.56) is identical to the equation in Thurman and Mote (1969). Without the translation in Eq.(5.55), equations of motion for rotating strings are for $I = 1, 2, 3$

$$u^I_{,tt} + 2\Omega u^K_{,t} e_{I1K} - \Omega^2 u^I(1-\delta^I_1) = q^I + \left[\frac{T(\eta)(\delta^I_1 + u^I_{,\eta})}{(1+\mathscr{E})}\right]_{,\eta}. \tag{5.58}$$

With Eq.(5.57), equation (5.58) is identical to the model in Luo et al. (1996).

Without rotation in Eq.(5.53), equations of motion for a traveling, sagged, elastic cable with constant traveling speed are

$$(X^I + u^I)_{,tt} + 2c(X^I + u^I)_{,\eta t} + c^2(X^I_{,\eta\eta} + u^I_{,\eta\eta})$$
$$= q^I + \left[\frac{T(\eta)(X^I_{,\eta} + u^I_{,\eta})}{(1+\mathscr{E})}\right]_{,\eta}. \tag{5.59}$$

With Eq.(5.57), equation (5.59) is identical to Luo and Mote (2000).

As in Luo and Mote (2000), the generalized loading on the cable is considered in Fig.5.4. $\bar{\mathbf{q}}_k = \bar{q}^I_k \mathbf{I}_I$ ($k = 1, 2, \ldots, m$) and $\bar{\mathbf{F}}_k = \bar{F}^I_k \mathbf{I}_I$ for the kth-segment are distributed and concentrated forces on the cable. Similarly, the corresponding displacement $\bar{\mathbf{u}}_k = \bar{u}^I_k \mathbf{I}_I$ is from the initial configuration $\bar{\mathbf{X}}_k = \bar{X}^I_k \mathbf{I}_I$ to the equilibrium configuration $\bar{\mathbf{x}}_k = \bar{x}^I_k \mathbf{I}_I$. For the kth-segment, $\bar{u}^I_k = \bar{x}^I_k - \bar{X}^I_k$ and $\sqrt{\bar{X}^I_{k,\bar{S}} \bar{X}^I_{k,\bar{S}}} = 1$. For straight cables, $\bar{S} = \bar{X}^1_k(\bar{S})$ and $\bar{X}^2_k = \bar{X}^3_k = 0$. For the kth-segment, as in Eq.(5.43), the strain of equilibrium under initial tension $\bar{T}^0$ with summation on I is

$$\varepsilon_k = \mathscr{E}^0_k(\bar{S}) + (\sqrt{(\bar{X}^I_{k,\bar{S}} + \bar{u}^I_{k,\bar{S}})(\bar{X}^I_{k,\bar{S}} + \bar{u}^I_{k,\bar{S}})} - 1) \tag{5.60}$$

and the corresponding tension in the cable at equilibrium is

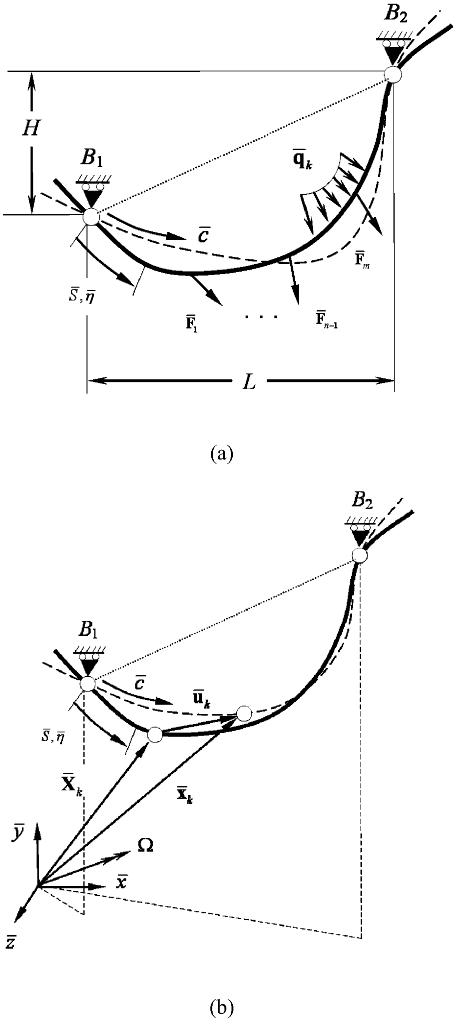


Fig. 5.4 Equilibrium and deformation of a traveling, sagged cable under arbitrary loading. (a) loading and (b) displacement.

$$\begin{aligned} &\overline{T}_k(\overline{S}) = \int_A \overline{f}(\varepsilon_k)\, dA \qquad \text{for any materials,} \\ &\overline{T}_k(\overline{S}) = \overline{T}_k^0(\overline{S}) + EA\,\left(\sqrt{(\overline{X}^I_{k,\overline{S}} + \overline{u}^I_{k,\overline{S}})(\overline{X}^I_{k,\overline{S}} + \overline{u}^I_{k,\overline{S}})} - 1\right) \\ &\qquad\qquad\qquad\qquad\qquad \text{for elastic materials} \end{aligned} \tag{5.61}$$

where $\mathscr{E}_k^0 = \overline{T}_k^0 / EA$. For inextensible cables ($EA \to \infty$), $\varepsilon_k = 0$ because of $\sqrt{\overline{X}^I_{k,s}\overline{X}^I_{k,s}} = 1$ and $u_k^I = 0$. $\{\overline{X}_k^I\}$ depends on the external loading except the initial tension. As in Eq.(5.45), the non-dimensional variables are

$$S=\frac{\overline{S}}{\mathscr{L}},\ t=\frac{c_q\overline{t}}{\mathscr{L}},\ x_k^I=\frac{\overline{x}_k^I}{\mathscr{L}},\ X_k^I=\frac{\overline{X}_k^I}{\mathscr{L}},\ u_k^I=\frac{\overline{u}_k^I}{\mathscr{L}},$$
$$q_k^I=\frac{\overline{q}_k^I}{\rho Ag},\ F_k^I=\frac{\overline{F}_k^I}{\rho Ag\mathscr{L}},\ T_k=\frac{\overline{T}_k}{\rho Ag\mathscr{L}},\ T_k^{\ 0}=\frac{\overline{T}_k^0}{\rho Ag\mathscr{L}}. \tag{5.62}$$

Further, the equations of motion for a cable segment $[S_{k-1}, S_k]$ between concentrated forces $\mathbf{F}_{k-1}$ and $\mathbf{F}_k$ ($k=1,2,\ldots,m+1$) are obtained via force balances of the deformed cable. $\mathbf{T}_k(\eta)=T_k\mathbf{n}_1$ and $\mathbf{n}_1$ is the normal direction of cross section of cable, i.e.,

$$\mathbf{n}_1=\cos\theta_{(\mathbf{n}_1,\mathbf{I}_I)}\mathbf{I}_I \quad (\text{sum on } I=1,2,3). \tag{5.63}$$

The corresponding components of tension is computed by

$$T_k^I=T_k\cos\theta_{(\mathbf{n}_1,\mathbf{I}_I)}=\frac{T_k(S)(X_{k,S}^I+u_{k,S}^I)}{(1+\mathscr{E}_k)}. \tag{5.64}$$

Using the geometrical relations, equations of motion without traveling become

$$(X_{k,tt}^I+u_{k,tt}^I)+\Omega_{,t}^J(X_k^K+u_k^K)e_{IJK}+2\Omega^J(X_{k,t}^K+u_{k,t}^K)e_{IJK}$$
$$+\Omega^J\Omega^K(X_k^L+u_k^L)e_{IJM}e_{MKL}=q_k^I+\left[\frac{T_k(S)(X_{k,S}^I+u_{k,s}^I)}{(1+\mathscr{E}_k)}\right]_{,S}. \tag{5.65}$$

The reactions at ends B_1 ($s_0=s_{B_1}=0$) and B_2 ($s_{m+1}=s_{B_2}=1$) are $\mathbf{F}_0$ and $\mathbf{F}_{m+1}$. For traveling cables, the variable η is used as a fixed coordinate system and s as a traveling coordinate system. Consider a constant traveling speed, and transformation of s to η:

$$\eta=s+ct, \tag{5.66}$$

results in equations of motion for segment $[\eta_{k-1},\eta_k]$ mapped from $[S_{k-1},S_k]$:

$$(X_{k,tt}^I+u_{k,tt}^I)+2c(X_{k,\eta t}^I+u_{k,\eta t}^I)+c^2(X_{k,\eta\eta}^I+u_{k,\eta\eta}^I)$$
$$+\Omega_{,t}^J(X_k^K+u_k^K)e_{IJK}+2\Omega_{,t}^J[(X_{k,t}^K+u_{k,t}^K)+c(X_{k,\eta}^K+u_{k,\eta}^K)]e_{IJK}$$
$$+\Omega^J\Omega^K(X_k^L+u_k^L)e_{IJM}e_{MKL}=q_k^I+\left[\frac{T_k(\eta)(X_{k,\eta}^I+u_{k,\eta}^I)}{(1+\mathscr{E}_k)}\right]_{,\eta}. \tag{5.67}$$

The corresponding boundary conditions for the equilibrium displacement and initial configuration become

$$u_0^I\,|_{\eta=0}=a^I,\;\; u_{m+1}^I\,|_{\eta=1}=b^I \;\text{ and }\; X_0^I\,|_{\eta=0}=A^I,\; X_{m+1}^I\,|_{\eta=1}=B^I. \tag{5.68}$$

Continuity for two segments requires

$$u_k^I\,|_{\eta=\eta_k}=u_{k+1}^I\,|_{\eta=\eta_k} \;\text{ and }\; X_k^I\,|_{\eta=\eta_k}=X_{k+1}^I\,|_{\eta=\eta_k} \;\text{ for } I=1,2,3. \tag{5.69}$$

The force balance conditions at the discontinuous point $\eta=\eta_k$ are

$$\begin{aligned} &\mathbf{T}_k(\eta_k)+\mathbf{F}_k+\mathbf{T}_{k+1}(\eta_k)=0,\\ &\text{or }\; T_k^I\,|_{\eta=\eta_k}=F_k^I+T_{k+1}^I\,|_{\eta=\eta_k}\;. \end{aligned} \tag{5.70}$$

Without rotation, equation (5.67) becomes

$$\begin{aligned} &(X_{k,tt}^I+u_{k,tt}^I)+2c(X_{k,\eta t}^I+u_{k,\eta t}^I)+c^2(X_{k,\eta\eta}^I+u_{k,\eta\eta}^I)\\ &=q_k^I+\left[T_k\frac{(X_{k,\eta}^I+u_{k,\eta}^I)}{(1+\mathscr{E}_k)}\right]_{,\eta}. \end{aligned} \tag{5.71}$$

5.3. Equilibrium of traveling elastic cables

In this section, equilibrium solutions for a traveling cable will be discussed. The closed-form solution for such equilibrium will be presented, and the existing solution with be discussed.

5.3.1. Existence conditions

For the elastic material, the tension is given by

$$T_k(\eta)=[c_k^0(\eta)]^2+c_p^2\left(\sqrt{(X_{k,\eta}^I+u_{k,\eta}^I)(X_{k,\eta}^I+u_{k,\eta}^I)}-1\right). \tag{5.72}$$

With vanishing of the time variations in Eq.(5.71), integration over $[\eta_{k-1},\eta_k]$ gives

$$\begin{aligned} &\left[\frac{[c_k^0(\eta)]^2-c_p^2}{\sqrt{(X_{k,\eta}^K+u_{k,\eta}^K)(X_{k,\eta}^K+u_{k,\eta}^K)}}+(c_p^2-c^2)\right](X_{k,\eta}^I+u_{k,\eta}^I)\\ &=-\int q_k^I d\eta+C_k^I, \end{aligned} \tag{5.73}$$

where C_k^I $(I=1,2,3)$ are integration constants. With the tension in Eq.(5.72), equation (5.73) reduces to the model of Yu et al (1995) when $q_k^1 = c = 0$ and q_k^I $(I = 2, 3)$ are uniformly distributed. The 2-D model in Irvine (1981) is a special case of Eq.(5.73) obtained by $q_k^1 = q_k^3 = c = 0$.

The squaring of both sides of Eq.(5.73) for each I and summing of them for all I leads to

$$\sqrt{(X_{k,\eta}^I + u_{k,\eta}^I)(X_{k,\eta}^I + u_{k,\eta}^I)}$$
$$= \frac{1}{c_p^2 - c^2}\left[\left(c_p^2 - [c_k^0(\eta)]^2\right) \pm \sqrt{\left(\int q_k^I d\eta - C_k^I\right)\left(\int q_k^I d\eta - C_k^I\right)}\right]. \tag{5.74}$$

The translating cable in Eq.(5.74) possesses two equilibria. When the translation speed equals the wave speed (i.e., $\overline{c} = \sqrt{E/\rho}$), the resonance occurs and the stretch ratio $\sqrt{(X_{k,\eta}^I + u_{k,\eta}^I)(X_{k,\eta}^I + u_{k,\eta}^I)}$ becomes infinite. In Eq.(5.74), the stretch ratio increases with increasing c for $c < c_p$.

Substitution of Eq.(5.74) into Eq.(5.71) leads to

$$T_k(\eta) = \frac{1}{c_p^2 - c^2}\left[c^2(c_p^2 - [c_k^0(\eta)]^2) \right.$$
$$\left. \pm c_p^2\sqrt{\left(\int q_k^I d\eta - C_k^I\right)\left(\int q_k^I d\eta - C_k^I\right)}\right], \tag{5.75}$$

showing that the tension increases monotonically with c for $c < c_p$. For stationary cables, setting $c = 0$ in Eq. (5.75) and choosing $T_k > 0$ for all the segments gives

$$T_k(\eta) = \sqrt{\left(\int q_k^I d\eta - C_k^I\right)\left(\int q_k^I d\eta - C_k^I\right)}. \tag{5.76}$$

In the inextensible cable, $\sqrt{(X_{k,\eta}^I + u_{k,\eta}^I)(X_{k,\eta}^I + u_{k,\eta}^I)} = 1$ (or $u_k^I = 0$), the stiffness in Eq.(5.73) becomes $(T_0 - \rho A c^2)$ identical to the results in Routh (1884), and the tension is

$$T_k(\eta) = c^2 \pm \sqrt{\left(\int q_k^I d\eta - C_k^I\right)\left(\int q_k^I d\eta - C_k^I\right)}. \tag{5.77}$$

Cable/string models require positive tension, i.e., $T_k > 0$. Therefore, a condition of existence of steady motion is from Eq.(5.75), i.e.,

$$[c_k^0(\eta)]^2 < c_p^2\left[1 \pm \frac{1}{c^2}\sqrt{\left(\int q_k^I d\eta - C_k^I\right)\left(\int q_k^I d\eta - C_k^I\right)}\right]. \tag{5.78}$$

A critical condition for the existence of steady motion is obtained at $T_k(\eta) = 0$ or equality in Eq.(5.78).

5.3.2. Displacements

Substitution of Eq.(5.74) into Eq.(5.73) and integration gives three components of displacement

$$u_k^I = -X_k^I + D_k^I$$
$$+ \int \frac{-\int q_k^I d\eta + C_k^I}{(c_p^2 - c^2)} \left[1 \pm \frac{c_p^2 - [c_k^0(\eta)]^2}{\sqrt{(\int q_k^I d\eta - C_k^I)(\int q_k^I d\eta - C_k^I)}} \right] d\eta, \tag{5.79}$$

where D_k^I (I=1, 2, 3) are constants. The boundary conditions in Eq. (5.68), displacement continuity in Eq.(5.69) and force balances at each $\mathbf{F}_k$ are used to determine all coefficients in Eqs.(5.73) and (5.79). The force balances are

$$(-\int q_k^I d\eta + C_k^I)\Big|_{\eta=\eta_k} - F_k^I = (-\int q_{k+1}^I d\eta + C_{k+1}^I)\Big|_{\eta=\eta_k}. \tag{5.80}$$

The inextensible cable requires $c_p \to \infty$, and Eq. (5.74) gives $\sqrt{X_{k,\eta}^I X_{k,\eta}^I} = 1$ indicating $u_k^{(\alpha)} = 0$. Substitution of the c_p and u_k^I into Eq.(5.79) gives the equilibrium configuration $\hat{X}_k^I$:

$$\hat{X}_k^I = \pm \int \frac{-\int q_k^I d\eta + \hat{C}_k^I}{\sqrt{(\int q_k^K d\eta - \hat{C}_k^K)(\int q_k^K d\eta - \hat{C}_k^K)}} d\eta + \hat{D}_k^I. \tag{5.81}$$

Similarly, $\hat{C}_k^I$ and $\hat{D}_k^I$ can be determined. Equations (5.80) and (5.81) show that $\hat{X}_k^I$ is independent of the initial tension. For the inextensible and deformable cables under the same loading, it is assumed that the two cables possess the same initial configuration. For the inextensible cable, its displacement is zero (i.e., $u_k^I = 0$), which implies that the equilibrium of the inextensible cable is an initial configuration (i.e., $X_k^I(\eta) = \hat{X}_k^I(\eta)$). Therefore, from Eqs.(5.79) and (5.81), the displacement of the elastic cable is

$$u_k^{(\alpha)} = \int \frac{-\int q_k^I d\eta + C_k^I}{(c_p^2 - c^2)} \left[1 \pm \frac{c_p^2 - [c_k^0(\eta)]^2}{\sqrt{(\int q_k^I d\eta - C_k^I)(\int q_k^I d\eta - C_k^I)}} \right] d\eta$$
$$+ D_k^I \mp \int \frac{-\int q_k^I d\eta + \hat{C}_k^I}{\sqrt{(\int q_k^K d\eta - \hat{C}_k^K)(\int q_k^K d\eta - \hat{C}_k^K)}} d\eta - \hat{D}_k^I. \tag{5.82}$$

5.3.3. Applications

In this subsection, several examples will be presented, which can be reduced to the classic approximate solutions of sagged cables.

5.3.3a Uniformly distributed loading

Consider a sagged cable traveling at constant speed c. The cable is subject to a uniformly distributed load $\mathbf{q} = q^I \mathbf{I}_I$. The chord ratio of the cable is x_B and a constant, initial tension is T^0. The subscripts denoting the particular segment have been dropped. The boundary conditions are

$$u^I\Big|_{\eta=0} = u^I\Big|_{\eta=1} = 0, \quad X^I\Big|_{\eta=0} = A^I = 0 \text{ and } X^I\Big|_{\eta=1} = B^I. \tag{5.83}$$

The displacement is given by Eq.(5.82):

$$\begin{aligned} u^I &= \frac{1}{c_p^2 - c^2}\left(-\frac{1}{2} q^I \eta^2 + C^I \eta\right) \\ &\pm \frac{c_p^2 - (c^0)^2}{c_p^2 - c^2}\left\{-\frac{q^I}{q}\Theta(\eta) + \frac{(C^I q^J - C^J q^I) q^J}{q^3}\log\left[\Xi(\eta) + \Theta(\eta)\right]\right\} \\ &+ D^I \mp \left\{-\frac{q^I}{q}\hat{\Theta}(\eta) + \frac{(\hat{C}^I q^J - \hat{C}^J q^I) q^J}{q^3}\log\left[\hat{\Xi}(\eta) + \hat{\Theta}(\eta)\right]\right\} - \hat{D}^I, \end{aligned} \tag{5.84}$$

where

$$\begin{aligned} \Theta(\eta) &= \sqrt{\eta^2 - 2\frac{C^I q^I}{q^2}\eta + \frac{C^I C^I}{q^2}}, \quad \Xi(\eta) = \eta - \frac{C^I q^I}{q^2}, \quad q = \sqrt{q^I q^I}; \\ \hat{\Theta}(\eta) &= \sqrt{\eta^2 - 2\frac{\hat{C}^I q^I}{q^2}\eta + \frac{\hat{C}^I \hat{C}^I}{q^2}}, \quad \hat{\Xi}(\eta) = \eta - \frac{\hat{C}^I q^I}{q^2}. \end{aligned} \tag{5.85}$$

The boundary conditions in Eqs.(5.83) with (5.84) give

$$D^I = \mp \frac{c_p^2 - (c^0)^2}{c_p^2 - c^2}\left\{-\frac{q^I}{q}\Theta(0) + \frac{(C^I q^J - C^J q^I) q^J}{q^3}\log\left[\Xi(0) + \Theta(0)\right]\right\}, \tag{5.86}$$

$$\begin{aligned} &C^I \pm \frac{[c_p^2 - (c^0)^2](C^I q^J - C^J q^I) q^J}{q^3}\log\frac{\Xi(1) + \Theta(1)}{\Xi(0) + \Theta(0)} \\ &= B^I\left(c_p^2 - c^2\right) + \frac{1}{2} q^I \pm \frac{[c_p^2 - (c^0)^2] q^I}{q}\left[\Theta(1) - \Theta(0)\right]. \end{aligned} \tag{5.87}$$

The constants C^I and D^I in Eq.(5.84) are determined through solution of the nonlinear algebraic equations in Eqs.(5.86) and (5.87), and the $\hat{C}^I$ and $\hat{D}^I$ for the inextensible cable are determined from

$$\hat{D}_k = \mp\left\{-\frac{q^I}{q}\hat{\Theta}(0) + \frac{(\hat{C}^I q^J - \hat{C}^J q^I)q^J}{q^3}\log\left[\hat{\Xi}(0)+\hat{\Theta}(0)\right]\right\}, \tag{5.88}$$

$$\pm\frac{(\hat{C}^I q^J - \hat{C}^J q^I)q^J}{q^3}\log\frac{\hat{\Xi}(1)+\hat{\Theta}(1)}{\hat{\Xi}(0)+\hat{\Theta}(0)} = B^I \pm \frac{q^I}{q}\left[\hat{\Theta}(1)-\hat{\Theta}(0)\right]. \tag{5.89}$$

The exact displacement solution is obtained. The tension and the equilibrium configuration are:

$$T(\eta) = \frac{1}{c_p^2 - c^2}\left\{c^2[c_p^2 - (c^0)^2] \pm c_p^2 q\Theta(\eta)\right\}. \tag{5.90}$$

$$x^I = X^I + u^I = \frac{1}{c_p^2 - c^2}\left(-\frac{1}{2}q^I\eta^2 + C^I\eta\right) + D^I \pm \frac{c_p^2 - (c^0)^2}{c_p^2 - c^2}$$
$$\times\left\{-\frac{q^I}{q}\Theta(\eta) + \frac{(C^I q^J - C^J q^I)q^J}{q^3}\log\left[\Xi(\eta)+\Theta(\eta)\right]\right\}. \tag{5.91}$$

For the inextensible cable, the equilibrium configuration is

$$\hat{X}^I(\eta) = \hat{D}^I \pm \left\{-\frac{q^I}{q}\hat{\Theta}(\eta) + \frac{(\hat{C}^I q^J - \hat{C}^J q^I)q^J}{q^3}\log\left[\hat{\Xi}(\eta)+\hat{\Theta}(\eta)\right]\right\}. \tag{5.92}$$

5.3.3b Special cases

Consider a 2-D traveling cable with $\bar{q}_x = 0$, $\bar{q}_y = -\rho A g = W/\mathscr{L}$, $B_x = L/\mathscr{L}$ and $B_y = H/\mathscr{L}$. Substitution of Eqs.(5.62), (5.86) and (5.87) into Eq.(5.91) and use of inverse hyperbolic functions gives an equilibrium of the 2-D deformed cable,

$$\bar{x}(\bar{\eta}) = \frac{\tilde{C}^1}{(E-\rho\bar{c}^2)\,A}\left\{\bar{\eta} \pm \frac{EA-\bar{T}^0}{\bar{q}_y}\left[\sinh^{-1}\frac{\tilde{C}^2}{\tilde{C}^1} - \sinh^{-1}\frac{\tilde{C}^2 - \bar{q}_y\bar{\eta}}{\tilde{C}^1}\right]\right\},$$
$$\bar{y}(\bar{\eta}) = \frac{1}{(E-\rho\bar{c}^2)\,A}\left\{\left(-\frac{1}{2}\bar{q}_y\bar{\eta}^2 + \tilde{C}^2\bar{\eta}\right)\right. \tag{5.93}$$
$$\left.\pm\frac{(EA-\bar{T}^0)\,\tilde{C}^1}{\bar{q}_y}\left[\sqrt{1+\left(\frac{\tilde{C}^2}{\tilde{C}^1}\right)^2} - \sqrt{1+\left(\frac{\tilde{C}^2-\bar{q}_y\bar{\eta}}{\tilde{C}^1}\right)^2}\right]\right\}.$$

The bar indicates the dimensionalized variables and parameters. For the stationary cable ($\bar{c}=0$), let $\bar{T}_0=0$ and neglect the solution with $\bar{T}\le 0$, and Eq. (5.93) becomes

$$\bar{x}(\bar{\eta})=\frac{\tilde{C}^1\bar{\eta}}{EA}+\frac{\tilde{C}^1\mathscr{L}}{W}\left[\sinh^{-1}\frac{\tilde{C}^2}{\tilde{C}^1}-\sinh^{-1}\frac{\tilde{C}^2\mathscr{L}-W\bar{\eta}}{\tilde{C}^1\mathscr{L}}\right],$$
$$\bar{y}(\bar{\eta})=\frac{W\bar{\eta}}{EA}\left(\frac{\tilde{C}^2}{W}-\frac{\bar{\eta}}{2\mathscr{L}}\right)+\frac{\tilde{C}^1\mathscr{L}}{W}\left[\sqrt{1+(\frac{\tilde{C}^2}{\tilde{C}^1})^2}-\sqrt{1+\left(\frac{\tilde{C}^2\mathscr{L}-W\bar{\eta}}{\tilde{C}^1\mathscr{L}}\right)^2}\right]. \tag{5.94}$$

which is the solution of Irvine (1981).

The inextensible, axially moving cable is obtained from Eq.(5.93) by setting $EA\to\infty$.

$$\hat{\bar{X}}(\eta)=\pm\frac{\tilde{C}^1}{\bar{q}_y}\left[\sinh^{-1}\frac{\tilde{C}^2}{\tilde{C}^1}-\sinh^{-1}\frac{\tilde{C}_2-\bar{q}_y\bar{\eta}}{\tilde{C}^1}\right],$$
$$\hat{Y}(\eta)=\pm\frac{\tilde{C}^1}{\bar{q}_y}\left[\sqrt{1+\left(\frac{\tilde{C}^2}{\tilde{C}^1}\right)^2}-\sqrt{1+\left(\frac{\tilde{C}_2-\bar{q}_y\bar{\eta}}{\tilde{C}^1}\right)^2}\right]; \tag{5.95}$$

where $\tilde{C}^1$ and $\tilde{C}^2$ are determined through the boundary condition at $\bar{\eta}=\mathscr{L}$. The equilibrium under the non-positive tension in Eq.(5.95) is unstable. Setting $\hat{\bar{X}}_0=\tilde{C}^1/\bar{q}_y\sinh^{-1}(\tilde{C}^2/\tilde{C}^1)$, equation (5.95) becomes

$$\hat{\bar{Y}}(\hat{\bar{X}})=\pm\frac{\tilde{C}^1}{\rho Ag}\left\{\cosh[\frac{\rho Ag}{\tilde{C}^1}(\hat{\bar{X}}-\hat{\bar{X}}_0)]-\cosh(\frac{\rho Ag}{\tilde{C}^1}\hat{\bar{X}}_0)\right\}. \tag{5.96}$$

Letting $\hat{\bar{T}}_1(0)=\pm\hat{\tilde{C}}^1=\pm[\hat{\bar{T}}_0(0)-\rho A\bar{c}^2]\cos\theta(0)$ where $d\hat{\bar{X}}/d\bar{\eta}=\cos\theta(\bar{\eta})$, equation (5.96) is given by O'Reilly (1996). $\hat{\bar{X}}_0$ is determined by the boundary condition at $\bar{\eta}=\mathscr{L}$. If the inextensible cable is sufficiently straight that $\cos\theta\approx 1$ and $\tilde{C}^1>>\tilde{C}^2$, then $\hat{\bar{X}}_0\approx 0$ and Eq. (5.96) becomes

$$\hat{\bar{X}}(\bar{\eta})=\pm\frac{\tilde{C}^1}{\bar{q}_y}\sinh^{-1}\frac{\bar{q}_y\bar{\eta}}{\tilde{C}^1},$$
$$\hat{\bar{Y}}(\bar{\eta})=\pm\frac{\tilde{C}^1}{\bar{q}_y}\left[1-\sqrt{1+\left(\frac{\bar{q}_y\bar{\eta}}{\tilde{C}^1}\right)^2}\right]=\mp\frac{\tilde{C}^1}{\bar{q}_y}\left[\cosh\frac{\bar{q}_y\bar{\eta}}{\tilde{C}^1}-1\right]. \tag{5.97}$$

This stable solution is given by Simpson (1972). However, Equation (5.97) does not provide the equilibrium solutions of the inextensible cable because the boundary conditions are not satisfied. The linear model of the straight cable gives

$$u = -\frac{q_x}{2(c_p^2 - c^2)}(\eta - 1)\eta,\quad v = -\frac{q_y}{2[(c^0)^2 - c^2]}(\eta - 1)\eta. \tag{5.98}$$

In all the plots, the solid and dash lines represent the upper (+) and lower (−) branches of the equilibrium configurations, respectively, and the dotted line denotes the unstable equilibrium of the cable as $T \le 0$. The longitudinal and transverse wave speeds are $c_p = 740.87$ and $c^0 = 0$.

Consider a 2-D horizontal straight cable ($B_x = x_{\rm B} = 1, B_y = B_z = 0$) hanging under its own weight ($q_x = 0,\ q_y = -1$). Two components of displacement $u = x(\eta) - \eta$ and $v = y(\eta)$ computed from Eq.(5.84) at $c = 10$ are shown in Fig.5.5. The chain curves denote the linear prediction of displacement from Eq.(5.98). The maximum longitudinal displacement is 3.998×10^{-5} (lower branch) and 2.923×10^{-5} (upper branch) at $\eta = 0.21$ and 0.79. The longitudinal linear displacement is zero due to $q_x = 0$. The maximum transverse displacement is 8.828×10^{-3} (lower branch) and 7.547×10^{-3} (upper branch) but 1.25×10^{-3} for the linear prediction. For the inextensible cable, two components of displacement are zero. The maximum transverse displacement and minimum tension in the 2-D sagged elastic cable ($B_x = x_{\rm B}, B_y = B_z = 0$) for $q_x = -1$ is plotted in Fig.5.6 when the chord ratio is $x_{\rm B} = 0.8$. The lower branch of the equilibrium configuration for any traveling speed always exists, and the displacement and tension increase with the transverse load and transport speed. The upper branch of equilibrium configuration is stable only when the transport creates positive tension. The equilibrium configuration and tension of a cable under its own weight ($q_y = -1$) and the longitudinal loads ($q_x = 0, -1$) are illustrated in Fig.5.7 at $x_{\rm B} = 0.8$ and $c = 1$. The equilibrium configurations and tension distributions are symmetric $q_x = 0$. The longitudinal and transverse displacements from the initial configuration to equilibrium of the sagged elastic cable are shown in Fig.5.8. Unlike the straight cable in Fig.5.5, the longitudinal displacement of the sagged cable is the same order of magnitude as the transverse displacement. For this problem, when the tension of the partial cable is zero, the configuration of cables will be changed, and a new configuration should be determined. How to determine the zero-tension boundary and such a new configuration is unsolved, and a further investigation should be conducted. The maximum transverse displacement and the related tension versus the chord ratio are shown in Fig.5.9 at $c = 10$. The maxima occur at $x_{\rm B} = 1$. The results indicates that slightly sagged cables ($x_{\rm B} = 0.9 \sim 1$) must be modeled as extensible to achieve the accuracy for most applications. The sagged cable model reduces to the straight cable model at $x_{\rm B} = 1$.

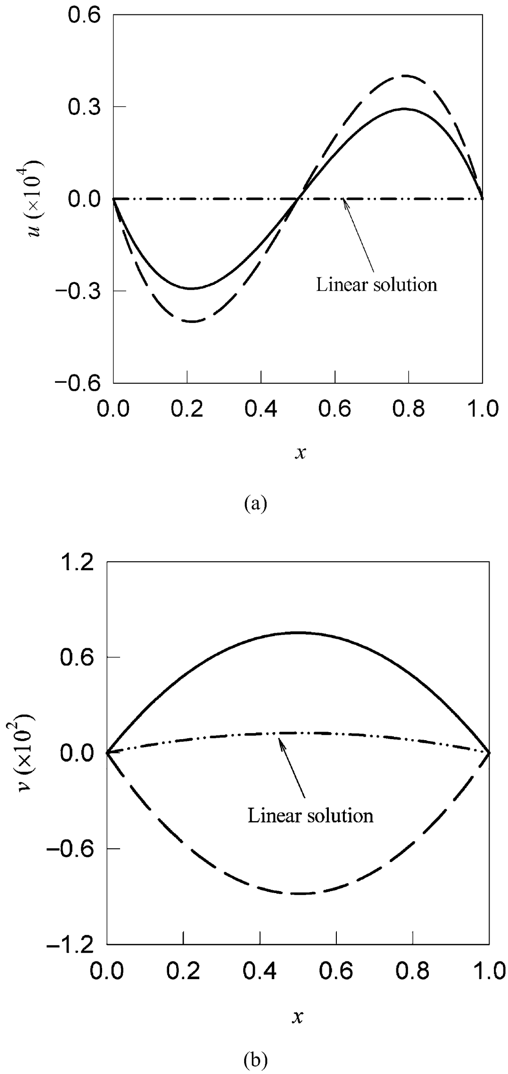


Fig. 5.5 (a) Longitudinal and (b) transverse equilibrium displacements of *straight*, *elastic* cable under $q_x = 0$ and $q_y = -1$ for $c = 10$, $c_p = 740.87$ and $c^0 = 0$.

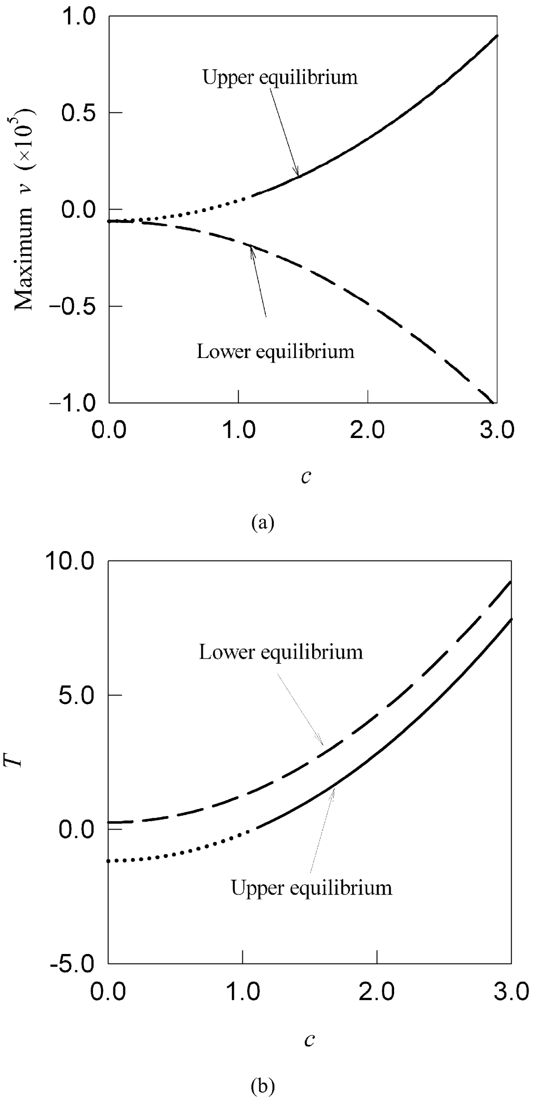


Fig. 5.6 (a)Displacement and (b) tension versus traveling speed of sagged, *elastic* cable ($x_B = 0.8$) for $q_x = q_y = -1 : c_p = 740.87$ and $c^0 = 0$.

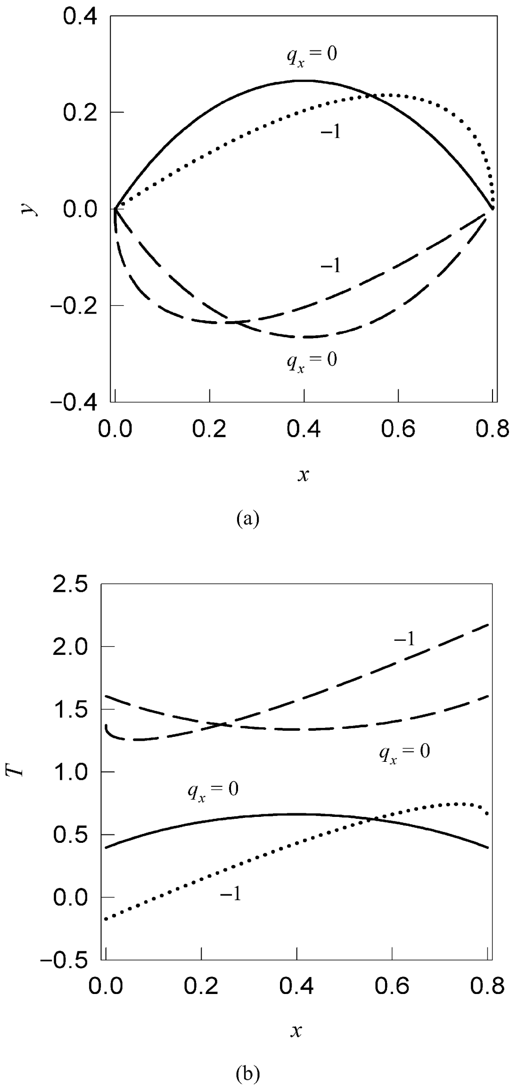
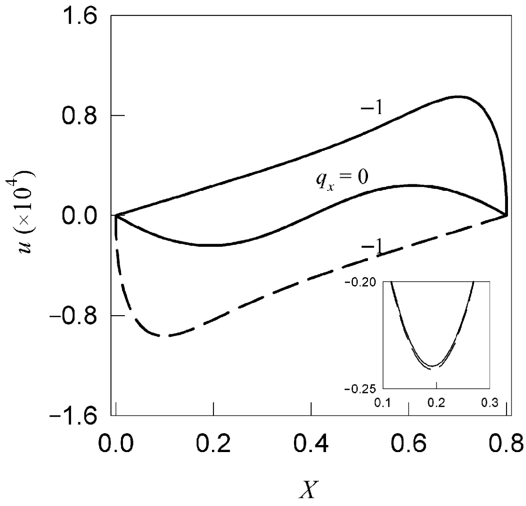
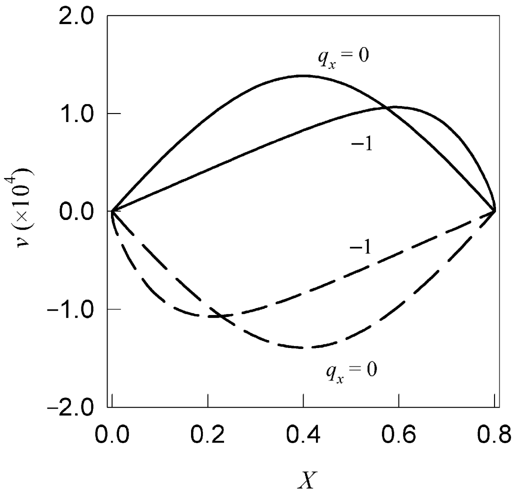


Fig. 5.7 (a) Multiple equilibrium configurations and (b) tension distributions of sagged, *elastic* cable at equilibrium ($x_B = 0.8$) under $q_y = -1$ and q_x for $c = 1 : c_p = 740.87$, and $c^0 = 0$.

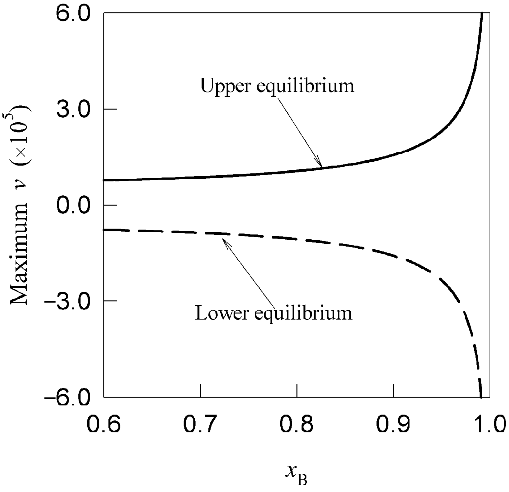


(a)

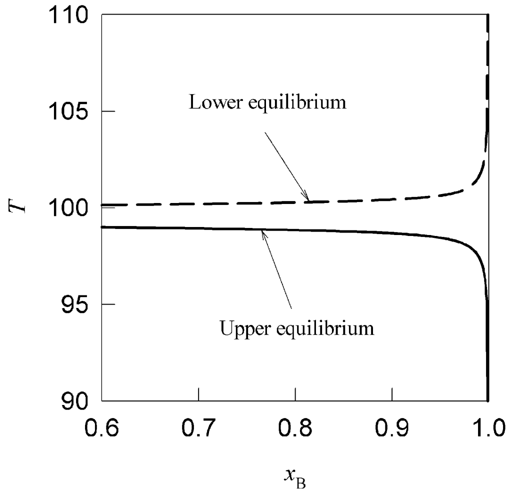


(b)

Fig. 5.8 (a) Longitudinal and (b) transverse displacements of sagged, *elastic* cable (x_B = 0.8)under its own weight ($q_y = -1$) and longitudinal loading for $c = 10$: $c_p = 740.87$ and $c^0 = 0$.



(a)



(b)

Fig. 5.9 (a) Displacement and (b) tension versus chord ratio of sagged, *elastic* cable with $q_y = -1$, $c = 10$, $c_p = 740.87$ and $c^0 = 0$.

5.3.3c Concentrated loading

Consider a 3-D sagged traveling cable carrying $m-1$ concentrated loads F_k^I ($k=1,2,\ldots,m-1$ and $I=1,2,3$ for *x, y, z*) which divide the cable into *m* segments. q_k^I ($k=1,2,\ldots,m$) are uniformly distributed loads and c_0^2 is a constant initial tension. The boundary conditions satisfy Eq.(5.68). Three components of displacement from Eq.(5.83) are computed through Eqs.(5.84) and (5.85) when $\{u^I, q^I, C^I, D^I, \Xi, \Theta, q\}$ and $\{\hat{C}^I, \hat{D}^I, \hat{\Xi}, \hat{\Theta}\}$ are replaced by $\{u_k^I, q_k^I, C_k^I, D_k^I, \Xi_k, \Theta_k, q_k\}$ and $\{\hat{C}_k^I, \hat{D}_k^I, \hat{\Xi}_k, \hat{\Theta}_k\}$. The corresponding boundary condition in Eq.(5.83) at $\eta=0$ becomes

$$D_1^I = \mp\frac{c_p^2-(c^0)^2}{c_p^2-c^2}\left\{-\frac{q_1^I}{q_1}\Theta(0)+\frac{(C_1^I q_1^J - C_1^J q_1^I)q_1^J}{(q_1)^3}\log\left[\Xi_1(0)+\Theta_1(0)\right]\right\}. \tag{5.99}$$

The displacement continuity between the *k*th and $(k+1)$th segments is for ($k=1,2,\ldots,m-1$ and $I=1,2,3$):

$$\begin{aligned}
&\pm\frac{[c_p^2-(c^0)^2](C_k^I q_k^J - C_k^J q_k^I)q_k^I}{(q_k)^3}\log\left[\Xi_k(\eta_k)+\Theta_k(\eta_k)\right] \\
&\mp\frac{[c_p^2-(c^0)^2]q_k^I}{q_k}\Theta_k(\eta_k)+C_k^I\eta_k-\frac{1}{2}q_k^I\eta_k^2 \\
&=\pm\frac{[c_p^2-(c^0)^2](C_{k+1}^I q_{k+1}^J - C_{k+1}^J q_{k+1}^I)q_{k+1}^J}{(q_{k+1})^3} \\
&\times\log\left[\Xi_{k+1}(\eta_k)+\Theta_{k+1}(\eta_k)\right]\mp\frac{[c_p^2-(c^0)^2]q_{k+1}^I}{q^{(\alpha+1)}}\Theta_{k+1}(\eta_k) \\
&+C_{k+1}^I\eta_k-\frac{1}{2}q_{k+1}^I\eta_k^2+(c_p^2-c^2)(D_{k+1}^I-D_{k+1}^I),
\end{aligned} \tag{5.100}$$

and force balances in Eq.(5.80) give:

$$-q_k^I\eta_k + C_k^I - F_k^I = -q_{k+1}^I\eta_k + C_{k+1}^I. \tag{5.101}$$

The boundary condition in Eq.(5.83) at $\eta=1$ produces

$$\begin{aligned}
&C_m^I \pm \frac{[c_p^2-(c^0)^2](C_m^I q_m^J - C_m^J q_m^I)q_m^J}{(q_m)^3}\log\left[\Xi_m(1)+\Theta_m(1)\right] \\
&+d_m^I = B^I(c_p^2-c^2)+\frac{1}{2}q_m^I \pm \frac{[c_p^2-(c^0)^2]q_m^I}{q_m}\Theta_m(1).
\end{aligned} \tag{5.102}$$

The C_k^I and D_k^I are determined by solving $6\times n$ nonlinear algebraic equations

(5.99)−(5.102). Similarly, the constants $\hat{C}_k$ and $\hat{D}_k$ for the inextensible cable are determined by the following $6\times n$ nonlinear equations.

$$\hat{D}_1^I = \mp\left\{-\frac{q_1^I}{q_1}\hat{\Theta}(0) + \frac{(\hat{C}_1^I q_1^J - \hat{C}_1^J q_1^I)q_1^J}{(q_1)^3}\log\left[\hat{\Xi}_1(0) + \hat{\Theta}_1(0)\right]\right\}, \tag{5.103}$$

$$\begin{aligned}&\pm\frac{(\hat{C}_k^I q_k^J - \hat{C}_k^J q_k^I)q_k^J}{(q_k)^3}\log\left[\hat{\Xi}_k(\eta_k) + \hat{\Theta}_k(\eta_k)\right] \mp \frac{q_k^I}{q_k}\hat{\Theta}_k(\eta_k) + \hat{D}_k^I \\ &= \pm\frac{(\hat{C}_{k+1}^I q_{k+1}^J - \hat{C}_{k+1}^J q_{k+1}^I)q_{k+1}^J}{(q_{k+1})^3}\log\left[\hat{\Xi}_{k+1}(\eta_k) + \hat{\Theta}_{k+1}(\eta_k)\right] \\ &\quad \mp \frac{q_{k+1}^I}{q_{k+1}}\hat{\Theta}_{k+1}(\eta_k) + \hat{D}_{k+1}^I,\end{aligned} \tag{5.104}$$

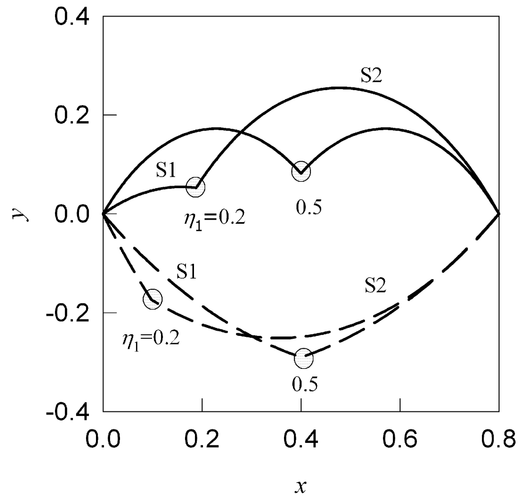
$$-q_k^I\eta_k + \hat{C}_k^I - F_k^I = -q_{k+1}^I\eta_k + \hat{C}_{k+1}^I, \tag{5.105}$$

$$\pm\frac{(\hat{C}_m^I q_m^J - \hat{C}_m^J q_m^I)q_m^J}{(q_m)^3}\log\left[\hat{\Xi}_m(1) + \hat{\Theta}_m(1)\right] + \hat{D}_m^I = B^I \pm \frac{q_m^I}{q_m}\hat{\Theta}_m(1). \tag{5.106}$$

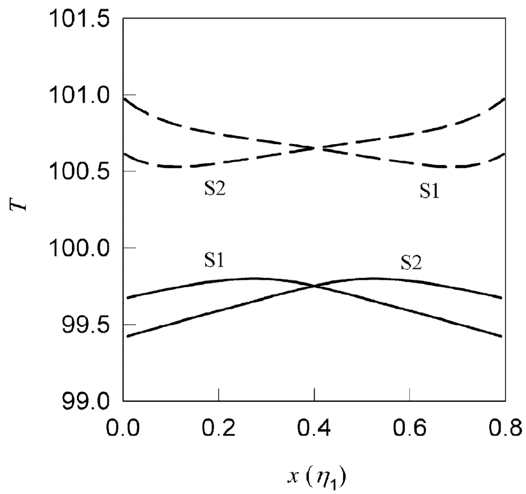
The coefficients can be obtained from Eqs.(5.99)−(5.106). The exact closed form displacement for the equilibrium configuration of traveling, sagged cables under the uniformly distributed and concentrated loading is completed. The computation of tension, equilibrium configuration for each segment of elastic and inextensible cables can be carried out. To illustrate the equilibrium configuration and tension distribution under concentrated loading, consider a 2-D sagged cable ($B_x = x_{\mathrm{B}}$, $B_y = B_z = 0$) under its own weight and a concentrated load $F_x = 0$, $F_y = -0.4$ when $c = 10$ and $x_{\mathrm{B}} = 0.8$. The equilibrium configurations and the tension segment 1 (S1) to segment 2 (S2) for the concentrated loading at $\eta_1 = 0.2$ and 0.5 are plotted in Fig.5.10. If the concentrated force is at the middle of cable ($\eta_1 = 0.5$), the jump in tension vanishes. The slopes of the configuration and tension are discontinuous at the location of the concentrated force.

5.4. Nonlinear dynamics of cables

In this section, the dynamics of extensible cables will be discussed. The corresponding equations of motion for the traveling and rotating cable will be discussed. The analytical solutions for the motion of the traveling inextensible cables will be presented, and numerical illustrations for the motion of inextensible cables will be given. Equations for pure deformable motions of the deformable cables will be presented.



(a)



(b)

Fig. 5.10 (a) Equilibrium configuration and (b) tension-jump from segment 1 to segment 2 of sagged, *elastic* cable ($x_B = 0.8$) under its own weight ($q_y = -1$) and a concentrated force ($F_y = -0.4$) for $c = 10 : c_p = 740.87$ and $c^0 = 0$.

5.4.1. Equations of motion

To describe the motion of the cable, as in Section 5.2, an initial equilibrium is required. Consider the initial configuration of the cable expressed through ${}^{(0)}\bar{\mathbf{X}}_k = {}^{(0)}\bar{X}_k^I \mathbf{I}_I$ (summation on $I = 1, 2, 3$). $\mathbf{I}_I$ is the unit vector in the three directions of the fixed coordinates. For the rigid body configuration, there is a relation $\sqrt{{}^{(0)}X_{k,\eta}^I \, {}^{(0)}X_{k,\eta}^I} = 1$ (summation for index *I*). The superscript "0" denotes the static configuration related to the static rigid-body equilibrium. Since the rigid body motion exists in sagged cables, for some time $\bar{t}$, a cable configuration relative to the rigid-body motion is defined by $\bar{\mathbf{X}}(t) = \bar{X}_k^I(t)\mathbf{I}_I$. The corresponding rigid-body displacement $\bar{\mathbf{U}}_k = \bar{U}_k^I \mathbf{I}_I$ is defined by $\bar{\mathbf{U}}_k = \bar{\mathbf{X}}_k - {}^{(0)}\bar{\mathbf{X}}_k$. The dynamical rigid-body configuration still requires $\sqrt{\bar{X}_{k,\bar{s}}^I \bar{X}_{k,\bar{s}}^I} = 1$. From the rigid-body configuration, the total configuration of the elastic cable is $\bar{\mathbf{x}}_k(\bar{t}) = x_k^I \mathbf{I}_I$ and the corresponding deformation displacement is defined as $\bar{u}_k^I = \bar{x}_k^I - \bar{X}_k^I$ with $\bar{\mathbf{u}}_k = \bar{u}_k^I \mathbf{I}_I$, as shown in Fig.5.11. For inextensible cables, equations of motion are given from Eq.(5.49), i.e., for $(I, J, K = 1,2,3)$,

$$\begin{aligned} &X_{k,tt}^I + \Omega_{,t}^J X_k^K e_{IJK} + 2\Omega^J X_{k,t}^K e_{IJK} + \Omega^J \Omega^K X_k^L e_{IJM} e_{MKL} \\ &= q_k^I + (c^0)^2 X_{k,SS}^I. \end{aligned} \tag{5.107}$$

With Eq.(5.50), the equations of motion for traveling and rotating inextensible cables are as in Eq.(5.51).

$$\begin{aligned} &X_{k,tt}^I + 2cX_{k,\eta t}^I + c^2 X_{k,\eta\eta}^I + \Omega_{,t}^J X_k^K e_{IJK} + 2\Omega^J (X_{k,t}^K + cX_{k,\eta}^K) e_{IJK} \\ &+\Omega^J \Omega^K X_k^L e_{IJM} e_{MKL} = q_k^I + (c^0)^2 X_{,\eta\eta}^I. \end{aligned} \tag{5.108}$$

Without rotation, the equations of motion for traveling cables are

$$X_{k,tt}^I + 2cX_{k,\eta t}^I + c^2 X_{k,\eta\eta}^I = q_k^i + (c^0)^2 X_{,\eta\eta}^I. \tag{5.109}$$

The corresponding boundary and initial conditions for the inextensible cables are

$$\begin{aligned} &X_0^I(0,t) = A^I, X_m^I(1,t) = B^I, \\ &X_k^I(\eta,0) = \phi_k^I(\eta), \; X_{k,t}^I(\eta,0) = \varphi_k^I(\eta). \end{aligned} \tag{5.110}$$

Continuity for displacement and velocity for two adjacent segments requires

$$\left.\begin{aligned} &X_k^I(\eta_k,t) = X_{k+1}^I(\eta_k,t), \\ &\phi_k^I(\eta_k) = \phi_{k+1}^I(\eta_k), \; \varphi_k^I(\eta_k) = \varphi_{k+1}^I(\eta_k) \end{aligned}\right\} \text{for } I = 1, 2, 3. \tag{5.111}$$

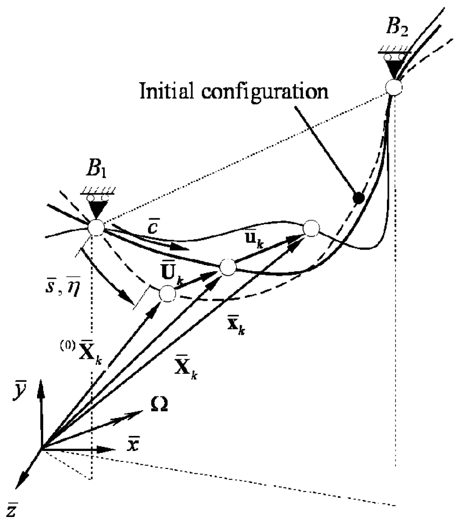


Fig. 5.11 Decomposition of rigid and elastic displacements of sagged cables.

Consider the inextensible cable as an initial configuration and its rigid-body motion (i.e., $X_k^I = {}^{(0)}X_k^I + U_k^I$). Thus, equation (5.107) becomes

$$U_{k,tt}^I + \Omega_{,t}^J({}^{(0)}X_k^K + U_k^K)e_{IJK} + 2\Omega^J U_{k,t}^K e_{IJK} + \Omega^J\Omega^K(X_k^L + U_k^L)e_{IJM}e_{MKL}$$
$$= q_k^I + (c^0)^2({}^{(0)}X_{k,\eta\eta}^I + U_{k,\eta\eta}^I). \tag{5.112}$$

The equation of motion in Eq.(5.108) for traveling and rotating cables becomes

$$U_{k,tt}^I + 2cU_{k,\eta t}^I + c^2({}^{(0)}X_{k,\eta\eta}^I + U_{k,\eta\eta}^I) + \Omega_{,t}^J({}^{(0)}X_k^K + U_k^K)e_{IJK}$$
$$+ 2\Omega^J[U_{k,t}^K + c({}^{(0)}X_{k,\eta}^K + U_{k,\eta}^K)]e_{IJK} + \Omega^J\Omega^K(X_k^L + U_k^L)e_{IJM}e_{MKL}$$
$$= q_k^I + (c^0)^2({}^{(0)}X_{k,\eta\eta}^I + U_{k,\eta\eta}^I). \tag{5.113}$$

Without any motion of the inextensible cable $U_k^I = U_{k,t}^I = U_{k,tt}^I = 0$, the initial configuration for rotating cables can be determined by

$$2c\Omega^J\, {}^{(0)}X_{k,\eta}^K e_{IJK} + \Omega^J\Omega^K X_k^L e_{IJM}e_{MKL} = q_k^I + (c^0)^2\, {}^{(0)}X_{k,\eta\eta}^I. \tag{5.114}$$

The initial configuration for traveling and rotating cables is given by

$$c^2\, {}^{(0)}X_{k,\eta\eta}^I + 2c\Omega^J\, {}^{(0)}X_{k,\eta}^K e_{IJK} + \Omega_{,t}^J\, {}^{(0)}X_k^K e_{IJK} + \Omega^J\Omega^K X_k^L e_{IJM}e_{MKL}$$
$$= q_k^I + (c^0)^2\, {}^{(0)}X_{k,\eta\eta}^I. \tag{5.115}$$

Without rotation, the initial configuration for traveling cables is

$$c^2\, {}^{(0)}X_{k,\eta\eta}^I = q_k^I + (c^0)^2\, {}^{(0)}X_{k,\eta\eta}^I. \tag{5.116}$$

The corresponding boundary and continuity conditions are

$$
\begin{aligned}
&{}^{(0)}X_1^I(0) = A^I,\ {}^{(0)}X_m^I(1) = B^I,\\
&{}^{(0)}X_k^I(\eta_k) = {}^{(0)}X_{k+1}^I(\eta_k).
\end{aligned}
\tag{5.117}
$$

The analytical solution of such an initial configuration of the inextensible cable can be given by Eqs.(5.114)−(5.116) with the boundary condition in Eq.(5.117). For traveling cables, the initial configuration can be referred to Section 5.3 (also see, Luo and Mote, 2000).

Substitution of Eq.(5.114) into Eq.(5.112) yields

$$
U_{k,tt}^I + \Omega_{,t}^J U_k^K e_{IJK} + 2\Omega^J U_{k,t}^K e_{IJK} + \Omega^J \Omega^K U_k^L e_{IJM} e_{MKL} = q_k^I + (c^0)^2 U_{k,\eta\eta}^I. \tag{5.118}
$$

Substitution of Eq.(5.115) into Eq.(5.113) yields

$$
\begin{aligned}
&U_{k,tt}^I + 2cU_{k,\eta t}^I + c^2 U_{k,\eta\eta}^I + 2\Omega^J [U_{k,t}^K + cU_{k,\eta}^K] e_{IJK} + \Omega^J \Omega^K U_k^L e_{IJM} e_{MKL}\\
&= (c^0)^2 U_{k,\eta\eta}^I.
\end{aligned}
\tag{5.119}
$$

Without rotation, one obtains

$$
U_{k,tt}^I + 2cU_{k,\eta t}^I + (c^2 - (c^0)^2) U_{k,\eta\eta}^I = 0. \tag{5.120}
$$

The corresponding boundary and initial conditions for the rigid body motion are

$$
\begin{aligned}
&U_1^I(0,t) = U_m^I(1,t) = 0;\\
&U_k^I(\eta,0) = \Phi_k^I(\eta),\ U_{k,t}^I(\eta,0) = \Gamma_k^I(\eta).
\end{aligned}
\tag{5.121}
$$

The continuity of displacement and velocity for two adjacent segments requires

$$
\left.
\begin{aligned}
&U_k^I(\eta_k,t) = U_{k+1}^I(\eta_k,t),\\
&\Phi_k^I(\eta_k) = \Phi_{k+1}^I(\eta_k),\ \Gamma_k^I(\eta_k) = \Gamma_{k+1}^I(\eta_k)
\end{aligned}
\right\}\text{for } I = 1, 2, 3.
\tag{5.122}
$$

From the above equations, the motion of inextensible cables can be determined.

5.4.2. Motions of inextensible cables

Before determining the motion of the inextensible cable, its initial (static) configuration is determined by formulas in Section 5.2 (also see, Luo and Mote, 2000). The traveling cable will be considered as an example. The solution presented in this section is based on Luo and Wang (2002). The rigid-body motion of cable (or the motion of inextensible cable) is solved by Eqs.(5.120) and (5.121). One segment of cable with distributed loading is considered. So the subscript "*k*" will be dropped. Therefore, the rigid body motion is assumed as

$$U^I = \Phi_1^I(\eta)e^{\mathrm{i}\omega^I t} + \Phi_2^I(\eta)e^{-\mathrm{i}\omega^I t}, \tag{5.123}$$

where $\mathrm{i} = \sqrt{-1}$. Substitution of Eq.(5.123) into Eq.(5.121) leads to

$$\begin{aligned}&(c^2-(c^0)^2)(\Phi_{1,\eta\eta}^I+\Phi_{2,\eta\eta}^I)-(\omega^I)^2(\Phi_1^I+\Phi_2^I)+2\mathrm{i}c\omega^I(\Phi_1^I-\Phi_2^I)=0,\\ &2\mathrm{i}c\omega^I(\Phi_1^I+\Phi_2^I)+(c^2-(c^0)^2)(\Phi_{1,\eta\eta}^I-\Phi_{2,\eta\eta}^I)-(\omega^I)^2(\Phi_1^I-\Phi_2^I)=0.\end{aligned} \tag{5.124}$$

To solve Eq.(5.124), consider the following functions:

$$\Phi_1^I+\Phi_2^I = B_1^I e^{\lambda_I\eta} \text{ and } \Phi_1^I-\Phi_2^I = B_2^I e^{\lambda_I\eta}, \tag{5.125}$$

where λ_I is constant. From Eq.(5.125), equation (5.124) becomes

$$\begin{bmatrix}\lambda_I^2(c^2-(c^0)^2)-(\omega^I)^2 & 2\mathrm{i}c\omega^I\lambda_I\\ 2\mathrm{i}c\omega^I\lambda_I & \lambda_I^2(c^2-(c^0)^2)-(\omega^I)^2\end{bmatrix}\begin{Bmatrix}B_1^I\\ B_2^I\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}. \tag{5.126}$$

For a non-trivial solution,

$$\begin{vmatrix}\lambda_I^2(c^2-(c^0)^2)-(\omega^I)^2 & 2\mathrm{i}c\omega^I\lambda_I\\ 2\mathrm{i}c\omega^I\lambda_I & \lambda_I^2(c^2-(c^0)^2)-(\omega^I)^2\end{vmatrix}=0. \tag{5.127}$$

The four eigenvalues (λ_I) are

$$\lambda_{I(1,2)} = \pm\mathrm{i}\frac{\omega^I}{c+c^0} \equiv \pm\mathrm{i}\alpha_1^I,\ \lambda_{I(3,4)} = \pm\mathrm{i}\frac{\omega^I}{c-c^0} \equiv \pm\mathrm{i}\alpha_2^I. \tag{5.128}$$

Using Euler's formula and eigenvectors, the rigid-body displacement is

$$\begin{aligned}U^I &= \sin\omega^I t\big[D_1^I\cos(\alpha_1^I\eta)-D_2^I\sin(\alpha_1^I\eta)\\ &\quad +D_3^I\cos(\alpha_2^I\eta)+D_4^I\sin(\alpha_2^I\eta)\big]\\ &\quad +\cos\omega^I t\big[D_1^I\sin(\alpha_1^I\eta)+D_2^I\cos(\alpha_1^I\eta)\\ &\quad -D_3^I\sin(\alpha_2^I\eta)+D_4^I\cos(\alpha_2^I\eta)\big].\end{aligned} \tag{5.129}$$

Satisfying the first equation of Eq.(5.121) gives $D_3^I = -D_1^I$, $D_4^I = D_2^I$. Application of the second equation of Eq.(5.121) to Eq.(5.129) generates

$$\begin{bmatrix}\cos\alpha_1^I-\cos\alpha_2^I & -(\sin\alpha_1^I+\sin\alpha_2^I)\\ \sin\alpha_1^I+\sin\alpha_2^I & \cos\alpha_1^I-\cos\alpha_2^I\end{bmatrix}\begin{Bmatrix}D_1^I\\ D_2^I\end{Bmatrix}=0. \tag{5.130}$$

Therefore, for any non-trivial solution, the frequency equation is obtained through

$$\begin{vmatrix}\cos\alpha_1^I-\cos\alpha_2^I & -(\sin\alpha_1^I+\sin\alpha_2^I)\\ \sin\alpha_1^I+\sin\alpha_2^I & \cos\alpha_1^I-\cos\alpha_2^I\end{vmatrix}=0, \tag{5.131}$$

which leads to the natural frequencies of the inextensible cables, i.e.,

$$\omega_p^I = p\omega^I,\ \ p = 1, 2, ...;\ \text{and}\ \omega^I = \frac{\pi[(c^0)^2 - c^2]}{c^0}. \tag{5.132}$$

The rigid-body displacement of the inextensible cable is thus written in a form of

$$\begin{aligned} U^I(\eta,t) = \sum\nolimits_{p=1}^{\infty}\Big\{&\sin(p\omega^I t)\left[C^I_{(1,p)}\varphi_{(1,p)}(\eta) + C^I_{(2,p)}\varphi_{(2,p)}(\eta)\right] \\ &+\cos(p\omega^I t)\left[C^I_{(2,p)}\varphi_{(1,p)}(\eta) - C^I_{(1,p)}\varphi_{(2,p)}(\eta)\right]\Big\}, \end{aligned} \tag{5.133}$$

where

$$\varphi_{(1,p)} = \sin\,(\frac{p\pi c}{c_1}\eta)\,\sin\,(p\pi\eta)\,,\ \ \varphi_{(2,p)} = \cos\,(\frac{p\pi c}{c_1}\eta)\,\sin\,(p\pi\eta). \tag{5.134}$$

Substitution of $U^I(\eta,t)$ into the second equation of Eq.(5.121) and truncation of the displacement yields

$$\begin{aligned} \sum\nolimits_{p=1}^{M}\left[C^I_{(2,p)}\varphi_{(1,p)}(\eta) - C^I_{(1,p)}\varphi_{(2,p)}(\eta)\right] &= \mathbf{\Phi}^I(\eta), \\ \sum\nolimits_{p=1}^{M} p\omega^I\left[C^I_{(1,p)}\varphi_{(1,p)}(\eta) + C^I_{(2,p)}\varphi_{(2,p)}(\eta)\right] &= \mathbf{\Gamma}^I(\eta), \end{aligned} \tag{5.135}$$

where M is a positive integer. For convenience, equation (5.135) is rewritten in a matrix form of

$$\mathbf{\Phi}^I(\eta)\mathbf{C}^I = \mathbf{U}_0^I(\eta), \tag{5.136}$$

where

$$\begin{aligned} \mathbf{\Phi}^I(\eta) = \Bigg[&\begin{matrix} -\varphi_{(2,1)} & \varphi_{(1,1)} & \cdots & -\varphi_{(2,p)} & \varphi_{(1,p)} & \cdots \\ \omega^I\varphi_{(1,1)} & \omega^I\varphi_{(2,1)} & \cdots & p\omega^I\varphi_{(1,p)} & p\omega^I\varphi_{(2,p)} & \cdots \end{matrix} \\ &\quad\begin{matrix} -\varphi_{(2,M)} & \varphi_{(1,M)} \\ M\omega^I\omega_k^2\varphi_{(1,M)} & M\omega^I\varphi_{(2,M)} \end{matrix}\Bigg], \end{aligned} \tag{5.137}$$

and

$$\left.\begin{aligned} &\mathbf{C}^I = (C^I_{(1,1)}, C^I_{(2,1)}, \ldots, C^I_{(1,2)}, C^I_{(2,2)}, \ldots, C^I_{(1,M)}, C^I_{(2,M)})^{\mathrm{T}}, \\ &\mathbf{U}_0^I(\eta) = (\mathbf{\Phi}^I, \mathbf{\Gamma}^I)^{\mathrm{T}}. \end{aligned}\right\} \tag{5.138}$$

To make a square matrix for determining all the coefficients, a function base is introduced by

$$\mathbf{\Theta} = \left(\sin\,(\pi\eta)\,, \sin\,(2\pi\eta)\,, ..., \sin\,(M\pi\eta)\right)^{\mathrm{T}} \tag{5.139}$$

The $2M$-linear algebraic equations of $C^I_{(1,p)}$ and $C^I_{(2,p)}$ are obtained by

$$\tilde{\mathbf{\Phi}}^I \mathbf{C}^I = \tilde{\mathbf{U}}_0^I, \tag{5.140}$$

where

$$[\tilde{\mathbf{\Phi}}^I]_{(p,q)} = \int_0^1 \{\mathbf{\Theta}^{\mathrm{T}}\}_p [\mathbf{\Phi}^I]_{(1,q)} \mathrm{d}\eta \text{ and } \{\tilde{\mathbf{U}}_0^I\}_p = \int_0^1 \{\mathbf{\Theta}^{\mathrm{T}}\}_p \Phi^I \mathrm{d}\eta \tag{5.141}$$

for $1 \le p \le M$ and $1 \le q \le 2M$,

$$[\tilde{\mathbf{\Phi}}^I]_{p,q} = \int_0^1 \{\mathbf{\Theta}^{\mathrm{T}}\}_p [\mathbf{\Phi}^I]_{(2,q)} \mathrm{d}\eta \text{ and } \{\tilde{\mathbf{U}}_0^I\}_p = \int_0^1 \{\mathbf{\Theta}^{\mathrm{T}}\}_p \Gamma^I \mathrm{d}\eta \tag{5.142}$$

for $M+1 \le p \le 2M, 1 \le q \le 2M$.

The other elements of $\tilde{\mathbf{\Phi}}^I$ are zero. The unknowns $\mathbf{C}^I$ can be obtained by solving Eq.(5.140). Substitution of $\mathbf{C}^I$ into Eq.(5.133) and neglecting terms with order higher than M gives an approximate expression of the rigid-body displacement of the cable at any time. The dynamic configuration at time t is expressed by

$$X^I(\eta,t) = {}^{(0)}X^I(\eta) + U^I(\eta,t) \tag{5.143}$$

From the above analysis, the initial rigid body configuration is very important to determine the rigid-body motion and the deformation motion. A two-dimensional traveling, sagged cable is illustrated as an example. The material properties of the cable are: $\rho = 6000\ \mathrm{kg/m^3}$, $S = 1\mathrm{m}$, $\bar{T}_0 = 100\mathrm{N}$, $A = 2.5\times10^{-5}\mathrm{m}^2$, $\bar{c} = 3.16\mathrm{m/s}$. Thus, the non-dimensional parameters are $c = 1$ and $c^0 \approx 8.17$. The boundary locations are $A_x = A_y = 0$, $B_x = 0.8$, and $B_y = 0$. Consider the vibration of the uniform cable near its rigid-body equilibrium under the self-weight, i.e., the loading is $(q_x, q_y) = (0, -1)$. The initial configuration is given through the equilibrium configuration determined with loading $(q_x, q_y) = (1, -1)$ at $t = 0$. For such an initial configuration, the traveling cable under $(q_x, q_y) = (0, -1)$ will be vibrated in the vicinity of its rigid-body equilibrium configuration. The free vibration responses of the rigid-displacement are computed and presented in Fig.5.12. From the given parameters, the oscillation period is $T = 0.2485$. The rigid-body configuration and displacements versus coordinate X are illustrated for time $t =$ $0T, 0.25T, \ldots, 1T$ in Fig.5.12(a)–(c), respectively. In Fig.5.12(a), the instant configuration of the traveling inextensible cable is presented. The configuration motion is like a pendulum swing motion. The rigid-body displacements of the inextensible cable in the vicinity of the equilibrium of $(q_x, q_y) = (0, -1)$ with $c = 1$ are presented in Fig.5.12(b) and (c). To demonstrate the time-histories of displacements, the rigid-body displacements at the midpoint of the cable are illustrated for the X and Y–directions in Fig.5.12 (d) and (e), respectively. The displacement responses are not simply harmonic. In addition, a plane of two rigid-body displacements at the middle point is illustrated in Fig.5.12(f). The acronyms "I.C." denotes initial displacement of the inextensible cable. The natural frequency versus the ratio of the translational speed to the constant wave speed is illustrated in Fig.5.13. It is observed that $\omega^I = 0$ at $c = c_0$. For $c > c_0$, the solution presented herein is not applicable. However, the solutions for the motion of the in-plane inextensible cable for $c \in [0, +\infty)$ can be referred to Wang and Luo (2004).

5.4.3. Motions of deformable cables

After discussion of equation of motion for the inextensible cable, the motion of deformable cables can be discussed. With Eq.(5.107), Equation (5.65) can be expressed through the deformation displacement, i.e.,

$$\begin{aligned} & u^I_{k,tt} + \Omega^J_{,t} u^K_k e_{IJK} + 2\Omega^J u^K_{k,t} e_{IJK} + \Omega^J \Omega^K u^L_k e_{IJM} e_{MKL} \\ & = \left[\frac{T_k(S)(X^I_{k,S} + u^I_{k,s})}{1 + \mathscr{E}_k} \right]_{,S} - (c^0)^2 X^I_{k,SS}. \end{aligned} \tag{5.144}$$

The deformation motion for traveling and rotating cables in Eq.(5.67) is written as

$$\begin{aligned} & u^I_{k,tt} + 2cu^I_{k,\eta t} + c^2 u^I_{k,\eta\eta} + \Omega^J_{,t} u^K_k e_{IJK} + 2\Omega^J_{,t}(u^K_{k,t} + cu^K_{k,\eta}) e_{IJK} \\ & + \Omega^J \Omega^K u^L_k e_{IJM} e_{MKL} = \left[\frac{T_k(\eta)(X^I_{k,\eta} + u^I_{k,\eta})}{1 + \mathscr{E}_k} \right]_{,\eta} - (c^0)^2 X^I_{k,\eta\eta}. \end{aligned} \tag{5.145}$$

Without rotation,

$$u^I_{k,tt} + 2cu^I_{k,\eta t} + c^2 u^I_{k,\eta\eta} = \left[\frac{T_k(\eta)(X^I_{k,\eta} + u^I_{k,\eta})}{1 + \mathscr{E}_k} \right]_{,\eta} - (c^0)^2 X^I_{k,\eta\eta}. \tag{5.146}$$

The boundary and initial conditions for the elastic deformation are:

$$\begin{aligned} & u^I_k(0,t) = 0, \; u^I_k(1,t) = 0; \\ & u^I_k(\eta,0) = \phi^I_k(\eta), \; u^I_{k,t}(\eta,0) = \gamma^I_k(\eta). \end{aligned} \tag{5.147}$$

The continuity conditions are

$$\left. \begin{aligned} & u^I_k(\eta_k,t) = u^I_{k+1}(\eta_k,t), \\ & \phi^I_k(\eta_k) = \phi^I_{k+1}(\eta_k), \; \gamma^I_k(\eta_k) = \gamma^I_{k+1}(\eta_k) \end{aligned} \right\} \text{ for } I = 1,2,3. \tag{5.148}$$

With $\mathscr{E}_k = 0$ and $T_k(S) = T^0(S) \equiv (c^0)^2$, Equations (5.144)–(8.146) become

$$u^I_{k,tt} + \Omega^J_{,t} u^K_k e_{IJK} + 2\Omega^J u^K_{k,t} e_{IJK} + \Omega^J \Omega^K u^L_k e_{IJM} e_{MKL} = (c^0)^2 u^I_{k,SS}. \tag{5.149}$$

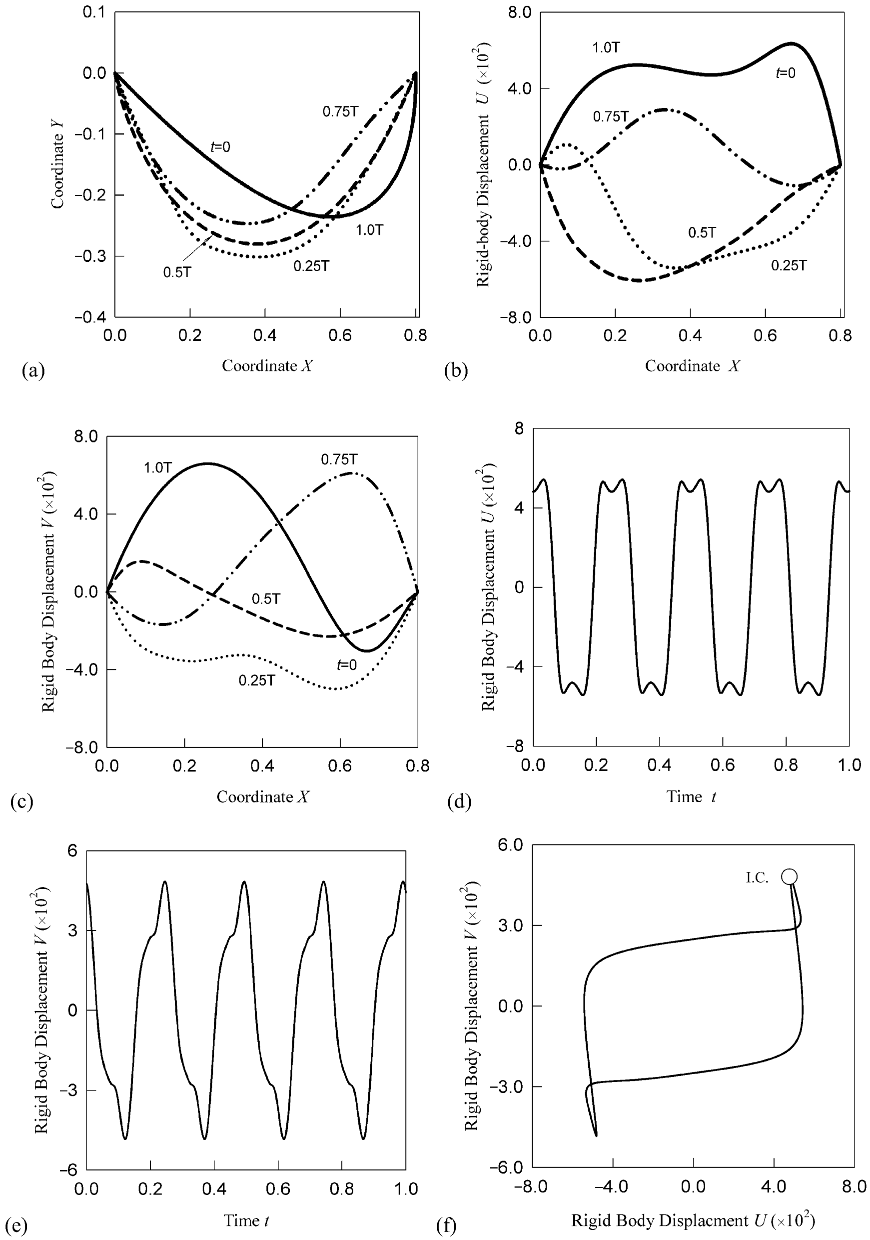


Fig. 5.12 The rigid-body configuration and displacements varying with coordinate X for $(q_x, q_y) = (0, -1)$: (a) configuration, (b) X-component, and (c) Y-component. The rigid-body responses of the middle point of the cable: (d) X-component, (e) Y-component, and (f) displacement plane. ($c = 1$, $c^0 = 8.17$ and $T \approx 0.2485$). Initial configuration is given by the equilibrium for $(q_x, q_y) = (1, -1)$.

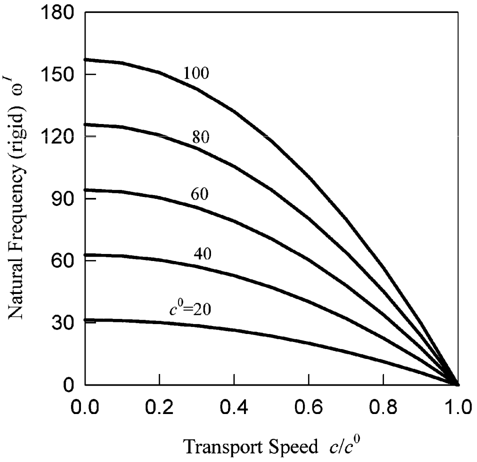


Fig. 5.13 Natural frequency varying with the transport speed. ($c=1$, $c^0 \approx 8.17$).

$$
\begin{aligned}
&u^I_{k,tt} + 2cu^I_{k,\eta t} + c^2 u^I_{k,\eta\eta} + \Omega^J_{,t} u^K_k e_{IJK} + 2\Omega^J_{,t}(u^K_{k,t} + cu^K_{k,\eta})e_{IJK} \\
&+\Omega^J \Omega^K u^L_k e_{IJM} e_{MKL} = (c^0)^2 u^I_{k,\eta\eta},
\end{aligned}
\tag{5.150}
$$

$$
u^I_{k,tt} + 2cu^I_{k,\eta t} + [c^2 - (c^0)^2] u^I_{k,\eta\eta} = 0. \tag{5.151}
$$

identical to Eqs.(5.109)–(5.111). The equation of deformation motion becomes the equation of motion for the inextensible cable. Based on the dynamical configuration of the inextensible cable, the pure deformation motion of the cable can be determined by Eq.(5.144) (or Eq.(5.145) or Eq.(5.146)) and Eqs.(5.147) and (5.148).

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Chapter 6
Nonlinear Plates and Waves

This chapter will present a nonlinear plate theory from three-dimensional elastic theory, and the approximate theories of thin plates will be discussed. From such a theory, approximate solutions for nonlinear waves in traveling plates and rotating disks will be presented. In addition, stationary and resonant waves in the traveling plates and rotating disks will be discussed. Finally, chaotic waves in traveling plates under periodic excitation will be presented.

6.1. A nonlinear theory of plates

In this section, a nonlinear plate theory of thin plate will be proposed from the theory of 3-D deformable body. In this theory, the exact geometry of the deformed middle surface is used to derive the strains and equilibrium of the plate. This theory reduces to existing nonlinear theories through imposition of constraints. Application of this theory does not depend on the constitutive law because the physical deformation measure is used. The comprehensive discussion can be referred to Luo (2000).

Definition 6.1. If a flat deformable body on the two principal directions of fibers resists the internal forces and bending moments, the deformable body is called a *deformable* plate.

6.1.1. Deformation of a 3-D body

Consider a particle $P(Y^1, Y^2, Y^3)$ in a deformable body $\mathfrak{B}$ at the initial configuration in Fig.6.1. The particle position **R** is described by Y^I:

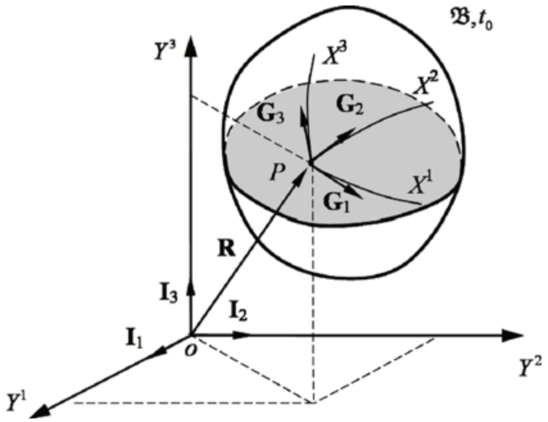


Fig. 6.1 A material particle P .

$$\mathbf{R} = Y^I \mathbf{I}_I \equiv Y^1 \mathbf{I}_1 + Y^2 \mathbf{I}_2 + Y^3 \mathbf{I}_3, \tag{6.1}$$

where $\mathbf{I}_I$ $(I = 1,2,3)$ are unit vectors in the fixed coordinates. In the local curvilinear refe-rence frame, $\mathbf{R}$ is represented by

$$\mathbf{R} = X^\Lambda \mathbf{G}_\Lambda \equiv X^1 \mathbf{G}_1 + X^2 \mathbf{G}_2 + X^3 \mathbf{G}_3, \tag{6.2}$$

where the component $X^I = \mathbf{R} \cdot \mathbf{G}^I$ in Eringen (1962) (also see, Chapter 3) and the initial base vectors $\mathbf{G}_\Lambda \equiv \mathbf{G}_\Lambda (X^1, X^2, X^3, t_0)$ are

$$\mathbf{G}_\Lambda = \frac{\partial \mathbf{R}}{\partial X^\Lambda} = \frac{\partial Y^I (X^1, X^2, X^3, t_0)}{\partial X^\Lambda} \mathbf{I}_I = Y^I_{,\alpha} \mathbf{I}_I \tag{6.3}$$

with magnitudes

$$|\mathbf{G}_\Lambda (\mathbf{X})| = \sqrt{G_{\Lambda\Lambda}} = \sqrt{Y^I_{,\Lambda} Y^I_{,\Lambda}} \tag{6.4}$$

without summation on Λ and $G_{\Lambda\Gamma} = \mathbf{G}_\Lambda \cdot \mathbf{G}_\Gamma$ are metric coefficients in $\mathfrak{B}$.

On deformation of $\mathfrak{B}$, the particle at point P moves through displacement $\mathbf{u}$ to position p, and the particle Q, infinitesimally close to $P(X^1, X^2, X^3, t_0)$, moves by $\mathbf{u} + d\mathbf{u}$ to q in the neighborhood of $p(X^1, X^2, X^3, t)$, as illustrated in Fig.6.2.

The position of point p is

$$\mathbf{r} = \mathbf{R} + \mathbf{u} = (X^\Lambda + u^\Lambda) \mathbf{G}_\Lambda, \tag{6.5}$$

where the displacement is $\mathbf{u} = u^\Lambda \mathbf{G}_\Lambda$. Thus, $\overline{PQ} = d\mathbf{R}$ and $\overline{pq} = d\mathbf{r}$ are

$$d\mathbf{R} = \mathbf{G}_\Lambda dX^\Lambda, \; d\mathbf{r} = \mathbf{G}_\Lambda dX^\Lambda + d\mathbf{u}; \tag{6.6}$$

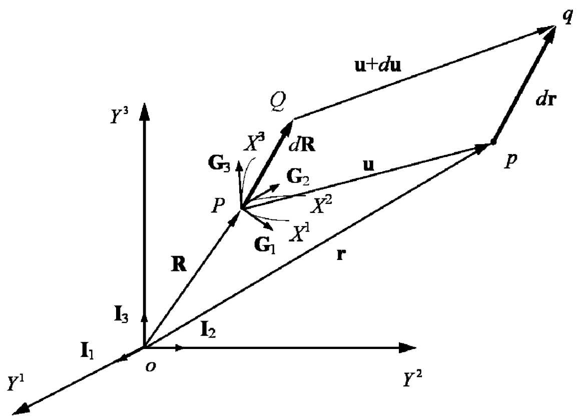


Fig. 6.2 Deformation of a differential linear element.

and the infinitesimal displacement becomes

$$d\mathbf{u} = u^{\Lambda}_{;\alpha} dX^{\alpha} \mathbf{G}_{\Lambda} = u_{\Lambda;\alpha} dX^{\alpha} \mathbf{G}^{\Lambda}, \tag{6.7}$$

where $u^{\Lambda}_{;\alpha} = \partial u^{\Lambda} / \partial X^{\alpha} + \Gamma^{\Lambda}_{\alpha\beta} u^{\beta}$ and $\Gamma^{\Lambda}_{\alpha\beta}$ is the Christoffel symbol defined in Chapter 2 (e.g., Eringen, 1967). The semicolon represents covariant partial differentiation. From Eqs.(6.6) and (6.7),

$$d\mathbf{r} = (u^{\Lambda}_{;\alpha} + \delta^{\Lambda}_{\alpha}) \mathbf{G}_{\Lambda} dX^{\alpha}. \tag{6.8}$$

The Lagrangian strain tensor $E_{\Lambda\Gamma}$ referred to the initial configuration is from Eringen (1962), i.e.,

$$E_{\alpha\beta} = \frac{1}{2}(u_{\alpha;\beta} + u_{\beta;\alpha} + u^{\Lambda}_{;\alpha} u_{\Lambda;\beta}), \tag{6.9}$$

As in Malvern (1969), the change in length of $d\mathbf{R}$ per unit length gives

$$\varepsilon_{\alpha} = \frac{|\underset{\alpha}{d\mathbf{r}}| - |\underset{\alpha}{d\mathbf{R}}|}{|\underset{\alpha}{d\mathbf{R}}|} = \frac{1}{\sqrt{G_{\alpha\alpha}}} \sqrt{G_{\alpha\alpha} + 2E_{\alpha\alpha}} - 1 \tag{6.10}$$

where ε_{α} is the relative elongation along $\mathbf{G}_{\alpha}$. The unit vectors along $d\mathbf{R}$ and $d\mathbf{r}$ in Chapter 3 (also see, Eringen, 1967) are

$$\mathbf{N}_{\alpha} = \frac{\underset{\alpha}{d\mathbf{R}}}{|\underset{\alpha}{d\mathbf{R}}|} = \frac{1}{\sqrt{G_{\alpha\alpha}}} \mathbf{G}_{\alpha}, \quad \mathbf{n}_{\alpha} = \frac{\underset{\alpha}{d\mathbf{r}}}{|\underset{\alpha}{d\mathbf{r}}|} = \frac{u^{\Lambda}_{;\alpha} + \delta^{\Lambda}_{\alpha}}{\sqrt{G_{\alpha\alpha} + 2E_{\alpha\alpha}}} \mathbf{G}_{\Lambda}. \tag{6.11}$$

The unit vectors of the deformed configuration in the directions $\mathbf{g}_1$ and $\mathbf{g}_2$ are

$$\mathbf{n}_1 = \frac{d\mathbf{r}_1}{|d\mathbf{r}_1|} \text{ and } \mathbf{n}_2 = \frac{d\mathbf{r}_2}{|d\mathbf{r}_2|}. \tag{6.12}$$

Let Θ_{12} and θ_{12} be the angles between $\mathbf{n}_1$ and $\mathbf{n}_2$ before and after deformation,

$$\begin{aligned}\cos\theta_{12} &\equiv \cos\left(\Theta_{12} - \gamma_{12}\right)\\ &= \frac{d\mathbf{r}_1 \cdot d\mathbf{r}_2}{|d\mathbf{r}_1||d\mathbf{r}_2|} = \frac{G_{12} + 2E_{12}}{\sqrt{\left(G_{11} + 2E_{11}\right)\left(G_{22} + 2E_{22}\right)}},\\ \cos\Theta_{12} &= \frac{d\mathbf{R}_1 \cdot d\mathbf{R}_2}{|d\mathbf{R}_1||d\mathbf{R}_2|} = \frac{G_{12}}{\sqrt{G_{11}G_{22}}};\end{aligned} \tag{6.13}$$

and the shear strain is

$$\begin{aligned}\gamma_{12} &\equiv \Theta_{12} - \theta_{12}\\ &= \cos^{-1}\frac{G_{12}}{\sqrt{G_{11}G_{22}}} - \cos^{-1}\frac{G_{12} + 2E_{12}}{\sqrt{(G_{11} + 2E_{11})(G_{22} + 2E_{22})}}.\end{aligned} \tag{6.14}$$

The other shear stains are obtained in a similar manner.

From Eq.(6.11), the direction cosine of the rotation without summation on α and β is

$$\cos\theta_{(\mathbf{N}_\alpha,\mathbf{n}_\beta)} = \frac{d\mathbf{R}_\alpha \cdot d\mathbf{r}_\beta}{|d\mathbf{R}_\alpha||d\mathbf{r}_\beta|} = \frac{(\delta_\beta^\Lambda + u_{;\beta}^\Lambda)G_{\alpha\Lambda}}{\sqrt{G_{\alpha\alpha}}\sqrt{G_{\beta\beta} + 2E_{\beta\beta}}} = \frac{G_{\alpha\beta} + u_{\alpha;\beta}}{\sqrt{G_{\alpha\alpha}}\sqrt{G_{\beta\beta} + 2E_{\alpha\alpha}}}. \tag{6.15}$$

In addition, the area and volume changes from Chapter 3 are given by

$$\frac{da_{\alpha\beta}}{dA_{\alpha\beta}} = \frac{(1+\varepsilon_\alpha)(1+\varepsilon_\beta)\sin\theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)}}{\sin\Theta_{(\mathbf{N}_\alpha,\mathbf{N}_\beta)}}, \quad \frac{dv}{dV} = |\delta_\alpha^\Lambda + u_{;\alpha}^\Lambda|, \tag{6.16}$$

where dv and dV are the infinitesimal material volumes after and before deformation, respectively; $|\cdot|$ represents the determinant. The areas after and before deformation are $da_{\alpha\beta} = |d\mathbf{r}_\alpha \times d\mathbf{r}_\beta|$ and $dA_{\alpha\beta} = |d\mathbf{R}_\alpha \times d\mathbf{R}_\beta|$, where $\alpha \neq \beta$.

6.1.2. Strains in thin plates

Consider the Lagrangian coordinates to be a Cartesian system:

$$Y^1 = X^1 = x, \; Y^2 = X^2 = y, \; Y^3 = X^3 = z; \tag{6.17}$$

and

$$\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}, \tag{6.18}$$

where $\mathbf{I}_1 = \mathbf{i},\ \mathbf{I}_2 = \mathbf{j}$ and $\mathbf{I}_3 = \mathbf{k}$. Therefore, from Eqs.(6.17) and (6.18),

$$G_{\alpha\alpha} = 1,\ \ G_{\alpha\beta} = 0,\ \Gamma^{\Lambda}_{\alpha\beta} = 0,\ \Theta_{\alpha\beta} = \frac{\pi}{2} \quad (\alpha \neq \beta). \tag{6.19}$$

Then the physical strains in Eqs.(6.10) and (6.14) become

$$\varepsilon_\alpha = \sqrt{(\delta^I_\alpha + u^I_{,\alpha})(\delta^I_\alpha + u^I_{,\alpha})} - 1, \tag{6.20}$$

$$\gamma_{\alpha\beta} = \sin^{-1}\frac{(\delta^I_\alpha + u^I_{,\alpha})(\delta^I_\beta + u^I_{,\beta})}{(1+\varepsilon_\alpha)(1+\varepsilon_\beta)}, \tag{6.21}$$

where $\alpha, \beta, \Lambda \in \{1,2,3\}$, $\{1 \equiv x,\ 2 \equiv y,\ 3 \equiv z\}$ and $\{u_1 \equiv u,\ u_2 \equiv v,\ u_3 \equiv w\}$ with summation on I.

For reduction of a three dimensional deformable body theory to a two-dimensional plate theory, displacements can be expressed in a Taylor series expanded about the displacement of the middle surface. Thus, As similar to the basic kinematics hypothesis in Wempner (1973), the displacement field is assumed as

$$u^I = u^I_0(x, y, t) + \sum\nolimits_{n=1}^{\infty} z^n \varphi^{(I)}_n (x, y, t), \tag{6.22}$$

where u^I_0 ($I = 1, 2, 3$) denotes displacements of the middle surface, and the $\varphi^{(I)}_n$ ($n = 1, 2, \cdots$) are rotations. Due to

$$\begin{aligned} &\frac{\partial \gamma_{IJ}}{\partial z} = \frac{1}{\cos\gamma_{IJ}}\frac{\partial \sin\gamma_{IJ}}{\partial z}, \\ &\frac{\partial^2 \gamma_{IJ}}{\partial z^2} = \frac{1}{\cos\gamma_{IJ}}\frac{\partial^2 \sin\gamma_{IJ}}{\partial z^2} + (\frac{\partial \gamma_{IJ}}{\partial z})^2 \tan\gamma_{IJ}, \ldots, \end{aligned} \tag{6.23}$$

substitution of Eq.(6.22) into Eqs.(6.20) and (6.21) and collection of like powers of z gives

$$\begin{aligned} \varepsilon_\alpha &\approx \varepsilon^{(0)}_\alpha + \left.\frac{\partial \varepsilon_\alpha}{\partial z}\right|_{z=0} z + \frac{1}{2!}\left.\frac{\partial^2 \varepsilon_\alpha}{\partial z^2}\right|_{z=0} z^2 + \cdots \\ &= \varepsilon^{(0)}_\alpha + \frac{(\delta^I_\alpha + u^I_{0,\alpha})\varphi^{(I)}_{1,\alpha}}{1+\varepsilon^{(0)}_\alpha} z \\ &+ \frac{1}{2}\left\{ \frac{2[(\delta^I_\alpha + u^I_{0,\alpha})\varphi^{(I)}_{2,\alpha}] + \varphi^{(I)}_{1,\alpha}\varphi^{(I)}_{1,\alpha}}{1+\varepsilon^{(0)}_\alpha} - \frac{[(\delta^I_\alpha + u^I_{0,\alpha})\varphi^{(I)}_{1,\alpha}]^2}{(1+\varepsilon^{(0)}_\alpha)^3} \right\} z^2 + \cdots, \end{aligned} \tag{6.24}$$

$$\varepsilon_3 \approx \varepsilon_3^{(0)} + \left.\frac{\partial \varepsilon_3}{\partial z}\right|_{z=0} z + \frac{1}{2!}\left.\frac{\partial^2 \varepsilon_3}{\partial z^2}\right|_{z=0} z^2 + \cdots$$
$$= \varepsilon_3^{(0)} + \frac{2(\delta_3^I + \varphi_1^{(I)})\varphi_2^{(I)}}{1+\varepsilon_3^{(0)}} z$$
$$+ \left\{ \frac{[2\varphi_2^{(I)}\varphi_2^{(I)} + 3(\delta_3^I + \varphi_1^{(I)})\varphi_3^{(I)}]}{1+\varepsilon_3^{(0)}} - \frac{2[(\delta_3^I + \varphi_1^{(I)})\varphi_2^{(I)}]^2}{(1+\varepsilon_3^{(0)})^3} \right\} z^2 + \cdots, \tag{6.25}$$

$$\gamma_{12} \approx \gamma_{12}^{(0)} + \left.\frac{\partial \gamma_{12}}{\partial z}\right|_{z=0} z + \frac{1}{2!}\left.\frac{\partial^2 \gamma_{12}}{\partial z^2}\right|_{z=0} z^2 + \cdots$$
$$= \gamma_{12}^{(0)} + \frac{1}{\cos\gamma_{12}^{(0)}} \left\{ \frac{(\delta_I^1 + u_{I,1}^{(0)})\varphi_{I,2}^{(1)} + (\delta_I^2 + u_{I,2}^{(0)})\varphi_{I,1}^{(1)}}{(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})} \right.$$
$$\left. - \sin\gamma_{12}^{(0)} \left[\frac{(\delta_I^1 + u_{I,1}^{(0)})\varphi_{I,1}^{(1)}}{(1+\varepsilon_1^{(0)})^2} + \frac{(\delta_I^2 + u_{I,2}^{(0)})\varphi_{I,2}^{(1)}}{(1+\varepsilon_2^{(0)})^2} \right] \right\} z + \cdots, \tag{6.26}$$

$$\gamma_{\alpha 3} \approx \gamma_{\alpha 3}^{(0)} + \left.\frac{\partial \gamma_{\alpha 3}}{\partial z}\right|_{z=0} z + \frac{1}{2!}\left.\frac{\partial^2 \gamma_{\alpha 3}}{\partial z^2}\right|_{z=0} z^2 + \cdots$$
$$= \gamma_{\alpha 3}^{(0)} + \frac{1}{\cos\gamma_{\alpha 3}^{(0)}} \left\{ \frac{(\delta_\alpha^I + u_{0,\alpha}^I)\varphi_1^{(I)} + (\delta_3^I + \varphi_1^{(I)})\varphi_{1,\alpha}^{(I)}}{\left(1+\varepsilon_\alpha^{(0)}\right)\left(1+\varepsilon_3^{(0)}\right)} \right.$$
$$\left. - \sin\gamma_{\alpha 3}^{0} \left[\frac{(\delta_\alpha^I + u_{0,\alpha}^I)\varphi_{1,\alpha}^{(I)}}{(1+\varepsilon_\alpha^{(0)})^2} + \frac{2(\delta_3^I + \varphi_1^{(I)})\varphi_2^{(I)}}{(1+\varepsilon_3^{(0)})^2} \right] \right\} z + \cdots, \tag{6.27}$$

where $\alpha = 1, 2$. The strains of the middle surface following Eqs.(6.20)–(6.22) at $z = 0$ are

$$\begin{aligned}
\varepsilon_\alpha^{(0)} &= \sqrt{(\delta_\alpha^I + u_{0,\alpha}^I)(\delta_\alpha^I + u_{0,\alpha}^I)} - 1, \\
\varepsilon_3^{(0)} &= \sqrt{(\delta_3^I + \varphi_1^{(I)})(\delta_3^I + \varphi_1^{(I)})} - 1; \\
\gamma_{12}^{(0)} &= \sin^{-1}\left[\frac{(\delta_1^I + u_{0,1}^I)(\delta_2^I + u_{0,2}^I)}{(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})} \right], \\
\gamma_{\alpha 3}^{(0)} &= \sin^{-1}\left[\frac{(\delta_\alpha^I + u_{0,\alpha}^I)(\delta_3^I + \varphi_1^{(I)})}{(1+\varepsilon_\alpha^{(0)})(1+\varepsilon_3^{(0)})} \right].
\end{aligned} \tag{6.28}$$

In Eqs.(6.24)–(6.27) prediction of strain requires specification of three constraints for determination of the three sets $\varphi_n^{(I)}$ ($I = 1, 2, 3;\ n = 1, 2, \ldots$) like the assumptions ($\gamma_{\alpha 3} = \varepsilon_3 = 0$) as in Kirchhoff (1850a,b).

6.1.3. Equations of motion

Consider a thin plate subjected to the inertia force $\rho u_{I,tt}$ and ρ_0 is the density of plate with $\rho = \int_{-h^-}^{h^+} \rho_0 dz$; body force $\mathbf{f} = f^I \mathbf{I}_I$ ($I = 1, 2, 3$); surface loading $\{p_I^+, p_I^-\}$ where the superscripts + and − denote the upper and lower surfaces; external moment m_0^α ($\alpha = 1, 2$) before deformation. The sign convention is adopted herein. That is, on the positive (or negative) surface, the forces in the positive (or negative) direction are positive and vice versa. The components for the distributed forces and moments $\mathbf{q} = q^I \mathbf{I}_I$ and $\mathbf{m} = m^I \mathbf{I}_I = (-1)^\beta m^\alpha \mathbf{I}_\beta$ are

$$
\begin{aligned}
q^I &= p_I^+ - p_I^- + \int_{-h^-}^{h^+} f^I dz \qquad (I = 1, 2, 3),\\
m^\alpha &= m_0^\alpha + h^+ p_\beta^+ + h^- p_\beta^- + \int_{-h^-}^{h^+} f^\beta z dz \quad (\alpha, \beta \in \{1, 2\}, \alpha \neq \beta);
\end{aligned}
\tag{6.29}
$$

where $m^3 = 0$ and $h = h^+ + h^-$. The distributed loading after deformation becomes

$$
q_I^* = \frac{q^I}{(1+\varepsilon_1)(1+\varepsilon_2)\cos\gamma_{12}}. \tag{6.30}
$$

Since the physical deformation measure is based on the Lagrangian coordinates, the constitutive laws for such deformation measure is

$$
\sigma_{\alpha\beta} = f_{\alpha\beta}(\Lambda_{\alpha\beta MN}, \varepsilon_{MN}, t) \tag{6.31}
$$

for determination of the physical stresses directly (or sometimes, termed the Cauchy stress), where $\Lambda_{\alpha\beta MN}$ are material properties (e.g., Young's modulus, Poisson ratio). Equation (6.31) can be the Hooke's law for linear elastic materials or the other similar laws for the plasticity and the others. Based on such physical stresses, the internal, resultant forces and moments can be determined in the Lagrangian coordinates. As in Wempner (1973), the stress resultant forces and couples in the deformed plate are defined as follows:

$$
\begin{aligned}
N_{\alpha\beta} &= \int_{-h^-}^{h^+} \sigma_{\alpha\beta} \left[(1+\varepsilon_{\alpha'})(1+\varepsilon_3)\cos\gamma_{\alpha'3}\right] dz,\\
M_{\alpha\beta} &= \int_{-h^-}^{h^+} \sigma_{\alpha\beta} \frac{z}{1+\varphi_3^{(1)}} \left[(1+\varepsilon_{\alpha'})(1+\varepsilon_3)^2\cos\gamma_{\alpha'3}\right] dz,\\
Q_\alpha &= \int_{-h^-}^{h^+} \sigma_{\alpha 3} \left[(1+\varepsilon_{\alpha'})(1+\varepsilon_3)\cos\gamma_{\alpha'3}\right] dz;\\
&\text{with } \alpha' = \mathrm{mod}(\alpha, 2) + 1,
\end{aligned}
\tag{6.32}
$$

where $N_{\alpha\beta}$ are membrane forces and $M_{\alpha\beta}$ are bending and twisting moments per unit length and $\alpha, \beta \in \{1,2\}$. If the Kirchhoff assumption ($\varepsilon_3 = 0$) is used, equation (6.32) reduces to the form as in the textbook. Before the equation of motion for the deformed plate is developed, the internal force vectors are introduced, i.e.,

$$\begin{aligned}
\mathbf{M}_\alpha &\equiv M_\alpha^I \mathbf{I}_I = (-1)^\beta M_\alpha \mathbf{n}_\beta + (-1)^\alpha M_{\alpha\beta} \mathbf{n}_\alpha, \\
\mathcal{N}_\alpha &\equiv N_\alpha^I \mathbf{I}_I = N_\alpha \mathbf{n}_\alpha + N_{\alpha\beta} \mathbf{n}_\beta + Q_\alpha \mathbf{n}_3, \\
{}^{\mathrm{N}}\mathbf{M}_\alpha &\equiv {}^{\mathrm{N}}M_\alpha^I \mathbf{I}_I = \mathbf{g}_\alpha \times \mathcal{N}_\alpha,
\end{aligned} \tag{6.33}$$

where

$$\mathbf{g}_\alpha \equiv \frac{d\mathbf{r}_\alpha}{dX^\alpha} = (\delta_\alpha^I + u_{,\alpha}^I)\mathbf{I}_I \text{ and } {}^{\mathrm{N}}\mathbf{M}_\alpha \equiv \frac{1}{dX^\alpha} d\mathbf{r}_\alpha \times \mathbf{N}_\alpha. \tag{6.34}$$

The components of the internal forces in the X^I-direction are

$$\begin{aligned}
N_\alpha^I &= N_\alpha \mathbf{n}_\alpha \cdot \mathbf{I}_I + N_{\alpha\beta} \mathbf{n}_\beta \cdot \mathbf{I}_I + Q_\alpha \mathbf{n}_3 \cdot \mathbf{I}_I \\
&= N_\alpha \cos\theta_{(\mathbf{n}_\alpha, \mathbf{I}_I)} + N_{\alpha\beta} \cos\theta_{(\mathbf{n}_\beta, \mathbf{I}_I)} + Q_\alpha \cos\theta_{(\mathbf{n}_3, \mathbf{I}_I)} \\
&= \frac{N_\alpha(\delta_\alpha^I + u_{0,\alpha}^I)}{1+\varepsilon_\alpha^{(0)}} + \frac{N_{\alpha\beta}(\delta_\beta^I + u_{0,\beta}^I)}{1+\varepsilon_\beta^{(0)}} + \frac{Q_\alpha(\delta_3^I + \varphi_1^{(I)})}{1+\varepsilon_3^{(0)}},
\end{aligned} \tag{6.35}$$

$$\begin{aligned}
M_\alpha^I &= (-1)^\beta M_\alpha \mathbf{n}_\beta \cdot \mathbf{I}_I + (-1)^\alpha M_{\alpha\beta} \mathbf{n}_\alpha \cdot \mathbf{I}_I \\
&= (-1)^\beta M_\alpha \cos\theta_{(\mathbf{n}_\beta, \mathbf{I}_I)} + (-1)^\alpha M_{\alpha\beta} \cos\theta_{(\mathbf{n}_\alpha, \mathbf{I}_I)} \\
&= (-1)^\beta \frac{M_\alpha(\delta_\beta^I + u_{0,\beta}^I)}{1+\varepsilon_\beta^{(0)}} + (-1)^\alpha \frac{M_{\alpha\beta}(\delta_\alpha^I + u_{0,\alpha}^I)}{1+\varepsilon_\alpha^{(0)}},
\end{aligned} \tag{6.36}$$

$$\begin{aligned}
{}^{\mathrm{N}}M_\alpha^I &= (\mathbf{g}_\alpha \times \mathcal{N}_\alpha) \cdot \mathbf{I}_I = (\delta_\alpha^J + u_{0,\alpha}^J) N_\alpha^K - (X_{,\alpha}^K + u_{0,\alpha}^K) N_\alpha^J \\
&= (\delta_\alpha^J + u_{0,\alpha}^J) \left[\frac{N_{\alpha\beta}(\delta_\beta^K + u_{0,\beta}^K)}{1+\varepsilon_\beta^{(0)}} + \frac{Q_\alpha(\delta_3^K + \varphi_1^{(K)})}{1+\varepsilon_3^{(0)}} \right] \\
&\quad - (\delta_\alpha^K + u_{0,\alpha}^K) \left[\frac{N_{\alpha\beta}(\delta_\beta^J + u_{0,\beta}^J)}{1+\varepsilon_\beta^{(0)}} + \frac{Q_\alpha(\delta_3^J + \varphi_1^{(J)})}{1+\varepsilon_3^{(0)}} \right]
\end{aligned} \tag{6.37}$$

for $(I, J, K \in \{1, 2, 3\},\ I \neq J \neq K \neq I)$ and the indices (I, J, K) rotate clockwise. Based on the deformed middle surface in the Lagrangian coordinates, the equations of motion for the deformed plates are

$$\begin{aligned}
\mathcal{N}_{\alpha,\alpha} + \mathbf{q} &= \rho \mathbf{u}_{0,tt} + I_3 \boldsymbol{\varphi}_{1,tt}, \\
\mathbf{M}_{\alpha,\alpha} + {}^{\mathrm{N}}\mathbf{M}_\alpha + \mathbf{m} &= I_3 \mathbf{u}_{0,tt} + J_3 \boldsymbol{\varphi}_{1,tt},
\end{aligned} \tag{6.38}$$

or for $I = 1, 2, 3$ and with summation on $\alpha = 1, 2$,

$$
\begin{aligned}
N^I_{\alpha,\alpha} + q^I &= \rho u^I_{0,tt} + I_3 \varphi^{(I)}_{1,tt}, \\
M^I_{\alpha,\alpha} + {}^{\mathrm{N}}M^I_{\alpha} + m^I &= I_3 u^I_{0,tt} + J_3 \varphi^I_{(1),tt},
\end{aligned}
\tag{6.39}
$$

where summation on $\alpha = 1, 2$ should be completed, and $I_3 = \int_{-h^-}^{h^+} \rho_0 z dz, \quad J_3 = \int_{-h^-}^{h^+} \rho_0 z^2 dz$. If $h^+ = h^- = h/2$, then $I_3 = 0$ and $J_3 = \rho_0 h^3/12$. The equation of motion in Eq.(6.38) can be written in a form of

$$
\begin{aligned}
&\left[\frac{N_{11}(\delta^I_1 + u^I_{0,1})}{1+\varepsilon^{(0)}_1} + \frac{N_{12}(\delta^I_2 + u^I_{0,2})}{1+\varepsilon^{(0)}_2} + \frac{Q_1(\delta^I_3 + \varphi^{(I)}_1)}{1+\varepsilon^{(0)}_3}\right]_{,1} \\
&+\left[\frac{N_{21}(\delta^I_1 + u^I_{0,1})}{1+\varepsilon^{(0)}_1} + \frac{N_{22}(\delta^I_2 + u^I_{0,2})}{1+\varepsilon^{(0)}_2} + \frac{Q_2(\delta^I_3 + \varphi^{(I)}_1)}{1+\varepsilon^{(0)}_3}\right]_{,2} + q^I \\
&= \rho u^I_{0,tt} + I_3 \varphi^{(I)}_{1,tt},
\end{aligned}
\tag{6.40}
$$

$$
\begin{aligned}
&\left[\frac{M_1(\delta^I_2 + u^I_{0,2})}{1+\varepsilon^{(0)}_2} - \frac{M_{12}(\delta^I_1 + u^I_{0,1})}{1+\varepsilon^{(0)}_1}\right]_{,1} \\
&+\left[\frac{M_{12}(\delta^I_2 + u^I_{0,2})}{1+\varepsilon^{(0)}_2} - \frac{M_2(\delta^I_1 + u^I_{0,1})}{1+\varepsilon^{(0)}_1}\right]_{,2} \\
&+ (\delta^J_1 + u^J_{0,1})\left[\frac{N_{12}(\delta^K_2 + u^K_{0,2})}{1+\varepsilon^{(0)}_2} + \frac{Q_1(\delta^K_3 + \varphi^{(K)}_1)}{1+\varepsilon^{(0)}_3}\right] \\
&- (\delta^K_1 + u^K_{0,1})\left[\frac{N_{21}(\delta^J_2 + u^J_{0,2})}{1+\varepsilon^{(0)}_2} + \frac{Q_1(\delta^J_3 + \varphi^{(J)}_1)}{1+\varepsilon^{(0)}_3}\right] \\
&+ (\delta^J_2 + u^J_{0,2})\left[\frac{N_{12}(\delta^K_1 + u^K_{0,1})}{1+\varepsilon^{(0)}_1} + \frac{Q_2(\delta^K_3 + \varphi^{(K)}_1)}{1+\varepsilon^{(0)}_3}\right] \\
&- (\delta^K_2 + u^K_{0,2})\left[\frac{N_{21}(\delta^J_1 + u^J_{0,1})}{1+\varepsilon^{(0)}_1} + \frac{Q_2(X^J_{,3} + \varphi^{(J)}_1)}{1+\varepsilon^{(0)}_3}\right] + m^I \\
&= I_3 u^I_{0,tt} + J_3 \varphi^{(I)}_{1,tt}.
\end{aligned}
\tag{6.41}
$$

The equations of motion for thin plates are given in Eq.(6.38) or Eq.(6.39). They together with Eqs.(6.25)–(6.28) constitute an approximate nonlinear theory for thin plates. The equations of motion are based on the deformed middle surface in the Lagrangian coordinates. The alternative approach presented in Wempner (1973) can derive the similar equilibrium equations, and also equations (6.33)–(6.41) can be derived through Eq.(6.32) and Boussinesq-Kirchhoff equations in Eq.(3.201) (e.g., Guo, 1980). The other theories of plates and shells can be referenced as in Chien (1944a,b).

6.1.4. Reduction to established theories

In this section, several existing plate theories will be presented from the aforesaid nonlinear plate theory under certain assumptions.

6.1.4a Kirchhoff plate theory

Kirchhoff's assumptions specify $\varepsilon_3 = \gamma_{\alpha 3} = 0$. From Eqs.(6.20) and (6.21), these constraints for $(\alpha = 1, 2)$ become

$$
\begin{aligned}
(\delta_3^I + u_{,3}^I)(\delta_3^I + u_{,3}^I) &= 1, \\
(\delta_\alpha^I + u_{,\alpha}^I)(\delta_3^I + u_{,3}^I) &= 0.
\end{aligned} \tag{6.42}
$$

Substitution of Eq.(6.22) into Eq.(6.42), expansion of them in Taylor series in z and the vanishing of the zero-order terms in z gives

$$
\begin{aligned}
(\delta_3^I + \varphi_1^{(I)})(\delta_3^I + \varphi_1^{(I)}) &= 1, \\
(\delta_\alpha^I + u_{0,\alpha}^I)(\delta_3^I + \varphi_1^{(I)}) &= 0.
\end{aligned} \tag{6.43}
$$

Further, $\varphi_1^{(I)}$ can be obtained from Eqs.(6.43) first. From the Taylor series, vanishing of the first order terms in z gives three equations in $\varphi_2^{(I)}$ similar to Eqs.(6.43). The three equations plus $\varphi_1^{(I)}$ give $\varphi_2^{(I)}$; $\varphi_n^{(I)}$ for $n = 3, 4, \ldots$ can be determined in a like manner. With the zero-order approximation in Eqs.(6.42), the solution to Eqs.(6.43) for $\{1 \equiv x,\ 2 \equiv y,\ 3 \equiv z\}$ and $\{u^1 \equiv u,\ u^2 \equiv v,\ u^3 \equiv w\}$ with $\{u_0^1 \equiv u^0$, $u_0^2 \equiv v^0,\ u_0^3 \equiv w^0\}$ is

$$
\delta_3^I + \varphi_1^{(I)} = \pm \frac{\Delta_I}{\Delta} \quad (I = 1, 2, 3), \tag{6.44}
$$

where

$$
\begin{aligned}
\Delta_1 &= v_{,x}^0 w_{,y}^0 - (1 + v_{,y}^0) w_{,x}^0, \\
\Delta_2 &= u_{,y}^0 w_{,x}^0 - (1 + u_{,x}^0) w_{,y}^0, \\
\Delta_3 &= (1 + u_{,x}^0)(1 + v_{,y}^0) - v_{,x}^0 u_{,y}^0, \\
\Delta &= \sqrt{\Delta_1^2 + \Delta_2^2 + \Delta_3^2}.
\end{aligned} \tag{6.45}
$$

In application, only the positive (+) in Eq.(6.44) is used. With ignoring $\varphi_2^{(I)}$, substitution of Eqs.(6.43)−(6.45) into Eqs.(6.25) and (6.27) gives

$$\begin{aligned}\varepsilon_\alpha &\approx \varepsilon_\alpha^{(0)} + \left.\frac{\partial \varepsilon_\alpha}{\partial z}\right|_{z=0} z + \frac{1}{2!}\left.\frac{\partial^2 \varepsilon_\alpha}{\partial z^2}\right|_{z=0} z^2 + \cdots \\ &= \varepsilon_\alpha^{(0)} + \frac{(\delta_\alpha^I + u_{0,\alpha}^I)\varphi_{1,\alpha}^{(I)}}{1+\varepsilon_\alpha^{(0)}} z \\ &\quad + \frac{1}{2}\left\{\frac{\varphi_{1,\alpha}^{(I)}\varphi_{1,\alpha}^{(I)}}{1+\varepsilon_\alpha^{(0)}} - \frac{[(\delta_\alpha^I + u_{0,\alpha}^I)\varphi_{1,\alpha}^{(I)}]^2}{(1+\varepsilon_\alpha^{(0)})^3}\right\} z^2 + \cdots,\end{aligned} \tag{6.46}$$

$$\begin{aligned}\gamma_{12} &\approx \gamma_{12}^{(0)} + \left.\frac{\partial \gamma_{12}}{\partial z}\right|_{z=0} z + \frac{1}{2!}\left.\frac{\partial^2 \gamma_{12}}{\partial z^2}\right|_{z=0} z^2 + \cdots \\ &= \gamma_{12}^{(0)} + \frac{1}{\cos\gamma_{12}^{(0)}}\left\{\frac{(\delta_1^I + u_{0,1}^I)\varphi_{1,2}^{(I)} + (\delta_2^I + u_{0,2}^I)\varphi_{1,1}^{(I)}}{(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})}\right. \\ &\quad \left. - \sin\gamma_{12}^{(0)}\left[\frac{(\delta_1^I + u_{0,1}^I)\varphi_{1,1}^{(I)}}{(1+\varepsilon_1^{(0)})^2} + \frac{(\delta_2^I + u_{0,2}^I)\varphi_{1,2}^{(I)}}{(1+\varepsilon_2^{(0)})^2}\right]\right\} z + \cdots,\end{aligned} \tag{6.47}$$

Substitution of Eq.(6.42) into Eqs.(6.33)–(6.41) yields the equilibrium balance for Kirchhoff's plates.

6.1.4b Thin plates with moderately large deflections

For moderately large transverse deflection, assumptions for the middle surface are for $\alpha = 1, 2$,

$$u_{0,\alpha}^1 \approx O(u_{0,\alpha}^2) \approx O[(w_{,\alpha}^0)^2] << 1,\ 1+\varepsilon_\alpha^{(0)} \approx 1. \tag{6.48}$$

Therefore, the strains of the middle surface become

$$\begin{aligned}&\varepsilon_\alpha^{(0)} \approx u_{0,\alpha}^\alpha + \frac{1}{2}(w_{,\alpha}^0)^2 \quad \text{for } \alpha = 1, 2; \\ &\gamma_{12}^0 \approx u_{0,1}^2 + u_{0,2}^2 + w_{,1}^0 w_{,2}^0.\end{aligned} \tag{6.49}$$

Substitution of Eqs.(6.48) and (6.49) into Eqs.(6.44) and (6.45) generates

$$\varphi_1^{(\alpha)} \approx -w_{,\alpha}^0 \text{ and } \varphi_1^{(3)} \approx 0. \tag{6.50}$$

Substitution of Eqs.(6.48)–(6.50) into Eqs.(6.30), (6.46) and (6.47), and retention of terms that are the first order in *z*, leads to

$$\begin{aligned}&\varepsilon_\alpha \approx u_{0,\alpha}^\alpha + \frac{1}{2}(w_{,\alpha}^0)^2 - w_{,\alpha\alpha}^0 z \quad \text{for } \alpha = 1, 2; \\ &\gamma_{12} \approx u_{0,1}^2 + u_{0,2}^1 + w_{,1}^0 w_{,2}^0 - 2w_{,12}^0 z.\end{aligned} \tag{6.51}$$

From Eqs.(6.40), (6.49) and (6.51), the equilibrium balance in Eq.(6.39) gives for $\alpha = 1, 2$,

$$\begin{aligned} \left[N_{\alpha\beta} - Q_\beta w^0_{,\alpha}\right]_{,\beta} + q^\alpha &= \rho u^\alpha_{0,tt} - I_3 w^0_{,\alpha tt}, \\ \left[N_{\alpha\beta} w^0_{,\beta} + Q_\alpha\right]_{,\alpha} + q^3 &\approx \rho w^0_{,tt}; \\ M_{\alpha\beta,\beta} + N_{12} w^0_{,\alpha+1} - Q_\alpha + m^\alpha &\approx I_3 u^\alpha_{0,tt} - J_3 w^0_{,\alpha tt}, \end{aligned} \tag{6.52}$$

with summation on $\beta = 1,2$. The density $\rho = \rho_0 h$ is constant. For $h^+ = h^- = h/2$, one obtains $I_3 = 0$ and $J_3 = \rho_0 h^3/12$. When the rotary inertia is neglected, the third equation of Eq.(6.52) becomes

$$M_{\alpha\beta,\beta} + N_{12} w^0_{,\alpha+1} - Q_\alpha + m^\alpha \approx 0, \tag{6.53}$$

When $m^\alpha = q^\alpha = 0$, the shear Q_α in the first equation of Eq.(6.52) and N_{12} in Eq.(6.53) vanish, and the first of two equations of Eq.(6.49) plus (6.53) at $h^+ = h^- = h/2$ reduce to the nonlinear plate theory of Herrmann (1955).

The von Karman plate theory (von Karman, 1910) is recovered by letting $m^\alpha = q^\alpha = 0$ and neglecting the terms $w^0_{,\alpha} << 1$ in Eqs.(6.52) and (6.53), i.e., with summation on $\beta = 1,2$, one obtains

$$N_{\alpha\beta,\beta} \approx 0 \text{ and } M_{\alpha\beta,\beta} - Q_\alpha \approx 0 \quad (\alpha = 1, 2). \tag{6.54}$$

Substitution of Eqs.(6.53) and (6.54) into the third equation Eq.(6.52) gives

$$N_{\alpha\beta} w^0_{,\alpha\beta} + M_{\alpha\beta,\alpha\beta} + q_3 = \rho w^0_{,tt}. \tag{6.55}$$

The von Karman theory is applicable to plates of the moderately large deflection and small rotation, while, the theory in Eqs.(6.49)−(6.52) is applicable to plates of moderately large deflection and rotation because of Eq.(6.48).

6.1.4c Linear plate theory

The linear theory for thin plates is recovered from Eqs.(6.25)−(6.28) and (6.39), when the Kirchhoff constraints are imposed: the elongation and shear in the plate are small compared to unity; the rotations are negligible compared to the elongation and shear, i.e.,

$$u^1_{0,\alpha} \approx O(u^2_{0,\alpha}) \approx O(w^{(0)}_{,\alpha}) << 1 \text{ and } 1 + \varepsilon^{(0)}_\alpha \approx 1. \tag{6.56}$$

From the foregoing, the strains of the middle surface become

$$\varepsilon^{(0)}_\alpha \approx u^\alpha_{0,\alpha} \text{ and } \gamma^{(0)}_{12} \approx u^2_{0,1} + u^1_{0,2}. \tag{6.57}$$

Substitution of Eqs.(6.56) and (6.57) into Eqs.(6.43)-(6.45) generates the rotation angles given by Eq.(6.50). With Eqs.(6.50), (6.56) and (6.57), equations (6.39) or (6.40) and (6.41) reduce to the linear plate theory (e.g., Reissener, 1944, 1957)

$$N_{\alpha\beta,\beta} + q^{\alpha} \approx \rho u^{\alpha}_{0,tt} \text{ and } M_{\alpha\beta,\alpha\beta} + q_3 \approx \rho w^{0}_{,tt}. \tag{6.58}$$

The shear forces are given by Eq.(6.54).

6.2. Waves in traveling plates

In this section, the nonlinear waves in traveling plates will be presented. The static waves in the traveling plates will be discussed first. The resonant waves and stationary waves will be investigated. Finally, chaotic waves in the forced traveling plates will be discussed.

6.2.1. An approximate theory

As shown in Fig.6.3, a thin rectangular plate simply supported along the four edge moves axially with a constant speed c and are subject to a distributed surface loading $Q(x, y)$ and a longitudinal force $P(y)$ applied at the edges. The length, width and thickness of the moving plate are l, b and h, respectively. Transformation of the coordinate system (x, y, t) traveling with the plate to a fixed coordinate system (ς, y, t) needs

$$\varsigma = ct + x, \tag{6.59}$$

where t denotes time. The geometric constraints for moderately large deflection plates in Eq.(6.45) become

$$u^{0}_{,\varsigma} \approx O(v^{0}_{,\varsigma}) \approx O[(w^{0}_{,\varsigma})^2] << 1,\ 1+\varepsilon^{(0)}_{\varsigma} \approx 1;$$

$$u^{0}_{,y} \approx O(v^{0}_{,y}) \approx O[(w^{0}_{,y})^2] << 1,\ 1+\varepsilon^{(0)}_{y} \approx 1, \tag{6.60}$$

where $(\cdot)_{,\varsigma}$ and $(\cdot)_{,y}$ are derivatives with respect to ς and y. u^0, v^0 and w^0 are displacements of the middle surface in the ς-, y- and z-directions, respectively; and $\varepsilon^0_{\varsigma}, \varepsilon^0_{y}$ denote the corresponding strains in the middle surface. Owing to Eq.(6.60), the strains in Eq.(6.46) and (6.47) are

$$\begin{aligned} \varepsilon_{\varsigma} &\approx \varepsilon^{(0)}_{\varsigma} - w^{0}_{,\varsigma\varsigma} z, \\ \varepsilon_{y} &\approx \varepsilon^{(0)}_{y} - w^{0}_{,yy} z, \\ \gamma_{\varsigma y} &\approx \gamma^{(0)}_{\varsigma y} - 2w^{0}_{,\varsigma y} z, \end{aligned} \tag{6.61}$$

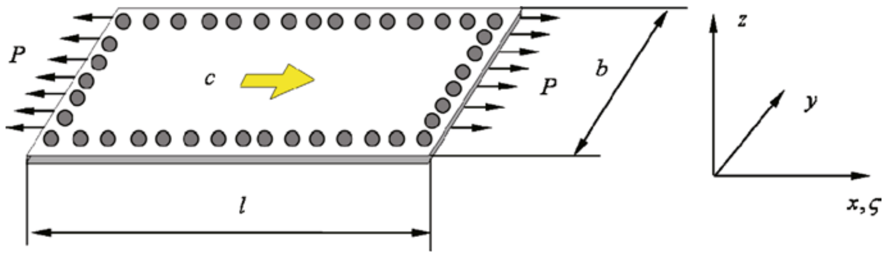


Fig. 6.3 A simply supported plate traveling with speed c in the x-direction.

where the strains of the middle surface are

$$\begin{aligned}
\varepsilon_{\varsigma}^{(0)} &\approx u_{,\varsigma}^{0} + \frac{1}{2}(w_{,\varsigma}^{0})^{2}, \\
\varepsilon_{y}^{(0)} &\approx v_{,y}^{0} + \frac{1}{2}(w_{,y}^{0})^{2}, \\
\gamma_{\varsigma y}^{(0)} &\approx v_{,\varsigma}^{0} + u_{,y}^{0} + w_{,\varsigma}^{0} w_{,y}^{0}.
\end{aligned} \tag{6.62}$$

From Eqs.(6.52) and (6.60), equations of motion for axially moving plates are:

$$[N_{\varsigma} - Q_{\varsigma} w_{,\varsigma}^{0}]_{,\varsigma} + [N_{\varsigma y} - Q_{y} w_{,\varsigma}^{0}]_{,y} \approx \rho_{0} h(\ddot{u}^{0} + 2c\dot{u}_{,\varsigma}^{0} + c^{2} u_{,\varsigma\varsigma}^{0}), \tag{6.63}$$

$$[N_{\varsigma y} - Q_{\varsigma} w_{,y}^{0}]_{,\varsigma} + [N_{y} - Q_{y} w_{,y}^{0}]_{,y} \approx \rho_{0} h(\ddot{v}^{0} + 2c\dot{v}_{,\varsigma}^{0} + c^{2} v_{,\varsigma\varsigma}^{0}), \tag{6.64}$$

$$\begin{aligned}
&[N_{\varsigma} w_{,\varsigma}^{0} + N_{\varsigma y} w_{,y}^{0} + Q_{\varsigma}]_{,\varsigma} + [N_{\varsigma y} w_{,\varsigma}^{0} + N_{y} w_{,y}^{0} + Q_{y}]_{,y} \\
&+ Q(\varsigma, y, t) \approx \rho_{0} h(\ddot{w}^{0} + 2c\dot{w}_{,\varsigma}^{0} + c^{2} w_{,\varsigma\varsigma}^{0});
\end{aligned} \tag{6.65}$$

$$M_{\varsigma,\varsigma} + M_{\varsigma y,y} + N_{\varsigma y} w_{,y}^{0} - Q_{\varsigma} \approx 0, \tag{6.66}$$

$$M_{\varsigma y,\varsigma} + M_{y,y} + N_{\varsigma y} w_{,\varsigma}^{0} - Q_{y} \approx 0, \tag{6.67}$$

where superscript "dot" (e.g., $\dot{w} \equiv dw/dt$) denotes the derivative with respect time t, and ρ_0 is the density. The membrane forces $\{N_{\varsigma}, N_{y}, N_{\varsigma y}\}$, bending moments and twisting moment $\{M_{\varsigma}, M_{y}, M_{\varsigma y}\}$ are

$$\begin{aligned}
N_{\varsigma} &= P(y) + \frac{Eh}{1-\mu^{2}}[(u_{,\varsigma}^{0} + \frac{1}{2}(w_{,\varsigma}^{0})^{2}) + \mu(v_{,y}^{0} + \frac{1}{2}(w_{,y}^{0})^{2})], \\
N_{y} &= \frac{Eh}{1-\mu^{2}}[(v_{,y}^{0} + \frac{1}{2}(w_{,y}^{0})^{2}) + \mu(u_{,\varsigma}^{0} + \frac{1}{2}(w_{,\varsigma}^{0})^{2})], \\
N_{\varsigma y} &= \frac{Eh}{1-\mu^{2}}(v_{,\varsigma}^{0} + u_{,y}^{0} + w_{,\varsigma}^{0} w_{,y}^{0});
\end{aligned} \tag{6.68}$$

$$\begin{aligned} M_x &= -D(w^0_{,\varsigma\varsigma} + \mu w^0_{,yy}), \\ M_y &= -D(w^0_{,yy} + \mu w^0_{,\varsigma\varsigma}), \\ M_{\varsigma y} &= M_{y\varsigma} = -D(1-\mu) w^0_{,\varsigma y}, \end{aligned} \tag{6.69}$$

where E and μ are Young's modulus and Poisson ratio, respectively. The flexural rigidity is $D = Eh^3/12(1-\mu^2)$. Substitution of Q_ς and Q_y with Eqs.(6.68) and (6.69) into Eqs.(6.63)–(6.65) gives

$$\begin{aligned} & u^0_{,\varsigma\varsigma} + \frac{1}{2}(1-\mu)u^0_{,yy} + \frac{1}{2}(1+\mu)v^0_{,\varsigma y} \\ & + w^0_{,\varsigma}[w^0_{,\varsigma\varsigma} + \frac{1}{2}(1-\mu)w^0_{,yy}] + \frac{1}{2}(1+\mu)w^0_{,\varsigma y}w^0_{,y} \\ & + \frac{1}{12}h^2\{[(\nabla^2 w^0)_{,\varsigma} w^0_{,\varsigma}]_{,\varsigma} + [(\nabla^2 w^0)_{,y} w^0_{,\varsigma}]_{,y}\} \\ & \approx \frac{\rho_0(1-\mu^2)}{E}(\ddot{u}^0 + 2c\dot{u}^0_{,\varsigma} + c^2 u^0_{,\varsigma\varsigma}), \end{aligned} \tag{6.70}$$

$$\begin{aligned} & v^0_{,yy} + \frac{1}{2}(1-\mu)v^0_{,\varsigma\varsigma} + \frac{1}{2}(1+\mu)u^0_{,\varsigma y} \\ & + w^0_{,y}[w^0_{,yy} + \frac{1}{2}(1-\mu)w^0_{,\varsigma\varsigma}] + \frac{1}{2}(1-\mu)w^0_{,\varsigma y}w^0_{,\varsigma} \\ & + \frac{1}{12}h^2\{[(\nabla^2 w^0)_{,y} w^0_{,y}]_{,y} + [(\nabla^2 w^0)_{,\varsigma} w^0_{,y}]_{,\varsigma}\} \\ & \approx \frac{\rho_0(1-\mu^2)}{E}(\ddot{v}^0 + 2c\dot{v}^0_{,\varsigma} + c^2 v^0_{,\varsigma\varsigma}), \end{aligned} \tag{6.71}$$

$$\begin{aligned} & \left[N_\varsigma w^0_{,\varsigma} + 2N_{\varsigma y} w^0_{,y}\right]_{,\varsigma} + \left[N_y w^0_{,y} + 2N_{\varsigma y} w^0_{,\varsigma}\right]_{,y} - D\nabla^4 w^0 + Q(\varsigma, y, t) \\ & \approx \rho_0 h(\ddot{w}^0 + 2c\dot{w}^0_{,\varsigma} + c^2 w^0_{,\varsigma\varsigma}), \end{aligned} \tag{6.72}$$

where ∇^2 and ∇^4 are the two-dimensional Laplace and biharmonic operators, respectively.

The displacement boundary conditions are

$$\begin{aligned} & u^0(0,y,t) = u^0(l,y,t) = w^0(0,y,t) = w^0(l,y,t) = 0, \\ & w^0_{,\varsigma\varsigma}(0,y,t) = w^0_{,\varsigma\varsigma}(l,y,t) = 0; \\ & v^0(\varsigma,0,t) = v^0(\varsigma,b,t) = w^0(\varsigma,0,t) = w^0(\varsigma,b,t) = 0, \\ & w^0_{,yy}(\varsigma,0,t) = w^0_{,yy}(\varsigma,b,t) = 0. \end{aligned} \tag{6.73}$$

The force boundary conditions are

$$\int_0^b N_\varsigma dy = \int_0^b P(y,t)dy \quad \text{at } \varsigma=0,\, l;$$
$$\int_0^l N_y d\varsigma = 0 \quad \text{at } y=0,\, b. \tag{6.74}$$

The specified initial conditions are

$$\begin{aligned} u^0(\varsigma,y,\tau)\,|_{\tau=0} &= \varphi_1^0(\varsigma,y), \quad \dot{u}^0(\varsigma,y,\tau)\,|_{\tau=0} = \psi_1^0(\varsigma,y), \\ v^0(\varsigma,y,\tau)\,|_{\tau=0} &= \varphi_2^0(\varsigma,y), \quad \dot{v}^0(\varsigma,y,\tau)\,|_{\tau=0} = \psi_2^0(\varsigma,y), \\ w^0(\varsigma,y,\tau)\,|_{\tau=0} &= \varphi_3^0(\varsigma,y), \quad \dot{w}^0(\varsigma,y,\tau)\,|_{\tau=0} = \psi_3^0(\varsigma,y). \end{aligned} \tag{6.75}$$

Following Chu and Herrmann (1956), non-dimensional variables are

$$X = \frac{\varsigma}{b},\; Y = \frac{y}{b},\; u = \frac{u^0}{b},\; v = \frac{v^0}{b},\; w = \frac{w^0}{b};\; \varepsilon = \frac{h}{b}, \tag{6.76}$$

where $b \le l$. Other new notations are

$$\begin{aligned} &r = \frac{b}{l},\; t = \frac{\varepsilon c_p \tau}{\sqrt{12}b};\; c_1 = \frac{\sqrt{12}c}{c_p \varepsilon},\; c_p^2 = \frac{E}{\rho_0(1-\mu^2)};\; P^* = \frac{12P(y,t)}{c_p^2 \rho_0 h \varepsilon^2}, \\ &q^* = \frac{bQ(\varsigma,y,t)}{c_p^2 \rho_0 h \varepsilon^3},\; \varphi_i = \frac{\varphi_i^0}{b},\; \psi_i = \frac{\sqrt{12}\psi_i^0}{\varepsilon c_p}\; (i=1,2,3). \end{aligned} \tag{6.77}$$

With transformation to these variables, the equations of motion in Eqs.(6.70)–(6.72) become

$$\begin{aligned} &u_{,XX} + \frac{1}{2}(1-\mu)u_{,YY} + \frac{1}{2}(1+\mu)v_{,XY} + w_{,X}[w_{,XX} + \frac{1}{2}(1-\mu)w_{,YY}] \\ &+ \frac{1}{2}(1+\mu)w_{,XY}w_{,Y} + \frac{\varepsilon^2}{12}\{[(\nabla^2 w)_{,X} w_{,X}]_{,X} + [(\nabla^2 w)_{,Y} w_{,X}]_{,Y}\} \\ &\approx \frac{\varepsilon^2}{12}(\ddot{u} + 2c_1\dot{u}_{,X} + c_1^2 u_{,XX}), \end{aligned} \tag{6.78}$$

$$\begin{aligned} &v_{,YY} + \frac{1}{2}(1-\mu)v_{,XX} + \frac{1}{2}(1+\mu)u_{,XY} + w_{,Y}[w_{,YY} + \frac{1}{2}(1-\mu)w_{,XX}] \\ &+ \frac{1}{2}(1+\mu)w_{,YX}w_{,XX} + \frac{\varepsilon^2}{12}\{[(\nabla^2 w)_{,YY} w_{,Y}]_{,Y} + [(\nabla^2 w)_{,X} w_{,Y}]_{,X}\} \\ &\approx \frac{\varepsilon^2}{12}(\ddot{v} + 2c_1\dot{v}_{,X} + c_1^2 v_{,XX}), \end{aligned} \tag{6.79}$$

$$\begin{aligned} &[N_X w_{,X} + 2N_{XY} w_{,Y}]_{,X} + [N_Y w_{,Y} + 2N_{XY} w_{,X}]_{,Y} \\ &+ \varepsilon^3 q^* \approx \frac{\varepsilon^2}{12}\nabla^4 w + \frac{\varepsilon^2}{12}(\ddot{w} + 2c_1\dot{w}_{,X} + c_1^2 w_{,XX}). \end{aligned} \tag{6.80}$$

Accordingly, the membrane forces are

$$N_X \equiv \frac{N_\varsigma}{c_p^2 \rho_0 h} = \frac{\varepsilon^2}{12} P^* + [u_{,X} + \frac{1}{2}(w_{,X})^2] + \mu[v_{,Y} + \frac{1}{2}(w_{,Y})^2],$$

$$N_Y \equiv \frac{N_y}{c_p^2 \rho_0 h} = [v_{,Y} + \frac{1}{2}(w_{,Y})^2] + \mu[u_{,X} + \frac{1}{2}(w_{,X})^2], \tag{6.81}$$

$$N_{XY} \equiv \frac{N_{\varsigma y}}{c_p^2 \rho_0 h} = \frac{1}{2}(1-\mu)(v_{,X} + u_{,Y} + w_{,X} w_{,Y}).$$

The boundary conditions in Eqs.(6.73) and (6.74) are

$$\begin{aligned} &u(0,Y,t) = u(1/r,Y,t) = 0; \\ &w(0,Y,t) = w(1/r,Y,t) = w_{,XX}(0,Y,t) = w_{,XX}(1/r,Y,t) = 0; \\ &v(X,0,t) = v(X,1,t) = 0; \\ &w(X,0,t) = w(X,1,t) = w_{,YY}(X,0,t) = w_{,YY}(X,1,t) = 0; \\ &\int_0^1 N_X dY = \frac{\varepsilon^2}{12}\int_0^1 P^*(Y,t) dY \quad \text{at } X\text{=0, } 1/r; \\ &\int_0^{1/r} N_Y dX = 0 \quad \text{at } Y\text{=0, 1}. \end{aligned} \tag{6.82}$$

The initial conditions from Eqs.(6.75)-(6.77) are

$$\begin{aligned} &u(X,Y,t)|_{t=0} = \varphi_1(X,Y),\ \dot{u}(X,Y,t)|_{t=0} = \psi_1(X,Y); \\ &v(X,Y,t)|_{t=0} = \varphi_2(X,Y),\ \dot{v}(X,Y,t)|_{t=0} = \psi_2(X,Y); \\ &w(X,Y,t)|_{t=0} = \varphi_3(X,Y),\ \dot{w}(X,Y,t)|_{t=0} = \psi_3(X,Y). \end{aligned} \tag{6.83}$$

From the above non-dimensionalized equations, the waves in traveling plates can be obtained. However, it is very difficult to get closed-form solutions. The perturbation analysis will be adopted herein which may not be proper for chaotic waves, but it can be as an approximate estimate.

6.2.2. Perturbation analysis

To balance the $\varepsilon \ll 1$ in Eqs.(6.78)–(6.80), the perturbation expansions of the displacement from Eq.(6.60) are

$$\begin{aligned} w &= \varepsilon w_1 + \varepsilon^3 w_3 + \cdots, \\ u &= \varepsilon^2 u_2 + \varepsilon^4 u_4 + \cdots, \\ v &= \varepsilon^2 v_2 + \varepsilon^4 v_4 + \cdots. \end{aligned} \tag{6.84}$$

Without loss of generality, set $w_3 = w_5 = \cdots = 0$ to avoid the tedious work of derivation. The membrane forces are represented by

$$\begin{aligned} N_X &= \varepsilon^2(\frac{1}{12}P^* + N_{2X}) + \varepsilon^4 N_{4Y} + \cdots, \\ N_Y &= \varepsilon^2 N_{2Y} + \varepsilon^4 N_{4Y} + \cdots, \\ N_{XY} &= \varepsilon^2 N_{2XY} + \varepsilon^4 N_{4XY} + \cdots, \end{aligned} \tag{6.85}$$

where

$$\begin{aligned} N_{2X} &= [u_{2,X} + \frac{1}{2}(w_{1,X})^2] + \mu[v_{2,Y} + \frac{1}{2}(w_{1,Y})^2], \\ N_{2Y} &= [v_{2,Y} + \frac{1}{2}(w_{1,Y})^2] + \mu[u_{2,X} + \frac{1}{2}(w_{1,X})^2], \\ N_{2XY} &= \frac{1}{2}(1-\mu)(v_{2,X} + u_{2,Y} + w_{1,X}w_{1,Y}); \end{aligned} \tag{6.86}$$

$$\begin{aligned} N_{(2\sigma)X} &= u_{(2\sigma),X} + \mu v_{(2\sigma),Y}, \\ N_{(2\sigma)Y} &= v_{(2\sigma),Y} + \mu u_{(2\sigma),X}, \\ N_{(2\sigma)XY} &= \frac{1}{2}(1-\mu)(v_{(2\sigma),X} + u_{(2\sigma),Y}), \end{aligned} \tag{6.87}$$

where $\sigma = 2, 3, 4, \ldots$.

Substitution of Eqs.(6.84)–(6.87) into Eqs.(6.79)–(6.81) gives:
for order ε^2:

$$\begin{aligned} &u_{2,XX} + \frac{1}{2}(1-\mu)u_{2,YY} + \frac{1}{2}(1+\mu)v_{2,XY} \\ &+ w_{1,X}[w_{1,XX} + \frac{1}{2}(1-\mu)w_{1,YY}] + \frac{1}{2}(1+\mu)w_{1,XY}w_{1,Y} = 0, \\ &v_{2,YY} + \frac{1}{2}(1-\mu)v_{2,XX} + \frac{1}{2}(1+\mu)u_{2,XY} \\ &+ w_{1,Y}[w_{1,YY} + \frac{1}{2}(1-\mu)w_{1,XX}] + \frac{1}{2}(1+\mu)w_{1,YX}w_{1,XX} = 0; \end{aligned} \tag{6.88}$$

for order ε^3:

$$\begin{aligned} &[(P^* + N_{2X})w_{1,X} + 2N_{2XY}w_{1,Y}]_{,X} + [N_{2Y}w_{1,Y} + 2N_{2XY}w_{1,X}]_{,Y} + q^* \\ &= \frac{1}{12}\nabla^4 w_1 + \frac{1}{12}(\ddot{w}_1 + 2c_1\dot{w}_{1,X} + c_1^2 w_{1,XX}); \end{aligned} \tag{6.89}$$

for order ε^4:

$$\begin{aligned}
&u_{4,XX}+\frac{1}{2}(1-\mu)u_{4,YY}+\frac{1}{2}(1+\mu)v_{4,XY}\\
&-\frac{1}{12}\{[(\nabla^2 w_1)_{,X}w_{1,X}]_{,X}+[(\nabla^2 w_1)_{,Y}w_{1,X}]_{,Y}\}\\
&=\frac{1}{12}(\ddot{u}_2+2c_1\dot{u}_{2,X}+c_1^2u_{2,XX}),\\
&v_{4,YY}+\frac{1}{2}(1-\mu)v_{4,XX}+\frac{1}{2}(1+\mu)u_{4,XY}\\
&-\frac{1}{12}\{[(\nabla^2 w_1)_{,Y}w_{1,Y}]_{,Y}+[(\nabla^2 w_1)_{,X}w_{1,Y}]_{,X}\}\\
&=\frac{1}{12}(\ddot{v}_2+2c_1\dot{v}_{2,X}+c_1^2v_{2,XX});
\end{aligned}\tag{6.90}$$

for order $\varepsilon^{2\sigma}$:

$$\begin{aligned}
&u_{(2\sigma),XX}+\frac{1}{2}(1-\mu)u_{(2\sigma),YY}+\frac{1}{2}(1+\mu)v_{(2\sigma),XY}\\
&=\frac{1}{12}(\ddot{u}_{(2\sigma-1)}+2c_1\dot{u}_{(2\sigma-1),X}+c_1^2u_{(2\sigma-1),XX}),\\
&v_{(2\sigma),YY}+\frac{1}{2}(1-\mu)v_{(2\sigma),XX}+\frac{1}{2}(1+\mu)u_{(2\sigma),XY}\\
&=\frac{1}{12}(\ddot{v}_{(2\sigma-1)}+2c_1\dot{v}_{(2\sigma-1),X}+c_1^2v_{(2\sigma-1),XX});
\end{aligned}\tag{6.91}$$

where $\sigma=3,4,\ldots$. The boundary conditions in Eq.(6.83) become

$$\begin{aligned}
&u_2(0,Y)=u_2(1/r,Y)=0;\\
&w_1(0,Y)=w_1(1/r,Y)=w_{1,XX}(0,Y)=w_{1,XX}(1/r,Y)=0;\\
&v_2(X,0)=v_2(X,1)=0;\\
&w_1(X,0)=w_1(X,1)=w_{1,YY}(X,0)=w_{1,YY}(X,1)=0;\\
&\int_0^1 N_{2X}dY=0\quad \text{at } X=0,1/r;\\
&\int_0^{1/r} N_{1Y}dY=0\quad \text{at } Y=0,1;
\end{aligned}\tag{6.92}$$

and

$$\begin{aligned}
&u_{(2\sigma)}(0,Y)=u_{(2\sigma)}(1/r,Y)=0;\ v_{(2\sigma)}(X,0)=v_{(2\sigma)}(X,1)=0;\\
&\int_0^1 N_{(2\sigma)X}dY=0\quad \text{at } X=0,1/r;\ \int_0^{1/r} N_{(2\sigma)Y}dX=0\quad \text{at } Y=0,1;
\end{aligned}\tag{6.93}$$

where $\sigma=2,3,\ldots$.

Through this perturbation analysis, the initial conditions in the in-plane directions will be dropped. This analysis just provides an effect of inertia on the deflection of plate in the transverse direction. For accurate analysis, the perturbation analysis may not be adequate. As in Levy (1942), a solution for the transverse displacement in Eqs.(6.88)–(6.91) that enforces the displacement boundary conditions is

$$w_1 = \sum_m \sum_n f_{1mn}(t)\sin(m\pi Xr)\sin(n\pi Y). \tag{6.94}$$

Only the single mode (m,n) motion of plate is investigated herein. For the multiple mode solution, the mode-interaction should be investigated owing to the plate nonlinearity. The mode-interaction for the static waves can be referred to Luo and Hamidzadeh (2004). Substitution of Eq.(6.94) into Eq.(6.88)–(6.91) yields

$$\begin{aligned} u_{2,XX} &+ \frac{1}{2}(1-\mu)u_{2,YY} + \frac{1}{2}(1+\mu)v_{2,XY} = \frac{1}{4}(m\pi^3 r)f_{1mn}^2\sin(2m\pi rX) \\ &\times\{(mr)^2 - \mu n^2 - [(mr)^2 + n^2]\cos(2n\pi Y)\}, \end{aligned} \tag{6.95}$$

$$\begin{aligned} v_{2,YY} &+ \frac{1}{2}(1-\mu)v_{2,XX} + \frac{1}{2}(1+\mu)u_{2,XY} = \frac{1}{4}(n\pi^3)f_{1mn}^2\sin(2n\pi Y) \\ &\times\{n^2 - \mu(mr)^2 - [n^2 + (mr)^2]\cos(2m\pi rX)\}. \end{aligned} \tag{6.96}$$

The solutions to Eqs.(6.95) and (6.96) with the boundary conditions in Eq.(6.92) are given by

$$u_2 = \frac{\pi}{16mr}f_{1mn}^2\sin(2m\pi rX)\{(mr)^2[\cos(2n\pi Y)-1] + \mu n^2\}, \tag{6.97}$$

$$v_2 = \frac{\pi}{16n}f_{1mn}^2\sin(2n\pi Y)\{n^2[\cos(2m\pi rX)-1] + \mu(mr)^2\}. \tag{6.98}$$

Substitution of Eqs.(6.94), (6.97) and (6.98) into Eqs.(6.90) leads to

$$\begin{aligned} &u_{4,XX} + \frac{1}{2}(1-\mu)u_{4,YY} + \frac{1}{2}(1+\mu)v_{4,XY} \\ &= \frac{\pi}{96(mr)}[(\dot{f}_{1mn}^2 + f_{1mn}\ddot{f}_{1mn}) - 2c_1^2(m\pi r)^2 f_{1mn}^2] \\ &\times\{(mr)^2[\cos(2n\pi Y)-1] + \mu n^2\}\sin(2m\pi rX) \\ &+ \frac{c_1(mr)\pi^2}{48}f_{1mn}\dot{f}_{1mn}\{(mr)^2[\cos(2n\pi Y)-1] + \mu n^2\}\cos(2m\pi rX) \\ &+ \frac{(mr)\pi^5}{24}[(mr)^2 + n^2]f_{1mn}^2\{(mr)^2\sin(2m\pi rX)[1-\cos(2n\pi Y)] \\ &+ n^2\sin(2m\pi rX)\sin(2n\pi Y)\}, \end{aligned} \tag{6.99}$$

$$v_{4,XX} + \frac{1}{2}(1-\mu)v_{4,YY} + \frac{1}{2}(1+\mu)v_{4,XY}$$

$$
\begin{aligned}
&= \frac{\pi}{96n}[(\dot{f}_{1mn}^2 + f_{1mn}\ddot{f}_{1mn}) - 2c_1^2(m\pi r)^2 f_{1mn}^2] \\
&\quad \times \{n^2[\cos(2mr\pi X) - 1] + \mu(mr)^2\}\sin(2n\pi Y) \\
&\quad - \frac{(mnr)\pi^2}{48} c_1 f_{1mn}\dot{f}_{1mn} \sin(2mr\pi X)\sin(2n\pi Y) \\
&\quad + \frac{n\pi^5}{24}[(mr)^2 + n^2] f_{1mn}^2 \{n^2 \sin(2n\pi Y)[1 - \cos(2m\pi rX)] \\
&\quad + (mr)^2 \sin(2m\pi rX)\sin(2n\pi Y)\}.
\end{aligned} \tag{6.100}
$$

From the boundary conditions in Eq.(6.93), the in-plane displacements u_4, v_4 are

$$
\begin{aligned}
u_4 &= \frac{1}{384\pi}[\ddot{f}_{1mn} f_{1mn} + \dot{f}_{1mn}^2 - 2(c_1 m\pi r)^2 f_{1mn}^2]\Bigg[\frac{(mr)^2 - \mu n^2}{(mr)^3} \\
&\quad \times \sin(2m\pi rX) - \frac{mr}{(mr)^2 + n^2}\sin(2m\pi rX)\cos(2n\pi Y)\Bigg] \\
&\quad + \frac{1}{192}\dot{f}_{1mn} f_{1mn}\Bigg\{\frac{2}{(mr)^2}[(mr)^2 - \mu n^2][\cos(2m\pi rX) - 1] \\
&\quad + \frac{c_1 m^2 r[(1-\mu)(mr)^2 + (3+\mu)n^2]}{(1-\mu)[(mr)^4 + 3n^4 + (3-\mu)n^2(mr)^2]}[\cos(2m\pi rX) - 1]\cos(2n\pi Y)\Bigg\} \\
&\quad + \frac{mr\pi^3}{96}[(mr)^2 + n^2] f_{1mn}^2 [1 - \cos(2n\pi Y)]\sin(2m\pi rX),
\end{aligned} \tag{6.101}
$$

$$
\begin{aligned}
v_4 &= \frac{n^2 - \mu(mr)^2}{384\pi n^3}[\ddot{f}_{1mn} f_{1mn} + \dot{f}_{1mn}^2]\sin(2n\pi Y) \\
&\quad - \frac{n}{384\pi[(mr)^2 + n^2]}[\ddot{f}_{1mn} f_{1mn} + \dot{f}_{1mn}^2 - 2(c_1 m\pi r)^2 f_{1mn}^2] \\
&\quad \times \sin(2n\pi Y)\cos(2m\pi rX) \\
&\quad - \frac{c_1 mnr[(3+\mu)(mr)^2 + (1-\mu)n^2]}{192(1-\mu)[(mr)^4 + 3n^4 + (3-\mu)n^2(mr)^2]}\dot{f}_{1mn} f_{1mn} \\
&\quad \times \sin(2n\pi Y)\sin(2m\pi rX) \\
&\quad + \frac{n\pi^3[n^2 + (mr)^2]}{96} f_{1mn}^2 [1 - \cos(2m\pi rX)]\sin(2n\pi Y).
\end{aligned} \tag{6.102}
$$

Substitution of the displacement solutions into Eqs.(6.86) and (6.87) generates the membrane forces (e.g., Luo and Hamidzadeh, 2004). Furthermore, substitution of both of them into Eq.(6.80) and use of the Galerkin method gives

$$
\ddot{f}_{mn} + \alpha_{mn} f_{mn} + \beta_{mn} f_{mn}^3 + \varepsilon^2 \gamma_{mn}(\ddot{f}_{mn} f_{mn} + \dot{f}_{mn}^2) f_{mn} = q_{mn}^*, \tag{6.103}
$$

where the coefficients are determined by

$$
\begin{aligned}
\alpha_{mn} &= \pi^4[(mr)^2+n^2]^2+\pi^2(P^*-c_1^2)(mr)^2,\\
\beta_{mn} &= \beta_{mn}^{(1)}+\beta_{mn}^{(2)}+\varepsilon^2\beta_{mn}^{(3)},\\
\beta_{mn}^{(1)} &= \frac{3\pi^4}{4}[(mr)^4+n^4](3-\mu^2)+3\mu\pi^2(mr)^2,\\
\beta_{mn}^{(2)} &= -\frac{\pi^4}{16}(\frac{c}{c_p})^2\Big\{[(mr)^4-(\mu n^2)^2]\\
&\quad +\frac{(mr)^2}{2[(mr)^2+n^2]}[(mr)^4+n^4+2\mu(mr)^2n^2]\Big\},\\
\beta_{mn}^{(3)} &= -\frac{\pi^6}{192}[(mr)^2+n^2]\{3[(mr)^4+n^4]-2\mu n^2(mr)^2\},\\
\gamma_{mn} &= \frac{\pi^2}{192}\Bigg\{(mr)^2+n^2-\mu^2(\frac{n^4}{(mr)^2}+\frac{(mr)^4}{n^2})\\
&\quad +\frac{\pi^2}{2[(mr)^2+n^2]}[(mr)^4+n^4+2\mu(mr)^2n^2]\Bigg\},\\
q_{mn}^* &= 192q^*.
\end{aligned}
\tag{6.104}
$$

The subscript "1" has been dropped from now on without loss of generality. Note that the terms $\beta_{mn}^{(1)},\beta_{mn}^{(2)},\beta_{mn}^{(3)}$ and γ_{mn} are related to the nonlinear transverse deflection, the in-plane Coriolis acceleration, the shear forces at the in-plane equilibrium and the in-plane inertia, respectively. The vibration solutions are determined by solving Eq.(6.103). From Eq.(6.103), the in-plane inertia term is of ε^2-term. For thin plates, such terms can be ignored. Such a discussion can be referred to Pasic and Hermann (1983).

6.2.3. Static waves

Let time-relative terms vanish, the *static waves* can be discussed from the above solutions:

$$\alpha_{mn}f_{mn}+\beta_{mn}f_{mn}^3=q_{mn}^*. \tag{6.105}$$

The buckling stability for the mode (m,n) of the axially moving plate is by $\alpha_{mn}=0$. In other words, a critical speed is determined by

$$c_{cr}=\sqrt{\frac{E\pi^2}{12\rho_0(1-\mu^2)(mr)^2}\frac{h^2}{b^2}((mr)^2+n^2)^2+\frac{P}{\rho_0 h}}. \tag{6.106}$$

For $c<c_{cr}$, the traveling plate will not be buckled. For $c>c_{cr}$, the traveling plate will be buckled. If the plate is stationary, i.e., $c = 0$, the buckling stability for compressed plates is recovered,

$$P_{cr} = -\frac{Eh\pi^2}{12(1-\mu^2)(mr)^2}\frac{h^2}{b^2}[(mr)^2 + n^2]^2 < 0 \tag{6.107}$$

being identical to that of Timoshenko (1940). If axially moving plates are very thin, they can be considered to be axially moving flat membranes. Hence, a critical tension of axially moving membranes is

$$P_{cr} = \rho_0 c^2 h > 0. \tag{6.108}$$

For $P < P_{cr} = \rho_0 c^2 h,$ the axially moving membrane wrinkles.

Compared with the existing results, equation (6.105) reduces directly to several specific cases:

(i) *linear theory*

$$\{\pi^4[(mr)^2 + n^2]^2 + \pi^2(P^* - c_1^2)(mr)^2\} f_{mn} = 192q^*. \tag{6.109}$$

If $c_1 = 0$, equation (6.109) is recovered to the linear result in Timoshenko (1940).

(ii) *von Karman theory*

$$\begin{aligned} &\{\pi^4[(mr)^2 + n^2]^2 + \pi^2(P^* - c_1^2)(mr)^2\} f_{mn} \\ &+ \left\{\frac{3\pi^4}{4}[(mr)^4 + n^4](3-\mu^2) + 3\mu\pi^2(mr)^2\right\} f_{mn}^3 = 192q^*. \end{aligned} \tag{6.110}$$

At $c_1 = 0$ and $m = n = 1$, equation (6.110) is identical to the result of Chu and Herrmann (1956).

(iii) *nonlinear membrane theory*

$$\begin{aligned} &\pi^2(P^* - c_1^2)(mr)^2 f_{mn} + f_{mn}^3 \Bigg(\frac{3\pi^4}{4}[(mr)^4 + n^4](3-\mu^2) \\ &+ 3\mu\pi^2(mr)^2 - \varepsilon^2 \frac{c_1^2\pi^4}{192}\Big\{[(mr)^4 - (\mu n^2)^2] \\ &+ \frac{(mr)^2}{2[(mr)^2 + n^2]}[(mr)^4 + n^4 + 2\mu(mr)^2 n^2]\Big\}\Bigg) = 192q^*. \end{aligned} \tag{6.111}$$

Consider material properties ($E = 2\times10^{11}\,\mathrm{N/m^2}$, $\rho_0 = 7.8\times10^3\,\mathrm{kg/m^3}$, $\mu = 0.3$) with geometrical properties ($l = 2.0$m, $b = 1.0$m , $h = 2$mm). If $Q = 39.11\,\mathrm{N/m^2}$ and $c = 1$ m/s, we obtain f_{1mn}=2 from Eqs.(6.76) and (6.103) at $m = n = 1$, $P = 200$N. The displacements of static waves in the three-directions from

Eqs.(6.76), (6.77), (6.94), (6.97) and (6.98) are plotted in Fig.6.4(a)−(c). The in-plane displacements are much smaller than the deflection in the transverse direction satisfying Eq.(6.60).

6.2.4. Nonlinear waves

For the free vibration of thin plates, let $q^* = 0$ and ignore the very small in-plane inertia compared to the transverse inertia for $\varepsilon = h/b << 1$, but the in-plane Coriolis acceleration increasing with increasing translation speed is retained in the nonlinear term. Equation (6.104) becomes

$$\ddot{f}_{mn} + \alpha_{mn} f_{mn} + \beta_{mn} f_{mn}^3 = 0. \tag{6.112}$$

The Hamiltonian of the periodic orbit of Eq.(6.112) is

$$H_0 \equiv \frac{1}{2}\dot{f}_{mn}^2 + \frac{1}{2}\alpha_{mn} f_{mn}^2 + \frac{1}{4}\beta_{mn} f_{mn}^4. \tag{6.113}$$

Evaluation of Eq.(6.113) with appropriate initial conditions in Eq.(6.83) leads to $H_0 = E_0$ and $E_0 =$ constant. The comprehensive discussion was presented in Luo (2003). From the energy H_0, the wave solutions of the pre-buckled and post-buckled plates are discussed as follows.

6.2.4a Pre-buckled plates

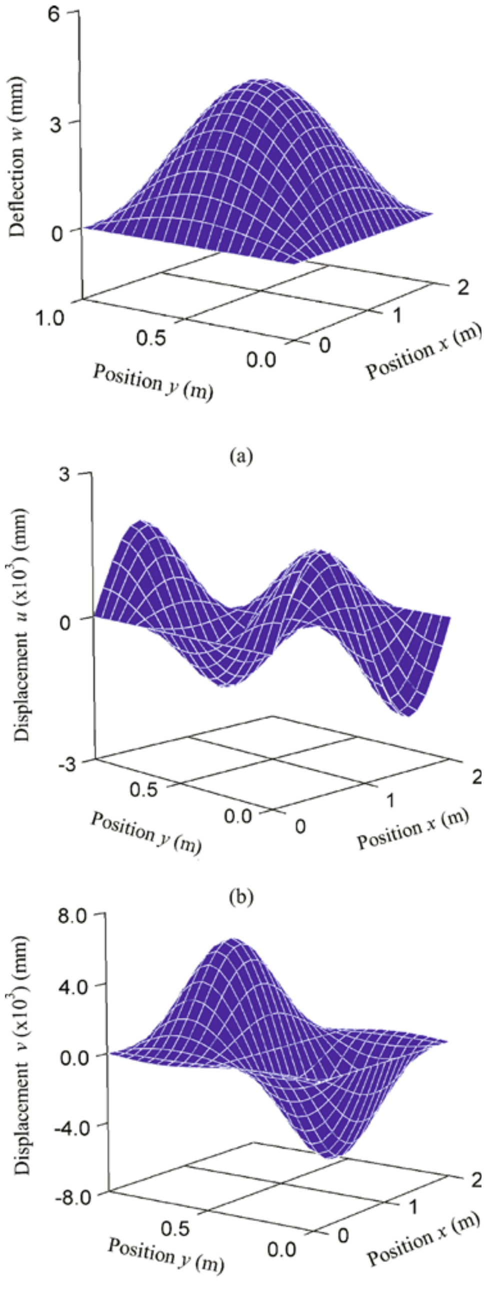
For the pre-buckled plate ($\alpha_{mn} \geq 0$), the solutions to Eq.(6.112) are

$$\begin{aligned} f_{mn}(t) &= A_{mn}\,\mathrm{cn}\left[\frac{2K(k_{mn})\omega_{mn}t}{\pi}, k_{mn}\right], \\ \dot{f}_{mn}(t) &= \frac{\sqrt{\beta_{mn}}}{\sqrt{2}k_{mn}}(A_{mn})^2\,\mathrm{sn}\left[\frac{2K(k_{mn})\omega_{mn}t}{\pi}, k_{mn}\right]\mathrm{dn}\left[\frac{2K(k_{mn})\omega_{mn}t}{\pi}, k_{mn}\right]; \end{aligned} \tag{6.114}$$

where cn, sn and dn are Jacobi-elliptic functions, $K(k_{mn})$ is the complete elliptic integral of the first kind, and A_{mn} and k_{mn} are the amplitude of $f_{mn}(t)$ and the modulus of the elliptic function for the (m,n) -mode wave:

$$A_{mn} = k_{mn}\sqrt{\frac{2\alpha_{mn}}{\beta_{mn}(1-2k_{mn}^2)}},\;\; k_{mn} = \frac{1}{\sqrt{2}}\sqrt{1-\frac{\alpha_{mn}}{\sqrt{4\beta_{mn}E_0+\alpha_{mn}^2}}}. \tag{6.115}$$

The natural frequency is



(c)

Fig. 6.4 Static waves in traveling plates: (a) transverse displacement *w*, (b) longitudinal displacement *u*, and (c) longitudinal displacement *v*. $E = 2\times10^{11}\,\text{N/m}^2$, $\rho_0 = 7.8\times10^3\,kg/\text{m}^3$, $\mu = 0.3$, $l = 2.0$ m, $b = 1.0$ m, $P = 200$ N/m, $c = 1$m/s, $Q = 39.11\,\text{N}/\text{m}^2$, h = 2 mm and $m = n = 1$.

$$\omega_{mn} = \frac{1}{2}\sqrt{\frac{\beta_{mn}}{2}}\frac{A_{mn}\pi}{k_{mn}K(k_{mn})}. \tag{6.116}$$

For $\beta_{mn} = 0$ (or $E_0 = 0$), $k_{mn} = 0$ in Eq.(6.115) results in $K(k_{mn}) = \pi/2$. Furthermore, the linear natural frequency is recovered from (6.116), i.e.,

$$\omega_{mn} = \sqrt{\alpha_{mn}} = \sqrt{[(m\pi r)^2 + (n\pi)^2]^2 + (P^* - c_1^2)(m\pi r)^2}. \tag{6.117}$$

$\omega_{mn} = 0$ gives the critical speed of buckling plates determined. Substitution of Eq.(6.114) into Eq.(6.84) with Eqs.(6.60), (6.76) and (6.77) generates the transverse displacement of the nonlinear wave:

$$\begin{aligned} w^0 &= hA_{mn}\mathrm{cn}\left[\frac{hc_p K(k_{mn})\omega_{mn}\tau}{\sqrt{3}\pi b^2}, k_{mn}\right]\sin\frac{m\pi(x+c\tau)}{l}\sin\frac{n\pi y}{b} \\ &= \sum_{p=1} U_{(2p-1)mn}\left\{\sin\left[\frac{m\pi x}{l} + (\frac{m\pi c}{l} + (2p-1)\frac{hc_p\omega_{mn}}{\sqrt{12}b^2})\tau\right]\right. \\ &\quad \left. -\sin\left[((2p-1)\frac{hc_p\omega_{mn}}{\sqrt{12}b^2} - \frac{m\pi c}{l})\tau - \frac{m\pi x}{l}\right]\right\}\sin\frac{n\pi y}{b}, \end{aligned} \tag{6.118}$$

where the $(2p-1)^{\text{th}}$ order subharmonic wave amplitude is

$$U_{(2p-1)mn} = \frac{\pi h A_{mn}}{2k_{mn}K(k_{mn})\cosh\left[\pi(2p-1)\frac{K'(k_{mn})}{2K(k_{mn})}\right]}. \tag{6.119}$$

6.2.4b Post-buckled plates

For the post-buckled plate ($\alpha_{mn} < 0$), either $E_0 \geq 0$ or $E_0 \leq 0$ exists. For the case of $E_0 \geq 0$, the free vibration wave solution, natural frequency and subharmonic wave conditions are the same forms as in Eqs.(6.114) and (6.116) for the pre-buckled plate, but the magnitude A_{mn} and k_{mn} are

$$A_{mn} = \sqrt{2}k_{mn}\sqrt{\frac{|\alpha_{mn}|}{\beta_{mn}(2k_{mn}^2 - 1)}},\ k_{mn} = \frac{1}{\sqrt{2}}\sqrt{1 + \frac{|\alpha_{mn}|}{\sqrt{4\beta_{mn}E_0 + \alpha_{mn}^2}}}. \tag{6.120}$$

The linear model gives $\omega_{mn} = 0$ at either $\beta_{mn} = 0$ or $E_0 = 0$.

In the case of $E_0 \leq 0$, the solution of Eq.(6.112) is

$$f_{mn}(t) = A_{mn}\mathrm{dn}\left[\frac{K(k_{mn})\omega_{mn}t}{\pi}, k_{mn}\right],$$
$$\dot{f}_{mn}(t) = \sqrt{\frac{\beta_{mn}}{2}}(A_{mn}k_{mn})^2 \mathrm{sn}\left[\frac{K(k_{mn})\omega_{mn}t}{\pi}, k_{mn}\right]\mathrm{cn}\left[\frac{K(k_{mn})\omega_{mn}t}{\pi}, k_{mn}\right], \tag{6.121}$$

where A_{mn} and k_{mn} are determined by

$$A_{mn} = \sqrt{\frac{2|\alpha_{mn}|}{\beta_{mn}(2-k_{mn}^2)}},\ \ k_{mn} = \sqrt{\frac{2\sqrt{4\beta_{mn}E_0+\alpha_{mn}^2}}{|\alpha_{mn}|+\sqrt{4\beta_{mn}E_0+\alpha_{mn}^2}}}. \tag{6.122}$$

The corresponding nonlinear frequency is

$$\omega_{mn} = \sqrt{\frac{\beta_{mn}}{2}}\frac{A_{mn}\pi}{K(k_{mn})}. \tag{6.123}$$

As $k_{mn} \to 0$, $K(k_{mn}) \to \pi/2$, and then the nonlinear frequency ω_{mn} in Eq.(6.123) converges to the linear natural frequency. That is,

$$\omega_{mn} = \sqrt{2|\alpha_{mn}|} = \sqrt{2}\pi\sqrt{(c_1^2 - P^*)(mr)^2 - [(mr)^2+n^2]^2}. \tag{6.124}$$

Note that $k_{mn} = 1$ in Eq.(6.123) at $\beta_{mn} = 0$ (or $E_0 = 0$) gives $K(k_{mn}) \to \infty$, so that $\omega_{mn} = 0$ in Eq.(6.123) and Eq. (6.121) becomes

$$f_{mn}(t) = \pm\frac{1}{\sqrt{\beta_{mn}}}\sqrt{2|\alpha_{mn}|}\ \mathrm{sech}(\sqrt{\beta_{mn}}t),$$
$$\dot{f}_{mn}(t) = \pm\frac{1}{\sqrt{\beta_{mn}}}\sqrt{2}|\alpha_{mn}|\ \mathrm{sech}(\sqrt{\beta_{mn}}t)\tanh(\sqrt{\beta_{mn}}t). \tag{6.125}$$

However, for linear analysis, the system will become unstable. Similarly, substitution of Eq.(6.121) into Eq.(6.84) with Eqs.(6.60), (6.76) and (6.77) results in an expression for the transverse displacement,

$$\begin{aligned} w^0 &= hA_{mn}\mathrm{dn}\left[\frac{hc_pK(k_{mn})\omega_{mn}\tau}{\pi\sqrt{12}b^2}, k_{mn}\right]\sin\frac{m\pi(x+c\tau)}{l}\sin\frac{n\pi y}{b} \\ &= \sum_{p=1} U_{pmn}\left\{\sin\left[\frac{m\pi x}{l}+\left(\frac{m\pi c}{l}+\frac{phc_p\omega_{mn}}{\sqrt{12}b^2}\right)\tau\right]\right. \\ &\quad \left. -\sin\left[\left(\frac{phc_p\omega_{mn}}{\sqrt{12}b^2}-\frac{m\pi c}{l}\right)\tau-\frac{m\pi x}{l}\right]\right\}\sin\frac{n\pi y}{b}, \end{aligned} \tag{6.126}$$

where the p^{th} -order subharmonic wave amplitude is determined by

$$U_{pmn} = \frac{\pi h A_{mn}}{2K(k_{mn})\cosh\left[p\pi \dfrac{K'(k_{mn})}{K(k_{mn})}\right]}. \tag{6.127}$$

The following parameters ($E = 2\times10^{11}\,\text{N/m}^2$, $\rho_0 = 7.8\times10^3\,\text{kg/m}^3$, $\mu = 0.3$, $l = 2.0$ m, $b = 1.0$ m, $h = 2$ mm and $P = 200$ N/m) are used in Fig. 6.5 for illustration of natural frequency. The $(1,1)$ -mode wave natural frequency versus the plate translation speed is shown in Fig.6.5(a) for specific conservative energies given by the initial conditions. The natural frequencies for the pre-buckled and post-buckled plates are plotted on the left and right sides of the dot-line. For the post-buckled plates, two natural frequencies are generated by the positive and negative conservative energy. The two frequencies merge essentially at critical point with zero natural frequency which can be obtained from Eqs.(6.116) and (6.117), (6.123) and (6.124) directly. The computational accuracy of the complete elliptic integral of the first kind at $k \to 1$ cannot provide an accurate solution in Fig.6.5(a). The results for $E_0 = 0$ give the natural frequency determined by the linear model. The critical translation speed at the pre-buckled plate $\omega = 0$ is noted. In the linear model, the instability occurs. The critical translation speeds for the nonlinear frequency of mode (m,n) at $E_0 = 0.0$ are plotted in Fig.6.5(b). For the plate parameters specified above, the minimum critical translation speed occurs in the (2,1)-mode. The buckling instability first occurs in the (2,1)-mode as the translation speed increases from zero. To illustrate the nonlinear natural frequency varying with the translation speed, the (m,n) -mode wave frequency versus the translation speed for specified conservative energy are presented in Fig.6.6.

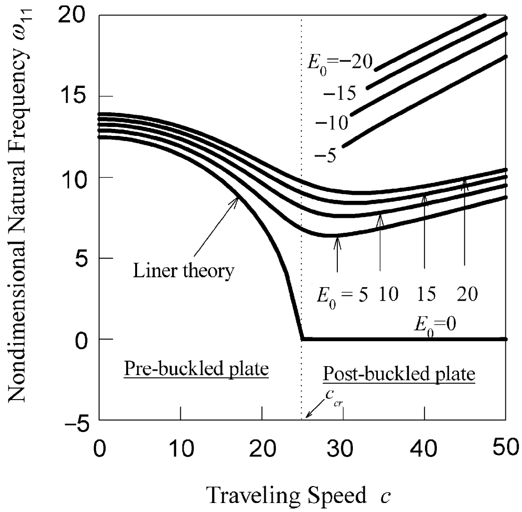
6.2.4c Resonant and stationary waves

Resonant wave: From the time-dependent terms in Eq.(6.118), there are two mode-frequencies for a specified (m, n):

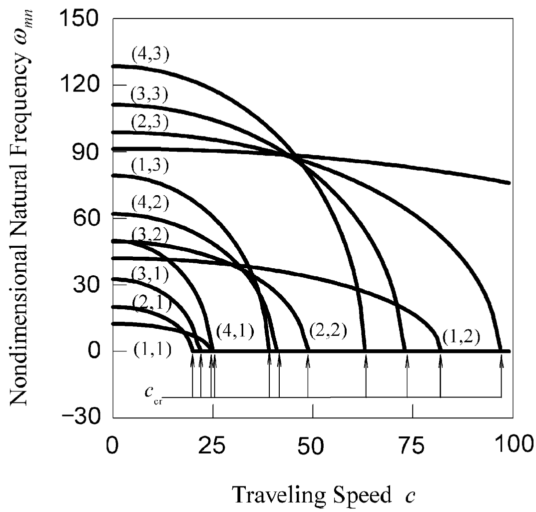
$$\begin{aligned}
&\Omega_{1,2} \equiv \frac{(2p-1)hc_p\omega_{mn}}{\sqrt{12}b^2} \mp \frac{m\pi c}{l} \geq 0, \quad \text{for subcritical wave;}\\
&\Omega_{1,2} \equiv \frac{m\pi c}{l} \mp \frac{(2p-1)hc_p\omega_{mn}}{\sqrt{12}b^2} \geq 0, \quad \text{for supercritical wave.}
\end{aligned} \tag{6.128}$$

The resonant condition for a two-frequency system is $m_1\Omega_1 = m_2\Omega_2$. With Eq.(6.128), the resonant condition gives the translation speed:

$$c = \frac{m_1 - m_2}{m_1 + m_2}\frac{(2p-1)hlc_p\omega_{mn}}{m\pi\sqrt{12}b^2}, \quad \text{for subcritical motion;}$$

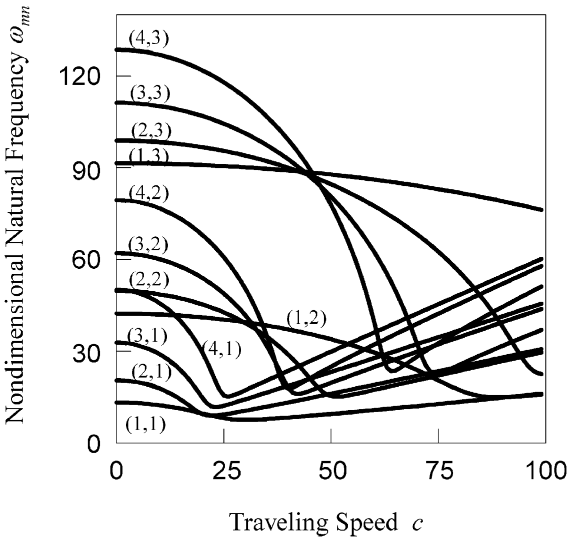


(a)

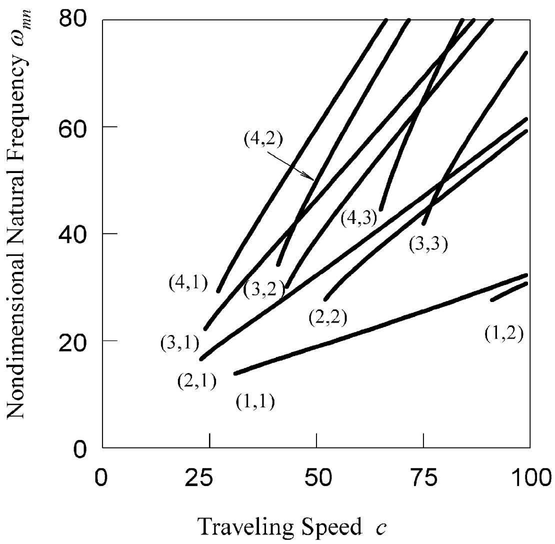


(b)

Fig. 6.5 (a) Natural frequency ω_{11} versus translation speed for specified initial energy E_0, (b) critical translation speed for (m,n)-mode waves predicted by the linear model at $E_0 = 0.0$. The linear critical speed is denoted c_{cr}. ($E = 2\times10^{11}\text{N/m}^2$, $\rho_0 = 7.8\times10^3\text{kg/m}^3$, $\mu = 0.3$, $l = 2.0\text{m}$, $b = 1.0$ m, $h = 2$ mm, $P = 200$ N/m).



(a)



(b)

Fig. 6.6 Natural frequency of (m,n) -mode waves predicted by the nonlinear model: (a) $E_0 = 10$ and (b) $E_0 = -10.0$. ($E = 2\times10^{11}$N/m^2, $\rho_0 = 7.8\times10^3$kg/m^3, $\mu = 0.3$, $l = 2.0$m, $b = 1.0$ m, $h = 2$ mm, $P = 200$ N/m).

$$c = \frac{m_1 + m_2}{m_1 - m_2} \frac{(2p-1)hlc_p\omega_{mn}}{m\pi\sqrt{12b^2}}, \text{ for supercritical motion.} \tag{6.129}$$

where integers m_1, m_2 are positive and irreducible and $m_1 > m_2$. Equation (6.129) gives the $(m_1 : m_2)$ resonance of harmonics at $p = 1$ and of the $(2p-1)$th subharmonics for $p > 1$. The strength of resonance in Eq.(6.119) is the strongest since the hyperbolic function "cosh" increases exponentially as the order of subharmonics (i.e., $2p-1$) increases.

For the linear model, the natural frequency in Eq.(6.117) with the resonant condition give

$$\begin{aligned} c &= \frac{m_1 - m_2}{m_1 + m_2} \frac{l}{\sqrt{12b^2}} \sqrt{(r^2 + \frac{n^2}{m^2})^2 c_p^2 h^2 + (P^* c_p^2 h^2 - 12c^2 b^2) r^2}, \\ &\text{for subcritical motion;} \\ c &= \frac{m_1 + m_2}{m_1 - m_2} \frac{l}{\sqrt{12b^2}} \sqrt{(r^2 + \frac{n^2}{m^2})^2 c_p^2 h^2 + (P^* c_p^2 h^2 - 12c^2 b^2) r^2}, \\ &\text{for supercritical motion.} \end{aligned} \tag{6.130}$$

Note that the resonant translation speed in Eqs.(6.129) and (6.130) are applicable to the post-buckled plates for $E_0 > 0$. For the traveling, post-buckled plate, the resonant translation speed for $E_0 < 0$ is

$$\begin{aligned} c &= \frac{m_1 - m_2}{m_1 + m_2} \frac{phlc_p\omega_{mn}}{m\pi\sqrt{12b^2}}, \text{ for subcritical motion;} \\ c &= \frac{m_1 + m_2}{m_1 - m_2} \frac{phlc_p\omega_{mn}}{m\pi\sqrt{12b^2}}, \text{ for supercritical motion.} \end{aligned} \tag{6.131}$$

Stationary waves: The stationary wave requires $\Omega_1 = 0$ which gives

$$c = \frac{(2p-1)hc_p l\omega_{mn}}{\sqrt{12}m\pi b^2}, \tag{6.132}$$

for the $(2p-1)^{\text{th}}$ order, subharmonic stationary transverse wave of the transverse displacement of the pre-buckled plate. Likewise, substitution of Eq.(6.114) into Eqs.(6.97), (6.98), (6.101) and (6.102) yields the longitudinal free vibration wave solutions and the corresponding stationary wave conditions for the in-plane motions. The condition for the $(2p-1)^{\text{th}}$ order subharmonic stationary wave of the longitudinal displacement is one-half c in Eq.(6.132).

For the post-buckled plate, the translation speed for the stationary wave of $E_0 > 0$ is the same expression as in Eq.(6.132) only except for the A_{mn} and k_{mn} given in Eq.(6.120). However, for $E_0 < 0$, the translation speed for the pth order subharmonic transverse stationary wave is

$$c = \frac{phc_p l\omega_{mn}}{2\sqrt{3}m\pi b^2}. \tag{6.133}$$

The longitudinal stationary wave translation speed is one-half in Eq.(6.133). For the linear vibration of traveling plates, the translation speed for the stationary wave is

$$c = \frac{c_p \varepsilon}{\sqrt{12}} \sqrt{(1+\frac{n^2}{(mr)^2})^2 (mr)^2 + P^*}. \tag{6.134}$$

Such a detailed analysis can be referred to Luo (2003).

6.2.5 Chaotic waves

As in Luo (2005), consider $q^*_{mn}(t) = Q_0 \cos\Omega t$ in Eq.(6.103). For a thin plate in forced vibration, the in-plane inertia is ignored because it is very small compared to the transverse inertia as $\varepsilon = h/b << 1$, but the in-plane Coriolis acceleration is retained in the nonlinear term due to the transport speed. Equation (6.103) becomes

$$\ddot{f}_{mn} + \alpha_{mn} f_{mn} + \beta_{mn} f^3_{mn} = Q_0 \cos\Omega t. \tag{6.135}$$

A Hamiltonian of Eq. (6.135) is

$$H = \frac{1}{2}\dot{f}^2_{mn} + \frac{1}{2}\alpha_{mn} f^2_{mn} + \frac{1}{4}\beta_{mn} f^4_{mn} - f_{mn} Q_0 \cos\Omega t. \tag{6.136}$$

where the Hamiltonian can be divided into the time-independent and time-dependent parts, i.e., $H = H_0 + H_1$, where H_0 and H_1 are the conservative energy and the work done by the excitation:

$$H_0 = \frac{1}{2}\dot{f}^2_{mn} + \frac{1}{2}\alpha_{mn} f^2_{mn} + \frac{1}{4}\beta_{mn} f^4_{mn} \text{ and } H_1 = -f_{mn} Q_0 \cos\Omega t. \tag{6.137}$$

Evaluation of the conservative energy with appropriate initial conditions $(f^0_{mn}, \dot{f}^0_{mn})$ reduced from Eq.(6.83) leads to $H_0 = E_0 = \text{constant}$. From H_0, the periodic solution and the nonlinear natural frequency of the conservative system of the axially traveling plate are estimated in Section 6.2.4, and using the solution and frequency of the conservative system, the analytical condition for chaotic wave motion in the axially traveling plate can be achieved.

6.2.5a Pre-buckled plates

If the axially traveling speed of the plate is less than the critical traveling speed, the axially traveling plate is not buckled, i.e., $\alpha_{mn} \geq 0$. Substitution of Eq.(6.114) into H_1 in Eq.(6.137) yields the subharmonic resonant condition, i.e.,

$$\Omega = (2p-1)\omega_{mn}, \tag{6.138}$$

where p is a positive integer. From H_1, a separatrix generated by the $(2p-1)$th subharmonic resonance has the similar property as the heteroclinic orbit, thus, the chaotic wave motion might appear in vicinity of the separatrix. From the theory of Luo (1995) (also see, Han and Luo, 1998; Luo, 2002, 2008), the energy increment of an orbit of Eq.(6.135) in the neighborhood of the $(2p-1)$th resonant separatrix during one period $T(E_0) = 2\pi / \omega_{mn}$ can be approximated by the work done by the excitation traveling along one cycle of a conservative orbit related to the $(2p-1)$th resonant condition, namely,

$$\Delta H \approx \int_{t_0}^{t_0+T(E_0)} \left[\frac{\partial H_0}{\partial \dot{f}_{mn}} \frac{\partial H_1}{\partial f_{mn}} - \frac{\partial H_1}{\partial \dot{f}_{mn}} \frac{\partial H_0}{\partial f_{mn}} \right] dt = U_{mn} \sin \Omega t_0, \tag{6.139}$$

where

$$U_{mn}^{(2p-1)} = \frac{2\sqrt{2}\pi\Omega Q_0}{\sqrt{\beta_{mn}}} \operatorname{sech}\left[(2p-1) \frac{\pi K(k'_{mn})}{2K(k_{mn})} \right] \tag{6.140}$$

where $k'_{mn} = \sqrt{1-k_{mn}^2}$. The minimum condition for the chaotic motion related to the $(2p-1)$th resonance in Luo (1995, 2002, 2008) is

$$G_{mn} U_{mn}^{(2p-1)} \approx 0.9716354, \tag{6.141}$$

where

$$G_{mn} = \frac{8\sqrt{2}\Omega k_{mn}^3}{\beta_{mn}^{3/2} A_{mn}^5} \left[K(k_{mn}) - \frac{1-2k_{mn}^2}{1-k_{mn}^2} E(k_{mn}) \right]. \tag{6.142}$$

Once any other primary resonance of the system is involved in the chaotic wave motion near the assigned resonant separatrix, the chaotic wave motion relative to the $(2p-1)$th resonance is destroyed. Thus, the maximum condition for chaotic motion related to the $(2p-1)$th resonance is

$$\frac{2\sqrt{2}\pi\Omega Q_0}{\sqrt{\beta_{mn}}} \operatorname{sech}\left[(2p-1) \frac{\pi K(k'_{mn})}{2K(k_{mn})} \right] = E_0^{2p-1} - E_0^{2p+1}. \tag{6.143}$$

6.2.5b Post-buckled plates

If the axially traveling speed of the plate is over the critical traveling speed, the axially traveling plate is buckled (i.e., $\alpha_{mn} < 0$). For the axially traveling, post-buckled plate, the chaotic conditions for cases of $E_0 > 0$, $E_0 < 0$ and $E_0 = 0$ are discussed.

For the case of $E_0 > 0$, the solution and natural frequency of the conservative system of Eq.(6.135), subharmonic resonant and chaotic conditions for Eq. (6.135) are of the same as Eqs.(6.114), (6.116), (6.138), (6.141), (6.143), but the magnitude A_{mn} and k_{mn} are given as in Eq.(6.120).

For the case of $E_0 < 0$, the solution and natural frequency of the conservative system of Eq.(6.135) for the axially traveling, post-buckled plate are given by Eq. (6.121). Substitution of Eq.(6.121) into H_1 gives the sub-harmonic resonant condition:

$$\Omega = p\omega_{mn}. \tag{6.144}$$

As in the axially traveling, pre-buckled plate, the minimum and maximum conditions for the chaotic motion relative to Eq.(6.144) are

$$G_{mn}^{(p)} U_{mn}^{(p)} \approx 0.9716354, \tag{6.145}$$

and

$$\frac{\sqrt{2}\pi\Omega Q_0}{\sqrt{\beta_{mn}}} \operatorname{sech} \frac{p\pi K(k'_{mn})}{K(k_{mn})} = E_0^{p+1} - E_0^{p}, \tag{6.146}$$

where

$$\begin{aligned} U_{mn}^{(p)} &= \frac{\sqrt{2}\pi\Omega Q_0}{\sqrt{\beta_{mn}}} \operatorname{sech} \frac{p\pi K(k'_{mn})}{K(k_{mn})}, \\ G_{mn}^{(p)} &= \frac{4\sqrt{2}\Omega}{\sqrt{\beta_{mn}} k_{mn}^4 A_{mn}^5} \left[2K(k_{mn}) - \frac{2 - k_{mn}^2}{1 - k_{mn}^2} E(k_{mn}) \right]. \end{aligned} \tag{6.147}$$

For the case of $E_0 = 0$, the solution of the conservative system of Eq.(6.135) is given by Eq.(6.125). The solution describes the homoclinic separatrix. The chaotic wave motion in vicinity of the homoclinic separatrix is associated with the motion of both $E_0 > 0$ and $E_0 < 0$. The chaotic wave motion near the homoclinic separatrix involves many resonant motions of both $E_0 > 0$ and $E_0 < 0$. Therefore, the analytical conditions for the $(2p-1)$th resonance of $E_0 > 0$ and the pth resonance of $E_0 < 0$ embedded in chaotic motion near the homoclinic separatrix are

$$2Q_0\pi\Omega\sqrt{\frac{2}{\beta_{mn}}}\operatorname{sech}\frac{\pi\Omega}{2\sqrt{\alpha_{mn}}}=E_0^{2p-1}, \tag{6.148}$$

and

$$2Q_0\pi\Omega\sqrt{\frac{2}{\beta_{mn}}}\operatorname{sech}\frac{\pi\Omega}{2\sqrt{\alpha_{mn}}}=E_0^{p}. \tag{6.149}$$

6.2.5c Prediction comparisons

Using the energy spectrum methods in Luo (2002, 2008), the numerical prediction of the appearance and disappearance of the resonant separatrix can be carried out. The analytical and numerical conditions for chaotic wave motions will be illustrated from the parameters (i.e., $E=2\times10^{11}\,\text{N/m}^2$, $\rho_0=7.8\times10^3\,\text{kg/m}^3$, $\mu=0.3$, $l=2.0$ m, $h=2$ mm, $P=200$ N, $m=n=1$). For the axially traveling, pre-buckled plate at the transport speed $c=15\text{m/s}<c_{\text{cr}}$, the analytical and numerical conditions for chaotic motions near the primary resonant separatrix are shown in Fig.6.7, where the solid and dash curves represent the maximum and minimum *analytical* conditions, and the triangle and circle lines denote the corresponding *numerical* prediction. Only the maximum condition for chaotic wave motion in the primary resonant separatrix zone of the first order is found in Fig.6.7 because the sub-resonance of the primary resonance of the first order is very strong. Once the chaotic motion in the resonant separatrix zone of the first order appears, the chaotic wave motion includes other primary resonance. For the axially traveling, post-buckled plate at the transport speed $c=35\text{m/s}>c_{\text{cr}}$, the conditions for chaotic wave motions in the primary resonant separatrix for $E_0>0$ and $E_0<0$ are shown in Fig.6.8(a) and (b). The numerical and analytical conditions for the primary resonance of both $E_0>0$ and $E_0<0$ embedded in the chaotic wave motion in the zone of the homoclinic separatrix are plotted in Fig.6.8(c)–(d). Note that the solid and circular symbol curves depict the analytical and numerical conditions.

From Fig.6.7 and Fig.6.8(a)–(b), the analytical and numerical conditions for the chaotic waves in the resonant separatrix zone are not in very good agreement because the energy increment is approximated (i.e., Eq.(6.139)). For this reason, if the excitation is very strong, then the agreement of the analytical prediction with the numerical prediction becomes worst. In additions, the sub-resonance in chaotic motions near the primary resonant separatrix zone should be modeled to improve the analytical prediction of chaos. However, the analytical and numerical predictions for the primary resonances embedded in chaotic motion near the homoclinic orbit are in very good agreement. The comprehensive discussion can be found in Luo (2008).

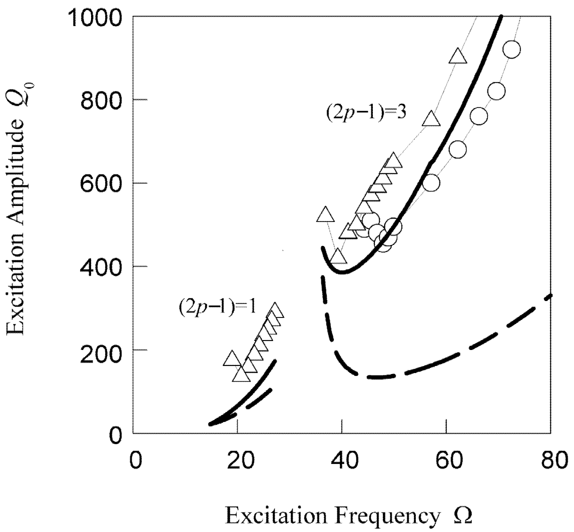


Fig. 6.7 Analytical and numerical predictions for chaotic wave motions in the axially traveling, pre-buckled plate. ($E = 2\times10^{11}\text{N/m}^2$, $\rho_0 = 7.8\times10^3\text{kg/m}^3$, $\mu = 0.3$, $l = 2.0$ m, $b = 1.0$ m, $h = 2$ mm, $m = n = 1$, P =200 N, $c = 15$m/s.)

6.2.5d Chaotic waves in resonant zones

The Poincaré mapping sections of chaotic wave motions near the resonant separatrix and the homoclinic orbit in the axially traveling plate are plotted in Figs.6.9–6.11 for the running time $3\times10^5 T$ where $T = 2\pi/\Omega$. The large circular symbols are equilibrium points for conservative system in Eq.(6.135). The small circular symbols are equilibrium points for the resonant separatrix, which is determined by the resonant layer dynamics in Luo (2008). The red and green circular symbols represent stable and unstable equilibriums, respectively. The initial condition ($f_{mn}^0 = 0.5398$, $\dot{f}_{mn}^0 = 18.7908$), the excitation frequency and amplitude ($\Omega = 57.1181, Q_0 = 610 > Q_0^{\min} \approx 600$), the transport speed ($c = 15\text{m/s}$) and other parameters given as before are used, and the corresponding Poincaré mapping section of the chaotic wave motion in the resonant separatrix zone of the third order for the axially traveling, pre-buckled plate is produced, as shown in Fig.6.9. The maximum non-dimensional displacement $f_{mn} = w/h$ is about 1.5. This implies that the chaotic motion might occur in the small amplitude oscillation of plates when the corresponding geometrical nonlinearity is invoked in the governing equations of motion.

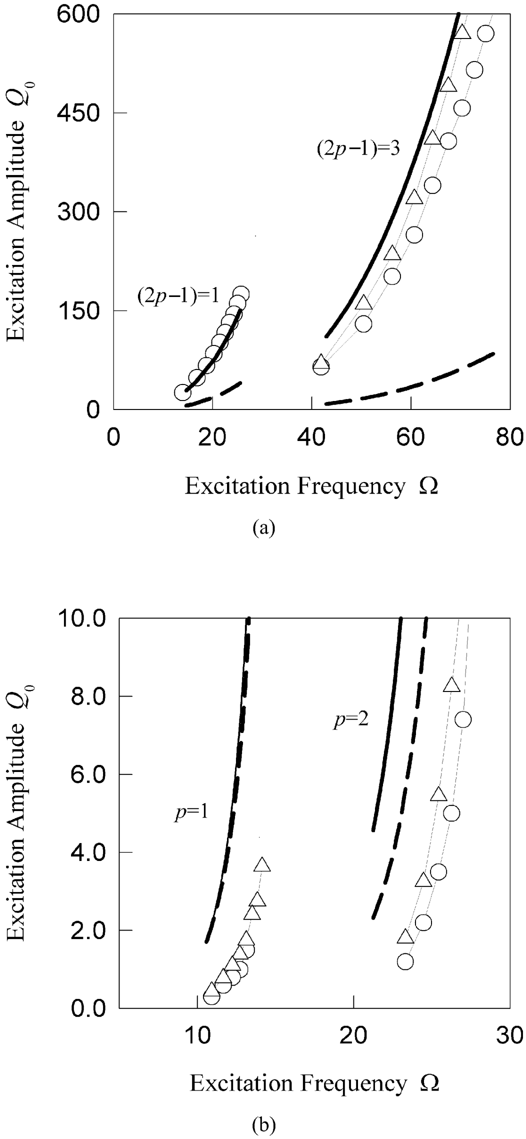
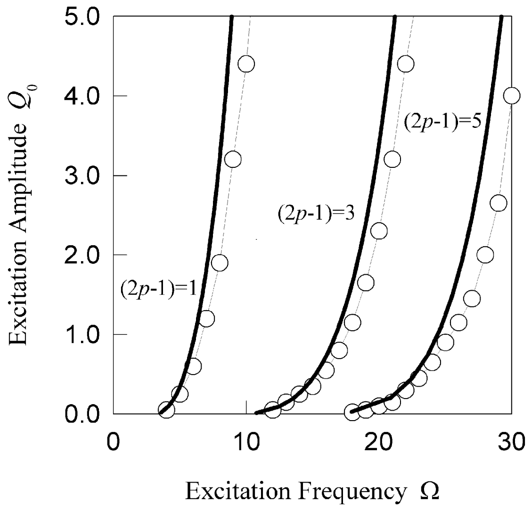
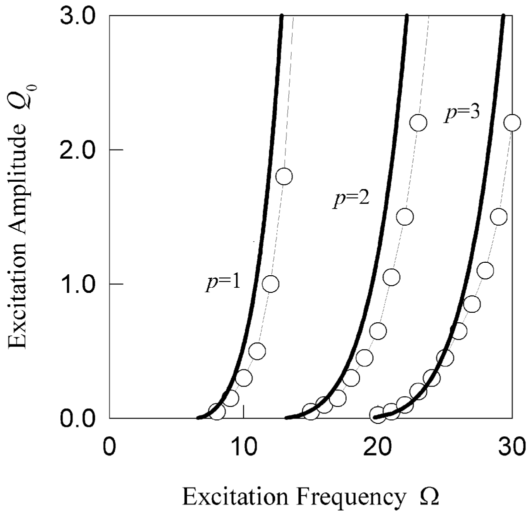


Fig. 6.8 Analytical and numerical predictions for chaotic wave motions in the axially traveling, post-buckled plate: (a) $E_0 > 0$, (b) $E_0 < 0$, (c) $E_0 = 0$ relative to $E_0 > 0$ and (d) $E_0 = 0$ relative to $E_0 < 0$.($E = 2\times10^{11}\text{N/m}^2$, $\rho_0 = 7.8\times10^3\text{kg/m}^3$, $\mu = 0.3$, $l = 2.0$ m, $b = 1.0$ m, $h = 2$ mm, $m = n = 1$, P =200 N, $c = 35$m/s.)



(c)



(d)

Fig. 6.8 (continued).

In Fig.6.10(a), the chaotic wave motion near the primary resonant separatrix zone of the third order for the axially traveling, post-buckled plate is illustrated through the Poincaré mapping section for ($f_{mn}^0 = 0.7376$, $\dot{f}_{mn}^0 = 28.6717$, $\Omega = 60.6762$, $Q_0 = 270 > Q_0^{\min} \approx 265$, $c = 35\text{m/s}$) and other parameters are given as before. To stimulate the same type of resonant separatrix, the excitation amplitude for the post-buckled plate is much smaller than for the pre-buckled plate.

Two Poincaré mapping sections of chaotic motions near the primary resonant separatrix zone of the second order in the left and right potential wells of the axially traveling, post-buckled plate are plotted in Fig.6.10(b). Two illustrations use the same excitation ($\Omega = 25.4274$, $Q_0 = 4.5 > Q_0^{\min} \approx 3.5$) and the same conservative energy $E_0 = -5.7963$, but they are totally different because of signs of displacement in the two wells. The hyperbolic points selected as initial conditions for numerical simulations are ($f_{mn}^0 = -0.5617$ and $\dot{f}_{mn}^0 = -0.3099$) in the left well and ($f_{mn}^0 = 0.3615$ and $\dot{f}_{mn}^0 = 0$) in the right well. From Fig.6.10(b), chaotic wave motions near the primary resonant separatrix in two wells are not symmetric even if phase portraits for the non-excited case of Eq. (6.135) are symmetric. The Poincaré mapping sections of chaotic wave motions near the homoclinic separatrix are shown in Fig.6.11(a) and (b) for ($Q_0 = 1.2 > Q_0^{cr} \approx 1.05$, $Q_0 = 3.5 > Q_0^{cr} \approx 3.2$) at $\Omega = 20.0$ and ($f_{mn}^0 = \dot{f}_{mn}^0 = 0$). In Fig.6.11(a) the primary resonance of the second order for $E_0 < 0$ is embedded in chaotic wave motion near the homoclinic separatrix. With the excitation amplitude increase, the primary resonance of the third order for $E_0 > 0$ are implanted in the chaotic wave motion near the homoclinic separatrix, as clearly shown in Fig.6.11(b).

6.3. Waves in rotating disks

In this section, waves in rotating disks will be presented. As in traveling plates, the perturbation analysis for rotating disks will be carried. Through the energy analysis, an approximate solution for waves in rotating disks will be presented (also see, Luo and Mote, 2000). Finally, the resonant and stationary waves in the rotating disks will be discussed.

6.3.1. Equations of motions

Consider a flexible, circular disk rotating with constant angular speed Ω, as sketched in Fig.6.12. The disk is clamped at the hub $r = a$, free at the outer edge $r = b$, and is of uniform thickness *h*. Both the rotating and stationary coordinate systems (r, ϑ, τ) and (r, θ, τ) satisfy

$$\theta = \vartheta + \Omega\tau. \tag{6.150}$$

For large deflection plates, the accurate plate theory in Section 6.2.1 (also see, Luo, 2000) requires:

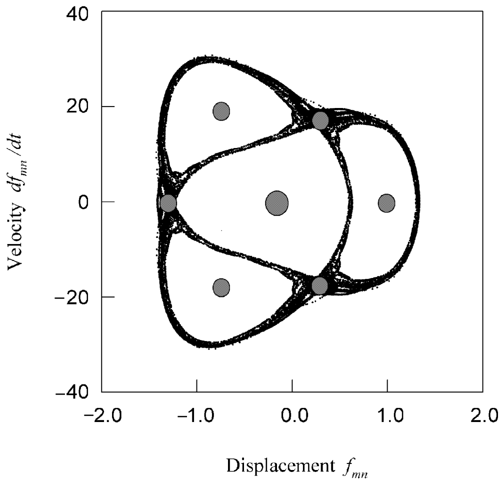


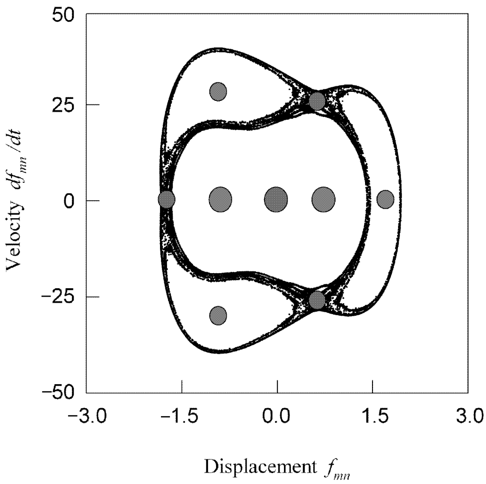
Fig. 6.9 Chaotic wave motion in the resonant separatrix zone of the third order of the axially traveling, pre-buckled plate. Red and green points are center and hyperbolic equilibrium points for resonant separatrix, respectively. ($f^0_{mn} = 0.5398$, $\dot{f}^0_{mn} = 18.7908$, $\Omega = 57.1181$, $Q_0 = 610 >$ $Q_0^{\min} \approx 600$, $E = 2\times10^{11}\,\text{N/m}^2$, $\rho_0 = 7.8\times10^3\,\text{kg/m}^3$, $\mu = 0.3$, $l = 2.0$ m, $b = 1.0$m, $h = 2$ mm, $m = n = 1$, $P = 200$ N, $c = 15$ m/s.) (color plot in the book end)

$$
\begin{aligned}
&u_{r,r} \approx O(u_{\vartheta,r}) \approx O[(u_{z,r})^2],\ \frac{1}{r}u_{r,\theta} \approx O(\frac{1}{r}u_{\theta,\theta}) \approx O[(\frac{1}{r}u_{z,\theta})^2];\\
&u_{r,r} + \frac{1}{2}(u_{z,r})^2 << 1,\ \frac{1}{r}u_{r,\theta} + \frac{1}{r}u_{\theta,\theta} + \frac{1}{2}(\frac{1}{r}u_{z,\theta})^2 << 1;\\
&1+\varepsilon_r \approx 1,\ 1+\varepsilon_\theta \approx 1,
\end{aligned}
\tag{6.151}
$$

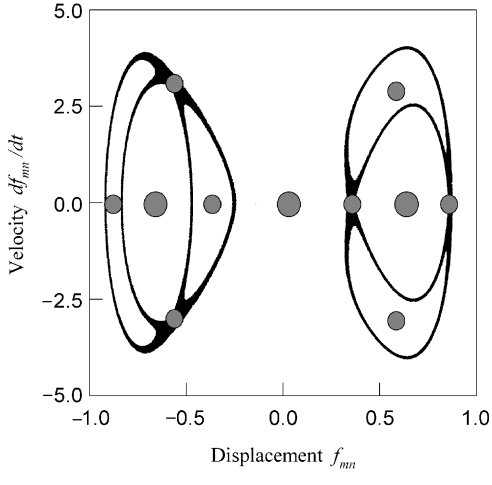
where a comma in the subscript denotes partial differentiation, and three displacement components of a material point on the disk are represented by u_r , u_θ and u_z .

Under Eq.(6.151), three components of strain in the middle surface are approximated by

$$
\begin{aligned}
\varepsilon_r &\approx u_{r,r} + \frac{1}{2}(u_{z,r})^2,\\
\varepsilon_\theta &\approx \frac{1}{r}u_r + \frac{1}{r}u_{\theta,\theta} + \frac{1}{2r^2}(u_{z,\theta})^2,\\
\gamma_{r\theta} &\approx \frac{1}{r}u_{r,\theta} + u_{\theta,r} - \frac{1}{r}u_\theta + \frac{1}{r}u_{z,r}u_{z,\theta}.
\end{aligned}
\tag{6.152}
$$

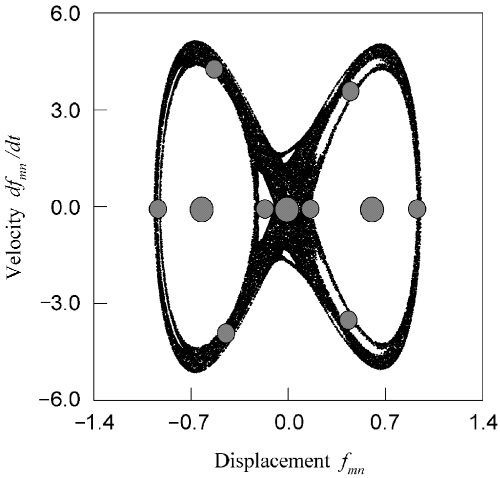


(a)

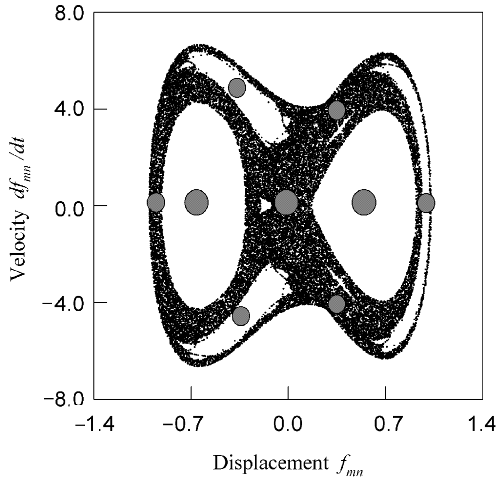


(b)

Fig. 6.10 Chaotic wave motion in the resonant separatrix zone of the axially traveling, post-buckled plate: (a) $E_0 > 0$ ($f_{mn}^0 = 0.7376$, $\dot{f}_{mn}^0 = 28.6717$, $\Omega = 60.6762$, $Q_0 = 270 > Q_0^{\min} \approx 265$) and (b) $E_0 < 0$ ($f_{mn}^0 = -0.5617$, $\dot{f}_{mn}^0 = -0.3099$) in the left well and ($f_{mn}^0 = 0.3615$, $\dot{f}_{mn}^0 = 0$) in the right well. ($\Omega = 25.4274$, $Q_0 = 4.5 > Q_0^{\min} \approx 3.5$.) Red and green points are center and hyperbolic equilibrium points for resonant separatrix, respectively. ($E = 2\times10^{11}$ N/m^2, $\rho_0 = 7.8\times10^3$ kg/m^3, $\mu = 0.3$, $l = 2.0$ m, $b = 1.0$ m, $h = 2$ mm, $m = n = 1$, $P = 200$ N, $c = 35$ m/s.) (color plot in the book end)



(a)



(b)

Fig. 6.11 Chaotic wave motions in the homoclinic separatrix zone of the axially traveling, post-buckled plate: (a) for the resonance of the second order at $E_0 < 0$ embedded and (b) for the resonance of the third order at $E_0 > 0$ embedded. Red and green symbols are center and hyperbolic equilibrium points for resonant separatrix, respectively. ($Q_0 = 1.2 > Q_0^{cr} \approx 1.05$, $Q_0 = 3.5 > Q_0^{cr}$, $Q_0^{cr} \approx 3.2$, $\Omega = 20.0$ $f_{mn}^0 = \dot{f}_{mn}^0 = 0$, $E = 2\times10^{11}\text{N/m}^2$, $\rho_0 = 7.8\times10^3\text{kg/m}^3$, $\mu = 0.3$, $l = 2.0$ m, $b = 1.0$ m, $h = 2$ mm, $m = n = 1$, $P = 200$ N, $c = 35$m/s.) (color plot in the book end)

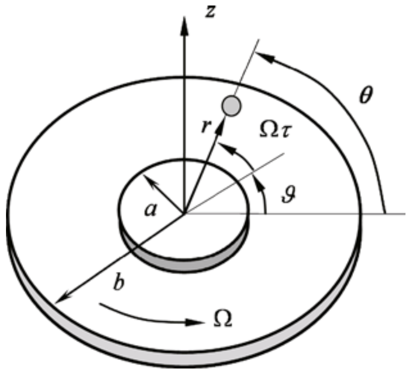


Fig. 6.12 An axisymmetric, rotating disk with clamped-free boundaries.

From Eqs.(6.151) and (6.152), the accurate plate theory in Section 6.1 (e.g., Luo, 2000) gives force and moment balances:

$$[N_r - (Q_r u_{z,r})]_{,r} + \frac{1}{r}[N_{r\theta} - (Q_\theta u_{z,r})]_{,\theta} + \frac{1}{r}(N_r - N_\theta)$$
$$+\rho_0 h \Omega^2 r = \rho_0 h(\ddot{u}_r + 2\Omega \dot{u}_{r,\theta} + \Omega^2 u_{r,\theta\theta}), \tag{6.153}$$

$$[N_{r\theta} - \frac{1}{r}(Q_r u_{z,\theta})]_{,r} + \frac{1}{r}[N_\theta - \frac{1}{r}(Q_\theta u_{z,\theta})]_{,\theta} + \frac{2}{r}N_{r\theta}$$
$$= \rho_0 h(\ddot{u}_\theta + 2\Omega \dot{u}_{\theta,\theta} + \Omega^2 u_{\theta,\theta\theta}), \tag{6.154}$$

$$\frac{1}{r}[r(N_r u_{z,r}) + N_{r\theta} u_{z,\theta} + rQ_r]_{,r} + \frac{1}{r}[\frac{1}{r}(N_\theta u_{z,\theta}) + N_{r\theta} u_{z,r} + Q_\theta]_{,\theta}$$
$$= \rho_0 h(\ddot{u}_z + 2\Omega \dot{u}_{z,\theta} + \Omega^2 u_{z,\theta\theta}), \tag{6.155}$$

$$M_{r,r} + \frac{1}{r}M_{r\theta,\theta} + \frac{1}{r}(M_r - M_\theta) + \frac{1}{r}(N_{r\theta} u_{z,\theta}) - Q_r = 0, \tag{6.156}$$

$$M_{r\theta,r} + \frac{1}{r}M_{\theta,\theta} + \frac{2}{r}M_{r\theta} + N_{r\theta} u_{z,r} - Q_\theta = 0, \tag{6.157}$$

where the superscript dot denotes derivative with respect to time τ. Q_r and Q_θ denote shear forces in radial and hoop cross sections. In the von Karman theory, shear force contributions in Eqs.(6.153) and (6.154) and membrane force contribution in Eqs.(6.156) and (6.157) were not considered. The membrane forces (N_r, N_θ, $N_{r\theta}$) are

$$N_r = \frac{Eh}{1-\mu^2}\{u_{r,r} + \frac{1}{2}(u_{z,r})^2 + \mu[\frac{1}{r}u_r + \frac{1}{r}u_{\theta,\theta} + \frac{1}{2r^2}(u_{z,\theta})^2]\},$$

$$N_\theta = \frac{Eh}{1-\mu^2}\{\frac{1}{r}u_r + \frac{1}{r}u_{\theta,\theta} + \frac{1}{2r^2}(u_{z,\theta})^2 + \mu[u_{r,r} + \frac{1}{2}(u_{z,r})^2]\},$$
$$N_{r\theta} = \frac{Eh}{2(1+\mu)}[u_{\theta,r} + \frac{1}{r}u_{r,\theta} - \frac{1}{r}u_\theta + \frac{1}{r}u_{z,r}u_{z,\theta}]; \tag{6.158}$$

and the bending and twisting moments (M_r, M_θ, $M_{r\theta}$) are

$$M_r = -D[u_{z,rr} + \mu(\frac{1}{r}u_{z,r} + \frac{1}{r^2}u_{z,\theta\theta})],$$
$$M_\theta = -D(\frac{1}{r}u_{z,r} + \frac{1}{r^2}u_{z,\theta\theta} + \mu u_{z,rr}), \tag{6.159}$$
$$M_{r\theta} = -(1-\mu)D(\frac{1}{r}u_{z,r\theta} - \frac{1}{r^2}u_{z,\theta}),$$

where E and μ are Young's modulus and Poisson ratio, and the stiffness for disks is $D = Eh^3/12(1-\mu^2)$.

For convenience, the following dimensionless variables are introduced.

$$R = \frac{r}{b},\ U_R = \frac{u_r}{b},\ U_\theta = \frac{u_\theta}{b},\ U_z = \frac{u_z}{b},\ \kappa = \frac{a}{b},\ \varepsilon = \frac{h}{b},$$
$$t = \frac{\varepsilon c_p \tau}{\sqrt{12}b},\ \Omega^* = \frac{\sqrt{12}b\Omega}{\varepsilon c_p},\ c_p^2 = \frac{E}{\rho_0(1-\mu^2)}, \tag{6.160}$$

With Eq.(6.160), substitution of Eqs.(6.156)−(6.159) into Eqs.(6.153)−(6.165) yields

$$U_{R,RR} + \frac{1}{R}U_{R,R} - \frac{1}{R^2}U_R + \frac{1-\mu}{2R^2}U_{R,\theta\theta} + \frac{1+\mu}{2R}U_{\theta,R\theta} - \frac{3-\mu}{2R^2}U_{\theta,\theta}$$
$$+ U_{Z,R}(U_{Z,RR} + \frac{1-\mu}{2R^2}U_{Z,\theta\theta}) + \frac{1+\mu}{2R^2}U_{Z,R\theta}U_{Z,\theta} + \frac{1-\mu}{2R}\Big[(U_{Z,R})^2$$
$$- \frac{1}{R^2}(U_{Z,\theta})^2\Big] + \frac{\varepsilon^2}{12}\{[(\nabla^2 U_R)_{,R}U_{Z,R}]_{,R} + \frac{1}{R^2}[(\nabla^2 U_Z)_{,\theta}U_{Z,R}]_{,\theta}\}$$
$$+ \frac{\varepsilon^2}{12}\Omega^{*2}R = \frac{\varepsilon^2}{12}(\ddot{U}_R + 2\Omega^* U_{R,\theta} + \Omega^{*2}U_{R,\theta\theta}), \tag{6.161}$$

$$\frac{1}{R^2}U_{\theta,\theta\theta} + \frac{1-\mu}{2}(U_{\theta,RR} + \frac{1}{R}U_{\theta,R} - \frac{1}{R^2}U_\theta)$$
$$+ \frac{1+\mu}{2R}U_{R,R\theta} + \frac{3-\mu}{2R^2}U_{R,\theta} + \frac{1+\mu}{2R}U_{Z,R}U_{Z,R\theta}$$
$$+ \frac{1}{R}U_{Z,\theta}[\frac{1}{R^2}U_{Z,\theta\theta} + \frac{1-\mu}{2}(U_{Z,RR} + \frac{1}{R}U_{Z,R})]$$

$$+\frac{\varepsilon^2}{12}\{[\frac{1}{R}(\nabla^2 U_Z)_{,R}U_{Z,\theta}]_{,R}+\frac{1}{R^3}[(\nabla^2 U_Z)_{,\theta}U_{Z,\theta}]_{,\theta}\}$$
$$=\frac{\varepsilon^2}{12}(\ddot{U}_\theta+2\Omega^*\dot{U}_{\theta,\theta}+\Omega^{*2}U_{\theta,\theta\theta}), \tag{6.162}$$

$$\frac{1}{R}(R\{U_{R,R}+\frac{1}{2}(U_{Z,R})^2+\mu[\frac{1}{R}U_R+\frac{1}{R}U_{\theta,\theta}+\frac{1}{2R^2}(U_{z,\theta})^2]\}U_{Z,R}$$
$$+(1-\mu)[U_{\theta,R}+\frac{1}{R}U_{R,\theta}-\frac{1}{R}U_\theta+\frac{1}{R}U_{Z,R}U_{Z,\theta}]U_{Z,\theta})_{,R}$$
$$+\frac{1}{R}(\frac{1}{R}\{\frac{1}{R}U_r+\frac{1}{R}U_{\theta,\theta}+\frac{1}{2R^2}(U_{Z,\theta})^2+\mu[U_{R,R}+\frac{1}{2}(U_{Z,R})^2]\}U_{Z,\theta}$$
$$+(1-\mu)[U_{\theta,R}+\frac{1}{R}U_{R,\theta}-\frac{1}{R}U_\theta+\frac{1}{R}U_{Z,R}U_{Z,\theta}]U_{Z,R})_{,\theta}$$
$$=\frac{\varepsilon^2}{12}(\nabla^4 U_Z+\ddot{U}_Z+2\Omega^*\dot{U}_{Z,\theta}+\Omega^{*2}U_{Z,\theta\theta}). \tag{6.163}$$

Where

$$\nabla^2=\frac{\partial^2}{\partial R^2}+\frac{1}{R}\frac{\partial^2}{\partial R\partial\theta}+\frac{1}{R^2}\frac{\partial^2}{\partial\theta^2},$$
$$\nabla^4=(\frac{\partial^2}{\partial R^2}+\frac{1}{R}\frac{\partial^2}{\partial R\partial\theta}+\frac{1}{R^2}\frac{\partial^2}{\partial\theta^2})(\frac{\partial^2}{\partial R^2}+\frac{1}{R}\frac{\partial^2}{\partial R\partial\theta}+\frac{1}{R^2}\frac{\partial^2}{\partial\theta^2}). \tag{6.164}$$

Note that the ε^2 terms in Eqs.(6.161) and (6.162) represent the shear force contributions computed by the bending and twisting moments and in-plane membrane forces, centrifugal forces and the rotation-induced Coriolis inertia and the other nonlinear terms are directly contributed by the in-plane nonlinear membrane forces. The $(1-\mu)$ terms in Eq.(6.163) are the contribution of the shear membrane forces and shear forces in the transverse direction. In the von Karman theory of rotating disk (Tobias and Arnold, 1957; Nowinski,1964; Advani, 1967), the shear force contribution to the in-plane force balances and the contributions of in-plane membrane forces to the moment balances were not considered. The detailed discussion on the aforementioned effects can be referred to Luo (2000).

Similarly, the membrane forces in (6.158) become

$$\bar{N}_R=U_{R,R}+\frac{1}{2}(U_{Z,R})^2+\mu[\frac{1}{R}U_R+\frac{1}{R}U_{\theta,\theta}+\frac{1}{2R^2}(U_{z,\theta})^2],$$
$$\bar{N}_\theta=\frac{1}{R}U_R+\frac{1}{R}U_{\theta,\theta}+\frac{1}{2R^2}(U_{z,\theta})^2+\mu[U_{R,R}+\frac{1}{2}(U_{Z,R})^2], \tag{6.165}$$
$$\bar{N}_{R\theta}=\frac{1-\mu}{2}(U_{\theta,R}+\frac{1}{R}U_{R,\theta}-\frac{1}{R}U_\theta+\frac{1}{R}U_{Z,R}U_{Z,\theta}).$$

The corresponding boundary conditions become

$$\begin{aligned} &U_R = U_\theta = U_Z = 0, \quad U_{Z,R} = 0 \qquad \text{at } R=\kappa; \\ &\left.\begin{aligned} &U_{Z,RR} + \mu\left(\frac{1}{R}U_{Z,R} + \frac{1}{R^2}U_{Z,\theta\theta}\right) = 0, \\ &(\nabla^2 U_Z)_{,R} + \frac{1-\mu}{R^2}\left(U_{Z,R} - \frac{1}{R}U_Z\right)_{,\theta\theta} = 0 \end{aligned}\right\} \text{ at } R=1. \end{aligned} \tag{6.166}$$

In addition, the radial and shear membrane forces at $R=1$ require $\bar{N}_R = \bar{N}_{R\theta} = 0$, i.e.,

$$\left.\begin{aligned} &U_{R,R} + \frac{1}{2}(U_{Z,R})^2 + \mu\left[\frac{1}{R}U_R + \frac{1}{R}U_{\theta,\theta} + \frac{1}{2R^2}(U_{Z,\theta})^2\right] = 0, \\ &U_{\theta,R} + \frac{1}{R}U_{R,\theta} - \frac{1}{R}U_\theta + \frac{1}{R}U_{Z,R}U_{Z,\theta} = 0, \end{aligned}\right\} \text{ at } R=1. \tag{6.167}$$

If the waviness of the disk exists only in the transverse direction, the initial conditions are

$$\begin{aligned} &U_R = \dot{U}_R = U_\theta = \dot{U}_\theta = 0, \\ &U_Z = \Phi(R,\theta), \ \dot{U}_Z = \Psi(R,\theta), \text{ at } t=0. \end{aligned} \tag{6.168}$$

To proceed with analysis of Eqs.(6.161)–(6.163) in the small parameter of $0 < \varepsilon = h/b << 1$, series solutions satisfying Eq.(6.151) are proposed as

$$\begin{aligned} U_Z &= \varepsilon U_Z^{(1)} + \varepsilon^3 U_Z^{(3)} + \cdots, \\ U_R &= \varepsilon^2 U_R^{(2)} + \varepsilon^4 U_R^{(4)} + \cdots, \\ U_\theta &= \varepsilon^2 U_\theta^{(2)} + \varepsilon^4 U_\theta^{(4)} + \cdots. \end{aligned} \tag{6.169}$$

When the series solutions in Eq.(6.169) are substituted into Eqs.(6.161)–(6.163), it is found that the inertial forces in the radial and circumferential directions are of order ε^4 but the one in the transverse direction is of order ε^3. Therefore, in the following analysis, the inertial forces in the radial and hoop directions are ignored. In other words, without loss of generality, we only retain terms through ε^2 to avoid higher order calculation in radial and hoop direction. Substitution of Eq.(6.169) into Eqs.(6.161) and (6.162) gives for order ε^2:

$$\begin{aligned} &U_{R,RR}^{(2)} + \frac{1}{R}U_{R,R}^{(2)} - \frac{1}{R^2}U_R^{(2)} + \frac{1-\mu}{2R^2}U_{R,\theta\theta}^{(2)} + \frac{1+\mu}{2R}U_{\theta,R\theta}^{(2)} \\ &- \frac{3-\mu}{2R^2}U_{\theta,\theta}^{(2)} + U_{Z,R}^{(1)}\left(U_{Z,RR}^{(1)} + \frac{1-\mu}{2R^2}U_{Z,\theta\theta}^{(1)}\right) + \frac{1+\mu}{2R^2}U_{Z,R\theta}^{(1)}U_{Z,\theta}^{(1)} \\ &+ \frac{1-\mu}{2R}\left[(U_{Z,R}^{(1)})^2 - \frac{1}{R^2}(U_{Z,\theta}^{(1)})^2\right] + \frac{1}{12}\Omega^{*2}R = 0, \end{aligned} \tag{6.170}$$

$$\frac{1}{R^2}U^{(2)}_{\theta,\theta\theta}+\frac{1-\mu}{2}(U^{(2)}_{\theta,RR}+\frac{1}{R}U^{(2)}_{\theta,R}-\frac{1}{R^2}U^{(2)}_{\theta})$$
$$+\frac{1+\mu}{2R}U^{(2)}_{R,R\theta}+\frac{3-\mu}{2R^2}U^{(2)}_{R,\theta}+\frac{1+\mu}{2R}U^{(1)}_{Z,R}U^{(1)}_{Z,R\theta}$$
$$+\frac{1}{R}U^{(1)}_{Z,\theta}[\frac{1}{R^2}U^{(1)}_{Z,\theta\theta}+\frac{1-\mu}{2}(U^{(1)}_{Z,RR}+\frac{1}{R}U^{(1)}_{Z,R})]=0, \tag{6.171}$$

By Eqs.(6.166)–(6.168), the boundary and initial conditions for $U_R^{(2)}$, $U_\theta^{(2)}$ and $U_Z^{(1)}$ are

$$U_R^{(2)}=U_\theta^{(2)}=U_Z^{(2)}=0,\ \ U^{(1)}_{Z,R}=0,\quad \text{at } R=\kappa;$$
$$\left.\begin{aligned}&U^{(1)}_{Z,RR}+\mu(\frac{1}{R}U^{(1)}_{Z,R}+\frac{1}{R^2}U^{(1)}_{Z,\theta\theta})=0,\\ &(\nabla^2U_Z^{(1)})_{,R}+\frac{1-\mu}{R^2}(U^{(1)}_{,R}-\frac{1}{R}U_Z^{(1)})=0\end{aligned}\right\}\ \text{at } R=1. \tag{6.172}$$

$$\left.\begin{aligned}&U^{(2)}_{R,R}+\frac{1}{2}(U^{(1)}_{Z,R})^2+\mu[\frac{1}{R}U_R^{(2)}+\frac{1}{R}U^{(2)}_{\theta,\theta}+\frac{1}{2}(U^{(1)}_{Z,\theta})^2]=0,\\ &U^{(2)}_{\theta,R}+\frac{1}{R}U^{(2)}_{R,\theta}-\frac{1}{R}U_\theta^{(2)}+\frac{1}{R}U^{(1)}_{Z,R}U^{(1)}_{Z,\theta}=0,\end{aligned}\right\}\ \text{at } R=\ 1. \tag{6.173}$$

$$\begin{aligned}&U_R^{(2)}=\dot U_R^{(2)}=U_\theta^{(2)}=\dot U_\theta^{(2)}=0,\\ &U_Z^{(1)}=\Phi(R,\theta),\ \dot U_Z^{(1)}=\Psi(R,\theta),\ \text{at } t=0.\end{aligned} \tag{6.174}$$

6.3.2. Nonlinear waves

A solution for the transverse displacement satisfying the boundary condition in Eq.(6.172) is

$$U_Z^{(1)}=\sum_{s=0}^{\infty}\sum_{m=0}^{4}C_mR^{m+s}\left[f_{sc}(t)\cos(s\theta)+f_{ss}(t)\sin(s\theta)\right], \tag{6.175}$$

where $f_{sc}(t)$ and $f_{ss}(t)$ are generalized coordinates, s denotes the number of nodal diameters, and C_m are obtained from Eq.(6.172), i.e.,

$$\begin{aligned}&C_0=1,\ \ C_1=C_3\kappa^2+2C_4\kappa^3-\frac{2}{\kappa},\ C_2=-2C_3\kappa-3C_4\kappa^2+\frac{1}{\kappa^2},\\ &C_3=\frac{a_{10}a_{22}-a_{20}a_{12}}{a_{11}a_{22}-a_{21}a_{12}},\quad C_4=\frac{a_{20}a_{11}-a_{10}a_{21}}{a_{11}a_{22}-a_{21}a_{12}}.\end{aligned} \tag{6.176}$$

$$\begin{aligned}
&a_{10}=-(a_0-\frac{2}{\kappa}a_1+\frac{1}{\kappa^2}a_2),\ \ a_{11}=a_1\kappa^2-2a_2\kappa+a_3,\\
&a_{12}=2a_1\kappa^3-3a_2\kappa^2+a_4;\ \ a_{20}=-(b_0-\frac{2}{\kappa}b_1+\frac{1}{\kappa^2}b_2),\\
&a_{21}=b_1\kappa^2-2b_2\kappa+b_3,\ \ a_{22}=2b_1\kappa^3-3b_2\kappa^2+b_4;\\
&a_m=(m+s)(m+s-1)+\mu(m+s-s^2);\\
&b_m=(m+s-2)[(m+s)^2-s^2]-(1-\mu)(m+s-1)s^2.
\end{aligned} \tag{6.177}$$

Without loss of generality, in the analysis, that follows single-mode solutions for a specified s will be sought. Substitution of Eqs.(6.175) into (6.170) and (6.171) for specified s leads to

$$\begin{aligned}
U_R^{(2)}=&\Pi_0^{(2)}(f_{sc}^2+f_{ss}^2)+\Pi_1^{(2)}\left[(f_{sc}^2-f_{ss}^2)\cos(2s\theta)\ +2(f_{sc}f_{ss})\sin(2s\theta)\right]\\
&-\frac{\Omega^{*2}}{96}\left\{R^3+\frac{[(1+\mu)\kappa^4-(3+\mu)\kappa^2]R^{-1}}{(1-\mu)\kappa^2+(1+\mu)}+\frac{[(1-\mu)\kappa^4+3+\mu]R}{(1-\mu)\kappa^2+(1+\mu)}\right\},
\end{aligned} \tag{6.178}$$

$$U_\theta^{(2)}=\Pi_2^{(2)}\left[(f_{sc}^2-f_{ss}^2)\sin(2s\theta)-2(f_{sc}f_{ss})\cos(2s\theta)\right], \tag{6.179}$$

and the functions $\Pi_0^{(2)},\Pi_1^{(2)}$ and $\Pi_2^{(2)}$ are given by

$$\begin{aligned}
\Pi_0^{(2)}=&A_1^0R+A_2^0R^{-1}+\sum_{m_1,m_2=0}^{4}(\hat{A}_1^0\delta^2_{m_1+m_2+2s}R+\hat{A}_2^0\delta^0_{m_1+m_2+2s}R^{-1})\log R\\
&+\sum_{m_1,m_2=0}^{4}\left[\hat{A}_1^0\frac{1-\delta^2_{m_1+m_2+2s}}{m_1+m_2+2s-2}+\hat{A}_2^0\frac{1-\delta^0_{m_1+m_2+2s}}{m_1+m_2+2s}\right]R^{m_1+m_2+2s-1},
\end{aligned} \tag{6.180}$$

$$\begin{aligned}
\Pi_1^{(2)}=&-a_1A_1R^{2s+1}-A_2R^{2s-1}+b_1A_3R^{-2s+1}+A_4R^{-2s-1}\\
&+\sum_{m_1,m_2=0}^{4}\left[-a_1\hat{A}_1\frac{1-\delta^2_{m_1+m_2}}{m_1+m_2-2}-\hat{A}_2\frac{1-\delta^0_{m_1+m_2}}{m_1+m_2}\right.\\
&\left.+b_1\hat{A}_3\frac{1-\delta^2_{m_1+m_2+4s}}{m_1+m_2+4s-2}+\hat{A}_4\frac{1-\delta^0_{m_1+m_2+4s}}{m_1+m_2+4s}\right]R^{m_1+m_2+2s-1}\\
&+\sum_{m_1,m_2=0}^{4}[-a_1\hat{A}_1\delta^2_{m_1+m_2}R^{2s+1}-\hat{A}_2\delta^0_{m_1+m_2}R^{2s-1}\\
&+b_1\hat{A}_3\delta^2_{m_1+m_2+4s}R^{-2s+1}+\hat{A}_4\delta^0_{m_1+m_2+4s}R^{-2s-1}]\log R,
\end{aligned} \tag{6.181}$$

$$\begin{aligned}
\Pi_2^{(2)}=&A_1R^{2s+1}+A_2R^{2s-1}+A_3R^{-2s+1}+A_4R^{-2s-1}\\
&+\sum_{m_1,m_2=0}^{4}\left[\hat{A}_1\frac{1-\delta^2_{m_1+m_2}}{m_1+m_2-2}+\hat{A}_2\frac{1-\delta^0_{m_1+m_2}}{m_1+m_2}\right.
\end{aligned}$$

$$+\hat{A}_3\frac{1-\delta^2_{m_1+m_2+4s}}{m_1+m_2+4s-2}+\hat{A}_4\frac{1-\delta^0_{m_1+m_2+4s}}{m_1+m_2+4s}\Bigg]R^{m_1+m_2+2s-1}$$
$$+\sum_{m_1,m_2=0}^{4}[\hat{A}_1\delta^2_{m_1+m_2}R^{2s+1}+\hat{A}_2\delta^0_{m_1+m_2}R^{2s-1}$$
$$+\hat{A}_3\delta^2_{m_1+m_2+4s}R^{-2s+1}+\hat{A}_4\delta^0_{m_1+m_2+4s}R^{-2s-1}]\log R, \tag{6.182}$$

where all the coefficients $A_1^0, A_2^0, \ldots, A_4$, $\hat{A}_1^0, \hat{A}_2^0, \ldots, \hat{A}_4$, b_1 and b_2 are determined by Eqs.(6.170)–(6.173) with Eqs.(6.178) and (6.179), δ_i^j is the Kronecker delta.

Substitution of Eq.(6.169) into Eq.(6.165), retention of the terms ε^2 and use of $U_R^{(2)}$ and $U_\theta^{(2)}$ gives the membrane forces:

$$\begin{aligned}\bar{N}_R &= \frac{1}{96}\Omega^{*2}N_R^L+N_{R0}(f_{sc}^2+f_{ss}^2)\\ &\quad+N_{R1}[(f_{sc}^2-f_{ss}^2)\cos(2s\theta)-2(f_{sc}f_{ss})\sin(2s\theta)],\\ \bar{N}_\theta &= \frac{1}{96}\Omega^{*2}N_\theta^L+N_{\theta 0}(f_{sc}^2+f_{ss}^2)\\ &\quad+N_{\theta 1}[(f_{sc}^2-f_{ss}^2)\cos(2s\theta)-2(f_{sc}f_{ss})\sin(2s\theta)],\\ \bar{N}_{R\theta} &= N_{R\theta 1}[(f_{sc}^2-f_{ss}^2)\sin(2s\theta)+2(f_{sc}f_{ss})\cos(2s\theta)],\end{aligned} \tag{6.183}$$

where $N_{R0}, R_{\theta 0}, \ldots, N_{R\theta 1}$ are computed through $\Pi_0^{(2)}, \Pi_1^{(2)}$ and $\Pi_2^{(2)}$. The linear membrane forces are

$$\begin{aligned}N_R^L &= \frac{(1-\mu)[(3+\mu)\kappa^2-(1+\mu)\kappa^4]}{(1-\mu)\kappa^2+(1+\mu)}R^{-2}\\ &\quad+\frac{(1+\mu)[(1-\mu)\kappa^4+3+\mu]}{(1-\mu)\kappa^2+(1+\mu)}-(3+\mu)R^2,\\ N_\theta^L &= \frac{(1-\mu)[(1+\mu)\kappa^4-(3+\mu)\kappa^2]}{(1-\mu)\kappa^2+(1+\mu)}R^{-2}\\ &\quad+\frac{(1+\mu)[(1-\mu)\kappa^4+3+\mu]}{(1-\mu)\kappa^2+(1+\mu)}-(3\mu+1)R^2.\end{aligned} \tag{6.184}$$

For specified s, substitution of Eqs.(6.175), (6.178) and (6.179) into Eq.(6.163) and use of the Galerkin method yields:

$$\begin{aligned}&\ddot{f}_{sc}+2\Omega^* s\dot{f}_{ss}+(\frac{\alpha_s+\Omega^{*2}\gamma_0}{\beta_s}-\Omega^{*2}s^2)f_{sc}+\frac{\gamma_{sc}+\gamma_{s0}}{\beta_s}(f_{sc}^2+f_{ss}^2)f_{sc}=0,\\ &\ddot{f}_{ss}-2\Omega^* s\dot{f}_{sc}+(\frac{\alpha_s+\Omega^{*2}\gamma_0}{\beta_s}-\Omega^{*2}s^2)f_{ss}+\frac{\gamma_{sc}+\gamma_{s0}}{\beta_s}(f_{sc}^2+f_{ss}^2)f_{ss}=0;\end{aligned} \tag{6.185}$$

where

$$\alpha_s = \frac{1}{12}\sum_{m=0}^{4}\sum_{n=0}^{4}\frac{(1-\kappa^{m+n+2s-3})}{m+n+2s-3}[(m+s)^2 - s^2][(m+s-2)^2 - s^2]C_m C_n, \tag{6.186}$$

$$\beta_s = \frac{1}{12}\frac{1-\kappa^{m+n+2s+1}}{m+n+2s+1}C_m C_n, \tag{6.187}$$

$$\begin{aligned}\gamma_0 = \sum_{m=0}^{4}\sum_{n=0}^{4}\frac{1}{96}\Bigg\{&\frac{(1-\mu)[(1+\mu)\kappa^4 - 3\kappa^2]}{(1-\mu)\kappa^2 + (1+\mu)}[(m+s)(m+s-2)+s^2]\\ &\times\frac{1-\kappa^{m+n+2s-3}}{m+n+2s-3} - \frac{(1+\mu)[(1-\mu)\kappa^4+3]}{(1-\mu)\kappa^2+(1+\mu)}[(m+s)^2 - s^2]\\ &\times\frac{1-\kappa^{m+n+2s-1}}{m+n+2s-1} + [(m+s)(m+s+2)(3+\mu) - s^2(3\mu+1)]\\ &\times\frac{1-\kappa^{m+n+2s+1}}{m+n+2s+1}\Bigg\}C_m C_n,\end{aligned} \tag{6.188}$$

$$\begin{aligned}\gamma_{s0} = \sum_{m=0}^{4}\sum_{n=0}^{4}\Bigg[&(m+s)(n+s-1)\int_\kappa^1 N_{R0}R^{m+n+2s-2}dR\\ &+ s^2\int_\kappa^1 N_{\theta 0}R^{m+n+2s-2}dR - (m+s)(N_{R0}R^{m+n+2s-1})\Big|_\kappa^1\Bigg]C_m C_n,\end{aligned} \tag{6.189}$$

$$\begin{aligned}\gamma_{sc} = \sum_{m=0}^{4}\sum_{n=0}^{4}\Bigg[&(m+s)(n+s-1)\int_\kappa^1 N_{R1}R^{m+n+2s-2}dR\\ &+ s^2\int_\kappa^1 N_{\theta 1}R^{m+n+2s-2}dR - (m+s)(N_{R1}R^{m+n+2s-1})\Big|_\kappa^1\\ &-(2s)(m+n+2s-1)\int_\kappa^1 N_{R\theta 1}R^{m+n+2s-2}dR\\ &+(2s)(N_{R\theta 1}R^{m+n+2s-1})\Big|_\kappa^1\Bigg]C_m C_n.\end{aligned} \tag{6.190}$$

Notice that the coefficients $\alpha_s, \gamma_0, \beta_s, \gamma_{s0}, \gamma_{sc}$ are related to the plate stiffness, centrifugal forces, inertia forces and the membrane forces. The second term in Eq.(6.183) is caused by the Coriolis force. When $s=0$, two equations in Eq.(6.183) are identical. The problem reduces to the symmetrical one. For the symmetrical response, no Coriolis force contributes on the symmetrical response. Only the centrifugal force caused by rotation affects the symmetrical response. Therefore, the analyses given by Lamb and Southwell (1921) and Southwell (1922) are only for the symmetrical response of rotating disks.

The normalization of the initial conditions in Eq.(6.174) gives:

$$f_{sc}^0 = \frac{1}{\pi\Xi}\sum_{m=0}^{4}C_m\int_0^{2\pi}\int_\kappa^1 \Phi(R,\theta)R^{m+s}\cos(s\theta)dRd\theta,$$

$$\dot{f}_{sc}^{0} = \frac{1}{\pi\Xi}\sum_{m=0}^{4} C_m \int_0^{2\pi}\int_\kappa^1 \Psi(R,\theta)R^{m+s}\cos(s\theta)dRd\theta,$$
$$f_{ss}^{0} = \frac{1}{\pi\Xi}\sum_{m=0}^{4} C_m \int_0^{2\pi}\int_\kappa^1 \Phi(R,\theta)R^{m+s}\sin(s\theta)dRd\theta, \tag{6.191}$$
$$\dot{f}_{ss}^{0} = \frac{1}{\pi\Xi}\sum_{m=0}^{4} C_m \int_0^{2\pi}\int_\kappa^1 \Psi(R,\theta)R^{m+s}\sin(s\theta)dRd\theta,$$

where

$$\Xi = \sum_{m=0}^{4}\sum_{n=0}^{4}\frac{1-\kappa^{m+n+2s+1}}{m+n+2s+1}C_m C_n. \tag{6.192}$$

Integration of Eq.(6.185) leads to a constant energy function:

$$H = \frac{1}{2}(\dot{f}_{sc}^2 + \dot{f}_{ss}^2) + \frac{1}{2}[\frac{\alpha_s + \Omega^*\gamma_0}{\beta_s} - (\Omega^* s)^2](f_{sc}^2 + f_{ss}^2)$$
$$+\frac{1}{4}\frac{\gamma_{sc}+\gamma_{s0}}{\beta_s}(f_{sc}^2 + f_{ss}^2)^2 = E_0. \tag{6.193}$$

For the hardening disk $(\gamma_{sc}+\gamma_{s0})>0$ and for the softening disk $(\gamma_{sc}+\gamma_{s0})<0$. The integration constant E_0 can be determined by Eqs.(6.192) and (6.193).

(A) For $(\gamma_{sc}+\gamma_{s0})>0$, the solution to Eq.(6.192) with Eq.(6.191) is

$$f_{sc} = \frac{\pi\hat{A}_s}{2k_s K(k_s)}\sum_{l=0}^{\infty}\operatorname{sech}\left[(2l+1)\frac{\pi K(k_s')}{2K(k_s)}\right]$$
$$\times\Big(\cos\{[(2l+1)\omega_s + \Omega^* s]t + (2l+1)\varphi_0 + \phi_0\}$$
$$+\cos\{[(2l+1)\omega_s - \Omega^* s]t + (2l+1)\varphi_0 - \phi_0\}\Big),$$
$$f_{ss} = \frac{\pi\hat{A}_s}{2k_s K(k_s)}\sum_{l=0}^{\infty}\operatorname{sech}\left[(2l+1)\frac{\pi K(k_s')}{2K(k_s)}\right] \tag{6.194}$$
$$\times\Big(\sin\{[(2l+1)\omega_s + \Omega^* s]t + (2l+1)\varphi_0 + \phi_0\}$$
$$-\sin\{[(2l+1)\omega_s - \Omega^* s]t + (2l+1)\varphi_0 - \phi_0\}\Big),$$

where $K(k_s)$ is the complete elliptic integral of the first kind, $k_s' = \sqrt{1-k_s^2}$, $\hat{A}_s$ is the amplitude, ω_s is the natural frequency without Coriolis forces and φ_0 is the initial phase

$$\hat{A}_s = \frac{\sqrt{B-C}}{\sqrt{(\gamma_{sc}+\gamma_{s0})}},\ k_s = \sqrt{\frac{1}{2}-\frac{C}{2B}},\ \omega_s = \frac{\sqrt{B}\pi}{2\sqrt{\beta_s}K(k_s)};$$

$$
\begin{aligned}
& B=\sqrt{C^2+4(\gamma_{sc}+\gamma_{s0})\beta_s E_0}, \\
& C=\alpha_s+\Omega^{*2}\gamma;\ \tan\phi_0=\frac{f_{ss}^0}{f_{sc}^0}; \\
& \quad \mathrm{tn}\left[\frac{2K(k_s)\varphi_0}{\pi},k_s\right]\mathrm{dn}\left[\frac{2K(k_s)\varphi_0}{\pi},k_s\right] \\
& \quad =\frac{\pi\sqrt{[\dot{f}_{sc}^0-(\Omega^* s)f_{ss}^0]^2+[\dot{f}_{ss}^0+(\Omega^* s)f_{sc}^0]^2}}{2\omega_s K\left(k_s\right)\sqrt{f_{sc}^2+f_{ss}^2}},
\end{aligned}
\tag{6.195}
$$

where dn and tn are elliptic functions. Substitution of Eq.(6.194) into Eq.(6.175) gives transverse displacement:

$$
\begin{aligned}
U_Z^{(1)} &= \sum_{l=0}^{\infty}\sum_{m=0}^{4}\frac{\pi\hat{A}_s C_m R^{m+s}}{2k_s K(k_s)}\mathrm{sech}\left[(2l+1)\frac{\pi K(k_s')}{2K(k_s)}\right] \\
& \times\Big(\cos\{[(2l+1)\omega_s+\Omega^* s]t+(2l+1)\varphi_0+\phi_0-s\theta\} \\
& +\cos\{[(2l+1)\omega_s-\Omega^* s]t+(2l+1)\varphi_0-\phi_0+s\theta\}\Big).
\end{aligned}
\tag{6.196}
$$

(B) For $\left(\gamma_{sc}+\gamma_{s0}\right)<0$, the solution to Eq.(6.190) with Eq.(6.192) is

$$
\begin{aligned}
f_{sc} &= \frac{\pi\hat{A}_s}{2k_s K(k_s)}\sum_{l=0}^{\infty}\mathrm{csch}\left[(2l+1)\frac{\pi K(k_s')}{2K(k_s)}\right] \\
& \times\Big(\sin\{(2l+1)\omega_s+\Omega^* s]t+(2l+1)\varphi_0+\phi_0\} \\
& +\sin\{(2l+1)\omega_s-\Omega^* s]t+(2l+1)\varphi_0-\phi_0\}\Big), \\
f_{ss} &= \frac{\pi\hat{A}_s}{2k_s K(k_s)}\sum_{l=0}^{\infty}\mathrm{csch}\left[(2l+1)\frac{\pi K(k_s')}{2K(k_s)}\right] \\
& \times\Big(-\cos\{(2l+1)\omega_s+\Omega^* s]t+(2l+1)\varphi_0+\phi_0\} \\
& +\cos\{(2l+1)\omega_s-\Omega^* s]t+(2l+1)\varphi_0-\phi_0\}\Big),
\end{aligned}
\tag{6.197}
$$

where

$$
\begin{aligned}
& \hat{A}_s=\sqrt{\frac{C_1-B_1}{\left|\gamma_{sc}+\gamma_{s0}\right|}},k_s=\sqrt{\frac{C_1-B_1}{C_1+B_1}},\omega_s=\frac{\sqrt{C_1+B_1}}{2\sqrt{2}}\frac{\pi}{K(k_s)}; \\
& B_1=\sqrt{C_1^2-4\left|\gamma_{sc}+\gamma_{s0}\right|\beta_s E_0}, \\
& C_1=\alpha_s+\Omega^{*2}\gamma;\tan\phi_0=\frac{f_{ss}^0}{f_{sc}^0};
\end{aligned}
$$

$$\operatorname{cs}\left[\frac{2K(k_s)\varphi_0}{\pi},k_s\right]\operatorname{dn}\left[\frac{2K(k_s)\varphi_0}{\pi},k_s\right] = \frac{\pi\sqrt{\left[\dot{f}_{sc}^0-(\Omega^* s)f_{ss}^0\right]^2+\left[\dot{f}_{ss}^0+(\Omega^* s)f_{sc}^0\right]^2}}{2\omega_s K(k_s)\sqrt{f_{sc}^2+f_{ss}^2}}. \tag{6.198}$$

Substitution of Eq.(6.197) into Eq.(6.175) gives

$$\begin{aligned} U_Z^{(1)} &= \sum_{l=0}^{\infty}\sum_{m=0}^{4}\frac{\pi A_s C_m R^{m+s}}{2k_s K(k_s)}\operatorname{csch}\left[(2l+1)\frac{\pi K(k_s')}{2K(k_s)}\right] \\ &\times\big(\sin\{[(2l+1)\omega_s+\Omega^* s]t+(2l+1)\varphi_0+\phi_0-s\theta\} \\ &+\sin\{[(2l+1)\omega_s-\Omega^* s]t+(2l+1)\varphi_0-\phi_0+s\theta\}\big). \end{aligned} \tag{6.199}$$

$U_R^{(2)}$ and $U_\theta^{(2)}$ can be determined from Eqs.(6.178) and (6.179) in a similar manner. From time-dependent sine and cosine terms in Eq.(6.189) or Eq.(6.199), they indicate two dimensionless, modal frequencies:

$$\varpi_{1,2}^* = \begin{cases} (2l+1)\,\omega_s \pm \Omega^* s, & \text{if } (2l+1)\,\omega_s \ge \Omega^* s, \\ \Omega^* s \pm (2l+1)\,\omega_s, & \text{if } (2l+1)\,\omega_s \le \Omega^* s. \end{cases} \tag{6.200}$$

From the nonlinear waves, linear waves can be obtained from reduction of nonlinear waves. If the nonlinear terms in Eq.(6.190) vanish, the linear solution is obtained as

$$\begin{aligned} U_Z &= \sum_{s=0}^{\infty}\sum_{m=0}^{4}\frac{1}{2}C_m R^{m+s}\hat{A}_s\left\{\cos[(\omega_s+\Omega^* s)t+\varphi_0+\phi_0-s\theta]\right. \\ &\left.+\cos[(\omega_s-\Omega^* s)t+\varphi_0-\phi_0+s\theta]\right\}, \end{aligned} \tag{6.201}$$

where

$$\begin{aligned} &\omega_s = \sqrt{\frac{\alpha_s+\Omega^{*2}\gamma_0}{\beta_s}},\ \hat{A}_s = \frac{\sqrt{(\dot{A}_s^0)^2+(A_s^0)^2}}{\omega_s},\ \tan\phi_0 = \frac{f_{ss}^0}{f_{sc}^0}, \\ &\tan\varphi_0 = \frac{\sqrt{[\dot{f}_{sc}^0-(\Omega^* s)f_{ss}^0]^2+[\dot{f}_{ss}^0+(\Omega^* s)f_{sc}^0]^2}}{\omega_s\sqrt{f_{sc}^2+f_{ss}^2}}. \end{aligned} \tag{6.202}$$

Because $(\gamma_{sc}+\gamma_{s0})=0$ in Eqs.(6.195) and (6.198), $k_s=0$, $K(k_s)=\pi/2$ and Eq.(6.201) is recovered. $\operatorname{tn}[2K(k_s)\varphi_0/\pi,k_s]=\tan\varphi_0$, $\operatorname{sc}[2K(k_s)\varphi_0/\pi,k_s]=\cot\varphi_0$ and $\operatorname{dn}[2K(k_s)\varphi_0/\pi,k_s]=1$ as $k_s=0$. The Jacobi's nome in Byrd and Friedman (1954) gives for $k_s^2<1$,

$$e^{-\pi K(k_s')/K(k_s)} = \frac{k_s^2}{16}\left[1+2(\frac{k_s}{4})^2+15(\frac{k_s}{4})^4+150(\frac{k_s}{4})^6+\cdots\right]^4; \tag{6.203}$$

and

$$\lim_{k_s\to 0}\frac{1}{k_s}\operatorname{sech}\left[(2l+1)\frac{\pi K(k_s')}{K(k_s)}\right]$$
$$= \lim_{k_s\to 0}\frac{\frac{2k_s^{2l}}{16^{(l+1/2)}}[1+2(\frac{k_s}{4})^2+15(\frac{k_s}{4})^4+\cdots]^{4(l+1/2)}}{1+\frac{k_s^{2(2l+1)}}{16^{(2l+1)}}[1+2(\frac{k_s}{4})^2+15(\frac{k_s}{4})^4+\cdots]^{4(2l+1)}} = \begin{cases}\frac{1}{2}, & \text{for } l=0,\\ 0, & \text{for } l\neq 0;\end{cases} \tag{6.204}$$

$$\lim_{k_s\to 0}\frac{1}{k_s}\operatorname{csch}\left[(2l+1)\frac{\pi K(k_s')}{2K(k_s)}\right]$$
$$= \lim_{k_s\to 0}\frac{\frac{2k_s^{2l}}{16^{(l+1/2)}}[1+2(\frac{k_s}{4})^2+15(\frac{k_s}{4})^4+\cdots]^{4(l+1/2)}}{1-\frac{k_s^{2(2l+1)}}{16^{(2l+1)}}[1+2(\frac{k_s}{4})^2+15(\frac{k_s}{4})^4+\cdots]^{4(2l+1)}} = \begin{cases}\frac{1}{2}, & \text{for } l=0,\\ 0, & \text{for } l\neq 0.\end{cases} \tag{6.205}$$

Therefore, as $\gamma_{sc}+\gamma_{s0}=0$, the subharmonic terms in Eqs.(6.196) and (6.199) for $l\neq 0$ vanish. The linear solution in Eq.(6.201) is recovered.

Natural frequencies predicted through the linear analysis for specific s are shown in Fig.6.13. The frequency is independent of modal amplitude A_s. From Eqs.(6.188) and (6.191), the natural frequency in the nonlinear analysis depends on A_s and nodal-diameter number s because E_0 and k_s depend on A_s and s. The natural frequency in the symmetric mode ($s=0$, $l=0$) is plotted in Fig.6.14 for A_s. Vanishing amplitude $A_s=0.0$ returns the linear prediction. For $A_s\leq 1.0$, the nonlinear prediction of the natural frequencies differs from the linear prediction by less than 6.5%. The natural frequencies predicted by use of Eqs.(6.160)–(6.163) and the von Karman theory are identical for $s=0$. The natural frequency ($s=3$, $l=0$) for an asymmetric response of the hardening disk is depicted by the solid line through Eqs.(6.160)–(6.163) and the dash line through the von Karman theory in Fig.6.15. The natural frequencies for such disk increase with A_s. The natural frequency predicted by two nonlinear theories differs by less than 10% as $A_s<0.4$. The critical speeds, predicted by the linear theory, the von Karman theory and the new theory, are $\Omega_{\text{crL}}\approx 35{,}000$ rpm, $\Omega_{\text{crK}}\approx 39{,}600$ rpm and $\Omega_{\text{crN}}\approx$ 43,600 rpm at $A_s=0.4$, respectively.

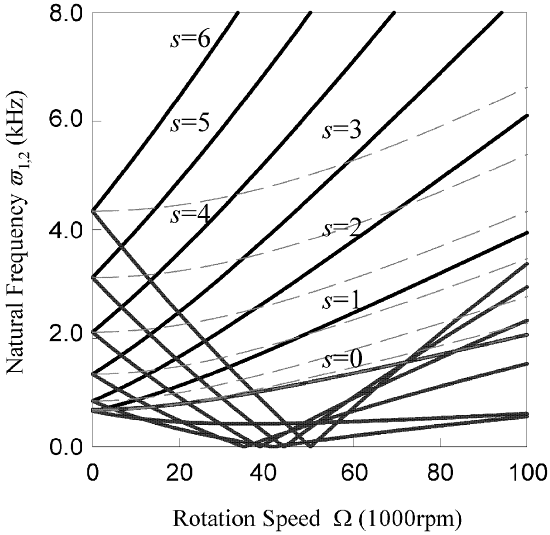


Fig. 6.13 Natural frequency of the computer disk predicted through the linear analysis. ($a = 15.5$ mm, $b = 43$ mm, $h = 0.775$ mm, $\rho_0 = 3641\text{kg/m}^3$, $E = 69$ GPa, $\mu = 0.33$.)

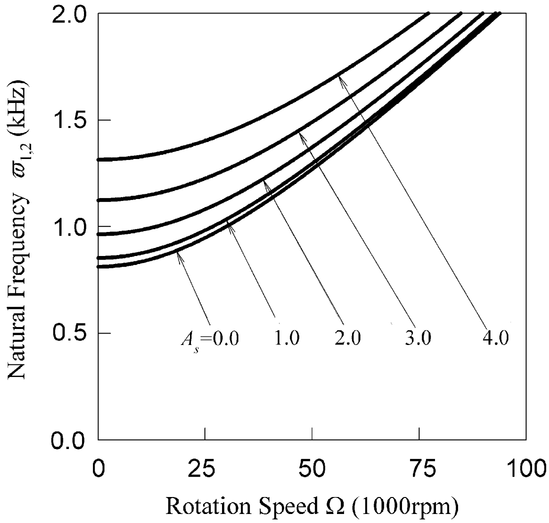


Fig. 6.14 Natural frequency ($s = 0,\ l = 0$)of the disk. $A_s = \sqrt{f_{sc}^2 + f_{ss}^2}$ is modal amplitude. Two nonlinear theories give the identical predictions. Two nonlinear models at $A_s = 0$ reduce to the linear model. ($a = 15.5$ mm, $b = 43$ mm, $h = 0.775$ mm, $\rho_0 = 3641\text{kg/m}^3$, $E = 69$ GPa, $\mu =$ 0.33 .)

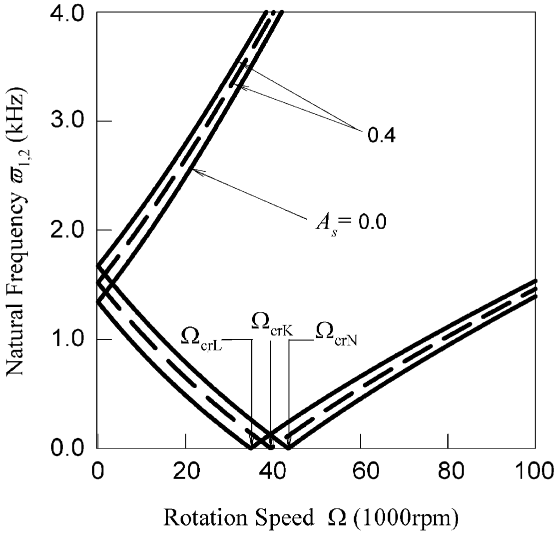


Fig. 6.15 Natural frequency ($s = 3$, $l = 0$) of the hardening disk. $A_s = \sqrt{f_{sc}^2 + f_{ss}^2}$ is modal amplitude. The solid and dash lines denote this theory and the von Karman theory. Two nonlinear models at $A_s = 0$ reduce to the linear model. $\Omega_{\mathrm{crL}}, \Omega_{\mathrm{crK}}$ and Ω_{crN} are the critical speeds predicted through the linear theory, the von Karman theory and the new theory, respectively. ($a = 15.5$ mm, $b = 43$ mm, $h = 0.775$ mm, $\rho_0 = 3641 \mathrm{kg/m^3}$, $E = 69$ GPa, $\mu = 0.33$.)

6.3.3. Resonant and stationary waves

Herein, from nonlinear and nonlinear wave solutions of the rotating disks, the resonant and stationary waves will be discussed. In this section, such materials can be referred to Luo and Tan (2001).

6.3.3a Resonant waves

Applying the resonant conditions for multiple degrees-of-freedom systems in Lichtenberg and Lieberman (1992), the two natural frequencies in Eq.(6.200) satisfying $m_1\varpi_1 = n_1\varpi_2$, require

$$\Omega^* s = \frac{m_1 + n_1}{m_1 - n_1}(2l+1)\omega_s, \quad \text{for } (2l+1)\omega_s < \Omega^* s,$$
$$\Omega^* s = \frac{m_1 - n_1}{m_1 + n_1}(2l+1)\omega_s, \quad \text{for } (2l+1)\omega_s > \Omega^* s; \tag{6.206}$$

where integers m_1 and n_1 are positive and irreducible and $m_1 > n_1$. Equation(6.206) gives the $(m_1 : n_1)$ resonance of harmonics at $l = 0$ and of the $(2l+1)$th sub-harmonics at $l \neq 0$. The amplitude of resonance in Eq.(6.195) or Eq.(6.198) at $l = 0$ is the largest because the hyperbolic function "sech" or "csch" decreases exponentially as l increases. The rotational speed at which the specific sub-harmonic $(m_1 : n_1)$ resonance occurs can be solved through Eq.(6.195) (or Eq.(6.199)) and Eq.(6.198).

For the linear vibration, the natural frequencies in Eq.(6.202) are

$$\varpi_{1,2} = \begin{cases} \omega_s \mp \Omega^* s, & \text{if } \omega_s \geq \Omega^* s, \\ \Omega^* s \mp \omega_s, & \text{if } \omega_s \leq \Omega^* s. \end{cases} \tag{6.207}$$

The resonant condition $m_1 \varpi_1 = n_1 \varpi_2$ gives:

$$\begin{aligned} \Omega^* s &= \frac{m_1 + n_1}{m_1 - n_1} \sqrt{\frac{\alpha_s + \Omega^{*2} \gamma_0}{\beta_s}}, \qquad \text{for } \omega_s < \Omega^* s; \\ \Omega^* s &= \frac{m_1 - n_1}{m_1 + n_1} \sqrt{\frac{\alpha_s + \Omega^{*2} \gamma_0}{\beta_s}}, \qquad \text{for } \omega_s > \Omega^* s. \end{aligned} \tag{6.208}$$

Consider a 3.5 inch diameter disk similar to a digital memory disk with inner and outer radii of $a = 15.5$ mm, $b = 43$ mm; thickness $h = 0.775$ mm, density $\rho_0 = 3641 \text{kg/m}^3$, Young's modulus E=69 GPa and Poisson's ratio $\nu = 0.33$. For an initial condition ($f_{sc}^0 = 0.1$, $f_{ss}^0 = 0.05$ and $\dot{f}_{sc}^0 = \dot{f}_{ss}^0 = 0.0$), the wave amplitude is $A = A_s = 0.1118$ and $\phi_0 = 26.565^\circ$. Using the resonant conditions and the given wave amplitude of generalized coordinates, the resonant spectrum based on the nonlinear plate theory is illustrated in Fig.6.16(a). The fractional numbers represent the $(m_1 : n_1)$-resonance defined by $\varpi_1 / \varpi_2 = n_1 / m_1$. From the second condition of Eq.(6.206) or (6.208), two limiting cases exist: $\Omega = 0$ at $n_1 / m_1 = 1$ and $\Omega = \Omega_{cr}$ at $n_1 / m_1 = 0$, where Ω_{cr} is determined by the stationary wave condition discussed later. However, from the first condition of Eq.(6.206) or (6.208), $\Omega = \Omega_{cr}$ at $n_1 / m_1 = 0$ and $\Omega = \infty$ at $n_1 / m_1 = 1$. In Fig.6.16(b), the resonant rotational speed varying with the wave amplitude is illustrated for a (2:1)-resonant condition. When the initial wave amplitude vanishes, the resonant condition reduces to the one predicted by linear model. For hardening (softening) modes of rotating disks, the resonant rotational speed increases (decreases) with increasing wave amplitude. For a linear (2:1)-resonant wave response with $s = 4$, the rotational speed is $\Omega = 10{,}7695$ rpm from Eq.(6.208). At this rotation speed, f_{sc}, f_{ss} as predicted by the linear theory are shown in Fig.6.17(a). A triangular locus of the f_{sc} versus f_{ss} diagram results from $m_1 + n_1 = 3$. In the nonlinear (2:1)-primary resonant response, the rotational speed for wave amplitude $A = 0.1118$ is

$\Omega = 10,941$ rpm computed from (6.206) with $l = 0$ and $s = 4$. As in Fig.6.17(b), as time increases, the locus of f_{sc} versus f_{ss} generates a rotating triangular form because of the sub-harmonic terms in the wave solution (e.g., Eq.(6.194)).

6.3.3b Stationary waves

For stationary waves, the natural frequency $\varpi_1 = (2l+1)\omega_s - \Omega^* s = 0$ is required, i.e.,

$$\Omega^* = \frac{(2l+1)}{2s}\left[\frac{4(\gamma_{sc}+\gamma_{s0})}{\beta_s}E_0 + (\frac{\alpha_s + \Omega^{*2}\gamma_0}{\beta_s})^2\right]^{1/4} \frac{\pi}{K(k_s)}, \quad \text{for } \gamma_{sc}+\gamma_{s0} \geq 0;$$

$$\Omega^* = \frac{(2l+1)}{2\sqrt{2}s}\left[\sqrt{(\frac{\alpha_s + \Omega^{*2}\gamma_0}{\beta_s})^2 + \frac{4(\gamma_{sc}+\gamma_{s0})}{\beta_s}E_0} + \frac{\alpha_s + \Omega^{*2}\gamma_0}{\beta_s}\right]^{1/2} \frac{\pi}{K(k_s)}, \quad \text{for } \gamma_{sc}+\gamma_{s0} \leq 0. \tag{6.209}$$

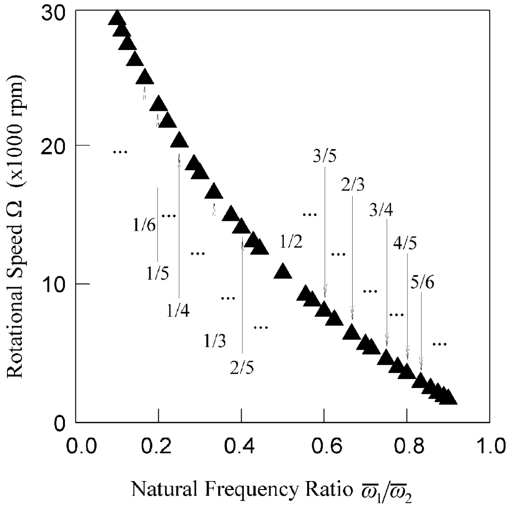
If the non-dimensional, rotation speed Ω^* satisfies Eq.(6.209), one of traveling waves becomes stationary. From Eq.(6.209) a stationary wave is possible for each integer l and a specified s. Note that the $l = 0$ terms give the primary harmonic waves and the $l \neq 0$ terms give the subharmonic waves. The rotation speeds for the stationary waves at $l = 0$ are referred to as the critical speed in the linear analysis. For linear vibration of the rotating disk, the stationary wave satisfies $\varpi_1 = \Omega^* s - \omega_s = 0$ that leads to

$$\Omega^* = \sqrt{\frac{\alpha_s}{\beta_s s^2 - \gamma_0}}. \tag{6.210}$$

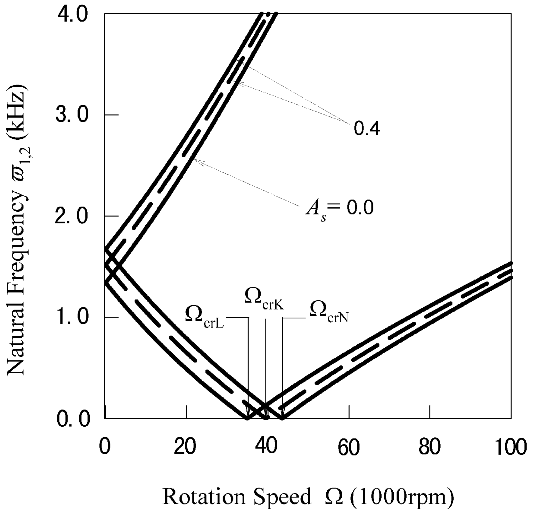
Moreover, with the condition $\omega_s - \Omega^* s = 0$ and the initial velocities $\dot{f}_{sc}^0 = \dot{f}_{ss}^0 = 0.0$, equation (6.183) gives $f_{sc} = f_{sc}^0$ and $f_{ss} = f_{ss}^0$. Thus one obtains

$$U_Z = \sum_{m=0}^{4} C_m R^{m+s}[f_{sc}^0 \cos(s\theta) + f_{sc}^0 \sin(s\theta)]. \tag{6.211}$$

The rotational speed for a stationary wave in the linear model is $\Omega = 38,546$ rpm, computed from Eq.(6.210) for $s = 4$ as an example. At this speed, $f_{sc} = 0.1$ and $f_{ss} = 0.05$ are independent of time. The rotation speed for the stationary wave of the first order (i.e., $(2l+1) = 1$) is $\Omega = 38,700$ rpm computed from Eq.(6.209) with $A = 0.1118$.



(a)



(b)

Fig.6.16 (a) Resonant spectrum ($s = 4$, $l = 0$) based on the nonlinear theory for the primary wave with amplitude $A \approx 0.1118$. (b) $(2:1)$ -resonant conditions represented as a function of wave amplitude and rotational speed for primary waves. The resonant condition for the linear model is at $A = 0$. The fractional numbers indicate the $(m_1 : n_1)$ -resonance defined by $\varpi_1 / \varpi_2 = n_1 / m_1$. ($a = 15.5$ mm, $b = 43$ mm, $h = 0.775$ mm, $\rho_0 = 3641$ kg/m^3, $E = 69$ GPa, $\nu = 0.33$.)

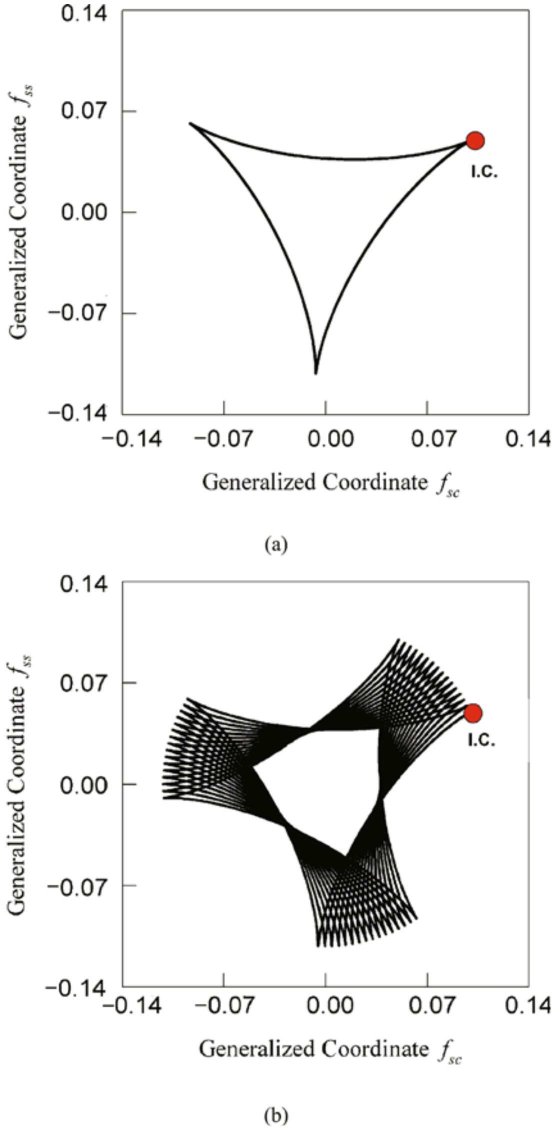


Fig. 6.17 A (2:1)- resonant response for ($s = 4, l = 0$) at $\Omega = 10{,}769$ rpm : (a) the linear theory and (b) the nonlinear theory. The initial condition is $f_{sc}^0 = 0.1$, $f_{ss}^0 = 0.05$, $\dot{f}_{sc}^0 = \dot{f}_{ss}^0 = 0.0$. ($a = 15.5$ mm, $b = 43$ mm, $h = 0.775$ mm, $\rho_0 = 3641$ kg/m^3, $E = 69$ GPa, $\nu = 0.33$.)

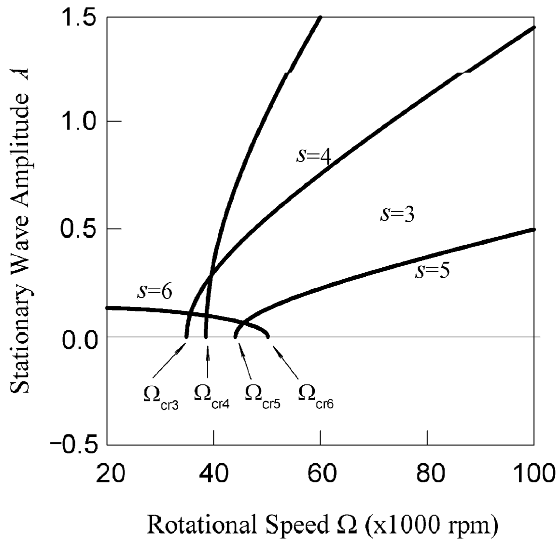


Fig.6.18 Stationary wave conditions represented as a function of wave amplitude and rotational speed for primary wave $(l = 0)$. The critical speed predicted by the linear model is at $A = 0$. ($a = 15.5$ mm, $b = 43$ mm, $h = 0.775$ mm, $\rho_0 = 3641$ kg/m^3, $E = 69$ GPa, $\nu = 0.33$.)

Because the backward waves in the nonlinear model include infinite numbers of terms in Eq.(6.189) or Eq.(6.199), the stationary wave condition (i.e., $(2l+1)\omega_s - \Omega^* s = 0$) is affected by the other non-zero terms of backward sub-harmonic waves. In Fig.6.18, the stationary conditions are illustrated as a function of the wave amplitude and rotational speed. The critical speed at $A = 0$ is denoted by Ω_{cr}. The rotation speed for stationary waves in the hardening (or softening), rotating disk modes increases (or decreases) with increasing wave amplitude. These hardening and softening characteristics are similar to those observed in resonant waves.

6.4. Conclusions

This chapter presented a frame work for approximate nonlinear plate theories from the theory of 3-dimensional deformed body. An approximate nonlinear plate theory was developed under the Kirchoff's assumptions. Such a theory can easily reduce to the traditional linear and nonlinear (von Karman) plate theories. This theory provides a possibility for one to further consider higher-order terms of nonlinearity in plates, and such a theory was applied for traveling plates and rotating disks. From the traditional perturbation and Galerkin analysis, the approximate solutions of nonlinear waves in the traveling plates and rotating disk were developed. Based on such nonlinear and linear solutions of waves, the resonant and stationary waves were discussed. In addition, under periodic excitation, the chaotic wave motions of traveling plates were discussed. For the other boundary conditions and loading, one can follow the similar procedure to investigate nonlinear and chaotic waves in traveling plates (or panels) and rotating disks. Nonlinear waves in thermal rotating disks, can be found in (e.g., Nowinski,1981; Saniei and Luo, 2001, 2007).

Even if the plate theory is still approximate, the wave solutions of the traveling and rotating disk are adequate in certain conditions. However, the exact solutions of nonlinear waves need to be further developed. One can adopt the Lie group analysis to find the exact solutions for such nonlinear plates. Of course, the best way is to solve such nonlinear waves in plates directly from the 3-dimensional theory, and the exact solution for nonlinear waves in plates can be achieved.

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Chapter 7
Nonlinear Webs, Membranes and Shells

This chapter will discuss the nonlinear theories of webs, membranes and shells. The webs and membranes are extensively used to model bio-tissues and membranes experiencing arbitrary initial configurations. Since any webs cannot resist any compressive forces, it is very difficult to determine the corresponding deformations and stresses in the webs. Traditionally, one used the membrane theory with certain constraints to determine the final configuration of webs. However, the webs can be of non-continuum and continuum. Thus, the network non-continuum web will be presented first from the cable theory, and the continuum web will be developed. Further, the nonlinear theory of membranes will be developed in an analogy way. The nonlinear theory of shells will be developed from the general theory of the 3-dimensional deformable body, and such a theory of shells can be easily reduced to the established theories.

7.1. Nonlinear webs

In this section, the theory for nonlinear webs will be discussed. Before the nonlinear web theory is discussed, the following concepts are introduced first.

Definition 7.1. If a deformable body on the two principal directions of fibers only resists the tensile forces, the deformable body is called a *deformable* web.

Definition 7.2. If a non-deformable body on the two principal directions of fibers only resists the tensile forces, the non-deformable body is called an *inextensible* web.

From the above two definitions, if the internal tensile forces of the web are compressive globally, the current configurations of the deformable and inextensible webs cannot exist. In other words, the deformable and inextensible webs cannot resist the compressive forces. If the internal tensile forces on the web become zero globally, the web configuration will keep the configuration of the inextensible web. If the internal tensile forces on the web become zero in a local zone, the configuration of the local zone of the web will keep the inextensible web configuration. On such a local region, the corresponding inextensible configuration may be changed with any small perturbation force to form a new configuration with tensile forces or to be in any wrinkling state without any configuration. Such a local new configuration is discontinuous to the non-wrinkled, global configuration. Such a phenomenon of the web is called *the local wrinkling* of web. After the web is locally wrinkled, the wrinkled zone cannot form any configuration. In other words, only when the tensile force exists, the web configuration can be formed.

Consider a nonlinear web with an initial configuration in coordinates (X^I, $I=1,2,3$) with unit vectors ($\mathbf{I}_I$, $I=1,2,3$), as shown in Fig.7.1. A point P on the initial configuration is given by (X^I, $I=1,2,3$) which is the function of a curvilinear coordinate (S^Λ, $\Lambda=1,2,3$) with base vectors $\mathbf{G}_\Lambda$. Therefore, assumes $X^I=X^I(S^1,S^2)$ and the point P on the initial configuration is

$$\mathbf{R}=X^I(S^1,S^2)\mathbf{I}_I. \tag{7.1}$$

Under external forces, the nonlinear web will form a new configuration in a coordinate (x^I, $I=1,2,3$) with unit vectors ($\mathbf{I}_I$, $I=1,2,3$), as shown in Fig.7.2. A point p on the new configuration is given by (x^I, $I=1,2,3$). For the new configuration, the corresponding curvilinear coordinates (s^Λ, $\Lambda=1,2,3$) with base vectors ($\mathbf{g}_\Lambda$, $\Lambda=1,2,3$) exist. Such a configuration is also called a final configuration under such external forces. To describe the deformation of the web, a point p (x^I, $I=1,2,3$) can also be described through the curvilinear coordinates (S^α, $\alpha=1,2$) with base vectors $\mathbf{G}_\alpha$. Therefore, let $x^I=x^I(S^1,S^2)$ and the vector $\mathbf{r}$ is

$$\mathbf{r}=x^I(S^1,S^2)\mathbf{I}_I. \tag{7.2}$$

The displacement between the point P and point p is given by

$$\mathbf{u}=u^I(S^1,S^2)\mathbf{I}_I. \tag{7.3}$$

Thus,

$$\left.\begin{aligned}
&\mathbf{r}=\mathbf{R}+\mathbf{u},\ \text{or}\ x^I(S^1,S^2)=X^I(S^1,S^2)+u^I(S^1,S^2);\\
&\mathbf{G}_\alpha\equiv\frac{\partial\mathbf{R}}{\partial S^\alpha}=X^I_{,\alpha}\mathbf{I}_I\ \text{and}\ \mathbf{g}_\alpha\equiv\frac{\partial\mathbf{r}}{\partial S^\alpha}=(X^I_{,\alpha}+u^I_{,\alpha})\mathbf{I}_I;\\
&\mathbf{N}_\alpha\equiv\frac{\mathbf{G}_\alpha}{|\mathbf{G}_\alpha|}=\frac{1}{\sqrt{G_{\alpha\alpha}}}X^I_{,\alpha}\mathbf{I}_I\ \text{and}\ \mathbf{n}_\alpha\equiv\frac{\mathbf{g}_\alpha}{|\mathbf{g}_\alpha|}=\frac{X^I_{,\alpha}+u^I_{,\alpha}}{\sqrt{G_{\alpha\alpha}+2E_{\alpha\alpha}}}\mathbf{I}_I.
\end{aligned}\right\} \tag{7.4}$$

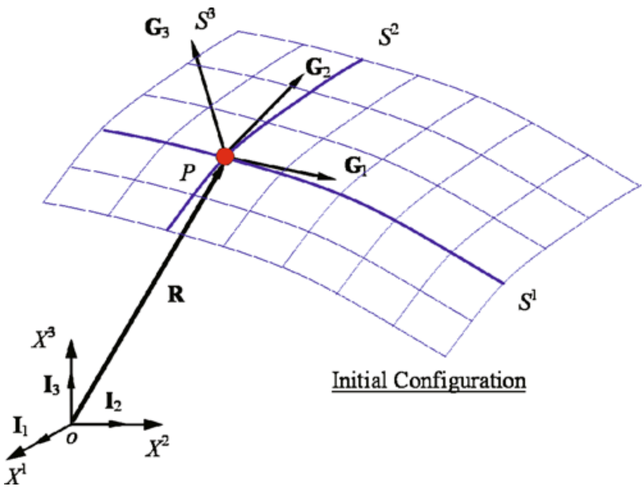

Fig. 7.1. A nonlinear web with an initial configuration.

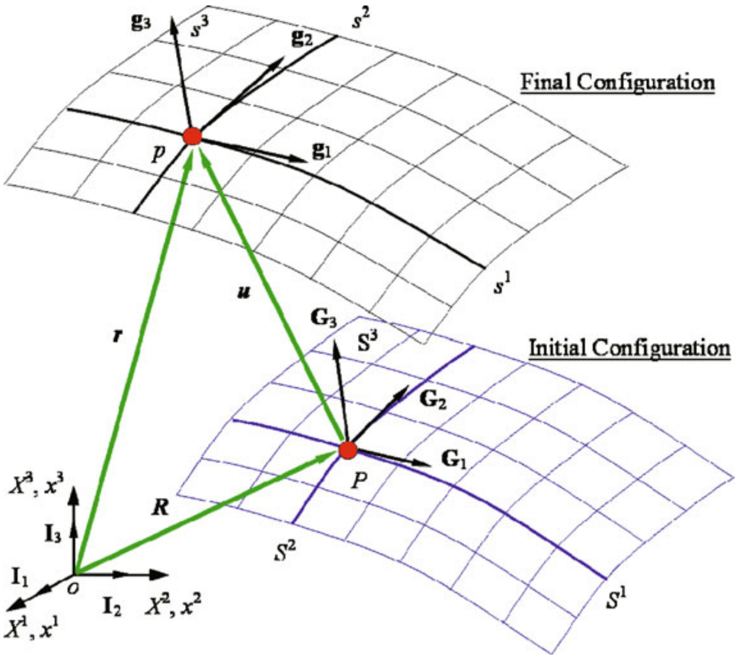

Fig. 7.2 Displacement on the webs with the initial and final configurations.

where $(\cdot)_{,\alpha} = \partial(\cdot)/\partial S^{\alpha}$ and summation on I should be completed, and

$$G_{\alpha\alpha} = X^{I}_{,\alpha} X^{I}_{,\alpha} \text{ and } E_{\alpha\beta} = X^{I}_{,\alpha} u^{I}_{,\beta} + X^{I}_{,\beta} u^{I}_{,\alpha} + u^{I}_{,\alpha} u^{I}_{,\beta}. \tag{7.5}$$

The strain on the direction of S^{α} ($\alpha = 1, 2$) is

$$\mathscr{E}_{\alpha}(S^1, S^2) = \frac{|\underset{\alpha}{d\mathbf{r}}| - |\underset{\alpha}{d\mathbf{R}}|}{|\underset{\alpha}{d\mathbf{R}}|} = \frac{1}{\sqrt{G_{\alpha\alpha}}} \sqrt{(X^{I}_{,\alpha} + u^{I}_{,\alpha})(X^{I}_{,\alpha} + u^{I}_{,\alpha})} - 1. \tag{7.6}$$

From material laws, the tension is

$$T_{\alpha} = \int_{A_{\alpha}} f_{\alpha}(\mathscr{E}_{\alpha}) dA_{\alpha}. \tag{7.7}$$

If $\mathscr{E}_{\alpha} = 0$ then $f(\mathscr{E}_{\alpha}) = 0$. For linear elastic materials, the tension is determined by

$$T_{\alpha} = \int_{A_{\alpha}} E_{\alpha} \mathscr{E}_{\alpha} dA_{\alpha} = \int_{A_{\alpha}} E_{\alpha} \Big(\frac{1}{\sqrt{G_{\alpha\alpha}}} \sqrt{(X^{I}_{,\alpha} + u^{I}_{,\alpha})(X^{I}_{,\alpha} + u^{I}_{,\alpha})} - 1\Big)\, dA_{\alpha}, \tag{7.8}$$

where Young's modulus and cross section in the S^{α}-direction are E_{α} and A_{α}, respectively. If the initial configuration is in the deformed state with the initial tension T^0_{α} in the S^{α}-direction, the corresponding strain is expressed by

$$\varepsilon_{\alpha} = \mathscr{E}^{0}_{\alpha}(S^1, S^2) + \frac{1}{\sqrt{G_{\alpha\alpha}}} \sqrt{(X^{I}_{,\alpha} + u^{I}_{,\alpha})(X^{I}_{,\alpha} + u^{I}_{,\alpha})} - 1, \tag{7.9}$$

where the initial tension $T^0_{\alpha}(S^1, S^2) = \int_{A_{\alpha}} f(\mathscr{E}^{0}_{\alpha}) dA_{\alpha}$ and the tension in the deformed configuration is

$$T_{\alpha} = \int_{A_{\alpha}} f_{\alpha}(\varepsilon_{\alpha}) dA_{\alpha} = \int_{A_{\alpha}} f_{\alpha}(\mathscr{E}^{0}_{\alpha} + \mathscr{E}_{\alpha}) dA_{\alpha}. \tag{7.10}$$

For the linear elasticity,

$$T_{\alpha} = T^0_{\alpha}(S^1, S^2) + E_{\alpha} A_{\alpha} \Big(\frac{1}{\sqrt{G_{\alpha\alpha}}} \sqrt{(X^{I}_{,\alpha} + u^{I}_{,\alpha})(X^{I}_{,\alpha} + u^{I}_{,\alpha})} - 1. \tag{7.11}$$

If the new displacement $u^I = 0$ ($I = 1, 2, 3$), then the strain in Eq. (7.9) becomes

$$\varepsilon_{\alpha} = \mathscr{E}^{0}_{\alpha}(S^1, S^2) + \Big(\frac{1}{\sqrt{G_{\alpha\alpha}}} \sqrt{X^{I}_{,\alpha} X^{I}_{,\alpha}} - 1\Big). \tag{7.12}$$

From the geometric relation, the following relation exists

$$\frac{1}{\sqrt{G_{\alpha\alpha}}} \sqrt{X^{I}_{,\alpha} X^{I}_{,\alpha}} = 1. \tag{7.13}$$

The initial strain and tension can be recovered (i.e., $\varepsilon_\alpha = \mathscr{E}^0_\alpha(S^1, S^2)$). In fact, the initial tension is very difficult to obtain. On the other hand, the inextensible web possesses the fact of $E_\alpha A_\alpha \to \infty$. Deformation of Eq.(7.11) gives

$$\frac{T_\alpha}{E_\alpha A_\alpha} = \frac{T^0_\alpha(S^1, S^2)}{E_\alpha A_\alpha} + \left(\frac{1}{\sqrt{G_{\alpha\alpha}}}\sqrt{(X^I_{,\alpha} + u^I_{,\alpha})(X^I_{,\alpha} + u^I_{,\alpha})} - 1\right). \tag{7.14}$$

As $E_\alpha A_\alpha \to \infty$, the foregoing equation gives for $\alpha = 1, 2$,

$$\frac{1}{\sqrt{G_{\alpha\alpha}}}\sqrt{(X^I_{,\alpha} + u^I_{,\alpha})(X^I_{,\alpha} + u^I_{,\alpha})} = 1 \quad \text{(summation on } I\text{)}. \tag{7.15}$$

However, compared with Eq.(7.13), equation (7.15) yields

$$u^I = 0 \quad (I = 1, 2, 3). \tag{7.16}$$

It means that the inextensible web can support any tension without stretch. Therefore, one can consider the inextensible web configuration as the initial configuration. There are two non-continuum webs: (i) network non-continuum webs and (ii) fabric non-continuum web.

Definition 7.3. A deformable web is called a *cable-network web* if the deformable web is formed through cable or strings networks.

Definition 7.4. A deformable web is called a *cable-fabric web* if the deformable web is weaved by cables or strings.

Definition 7.5. A deformable web is called a *continuum web* if the deformable web is made by continuum media.

Definition 7.6. A deformable web is called *a cable-reinforced, continuum web* if the deformable web is formed by cable-network web and continuum skin webs.

The cable-network and cable-fabric webs are of non-continuum. If a web consists of network non-continuum webs and continuum skin webs, the continuum skin webs can be investigated by the web theory and the reinforced cables can be investigated by the cable theories. The concept of finite elements should be adopted and the corresponding boundary conditions should be considered. The theory for cable-network webs will be first presented through the cable theory.

7.1.1. Cable-network webs

The network non-continuum web is composed of many separated cables in many directions with nodes. A network non-continuum web with cables in two directions is sketched in Fig.7.3. The filled symbols represent the nodes for physical connection and concentrated forces. The connection between two nodes is a cable segment. For generality, at a node of the non-continuum web, there can be many cables with different directions rather than two directions. In addition, the nodes include the physical nodes and force nodes. The physical nodes are the connection joints of the cable segments to form a web, and force nodes are where the concentrated forces are exerted. Of course, on the physical nodes, the concentrated forces can be exerted as well. Suppose that a node point is connected with finite l-cables ($l \in \mathbb{N}$). There is a cable segment between this node and one of its adjacent nodes in one of l-directions, and the distributed forces can be exerted on such a cable segment. Thus, to apply the external forces on a non-continuum web and to investigate the corresponding nonlinear dynamics, the point and segment sets on the initial configuration of the non-continuum webs are introduced first.

$$\begin{aligned} \mathscr{P} &= \bigcup_{k_1=0}^{m_1} \bigcup_{k_2=0}^{m_2} \cdots \bigcup_{k_l=0}^{m_l} \mathscr{P}_{(k_1,k_2,\ldots,k_l)}, \\ \mathscr{P}_{(k_1,k_2,\ldots,k_l)} &\equiv \left\{ (S^1, S^2, \ldots, S^l) \middle| S^\alpha = A^\alpha_{(k_1,k_2,\ldots,k_l)} \text{for } \alpha = 1, 2, \ldots, l \right\}. \end{aligned} \tag{7.17}$$

$$\begin{aligned} \mathscr{S} &= \bigcup_{k_1=1}^{m_1} \bigcup_{k_2=1}^{m_2} \cdots \bigcup_{k_l=1}^{m_l} \bigcup_{\alpha=1}^{l} \mathscr{S}^\alpha_{(k_1,k_2,\ldots,k_l)}, \\ \mathscr{S}^\alpha_{(k_1,k_2,\ldots,k_l)} &\equiv \left\{ S^\alpha \middle| S^\alpha \in [A^\alpha_{(k_1,\ldots,k_\alpha-1,\ldots,k_l)}, A^\alpha_{(k_1,\ldots,k_\alpha,\ldots,k_l)}] \right\}. \end{aligned} \tag{7.18}$$

The point and segment sets on the deformed configuration are defined as

$$\begin{aligned} \mathscr{p} &\equiv \bigcup_{k_1=1}^{m_1} \bigcup_{k_2=1}^{m_2} \cdots \bigcup_{k_l=1}^{m_l} \mathscr{p}_{(k_1,k_2,\ldots,k_l)}, \\ \mathscr{p}_{(k_1,k_2,\ldots,k_l)} &\equiv \left\{ (s^1, s^2, \ldots, s^l) \middle| s^\alpha = a^\alpha_{(k_1,k_2,\ldots,k_l)} \text{ for } \alpha = 1, 2, \ldots, l \right\}. \end{aligned} \tag{7.19}$$

$$\begin{aligned} \mathscr{s} &= \bigcup_{k_1=1}^{m_1} \bigcup_{k_2=1}^{m_2} \cdots \bigcup_{k_l=1}^{m_l} \bigcup_{\alpha=1}^{l} \mathscr{s}^\alpha_{(k_1,k_2,\ldots,k_l)}, \\ \mathscr{s}^\alpha_{(k_1,k_2,\ldots,k_l)} &\equiv \left\{ s^\alpha \middle| s^\alpha \in [a^\alpha_{(k_1,\ldots,k_\alpha-1,\ldots,k_l)}, a^\alpha_{(k_1,\ldots,k_\alpha,\ldots,k_l)}] \right\}. \end{aligned} \tag{7.20}$$

Consider distributed forces on segment $\mathscr{S}^\alpha_{(k_1,k_2,\ldots,k_l)}$ and concentrated forces on a point $\mathscr{P}_{(k_1,k_2,\ldots,k_l)}$ for $k_\alpha \in \mathscr{M}_\alpha$ and $\mathscr{M}_\alpha = \{1, 2, \ldots, m_\alpha\}$ with ($\alpha = 1, 2, \ldots, l$), i.e.,

$$\begin{aligned} \mathbf{F}_{(k_1,k_2,\ldots,k_l)} &\equiv F^I_{(k_1,k_2,\ldots,k_l)}(S^1, S^2, \ldots, S^l)\mathbf{I}_I \text{ at } \mathscr{P}_{(k_1,k_2,\ldots,k_l)}, \\ {}^\alpha\mathbf{q}_{(k_1,k_2,\ldots,k_l)} &\equiv {}^\alpha q^I_{(k_1,k_2,\ldots,k_l)}(S^1, S^2, \ldots, S^l)\mathbf{I}_I \text{ on } \mathscr{S}^\alpha_{(k_1,k_2,\ldots,k_l)}. \end{aligned} \tag{7.21}$$

The corresponding forces on the segment $\mathscr{s}^\alpha_{(k_1,k_2,\ldots,k_l)}$ and concentrated forces on point $\mathscr{p}_{(k_1,k_2,\ldots,k_l)}$ are expressed by

$$\begin{aligned} \mathbf{f}_{(k_1,k_2,\ldots,k_l)} &\equiv f^I_{(k_1,k_2,\ldots,k_l)}(s^1, s^2, \ldots, s^l)\mathbf{I}_I \text{ at } \mathscr{p}_{(k_1,k_2,\ldots,k_l)}, \\ {}^\alpha\mathbf{p}_{(k_1,k_2,\ldots,k_l)} &\equiv {}^\alpha p^I_{(k_1,k_2,\ldots,k_l)}(s^1, s^2, \ldots, s^l)\mathbf{I}_I \text{ on } \mathscr{s}^\alpha_{(k_1,k_2,\ldots,k_l)}. \end{aligned} \tag{7.22}$$

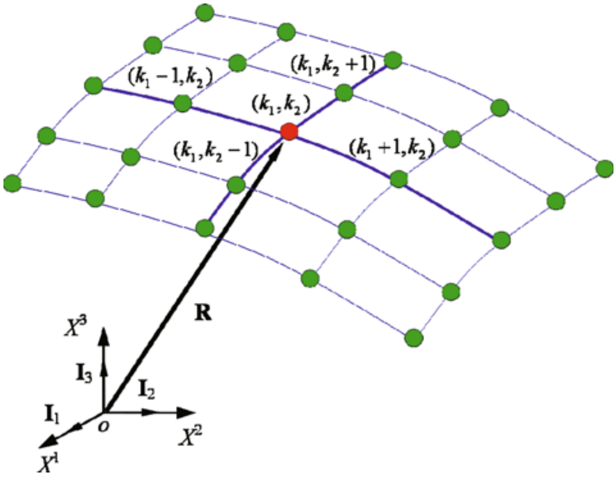

Fig. 7.3 Separated cables and nodes in a cable-network non-continuum web. The filled symbols represent the nodes for concentrated forces. The connection between two nodes is a cable segment.

From Eqs.(7.21) and (7.22), one obtains

$$\begin{gathered}\mathbf{F}_{(k_1,k_2,\ldots,k_l)}=\mathbf{f}_{(k_1,k_2,\ldots,k_l)},\\ F^I_{(k_1,k_2,\ldots,k_l)}(S^1,S^2,\ldots,S^l)=f^I_{(k_1,k_2,\ldots,k_l)}(s^1,s^2,\ldots,s^l);\\ {}^\alpha\mathbf{p}_{(k_1,k_2,\ldots,k_l)}\equiv\frac{1}{\sqrt{G_{\alpha\alpha}}(1+\mathscr{E}_\alpha)}\,{}^\alpha\mathbf{q}_{(k_1,k_2,\ldots,k_l)},\\ {}^\alpha p^I_{(k_1,k_2,\ldots,k_l)}(s^1,s^2,\ldots,s^l)=\frac{1}{\sqrt{G_{\alpha\alpha}}(1+\mathscr{E}_\alpha)}\,{}^\alpha q^I_{(k_1,k_2,\ldots,k_l)}(S^1,S^2,\ldots,S^l),\end{gathered}\tag{7.23}$$

where

$$\sqrt{G_{\alpha\alpha}}(1+\mathscr{E}_\alpha)\equiv\sqrt{(X^K_{\alpha,\alpha}+u^K_{\alpha,\alpha})(X^K_{\alpha,\alpha}+u^K_{\alpha,\alpha})}.\tag{7.24}$$

Therefore, equation of motion for segments $S^\alpha\in\mathscr{S}^\alpha_{(k_1,k_2,\ldots,k_l)}$ is given by

$$\begin{gathered}\rho^0_\alpha A_\alpha(\mathbf{X}_{\alpha,tt}+\mathbf{u}_{\alpha,tt})=\mathbf{q}_\alpha+{}^\alpha\mathbf{T}_{,\alpha},\ \text{or}\\ \rho^0_\alpha A_\alpha(X^I_{\alpha,tt}+u^I_{\alpha,tt})=q^I_\alpha+\left[\frac{{}^\alpha T(X^I_{\alpha,\alpha}+u^I_{\alpha,\alpha})}{\sqrt{G_{\alpha\alpha}}(1+\mathscr{E}_\alpha)}\right]_{,\alpha},\end{gathered}\tag{7.25}$$

where $\rho^0_\alpha A_\alpha$ is based on the initial configuration and no summation on α exists. In Eq. (7.25), the tension is computed by

$$
{}^{\alpha}T = \begin{cases} \int_{A_\alpha} f_\alpha(\varepsilon_\alpha) dA_\alpha & \text{for any materials,} \\ {}^{\alpha}T^0(S^\alpha) + \int_{A_\alpha} E_\alpha \mathscr{E}_\alpha dA_\alpha & \text{for linear elasticity,} \end{cases} \tag{7.26}
$$

where ${}^{\alpha}T^0(S^\alpha) = \int_{A_\alpha} E_\alpha \mathscr{E}_\alpha^0 dA_\alpha$ and $\varepsilon_\alpha = \mathscr{E}_\alpha^0 + \mathscr{E}_\alpha$.

The tension vector is defined as

$$
{}^{\alpha}\mathbf{T}(S^\alpha) = {}^{\alpha}T(S^\alpha)\, \mathbf{n}_\alpha \equiv {}^{\alpha}T^I(S^\alpha)\mathbf{I}_I, \tag{7.27}
$$

where $\mathbf{n}_\alpha$ is the unit normal direction of the cable cross section and

$$
{}^{\alpha}T^I(S^\alpha) = {}^{\alpha}T(S^\alpha)\, \mathbf{n}_\alpha \cdot \mathbf{I}_I = {}^{\alpha}T(S^\alpha) \cos\theta_{(\mathbf{n}_\alpha, \mathbf{I}_I)} \tag{7.28}
$$

and

$$
\mathbf{n}_\alpha = \frac{(X^I_{,\alpha} + u^I_{,\alpha})}{\sqrt{G_{\alpha\alpha} + 2E_{\alpha\alpha}}} \mathbf{I}_I \text{ and } \cos\theta_{(\mathbf{n}_\alpha, \mathbf{I}_I)} = \frac{X^I_{\alpha,\alpha} + u^I_{\alpha,\alpha}}{\sqrt{G_{\alpha\alpha}}(1 + \mathscr{E}_\alpha)}. \tag{7.29}
$$

Suppose that there are l-cables connected at a node point $\mathscr{P}_{(k_1, k_2, \ldots, k_l)}$. The force condition at the node $\mathscr{P}_{(k_1, k_2, \ldots, k_l)}$ with $S^\alpha = A^\alpha_{(k_1, k_2, \ldots, k_l)}$ is given by

$$
\begin{aligned}
\sum\nolimits_{\alpha=1}^{l} {}^{\alpha}\mathbf{T}_{(k_1, k_2, \ldots, k_l)}(S^\alpha)\Big|_{S^\alpha = A^\alpha_{(k_1, k_2, \ldots, k_l)}} + \mathbf{F}_{(k_1, k_2, \ldots, k_l)} = 0, \text{ or} \\
\sum\nolimits_{\alpha=1}^{l} {}^{\alpha}T_{(k_1, k_2, \ldots, k_l)}(S^\alpha) \cos\theta_{(\mathbf{n}_\alpha, \mathbf{I}_I)}\Big|_{S^\alpha = A^\alpha_{(k_1, k_2, \ldots, k_l)}} + F^I_{(k_1, k_2, \ldots, k_l)} = 0,
\end{aligned} \tag{7.30}
$$

where the tension vector on such a normal direction of $\mathbf{n}_\alpha$ can be given, i.e.,

$$
{}^{\alpha}\mathbf{T}_{(k_1, k_2, \ldots, k_l)}(S^\alpha) = {}^{\alpha}T_{(k_1, k_2, \ldots, k_l)}(S^\alpha)\mathbf{n}_\alpha \tag{7.31}
$$

and

$$
\begin{aligned}
\cos\theta_{(\mathbf{n}_\alpha, \mathbf{I}_I)} &= \frac{X^I_{\alpha,\alpha} + u^I_{\alpha,\alpha}}{\sqrt{G_{\alpha\alpha}}(1 + \mathscr{E}_\alpha)} && \text{for } 0 \le \theta_{(\mathbf{n}_\alpha, \mathbf{I}_I)} \le \pi, \\
\cos\theta_{(\mathbf{n}_\alpha, \mathbf{I}_I)} &= (-1)\frac{X^I_{\alpha,\alpha} + u^I_{\alpha,\alpha}}{\sqrt{G_{\alpha\alpha}}(1 + \mathscr{E}_\alpha)} && \text{for } \pi < \theta_{(\mathbf{n}_\alpha, \mathbf{I}_I)} \le 2\pi;
\end{aligned} \tag{7.32}
$$

and the displacement conditions are

$$
u^I_\alpha = u^I_\beta \text{ and } X^I_\alpha = X^I_\beta \text{ for } \alpha, \beta \in \{1, 2, \ldots, l\} \text{ and } \alpha \ne \beta. \tag{7.33}
$$

The displacement boundary condition at a node point $\mathscr{P}_{(r_1, r_2, \ldots, r_b)}$ is

$$
u^I_{(r_1, r_2, \ldots, r_{l_b})} = b^I_{(r_1, r_2, \ldots, r_{l_b})} \text{ and } X^I_{(r_1, r_2, \ldots, r_{l_b})} = B^I_{(r_1, r_2, \ldots, r_{l_b})} \text{ for } \beta \in \{1, 2, \ldots, l_b\}. \tag{7.34}
$$

For the force boundary, the boundary conditions can be given as in Eq.(7.25). That is,

$$\sum_{\alpha=1}^{l_b} {}^{\alpha}T_{(r_1,r_2,\ldots,r_{l_b})}(S^{\alpha})\cos\theta_{(\mathbf{n}_{\alpha},\mathbf{I}_I)}\Big|_{S^{\alpha}=a^{\alpha}_{(r_1,r_2,\ldots,r_{l_b})}} + F^{I}_{(r_1,r_2,\ldots,r_{l_b})} = 0. \tag{7.35}$$

The static and dynamic solutions for each cable segment of the non-continuum web can be obtained as in Chapter 5 (also see, Luo and Mote, 2000; Luo and Wang, 2002; Wang and Luo, 2004). From definition, the wrinkling conditions for the non-continuum web can be determined by each segment between the two adjacent nodes. In other words, one has the wrinkling conditions through the following definitions.

Definition 7.7. A network non-continuum web is called *a locally wrinkled network web* in $\mathcal{S}^{\alpha}_{(k_1,k_2,\ldots,k_l)}$ if the tension at any point $S^{\alpha}\in\mathcal{S}^{\alpha}_{(k_1,k_2,\ldots,k_l)}$ satisfies

$${}^{\alpha}T_{(k_1,k_2,\cdots,k_l)}(S^{\alpha})<0 \ \text{ for } S^{\alpha}\in[A^{\alpha}_{(k_1,\ldots,k_{\alpha}-1,\ldots,k_l)}, A^{\alpha}_{(k_1,\ldots,k_{\alpha},\ldots,k_l)}]. \tag{7.36}$$

A network non-continuum web is called a *globally wrinkled network web* if all segments $\mathcal{S}^{\alpha}_{(k_1,k_2,\ldots,k_l)}\subset\mathcal{S}$ ($k_{\alpha}\in\{1,2,\ldots,m_{\alpha}\}, \alpha=1,2,\ldots,l$) in such a non-continuum web satisfy Eq.(7.36). The two wrinkling boundaries on the segment $\mathcal{S}^{\alpha}_{(k_1,k_2,\ldots,k_l)}$ are determined by

$${}^{\alpha}T(S^{\alpha})=0 \ \text{ for } S^{\alpha}\in[A^{\alpha}_{(k_1,\ldots,k_{\alpha}-1,\ldots,k_l)}, A^{\alpha}_{(k_1,\ldots,k_{\alpha},\ldots,k_l)}]. \tag{7.37}$$

7.1.2. Cable-fabric webs

The cable-fabric web is a kind of non-continuum webs, and such a web is called *the continuously weaved web* as well. A fabric web consists of continuous nodes of fibers in two or more directions. The fibers going through nodes are continuous. Such a web can support the tension only. Consider a cable-fabric web with two main fiber directions $\mathbf{G}_{\alpha}$ of S^{α} ($\alpha=1,2$) and their angle $\Theta_{(S^1,S^2)}\in(0,\pi/2]$, as shown in Fig.7.4.

To exert the external force on the web, the point and domain sets on the initial configuration are defined as

$$\begin{gathered}\mathscr{P}=\bigcup_{k_2=0}^{m_2}\bigcup_{k_1=0}^{m_1}\mathscr{P}_{(k_1,k_2)} \text{ and}\\ \mathscr{P}_{(k_1,k_2)}=\left\{(S^1,S^2)\,\middle|\,S^i=A^i_{(k_1,k_2)},\ i=1,2\right\}\end{gathered} \tag{7.38}$$

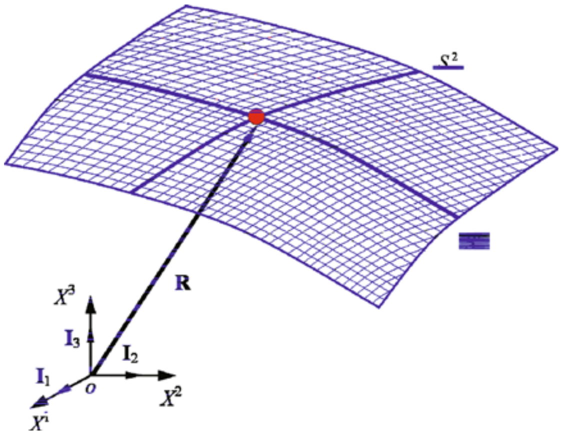

Fig. 7.4 Cable-fabric non-continuum web with fibers two directions.

$$\begin{aligned} &\mathscr{D} = \bigcup_{k_1=1}^{m_1} \bigcup_{k_2=1}^{m_2} \mathscr{D}_{(k_1,k_2)} \text{ and} \\ &\mathscr{D}_{(k_1,k_2)} \equiv \left\{ (S^1, S^2) \left| \begin{array}{l} A^1_{(k_1-1,k_2)} \le S^1 \le A^1_{(k_1,k_2)} \\ A^2_{(k_1,k_2-1)} \le S^2 \le A^2_{(k_1,k_2)} \end{array} \right. \right\}. \end{aligned} \tag{7.39}$$

The point and domain sets on the deformed configuration are defined as

$$\left. \begin{aligned} &\boldsymbol{\mathscr{p}} = \bigcup_{k_2=0}^{m_2} \bigcup_{k_1=0}^{m_1} \boldsymbol{\mathscr{p}}_{(k_1,k_2)} \text{ and} \\ &\boldsymbol{\mathscr{p}}_{(k_1,k_2)} \equiv \left\{ (s^1, s^2) \middle| s^i = a^i_{(k_1,k_2)},\ i = 1, 2 \right\} \end{aligned} \right\}, \tag{7.40}$$

$$\left. \begin{aligned} &\boldsymbol{\mathscr{d}} = \bigcup_{k_2=1}^{m_2} \bigcup_{k_1=1}^{m_1} \boldsymbol{\mathscr{d}}_{(k_1,k_2)} \text{and} \\ &\boldsymbol{\mathscr{d}}_{(k_1,k_2)} \equiv \left\{ (s^1, s^2) \left| \begin{array}{l} a^1_{(k_1-1,k_2)} \le s^1 \le a^1_{(k_1,k_2)}, \\ a^2_{(k_1,k_2-1)} \le s^2 \le a^2_{(k_1,k_2)} \end{array} \right. \right\} \end{aligned} \right\}. \tag{7.41}$$

On the initial configuration, the external concentrated force on point $\mathscr{P}_{(k_1,k_2)}$ and distributed force on domain $\mathscr{D}_{(k_1,k_2)}$ are defined as

$$\begin{aligned} &\mathbf{q}_{(k_1,k_2)} = q^I_{(k_1,k_2)}(S^1, S^2)\mathbf{I}_I\ ,\ (S^1, S^2) \in \mathscr{D}_{(k_1,k_2)}, \\ &\mathbf{F}_{(k_1,k_2)} = F^I_{(k_1,k_2)}(S^1, S^2)\mathbf{I}_I\ ,\ (S^1, S^2) \in \mathscr{P}_{(k_1,k_2)} \end{aligned} \tag{7.42}$$

and on the deformed configuration, the forces are expressed by

$$\begin{aligned} &\mathbf{f}_{(k_1,k_2)} = f^I_{(k_1,k_2)}(s^1, s^2)\mathbf{I}_I\ ,\ (s^1, s^2) \in \boldsymbol{\mathscr{p}}_{(k_1,k_2)}, \\ &\mathbf{p}_{(k_1,k_2)} = p^I_{(k_1,k_2)}(s^1, s^2)\mathbf{I}_I\ ,\ (s^1, s^2) \in \boldsymbol{\mathscr{d}}_{(k_1,k_2)}. \end{aligned} \tag{7.43}$$

Because no shear force exists on the fabric webs, the angle between the two fibers will not be changed. The corresponding relations between the two forces are

$$\mathbf{F}_{(k_1,k_2)} = \mathbf{f}_{(k_1,k_2)} \text{ and } \mathbf{p}_{(k_1,k_2)} = \frac{1}{(1+\mathscr{E}_1)(1+\mathscr{E}_2)}\mathbf{q}_{(k_1,k_2)}. \tag{7.44}$$

As in Eq.(7.24),

$$\sqrt{G_{\alpha\alpha}}(1+\mathscr{E}_\alpha) \equiv \sqrt{(X^K_{\alpha,\alpha}+u^K_{\alpha,\alpha})(X^K_{\alpha,\alpha}+u^K_{\alpha,\alpha})} \;\; (\alpha=1,2). \tag{7.45}$$

The tension is defined by $\overline{T}_\alpha$, and the tension per unit length can be defined as

$$T_\alpha = \frac{d\overline{T}_\alpha}{dS^\beta} = \int_{h_1}^{h_2} f_\alpha(\mathscr{E}_\alpha)dz \;\text{ for } \alpha, \beta \in \{1,2\} \text{ and } \alpha \neq \beta. \tag{7.46}$$

If $\mathscr{E}_\alpha = 0$ then $f_\alpha(\mathscr{E}_\alpha)=0$. Consider linear elastic materials as an example. From Eq.(7.8), the tension on dA_α in the α-direction is

$$d\overline{T}_\alpha = E_\alpha dA_\alpha \mathscr{E}_\alpha = E_\alpha dA_\alpha \Big(\frac{1}{\sqrt{G_{\alpha\alpha}}}\sqrt{(X^K_{\alpha,\alpha}+u^K_{\alpha,\alpha})(X^K_{\alpha,\alpha}+u^K_{\alpha,\alpha})}-1\Big), \tag{7.47}$$

where $dA_\alpha = h\times dS^\beta$ and h is the thickness of the web. Thus, equations (7.46) anf (7.47) yields

$$T_\alpha = \int_{h_1}^{h_2} E_\alpha \mathscr{E}_\alpha dz = E_\alpha h\Big(\frac{1}{\sqrt{G_{\alpha\alpha}}}\sqrt{(X^K_{,\alpha}+u^K_{,\alpha})(X^K_{,\alpha}+u^K_{,\alpha})}-1\Big). \tag{7.48}$$

The tension per unit length will be simply called *the tension* on the α-cross section. Similarly, if the initial configuration is in the deformed state with an initial tension T^0_α per unit length in the S^α-direction, the corresponding strain is expressed by

$$\varepsilon_\alpha = \mathscr{E}^0_\alpha(S^1,S^2) + \frac{1}{\sqrt{G_{\alpha\alpha}}}\sqrt{(X^K_{,\alpha}+u^K_{,\alpha})(X^K_{,\alpha}+u^K_{,\alpha})}-1, \tag{7.49}$$

where the initial tension per unit length $T^0_\alpha(S^1,S^2) = \int_{h_1}^{h_2} f_\alpha(\mathscr{E}^0_\alpha)dz$, and the tension per unit length on the configuration is

$$T_\alpha(S^1,S^2) = \begin{cases} \int_{h_1}^{h_2} f_\alpha(\varepsilon_\alpha)dz & \text{for any materials,} \\ E_\alpha h\mathscr{E}^0_\alpha + E_\alpha h\mathscr{E}_\alpha & \text{for linear elasticity.} \end{cases} \tag{7.50}$$

Therefore, equation of motion for a domain $(S^1, S^2) \in \mathscr{D}_{(k_1,k_2)}$ is

$$\rho_0 h(\mathbf{X}_{,tt} + \mathbf{u}_{,tt}) \sin\Theta_{(S^1,S^2)} = \mathbf{q}_{(k_1,k_2)} + \mathbf{T}_{\alpha,\alpha} \text{ or}$$
$$\rho_0 h(X^I_{,tt} + u^I_{,tt}) \sin\Theta_{(S^1,S^2)} = q^I_{(k_1,k_2)} + \left[\frac{T_\alpha (X^I_{,\alpha} + u^I_{,\alpha})}{\sqrt{G_{\alpha\alpha}}(1+\mathscr{E}_\alpha)}\right]_{,\alpha}, \tag{7.51}$$

where ρ_0 is based on the initial configuration and summation on $\alpha = 1,2$ should be done. As in the network non-continuum web, the tension vector on the S^α-surface ($\alpha = 1,2$) is

$$\mathbf{T}_\alpha(S^1, S^2) = T_\alpha(S^1, S^2)\mathbf{n}_\alpha \equiv T^I_\alpha(S^\alpha)\mathbf{I}_I, \tag{7.52}$$

where the vector $\mathbf{n}_\alpha$ is the normal direction of the S^α-surface, and the component of the tension vector on such S^α-surface can be given by

$$T^I_\alpha(S^1, S^2) = T_\alpha(S^1, S^2)\mathbf{n}_\alpha \cdot \mathbf{I}_I = T_\alpha(S^1, S^2)\cos\theta_{(\mathbf{n}_\alpha,\mathbf{I}_I)} \tag{7.53}$$

and the direction cosine is given by

$$\mathbf{n}_\alpha = (X^I_{,\alpha} + u^I_{,\alpha})\mathbf{I}_I \text{ and } \cos\theta_{(\mathbf{n}_\alpha,\mathbf{I}_I)} = \frac{X^I_{\alpha,\alpha} + u^I_{\alpha,\alpha}}{\sqrt{G_{\alpha\alpha}}(1+\mathscr{E}_\alpha)}. \tag{7.54}$$

At a point $\mathscr{P}_{(k_1,k_2)}$ with $S^i = A^i_{(k_1,k_2)}$, the force condition is given by

$$\sum\nolimits_{\alpha=1}^{2} {}^{-}\mathbf{T}_\alpha(S^1, S^2)\Big|_{(S^1,S^2)=(A^1_{(k_1,k_2)},A^2_{(k_1,k_2)})}$$
$$+\sum\nolimits_{\alpha=1}^{2} {}^{+}\mathbf{T}_\alpha(S^1, S^2)\Big|_{(S^1,S^2)=(A^1_{(k_1,k_2)},A^2_{(k_1,k_2)})} + \mathbf{F}_{(k_1,k_2)} = 0, \tag{7.55}$$

or

$$\sum\nolimits_{\alpha=1}^{2} {}^{-}T_\alpha(S^1, S^2)\cos\theta_{(\mathbf{n}_\alpha,\mathbf{I}_I)}\Big|_{(S^1,S^2)=(A^1_{(k_1,k_2)},A^2_{(k_1,k_2)})}$$
$$=\sum\nolimits_{\alpha=1}^{2} {}^{+}T_\alpha(S^1, S^2)\cos\theta_{(\mathbf{n}_\alpha,\mathbf{I}_I)}\Big|_{(S^1,S^2)=(A^1_{(k_1,k_2)},A^2_{(k_1,k_2)})} + F^I_{(k_1,k_2)}, \tag{7.56}$$

where superscript "$\pm$" represent tensions on the direction of the negative and positive cross sections.

The displacement continuity conditions are

$${}^{+}u^I_{(k_1,k_2)} = {}^{-}u^I_{(k_1,k_2)} \text{ and } {}^{+}X^I_{(k_1,k_2)} = {}^{-}X^I_{(k_1,k_2)}. \tag{7.57}$$

The displacement boundary condition at a boundary point $\mathscr{P}_{(r_1,r_2)}$ is

$$u^I_{(r_1,r_2)} = b^I_{(r_1,r_2)} \text{ and } X^I_{(r_1,r_2)} = B^I_{(r_1,r_2)}. \tag{7.58}$$

The force boundary condition at the boundary point $\mathscr{P}_{(r_1,r_2)}$ is

$$\sum_{\alpha=1}^{2}\mathbf{T}_{\alpha}(S^{1},S^{2})\Big|_{(S^{1},S^{2})=(B^{1}_{(r_1,r_2)},B^{2}_{(r_1,r_2)})}+\mathbf{F}_{(r_1,r_2)}=0 \tag{7.59}$$

or

$$\sum_{\alpha=1}^{2}T_{\alpha}(S^{1},S^{2})\cos\theta_{(\mathbf{n}_\alpha,\mathbf{I}_I)}\Big|_{(S^{1},S^{2})=(B^{1}_{(r_1,r_2)},B^{2}_{(r_1,r_2)})}+F^{I}_{(r_1,r_2)}=0. \tag{7.60}$$

The displacement boundary condition on the web edge is

$$\begin{aligned}u^{I}(S^{1},S^{2})\Big|_{S^{2}=S^{2}_{*}}&=U^{I}(S^{1})\text{ and }X^{I}(S^{1},S^{2})\Big|_{S^{2}=S^{2}_{*}}=B^{I}(S^{1}),\\ u^{I}(S^{1},S^{2})\Big|_{S^{1}=S^{1}_{*}}&=U^{I}(S^{2})\text{ and }X^{I}(S^{1},S^{2})\Big|_{S^{1}=S^{1}_{*}}=B^{I}(S^{2}).\end{aligned} \tag{7.61}$$

The force boundary condition on the web edge is

$$\begin{aligned}&(\mathbf{T}_{\alpha}+\frac{\partial}{\partial S^{\beta}}\int\mathbf{T}_{\beta}dS^{\alpha})\Big|_{S^{\alpha}=S^{\alpha}_{*}}+\mathbf{F}(S^{\beta})=0,\text{ or}\\ &\frac{\partial}{\partial S^{\beta}}\int T_{\beta}(S^{1},S^{2})\cos\theta_{(\mathbf{n}_\beta,\mathbf{I}_I)}dS^{\alpha}\Big|_{S^{\alpha}=S^{\alpha}_{*}}\\ &+T_{\alpha}(S^{1},S^{2})\cos\theta_{(\mathbf{n}_\alpha,\mathbf{I}_I)}\Big|_{S^{\alpha}=S^{\alpha}_{*}}+F^{I}(S^{\beta})=0.\end{aligned} \tag{7.62}$$

From the definition, the wrinkling for a fabric web can exist in one of two directions, and the corresponding definitions can be given as follows.

Definition 7.8. A fabric non-continuum web is called a *locally wrinkled fabric web* in the S^{α}-direction ($\alpha\in\{1,2\}$) in domain $\mathscr{D}_{(k_1,k_2)}$ if the tension at a point $(S^{1},S^{2})\in\mathscr{D}_{(k_1,k_2)}$ on the S^{α}-direction satisfies

$$T_{\alpha}(S^{1},S^{2})<0\text{ for }(S^{1},S^{2})\in\mathscr{D}_{(k_1,k_2)}\text{ and }\alpha\in\{1,2\}. \tag{7.63}$$

A fabric non-continuum web is called *a globally wrinkled fabric web* in the S^{α}-direction ($\alpha\in\{1,2\}$) if Eq.(7.63) holds for all domains $\mathscr{D}_{(k_1,k_2)}\subset\mathscr{D}$ in such a fabric web. The wrinkling boundary on the domain $\mathscr{D}_{(k_1,k_2)}$ is determined by

$$T_{\alpha}(S^{1},S^{2})=0\ \text{ for }(S^{1},S^{2})\in\mathscr{D}_{(k_1,k_2)}\text{ and }\alpha\in\{1,2\}. \tag{7.64}$$

Definition 7.9. A fabric non-continuum web is called a *locally free, fabric web* in domain $\mathscr{D}_{(k_1,k_2)}$ if two tensions at any point $(S^{1},S^{2})\in\mathscr{D}_{(k_1,k_2)}$ satisfy

$$T_{\alpha}(S^{1},S^{2})<0\text{ for }(S^{1},S^{2})\in\mathscr{D}_{(k_1,k_2)}\text{ and }\alpha=1,2. \tag{7.65}$$

A fabric non-continuum web is called a *globally free, fabric web* if all domains $\mathscr{D}_{(k_1,k_2)}\subset\mathscr{D}$ in such a fabric web satisfy Eq.(7.65). The boundary for the free web in the domain $\mathscr{D}_{(k_1,k_2)}$ is determined by

$$T_\alpha(S^1, S^2) = 0 \text{ for } (S^1, S^2) \in \mathscr{D}_{(k_1,k_2)} \text{ and } \alpha = 1, 2. \tag{7.66}$$

In the above discussion, the two main fibers for fabric webs possess an angle $\Theta_{(S^1,S^2)}$. If $\Theta_{(S^1,S^2)} = \pi/2$, the two main fibers in the fabric web are orthogonal. Equation of motion in Eq.(7.51) becomes

$$\begin{aligned}
&\rho_0 h(\mathbf{X}_{,tt} + \mathbf{u}_{,tt}) = \mathbf{q}_{(k_1,k_2)} + \mathbf{T}_{\alpha,\alpha} \text{ or}\\
&\rho_0 h(X^I_{,tt} + u^I_{,tt}) = q^I_{(k_1,k_2)} + \left[\frac{T_\alpha(X^I_{,\alpha} + u^I_{,\alpha})}{\sqrt{G_{\alpha\alpha}}(1+\mathscr{E}_\alpha)}\right]_{,\alpha}
\end{aligned} \tag{7.67}$$

and summation on $\alpha = 1, 2$ should be completed. If $\Theta_{(S^1,S^2)} = 0$, this case is singular, and the two main fiber directions are in one direction. The corresponding equation of motions becomes

$$\rho_0 h(\mathbf{X}_{,tt} + \mathbf{u}_{,tt}) = \mathbf{q}_{(k_1,k_2)} + \mathbf{T}_{\alpha,\alpha}, \tag{7.68}$$

$$\rho_0 h(X^I_{,tt} + u^I_{,tt}) = q^I_{(k_1,k_2)} + \left[\frac{T_\alpha(X^I_{,\alpha} + u^I_{,\alpha})}{\sqrt{G_{\alpha\alpha}}(1+\mathscr{E}_\alpha)}\right]_{,\alpha} \tag{7.69}$$

without summation on $\alpha = 1, 2$. For a flat web, $G_{\alpha\alpha}=1$ ($\alpha = 1, 2$) and $X^I_{,\alpha} = \delta^I_\alpha$.

Definition 7.10. A fabric non-continuum web is called

(i) a *skew weaving web* if $\Theta_{(S^1,S^2)} \in (0, \frac{\pi}{2})$,

(ii) an *orthogonal weaving web* if $\Theta_{(S^1,S^2)} = \frac{\pi}{2}$,

(iii) a *uniaxial weaving web* if $\Theta_{(S^1,S^2)} = 0$.

7.1.3. Continuum webs

A continuum web consists of continuous media. The continuum web does not have any specified fiber directions as in the fabric webs. It is assumed that such a web can support the principal tension only. Because no specified fiber directions exist in the continuum web, the two directions of the principal tensions are dependent on the external loading rather than the two specific directions. Thus for any coordinates, shear membrane forces in such a web may exist as in the traditional membrane theory. Thus, there is a sort of twisting in such a web. Such a twisting is caused by the two principal tensions. Therefore, consider a curvilinear coordinates (S^Λ, $\Lambda = 1, 2, 3$) for the initial web configuration, and the two curvilinear coordinates (S^α, $\alpha = 1, 2$) with $\Theta_{(S^1,S^2)} \in (0, \pi/2]$ are on the initial configuration, as shown in Fig.7.5. As in fabric non-continuous webs, the normal strains and tensions can be discussed in Eqs.(7.45)−(7.50). However, because the tensional twisting exists in such a continuum web, the shear strain should be introduced. Let $\Theta_{(S^1,S^2)}$ and $\theta_{(S^1,S^2)}$ be the induced angles between $\mathbf{G}_1$ and $\mathbf{G}_2$ *before* and *after* deformation. The shear strain is defined as

$$\gamma_{12} \equiv \Theta_{(\mathbf{N}_1,\mathbf{N}_2)} - \theta_{(\mathbf{n}_1,\mathbf{n}_2)} \tag{7.70}$$

and the two angles are computed by

$$\begin{aligned}\cos\theta_{(\mathbf{n}_1,\mathbf{n}_2)} &\equiv \cos(\Theta_{(\mathbf{N}_1,\mathbf{N}_2)} - \gamma_{12}) \\ &= \frac{d\underset{1}{\mathbf{r}} \cdot d\underset{2}{\mathbf{r}}}{|d\underset{1}{\mathbf{r}}||d\underset{2}{\mathbf{r}}|} = \frac{(X^K_{,1} + u^K_{,1})dS^1\mathbf{I}_K \cdot (X^I_{,2} + u^I_{,2})dS^2\mathbf{I}_I}{\sqrt{G_{11}}(1+\mathscr{E}_1)dS^1\sqrt{G_{22}}(1+\mathscr{E}_2)dS^2} \\ &= \frac{(X^K_{,1} + u^K_{,1})(X^K_{,2} + u^K_{,2})}{\sqrt{G_{11}G_{22}}(1+\mathscr{E}_1)(1+\mathscr{E}_2)} \text{ (summation on } K\text{)},\end{aligned} \tag{7.71}$$

$$\begin{aligned}\cos\Theta_{(\mathbf{N}_1,\mathbf{N}_2)} &= \frac{d\underset{1}{\mathbf{R}} \cdot d\underset{2}{\mathbf{R}}}{|d\underset{1}{\mathbf{R}}||d\underset{2}{\mathbf{R}}|} = \frac{X^K_{,1}dS^1\mathbf{I}_K \cdot X^I_{,2}dS^2\mathbf{I}_I}{\sqrt{G_{11}}\sqrt{G_{22}}dS^1dS^2} \\ &= \frac{X^K_{,1}X^K_{,2}}{\sqrt{G_{11}}\sqrt{G_{22}}} \text{ (summation on } K\text{)}.\end{aligned} \tag{7.72}$$

Finally, the shear strain is computed by

$$\begin{aligned}\gamma_{12} &\equiv \Theta_{(\mathbf{N}_1,\mathbf{N}_2)} - \theta_{(\mathbf{n}_1,\mathbf{n}_2)} \\ &= \cos^{-1}\left(\frac{X^K_{,1}X^K_{,2}}{\sqrt{G_{11}}\sqrt{G_{22}}}\right) - \cos^{-1}\left[\frac{(X^K_{,1} + u^K_{,1})(X^K_{,2} + u^K_{,2})}{\sqrt{G_{11}}\sqrt{G_{22}}(1+\mathscr{E}_1)(1+\mathscr{E}_2)}\right].\end{aligned} \tag{7.73}$$

Note that $\gamma_{12} = \gamma_{21}$. In addition, the area change from Chapter 3 is given by

$$\frac{d\underset{12}{a}}{d\underset{12}{A}} = \frac{(1+\mathscr{E}_1)(1+\mathscr{E}_2)\sin\theta_{(\mathbf{n}_1,\mathbf{n}_2)}}{\sin\Theta_{(\mathbf{N}_1,\mathbf{N}_2)}} \tag{7.74}$$

where $d\underset{\alpha\beta}{a} = |d\underset{\alpha}{\mathbf{r}} \times d\underset{\beta}{\mathbf{r}}|$ and $d\underset{\alpha\beta}{A} = |d\underset{\alpha}{\mathbf{R}} \times d\underset{\beta}{\mathbf{R}}|$ are the areas for after and before deformation, respectively. On the initial configuration, the external concentrated force on point $\mathscr{P}_{(k_1,k_2)}$ and distributed force on domain $\mathscr{D}_{(k_1,k_2)}$ are defined as in Eqs.(7.42) and (7.43). However, the relations between the two forces on initial and the deformed configurations are

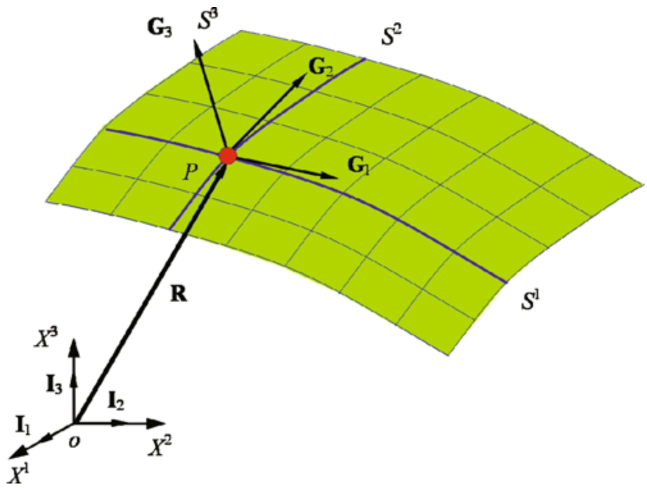

Fig. 7.5 A continuum web with an initial configuration.

$$\mathbf{F}_{(k_1,k_2)} = \mathbf{f}_{(k_1,k_2)} \text{ and } \mathbf{p}_{(k_1,k_2)} = \frac{\sin\Theta_{(\mathbf{N}_1,\mathbf{N}_2)}}{(1+\mathscr{E}_1)(1+\mathscr{E}_2)\sin\theta_{(\mathbf{n}_1,\mathbf{n}_2)}}\mathbf{q}_{(k_1,k_2)}. \tag{7.75}$$

The web shear force caused by the twisting is computed by

$$\bar{T}_{\alpha\beta} = \int_{A_\alpha} f_{\alpha\beta}(\mathscr{E}_{\alpha\beta})dA_\alpha \quad \text{for } \alpha, \beta \in \{1, 2\} \text{ and } \alpha \neq \beta, \tag{7.76}$$

where $\mathscr{E}_{\alpha\beta} \equiv \gamma_{\alpha\beta}$. $\bar{T}_{\alpha\beta}$ represents a tensional shear force on the deformed α-surface with the normal direction $\mathbf{n}_\alpha$ and its direction is the same as the direction of $\mathbf{g}_\beta$. If $\alpha = \beta$, $\mathscr{E}_{\alpha\beta} \equiv \mathscr{E}_\alpha$, and the foregoing equation reduces to Eq.(7.7) for the tension only. The shear force per unit length is defined by

$$T_{\alpha\beta} = \frac{d\bar{T}_{\alpha\beta}}{dS^\beta} = \int_{h_1}^{h_2} f_{\alpha\beta}(\mathscr{E}_{\alpha\beta})dz \quad \text{for } \alpha, \beta \in \{1, 2\} \text{ and } \alpha \neq \beta. \tag{7.77}$$

If $\mathscr{E}_{\alpha\beta} = 0$ then $f_{\alpha\beta}(\mathscr{E}_{\alpha\beta}) = 0$. Consider linear elastic materials as an example. The tensional shear force on dA_α in the α-direction is

$$dT_{\alpha\beta} = \mathcal{G}_{\alpha\beta}dA_\alpha\mathscr{E}_{\alpha\beta} = \mathcal{G}_{\alpha\beta}dA_\alpha\gamma_{\alpha\beta}, \tag{7.78}$$

where $dA_\alpha = h \times dS^\beta$ and h is the thickness of the web, $\mathcal{G}_{\alpha\beta}$ is the shear modulus. Thus, using Eq. (7.77), one obtains

$$T_{\alpha\beta} = \int_{h_1}^{h_2} \mathcal{G}_{\alpha\beta}\mathscr{E}_{\alpha\beta}dz = \mathcal{G}_{\alpha\beta}h\gamma_{\alpha\beta}. \tag{7.79}$$

The shear forces per unit length will be simply called *the shear* on the α-cross section. The web force vector on the S^α - surface is

$$\mathbf{T}_\alpha = T_\alpha \mathbf{n}_\alpha + T_{\alpha\beta}\mathbf{n}_\beta = T_\alpha^I \mathbf{I}_I, \tag{7.80}$$

where

$$T_\alpha^I = T_\alpha \mathbf{n}_\alpha \cdot \mathbf{I}_I + T_{\alpha\beta}\mathbf{n}_\beta \cdot \mathbf{I}_I = T_\alpha \cos\theta_{(\mathbf{n}_\alpha,\mathbf{I}_I)} + T_{\alpha\beta}\cos\theta_{(\mathbf{n}_\beta,\mathbf{I}_I)}, \tag{7.81}$$

$$\cos\theta_{(\mathbf{n}_\alpha,\mathbf{I}_I)} = \frac{(X_{,\alpha}^I + u_{,\alpha}^I)}{\sqrt{G_{\alpha\alpha}}(1+\mathscr{E}_\alpha)} \text{ and } \cos\theta_{(\mathbf{n}_\beta,\mathbf{I}_I)} = \frac{(X_{,\beta}^I + u_{,\beta}^I)}{\sqrt{G_{\beta\beta}}(1+\mathscr{E}_\beta)}. \tag{7.82}$$

The component of the web force of the S^α -surface in the direction of $\mathbf{I}_I$ is computed by

$$T_\alpha^I = \frac{T_\alpha (X_{,\alpha}^I + u_{,\alpha}^I)}{\sqrt{G_{\alpha\alpha}}(1+\mathscr{E}_\alpha)} + \frac{T_{\alpha\beta}(X_{,\beta}^I + u_{,\beta}^I)}{\sqrt{G_{\beta\beta}}(1+\mathscr{E}_\beta)}. \tag{7.83}$$

From the tension and the tensional shear forces, equation of motion for a domain $(S^1, S^2) \in \mathscr{D}_{(k_1,k_2)}$ is

$$\begin{aligned} &\rho_0 h(\mathbf{X}_{,tt} + \mathbf{u}_{,tt})\sin\Theta_{(S^1,S^2)} = \mathbf{q}_{(k_1,k_2)} + \mathbf{T}_{\alpha,\alpha}, \\ \text{or } &\rho_0 h(X_{,tt}^I + u_{,tt}^I)\sin\Theta_{(S^1,S^2)} = q_{(k_1,k_2)}^I + T_{\alpha,\alpha}^I \end{aligned} \tag{7.84}$$

with summation on $\alpha = 1, 2$, and the foregoing equation is rewritten as

$$\rho_0 h(X_{,tt}^I + u_{,tt}^I)\sin\Theta_{(S^1,S^2)} = q_{(k_1,k_2)}^I + \left[\frac{T_\alpha (X_{,\alpha}^I + u_{,\alpha}^I)}{\sqrt{G_{\alpha\alpha}}(1+\mathscr{E}_\alpha)} + \frac{T_{\alpha\beta}(X_{,\beta}^I + u_{,\beta}^I)}{\sqrt{G_{\beta\beta}}(1+\mathscr{E}_\beta)}\right]_{,\alpha}, \tag{7.85}$$

where ρ_0 is based on the initial configuration and summation on $\alpha, \beta = 1, 2$ and $\alpha \neq \beta$ should be done. For a flat web, one has $G_{\alpha\alpha} = 1$ ($\alpha = 1, 2$). The above equation can be reduced. As in Eq.(7.55), the corresponding force condition at a point $\mathscr{P}_{(k_1,k_2)}$ with $S^i = A_{(k_1,k_2)}^i$ is

$$\begin{aligned} &\sum\nolimits_{\alpha=1}^2 {}^-\mathbf{T}_\alpha(S^1,S^2)\Big|_{(S^1,S^2)=(A_{(k_1,k_2)}^1, A_{(k_1,k_2)}^2)} \\ &+\sum\nolimits_{\alpha=1}^2 {}^+\mathbf{T}_\alpha(S^1,S^2)\Big|_{(S^1,S^2)=(A_{(k_1,k_2)}^1, A_{(k_1,k_2)}^2)} + \mathbf{F}_{(k_1,k_2)} = 0, \end{aligned} \tag{7.86}$$

or

$$\begin{aligned} &\sum\nolimits_{\alpha=1}^2 [{}^-T_\alpha(S^1,S^2) + {}^-T_{\beta\alpha}(S^1,S^2)]\cos\theta_{(\mathbf{n}_\alpha,\mathbf{I}_I)}\Big|_{(S^1,S^2)=(A_{(k_1,k_2)}^1, A_{(k_1,k_2)}^2)} \\ &=\sum\nolimits_{\alpha=1}^2 [{}^+T_\alpha(S^1,S^2) + {}^+T_{\beta\alpha}(S^1,S^2)]\cos\theta_{(\mathbf{n}_\alpha,\mathbf{I}_I)}\Big|_{(S^1,S^2)=(A_{(k_1,k_2)}^1, A_{(k_1,k_2)}^2)} + F_{(k_1,k_2)}^I. \end{aligned} \tag{7.87}$$

The force boundary condition at the boundary point $\mathscr{P}_{(r_1,r_2)}$ is

$$\sum_{\alpha=1}^{2}\mathbf{T}_\alpha(S^1,S^2)\Big|_{(S^1,S^2)=(B^1_{(r_1,r_2)},B^2_{(r_1,r_2)})}+\mathbf{F}_{(r_1,r_2)}=0 \tag{7.88}$$

or

$$\sum_{\alpha=1}^{2}[T_\alpha(S^1,S^2)+T_{\beta\alpha}(S^1,S^2)]\cos\theta_{(\mathbf{n}_\alpha,\mathbf{I}_I)}\Big|_{(S^1,S^2)=(B^1_{(r_1,r_2)},B^2_{(r_1,r_2)})}$$
$$+F^I_{(r_1,r_2)}=0. \tag{7.89}$$

The force boundary condition on the web edge is

$$(\mathbf{T}_\alpha+\frac{\partial}{\partial S^\beta}\int\mathbf{T}_\beta dS^\alpha)\Big|_{S^\alpha=S^\alpha_*}+\mathbf{F}(S^\beta)=0, \tag{7.90}$$

or

$$\left[\frac{\partial}{\partial S^\beta}(\int T_\beta\cos\theta_{(\mathbf{n}_\beta,\mathbf{I}_I)}dS^\alpha)+\frac{\partial}{\partial S^\beta}(\int T_{\beta\alpha}\cos\theta_{(\mathbf{n}_\alpha,\mathbf{I}_I)}dS^\alpha)\right]\Big|_{S^\alpha=S^\alpha_*}$$
$$+(T_\alpha\cos\theta_{(\mathbf{n}_\alpha,\mathbf{I}_I)}+T_{\alpha\beta}\cos\theta_{(\mathbf{n}_\beta,\mathbf{I}_I)})\big|_{S^\alpha=S^\alpha_*}+F^I(S^\beta)=0. \tag{7.91}$$

The displacement continuity and boundary conditions are given as in Eqs.(7.58) and (7.61).

For a continuum web, because the tensional shear exists, the principal tensions ($T^p_{\min,\max}$) should be computed. For instance, if $\Theta_{(S^1,S^2)}=\pi/2$, one obtains

$$T^p_{\min,\max}=\frac{T_\alpha+T_\beta}{2}\pm\sqrt{(\frac{T_\alpha-T_\beta}{2})^2+T^2_{\alpha\beta}} \tag{7.92}$$

and the principal direction S^p with the minimum principal tension $T^p_{\min}$ is determined by

$$\tan 2\theta_{(S^p,S^\alpha)}=\frac{2T_{\alpha\beta}}{T_\alpha-T_\beta}. \tag{7.93}$$

Definition 7.11. A continuum web is called *a locally wrinkled continuum web* on the S^p-direction ($\alpha\in\{1,2\}$) in domain $\mathscr{D}_{(k_1,k_2)}$ if the minimum principal tension at a point $(S^1,S^2)\in\mathscr{D}_{(k_1,k_2)}$ on the S^p-direction satisfies

$$T^p_{\min}(S^1,S^2)<0 \text{ for } (S^1,S^2)\in\mathscr{D}_{(k_1,k_2)}. \tag{7.94}$$

A continuum web is called *a globally wrinkled continuum web* on the S^p-direction ($\alpha\in\{1,2\}$) if Eq.(7.94) holds for all domains $\mathscr{D}_{(k_1,k_2)}\subset\mathscr{D}$ in such a continuum web. The wrinkling boundary on the domain $\mathscr{D}_{(k_1,k_2)}$ is determined by

$$T^p_{\min}(S^1, S^2) = 0 \text{ for } (S^1, S^2) \in \mathscr{D}_{(k_1,k_2)}. \tag{7.95}$$

Definition 7.12. A continuum web is called *a locally free, continuum web* in domain $\mathscr{D}_{(k_1,k_2)}$ if the maximum principal tension at any point $(S^1, S^2) \in \mathscr{D}_{(k_1,k_2)}$ satisfies

$$T^p_{\max}(S^1, S^2) < 0 \text{ for } (S^1, S^2) \in \mathscr{D}_{(k_1,k_2)}. \tag{7.96}$$

A continuum web is called *a globally free continuum web* if all domains $\mathscr{D}_{(k_1,k_2)} \subset \mathscr{D}$ in such a continuum web satisfy Eq.(7.96). The boundary for the free web in the domain $\mathscr{D}_{(k_1,k_2)}$ is determined by

$$T^p_{\max}(S^1, S^2) = 0 \text{ for } (S^1, S^2) \in \mathscr{D}_{(k_1,k_2)}. \tag{7.97}$$

7.2. Nonlinear membranes

In the previous section, the web theory was presented. In this section, a theory for nonlinear membranes will be discussed.

Definition 7.13. If a deformable body on the two principal directions of fibers resists the tensile and compressive forces only, the deformable body is called *a deformable membrane.*

Generally speaking, there are three deformable membranes: (i) arch-network membranes, (ii) arch-fabric membranes, and (iii) continuum membranes. The arch-network membranes and arch-fabric membranes are of non-continuum. The continuum membrane is called usually the membrane in mechanics. To clarify the concepts, the corresponding definitions are given as follows.

Definition 7.14. A deformable membrane is called *an arch-network membrane* if the deformable membrane is formed by the deformable arch or truss network.

Definition 7.15. A deformable membrane is called *an arch-fabric membrane* if the deformable membrane is continuously formed by the deformable arches or trusses.

Definition 7.16. A deformable membrane is called *a continuum membrane* if the deformable membrane is made by continuous media.

Definition 7.17. A deformable body is called *an arch-reinforced web* if the deformable body is formed by the arch-network membranes and the continuum skin webs.

The arch-network reinforced web possesses the function of the membranes, so such a reinforced web can be called the *membrane-web*. For instance, the kite structure is a membrane-web structure or the arch-reinforced web. For the structure, the arch-network can be investigated by the arch theory and the continuum skin web can be investigated by the theory of the continuum webs.

To discuss the soft structure combined with continuous webs and non-continuum membrane, the concept for one-dimensional deformable membrane will be introduced. Such a membrane is a kind of arch-reinforced web, which is also called the *soft membrane*. The definition of the one-dimensional deformable membrane is given as follows.

Definition 7.18. If a deformable body resists the tensile and compressive forces in one principal direction of fibers and the tensile force in the other principal direction, the deformable body is called *a one-dimensional, deformable membrane.*

Definition 7.19. A one-dimensional, deformable membrane is called *a cable-arch network membrane* if the one-dimensional, deformable membrane is formed by the deformable cables (or strings) in a principal direction and the deformable arches (or trusses) in the other principal direction.

Definition 7.20. A one-dimensional, deformable membrane is called *a cable-arch fabric membrane* if the one-dimensional, deformable membrane is continuously formed by deformable cables (or strings) in a principal direction and deformable arches (or trusses) in the other principal direction.

Definition 7.21. A one-dimensional, deformable membrane is called *a one-dimensional, continuum membrane* if the one-dimensional, deformable membrane is made by continuous media.

Definition 7.22. A deformable body is called *a cable-arch reinforced web* if the deformable body is formed by the cable-arch network membranes and the continuous skin webs.

The cables and arches in cable-arch network membranes can be arranged either in order or randomly. Such membranes with the random arrangement will possess more web characteristics. Similarly, the cable-arch reinforced web possesses the more functions as the web with less membrane characteristics.

The cable-arch network and cable-arch fabric membranes are of non-continuum. For non-continuum membranes, the theory can be described as similar to the non-continuum webs. In Sections 7.1.1−7.1.2, $N_\alpha \equiv T_\alpha$ for deformable trusses and arches, and for cables and strings, the expression will be of the same. If the membrane consists of network non-continuum membranes and continuum skin webs, the continuum skin webs can be investigated by the web theory and the reinforced cables and arches can be investigated by the cable and arch theories. The concept of finite elements should be adopted and the corresponding boundary conditions should be considered.

Compared to the deformed webs, a deformable membrane is required to resist compressive internal forces. Thus, the membrane must possess a specific, initial configuration to support such compressive internal forces. This is a main difference between the membrane and continuum web. The continuum membrane can be described as a nonlinear continuous web in Section 7.1.3. Herein, such a description will not be repeated. One can use $N_\alpha \equiv T_\alpha$ and $N_{\alpha\beta} \equiv T_{\alpha\beta}$ for all equations in Section 7.1.3. In addition, $\ddot{X}^I = \dot{X}^I = 0$ is admitted because the initial configuration is fixed. However, *the continuum web possesses a dynamical initial dynamical configuration.* In other words, the non-deformable configuration of membrane is invariant with time, but the non-deformable configuration of the web will be changed with time, which is another difference between the continuum webs and membranes.

7.2.1. A membrane theory based on the Cartesian coordinates

As in nonlinear webs, from Eqs.(7.1)–(7.6), the normal membrane force for the nonlinear membrane can be expressed by

$$\bar{N}_\alpha = \int_{A_\alpha} f_\alpha(\mathscr{E}_\alpha) dA_\alpha. \tag{7.98}$$

The normal membrane force per unit length is defined as

$$N_\alpha = \frac{d\bar{N}_\alpha}{dS^\beta} = \int_{h_1}^{h_2} f_\alpha(\mathscr{E}_\alpha) dz \text{ for } \alpha, \beta \in \{1, 2\} \text{ and } \alpha \neq \beta. \tag{7.99}$$

If $\mathscr{E}_\alpha = 0$ then $f_\alpha(\mathscr{E}_\alpha) = 0$. For linear elastic materials,

$$N_\alpha = \int_{h_1}^{h_2} E_\alpha \mathscr{E}_\alpha dz = E_\alpha h(\frac{1}{\sqrt{G_{\alpha\alpha}}}\sqrt{(X^K_{,\alpha} + u^K_{,\alpha})(X^K_{,\alpha} + u^K_{,\alpha})} - 1). \tag{7.100}$$

The shear membrane force caused by the twisting is defined as

$$\bar{N}_{\alpha\beta} = \int_{A_\alpha} f_{\alpha\beta}(\mathscr{E}_{\alpha\beta}) dA_\alpha \text{ for } \alpha, \beta \in \{1, 2\} \text{ and } \alpha \neq \beta, \tag{7.101}$$

where $\mathscr{E}_{\alpha\beta} \equiv \gamma_{\alpha\beta}$. $\bar{N}_{\alpha\beta}$ represents a shear membrane force on the α-surface with a normal direction ($\mathbf{n}_\alpha$) and its direction is the same as the direction of $\mathbf{n}_\beta$. If $\alpha = \beta$, $\mathscr{E}_{\alpha\beta} \equiv \mathscr{E}_\alpha$ and the foregoing equation reduces to Eq.(7.98) for the normal membrane force only. The shear membrane force per unit length can be defined by

$$N_{\alpha\beta} = \frac{d\bar{N}_{\alpha\beta}}{dS^{\beta}} = \int_{h_1}^{h_2} f_{\alpha\beta}(\mathscr{E}_{\alpha\beta})dz \text{ for } \alpha, \beta \in \{1, 2\} \text{ and } \alpha \neq \beta. \tag{7.102}$$

For linear elastic materials,

$$N_{\alpha\beta} = \int_{h_1}^{h_2} \mathcal{G}_{\alpha\beta}\mathscr{E}_{\alpha\beta}dz = \mathcal{G}_{\alpha\beta}h\gamma_{\alpha\beta}. \tag{7.103}$$

The shear membrane forces per unit length will be simply called *the membrane shear* on the α -cross section.

As in Eqs.(7.80)–(7.83), the membrane force vector on the S^{α} -surface is

$$\boldsymbol{\mathcal{N}}_{\alpha} = N_{\alpha}\mathbf{n}_{\alpha} + N_{\alpha\beta}\mathbf{n}_{\beta} \equiv N_{\alpha}^{I}\mathbf{I}_{I}, \tag{7.104}$$

$$\begin{aligned} N_{\alpha}^{I} &= N_{\alpha}\mathbf{n}_{\alpha}\cdot\mathbf{I}_{I} + N_{\alpha\beta}\mathbf{n}_{\beta}\cdot\mathbf{I}_{I} = N_{\alpha}\cos\theta_{(\mathbf{n}_{\alpha},\mathbf{I}_{I})} + N_{\alpha\beta}\cos\theta_{(\mathbf{n}_{\beta},\mathbf{I}_{I})} \\ &= \frac{N_{\alpha}(X_{,\alpha}^{I} + u_{,\alpha}^{I})}{\sqrt{G_{\alpha\alpha}}(1+\mathscr{E}_{\alpha})} + \frac{N_{\alpha\beta}(X_{,\beta}^{I} + u_{,\beta}^{I})}{\sqrt{G_{\beta\beta}}(1+\mathscr{E}_{\beta})}. \end{aligned} \tag{7.105}$$

Equation of motion in a domain $(S^1, S^2) \in \mathscr{D}_{(k_1,k_2)}$ for ($I = 1, 2, 3$) is

$$\begin{aligned} \rho_0 h\mathbf{u}_{,tt}\sin\Theta_{(S^1,S^2)} &= \mathbf{q}_{(k_1,k_2)} + \boldsymbol{\mathcal{N}}_{\alpha,\alpha}, \quad \text{or} \\ \rho_0 h u_{,tt}^{I}\sin\Theta_{(S^1,S^2)} &= q_{(k_1,k_2)}^{I} + N_{\alpha,\alpha}^{I}. \end{aligned} \tag{7.106}$$

The foregoing equation is rewritten as

$$\rho_0 h u_{,tt}^{I}\sin\Theta_{(S^1,S^2)} = q_{(k_1,k_2)}^{I} + \left[\frac{N_{\alpha}(X_{,\alpha}^{I} + u_{,\alpha}^{I})}{\sqrt{G_{\alpha\alpha}}(1+\mathscr{E}_{\alpha})} + \frac{N_{\alpha\beta}(X_{,\beta}^{I} + u_{,\beta}^{I})}{\sqrt{G_{\beta\beta}}(1+\mathscr{E}_{\beta})}\right]_{,\alpha}, \tag{7.107}$$

where summation on $\alpha, \beta = 1, 2$ and $\alpha \neq \beta$ should be done.

The corresponding force condition at a point $\mathscr{P}_{(k_1,k_2)}$ with $S^i = A_{(k_1,k_2)}^{i}$ is

$$\begin{aligned} &\sum\nolimits_{\alpha=1}^{2} {}^{-}\boldsymbol{\mathcal{N}}_{\alpha}(S^1, S^2)\Big|_{(S^1,S^2)=(A_{(k_1,k_2)}^{1}, A_{(k_1,k_2)}^{2})} \\ &+\sum\nolimits_{\alpha=1}^{2} {}^{+}\boldsymbol{\mathcal{N}}_{\alpha}(S^1, S^2)\Big|_{(S^1,S^2)=(A_{(k_1,k_2)}^{1}, A_{(k_1,k_2)}^{2})} + \mathbf{F}_{(k_1,k_2)} = 0, \end{aligned} \tag{7.108}$$

or

$$\begin{aligned} &\sum\nolimits_{\alpha=1}^{2}[{}^{-}N_{\alpha}(S^1, S^2) + {}^{-}N_{\beta\alpha}(S^1, S^2)]\cos\theta_{(\mathbf{n}_{\alpha},\mathbf{I}_{I})}\Big|_{(S^1,S^2)=(A_{(k_1,k_2)}^{1}, A_{(k_1,k_2)}^{2})} \\ &=\sum\nolimits_{\alpha=1}^{2}[{}^{+}N_{\alpha}(S^1, S^2) + {}^{+}N_{\beta\alpha}(S^1, S^2)]\cos\theta_{(\mathbf{n}_{\alpha},\mathbf{I}_{I})}\Big|_{(S^1,S^2)=(A_{(k_1,k_2)}^{1}, A_{(k_1,k_2)}^{2})} + F_{(k_1,k_2)}^{I}. \end{aligned} \tag{7.109}$$

The force boundary condition at the boundary point $\mathscr{P}_{(r_1,r_2)}$ is

$$\sum_{\alpha=1}^{2} \mathcal{N}_{\alpha}(S^1, S^2)\Big|_{(S^1,S^2)=(B^1_{(r_1,r_2)},B^2_{(r_1,r_2)})} + \mathbf{F}_{(r_1,r_2)} = 0 \tag{7.110}$$

or

$$\sum_{\alpha=1}^{2} [N_{\alpha}(S^1, S^2) + N_{\beta\alpha}(S^1, S^2)]\cos\theta_{(\mathbf{n}_\alpha,\mathbf{I}_I)}\Big|_{(S^1,S^2)=(B^1_{(r_1,r_2)},B^2_{(r_1,r_2)})}$$
$$+ F^I_{(r_1,r_2)} = 0. \tag{7.111}$$

The force boundary conditions on the membrane edge is given by

$$(\mathcal{N}_{\alpha} + \frac{\partial}{\partial S^{\beta}} \int \mathcal{N}_{\beta} dS^{\alpha})\Big|_{S^{\alpha}=S^{\alpha}_{*}} + \mathbf{F}(S^{\beta}) = 0, \tag{7.112}$$

or

$$\frac{\partial}{\partial S^{\beta}} \int N_{\beta} \cos\theta_{(\mathbf{n}_\beta,\mathbf{I}_I)} dS^{\alpha}\Big|_{S^{\alpha}=S^{\alpha}_{*}} + N_{\alpha}\cos\theta_{(\mathbf{n}_\alpha,\mathbf{I}_I)}\Big|_{S^{\alpha}=S^{\alpha}_{*}} + F^I(S^{\beta}) = 0. \tag{7.113}$$

The displacement continuity and boundary conditions are given as in Eqs.(7.57) and (7.58).

If the coordinates (S^{α}, $\alpha = 1, 2$) are orthogonal, equation of motion is

$$\rho_0 h \mathbf{u}_{,tt} = \mathbf{q}_{(k_1,k_2)} + \mathcal{N}_{\alpha,\alpha}, \tag{7.114}$$

or

$$\rho h u^I_{,tt} = q^I_{(k_1,k_2)} + \left[\frac{N_{\alpha}(X^I_{,\alpha} + u^I_{,\alpha})}{\sqrt{G_{\alpha\alpha}}(1+\mathscr{E}_{\alpha})} + \frac{N_{\alpha\beta}(X^I_{,\beta} + u^I_{,\beta})}{\sqrt{G_{\beta\beta}}(1+\mathscr{E}_{\beta})} \right]_{,\alpha} \tag{7.115}$$

with summation on $\alpha, \beta = 1, 2$ and $\alpha \neq \beta$. For a flat membrane, $G_{\alpha\alpha} = 1$. Thus, equation (7.114) becomes

$$\rho h u^I_{,tt} = q^I_{(k_1,k_2)} + \left[\frac{N_{\alpha}(\delta^I_{\alpha} + u^I_{,\alpha})}{(1+\mathscr{E}_{\alpha})} + \frac{N_{\alpha\beta}(\delta^I_{\beta} + u^I_{,\beta})}{(1+\mathscr{E}_{\beta})} \right]_{,\alpha}. \tag{7.116}$$

7.2.2. A membrane theory based on the curvilinear coordinates

Consider a material particle $P(X^1, X^2, X^3)$ in an initial configuration of a membrane at the initial state shown in Fig.7.6. The position **R** of the material particle is described by X^I :

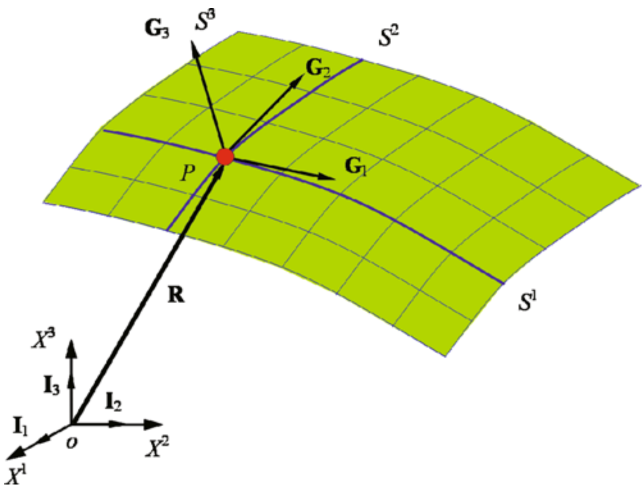

Fig.7.6 A material particle *P* on an initial configuration of a nonlinear membrane.

$$\mathbf{R} = X^I(S^1, S^2)\mathbf{I}_I \equiv X^1\mathbf{I}_1 + X^2\mathbf{I}_2 + X^3\mathbf{I}_3, \tag{7.117}$$

where $\mathbf{I}_K$ are unit vectors in the fixed coordinates. In the local curvilinear reference frame, the position $\mathbf{R}$ is represented by

$$\mathbf{R} = S^\alpha \mathbf{G}_\alpha \equiv S^1\mathbf{G}_1 + S^2\mathbf{G}_2, \tag{7.118}$$

where the component $S^\alpha = \mathbf{R}\cdot\mathbf{G}^\alpha$ (also see, Chapter 3) and the initial base vectors $\mathbf{G}_\alpha \equiv \mathbf{G}_\alpha(S^1, S^2)$ ($\alpha = 1, 2$) are

$$\mathbf{G}_\alpha = \frac{\partial \mathbf{R}}{\partial S^\alpha} = \frac{\partial X^M(S^1, S^2)}{\partial S^\alpha}\mathbf{I}_M = X^M_{,\alpha}\mathbf{I}_M, \tag{7.119}$$

with magnitudes

$$\left|\mathbf{G}_\alpha(\mathbf{X})\right| = \sqrt{G_{\alpha\alpha}} = \sqrt{X^I_{,\alpha}X^I_{,\alpha}} = \sqrt{(\frac{\partial X^1}{\partial S^\alpha})^2 + (\frac{\partial X^2}{\partial S^\alpha})^2 + (\frac{\partial X^3}{\partial S^\alpha})^2} \tag{7.120}$$

and

$$\mathbf{G}_3(S^1, S^2) = \frac{\mathbf{G}_1 \times \mathbf{G}_2}{\left|\mathbf{G}_1 \times \mathbf{G}_2\right|}, \tag{7.121}$$

without summation on α and

$$\begin{aligned} G_{\alpha\beta} &= \mathbf{G}_\alpha \cdot \mathbf{G}_\beta = X^I_{,\alpha}X^I_{,\beta} \\ &= \frac{\partial X^1}{\partial S^\alpha}\frac{\partial X^1}{\partial S^\beta} + \frac{\partial X^2}{\partial S^\alpha}\frac{\partial X^2}{\partial S^\beta} + \frac{\partial X^3}{\partial S^\alpha}\frac{\partial X^3}{\partial S^\beta} \end{aligned} \tag{7.122}$$

are metric tensors in the initial configuration. If the two base vectors are orthogonal, $G_{\alpha\beta} = 0$.

On the deformed configuration of the membrane, a particle at point P moves through displacement $\mathbf{u}$ to position p, and the particle Q, infinitesimally close to $P(S^1, S^2)$, moves through $\mathbf{u} + d\mathbf{u}$ to q in the neighborhood of $p(S^1, S^2)$, as illustrated in Fig.7.7. The position of point p is for $\alpha = 1,2$

$$\mathbf{r} = \mathbf{R} + \mathbf{u} = (S^\alpha + u^\alpha)\mathbf{G}_\alpha + u^3\mathbf{G}_3, \tag{7.123}$$

where $\mathbf{u} = u^\Lambda \mathbf{G}_\Lambda$ ($\Lambda = 1,2,3$). Thus, $\overline{PQ} = d\mathbf{R}$ and $\overline{pq} = d\mathbf{r}$ are

$$d\mathbf{R} = \mathbf{G}_\alpha dS^\alpha, \; d\mathbf{r} = \mathbf{G}_\alpha dS^\alpha + d\mathbf{u}; \tag{7.124}$$

and the infinitesimal displacement is

$$\begin{aligned} d\mathbf{u} &= u^\beta_{;\alpha} dS^\alpha \mathbf{G}_\beta = (u^\beta_{,\alpha} + \Gamma^\beta_{\alpha\gamma} u^\gamma) dS^\alpha \mathbf{G}_\beta \\ &= u_{\beta;\alpha} dS^\alpha \mathbf{G}^\beta = (u_{\beta,\alpha} - \Gamma^\gamma_{\alpha\beta} u_\gamma) dS^\alpha \mathbf{G}^\beta, \end{aligned} \tag{7.125}$$

where

$$\begin{aligned} &\mathbf{G}^\alpha = G^{\alpha\beta}\mathbf{G}_\beta, \; G^{\alpha\beta} = \frac{\text{cofactor } (G_{\alpha\beta})}{G}, \\ &G = \det(G_{\alpha\beta}), \; u_\alpha = G_{\alpha\beta} u^\beta \end{aligned} \tag{7.126}$$

and

$$\Gamma^\beta_{\alpha\gamma} = \frac{\partial^2 X^I}{\partial S^\alpha \partial S^\gamma} \frac{\partial S^\beta}{\partial X^I} \tag{7.127}$$

is the Christoffel symbol defined in Chapter 2 (also see, Eringen, 1967). The semicolon represents covariant partial differentiation.

From Eqs.(7.124) and (7.125),

$$\begin{aligned} &d\mathbf{r} = (u^\beta_{;\alpha}\mathbf{G}_\beta + \mathbf{G}_\alpha) dS^\alpha = dS^\alpha \mathbf{g}_\alpha; \\ &\mathbf{g}_\alpha \equiv \frac{\partial \mathbf{r}}{\partial S^\alpha} = u^\beta_{;\alpha}\mathbf{G}_\beta + \mathbf{G}_\alpha = (u^\beta_{;\alpha} + \delta^\beta_\alpha)\mathbf{G}_\beta. \end{aligned} \tag{7.128}$$

The Lagrangian strain tensor $E_{\alpha\beta}$ referred to the initial configuration is

$$E_{\alpha\beta} = \frac{1}{2}(u_{\alpha;\beta} + u_{\alpha;\beta} + u^\gamma_{;\alpha} u_{\gamma;\beta}). \tag{7.129}$$

As in Eringen (1962), the change in length of $d\mathbf{R}$ per unit length gives

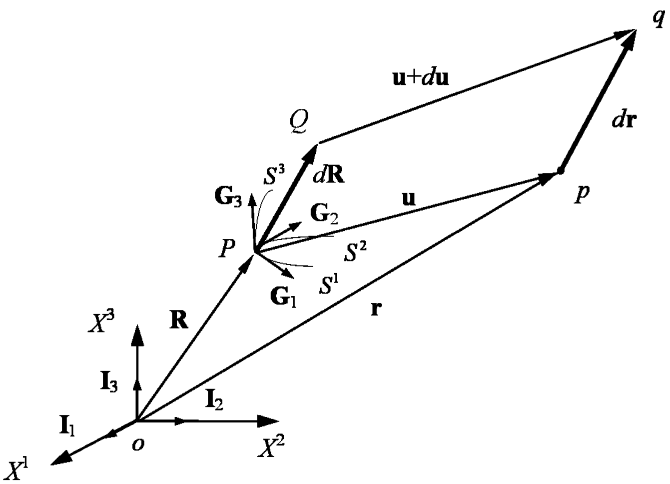

Fig. 7.7 Deformation of a differential linear element.

$$\mathscr{E}_\alpha = \frac{|\underset{\alpha}{d\mathbf{r}}| - |\underset{\alpha}{d\mathbf{R}}|}{|\underset{\alpha}{d\mathbf{R}}|} = \sqrt{1 + \frac{2E_{\alpha\alpha}}{G_{\alpha\alpha}}} - 1, \tag{7.130}$$

where $\mathscr{E}_\alpha$ is the relative elongation changing with dS^α. The unit vectors along $d\mathbf{R}$ and $d\mathbf{r}$ in Chapter 3 are

$$\begin{aligned} \mathbf{N}_\alpha &\equiv \frac{\mathbf{G}_\alpha}{|\mathbf{G}_\alpha|} = \frac{\underset{\alpha}{d\mathbf{R}}}{|\underset{\alpha}{d\mathbf{R}}|} = \frac{1}{\sqrt{G_{\alpha\alpha}}}\mathbf{G}_\alpha, \\ \mathbf{n}_\alpha &\equiv \frac{\mathbf{g}_\alpha}{|\mathbf{g}_\alpha|} = \frac{\underset{\alpha}{d\mathbf{r}}}{|\underset{\alpha}{d\mathbf{r}}|} = \frac{\delta_\alpha^\beta + u_{;\alpha}^\beta}{\sqrt{G_{\alpha\alpha} + 2E_{\alpha\alpha}}}\mathbf{G}_\beta. \end{aligned} \tag{7.131}$$

Let $\Theta_{(\mathbf{N}_\alpha,\mathbf{N}_\beta)}$ and $\theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)}$ be the induced angles between $\mathbf{n}_\alpha$ and $\mathbf{n}_\beta$ before and after deformation,

$$\begin{aligned} \cos\theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)} &\equiv \cos(\Theta_{(\mathbf{N}_\alpha,\mathbf{N}_\beta)} - \gamma_{\alpha\beta}) \\ &= \frac{\underset{\alpha}{d\mathbf{r}}\cdot\underset{\beta}{d\mathbf{r}}}{|\underset{\alpha}{d\mathbf{r}}||\underset{\beta}{d\mathbf{r}}|} = \frac{G_{\alpha\beta} + 2E_{\alpha\beta}}{\sqrt{(G_{\alpha\alpha} + 2E_{\alpha\alpha})(G_{\beta\beta} + 2E_{\beta\beta})}}, \\ \cos\Theta_{(\mathbf{N}_\alpha,\mathbf{N}_\beta)} &= \frac{\underset{\alpha}{d\mathbf{R}}\cdot\underset{\beta}{d\mathbf{R}}}{|\underset{\alpha}{d\mathbf{R}}||\underset{\beta}{d\mathbf{R}}|} = \frac{G_{\alpha\beta}}{\sqrt{G_{\alpha\alpha}G_{\beta\beta}}}; \end{aligned} \tag{7.132}$$

and the shear strain is

$$\gamma_{\alpha\beta} \equiv \Theta_{(\mathbf{N}_\alpha,\mathbf{N}_\beta)} - \theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)} = \cos^{-1}\frac{G_{\alpha\beta}}{\sqrt{G_{\alpha\alpha}G_{\beta\beta}}} - \cos^{-1}\frac{G_{\alpha\beta}+2E_{\alpha\beta}}{\sqrt{(G_{\alpha\alpha}+2E_{\alpha\alpha})(G_{\beta\beta}+2E_{\beta\beta})}}. \tag{7.133}$$

From Eq.(7.131), the direction cosine of the rotation without summation on α and β is

$$\cos\theta_{(\mathbf{N}_\alpha,\mathbf{n}_\beta)} = \frac{d\underset{\alpha}{\mathbf{R}}\cdot d\underset{\beta}{\mathbf{r}}}{|d\underset{\alpha}{\mathbf{R}}||d\underset{\beta}{\mathbf{r}}|} = \frac{G_{\alpha\beta}+u_{\alpha;\beta}}{\sqrt{G_{\alpha\alpha}}\sqrt{G_{\beta\beta}+2E_{\beta\beta}}}. \tag{7.134}$$

In addition, the change ratio of areas before and after deformation is

$$\frac{d\underset{\alpha\beta}{a}}{d\underset{\alpha\beta}{A}} = \frac{(1+\mathscr{E}_\alpha)(1+\mathscr{E}_\beta)\sin\theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)}}{\sin\Theta_{(\mathbf{N}_\alpha,\mathbf{N}_\beta)}}, \tag{7.135}$$

where $d\underset{\alpha\beta}{a} = |d\underset{\alpha}{\mathbf{r}}\times d\underset{\beta}{\mathbf{r}}|$ and $d\underset{\alpha\beta}{A} = |d\underset{\alpha}{\mathbf{R}}\times d\underset{\beta}{\mathbf{R}}|$.

Similarly, the membrane forces per unit length are given by Eq.(7.99) and (7.102). Only the strains are replaced by those in the curvilinear frame. The membrane force vector on the S^Λ-direction surface ($\Lambda = 1, 2, 3$ and $\alpha, \beta \in \{1, 2\}$) can be expressed by

$$\boldsymbol{\mathcal{N}}_\alpha = N_\alpha \mathbf{n}_\alpha + N_{\alpha\beta}\mathbf{n}_\beta \equiv N_\alpha^\Lambda \mathbf{N}_\Lambda, \tag{7.136}$$

and

$$\begin{aligned} N_\alpha^\Lambda &= N_\alpha \mathbf{n}_\alpha\cdot\mathbf{N}_\Lambda + N_{\alpha\beta}\mathbf{n}_\beta\cdot\mathbf{N}_\Lambda \\ &= N_\alpha\cos\theta_{(\mathbf{n}_\alpha,\mathbf{N}_\Lambda)} + N_{\alpha\beta}\cos\theta_{(\mathbf{n}_\beta,\mathbf{N}_\Lambda)} \\ &= \frac{N_\alpha(G_{\Lambda\alpha}+u_{\Lambda;\alpha})}{\sqrt{G_{\Lambda\Lambda}}\sqrt{G_{\alpha\alpha}}(1+\mathscr{E}_\alpha)} + \frac{N_{\alpha\beta}(G_{\Lambda\beta}+u_{\Lambda;\beta})}{\sqrt{G_{\Lambda\Lambda}}\sqrt{G_{\beta\beta}}(1+\mathscr{E}_\beta)}. \end{aligned} \tag{7.137}$$

On the initial configuration, the external concentrated force on point $\mathscr{P}_{(k_1,k_2)}$ and distributed force on domain $\mathscr{D}_{(k_1,k_2)}$ are defined as

$$\begin{aligned} \mathbf{q}_{(k_1,k_2)} &= q^\Lambda_{(k_1,k_2)}(S^1,S^2)\mathbf{N}_\Lambda \quad \text{for } (S^1,S^2)\in\mathscr{D}_{(k_1,k_2)}, \\ \mathbf{F}_{(k_1,k_2)} &= F^\Lambda_{(k_1,k_2)}(S^1,S^2)\mathbf{N}_\Lambda \quad \text{for } (S^1,S^2)\in\mathscr{P}_{(k_1,k_2)}, \end{aligned} \tag{7.138}$$

and on the deformed configuration, the forces are defined as

$$\begin{aligned} \mathbf{f}_{(k_1,k_2)} &= f^\Lambda_{(k_1,k_2)}(s^1,s^2)\mathbf{n}_\Lambda \quad \text{for } (s^1,s^2)\in\boldsymbol{p}_{(k_1,k_2)}, \\ \mathbf{p}_{(k_1,k_2)} &= p^\Lambda_{(k_1,k_2)}(s^1,s^2)\mathbf{n}_\Lambda \quad \text{for } (s^1,s^2)\in\boldsymbol{d}_{(k_1,k_2)}, \end{aligned} \tag{7.139}$$

The corresponding relations between the two forces are

$$\mathbf{F}_{(k_1,k_2)} = \mathbf{f}_{(k_1,k_2)} \text{ and } \mathbf{p}_{(k_1,k_2)} = \frac{\sin\Theta_{(\mathbf{N}_1,\mathbf{N}_2)}}{(1+\mathscr{E}_1)(1+\mathscr{E}_2)\sin\theta_{(\mathbf{n}_1,\mathbf{n}_2)}}\mathbf{q}_{(k_1,k_2)}. \tag{7.140}$$

From the normal and shear membrane forces, equation of motion for a domain $(S^1, S^2)\in \mathscr{D}_{(k_1,k_2)}$ for ($\Lambda = 1,2,3$ and $\alpha = 1, 2$) is

$$\begin{aligned} &\rho_0 h\mathbf{u}_{,tt}\sin\Theta_{(\mathbf{N}_1,\mathbf{N}_2)} = \mathbf{q}_{(k_1,k_2)} + \boldsymbol{\mathcal{N}}_{\alpha;\alpha}, \\ \text{or } &\rho_0 h u_{\Lambda,tt}\sin\Theta_{(\mathbf{N}_1,\mathbf{N}_2)} = q^{\Lambda}_{(k_1,k_2)} + N^{\Lambda}_{\alpha;\alpha}. \end{aligned} \tag{7.141}$$

The foregoing equation is rewritten as

$$\begin{aligned} &\rho_0 h u_{\Lambda,tt}\sin\Theta_{(\mathbf{N}_1,\mathbf{N}_2)} \\ &= q^{\Lambda}_{(k_1,k_2)} + \left[\frac{N_\alpha(G_{\Lambda\alpha}+u_{\Lambda;\alpha})}{\sqrt{G_{\Lambda\Lambda}}\sqrt{G_{\alpha\alpha}}(1+\mathscr{E}_\alpha)} + \frac{N_{\alpha\beta}(G_{\Lambda\beta}+u_{\Lambda;\beta})}{\sqrt{G_{\Lambda\Lambda}}\sqrt{G_{\beta\beta}}(1+\mathscr{E}_\beta)}\right]_{;\alpha}, \end{aligned} \tag{7.142}$$

where summation on $\alpha, \beta = 1, 2$ and $\alpha \neq \beta$ should be done. The foregoing equation is expanded for $\alpha = 1,2$ as

$$\begin{aligned} &\rho h u_{\alpha,tt}\sin\Theta_{(\mathbf{N}_1,\mathbf{N}_2)} \\ &= q^{\alpha}_{(k_1,k_2)} + \left[\frac{N_1(G_{\alpha 1}+u_{\alpha;1})}{\sqrt{G_{\alpha\alpha}}\sqrt{G_{11}}(1+\mathscr{E}_1)} + \frac{N_{12}(G_{\alpha 2}+u_{\alpha;2})}{\sqrt{G_{\alpha\alpha}}\sqrt{G_{22}}(1+\mathscr{E}_2)}\right]_{;1} \\ &\quad + \left[\frac{N_{12}(G_{\alpha 1}+u_{\alpha;1})}{\sqrt{G_{\alpha\alpha}}\sqrt{G_{11}}(1+\mathscr{E}_1)} + \frac{N_{22}(G_{\alpha 2}+u_{\alpha;2})}{\sqrt{G_{\alpha\alpha}}\sqrt{G_{22}}(1+\mathscr{E}_2)}\right]_{;2} \end{aligned} \tag{7.143}$$

$$\begin{aligned} \rho h u_{3,tt}\sin\Theta_{(\mathbf{N}_1,\mathbf{N}_2)} &= q^{3}_{(k_1,k_2)} + \left[\frac{N_1 u_{3;1}}{\sqrt{G_{11}}(1+\mathscr{E}_1)} + \frac{N_{12}u_{3;2}}{\sqrt{G_{22}}(1+\mathscr{E}_2)}\right]_{;1} \\ &\quad + \left[\frac{N_{12}u_{3;1}}{\sqrt{G_{11}}(1+\mathscr{E}_1)} + \frac{N_{22}u_{3;2}}{\sqrt{G_{22}}(1+\mathscr{E}_2)}\right]_{;2}. \end{aligned} \tag{7.144}$$

Similarly, the corresponding force condition at a point $\mathscr{P}_{(k_1,k_2)}$ with $S^i = A^i_{(k_1,k_2)}$ is

$$\begin{aligned} &\sum\nolimits_{\alpha=1}^{2} {}^{-}\boldsymbol{\mathcal{N}}_\alpha(S^1,S^2)\Big|_{(S^1,S^2)=(A^1_{(k_1,k_2)},A^2_{(k_1,k_2)})} \\ &+\sum\nolimits_{\alpha=1}^{2} {}^{+}\boldsymbol{\mathcal{N}}_\alpha(S^1,S^2)\Big|_{(S^1,S^2)=(A^1_{(k_1,k_2)},A^2_{(k_1,k_2)})} + \mathbf{F}_{(k_1,k_2)} = 0, \end{aligned} \tag{7.145}$$

or

$$\sum_{\alpha=1}^{2}[{}^{-}N_{\alpha}(S^1,S^2)+{}^{-}N_{\beta\alpha}(S^1,S^2)]\cos\theta_{(\mathbf{n}_\alpha,\mathbf{N}_\Lambda)}\Big|_{(S^1,S^2)=(A^1_{(k_1,k_2)},A^2_{(k_1,k_2)})}$$
$$=\sum_{\alpha=1}^{2}[{}^{+}N_{\alpha}(S^1,S^2)+{}^{+}N_{\beta\alpha}(S^1,S^2)]\cos\theta_{(\mathbf{n}_\alpha,\mathbf{N}_\Lambda)}\Big|_{(S^1,S^2)=(A^1_{(k_1,k_2)},A^2_{(k_1,k_2)})}+F^{\Lambda}_{(k_1,k_2)}. \quad (7.146)$$

The force boundary condition at the boundary point $\mathscr{P}_{(r_1,r_2)}$ is

$$\sum_{\alpha=1}^{2}\mathcal{N}_{\alpha}(S^1,S^2)\Big|_{(S^1,S^2)=(B^1_{(r_1,r_2)},B^2_{(r_1,r_2)})}+\mathbf{F}_{(r_1,r_2)}=0 \quad (7.147)$$

or

$$\sum_{\alpha=1}^{2}[N_{\alpha}(S^1,S^2)+N_{\beta\alpha}(S^1,S^2)]\cos\theta_{(\mathbf{n}_\alpha,\mathbf{N}_\Lambda)}\Big|_{(S^1,S^2)=(B^1_{(r_1,r_2)},B^2_{(r_1,r_2)})}$$
$$+F^{\Lambda}_{(r_1,r_2)}=0. \quad (7.148)$$

The force boundary condition on the membrane edge is

$$(\mathcal{N}_{\alpha}+\frac{\partial}{\partial S^{\beta}}\int\mathcal{N}_{\beta}dS^{\alpha})\Big|_{S^{\alpha}=S^{\alpha}_{*}}+\mathbf{F}(S^{\beta})=0, \quad (7.149)$$

or

$$\frac{\partial}{\partial S^{\beta}}\int N_{\beta}\cos\theta_{(\mathbf{n}_\beta,\mathbf{N}_\Lambda)}dS^{\alpha}\Big|_{S^{\alpha}=S^{\alpha}_{*}}+N_{\alpha}\cos\theta_{(\mathbf{n}_\alpha,\mathbf{N}_\Lambda)}\Big|_{S^{\alpha}=S^{\alpha}_{*}}+F^{\Lambda}(S^{\beta})=0. \quad (7.150)$$

The displacement continuity and boundary conditions are

$${}^{+}u^{\Lambda}_{(k_1,k_2)}={}^{-}u^{\Lambda}_{(k_1,k_2)}\ \text{and}\ {}^{+}X^{\Lambda}_{(k_1,k_2)}={}^{-}X^{\Lambda}_{(k_1,k_2)}. \quad (7.151)$$

The displacement boundary condition at a boundary point $\mathscr{P}_{(r_1,r_2)}$ is

$$u^{\Lambda}_{(r_1,r_2)}=b^{\Lambda}_{(r_1,r_2)}\ \text{and}\ X^{\Lambda}_{(r_1,r_2)}=B^{\Lambda}_{(r_1,r_2)}. \quad (7.152)$$

7.3. Nonlinear shells

The theories for webs and membranes were presented in the previous sections. In this section, a theory for nonlinear shells will be presented. As in the membrane theory, based on the Cartesian coordinates and a curvilinear frame on the initial configuration, the theory of nonlinear shells will be presented. The general theory will reduce to the existing theories.

Definition 7.23. If a deformable body on the two principal directions of fibers resists the internal forces and bending moments, the deformable body is called *a deformable shell.*

For the similar discussion of membranes, the shell-like structures can be classified. The five shell-like structures are: (i) rod-network shell, (ii) rod-fabric shell, (iii) continuum shell, (iv) rod-reinforced membranes and (v) rod-reinforced web.

Definition 7.24. A deformable shell is called *a rod-network shell* if the deformable shell is formed by the deformable rod or beam network.

Definition 7.25. A deformable shell is called *a rod-fabric shell* if the deformable shell is continuously formed by the deformable rods or beams.

Definition 7.26. A deformable shell is called *a continuum shell* if the deformable shell is made by continuous media.

Definition 7.27. A deformable body is called *a rod-reinforced membrane* if the deformable body is formed by rod-network shells and continuum membranes.

Definition 7.28. A deformable body is called *a rod-reinforced web* if the deformable body is formed by rod-network membranes and continuum skin webs.

The rod-network reinforced membrane possesses the functions of shells. Thus, such a reinforced membrane can be called the *shell-membrane*. Such a *shell-membrane* structure is often used in aircrafts and chemical containers. In such a structure, the rod-network can be investigated by the rod (or beam) theory, and the continuum membranes can be investigated by the theory of continuum membranes. The rod and beam theories will be presented in next chapter. The rod-network reinforced web possesses the functions of shells, but such a reinforced web is framed by the rod-network, which can be called the *shell-web*. For instance, the balloon structures or umbrella-type structures are of the shell-web structure or the rod-reinforced webs. In that structure, the rod-network can be investigated by the rod (or beam) theory, and the continuum skin web can be also investigated by the theory of the continuum webs.

To discuss the soft structure combined with continuum webs and shells, the concept for one-dimensional deformable shells will be introduced. Such a shell is a kind of rod-reinforced webs, as called the *soft shell*. The one-dimensional deformable shell is defined as follows.

Definition 7.29. If a deformable body resists the tensile, compressive forces, bending and torsion moments in one principal direction of fibers and the tensile force in the other principal direction, the deformable body is called *a one-dimensional, deformable shell.*

Definition 7.30. A one-dimensional, deformable *shell* is called *a cable-rod network shell* if the one-dimensional, deformable shell is formed by the deformable cables (or strings) in a principal direction and the deformable beams in the other principal direction.

Definition 7.31. A one-dimensional, deformable shell is called *a cable-rod fabric shell* if the one-dimensional, deformable shell is continuously formed by deformable cables (or strings) in a principal direction and deformable beams in the other principal direction.

Definition 7.32. A one-dimensional, deformable shell is called *a one-dimensional, continuum shell* if the one-dimensional, deformable cable-shell is made by continuous media.

Definition 7.33. A deformable body is called *a cable-rod reinforced web* if the deformable body is formed by cable-rod network shells and continuum skin webs.

Definition 7.34. A deformable body is called *a cable-arch-rod reinforced web* if the deformable body is formed by a cable-arch-rod network deformable body and continuum skin webs.

The cables and arches in cable-rod or cable-arch-rod network deformable body can be arranged either in order or randomly. Such a deformable body with the random arrangement has more web characteristics. Similarly, the reinforced deformable structures, formed by cables, arches, rods, continuous membranes and shells can be discussed. The reinforced deformable webs, membranes and shells can be investigated through finite elements. Herein, the theory for continuous shells will be developed only.

7.3.1. A shell theory based on the Cartesian coordinates

Since a deformable shell can support internal forces and bending moments, the shell like a membrane must possesses a specific, initial configuration to resist such internal forces and moments. But the membrane cannot resist any moments. Again, $\ddot{X} = \dot{X} = 0$ because the initial configuration of the shell is fixed. Consider a material particle $P(X^1, X^2, X^3)$ in an initial configuration of a shell at the initial state shown in Fig.7.8. As in Eqs.(7.117)−(7.121) for the membranes, the position **R** of the material particle in the initial configuration of the shell is described by X^K :

$$\mathbf{R} = X^I(S^1, S^2, S^3)\mathbf{I}_I \equiv X^1\mathbf{I}_1 + X^2\mathbf{I}_2 + X^3\mathbf{I}_3, \tag{7.153}$$

where $\mathbf{I}_I$ are unit vectors in the fixed coordinates. In the membrane theory, only two curvilinear coordinates are considered. However, for shells, three curvilinear should be considered for resisting the transverse shear forces and moments. Thus, in the local curvilinear reference frame, the position vector **R** is represented by

$$\mathbf{R} = S^\Lambda\mathbf{G}_\Lambda \equiv S^1\mathbf{G}_1 + S^2\mathbf{G}_2 + S^3\mathbf{G}_3, \tag{7.154}$$

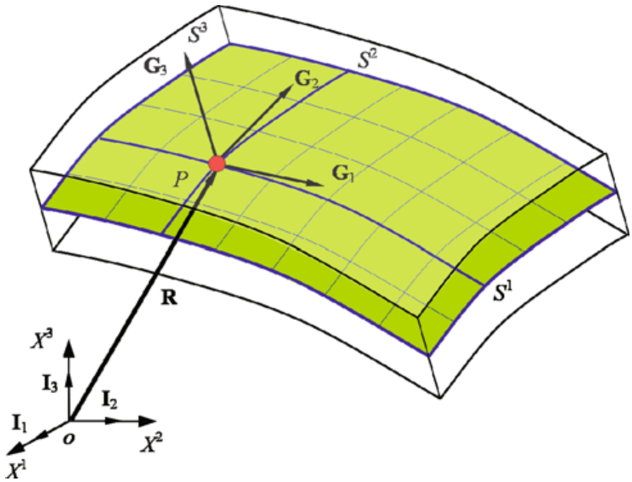

Fig. 7.8 A material particle P on an initial configuration of a nonlinear shell.

where the component $S^\alpha = \mathbf{R}\cdot\mathbf{G}^\alpha$ and the corresponding initial base vectors $\mathbf{G}_\Lambda \equiv \mathbf{G}_\Lambda(S^1, S^2, S^3)$ $(\Lambda = 1, 2, 3)$ are

$$\begin{aligned} \mathbf{G}_\alpha &= \frac{\partial \mathbf{R}}{\partial S^\alpha} = \frac{\partial X^I(S^1, S^2)}{\partial S^\alpha}\mathbf{I}_I = X^I_{,\alpha}\mathbf{I}_I, \\ \mathbf{N}_\alpha &\equiv \frac{\mathbf{G}_\alpha}{|\mathbf{G}_\alpha|} = \frac{1}{\sqrt{G_{\alpha\alpha}}}X^I_{,\alpha}\mathbf{I}_I \end{aligned} \tag{7.155}$$

with magnitude

$$|\mathbf{G}_\alpha| = \sqrt{G_{\alpha\alpha}} = \sqrt{X^I_{,\alpha}X^I_{,\alpha}} = \sqrt{(\frac{\partial X^1}{\partial S^\alpha})^2 + (\frac{\partial X^2}{\partial S^\alpha})^2 + (\frac{\partial X^3}{\partial S^\alpha})^2} \tag{7.156}$$

and the metric tensors in the initial configuration for $\alpha, \beta \in \{1, 2, 3\}$ are

$$\begin{aligned} G_{\alpha\beta} &= \mathbf{G}_\alpha \cdot \mathbf{G}_\beta = X^I_{,\alpha}X^I_{,\beta} \\ &= \frac{\partial X^1}{\partial S^\alpha}\frac{\partial X^1}{\partial S^\beta} + \frac{\partial X^2}{\partial S^\alpha}\frac{\partial X^2}{\partial S^\beta} + \frac{\partial X^3}{\partial S^\alpha}\frac{\partial X^3}{\partial S^\beta}. \end{aligned} \tag{7.157}$$

The equations in Eqs.(7.153)–(7.157) are similar to Eqs.(7.117)–(7.121).

7.3.1a Shell strains

On the deformed configuration, the particle at point P moves through displacement $\mathbf{u}$ to position p, and the particle Q, infinitesimally close to $P(S^1, S^2, S^3)$, moves through $\mathbf{u}+d\mathbf{u}$ to q in the neighborhood of $p(S^1, S^2, S^3)$, as illustrated in Fig.7.7. The corresponding position of point p is expressed from the rectangular coordinates instead of the curvilinear coordinates, i.e.,

$$\mathbf{r}=\mathbf{R}+\mathbf{u}=(X^I+u^I)\mathbf{I}_I. \tag{7.158}$$

The corresponding infinitesimal line elements $\overline{PQ}=d\mathbf{R}$ and $\overline{pq}=d\mathbf{r}$ are expressed ($\alpha=1, 2, 3$) by

$$d\mathbf{R}=X^I_{,\alpha}dS^\alpha\mathbf{I}_I,\ d\mathbf{u}=u^I_{,\alpha}dS^\alpha\mathbf{I}_I. \tag{7.159}$$

Thus,

$$d\mathbf{r}=(X^I_{,\alpha}+u^I_{,\alpha})dS^\alpha\mathbf{I}_I,\ \ \mathbf{g}_\alpha\equiv\frac{\partial\mathbf{r}}{\partial S^\alpha}=(X^I_{,\alpha}+u^I_{,\alpha})\mathbf{I}_I. \tag{7.160}$$

As in Eqs.(7.5) and (7.6), the strains based on the change in length of $d\mathbf{R}$ per unit length give

$$\begin{aligned}\varepsilon_\alpha&=\frac{|\underset{\alpha}{d\mathbf{r}}|-|\underset{\alpha}{d\mathbf{R}}|}{|\underset{\alpha}{d\mathbf{R}}|}=\frac{1}{\sqrt{G_{\alpha\alpha}}}\sqrt{(X^I_{,\alpha}+u^I_{,\alpha})(X^I_{,\alpha}+u^I_{,\alpha})}-1\\&=\frac{1}{\sqrt{G_{\alpha\alpha}}}\sqrt{G_{\alpha\alpha}+2E_{\alpha\alpha}}-1.\end{aligned} \tag{7.161}$$

The Lagrangian strain tensor $E_{\alpha\beta}$ referred to the initial configuration is

$$E_{\alpha\beta}=\frac{1}{2}(X^I_{,\alpha}u^I_{,\beta}+X^I_{,\beta}u^I_{,\alpha}+u^I_{,\alpha}u^I_{,\beta}). \tag{7.162}$$

Because of the shear forces, the shear deformation of the shell should be developed. As similar to Eqs.(7.131)–(7.134), the shear strains can be expressed except for the expression of $E_{\alpha\beta}$. Thus, the corresponding expressions are given as follows. The unit vectors along $d\mathbf{R}$ and $d\mathbf{r}$ in Chapter 3 are expressed by

$$\begin{aligned}\mathbf{N}_\alpha&\equiv\frac{\mathbf{G}_\alpha}{|\mathbf{G}_\alpha|}=\frac{\underset{\alpha}{d\mathbf{R}}}{|\underset{\alpha}{d\mathbf{R}}|}=\frac{X^I_{,\alpha}}{\sqrt{G_{\alpha\alpha}}}\mathbf{I}_I,\\\mathbf{n}_\alpha&\equiv\frac{\mathbf{g}_\alpha}{|\mathbf{g}_\alpha|}=\frac{\underset{\alpha}{d\mathbf{r}}}{|\underset{\alpha}{d\mathbf{r}}|}=\frac{X^I_{,\alpha}+u^I_{,\alpha}}{\sqrt{G_{\alpha\alpha}+2E_{\alpha\alpha}}}\mathbf{I}_I.\end{aligned} \tag{7.163}$$

In the similar fashion, the angles between $\mathbf{n}_\alpha$ and $\mathbf{n}_\beta$ *before* and *after* deformation are expressed by $\Theta_{(\mathbf{N}_\alpha,\mathbf{N}_\beta)}$ and $\theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)}$, i.e.,

$$\cos\theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)}\equiv\cos(\Theta_{(\mathbf{N}_\alpha,\mathbf{N}_\beta)}-\gamma_{\alpha\beta})$$

$$= \frac{d\underset{\alpha}{\mathbf{r}} \cdot d\underset{\beta}{\mathbf{r}}}{|d\underset{\alpha}{\mathbf{r}}||d\underset{\beta}{\mathbf{r}}|} = \frac{G_{\alpha\beta} + 2E_{\alpha\beta}}{\sqrt{\left(G_{\alpha\alpha} + 2E_{\alpha\alpha}\right)\left(G_{\beta\beta} + 2E_{\beta\beta}\right)}}$$

$$= \frac{(X^I_{,\alpha} + u^I_{,\alpha})(X^I_{,\beta} + u^I_{,\beta})}{\sqrt{\left(G_{\alpha\alpha} + 2E_{\alpha\alpha}\right)\left(G_{\beta\beta} + 2E_{\beta\beta}\right)}}, \tag{7.164}$$

$$\cos\Theta_{(\mathbf{N}_\alpha, \mathbf{N}_\beta)} = \frac{d\underset{\alpha}{\mathbf{R}} \cdot d\underset{\beta}{\mathbf{R}}}{|d\underset{\alpha}{\mathbf{R}}||d\underset{\beta}{\mathbf{R}}|} = \frac{G_{\alpha\beta}}{\sqrt{G_{\alpha\alpha}G_{\beta\beta}}} = \frac{X^I_{,\alpha}X^I_{,\beta}}{\sqrt{G_{\alpha\alpha}G_{\beta\beta}}}$$

and the corresponding shear strain is defined by

$$\gamma_{\alpha\beta} \equiv \Theta_{(\mathbf{N}_\alpha, \mathbf{N}_\beta)} - \theta_{(\mathbf{n}_\alpha, \mathbf{n}_\beta)}$$

$$= \cos^{-1}\frac{G_{\alpha\beta}}{\sqrt{G_{\alpha\alpha}G_{\beta\beta}}} - \cos^{-1}\frac{G_{\alpha\beta} + 2E_{\alpha\beta}}{\sqrt{(G_{\alpha\alpha} + 2E_{\alpha\alpha})(G_{\beta\beta} + 2E_{\beta\beta})}}$$

$$= \cos^{-1}\frac{X^I_{,\alpha}X^I_{,\beta}}{\sqrt{G_{\alpha\alpha}G_{\beta\beta}}} - \cos^{-1}\frac{(X^I_{,\alpha} + u^I_{,\alpha})(X^I_{,\beta} + u^I_{,\beta})}{\sqrt{(G_{\alpha\alpha} + 2E_{\alpha\alpha})(G_{\beta\beta} + 2E_{\beta\beta})}}. \tag{7.165}$$

Consider the base vector $\mathbf{G}_3$ to be normal to the plane of $\mathbf{G}_1$ and $\mathbf{G}_2$ in the curvilinear coordinates, which is defined by

$$\mathbf{G}_3 \equiv \frac{\mathbf{G}_1 \times \mathbf{G}_2}{G} \text{ and } G_{\alpha 3} = 0 \text{ for } \alpha = 1, 2. \tag{7.166}$$

From the foregoing definition,

$$\mathbf{G}_3 = X^I_{,3}\mathbf{I}_I = \frac{1}{G}(X^{\text{mod}(I,3)+1}_{,1} X^{\text{mod}(I+1,3)+1}_{,2} - X^{\text{mod}(I,3)+1}_{,2} X^{\text{mod}(I+1,3)+1}_{,1})\mathbf{I}_I \tag{7.167}$$

and

$$G = |\mathbf{G}_1 \times \mathbf{G}_2| = \sqrt{G_{11}G_{22} - G_{12}} \text{ and } G_{33} = 1. \tag{7.168}$$

If $\mathbf{G}_1$ and $\mathbf{G}_2$ are orthogonal, one obtains $G_{12} = 0$ and $G = \sqrt{G_{11}G_{22}}$.

From Eq.(7.163), the direction cosine of the rotation without summation on α and β is

$$\cos\theta_{(\mathbf{N}_\alpha, \mathbf{n}_\beta)} = \frac{d\underset{\alpha}{\mathbf{R}} \cdot d\underset{\beta}{\mathbf{r}}}{|d\underset{\alpha}{\mathbf{R}}||d\underset{\beta}{\mathbf{r}}|} = \frac{X^I_{,\alpha}X^I_{,\beta} + X^I_{,\alpha}u^I_{,\beta}}{\sqrt{G_{\alpha\alpha}}\sqrt{G_{\beta\beta} + 2E_{\beta\beta}}}. \tag{7.169}$$

In addition, the change ratio of areas before and after deformation is

$$\frac{\underset{\alpha\beta}{da}}{\underset{\alpha\beta}{dA}} = \frac{(1+\varepsilon_\alpha)(1+\varepsilon_\beta)\sin\theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)}}{\sin\Theta_{(\mathbf{N}_\alpha,\mathbf{N}_\beta)}}, \tag{7.170}$$

where $\underset{\alpha\beta}{da} = |\underset{\alpha}{d\mathbf{r}} \times \underset{\beta}{d\mathbf{r}}|$ and $\underset{\alpha\beta}{dA} = |\underset{\alpha}{d\mathbf{R}} \times \underset{\beta}{d\mathbf{R}}|$.

For reduction of a three dimensional deformable body theory to a shell theory, the displacements can be expressed in a Taylor series expanded about the displacement of the middle surface ($S^3 = 0$). The variable z is a variable in the $\mathbf{G}_3$-direction. Thus, the displacement field for the surface layer for a position $\mathbf{R}$ with $S^3 = z$ is assumed by

$$u^I = u_0^I(S^1, S^2, t) + \sum_{n=1}^{\infty} z^n \varphi_n^{(I)}(S^1, S^2, t), \tag{7.171}$$

with

$$\begin{aligned} \mathbf{R} &= S^1\mathbf{G}_1 + S^2\mathbf{G}_2 + z\mathbf{G}_3 = X^I(S^1, S^2, z)\mathbf{I}_I, \\ \mathbf{r} &= \mathbf{R} + \mathbf{u} = X^I(S^1, S^2, z)\mathbf{I}_I + u^I(S^1, S^2, z)\mathbf{I}_I, \end{aligned} \tag{7.172}$$

where u_0^I denotes displacements of the middle surface (or membrane surface), and $\varphi_n^{(I)}$ ($n = 1, 2, \ldots$) are rotations. With Eq.(7.157), substitution of Eq.(7.171) into Eqs.(7.161)–(7.165) and collection of like powers of z gives

$$\begin{aligned} \varepsilon_\alpha &\approx \varepsilon_\alpha^{(0)} + \left.\frac{\partial \varepsilon_\alpha}{\partial z}\right|_{z=0} z + \frac{1}{2!}\left.\frac{\partial^2 \varepsilon_\alpha}{\partial z^2}\right|_{z=0} z^2 + \cdots \\ &= \varepsilon_\alpha^{(0)} + \frac{(X_{,\alpha}^I + u_{0,\alpha}^I)\varphi_{1,\alpha}^{(I)}}{\sqrt{G_{\alpha\alpha}}(1+\varepsilon_\alpha^{(0)})} z + \frac{1}{2}\left\{ \frac{2[(X_{,\alpha}^I + u_{0,\alpha}^I)\varphi_{2,\alpha}^{(I)}] + \varphi_{1,\alpha}^{(I)}\varphi_{1,\alpha}^{(I)}}{G_{\alpha\alpha}(1+\varepsilon_\alpha^{(0)})} \right. \\ &\quad \left. - \frac{[(X_{,\alpha}^I + u_{0,\alpha}^I)\varphi_{1,\alpha}^{(I)}]^2}{G_{\alpha\alpha}^2[(1+\varepsilon_\alpha^{(0)})]^3} \right\} z^2 + \cdots, \end{aligned} \tag{7.173}$$

$$\begin{aligned} \varepsilon_3 &\approx \varepsilon_3^{(0)} + \left.\frac{\partial \varepsilon_3}{\partial z}\right|_{z=0} z + \frac{1}{2!}\left.\frac{\partial^2 \varepsilon_3}{\partial z^2}\right|_{z=0} z^2 + \cdots \\ &= \varepsilon_3^{(0)} + \frac{2(X_{,3}^I + \varphi_1^{(I)})\varphi_2^{(I)}}{1+\varepsilon_3^{(0)}} z + \left\{ \frac{[2\varphi_2^{(I)}\varphi_2^{(I)} + 3(X_{,3}^I + \varphi_1^{(I)})\varphi_3^{(I)}]}{1+\varepsilon_3^{(0)}} \right. \\ &\quad \left. - \frac{2[(X_{,3}^I + \varphi_1^{(I)})\varphi_2^{(I)}]^2}{(1+\varepsilon_3^{(0)})^3} \right\} z^2 + \cdots; \end{aligned} \tag{7.174}$$

$$\gamma_{12} \approx \gamma_{12}^{(0)} + \left.\frac{\partial \gamma_{12}}{\partial z}\right|_{z=0} z + \frac{1}{2!}\left.\frac{\partial^2 \gamma_{12}}{\partial z^2}\right|_{z=0} z^2 + \cdots$$

$$= \gamma_{12}^{(0)} - \frac{1}{\sin\theta^{(0)}_{(\mathbf{n}_1,\mathbf{n}_2)}}\left\{\frac{(X^I_{,1}+u^I_{0,1})\varphi^{(I)}_{1,2}+(X^I_{,2}+u^I_{0,2})\varphi^{(I)}_{1,1}}{\sqrt{G_{11}G_{22}}(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})}\right.$$
$$\left. - \cos\theta^{(0)}_{(\mathbf{n}_1,\mathbf{n}_2)}\left[\frac{(X^I_{,1}+u^I_{0,1})\varphi^{(I)}_{1,1}}{G_{11}(1+\varepsilon_1^{(0)})^2}+\frac{(X^I_{,2}+u^I_{0,2})\varphi^{(I)}_{1,2}}{G_{22}(1+\varepsilon_2^{(0)})^2}\right]\right\}z+\cdots, \tag{7.175}$$

$$\gamma_{\alpha 3} \approx \gamma_{\alpha 3}^{(0)} + \left.\frac{\partial\gamma_{\alpha 3}}{\partial z}\right|_{z=0} z + \frac{1}{2!}\left.\frac{\partial^2\gamma_{\alpha 3}}{\partial z^2}\right|_{z=0} z^2+\cdots$$
$$= \gamma_{\alpha 3}^{(0)} + \frac{1}{\cos\gamma_{\alpha 3}^{(0)}}\left\{\frac{(X^I_{,\alpha}+u^I_{0,\alpha})\varphi^{(I)}_{1}+(X^I_{,3}+\varphi^{(I)}_{1})\varphi^{(I)}_{1,\alpha}}{\sqrt{G_{\alpha\alpha}}(1+\varepsilon_\alpha^{(0)})(1+\varepsilon_3^{(0)})}\right.$$
$$\left. - \sin\gamma^0_{\alpha 3}\left[\frac{(X^I_{,\alpha}+u^I_{0,\alpha})\varphi^{(I)}_{1,\alpha}}{G_{\alpha\alpha}(1+\varepsilon_\alpha^{(0)})^2}+\frac{2(X^I_{,3}+\varphi^{(I)}_{1})\varphi^{(I)}_{2}}{(1+\varepsilon_3^{(0)})^2}\right]\right\}z+\cdots, \tag{7.176}$$

where $\alpha = 1, 2$ because of

$$\begin{aligned} &\frac{\partial\gamma_{\alpha\beta}}{\partial z} = \frac{1}{\cos\gamma_{\alpha\beta}}\frac{\partial\sin\gamma_{\alpha\beta}}{\partial z}, \\ &\frac{\partial^2\gamma_{\alpha\beta}}{\partial z^2} = \frac{1}{\cos\gamma_{IJ}}\frac{\partial^2\sin\gamma_{\alpha\beta}}{\partial z^2} + \left(\frac{\partial\gamma_{\alpha\beta}}{\partial z}\right)^2\tan\gamma_{\alpha\beta},\cdots. \end{aligned} \tag{7.177}$$

Equations (7.173)-(7.176) at $z = 0$ gives the strains of the middle surface, i.e.,

$$\begin{aligned} &\varepsilon_\alpha^{(0)} = \frac{1}{\sqrt{G_{\alpha\alpha}}}\sqrt{(X^I_{,\alpha}+u^I_{0,\alpha})(X^I_{,\alpha}+u^I_{0,\alpha})}-1, \\ &\varepsilon_3^{(0)} = \sqrt{(X^I_{,3}+\varphi^{(I)}_1)(X^{(I)}_{,3}+\varphi^I_1)}-1; \\ &\gamma_{12}^{(0)} = \cos^{-1}\frac{X^I_{,1}X^I_{,2}}{\sqrt{G_{11}G_{22}}} - \cos^{-1}\frac{(X^I_{,1}+u^I_{0,1})(X^I_{,2}+u^I_{0,2})}{\sqrt{G_{11}G_{22}}(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})}, \\ &\gamma_{\alpha 3}^{(0)} = \frac{\pi}{2} - \cos^{-1}\frac{(X^I_{,\alpha}+u^I_{0,\alpha})(X^I_{,3}+\varphi^{(I)}_1)}{\sqrt{G_{\alpha\alpha}}(1+\varepsilon_\alpha^{(0)})(1+\varepsilon_3^{(0)})}. \end{aligned} \tag{7.178}$$

In Eqs.(7.173)–(7.176), prediction of strain requires specification of three constraints for determination of the three sets $\varphi_n^{(I)}$ ($I = 1, 2, 3; n = 1, 2, \ldots$).

Consider Kirchhoff's assumptions ($\varepsilon_3 = \gamma_{\alpha 3} = 0$) as an example. From Eqs.(7.161) and (7.165), these constraints become

$$\begin{aligned} &(X^I_{,3}+u^I_{,3})(X^I_{,3}+u^I_{,3}) = 1, \\ &(X^I_{,\alpha}+u^I_{,\alpha})(X^I_{,3}+u^I_{,3}) = 0. \end{aligned} \tag{7.179}$$

Substitution of Eq.(7.171) into Eqs.(7.179), expansion of them in Taylor series in z and vanishing of the zero-order terms in z gives

$$
\begin{aligned}
&(X^I_{,3}+\varphi^{(I)}_1)(X^I_{,3}+\varphi^{(I)}_1)=1,\\
&(X^I_{,\alpha}+u^I_{0,\alpha})(X^I_{,3}+\varphi^{(I)}_1)=0.
\end{aligned}
\tag{7.180}
$$

Form the foregoing equations,

$$
X^I_{,3}+\varphi^{(I)}_1=\pm\frac{\Delta_I}{\Delta}\ (I=1,2,3),
\tag{7.181}
$$

where

$$
\begin{aligned}
\Delta_1&=(X^2_{,1}+u^2_{0,1})(X^3_{,2}+u^3_{0,2})-(X^2_{,2}+u^2_{0,2})(X^3_{,1}+u^3_{0,1})\\
\Delta_2&=(X^1_{,2}+u^1_{0,2})(X^3_{,1}+u^3_{0,1})-(X^1_{,1}+u^1_{0,1})(X^3_{,2}+u^3_{0,2})\\
\Delta_3&=(X^1_{,1}+u^1_{0,1})(X^2_{,2}+u^2_{0,2})-(X^2_{,1}+u^2_{0,1})(X^1_{,2}+u^1_{0,2})\\
\Delta&=\sqrt{\Delta_1^2+\Delta_2^2+\Delta_3^2}.
\end{aligned}
\tag{7.182}
$$

In application, only the positive (+) in Eq.(7.181) is adopted. From the Taylor series, vanishing of the first order terms in z gives three equations in $\varphi^{(I)}_2$ similar to Eqs.(7.180). The three equations plus $\varphi^{(I)}_1$ give $\varphi^{(I)}_2$; $\varphi^{(I)}_n$ for $n=3,4,\ldots$ can be determined in a like manner. Substitution of all $\varphi^{(I)}_n$ ($n=1,2,\ldots$) into Eqs.(7.173) and (7.175) generates the normal and shear strains ε_α and γ_{12}. The membrane strains are $\varepsilon^{(0)}_\alpha\equiv\mathscr{E}_\alpha$ and $\gamma^{(0)}_{12}$.

7.3.1b Equations of motion for shells

Consider a non-deformed shell subject to the inertia force per unit area $\rho u_{I,tt}$, where $\rho=\int_{-h^-}^{h^+}\rho_0 dz$ and ρ_0 is the density of shell, body force $\mathbf{f}=f^\Lambda\mathbf{N}_\Lambda$ ($\Lambda=1,2,3$), surface loading $\{p^+_\Lambda, p^-_\Lambda\}$, where the superscripts "+" and "−" denote the upper and lower surfaces, external moment m^α_0 ($\alpha=1,2$) before deformation. From the sign convention, the external and distributed forces are expressed by

$$
\begin{aligned}
\mathbf{m}&=m^I\mathbf{I}_I=(-1)^\beta m^\alpha\mathbf{N}_\beta\quad\text{for }\alpha,\beta\in\{1,2\},\alpha\neq\beta,\\
\mathbf{q}&=q^I\mathbf{I}_I=q^\Lambda\mathbf{N}_\Lambda\quad\text{for }\Lambda\in\{1,2,3\}.
\end{aligned}
\tag{7.183}
$$

The components of distributed force and moment are

$$\begin{aligned} q^{\Lambda} &= p_{\Lambda}^{+} - p_{\Lambda}^{-} + \int_{-h^{-}}^{h^{+}} f^{\Lambda} dz \ \ (\Lambda = 1, 2, 3), \\ m^{\alpha} &= m_{0}^{\alpha} + h^{+} p_{\beta}^{+} + h^{-} p_{\beta}^{-} + \int_{-h^{-}}^{h^{+}} f^{\beta} z dz \ (\alpha, \beta \in 1, 2, \alpha \neq \beta), \end{aligned} \tag{7.184}$$

where $m^3 = 0$ and $h = h^+ + h^-$. From Eq.(7.183), the following relations are obtained:

$$\begin{aligned} m^{I} &= m^{\Lambda} \mathbf{N}_{\Lambda} \cdot \mathbf{I}_{I} = m^{\Lambda} \cos\theta_{(\mathbf{N}_{\Lambda}, \mathbf{I}_{I})}, \\ q^{I} &= q^{\Lambda} \mathbf{N}_{\Lambda} \cdot \mathbf{I}_{I} = q^{\Lambda} \cos\theta_{(\mathbf{N}_{\Lambda}, \mathbf{I}_{I})}. \end{aligned} \tag{7.185}$$

In other words,

$$\begin{aligned} m^{I} &= \frac{X_{,1}^{I}}{\sqrt{G_{11}}} m^{1} + \frac{X_{,2}^{I}}{\sqrt{G_{22}}} m^{2}, \\ q^{I} &= \frac{X_{,1}^{I}}{\sqrt{G_{11}}} q^{1} + \frac{X_{,2}^{I}}{\sqrt{G_{22}}} q^{2} + X_{,3}^{I} q^{3}. \end{aligned} \tag{7.186}$$

The distributed loading after deformation becomes

$$\begin{bmatrix} q^{I*} \\ m^{I*} \end{bmatrix} = \frac{\sin\Theta_{(\mathbf{N}_1, \mathbf{N}_2)}}{(1+\varepsilon_1)(1+\varepsilon_2)\sin\theta_{(\mathbf{n}_1, \mathbf{n}_2)}} \begin{bmatrix} q^{I} \\ m^{I} \end{bmatrix}. \tag{7.187}$$

The constitutive laws give the stresses on the deformed configuration as

$$\sigma_{\alpha\beta} = f_{\alpha\beta}(\varepsilon_{MN}, t). \tag{7.188}$$

The internal forces and moments in the deformed shell are defined as

$$\begin{aligned} N_{\alpha\beta} &= \int_{-h^{-}}^{h^{+}} \sigma_{\alpha\beta} \left[\sqrt{G_{\alpha'\alpha'}} (1+\varepsilon_{\alpha'})(1+\varepsilon_3) \sin\theta_{(\mathbf{n}_{\alpha'}, \mathbf{n}_3)} \right] dz, \\ M_{\alpha\beta} &= \int_{-h^{-}}^{h^{+}} \sigma_{\alpha\beta} \frac{z}{1+\varphi_1^{(I)} X_{,3}^{I}} \left[\sqrt{G_{\alpha'\alpha'}} (1+\varepsilon_{\alpha'})(1+\varepsilon_3)^2 \sin\theta_{(\mathbf{n}_{\alpha'}, \mathbf{n}_3)} \right] dz, \\ Q_{\alpha} &= \int_{-h^{-}}^{h^{+}} \sigma_{\alpha 3} \left[\sqrt{G_{\alpha'\alpha'}} (1+\varepsilon_{\alpha'})(1+\varepsilon_3) \sin\theta_{(\mathbf{n}_{\alpha'}, \mathbf{n}_3)} \right] dz, \\ &\text{with } \alpha' = \mathrm{mod}(\alpha, 2) + 1, \end{aligned} \tag{7.189}$$

because of

$$\cos\theta_{(\mathbf{N}_3, \mathbf{n}_3)} = \frac{1 + X_{,3}^{I} u_{,3}^{I}}{\sqrt{1+2E_{33}}} \approx \frac{1 + X_{,3}^{I} \varphi_1^{(I)}}{1+\varepsilon_3^{(0)}}, \tag{7.190}$$

where $N_{\alpha\beta}$ are membrane forces and $M_{\alpha\beta}$ are bending and twisting moments per unit length and $\alpha, \beta \in \{1, 2\}$. Before the equation of motion for the shell is developed, the internal force vectors are introduced, i.e.,

$$\begin{aligned}\mathbf{M}_\alpha &\equiv M_\alpha^I \mathbf{I}_I = (-1)^\beta M_\alpha \mathbf{n}_\beta + (-1)^\alpha M_{\alpha\beta}\mathbf{n}_\alpha,\\ \mathcal{N}_\alpha &\equiv N_\alpha^I \mathbf{I}_I = N_\alpha \mathbf{n}_\alpha + N_{\alpha\beta}\mathbf{n}_\beta + Q_\alpha \mathbf{n}_3,\\ {}^{\mathrm{N}}\mathbf{M}_\alpha &\equiv {}^{\mathrm{N}}M_\alpha^I \mathbf{I}_I = \mathbf{g}_\alpha \times \mathcal{N}_\alpha,\end{aligned} \tag{7.191}$$

with out summation on $\alpha, \beta \in \{1,2\}$ and $\alpha \neq \beta$, where

$$\mathbf{g}_\alpha \equiv \frac{d\mathbf{r}_\alpha}{dS^\alpha} = (X_{,\alpha}^I + u_{,\alpha}^I)\mathbf{I}_I \text{ and } {}^{\mathrm{N}}\mathbf{M}_\alpha \equiv \frac{1}{dS^\alpha} d\mathbf{r}_\alpha \times \mathbf{N}_\alpha. \tag{7.192}$$

The components of the internal forces in the X^I -direction are

$$\begin{aligned}N_\alpha^I &= N_\alpha \mathbf{n}_\alpha \cdot \mathbf{I}_I + N_{\alpha\beta}\mathbf{n}_\beta \cdot \mathbf{I}_I + Q_\alpha \mathbf{n}_3 \cdot \mathbf{I}_I\\ &= N_\alpha \cos\theta_{(\mathbf{n}_\alpha,\mathbf{I}_I)} + N_{\alpha\beta}\cos\theta_{(\mathbf{n}_\beta,\mathbf{I}_I)} + Q_\alpha \cos\theta_{(\mathbf{n}_3,\mathbf{I}_I)}\\ &= \frac{N_\alpha (X_{,\alpha}^I + u_{0,\alpha}^I)}{\sqrt{G_{\alpha\alpha}}(1+\varepsilon_\alpha^{(0)})} + \frac{N_{\alpha\beta}(X_{,\beta}^I + u_{0,\beta}^I)}{\sqrt{G_{\beta\beta}}(1+\varepsilon_\beta^{(0)})} + \frac{Q_\alpha (X_{,3}^I + \varphi_1^{(I)})}{1+\varepsilon_3^{(0)}},\end{aligned} \tag{7.193}$$

$$\begin{aligned}M_\alpha^I &= (-1)^\beta M_\alpha \mathbf{n}_\beta \cdot \mathbf{I}_I + (-1)^\alpha M_{\alpha\beta}\mathbf{n}_\alpha \cdot \mathbf{I}_I\\ &= (-1)^\beta M_\alpha \cos\theta_{(\mathbf{n}_\beta,\mathbf{I}_I)} + (-1)^\alpha M_{\alpha\beta}\cos\theta_{(\mathbf{n}_\alpha,\mathbf{I}_I)}\\ &= (-1)^\beta \frac{M_\alpha (X_{,\beta}^I + u_{0,\beta}^I)}{\sqrt{G_{\beta\beta}}(1+\varepsilon_\beta^{(0)})} + (-1)^\alpha \frac{M_{\alpha\beta}(X_{,\alpha}^I + u_{0,\alpha}^I)}{\sqrt{G_{\alpha\alpha}}(1+\varepsilon_\alpha^{(0)})},\end{aligned} \tag{7.194}$$

$$\begin{aligned}{}^{\mathrm{N}}M_\alpha^I &= (\mathbf{g}_\alpha \times \mathcal{N}_\alpha)\cdot \mathbf{I}_I = (X_{,\alpha}^J + u_{0,\alpha}^J)N_\alpha^K - (X_{,\alpha}^K + u_{0,\alpha}^K)N_\alpha^J\\ &= (X_{,\alpha}^J + u_{0,\alpha}^J)\left[\frac{N_{\alpha\beta}(X_{,\beta}^K + u_{0,\beta}^K)}{\sqrt{G_{\beta\beta}}(1+\varepsilon_\beta^{(0)})} + \frac{Q_\alpha (X_{,3}^K + \varphi_1^{(K)})}{1+\varepsilon_3^{(0)}}\right]\\ &\quad - (X_{,\alpha}^K + u_{0,\alpha}^K)\left[\frac{N_{\alpha\beta}(X_{,\beta}^J + u_{0,\beta}^J)}{\sqrt{G_{\beta\beta}}(1+\varepsilon_\beta^{(0)})} + \frac{Q_\alpha (X_{,3}^J + \varphi_1^{(J)})}{1+\varepsilon_3^{(0)}}\right]\end{aligned} \tag{7.195}$$

for $(I, J, K \in \{1, 2, 3\},\ I \neq J \neq K \neq I)$ and the indices (I, J, K) rotate clockwise. Based on the deformed middle surface in the Lagrangian coordinates, the equations of motion for the deformed shell are

$$\begin{aligned}\mathcal{N}_{\alpha,\alpha} + \mathbf{q} &= (\rho \mathbf{u}_{0,tt} + I_0 \boldsymbol{\varphi}_{1,tt})\sin\Theta_{(\mathbf{N}_1,\mathbf{N}_2)},\\ \mathbf{M}_{\alpha,\alpha} + {}^{\mathrm{N}}\mathbf{M}_\alpha + \mathbf{m} &= (I_0 \mathbf{u}_{0,tt} + J_0 \boldsymbol{\varphi}_{1,tt})\sin\Theta_{(\mathbf{N}_1,\mathbf{N}_2)},\end{aligned} \tag{7.196}$$

or for $I = 1, 2, 3$ with summation on $\alpha = 1, 2$,

$$\begin{aligned}N_{\alpha,\alpha}^I + q^I &= (\rho u_{0,tt}^I + I_3 \varphi_{1,tt}^{(I)})\sin\Theta_{(\mathbf{N}_1,\mathbf{N}_2)},\\ M_{\alpha,\alpha}^I + {}^{\mathrm{N}}M_\alpha^I + m^I &= (I_3 u_{0,tt}^I + J_3 \varphi_{(1),tt}^I)\sin\Theta_{(\mathbf{N}_1,\mathbf{N}_2)},\end{aligned} \tag{7.197}$$

where $I_3 = \int_{-h^-}^{h^+} \rho_0 z dz$, $J_3 = \int_{-h^-}^{h^+} \rho_0 z^2 dz$. If $h^+ = h^- = h/2$, one obtains $I_3 = 0$ and $J_3 = \rho_0 h^3 / 12$. The equation of motion in Eq.(7.193) is in a form of

$$
\begin{aligned}
&\left[\frac{N_{11}(X_{,1}^I + u_{0,1}^I)}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})} + \frac{N_{12}(X_{,2}^I + u_{0,2}^I)}{\sqrt{G_{22}}(1+\varepsilon_2^{(0)})} + \frac{Q_1(X_{,3}^I + \varphi_1^{(I)})}{1+\varepsilon_3^{(0)}} \right]_{,1} \\
&+ \left[\frac{N_{21}(X_{,1}^I + u_{0,1}^I)}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})} + \frac{N_{22}(X_{,2}^I + u_{0,2}^I)}{\sqrt{G_{22}}(1+\varepsilon_2^{(0)})} + \frac{Q_2(X_{,3}^I + \varphi_1^{(I)})}{1+\varepsilon_3^{(0)}} \right]_{,2} + q^I \\
&= (\rho u_{0,tt}^I + I_3 \varphi_{1,tt}^{(I)}) \sin \Theta_{(\mathbf{N}_1, \mathbf{N}_2)},
\end{aligned} \tag{7.198}
$$

$$
\begin{aligned}
&\left[\frac{M_1(X_{,2}^I + u_{0,2}^I)}{\sqrt{G_{22}}(1+\varepsilon_2^{(0)})} - \frac{M_{12}(X_{,1}^I + u_{0,1}^I)}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})} \right]_{,1} \\
&+ \left[\frac{M_{12}(X_{,2}^I + u_{0,2}^I)}{\sqrt{G_{22}}(1+\varepsilon_2^{(0)})} - \frac{M_2(X_{,1}^I + u_{0,1}^I)}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})} \right]_{,2} \\
&+ (X_{,1}^J + u_{0,1}^J) \left[\frac{N_{12}(X_{,2}^K + u_{0,2}^K)}{\sqrt{G_{22}}(1+\varepsilon_2^{(0)})} + \frac{Q_1(X_{,3}^K + \varphi_1^{(K)})}{1+\varepsilon_3^{(0)}} \right] \\
&- (X_{,1}^K + u_{0,1}^K) \left[\frac{N_{21}(X_{,2}^J + u_{0,2}^J)}{\sqrt{G_{22}}(1+\varepsilon_2^{(0)})} + \frac{Q_1(X_{,3}^J + \varphi_1^{(J)})}{1+\varepsilon_3^{(0)}} \right] \\
&+ (X_{,2}^J + u_{0,2}^J) \left[\frac{N_{12}(X_{,1}^K + u_{0,1}^K)}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})} + \frac{Q_2(X_{,3}^K + \varphi_1^{(K)})}{1+\varepsilon_3^{(0)}} \right] \\
&- (X_{,2}^K + u_{0,2}^K) \left[\frac{N_{21}(X_{,1}^J + u_{0,1}^J)}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})} + \frac{Q_2(X_{,3}^J + \varphi_1^{(J)})}{1+\varepsilon_3^{(0)}} \right] + m^I \\
&= (I_3 u_{0,tt}^I + J_3 \varphi_{1,tt}^{(I)}) \sin \Theta_{(\mathbf{N}_1, \mathbf{N}_2)}.
\end{aligned} \tag{7.199}
$$

The balance of equilibrium for shells gives (7.192) or (7.193). They together with (7.173)–(7.178) constitute this approximate nonlinear theory for shells.

The force condition at a point $\mathscr{P}_{(k_1,k_2)}$ with $S^\alpha = A_{(k_1,k_2)}^\alpha$ ($\alpha = 1, 2$) is

$$
\begin{aligned}
&\sum\nolimits_{\alpha=1}^{2} {}^{-}\mathcal{N}_\alpha(S^1, S^2)\Big|_{(S^1,S^2)=(A_{(k_1,k_2)}^1, A_{(k_1,k_2)}^2)} \\
&+\sum\nolimits_{\alpha=1}^{2} {}^{+}\mathcal{N}_\alpha(S^1, S^2)\Big|_{(S^1,S^2)=(A_{(k_1,k_2)}^1, A_{(k_1,k_2)}^2)} + \mathbf{F}_{(k_1,k_2)} = 0,
\end{aligned} \tag{7.200}
$$

or

$$\sum_{\alpha=1}^{2} {}^{-}N_{\alpha}^{I}(S^1, S^2)\Big|_{(S^1, S^2)=(A^1_{(k_1,k_2)}, A^2_{(k_1,k_2)})}$$
$$=\sum_{\alpha=1}^{2} {}^{+}N_{\alpha}^{I}(S^1, S^2)\Big|_{(S^1, S^2)=(A^1_{(k_1,k_2)}, A^2_{(k_1,k_2)})} + F^I_{(k_1,k_2)}. \tag{7.201}$$

The force boundary condition at the boundary point $\mathscr{P}_{(r_1,r_2)}$ is

$$\sum_{\alpha=1}^{2} \mathcal{N}_{\alpha}(S^1, S^2)\Big|_{(S^1, S^2)=(B^1_{(r_1,r_2)}, B^2_{(r_1,r_2)})} + \mathbf{F}_{(r_1,r_2)} = 0, \tag{7.202}$$

or

$$\sum_{\alpha=1}^{2} N_{\alpha}^{I}(S^1, S^2)\Big|_{(S^1, S^2)=(B^1_{(r_1,r_2)}, B^2_{(r_1,r_2)})} + F^I_{(r_1,r_2)} = 0. \tag{7.203}$$

The force boundary conditions on the shell edge is for $\alpha, \beta \in \{1, 2\}$ and $\alpha \neq \beta$.

$$(\mathcal{N}_{\alpha}(S^1, S^2) + \frac{\partial}{\partial S^{\beta}} \int \mathcal{N}_{\beta}(S^1, S^2) dS^{\alpha})\Big|_{S^{\alpha}=S^{\alpha}_{*}} + \mathbf{F}(S^{\beta}) = 0, \tag{7.204}$$

or

$$\frac{\partial}{\partial S^{\beta}} \int N^I_{\beta}(S^1, S^2) dS^{\alpha}\Big|_{S^{\alpha}=S^{\alpha}_{*}} + N^I_{\alpha}(S^1, S^2)\Big|_{S^{\alpha}=S^{\alpha}_{*}} + F^I(S^{\beta}) = 0. \tag{7.205}$$

Without shear forces, the force conditions in Eqs.(7.200)–(7.205) are identical to Eqs.(7.145)–(7.150).

If there is a concentrated moment at a point $\mathscr{P}_{(k_1,k_2)}$ with $S^{\alpha} = A^{\alpha}_{(k_1,k_2)}$ ($\alpha = 1, 2$), the corresponding moment boundary condition is

$$\sum_{\alpha=1}^{2} {}^{-}\mathbf{M}_{\alpha}(S^1, S^2)\Big|_{(S^1, S^2)=(A^1_{(k_1,k_2)}, A^2_{(k_1,k_2)})}$$
$$=\sum_{\alpha=1}^{2} {}^{+}\mathbf{M}_{\alpha}(S^1, S^2)\Big|_{(S^1, S^2)=(A^1_{(k_1,k_2)}, A^2_{(k_1,k_2)})} + \mathbf{M}_{(k_1,k_2)}, \tag{7.206}$$

or

$$\sum_{\alpha=1}^{2} {}^{-}M_{\alpha}^{I}(S^1, S^2)\Big|_{(S^1, S^2)=(A^1_{(k_1,k_2)}, A^2_{(k_1,k_2)})}$$
$$=\sum_{\alpha=1}^{2} {}^{+}M_{\alpha}^{I}(S^1, S^2)\Big|_{(S^1, S^2)=(A^1_{(k_1,k_2)}, A^2_{(k_1,k_2)})} + M^I_{(k_1,k_2)}. \tag{7.207}$$

The moment boundary condition at the boundary point $\mathscr{P}_{(r_1,r_2)}$ is

$$\sum_{\alpha=1}^{2} \mathbf{M}_{\alpha}(S^1, S^2)\Big|_{(S^1, S^2)=(B^1_{(r_1,r_2)}, B^2_{(r_1,r_2)})} + \mathbf{M}_{(r_1,r_2)} = 0 \tag{7.208}$$

or

$$\sum_{\alpha=1}^{2} M_{\alpha}^{I}(S^1, S^2)\Big|_{(S^1, S^2)=(B^1_{(r_1,r_2)}, B^2_{(r_1,r_2)})} + M^I_{(r_1,r_2)} = 0. \tag{7.209}$$

The moment boundary condition on the shell edge for $\alpha, \beta \in \{1, 2\}$ and $\alpha \neq \beta$ is

$$
\begin{aligned}
&(\mathbf{M}_\alpha(S^1, S^2) + \int^{\mathrm{N}} \mathbf{M}_\alpha(S^1, S^2) dS^\alpha)\Big|_{S^\alpha = S^\alpha_*} \\
&+ (\int^{\mathrm{N}} \mathbf{M}_\beta(S^1, S^2) dS^\alpha + \frac{\partial}{\partial S^\beta} \int \mathbf{M}_\beta(S^1, S^2) dS^\alpha)\Big|_{S^\alpha = S^\alpha_*} \\
&+ \mathbf{M}(S^\beta) = 0,
\end{aligned} \tag{7.210}
$$

or

$$
\begin{aligned}
&(M^I_\alpha(S^1, S^2) + \int^{\mathrm{N}} M^I_\alpha(S^1, S^2) dS^\alpha)\Big|_{S^\alpha = S^\alpha_*} \\
&+ (\int^{\mathrm{N}} M^I_\beta(S^1, S^2) dS^\alpha + \frac{\partial}{\partial S^\beta} \int M^I_\beta(S^1, S^2) dS^\alpha)\Big|_{S^\alpha = S^\alpha_*} \\
&+ M^I(S^\beta) = 0.
\end{aligned} \tag{7.211}
$$

The displacement continuity and boundary conditions are the same as in Eqs.(7.57) and (7.58). The theory of nonlinear shells can easily reduce to the theory of thin plates in Chapter 6 (e.g., Luo, 2000). Such a nonlinear theory is very intuitive to understand.

7.3.2. A shell theory based on the curvilinear coordinates

Consider a material particle $P(X^1, X^2, X^3)$ in an initial configuration of a shell at the initial state. The base vectors at the position **R** of the material particle are described by Eqs.(7.117)–(7.122). On the deformed configuration of the shell, a particle at point P moves through displacement **u** to position p, and the particle Q, infinitesimally close to $P(S^1, S^2, S^3)$, moves through $\mathbf{u} + d\mathbf{u}$ to q in the vicinity of $p(S^1, S^2, S^3)$, as illustrated in Fig.7.7. The strain for three-dimensional body is similar to Eqs.(7.123)–(7.135). For clarity, the similar description is given as follows. The position of point p is described by

$$
\mathbf{r} = \mathbf{R} + \mathbf{u} = (S^\Lambda + u^\Lambda)\mathbf{G}_\Lambda , \tag{7.212}
$$

where the displacement is $\mathbf{u} = u^\Lambda \mathbf{G}_\Lambda$ ($\Lambda = 1, 2, 3$). The infinitesimal elements $d\mathbf{R}$ and $d\mathbf{r}$ are expressed by

$$
d\mathbf{R} = \mathbf{G}_\Lambda dS^\Lambda, \; d\mathbf{r} = \mathbf{G}_\Lambda dS^\Lambda + d\mathbf{u}; \tag{7.213}
$$

with the infinitesimal displacement is for $\Lambda, \Gamma = 1, 2, 3$,

$$
d\mathbf{u} = u^\Gamma_{;\Lambda} dS^\Lambda \mathbf{G}_\Gamma = (u^\Gamma_{,\Lambda} + \Gamma^\Gamma_{\Lambda \mathrm{K}} u^\mathrm{K}) dS^\Lambda \mathbf{G}_\Gamma . \tag{7.214}
$$

Further, the deformed infinitesimal element is

$$d\mathbf{r} = (u^{\Gamma}_{;\Lambda}\mathbf{G}_{\Gamma} + \mathbf{G}_{\Lambda})dS^{\Lambda} = dS^{\Lambda}\mathbf{g}_{\Lambda};$$
$$\mathbf{g}_{\Lambda} \equiv \frac{\partial \mathbf{r}}{\partial S^{\Lambda}} = (u^{\Gamma}_{;\Lambda}\mathbf{G}_{\Gamma} + \mathbf{G}_{\Lambda}) = (u^{\Gamma}_{;\Lambda} + \delta^{\Gamma}_{\Lambda})\mathbf{G}_{\Gamma}. \tag{7.215}$$

The length change of $d\mathbf{R}$ per unit length gives

$$\varepsilon_{\Lambda} = \frac{|\underset{\Lambda}{d\mathbf{r}}| - |\underset{\Lambda}{d\mathbf{R}}|}{|\underset{\Lambda}{d\mathbf{R}}|} = \sqrt{1 + \frac{2E_{\Lambda\Lambda}}{G_{\Lambda\Lambda}}} - 1 = \frac{1}{\sqrt{G_{\Lambda\Lambda}}}\sqrt{G_{\Lambda\Lambda} + 2E_{\Lambda\Lambda}} - 1$$
$$= \frac{\sqrt{G_{\Gamma\Gamma}}}{\sqrt{G_{\Lambda\Lambda}}}\sqrt{(\delta^{\Gamma}_{\Lambda} + u^{\Gamma}_{;\Lambda})(\delta^{\Gamma}_{\Lambda} + u^{\Gamma}_{;\Lambda})} - 1, \tag{7.216}$$

where the Lagrangian strain tensor $E_{\Lambda\Gamma}$ to the initial configuration is

$$E_{\Lambda\Gamma} = \frac{1}{2}(u_{\Lambda;\Gamma} + u_{\Gamma;\Lambda} + u^{K}_{;\Lambda}u_{K;\Gamma})$$
$$= \frac{1}{2}\left[(\delta^{K}_{\Lambda} + u^{K}_{;\Lambda})(\delta^{\Sigma}_{\Gamma} + u^{\Sigma}_{;\Gamma})G_{K\Sigma} - G_{\Lambda\Gamma}\right]. \tag{7.217}$$

The unit vectors along $\underset{\Lambda}{d\mathbf{R}}$ and $\underset{\Lambda}{d\mathbf{r}}$ are written as

$$\mathbf{N}_{\Lambda} \equiv \frac{\mathbf{G}_{\Lambda}}{|\mathbf{G}_{\Lambda}|} = \frac{\underset{\Lambda}{d\mathbf{R}}}{|\underset{\Lambda}{d\mathbf{R}}|} = \frac{1}{\sqrt{G_{\Lambda\Lambda}}}\mathbf{G}_{\Lambda},$$
$$\mathbf{n}_{\Lambda} \equiv \frac{\mathbf{g}_{\Lambda}}{|\mathbf{g}_{\Lambda}|} = \frac{\underset{\Lambda}{d\mathbf{r}}}{|\underset{\Lambda}{d\mathbf{r}}|} = \frac{\delta^{\Gamma}_{\Lambda} + u^{\Gamma}_{;\Lambda}}{\sqrt{G_{\Lambda\Lambda} + 2E_{\Lambda\Lambda}}}\mathbf{G}_{\Gamma} \tag{7.218}$$
$$= \frac{\delta^{\Gamma}_{\Lambda} + u^{\Gamma}_{;\Lambda}}{\sqrt{G_{KK}}\sqrt{(\delta^{K}_{\Lambda} + u^{K}_{;\Lambda})(\delta^{K}_{\Lambda} + u^{K}_{;\Lambda})}}\mathbf{G}_{\Gamma} = \frac{\delta^{\Gamma}_{\Lambda} + u^{\Gamma}_{;\Lambda}}{\sqrt{G_{\Lambda\Lambda}}(1 + \varepsilon_{\Lambda})}\mathbf{G}_{\Gamma}.$$

As similar to Eqs.(7.132)–(7.134), the shear strains can be defined in three-directions. Let $\Theta_{(\mathbf{N}_{\Lambda},\mathbf{N}_{\Gamma})}$ and $\theta_{(\mathbf{n}_{\Lambda},\mathbf{n}_{\Gamma})}$ be the induced angles between $\mathbf{n}_{\Lambda}$ and $\mathbf{n}_{\Gamma}$ *before* and *after* deformation.

$$\cos\theta_{(\mathbf{n}_{\Lambda},\mathbf{n}_{\Gamma})} \equiv \cos(\Theta_{(\mathbf{N}_{\Lambda},\mathbf{N}_{\Gamma})} - \gamma_{\Lambda\Gamma})$$
$$= \frac{\underset{\Lambda}{d\mathbf{r}}\cdot\underset{\Gamma}{d\mathbf{r}}}{|\underset{\Lambda}{d\mathbf{r}}||\underset{\Gamma}{d\mathbf{r}}|} = \frac{G_{\Lambda\Gamma} + 2E_{\Lambda\Gamma}}{\sqrt{(G_{\Lambda\Lambda} + 2E_{\Lambda\Lambda})(G_{\Gamma\Gamma} + 2E_{\Gamma\Gamma})}}$$
$$= \frac{(\delta^{K}_{\Lambda} + u^{K}_{;\Lambda})(\delta^{\Sigma}_{\Gamma} + u^{\Sigma}_{;\Gamma})G_{K\Sigma}}{\sqrt{G_{\Lambda\Lambda}G_{\Gamma\Gamma}}(1 + \varepsilon_{\Lambda})(1 + \varepsilon_{\Gamma})}, \tag{7.219a}$$

$$\cos\Theta_{(\mathbf{N}_\Lambda,\mathbf{N}_\Gamma)} = \frac{d\underset{\Lambda}{\mathbf{R}}\cdot d\underset{\Gamma}{\mathbf{R}}}{|d\underset{\Lambda}{\mathbf{R}}||d\underset{\Gamma}{\mathbf{R}}|} = \frac{G_{\Lambda\Gamma}}{\sqrt{G_{\Lambda\Lambda}G_{\Gamma\Gamma}}}; \tag{7.219b}$$

and the shear strain is

$$\begin{aligned}\gamma_{\Lambda\Gamma} &\equiv \Theta_{(\mathbf{N}_\Lambda,\mathbf{N}_\Gamma)} - \theta_{(\mathbf{n}_\Lambda,\mathbf{n}_\Gamma)}\\ &= \cos^{-1}\frac{G_{\Lambda\Gamma}}{\sqrt{G_{\Lambda\Lambda}G_{\Gamma\Gamma}}} - \cos^{-1}\left[\frac{G_{\Lambda\Gamma}+2E_{\Lambda\Gamma}}{\sqrt{(G_{\Lambda\Lambda}+2E_{\Lambda\Lambda})(G_{\Gamma\Gamma}+2E_{\Gamma\Gamma})}}\right],\\ &= \cos^{-1}\frac{G_{\Lambda\Gamma}}{\sqrt{G_{\Lambda\Lambda}G_{\Gamma\Gamma}}} - \cos^{-1}\left[\frac{(\delta_\Lambda^{\mathrm{K}}+u_{;\Lambda}^{\mathrm{K}})(\delta_\Gamma^{\Sigma}+u_{;\Gamma}^{\Sigma})G_{\mathrm{K}\Sigma}}{\sqrt{G_{\Lambda\Lambda}G_{\Gamma\Gamma}}(1+\varepsilon_\Lambda)(1+\varepsilon_\Gamma)}\right].\end{aligned} \tag{7.220}$$

From Eq.(7.218), the direction cosine of the rotation without summation on Λ and Γ is

$$\begin{aligned}\cos\theta_{(\mathbf{N}_\Lambda,\mathbf{n}_\Gamma)} &= \frac{d\underset{\Lambda}{\mathbf{R}}\cdot d\underset{\Gamma}{\mathbf{r}}}{|d\underset{\Lambda}{\mathbf{R}}||d\underset{\Gamma}{\mathbf{r}}|} = \frac{G_{\Lambda\Gamma}+u_{\Lambda;\Gamma}}{\sqrt{G_{\Lambda\Lambda}}\sqrt{G_{\Gamma\Gamma}+2E_{\Gamma\Gamma}}}\\ &= \frac{G_{\Lambda\Gamma}+u_{\Lambda;\Gamma}}{\sqrt{G_{\Lambda\Lambda}G_{\Gamma\Gamma}}\sqrt{(\delta_\Gamma^{\mathrm{K}}+u_{;\Gamma}^{\mathrm{K}})(\delta_\Gamma^{\mathrm{K}}+u_{;\Gamma}^{\mathrm{K}})}}\\ &= \frac{G_{\Lambda\Gamma}+u_{\Lambda;\Gamma}}{\sqrt{G_{\Lambda\Lambda}G_{\Gamma\Gamma}}(1+\varepsilon_\Gamma)} = \frac{(\delta_\Gamma^{\mathrm{K}}+u_{;\Gamma}^{\mathrm{K}})G_{\mathrm{K}\Lambda}}{\sqrt{G_{\Lambda\Lambda}G_{\Gamma\Gamma}}(1+\varepsilon_\Gamma)}.\end{aligned} \tag{7.221}$$

In addition, the change ratio of areas *before* and *after* deformation is

$$\frac{d\underset{\Lambda\Gamma}{a}}{d\underset{\Lambda\Gamma}{A}} = \frac{(1+\varepsilon_\Lambda)(1+\varepsilon_\Gamma)\sin\theta_{(\mathbf{n}_\Lambda,\mathbf{n}_\Gamma)}}{\sin\Theta_{(\mathbf{N}_\Lambda,\mathbf{N}_\Gamma)}}, \tag{7.222}$$

where $d\underset{\Lambda\Gamma}{a} = |d\underset{\Lambda}{\mathbf{r}}\times d\underset{\Gamma}{\mathbf{r}}|$ and $d\underset{\Lambda\Gamma}{A} = |d\underset{\Lambda}{\mathbf{R}}\times d\underset{\Gamma}{\mathbf{R}}|$.

7.3.2a Shell strains on curvilinear coordinates

Suppose the base vector $\mathbf{G}_3$ be normal to the plane of $\mathbf{G}_1$ and $\mathbf{G}_2$ in the curvilinear coordinates. Equations (7.166)−(7.168) can be used for the following derivatives. Again, if $\mathbf{G}_1$ and $\mathbf{G}_2$ are orthogonal, one obtains $G_{12}=0$ and $G=\sqrt{G_{11}G_{22}}$. For reduction of three dimensional strains to a two-dimensional form, displacements can be expressed in a Taylor series expansion about the displacement of the middle surface. Thus, the displacements are assumed for ($\Lambda = 1, 2, 3$ and $\alpha, \beta \in \{1, 2\}$) by

$$u^{\Lambda} = u_0^{\Lambda}(S^1, S^2, t) + \sum\nolimits_{n=1}^{\infty} z^n \varphi_n^{(\Lambda)}(S^1, S^2, t), \tag{7.223}$$

with

$$\begin{aligned} \mathbf{R} &= S^1\mathbf{G}_1 + S^2\mathbf{G}_2 + z\mathbf{G}_3 = S^{\alpha}\mathbf{G}_{\alpha} + z\mathbf{G}_3, \\ \mathbf{r} &= \mathbf{R} + \mathbf{u} = (S^{\alpha} + u^{\alpha})\mathbf{G}_{\alpha} + (z + u^3)\mathbf{G}_3; \end{aligned} \tag{7.224}$$

where u_0^{Λ} denotes displacements of the middle surface, and the $\varphi_n^{(\Lambda)}$ $(n = 1, 2, \ldots)$ are rotations. Substitution of Eq.(7.223) into Eqs.(7.217) and (7.220) and collection of the same powers of z gives

$$\begin{aligned} \varepsilon_{\alpha} &\approx \varepsilon_{\alpha}^{(0)} + \left.\frac{\partial \varepsilon_{\alpha}}{\partial z}\right|_{z=0} z + \frac{1}{2!}\left.\frac{\partial^2 \varepsilon_{\alpha}}{\partial z^2}\right|_{z=0} z^2 + \cdots \\ &= \varepsilon_{\alpha}^{(0)} + \frac{(\delta_{\alpha}^{\Lambda} + u_{0;\alpha}^{\Lambda})\varphi_{1;\alpha}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{\alpha\alpha}(1 + \varepsilon_{\alpha}^{(0)})} z \\ &\quad + \frac{[2(\delta_{\alpha}^{\Lambda} + u_{0;\alpha}^{\Lambda})\varphi_{2;\alpha}^{(\Lambda)} + \varphi_{1;\alpha}^{(\Lambda)}\varphi_{1;\alpha}^{(\Lambda)}]G_{\Lambda\Lambda}}{G_{\alpha\alpha}(1 + \varepsilon_{\alpha}^{(0)})} z^2 \\ &\quad - \frac{[(\delta_{\alpha}^{\Lambda} + u_{(0);\alpha}^{\Lambda})\varphi_{(1);\alpha}^{\Lambda}]^2 G_{\Lambda\Lambda}^2}{G_{\alpha\alpha}^2(1 + \varepsilon_{\alpha}^{(0)})^3} z^2 + \cdots, \end{aligned} \tag{7.225}$$

$$\begin{aligned} \varepsilon_3 &\approx \varepsilon_3^{(0)} + \left.\frac{\partial \varepsilon_3}{\partial z}\right|_{z=0} z + \frac{1}{2!}\left.\frac{\partial^2 \varepsilon_3}{\partial z^2}\right|_{z=0} z^2 + \cdots \\ &= \varepsilon_3^{(0)} + \frac{2(\delta_3^{\Lambda} + \varphi_1^{(\Lambda)})\varphi_2^{(\Lambda)} G_{\Lambda\Lambda}}{(1 + \varepsilon_3^{(0)})} z \\ &\quad + \frac{[2\varphi_2^{(\Lambda)}\varphi_2^{(\Lambda)} + 3(\delta_3^{\Lambda} + \varphi_1^{(\Lambda)})\varphi_3^{(\Lambda)}]G_{\Lambda\Lambda}}{1 + \varepsilon_3^{(0)}} z^2 \\ &\quad - \frac{2[(\delta_3^{\Lambda} + \varphi_1^{(\Lambda)})\varphi_2^{(\Lambda)}]^2 G_{\Lambda\Lambda}^2}{(1 + \varepsilon_3^{(0)})^3} z^2 + \cdots, \end{aligned} \tag{7.226}$$

$$\begin{aligned} \gamma_{12} &\approx \gamma_{12}^{(0)} + \left.\frac{\partial \gamma_{12}}{\partial z}\right|_{z=0} z + \frac{1}{2!}\left.\frac{\partial^2 \gamma_{12}}{\partial z^2}\right|_{z=0} z^2 + \cdots \\ &= \gamma_{12}^{(0)} + \frac{1}{\sin\theta_{(\mathbf{n}_1,\mathbf{n}_2)}^{(0)}} \left\{ \frac{(\delta_1^{\Lambda} + u_{0;1}^{\Lambda})\varphi_{1;2}^{(\Gamma)} + (\delta_2^{\Gamma} + u_{0;2}^{\Gamma})\varphi_{1;1}^{(\Lambda)}}{\sqrt{G_{11}G_{22}}(1 + \varepsilon_1^{(0)})(1 + \varepsilon_2^{(0)})} G_{\Lambda\Gamma} \right. \\ &\quad \left. - \cos\theta_{(\mathbf{n}_1,\mathbf{n}_2)}^{(0)} \left[\frac{(\delta_1^{\Lambda} + u_{0;1}^{\Lambda})\varphi_{1;1}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{11}(1 + \varepsilon_1^{(0)})^2} + \frac{(\delta_2^{\Lambda} + u_{0;2}^{\Lambda})\varphi_{1;2}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{22}(1 + \varepsilon_2^{(0)})^2} \right] \right\} z + \cdots, \end{aligned} \tag{7.227}$$

$$\begin{aligned}\gamma_{\alpha 3} &\approx \gamma_{\alpha 3}^{(0)} + \left.\frac{\partial \gamma_{\alpha 3}}{\partial z}\right|_{z=0} z + \frac{1}{2!}\left.\frac{\partial^2 \gamma_{\alpha 3}}{\partial z^2}\right|_{z=0} z^2 + \cdots \\ &= \gamma_{\alpha 3}^{(0)} + \frac{1}{\cos\gamma_{\alpha 3}^{(0)}}\left\{\frac{(\delta_\alpha^\Lambda + u_{0;\alpha}^\Lambda)\varphi_1^{(\Lambda)} + (\delta_3^\Lambda + \varphi_1^{(\Lambda)})\varphi_{1;\alpha}^{(\Lambda)}}{\sqrt{G_{\alpha\alpha}}(1+\varepsilon_\alpha^{(0)})(1+\varepsilon_3^{(0)})} G_{\Lambda\Lambda}\right. \\ &\quad \left. - \sin\gamma_{\alpha 3}^{(0)}\left[\frac{(\delta_\alpha^\Lambda + u_{0;\alpha}^\Lambda)\varphi_{1;\alpha}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{\alpha\alpha}(1+\varepsilon_\alpha^{(0)})^2} + \frac{2(\delta_3^\Lambda + \varphi_1^{(\Lambda)})\varphi_2^{(\Lambda)} G_{\Lambda\Lambda}}{(1+\varepsilon_3^{(0)})^2}\right]\right\} z + \cdots, \end{aligned} \tag{7.228}$$

for $\alpha = 1, 2$. The strains of the middle surface following Eqs.(7.217)-(7.220) at $z = 0$ are

$$\begin{aligned} \varepsilon_\alpha^{(0)} &= \frac{\sqrt{G_{\Lambda\Lambda}}}{\sqrt{G_{\alpha\alpha}}}\sqrt{(\delta_\alpha^\Lambda + u_{0;\alpha}^\Lambda)(\delta_\alpha^\Lambda + u_{0;\alpha}^\Lambda)} - 1, \\ \varepsilon_3^{(0)} &= \sqrt{G_{\Lambda\Lambda}}\sqrt{(\delta_3^\Lambda + \varphi_1^{(\Lambda)})(\delta_3^\Lambda + \varphi_1^{(\Lambda)})} - 1; \end{aligned} \tag{7.229}$$

$$\gamma_{12}^{(0)} \equiv \Theta_{(\mathbf{N}_1,\mathbf{N}_2)}^{(0)} - \theta_{(\mathbf{n}_1,\mathbf{n}_2)}^{(0)}, \ \gamma_{\alpha 3}^{(0)} \equiv \Theta_{(\mathbf{N}_\alpha,\mathbf{N}_3)}^{(0)} - \theta_{(\mathbf{n}_\alpha,\mathbf{n}_3)}^{(0)} \tag{7.230}$$

where

$$\begin{aligned} \theta_{(\mathbf{n}_1,\mathbf{n}_2)}^{(0)} &= \cos^{-1}\frac{G_{12} + 2E_{12}^{(0)}}{\sqrt{(G_{11} + 2E_{11}^{(0)})(G_{22} + 2E_{22}^{(0)})}} \\ &= \cos^{-1}\frac{(\delta_1^\Lambda + u_{0;1}^\Lambda)(\delta_2^\Gamma + u_{0;2}^\Gamma)G_{\Lambda\Gamma}}{\sqrt{G_{11}G_{22}}(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})}, \end{aligned} \tag{7.231}$$

$$\begin{aligned} \theta_{(\mathbf{n}_\alpha,\mathbf{n}_3)}^{(0)} &= \cos^{-1}\frac{G_{\alpha 3} + 2E_{\alpha 3}^{(0)}}{\sqrt{(G_{\alpha\alpha} + 2E_{\alpha\alpha}^{(0)})(G_{33} + 2E_{33}^{(0)})}} \\ &= \cos^{-1}\frac{2E_{\alpha 3}^{(0)}}{\sqrt{(G_{\alpha\alpha} + 2E_{\alpha\alpha}^{(0)})(1 + 2E_{33}^{(0)})}} \\ &= \cos^{-1}\frac{(\delta_\alpha^\Lambda + u_{0;\alpha}^\Lambda)(\delta_3^\Gamma + \varphi_1^{(\Gamma)})G_{\Lambda\Gamma}}{\sqrt{G_{\alpha\alpha}G_{33}}(1+\varepsilon_\alpha^{(0)})(1+\varepsilon_3^{(0)})} \\ &= \cos^{-1}\frac{(\delta_\alpha^\Lambda + u_{0;\alpha}^\Lambda)(\delta_3^\Gamma + \varphi_1^{(\Gamma)})G_{\Lambda\Gamma}}{\sqrt{G_{\alpha\alpha}}(1+\varepsilon_\alpha^{(0)})(1+\varepsilon_3^{(0)})}, \end{aligned} \tag{7.232}$$

$$\Theta_{(\mathbf{N}_\alpha,\mathbf{N}_\beta)} = \cos^{-1}\frac{G_{\alpha\beta}}{\sqrt{G_{\alpha\alpha}G_{\beta\beta}}} \text{ and } \Theta_{(\mathbf{N}_\alpha,\mathbf{N}_3)} = \frac{\pi}{2}. \tag{7.233}$$

$$E_{\alpha\beta}^{(0)} = \frac{1}{2}(u_{0\alpha;\beta} + u_{0\beta;\alpha} + u_{0;\alpha}^\Lambda u_{0\Lambda;\beta})$$

$$
\begin{aligned}
&= \frac{1}{2}[(\delta_\alpha^\Lambda + u_{0;\alpha}^\Lambda)(\delta_\beta^\Gamma + u_{0;\beta}^\Gamma)G_{\Lambda\Gamma} - G_{\alpha\beta}], \\
E_{\alpha 3}^{(0)} &= \frac{1}{2}(u_{0\alpha;3} + u_{03;\alpha} + u_{0;\alpha}^\Lambda u_{0\Lambda;3}) \\
&= \frac{1}{2}\left[(\delta_\alpha^\Lambda + u_{0;\alpha}^\Lambda)(\delta_3^\Gamma + \varphi_1^{(\Gamma)})G_{\Lambda\Gamma} - G_{\alpha 3}\right] \\
&= \frac{1}{2}(\delta_\alpha^\Lambda + u_{0;\alpha}^\Lambda)(\delta_3^\Gamma + \varphi_1^{(\Gamma)})G_{\Lambda\Gamma}.
\end{aligned} \tag{7.234}
$$

Because $\mathbf{G}_3$ is normal to the base vectors $\mathbf{G}_\alpha$ ($\alpha = 1, 2$), $G_{\alpha 3} = 0$ and $G_{33} = 1$. In Eqs.(7.225)−(7.228), the shell strain requires specification of three constraints for determination of three sets $\varphi_{(n)}^\Lambda$ ($\Lambda = 1, 2, 3;$ and $n = 1, 2, \ldots$) like the Kirchhoff assumptions ($\gamma_{\alpha 3} = \varepsilon_3 = 0$). From Eqs.(7.216) and (7.220), Kirchhoff's assumptions become

$$
\begin{aligned}
\sqrt{G_{\Lambda\Lambda}}\sqrt{(\delta_3^\Lambda + u_{;3}^\Lambda)(\delta_3^\Lambda + u_{;3}^\Lambda)} &= 1, \\
(\delta_\alpha^\Lambda + u_{;\alpha}^\Lambda)(\delta_3^\Gamma + u_{;3}^\Gamma)G_{\Lambda\Gamma} &= 0.
\end{aligned} \tag{7.235}
$$

Substitution of Eq.(7.223) into Eqs.(7.235), expansion of them in Taylor series in z and vanishing of the zero-order terms in z gives

$$
\begin{aligned}
\sqrt{G_{\Lambda\Lambda}}\sqrt{(\delta_3^\Lambda + \varphi_1^{(\Lambda)})(\delta_3^\Lambda + \varphi_1^{(\Lambda)})} &= 1, \\
(\delta_\alpha^\Lambda + u_{0;\alpha}^\Lambda)(\delta_3^\Gamma + \varphi_1^{(\Gamma)})G_{\Lambda\Gamma} &= 0.
\end{aligned} \tag{7.236}
$$

Form the foregoing equations,

$$
\delta_3^{(\Lambda)} + \varphi_1^{(\Lambda)} = \pm\frac{\Delta_\Lambda}{\Delta} \quad (\Lambda = 1, 2, 3) \tag{7.237}
$$

where only the positive (+) is adopted and

$$
\begin{aligned}
\Delta_1 &= \left[(\delta_1^\Lambda + u_{0;1}^\Lambda)G_{\Lambda 2}\right]\left[(\delta_2^\Lambda + u_{0;2}^\Lambda)G_{\Lambda 3}\right] \\
&\quad - \left[(\delta_1^\Lambda + u_{0;1}^\Lambda)G_{\Lambda 3}\right]\left[(\delta_2^\Lambda + u_{0;2}^\Lambda)G_{\Lambda 2}\right], \\
\Delta_2 &= \left[(\delta_1^\Lambda + u_{0;1}^\Lambda)G_{\Lambda 3}\right]\left[(\delta_2^\Lambda + u_{0;2}^\Lambda)G_{\Lambda 1}\right] \\
&\quad - \left[(\delta_1^\Lambda + u_{0;1}^\Lambda)G_{\Lambda 1}\right]\left[(\delta_2^\Lambda + u_{0;2}^\Lambda)G_{\Lambda 3}\right], \\
\Delta_3 &= \left[(\delta_1^\Lambda + u_{0;1}^\Lambda)G_{\Lambda 1}\right]\left[(\delta_2^\Lambda + u_{0;2}^\Lambda)G_{\Lambda 2}\right] \\
&\quad - \left[(\delta_1^\Lambda + u_{0;1}^\Lambda)G_{\Lambda 2}\right]\left[(\delta_2^\Lambda + u_{0;2}^\Lambda)G_{\Lambda 1}\right], \\
\Delta &= \sqrt{G_{11}\Delta_1^2 + G_{22}\Delta_2^2 + G_{33}\Delta_3^2}.
\end{aligned} \tag{7.238}
$$

From the Taylor series, vanishing of the first order terms in z gives three equations in $\varphi_2^{(\Lambda)}$ similar to Eqs.(7.236). The three equations plus $\varphi_1^{(\Lambda)}$ give $\varphi_2^{(\Lambda)}$; $\varphi_n^{(\Lambda)}$ for $n = 3, 4, \ldots$ can be determined in a like manner. Substitution of all φ_n^{Λ} ($n =$ $1, 2, 3, 4, \ldots$) into Eqs.(7.225) and (7.227) generates the normal and shear strains ε_α and γ_{12}.

7.3.2b Equations of motion on curvilinear coordinates

Consider the external and distributed forces on the curvilinear coordinates in the initial configuration as

$$\begin{aligned} &\mathbf{m} = (-1)^{\beta} m^{\alpha} \mathbf{N}_{\beta} \quad \text{for } \alpha, \beta \in \{1, 2\} \text{ and } \alpha \neq \beta, \\ &\mathbf{q} = q^{\Lambda} \mathbf{N}_{\Lambda} \quad \text{for } \Lambda \in \{1, 2, 3\}. \end{aligned} \tag{7.239}$$

The components of distributed force and moment are defined in Eq.(7.184), and the corresponding internal forces and moments in the deformed shell are defined as in Eq.(7.189), i.e.,

$$\begin{aligned} N_{\alpha\beta} &= \int_{-h^-}^{h^+} \sigma_{\alpha\beta} \left[\sqrt{G_{\alpha'\alpha'}} (1+\varepsilon_{\alpha'})(1+\varepsilon_3) \sin\theta_{(\mathbf{n}_{\alpha'}, \mathbf{n}_3)} \right] dz, \\ M_{\alpha\beta} &= \int_{-h^-}^{h^+} \sigma_{\alpha\beta} \frac{z}{1+\varphi_1^{(\Lambda)}} \left[\sqrt{G_{\alpha'\alpha'}} (1+\varepsilon_{\alpha'})(1+\varepsilon_3)^2 \sin\theta_{(\mathbf{n}_{\alpha'}, \mathbf{n}_3)} \right] dz, \\ Q_{\alpha} &= \int_{-h^-}^{h^+} \sigma_{\alpha 3} \left[\sqrt{G_{\alpha'\alpha'}} (1+\varepsilon_{\alpha'})(1+\varepsilon_3) \sin\theta_{(\mathbf{n}_{\alpha'}, \mathbf{n}_3)} \right] dz, \\ &\text{with } \alpha' = \operatorname{mod}(\alpha, 2) + 1, \end{aligned} \tag{7.240}$$

because of

$$\cos(\mathbf{N}_3, \mathbf{n}_3) = \frac{1 + u^{\Lambda}_{;3}}{\sqrt{1 + 2E_{33}}} \approx \frac{1 + \varphi_1^{(\Lambda)}}{1 + \varepsilon_3^{(0)}}, \tag{7.241}$$

where $N_{\alpha\beta}$ are membrane forces and $M_{\alpha\beta}$ are bending and twisting moments per unit length and $\alpha, \beta \in \{1, 2\}$. In a similar fashion, the internal force vectors are defined on the $\mathbf{G}_\Lambda$-surface, i.e.,

$$\begin{aligned} &\mathbf{M}_{\alpha} \equiv M_{\alpha}^{\Lambda} \mathbf{N}_{\Lambda} = (-1)^{\beta} M_{\alpha} \mathbf{n}_{\beta} + (-1)^{\alpha} M_{\alpha\beta} \mathbf{n}_{\alpha}, \\ &\boldsymbol{\mathcal{N}}_{\alpha} \equiv N_{\alpha}^{\Lambda} \mathbf{N}_{\Lambda} = N_{\alpha} \mathbf{n}_{\alpha} + N_{\alpha\beta} \mathbf{n}_{\beta} + Q_{\alpha} \mathbf{n}_3, \end{aligned} \tag{7.242}$$

$${}^{\mathrm{N}}\mathbf{M}_{\alpha} \equiv {}^{\mathrm{N}}M_{\alpha}^{\Lambda} \mathbf{N}_{\Lambda} = \mathbf{g}_{\alpha} \times \boldsymbol{\mathcal{N}}_{\alpha}, \tag{7.243}$$

where

$$\mathbf{g}_{\alpha} \equiv \frac{d\mathbf{r}_{\alpha}}{dS^{\alpha}} = (\delta_{\alpha}^{\Lambda} + u^{\Lambda}_{;\alpha}) \mathbf{G}_{\Lambda} \text{ and } {}^{\mathrm{N}}\mathbf{M}_{\alpha} \equiv \frac{1}{dS^{\alpha}} d\mathbf{r}_{\alpha} \times \boldsymbol{\mathcal{N}}_{\alpha}. \tag{7.244}$$

The components of the internal forces in the $\mathbf{G}_\Lambda$ -direction are

$$\begin{aligned} N_\alpha^\Lambda &= N_\alpha \mathbf{n}_\alpha \cdot \mathbf{N}_\Lambda + N_{\alpha\beta}\mathbf{n}_\beta \cdot \mathbf{N}_\Lambda + Q_\alpha \mathbf{n}_3 \cdot \mathbf{N}_\Lambda \\ &= N_\alpha \cos\theta_{(\mathbf{n}_\alpha,\mathbf{N}_\Lambda)} + N_{\alpha\beta}\cos\theta_{(\mathbf{n}_\beta,\mathbf{N}_\Lambda)} + Q_\alpha \cos_{(\mathbf{n}_3,\mathbf{N}_\Lambda)} \\ &= \frac{N_\alpha (G_{\Lambda\alpha} + u_{0;\alpha}^\Lambda)}{\sqrt{G_{\Lambda\Lambda}G_{\alpha\alpha}}(1+\varepsilon_\alpha^{(0)})} + \frac{N_{\alpha\beta}(G_{\Lambda\beta} + u_{0;\beta}^\Lambda)}{\sqrt{G_{\Lambda\Lambda}G_{\beta\beta}}(1+\varepsilon_\beta^{(0)})} + \frac{Q_\alpha (G_{\Lambda 3} + \varphi_1^{(\Lambda)})}{\sqrt{G_{\Lambda\Lambda}}(1+\varepsilon_3^{(0)})}, \end{aligned} \tag{7.245}$$

$$\begin{aligned} M_\alpha^\Lambda &= (-1)^\beta M_\alpha \mathbf{n}_\beta \cdot \mathbf{N}_\Lambda + (-1)^\alpha M_{\alpha\beta}\mathbf{n}_\alpha \cdot \mathbf{N}_\Lambda \\ &= (-1)^\beta M_\alpha \cos\theta_{(\mathbf{n}_\beta,\mathbf{N}_\Lambda)} + (-1)^\alpha M_{\alpha\beta}\cos\theta_{(\mathbf{n}_\alpha,\mathbf{N}_\Lambda)} \\ &= (-1)^\beta \frac{M_\alpha (G_{\Lambda\beta} + u_{0;\beta}^\Lambda)}{\sqrt{G_{\Lambda\Lambda}G_{\beta\beta}}(1+\varepsilon_\beta^{(0)})} + (-1)^\alpha \frac{M_{\alpha\beta}(G_{\Lambda\alpha} + u_{0;\alpha}^\Lambda)}{\sqrt{G_{\Lambda\Lambda}G_{\alpha\alpha}}(1+\varepsilon_\alpha^{(0)})}, \end{aligned} \tag{7.246}$$

$$\begin{aligned} {}^{\mathrm{N}}M_\alpha^\Lambda &= (\mathbf{g}_\alpha \times \mathcal{N}_\alpha)\cdot \mathbf{N}_\Lambda = (\delta_\alpha^\Gamma + u_{0;\alpha}^\Gamma) N_\alpha^\Sigma [\mathbf{N}_\Lambda \mathbf{N}_\Gamma \mathbf{N}_\Sigma] \\ &= (\delta_\alpha^\Gamma + u_{(0);\alpha}^\Gamma) N_\alpha^\Sigma \begin{vmatrix} \frac{X_{,\Lambda}^1}{\sqrt{G_{\Lambda\Lambda}}} & \frac{X_{,\Lambda}^2}{\sqrt{G_{\Lambda\Lambda}}} & \frac{X_{,\Lambda}^3}{\sqrt{G_{\Lambda\Lambda}}} \\ \frac{X_{,\Gamma}^1}{\sqrt{G_{\Gamma\Gamma}}} & \frac{X_{,\Gamma}^2}{\sqrt{G_{\Gamma\Gamma}}} & \frac{X_{,\Gamma}^3}{\sqrt{G_{\Gamma\Gamma}}} \\ \frac{X_{,\mathrm{K}}^1}{\sqrt{G_{\Sigma\Sigma}}} & \frac{X_{,\mathrm{K}}^2}{\sqrt{G_{\Sigma\Sigma}}} & \frac{X_{,\mathrm{K}}^3}{\sqrt{G_{\Sigma\Sigma}}} \end{vmatrix} \\ &= \frac{X_{,\Lambda}^I}{\sqrt{G_{\Lambda\Lambda}G_{\Gamma\Gamma}G_{\Sigma\Sigma}}} (\delta_\alpha^\Gamma + u_{0;\alpha}^\Gamma) N_\alpha^\Sigma (X_{,\Gamma}^J X_{,\Sigma}^K - X_{,\Gamma}^K X_{,\Sigma}^J) \end{aligned} \tag{7.247}$$

for $(I, J, K \in \{1, 2, 3\},\ I \neq J \neq K \neq I)$ and the indices (I, J, K) rotate clockwise; $(\Lambda, \Gamma, \Sigma \in \{1, 2, 3\},\ \Lambda \neq \Gamma \neq \Sigma \neq \Lambda)$. Based on the deformed middle surface in the Lagrangian coordinates, the equations of motion for the deformed shell are given by Eq. (7.196), i.e.,

$$\begin{aligned} \mathcal{N}_{\alpha;\alpha} + \mathbf{q} &= (\rho \mathbf{u}_{0,tt} + I_3 \boldsymbol{\varphi}_{1,tt})\sin\Theta_{(\mathbf{N}_1,\mathbf{N}_2)}, \\ \mathbf{M}_{\alpha;\alpha} + {}^{\mathrm{N}}\mathbf{M}_\alpha + \mathbf{m} &= (I_3 \mathbf{u}_{0,tt} + J_3 \boldsymbol{\varphi}_{1,tt})\sin\Theta_{(\mathbf{N}_1,\mathbf{N}_2)}, \end{aligned} \tag{7.248}$$

and the scalar expressions are for $\Lambda = 1, 2, 3$,

$$\begin{aligned} N_{\alpha;\alpha}^\Lambda + q^\Lambda &= (\rho u_{0,tt}^\Lambda + I_z \varphi_{1,tt}^{(\Lambda)})\sin\Theta_{(\mathbf{N}_1,\mathbf{N}_2)}, \\ M_{\alpha;\alpha}^\Lambda + {}^{\mathrm{N}}M_\alpha^\Lambda + m^\Lambda &= (I_z u_{0,tt}^\Lambda + J_3 \varphi_{1,tt}^{(\Lambda)})\sin\Theta_{(\mathbf{N}_1,\mathbf{N}_2)}, \end{aligned} \tag{7.249}$$

with summation on $\alpha = 1, 2$. The equation of motion in Eq.(7.249) is in a form of

$$\left[\frac{N_1(G_{\Lambda 1}+u^{\Lambda}_{0;1})}{\sqrt{G_{\Lambda\Lambda}G_{11}}(1+\varepsilon^{(0)}_1)}+\frac{N_{12}(G_{\Lambda 2}+u^{\Lambda}_{0;2})}{\sqrt{G_{\Lambda\Lambda}G_{22}}(1+\varepsilon^{(0)}_2)}+\frac{Q_1(G_{\Lambda 3}+\varphi^{(\Lambda)}_1)}{\sqrt{G_{\Lambda\Lambda}}(1+\varepsilon^{(0)}_3)}\right]_{;1}$$
$$+\left[\frac{N_{12}(G_{\Lambda 1}+u^{\Lambda}_{0;1})}{\sqrt{G_{\Lambda\Lambda}G_{11}}(1+\varepsilon^{(0)}_1)}+\frac{N_2(G_{\Lambda 2}+u^{\Lambda}_{0;2})}{\sqrt{G_{\Lambda\Lambda}G_{22}}(1+\varepsilon^{(0)}_2)}+\frac{Q_2(G_{\Lambda 3}+\varphi^{(\Lambda)}_1)}{\sqrt{G_{\Lambda\Lambda}}(1+\varepsilon^{(0)}_3)}\right]_{;2}$$
$$+q^{\Lambda}=(\rho u^{\Lambda}_{0,tt}+I_z\varphi^{(\Lambda)}_{1,tt})\sin\Theta_{(\mathbf{N}_1,\mathbf{N}_2)}, \tag{7.250}$$

$$\left[\frac{M_1(G_{\Lambda 2}+u^{\Lambda}_{0;2})}{\sqrt{G_{\Lambda\Lambda}G_{22}}(1+\varepsilon^{(0)}_2)}-\frac{M_{12}(G_{\Lambda 1}+u^{\Lambda}_{0;1})}{\sqrt{G_{\Lambda\Lambda}G_{11}}(1+\varepsilon^{(0)}_1)}\right]_{;1}$$
$$+\left[\frac{M_{12}(G_{\Lambda 2}+u^{\Lambda}_{0;2})}{\sqrt{G_{\Lambda\Lambda}G_{22}}(1+\varepsilon^{(0)}_2)}-\frac{M_2(G_{\Lambda 1}+u^{\Lambda}_{0;1})}{\sqrt{G_{\Lambda\Lambda}G_{11}}(1+\varepsilon^{(0)}_1)}\right]_{;2}$$
$$+\sum\nolimits_{I=1}^{3}\frac{X^I_{,\Lambda}}{\sqrt{G_{\Lambda\Lambda}G_{\Gamma\Gamma}G_{\Sigma\Sigma}}}(\delta^{\Gamma}_1+u^{\Gamma}_{0;1})N^{\Sigma}_1(X^J_{,\Gamma}X^K_{,\Sigma}-X^K_{,\Gamma}X^J_{,\Sigma})$$
$$+\sum\nolimits_{I=1}^{3}\frac{X^I_{,\Lambda}}{\sqrt{G_{\Lambda\Lambda}G_{\Gamma\Gamma}G_{\Sigma\Sigma}}}(\delta^{\Gamma}_2+u^{\Gamma}_{0;2})N^{\Sigma}_1(X^J_{,\Gamma}X^K_{,\Sigma}-X^K_{,\Gamma}X^J_{,\Sigma})$$
$$+m^{\Lambda}=\left(I_z u^{\Lambda}_{0,tt}+J_z\varphi^{(\Lambda)}_{1,tt}\right)\sin\Theta_{(\mathbf{N}_1,\mathbf{N}_2)}. \tag{7.251}$$

Notice that $(I, J, K\in\{1,2,3\},\ I\neq J\neq K\neq I)$ and the indices (I, J, K) rotate clockwise. $(\Lambda,\Gamma,\Sigma\in\{1,2,3\},\ \Lambda\neq\Gamma\neq\Sigma\neq\Lambda)$. The balance of equilibrium for shells in Eq.(7.248) or (7.249) with (7.225)–(7.230) constitute this approximate nonlinear theory for shells.

The force condition at a point $\mathscr{P}_{(k_1,k_2)}$ with $S^{\alpha}=A^{\alpha}_{(k_1,k_2)}$ ($\alpha=1,2$) can be developed as in Eq.(7.200) but the scalar expression is

$$\sum\nolimits_{\alpha=1}^{2}{}^{-}N^{\Lambda}_{\alpha}(S^1,S^2)\Big|_{(S^1,S^2)=(A^1_{(k_1,k_2)},A^2_{(k_1,k_2)})}$$
$$=\sum\nolimits_{\alpha=1}^{2}{}^{+}N^{\Lambda}_{\alpha}(S^1,S^2)\Big|_{(S^1,S^2)=(A^1_{(k_1,k_2)},A^2_{(k_1,k_2)})}+F^{\Lambda}_{(k_1,k_2)}. \tag{7.252}$$

The force boundary condition at the boundary point $\mathscr{P}_{(r_1,r_2)}$ is as in Eq.(7.202) and the scalar expression is

$$\sum\nolimits_{\alpha=1}^{2}N^{\Lambda}_{\alpha}(S^1,S^2)\Big|_{(S^1,S^2)=(B^1_{(r_1,r_2)},B^2_{(r_1,r_2)})}+F^{\Lambda}_{(r_1,r_2)}=0. \tag{7.253}$$

The force boundary conditions on the shell edge are given in Eq.(7.204) but the scalar expression is for $\alpha,\beta\in\{1,2\}$ and $\alpha\neq\beta$,

$$\frac{\partial}{\partial S^{\beta}}\int N^{\Lambda}_{\beta}(S^1,S^2)dS^{\alpha}\Big|_{S^{\alpha}=S^{\alpha}_{*}}+N^{\Lambda}_{\alpha}(S^1,S^2)\Big|_{S^{\alpha}=S^{\alpha}_{*}}+F^{\Lambda}(S^{\beta})=0. \tag{7.254}$$

If there is a concentrated moment at a point $\mathscr{P}_{(k_1,k_2)}$ with $S^\alpha = A^\alpha_{(k_1,k_2)}$ ($\alpha = 1, 2$), the corresponding moment boundary conditions are given in Eq.(7.206) but the scalar expression is

$$\sum_{\alpha=1}^{2} {}^{-}M_\alpha^\Lambda(S^1, S^2)\Big|_{(S^1, S^2)=(A^1_{(k_1,k_2)}, A^2_{(k_1,k_2)})}$$
$$=\sum_{\alpha=1}^{2} {}^{+}M_\alpha^\Lambda(S^1, S^2)\Big|_{(S^1, S^2)=(A^1_{(k_1,k_2)}, A^2_{(k_1,k_2)})} + M^\Lambda_{(k_1,k_2)}. \quad (7.255)$$

The moment boundary condition at the boundary point $\mathscr{P}_{(r_1,r_2)}$ is given in Eq.(7.208), but the scalar expression is

$$\sum_{\alpha=1}^{2} M_\alpha^\Lambda(S^1, S^2)\Big|_{(S^1, S^2)=(B^1_{(r_1,r_2)}, B^2_{(r_1,r_2)})} + M^\Lambda_{(r_1,r_2)} = 0. \quad (7.256)$$

The moment boundary conditions on the shell edge is given in Eq.(7.210), but the scalar expression is for $\alpha, \beta \in \{1, 2\}$ and $\alpha \neq \beta$.

$$(M_\alpha^\Lambda(S^1, S^2) + \int {}^{\mathrm{N}}M_\alpha^\Lambda(S^1, S^2)dS^\alpha)\Big|_{S^\alpha = S^\alpha_*}$$
$$+(\int {}^{\mathrm{N}}M_\beta^\Lambda(S^1, S^2)dS^\alpha + \frac{\partial}{\partial S^\beta}\int M_\beta^\Lambda(S^1, S^2)dS^\alpha)\Big|_{S^\alpha = S^\alpha_*}$$
$$+M^\Lambda(S^\beta) = 0. \quad (7.257)$$

The displacement continuity and boundary conditions are the same as in Eqs.(7.151) and (7.152).

Such a theory can easily reduce to the Kirshhoff theory (Kirchhoff, 1850a,b). The nonlinear theory of shells is different from the theories based on the Cosserat surface in E. and F. Cosserat (1896, 1909). Based on the three-dimensional deformable body, the Cosserat surface with one director was used to derive the nonlinear theory of shells (e.g., Green, Laws and Naghdi, 1968). The detailed presentation of such a shell theory can be referred to Naghdi (1972). In that theory, the general expressions of geometrical relations (or strains) were given with infinite directors, but how to determine the infinite directors was not presented. Only the existing theories can be included in such a generalized frame. However, the nonlinear theory of shells presented herein gives an intuitive and deterministic form to use. The unknown coefficients in the approximate displacement fields can be determined by the certain assumptions. Such a theory can be applied to any materials.

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Chapter 8
Nonlinear Beams and Rods

In this chapter, nonlinear theories for rods and beams will be discussed in the Cartesian coordinate frame and the curvilinear frame of the initial configuration. Without torsion, the theory for in-plane beams will be presented. The traditional treatises of nonlinear rods were based on the Cosserat's theory (e.g., E. and F. Cosserat, 1896) or the Kirchhoff assumptions (e.g., Kirchhoff, 1859; Love, 1944). This chapter will extend the ideas of Galerkin (1915), and the nonlinear theory of rods and beams will be developed from the general theory of the 3-dimensional deformable body. The definitions for beams and rods are given as follows.

Definition 8.1. If a 1-D deformable body on the three directions of fibers resists internal forces, *bending* and *twisting* moments, the 1-D deformable body is called a *deformable* rod.

Definition 8.2. If a 1-D deformable body on the three directions of fibers resists internal forces and *bending* moments, the 1-D deformable body is called a *deformable* beam.

8.1. Differential geometry of curves

Consider an initial configuration of a nonlinear rod as shown in Fig.8.1. The unit vectors $\mathbf{I}_I$ ($I = 1, 2, 3$) are the base vectors for the Cartesian coordinates and the based vectors $\mathbf{G}_\alpha$ ($\alpha = 1, 2, 3$) for the curvilinear coordinates are defined later. To present the nonlinear rod theory, it is assumed that the base vector $\mathbf{G}_1$ is normal to the surface formed by the other base vectors $\mathbf{G}_2$ and $\mathbf{G}_3$. The surface formed by the vectors $\mathbf{G}_2$ and $\mathbf{G}_3$ is called the *cross section* of the rod. The material particle on the central curves of the intersections of two neutral surfaces in the initial configuration is

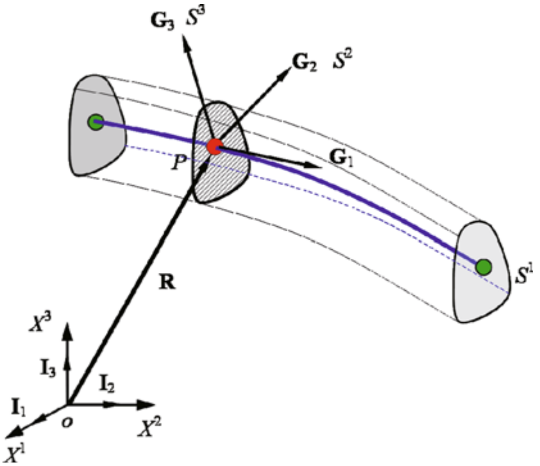

Fig. 8.1 A material particle *P* on an initial configuration of a nonlinear rod.

$$\mathbf{R} = X^I(S)\mathbf{I}_I, \tag{8.1}$$

where $S^1 = S$ and $S^2 = S^3 = 0$. From Eq.(8.1), the base vectors for the rod can be obtained. The base vector in the tangential direction of the rod is defined by

$$\mathbf{G}_1 = \frac{d\mathbf{R}}{dS} = \frac{\partial X^I}{\partial S}\mathbf{I}_I = X^I_{,S}\mathbf{I}_I \equiv G_1^I \mathbf{I}_I. \tag{8.2}$$

Note that $(\cdot)_{,S} = (\cdot)_{,1}$ and the metric measure is given by

$$G_{11} = \frac{\partial X^I}{\partial S}\frac{\partial X^I}{\partial S} = X^I_{,1}X^I_{,1} \equiv G_1^I G_1^I \text{ (summation on } I). \tag{8.3}$$

and

$$\mathbf{N}_1 = \frac{\mathbf{G}_1}{\sqrt{G_{11}}} = \frac{X^I_{,1}}{\sqrt{G_{11}}}\mathbf{I}_I, \tag{8.4}$$

while an arc length variable s is defined by

$$ds = \sqrt{G_{11}}\,dS, \tag{8.5}$$

$$\mathbf{N}_1 = X^I_{,s}\mathbf{I}_I \text{ and } \mathbf{G}_1 = \sqrt{G_{11}}\; X^I_{,s}\mathbf{I}_I \equiv G_1^I\mathbf{I}_I, \tag{8.6}$$

with $\sqrt{X^I_{,s}X^I_{,s}} = 1$. The direction of $\mathbf{G}_1$ is the tangential direction of the initial configuration of the rod curve. From differential geometry in Kreyszig (1968), the curvature vector can be determined by

$$\mathbf{G}_2 \equiv G_2^I \mathbf{I}_I = \frac{d\mathbf{N}_1}{ds} = X_{,ss}^I \mathbf{I}_I$$
$$= \frac{d\mathbf{N}_1}{dS}\frac{dS}{ds} = \frac{1}{G_{11}^2}\left[X_{,11}^I (X_{,1}^K X_{,1}^K) - X_{,1}^I X_{,1}^K X_{,11}^K\right]\mathbf{I}_I \tag{8.7}$$

and

$$\begin{aligned} G_2^I &\equiv \frac{1}{G_{11}^2}\left[X_{,11}^I (X_{,1}^K X_{,1}^K) - X_{,1}^I X_{,1}^K X_{,11}^K\right], \\ G_{22} &\equiv G_2^I G_2^I = \frac{1}{G_{11}^3}\left[(X_{,11}^I X_{,11}^I)(X_{,1}^K X_{,1}^K) - (X_{,1}^I X_{,11}^I)^2\right]. \end{aligned} \tag{8.8}$$

The *curvature of the rod* in the initial configuration is

$$\kappa = |\mathbf{G}_2| = \sqrt{G_{22}} = \sqrt{X_{,ss}^I X_{,ss}^I}$$
$$= \frac{\sqrt{(X_{,11}^I X_{,11}^I)(X_{,1}^K X_{,1}^K) - (X_{,1}^I X_{,11}^I)^2}}{G_{11}^{3/2}}. \tag{8.9}$$

The *unit principal normal* vector is given by

$$\mathbf{N}_2 = \frac{\mathbf{G}_2}{\sqrt{G_{22}}} = \frac{\mathbf{G}_2}{\kappa} = \frac{G_2^I}{\kappa}\mathbf{I}_I, \quad \frac{d\mathbf{N}_1}{ds} = \mathbf{G}_2 = \kappa \mathbf{N}_2. \tag{8.10}$$

The *unit bi-normal vector* is defined by

$$\mathbf{N}_3 = \mathbf{N}_1 \times \mathbf{N}_2 \tag{8.11}$$

and let

$$\mathbf{N}_3 = G_3^I \mathbf{I}_I \ \text{ with } G_3^I = e_{IJK}\frac{G_1^J}{\sqrt{G_{11}}}\frac{G_2^K}{\kappa(S)}, \tag{8.12}$$

where e_{IJK} is the Ricci symbol in Eq.(2.105). Therefore, $G_{33} = G_3^I G_3^I = 1$ (*summation on I*)

Consider the change rate of the unit bi-normal direction with respect to the arc length (s), which gives

$$\frac{d\mathbf{N}_3}{ds} = -\tau \mathbf{N}_2 \ \Rightarrow \ \tau = -\mathbf{N}_2 \cdot \frac{d\mathbf{N}_3}{ds}. \tag{8.13}$$

The *torsional curvature of the rod* (or torsion of the curve called in mathematics) is

$$\tau = \frac{1}{\kappa^2}[\mathbf{R}_{,s}\mathbf{R}_{,ss}\mathbf{R}_{,sss}]$$

$$= \frac{[\mathbf{R}_{,1}\mathbf{R}_{,11}\mathbf{R}_{,111}]}{(\mathbf{R}_{,11}\cdot\mathbf{R}_{,11})(\mathbf{R}_{,1}\cdot\mathbf{R}_{,1})-(\mathbf{R}_{,1}\cdot\mathbf{R}_{,11})^2}$$

$$= \frac{e_{IJK}X^I_{,1}X^J_{,11}X^K_{,111}}{(X^I_{,11}X^I_{,11})(X^K_{,1}X^K_{,1})-(X^I_{,1}X^I_{,11})^2}. \tag{8.14}$$

Based on the definition of unit based vector, the vector product gives

$$\mathbf{N}_2 = \mathbf{N}_3\times\mathbf{N}_1,\ \mathbf{N}_3 = -\mathbf{N}_2\times\mathbf{N}_1,\ \mathbf{N}_1 = -\mathbf{N}_3\times\mathbf{N}_2. \tag{8.15}$$

With Eqs.(8.10) and (8.13),

$$\begin{aligned}\frac{d\mathbf{N}_2}{ds} &= \frac{d\mathbf{N}_3}{ds}\times\mathbf{N}_1 + \mathbf{N}_3\times\frac{d\mathbf{N}_1}{ds}\\ &= -\tau\mathbf{N}_2\times\mathbf{N}_1 + \kappa\mathbf{N}_3\times\mathbf{N}_2\\ &= -\kappa\mathbf{N}_1 + \tau\mathbf{N}_3.\end{aligned} \tag{8.16}$$

Thus, from the formulae of Frenet (1847),

$$\begin{bmatrix}\dfrac{d\mathbf{N}_1}{ds}\\ \dfrac{d\mathbf{N}_2}{ds}\\ \dfrac{d\mathbf{N}_3}{ds}\end{bmatrix} = \begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\mathbf{N}_1\\ \mathbf{N}_2\\ \mathbf{N}_3\end{bmatrix}, \tag{8.17}$$

$$\begin{bmatrix}\dfrac{d\mathbf{N}_1}{dS}\\ \dfrac{d\mathbf{N}_2}{dS}\\ \dfrac{d\mathbf{N}_3}{dS}\end{bmatrix} = \begin{bmatrix}0 & \sqrt{G_{11}}\kappa & 0\\ -\sqrt{G_{11}}\kappa & 0 & \sqrt{G_{11}}\tau\\ 0 & -\sqrt{G_{11}}\tau & 0\end{bmatrix}\begin{bmatrix}\mathbf{N}_1\\ \mathbf{N}_2\\ \mathbf{N}_3\end{bmatrix}. \tag{8.18}$$

Consider a rotation vector (or the vector of Darboux)

$$\boldsymbol{\omega} = \tau\mathbf{N}_1 + \kappa\mathbf{N}_3 \tag{8.19}$$

Equations (8.17) and (8.18) become

$$\frac{d\mathbf{N}_1}{ds} = \boldsymbol{\omega}\times\mathbf{N}_1,\ \frac{d\mathbf{N}_2}{ds} = \boldsymbol{\omega}\times\mathbf{N}_2, \frac{d\mathbf{N}_3}{ds} = \boldsymbol{\omega}\times\mathbf{N}_3, \tag{8.20}$$

$$\frac{d\mathbf{N}_1}{dS} = \sqrt{G_{11}}\boldsymbol{\omega}\times\mathbf{N}_1,\ \frac{d\mathbf{N}_2}{dS} = \sqrt{G_{11}}\boldsymbol{\omega}\times\mathbf{N}_2,\ \frac{d\mathbf{N}_3}{dS} = \sqrt{G_{11}}\boldsymbol{\omega}\times\mathbf{N}_3. \tag{8.21}$$

Consider a material point $\mathbf{R}$ on the cross section of the rod

$$\mathbf{R} = X^I(S^1, S^2, S^2)\mathbf{I}_I. \tag{8.22}$$

Without loss of generality, S^1, S^2 and S^3 are collinear to the directions of $\mathbf{N}_1$, $\mathbf{N}_2$ and $\mathbf{N}_3$, respectively. On the cross section of $\mathbf{N}_1$, any variable can be converted on the two directions of $\mathbf{N}_2$ and $\mathbf{N}_3$.

Consider a displacement vector field at the point $\mathbf{R}$ to be

$$\mathbf{u} = u^I(S^1, S^2, S^3)\mathbf{I}_I \quad \text{or} \quad \mathbf{u} = u^\Lambda(S^1, S^2, S^3)\mathbf{G}_\Lambda. \tag{8.23}$$

From the previous definitions,

$$\mathbf{N}_\Lambda = \frac{\mathbf{G}_\Lambda}{\sqrt{G_{\Lambda\Lambda}}} = \frac{G_\Lambda^I}{\sqrt{G_{\Lambda\Lambda}}}\mathbf{I}_I. \tag{8.24}$$

The particle in the deformed configuration is expressed by the location and displacement vectors, i.e.,

$$\begin{aligned} \mathbf{r} &= \mathbf{R} + \mathbf{u} = (X^I + u^I(S^1, S^2, S^3))\mathbf{I}_I, \text{ or} \\ \mathbf{r} &= \mathbf{R} + \mathbf{u} = (S^\Lambda + u^\Lambda(S^1, S^2, S^3))\mathbf{G}_\Lambda \end{aligned} \tag{8.25}$$

and the corresponding infinitesimal line element of the deformed rod is

$$\begin{aligned} d\mathbf{r} &= d\mathbf{R} + d\mathbf{u} = (X^I_{,\alpha} + u^I_{,\alpha})dS^\alpha \mathbf{I}_I \\ &= (\delta^\Lambda_\alpha + u^\Lambda_{;\alpha})dS^\alpha \mathbf{G}_\Lambda. \end{aligned} \tag{8.26}$$

The base vector for the deformed rod becomes

$$\mathbf{g}_\alpha = (X^I_{,\alpha} + u^I_{,\alpha})\mathbf{I}_I = (\delta^\Lambda_\alpha + u^\Lambda_{;\alpha})\mathbf{G}_\Lambda \tag{8.27}$$

and the corresponding unit vector is

$$\begin{aligned} \mathbf{n}_\alpha &= \frac{X^I_{,\alpha} + u^I_{,\alpha}}{\sqrt{(X^K_{,\alpha} + u^K_{,\alpha})(X^K_{,\alpha} + u^K_{,\alpha})}}\mathbf{I}_I \\ &= \frac{\delta^\Lambda_\alpha + u^\Lambda_{;\alpha}}{\sqrt{G_{\Gamma\Gamma}}\sqrt{(\delta^\Gamma_\alpha + u^\Gamma_{;\alpha})(\delta^\Gamma_\alpha + u^\Gamma_{;\alpha})}}\mathbf{G}_\Lambda. \end{aligned} \tag{8.28}$$

8.2. A nonlinear theory of straight beams

Consider a *beam* in the initial configuration to be straight. This requires that the curvature and torsion should be zero ($\kappa(S) = 0$ and $\tau(S) = 0$). Thus, $S^I = X^I$, $G_{\alpha\beta} = 0$ and $G_{\alpha\alpha} = 1$ ($\alpha, \beta = 1, 2, 3$).

$$d\mathbf{R} = dX^I \mathbf{I}_I, \; d\mathbf{r} = d\mathbf{R} + d\mathbf{u} = (\delta_\alpha^I + u_{,\alpha}^I) dX^\alpha \mathbf{I}_I. \tag{8.29}$$

The strain based on the change in length of $d\mathbf{R}$ per unit length gives

$$\begin{aligned}\varepsilon_\alpha &= \frac{|d\underset{\alpha}{\mathbf{r}}| - |d\underset{\alpha}{\mathbf{R}}|}{|d\underset{\alpha}{\mathbf{R}}|} = \sqrt{1+2E_{\alpha\alpha}} - 1 \\ &= \sqrt{(\delta_\alpha^I + u_{,\alpha}^I)(\delta_\alpha^I + u_{,\alpha}^I)} - 1.\end{aligned} \tag{8.30}$$

The Lagrangian strain tensor $E_{\alpha\beta}$ referred to the initial configuration is

$$E_{\alpha\beta} = \frac{1}{2}(\delta_\alpha^I u_{,\beta}^I + \delta_\beta^I u_{,\alpha}^I + u_{,\alpha}^I u_{,\beta}^I). \tag{8.31}$$

In the similar fashion, the angles between $\mathbf{n}_\alpha$ and $\mathbf{n}_\beta$ *before* and *after* deformation are expressed by $\Theta_{(\mathbf{N}_\alpha,\mathbf{N}_\beta)} = \pi/2$ and $\theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)}$, i.e.,

$$\begin{aligned}\cos\theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)} &\equiv \cos(\Theta_{(\mathbf{N}_\alpha,\mathbf{N}_\beta)} - \gamma_{\alpha\beta}) \\ &= \frac{d\underset{\alpha}{\mathbf{r}} \cdot d\underset{\beta}{\mathbf{r}}}{|d\underset{\alpha}{\mathbf{r}}||d\underset{\beta}{\mathbf{r}}|} = \frac{(\delta_\alpha^I + u_{,\alpha}^I)(\delta_\beta^I + u_{,\beta}^I)}{\sqrt{(1+2E_{\alpha\alpha})(1+2E_{\beta\beta})}} \\ &= \frac{(\delta_\alpha^I + u_{,\alpha}^I)(\delta_\beta^I + u_{,\beta}^I)}{(1+\varepsilon_\alpha)(1+\varepsilon_\beta)}\end{aligned} \tag{8.32}$$

and the corresponding shear strain is defined by

$$\begin{aligned}\gamma_{\alpha\beta} &\equiv \Theta_{(\mathbf{N}_\alpha,\mathbf{N}_\beta)} - \theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)} \\ &= \sin^{-1}\frac{(\delta_\alpha^I + u_{,\alpha}^I)(\delta_\beta^I + u_{,\beta}^I)}{\sqrt{(1+2E_{\alpha\alpha})(1+2E_{\beta\beta})}} \\ &= \sin^{-1}\frac{(\delta_\alpha^I + u_{,\alpha}^I)(\delta_\beta^I + u_{,\beta}^I)}{(1+\varepsilon_\alpha)(1+\varepsilon_\beta)}.\end{aligned} \tag{8.33}$$

From Eq.(8.28), the direction cosine of the rotation is

$$\cos\theta_{(\mathbf{N}_\alpha,\mathbf{n}_\beta)} = \frac{d\underset{\alpha}{\mathbf{R}} \cdot d\underset{\beta}{\mathbf{r}}}{|d\underset{\alpha}{\mathbf{R}}||d\underset{\beta}{\mathbf{r}}|} = \frac{\delta_\beta^\alpha + u_{,\beta}^\alpha}{\sqrt{1+2E_{\beta\beta}}} = \frac{\delta_\beta^\alpha + u_{,\beta}^\alpha}{1+\varepsilon_\beta}. \tag{8.34}$$

In addition, the area changes before and after deformation are given by

$$\frac{d\underset{\alpha\beta}{a}}{d\underset{\alpha\beta}{A}} = (1+\varepsilon_\alpha)(1+\varepsilon_\beta)\sin\theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)}, \tag{8.35}$$

where $\underset{\alpha\beta}{da} = |\underset{\alpha}{d\mathbf{r}} \times \underset{\beta}{d\mathbf{r}}|$ and $\underset{\alpha\beta}{dA} = |\underset{\alpha}{d\mathbf{R}} \times \underset{\beta}{d\mathbf{R}}|$.

Consider the coordinate X^1 to be along the longitudinal direction of the beam and the other two coordinates X^2 and X^3 on the cross section of beam on the direction of X^1. The coordinates for the deformed straight beam are (s^1, s^2, s^3). Because the initial configuration of the beam is a straight beam, under external force, the deformed configuration of the beam does not experience any torsion ($\tau(s^1) \equiv 0$). Thus, the deformed configuration of the beam is a plane curve. Without loss of generality, the curvature direction of the deformed configuration can be assumed to be collinear to X^2. Because the widths of beam in two directions of X^2 and X^3 are very small compared to the length of the beam in direction of X^1, the elongation in the two directions of X^2 and X^3 should be very small, which can be neglected. From the aforementioned discussions, the following assumptions are adopted:

(i) The deformed configuration of the beam does not experience any torsion ($\tau(s^1) \equiv 0$).
(ii) The curvature direction of the deformed configuration is collinear to s^2.
(iii) The elongations in the two directions of X^2 and X^3 are very small (i.e., $\varepsilon_2 \approx 0$ and $\varepsilon_3 = 0$).
(iv) Under bending only, the neutral surface is not deformed (i.e., $\gamma_{12} \approx 0$ and $\gamma_{13} = 0$).

Consider an arbitrary coordinates as (X^1, Y, Z) at the *centroid* on the cross section of the beam. Under the resultant forces, the bending of beam is in the curvature direction of s^2. The deformed curve of the beam is on the plane of (X^1, X^2), as shown in Fig.8.2. In other words, the neutral surface of the deformed beam is on the plane of (X^1, X^3). When the transversal forces act at a point on the cross section of the beam and if the beam will not be twisted, such a point on the cross section of the beam is called the *shear center* (*or flexural center*). From Assumption (i), no torsion exists. In addition, the transversal forces should be placed to the shear center. Because the transversal forces are applied to the beam off the shear center, the beam will be twisted and bent. To explain this case, consider external distributed forces and moments at *the shear center* on the initial configuration to be

$$\mathbf{q} = q^I \mathbf{I}_I \text{ and } \mathbf{m} = m^I \mathbf{I}_I \ (I = 1, 2, 3) \tag{8.36}$$

and concentrated forces on the initial configuration at a point $X^1 = S_k$,

$${}^k\mathbf{F} = {}^kF^I \mathbf{I}_I \text{ and } {}^k\mathbf{M} = {}^kM^I \mathbf{I}_I \ (I = 1, 2, 3). \tag{8.37}$$

The displacement vectors on the initial configuration are

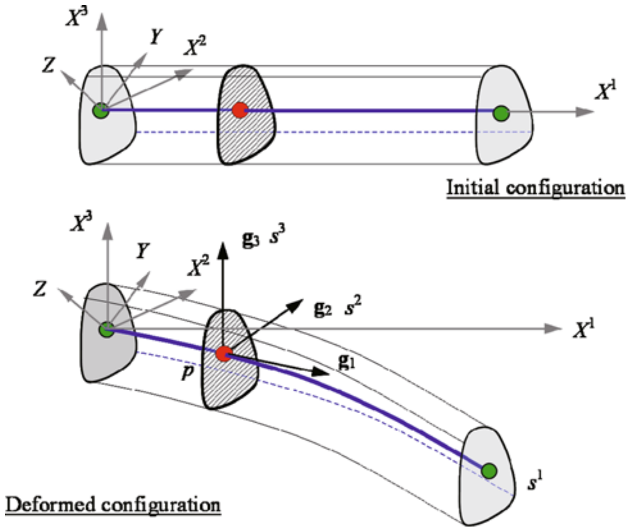

Fig. 8.2 A straight beam with initial and deformed configuration.

$$\mathbf{R} = X^1\mathbf{I}_1,\ {}^S\mathbf{R} = S\mathbf{I}_1 \text{ and } {}^k\mathbf{R} = X_k^1\mathbf{I}_1. \tag{8.38}$$

The internal forces and moments for $S > X_k^1$ are

$$\begin{aligned}
\mathbf{F}\,|_{X^1=S} &= \sum\nolimits_{k=1} {}^k\mathbf{F} + \int_0^S \mathbf{q}\,dX^1, \\
\mathbf{M}\,|_{X^1=S} &= \sum\nolimits_{k=1} {}^k\mathbf{M} + \int_0^S \mathbf{m}\,dX^1, \\
&\quad + \sum\nolimits_{k=1} ({}^S\mathbf{R} - {}^k\mathbf{R}) \times {}^k\mathbf{F} + \int_0^S ({}^S\mathbf{R} - \mathbf{R}) \times \mathbf{q}\,dX^1;
\end{aligned} \tag{8.39}$$

or for $I = 1, 2, 3$,

$$\begin{aligned}
F^I\,|_{X^1=S} &= \sum\nolimits_{k=1} {}^kF^I + \int_0^S q^I\,dX^1, \\
M^1\,|_{X^1=S} &= \sum\nolimits_{k=1} {}^kM^1 + \int_0^S m^1\,dX^1, \\
M^2\,|_{X^1=S} &= \sum\nolimits_{k=1} {}^kM^2 + \int_0^S m^2\,dX^1 \\
&\quad - \sum\nolimits_{k=1} {}^kF^3(S - X_k^1) - \int_0^S X^1 q^3\,dX^1, \\
M^3\,|_{X^1=S} &= \sum\nolimits_{k=1} {}^kM^3 + \int_0^S m^3\,dX^1 \\
&\quad + \sum\nolimits_{k=1} {}^kF^2(S - X_k^1) + \int_0^S (S - X^1) q^2\,dX^1.
\end{aligned} \tag{8.40}$$

From assumptions (i) and (ii), the following conditions exist:

$$\begin{aligned}
F^3 |_{X^1=S} &= \sum_{k=1} {}^k F^3 + \int_0^S q^3 dX^1 = 0, \\
M^1 |_{X^1=S} &= \sum_{k=1} {}^k M^1 + \int_0^S m^1 dX^1 = 0, \\
M^2 |_{X^1=S} &= \sum_{k=1} {}^k M^2 + \int_0^S m^2 dX^1 \\
&\quad - \sum_{k=1} {}^k F^3 (S - X_k^1) - \int_0^S (S - X^1) q^3 dX^1 = 0.
\end{aligned} \tag{8.41}$$

For all points on the beam to satisfy Eq.(8.41),

$$\begin{aligned}
{}^k F^3 &= 0 \ \text{ and } \ q^3 = 0, \\
{}^k M^1 &= 0 \ \text{ and } \ m^1 = 0, \\
{}^k M^2 &= 0 \ \text{ and } \ m^2 = 0.
\end{aligned} \tag{8.42}$$

If the external forcing exerts on the three directions of (X^1, Y, Z), the resultant forces and moments on three directions of (X^1, X^2, X^3) should satisfy Eq.(8.42). Such projection of the forces can be done through the rotation angle between the two coordinates (X^1, Y, Z) and (X^1, X^2, X^3).

From assumption (ii),

$$u^I = u_0^I(S, t) + \sum_{n=1}^{\infty} (X^2)^n \varphi_n^{(I)}(S, t) \ \text{ for } I = 1, 2, \tag{8.43}$$

where $X^2 = \sqrt{(Y)^2 + (Z)^2}$ is a distance to the neutral surface along the direction of curvature. From assumption (ii), no displacements exist in the direction of X^3 (i.e., $u^3 = 0$). From *Kirchhoff's assumptions,* under bending only, if the cross section is normal to the neutral surface before deformation, then after deformation, the deformed cross section is still normal to the deformed neutral surface. Thus, equation (8.43) becomes

$$u^I = u_0^I(S,t) + X^2 \varphi^{(I)}(S,t) \quad (I = 1, 2). \tag{8.44}$$

From assumptions (iii) and (iv),

$$\begin{aligned}
(\delta_2^I + u_{,2}^I)(\delta_2^I + u_{,2}^I) &= 1, \\
(\delta_1^I + u_{,1}^I)(\delta_2^I + u_{,2}^I) &= 0.
\end{aligned} \tag{8.45}$$

With $u^3 = 0$ and Eq.(8.43), the Taylor series expansion of Eq.(8.45) give for the zero-order of X^2,

$$\begin{aligned}
(\delta_2^I + \varphi_1^{(I)})(\delta_2^I + \varphi_1^{(I)}) &= 1, \\
(\delta_1^I + u_{0,1}^I)(\delta_2^I + \varphi_1^{(I)}) &= 0.
\end{aligned} \tag{8.46}$$

From Eq.(8.46),

$$\varphi_1^{(1)} = \mp \frac{u_{0,1}^2}{\sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^2)^2}},$$
$$\varphi_1^{(2)} = \pm \frac{1+u_{0,1}^1}{\sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^2)^2}} - 1. \tag{8.47}$$

From the sign convention, the positive "+" in the second equation of Eq.(8.47) will be adopted. Following the similar fashion, one can obtain $\varphi_n^{(I)}$ ($n = 1, 2, \ldots$ and $I = 1, 2$). Further, using the Taylor series expansion, the approximations of three strains on the cross section of the deformed beam are

$$\begin{aligned}\varepsilon_1 &\approx \varepsilon_1^{(0)} + \left.\frac{d\varepsilon_1}{dX^2}\right|_{X^2=0} X^2 + \frac{1}{2}\left.\frac{d^2\varepsilon_1}{d(X^2)^2}\right|_{X^2=0} (X^2)^2 + \cdots \\ &= \varepsilon_1^{(0)} + \frac{(\delta_1^I + u_{0,1}^I)\varphi_{1,1}^{(I)}}{1+\varepsilon_1^{(0)}} X^2 + \frac{1}{2}\left\{\frac{[2(\delta_1^I + u_{0,1}^I)\varphi_{2,1}^{(I)}] + \varphi_{1,1}^{(I)}\varphi_{1,1}^{(I)}}{1+\varepsilon_1^{(0)}} \right. \\ &\quad \left. - \frac{[(\delta_1^I + u_{0,1}^I)\varphi_{1,1}^{(I)}]^2}{(1+\varepsilon_1^{(0)})^3}\right\}(X^2)^2 + \cdots,\end{aligned} \tag{8.48}$$

$$\begin{aligned}\varepsilon_2 &\approx \varepsilon_2^{(0)} + \left.\frac{d\varepsilon_2}{dX^2}\right|_{X^2=0} X^2 + \frac{1}{2}\left.\frac{d^2\varepsilon_2}{d(X^2)^2}\right|_{X^2=0} (X^2)^2 + \cdots \\ &= \varepsilon_2^{(0)} + \frac{2(\delta_2^I + \varphi_1^{(I)})\varphi_2^{(I)}}{1+\varepsilon_2^{(0)}} X^2 + \left\{\frac{[2\varphi_2^{(I)}\varphi_2^{(I)} + 3(\delta_2^I + \varphi_1^{(I)})\varphi_3^{(I)}]}{1+\varepsilon_2^{(0)}} \right. \\ &\quad \left. - \frac{2[(\delta_2^I + \varphi_1^{(I)})\varphi_2^{(I)}]^2}{(1+\varepsilon_2^{(0)})^3}\right\}(X^2)^2 + \cdots;\end{aligned} \tag{8.49}$$

$$\begin{aligned}\gamma_{12} &\approx \gamma_{12}^{(0)} + \left.\frac{d\gamma_{12}}{dX^2}\right|_{X^2=0} X^2 + \frac{1}{2}\left.\frac{d^2\gamma_{12}}{d(X^2)^2}\right|_{X^2=0} (X^2)^2 + \cdots \\ &= \gamma_{12}^{(0)} + \frac{1}{\cos\gamma_{12}^{(0)}}\left\{\frac{2(\delta_1^I + u_{0,1}^{(I)})\varphi_2^{(I)} + (\delta_2^I + \varphi_1^{(I)})\varphi_{1,1}^{(I)}}{(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})} \right. \\ &\quad \left. - \sin\gamma_{12}^0\left[\frac{(\delta_1^I + u_{0,1}^{(I)})\varphi_{1,1}^{(I)}}{(1+\varepsilon_1^{(0)})^2} + \frac{2(\delta_2^I + \varphi_1^{(I)})\varphi_2^{(I)}}{(1+\varepsilon_2^{(0)})^2}\right]\right\} X^2 + \cdots,\end{aligned} \tag{8.50}$$

where for $I = 1, 2$,

$$\begin{aligned}\varepsilon_1^{(0)} &= \sqrt{(\delta_1^I + u_{0,1}^I)(\delta_1^I + u_{0,1}^I)} - 1 \\ &= \sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^2)^2} - 1,\end{aligned} \tag{8.51}$$

$$\begin{aligned}\varepsilon_2^{(0)} &= \sqrt{(\delta_2^I + \varphi_1^{(I)})(\delta_2^I + \varphi_1^{(I)})} - 1 \\ &= \sqrt{(\varphi_1^{(1)})^2 + (1+\varphi_1^2)^2} - 1,\end{aligned} \tag{8.52}$$

$$\begin{aligned}\gamma_{12}^{(0)} &= \sin^{-1}\frac{(\delta_1^I + u_{0,1}^I)(\delta_2^I + \varphi_1^{(I)})}{(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})} \\ &= \sin^{-1}\frac{(1+u_{0,1}^1)\varphi_1^{(1)} + u_{0,1}^2(1+\varphi_1^{(2)})}{(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})}.\end{aligned} \tag{8.53}$$

The constitutive laws give the stresses on the deformed configuration as

$$\sigma_1 = f(\varepsilon_1, t) \text{ and } \sigma_{12} = g(\gamma_{12}, t). \tag{8.54}$$

The internal forces and moments in the deformed beam are defined as

$$\begin{aligned} N_1 &= \int_A \sigma_1 \left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23}\right] dA, \\ M_3 &= \int_A \sigma_1 \frac{X^2}{1+\varphi_1^{(2)}}\left[(1+\varepsilon_3)(1+\varepsilon_2)^2 \cos\gamma_{23}\right] dA, \\ Q_1 &= \int_A \sigma_{12}\left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23}\right] dA. \end{aligned} \tag{8.55}$$

For convenience, the subscripts of the internal forces can be dropped. The internal force vectors can be defined as

$$\begin{aligned} \mathbf{M} &\equiv M^3 \mathbf{I}_3 = M\mathbf{n}_3, \\ \boldsymbol{\mathcal{N}} &\equiv N^I \mathbf{I}_I = N\mathbf{n}_1 + Q\mathbf{n}_2, \\ {}^{\mathrm{N}}\mathbf{M} &\equiv {}^{\mathrm{N}}M^I \mathbf{I}_I = \mathbf{g}_1 \times \boldsymbol{\mathcal{N}}, \end{aligned} \tag{8.56}$$

where

$$\mathbf{g}_1 \equiv \frac{d\mathbf{r}}{dS} = (\delta_1^I + u_{0,1}^I)\mathbf{I}_I \text{ and } {}^{\mathrm{N}}\mathbf{M} \equiv \frac{1}{dS} d\mathbf{r} \times \boldsymbol{\mathcal{N}}. \tag{8.57}$$

The components of the internal forces in the $\mathbf{I}_I$-direction are

$$\begin{aligned} N^I &= N\mathbf{n}_1 \cdot \mathbf{I}_I + Q\mathbf{n}_2 \cdot \mathbf{I}_I = N\cos\theta_{(\mathbf{n}_1, \mathbf{I}_I)} + Q\cos\theta_{(\mathbf{n}_2, \mathbf{I}_I)} \\ &= \frac{N(\delta_1^I + u_{0,1}^I)}{1+\varepsilon_1^{(0)}} + \frac{Q(\delta_2^I + \varphi_1^I)}{1+\varepsilon_2^{(0)}}, \end{aligned} \tag{8.58}$$

$$\begin{aligned} M^I &= M\mathbf{n}_3 \cdot \mathbf{I}_I = M\cos\theta_{(\mathbf{n}_3, \mathbf{I}_I)} = \frac{M(\delta_3^I + u_{0,3}^I)}{1+\varepsilon_3^{(0)}}, \\ {}^{\mathrm{N}}M^3 &= (\mathbf{g}_1 \times \boldsymbol{\mathcal{N}}) \cdot \mathbf{I}_3 = (1+u_{0,1}^1)N^2 - u_{0,1}^2 N^1. \end{aligned} \tag{8.59}$$

Because of $u^I_{0,3} = 0$ and $\varepsilon^{(0)}_3 = 0$, one obtains

$$\begin{aligned} {}^{\mathrm{N}}M^3 &= Q(1+\varepsilon^{(0)}_1),\ {}^{\mathrm{N}}M^1 = {}^{\mathrm{N}}M^2 = 0, \\ M^1 &= M^2 = 0 \text{ and } M^3 = M. \end{aligned} \tag{8.60}$$

Equations of motion on the deformed beam are given by

$$\begin{aligned} \boldsymbol{\mathcal{N}}_{,1} + \mathbf{q} &= \rho \mathbf{u}_{0,tt} + I_3 \boldsymbol{\varphi}_{1,tt}, \\ \mathbf{M}_{,1} + {}^{\mathrm{N}}\mathbf{M} + \mathbf{m} &= I_3 \mathbf{u}_{0,tt} + J_3 \boldsymbol{\varphi}_{1,tt}; \end{aligned} \tag{8.61}$$

and the corresponding scalar expressions are for $I = 1,2$,

$$\begin{aligned} N^I_{,1} + q^I &= \rho u^I_{(0),tt} + I_3 \varphi^{(I)}_{1,tt}, \\ M^3_{,1} + {}^{\mathrm{N}}M^3 + m^3 &= I_3 u^1_{(0),tt} + J_3 \varphi^{(1)}_{1,tt}; \end{aligned} \tag{8.62}$$

or

$$\begin{aligned} &\left[\frac{N(1+u^1_{0,1})}{1+\varepsilon^{(0)}_1} - \frac{Q u^2_{0,1}}{1+\varepsilon^{(0)}_1} \right]_{,1} + q^1 = \rho u^1_{(0),tt} + I_3 \varphi^{(1)}_{1,tt}; \\ &\left[\frac{N u^2_{0,1}}{1+\varepsilon^{(0)}_1} + \frac{Q(1+u^1_{0,1})}{1+\varepsilon^{(0)}_1} \right]_{,1} + q^2 = \rho u^2_{(0),tt} + I_3 \varphi^{(2)}_{1,tt}; \\ &M_{,1} + Q(1+\varepsilon^{(0)}_1) + m^3 = I_3 u^1_{(0),tt} + J_3 \varphi^{(1)}_{1,tt}, \end{aligned} \tag{8.63}$$

where $\rho = \int_A \rho_0 dA$, $I_3 = \int_A \rho_0 X^2 dA$ and $J_3 = \int_A \rho_0 (X^2)^2 dA$.

The force condition at a point $\mathscr{P}_k$ with $X^1 = X^1_k$ is

$$\begin{aligned} {}^{-}\mathbf{N}(X^1_k) + {}^{+}\mathbf{N}(X^1_k) + \mathbf{F}_k &= 0, \\ {}^{-}N^I(X^1_k) = {}^{+}N^I(X^1_k) + F^I_k \quad (I &= 1,2). \end{aligned} \tag{8.64}$$

The force boundary condition at the boundary point $\mathscr{P}_r$ is

$$\mathbf{N}(X^1_r) + \mathbf{F}_r = 0 \ \text{ or } \ N^I(X^1_r) + F^I_r = 0 \quad (I = 1,2). \tag{8.65}$$

If there is a concentrated moment at a point $\mathscr{P}_k$ with $X^1 = X^1_k$, the corresponding moment boundary condition is

$$\begin{aligned} {}^{-}\mathbf{M}(X^1_k) + {}^{+}\mathbf{M}(X^1_k) + \mathbf{M}_k &= 0, \\ {}^{-}M^3(X^1_k) = {}^{+}M^3(X^1_k) + M^3_k &. \end{aligned} \tag{8.66}$$

The moment boundary condition at the boundary point $\mathscr{P}_k$ is

$$\mathbf{M}(X^1_r) + \mathbf{M}_r = 0 \ \text{ or } \ M^3(X^1_r) + M^3_r = 0. \tag{8.67}$$

The displacement continuity and boundary conditions are

$$u^I_{k-} = u^I_{k+} \text{ and } u^I_r = B^I_r . \tag{8.68}$$

The afore-developed beam theory can be reduced to the beam theory given by Reissner (1972). The nonlinear vibration and chaos of a beam were extensively investigated (e.g., Verma, 1972; Luo and Han, 1999).

8.3. Nonlinear curved beams

Consider an arbitrary coordinate system as (X^1, Y, Z) at the *centroid* on the cross section of the beam. The central curve of the deformed beam is on the plane of (X^1, X^2), as shown in Fig.8.3. In other words, the neutral surface of the deformed beam is on the plane of (X^1, X^3). Let the coordinate S^1 be along the longitudinal direction of beam and the other two coordinates S^2 and S^3 be on the cross section of beam with the direction of S^1. The coordinates for the deformed, curved beam are (s^1, s^2, s^3). Because the initial configuration of the beam is a curved beam, under external force, the deformed configuration of the beam to the initial configuration does not experience any torsion ($\tau(s^1) \equiv 0$). In other words, under the resultant forces, the bending of beam is in the plane of (S^1, S^2). Thus, the configuration of the deformed beam is still a plane curve. Without loss of generality, the curvature direction of the deformed configuration can be assumed to be collinear to S^2. Because the widths of beam in two directions of S^2 and S^3 are very small compared to the length of the beam in direction of S^1, the elongation in the two directions of S^2 and S^3 should be very small, which can be neglected. Thus, as in the straight beam, the following assumptions are enforced.

(i) The configuration of the deformed beam to the initial curved beam does not experience any torsion ($\tau(s^1) \equiv 0$).
(ii) The curvature direction of the deformed beam is collinear to s^2.
(iii) The elongations in the two directions of S^2 and S^3 are very small (i.e., $\varepsilon_2 \approx 0$ and $\varepsilon_3 = 0$).
(iv) For bending only, the neutral surface is not deformed (i.e., $\gamma_{12} \approx 0$ and $\gamma_{13} = 0$).

From Assumption (i), no torque exists, and the transversal external forces should be added at *the shear center*. Similar to Eqs.(8.36) and (8.37), the external distributed forces and moments on the initial configuration are for ($I, \Lambda = 1, 2, 3$)

$$\mathbf{q} = q^I \mathbf{I}_I = q^\Lambda \mathbf{N}_\Lambda \text{ and } \mathbf{m} = m^I \mathbf{I}_I = m^\Lambda \mathbf{N}_\Lambda \tag{8.69}$$

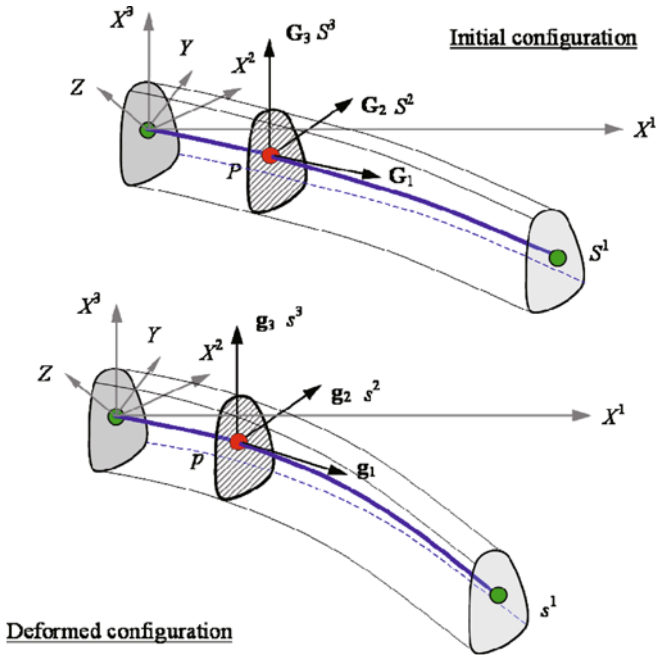

Fig. 8.3 A curved beam with initial and deformed configuration.

and concentrated forces on the initial configuration at a point $S^1 = S_k$,

$$\mathbf{F}_k = F_k^I \mathbf{I}_I = F_k^\Lambda \mathbf{N}_\Lambda \text{ and } \mathbf{M}_k = M_k^I \mathbf{I}_I = M_k^\Lambda \mathbf{N}_\Lambda. \tag{8.70}$$

Thus,

$$\begin{aligned} F^\Lambda |_{S^1=S} &= F^I \mathbf{I}_I \cdot \mathbf{N}_\Lambda = F^I \cos\theta_{(\mathbf{I}_I,\mathbf{N}_\Lambda)}, \\ M^\Lambda |_{S^1=S} &= M^I \mathbf{I}_I \cdot \mathbf{N}_\Lambda = M^I \cos\theta_{(\mathbf{I}_I,\mathbf{N}_\Lambda)}. \end{aligned} \tag{8.71}$$

The displacement vectors on the initial configuration are

$$\mathbf{R}(S^1) = X^I(S^1)\mathbf{I}_I,\ \mathbf{R}(S) = X_S^I \mathbf{I}_I \text{ and } \mathbf{R}_k = X_k^I \mathbf{I}_I. \tag{8.72}$$

The internal forces and moments for $(S^1 > S_k^1)$ are

$$\begin{aligned} \mathbf{F}|_{S^1=S} &= \sum_{k=1} \mathbf{F}_k + \int_0^S \mathbf{q} dS^1, \\ \mathbf{M}|_{S^1=S} &= \sum_{k=1} \mathbf{M}_k + \int_0^S \mathbf{m} dS^1 \\ &\quad + \sum_{k=1} (\mathbf{R}(S) - \mathbf{R}_k) \times {}^k\mathbf{F} + \int_0^S (\mathbf{R}(S) - \mathbf{R}(S^1)) \times \mathbf{q} dS^1; \end{aligned} \tag{8.73}$$

or for $I = 1, 2, 3$,

$$
\begin{aligned}
F^I \,|_{S^1=S} &= \sum\nolimits_{k=1} {}^kF^I + \int_0^S q^I dS^1, \\
M^I \,|_{S^1=S} &= \sum\nolimits_{k=1} M_k^I + \int_0^S m^I dS^1 \\
&\quad + \sum\nolimits_{k=1} e_{IJK}(X_S^J - X_k^J)F_k^K + \int_0^S e_{IJK}(X_S^J - X_k^J)q^K dS^1.
\end{aligned} \tag{8.74}
$$

Assumptions (i) and (ii) requires the following conditions:

$$
F^\Lambda \,|_{S^1=S} = 0 \;\text{ for } \Lambda=3 \;\text{ and }\; M^\Lambda \,|_{S^1=S} = 0 \;\text{ for } \Lambda=1,2. \tag{8.75}
$$

Since the vectors $\mathbf{I}_3$ and $\mathbf{N}_3$ ($\mathbf{G}_3$) are collinear and all points on the beam satisfy Eq.(8.75), one obtains

$$
\begin{aligned}
&{}^kF^I = 0 \;\text{ and }\; q^I = 0 \;\text{ for } I = 3, \\
&M^1 \cos\theta_{(\mathbf{I}_1,\mathbf{N}_\Lambda)} + M^2 \cos\theta_{(\mathbf{I}_2,\mathbf{N}_\Lambda)} = 0 \text{ for } \Lambda = 1, 2.
\end{aligned} \tag{8.76}
$$

Because

$$
\begin{vmatrix} \cos\theta_{(\mathbf{I}_1,\mathbf{N}_1)} & \cos\theta_{(\mathbf{I}_2,\mathbf{N}_1)} \\ \cos\theta_{(\mathbf{I}_1,\mathbf{N}_2)} & \cos\theta_{(\mathbf{I}_2,\mathbf{N}_2)} \end{vmatrix} \neq 0, \tag{8.77}
$$

the second equation of Eq.(8.76) gives

$$
M^1 = M^2 = 0. \tag{8.78}
$$

Thus, the external force conditions for the curved beam without twisting are given by the first equation of Eqs.(8.76) and (8.78). In other words, no external distributed and concentrated forces are in the direction of X^3 and the resultant external moments in the directions of X^1 and X^3 are zero.

8.3.1. A nonlinear theory based on the Cartesian coordinates

The strain based on the change in length of $d\mathbf{R}$ per unit length for a curved beam in the Cartesian coordinates gives

$$
\begin{aligned}
\varepsilon_\alpha &= \frac{|\underset{\alpha}{d\mathbf{r}}| - |\underset{\alpha}{d\mathbf{R}}|}{|\underset{\alpha}{d\mathbf{R}}|} = \frac{1}{\sqrt{G_{\alpha\alpha}}}\sqrt{G_{\alpha\alpha} + 2E_{\alpha\alpha}} - 1 \\
&= \frac{1}{\sqrt{G_{\alpha\alpha}}}\sqrt{(X_{,\alpha}^I + u_{,\alpha}^I)(X_{,\alpha}^I + u_{,\alpha}^I)} - 1
\end{aligned} \tag{8.79}
$$

in which no summation on α can be completed. The Lagrangian strain tensor $E_{\alpha\beta}$ referred to the initial configuration is

$$E_{\alpha\beta}=\frac{1}{2}(X^I_{,\alpha}u^I_{,\beta}+X^I_{,\beta}u^I_{,\alpha}+u^I_{,\alpha}u^I_{,\beta}). \tag{8.80}$$

In the similar fashion, the angles between $\mathbf{n}_\alpha$ and $\mathbf{n}_\beta$ *before* and *after* deformation are expressed by $\Theta_{(\mathbf{N}_\alpha,\mathbf{N}_\beta)}=\pi/2$ and $\theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)}$, i.e.,

$$\begin{aligned}\cos\theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)}&=\frac{d\underset{\alpha}{\mathbf{r}}\cdot d\underset{\beta}{\mathbf{r}}}{|d\underset{\alpha}{\mathbf{r}}||d\underset{\beta}{\mathbf{r}}|}\\&=\frac{(X^I_{,\alpha}+u^I_{,\alpha})(X^I_{,\beta}+u^I_{,\beta})}{\sqrt{(G_{\alpha\alpha}+2E_{\alpha\alpha})(G_{\beta\beta}+2E_{\beta\beta})}}\\&=\frac{(X^I_{,\alpha}+u^I_{,\alpha})(X^I_{,\beta}+u^I_{,\beta})}{\sqrt{G_{\alpha\alpha}G_{\beta\beta}}(1+\varepsilon_\alpha)(1+\varepsilon_\beta)},\end{aligned} \tag{8.81}$$

and the corresponding shear strain is defined as

$$\begin{aligned}\gamma_{\alpha\beta}&\equiv\Theta_{(\mathbf{N}_\alpha,\mathbf{N}_\beta)}-\theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)}\\&=\sin^{-1}\frac{(X^I_{,\alpha}+u^I_{,\alpha})(X^I_{,\beta}+u^I_{,\beta})}{\sqrt{(G_{\alpha\alpha}+2E_{\alpha\alpha})(G_{\beta\beta}+2E_{\beta\beta})}}\\&=\sin^{-1}\frac{(X^I_{,\alpha}+u^I_{,\alpha})(X^I_{,\beta}+u^I_{,\beta})}{\sqrt{G_{\alpha\alpha}G_{\beta\beta}}(1+\varepsilon_\alpha)(1+\varepsilon_\beta)}.\end{aligned} \tag{8.82}$$

From Eqs.(8.24) and (8.28), the direction cosine of the rotation without summation on α and β is

$$\begin{aligned}\cos\theta_{(\mathbf{N}_\alpha,\mathbf{n}_\beta)}&=\frac{d\underset{\alpha}{\mathbf{R}}\cdot d\underset{\beta}{\mathbf{r}}}{|d\underset{\alpha}{\mathbf{R}}||d\underset{\beta}{\mathbf{r}}|}=\frac{X^I_{,\alpha}(X^I_{,\beta}+u^I_{,\beta})}{\sqrt{G_{\alpha\alpha}}\sqrt{G_{\beta\beta}+2E_{\beta\beta}}}\\&=\frac{X^I_{,\alpha}(X^I_{,\beta}+u^I_{,\beta})}{\sqrt{G_{\alpha\alpha}G_{\beta\beta}}(1+\varepsilon_\beta)}.\end{aligned} \tag{8.83}$$

Finally, the change ratio of areas before and after deformation is

$$\frac{d\underset{\alpha\beta}{a}}{d\underset{\alpha\beta}{A}}=(1+\varepsilon_\alpha)(1+\varepsilon_\beta)\sin\theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)}, \tag{8.84}$$

where $d\underset{\alpha\beta}{a}=|d\underset{\alpha}{\mathbf{r}}\times d\underset{\beta}{\mathbf{r}}|$ and $d\underset{\alpha\beta}{A}=|d\underset{\alpha}{\mathbf{R}}\times d\underset{\beta}{\mathbf{R}}|$.

From assumption (ii),

$$u^I = u_0^I(S,t) + \sum_{n=1}^{\infty} (S^2)^n \varphi_n^{(I)}(S,t) \text{ for } I=1, 2. \tag{8.85}$$

From Assumption (ii), no displacements exist in the direction of S^3 (i.e., $u^3 = 0$). From the Kirchhoff's assumptions, under bending only, if the cross section is normal to the neutral surface before deformation, then after deformation, the deformed cross section is still normal to the deformed neutral surface. Thus, equation (8.85) can be assumed as

$$u^I = u_0^I(S,t) + S^2 \varphi^{(I)}(S,t) \quad (\alpha = 1, 2). \tag{8.86}$$

From assumptions (iii) and (iv),

$$\begin{gathered} \frac{1}{G_{22}}(X_{,2}^I + u_{,2}^I)(X_{,2}^I + u_{,2}^I) = 1, \\ (X_{,1}^I + u_{,1}^I)(X_{,2}^I + u_{,2}^I) = 0. \end{gathered} \tag{8.87}$$

With $u^3 = 0$ and Eq.(8.85), the Taylor series expansion of Eq.(8.87) give for the zero-order of S^2,

$$\begin{gathered} \frac{1}{G_{22}}(X_{,2}^I + \varphi_1^{(I)})(X_{,2}^I + \varphi_1^{(I)}) = 1, \\ (X_{,1}^I + u_{0,1}^I)(X_{,2}^I + \varphi_1^{(I)}) = 0, \end{gathered} \tag{8.88}$$

where $X_{,2}^I = G_2^I$. From Eq.(8.88),

$$\begin{aligned} \varphi_1^{(1)} &= \mp \frac{(X_{,1}^2 + u_{0,1}^2)\sqrt{G_{22}}}{\sqrt{(X_{,1}^1 + u_{0,1}^1)^2 + (X_{,1}^2 + u_{0,1}^2)^2}} - X_{,2}^1, \\ \varphi_1^{(2)} &= \pm \frac{(X_{,1}^1 + u_{0,1}^1)\sqrt{G_{22}}}{\sqrt{(X_{,1}^1 + u_{0,1}^1)^2 + (X_{,1}^2 + u_{0,1}^2)^2}} - X_{,2}^2. \end{aligned} \tag{8.89}$$

Similarly, one can obtain $\varphi_n^{(I)}$ ($n = 1, 2, \ldots$ and $I = 1, 2$). Using the Taylor series expansion gives the approximate strains, i.e.,

$$\begin{aligned} \varepsilon_1 &\approx \varepsilon_1^{(0)} + \left.\frac{d\varepsilon_1}{dS^2}\right|_{S^2=0} S^2 + \frac{1}{2}\left.\frac{d^2\varepsilon_1}{d(S^2)^2}\right|_{S^2=0} (S^2)^2 + \cdots \\ &= \varepsilon_1^{(0)} + \frac{(X_{,1}^I + u_{0,1}^I)\varphi_{1,1}^{(I)}}{G_{11}(1+\varepsilon_1^{(0)})} S^2 \\ &\quad + \frac{1}{2}\frac{[2(X_{,1}^I + u_{0,1}^I)\varphi_{2,1}^{(I)}] + \varphi_{1,1}^{(I)}\varphi_{1,1}^{(I)}}{G_{11}(1+\varepsilon_1^{(0)})}(S^2)^2 \\ &\quad - \frac{1}{2}\frac{[(X_{,1}^I + u_{0,1}^I)\varphi_{1,1}^{(I)}]^2}{G_{11}^2(1+\varepsilon_1^{(0)})^3}(S^2)^2 + \cdots, \end{aligned} \tag{8.90}$$

$$
\begin{aligned}
\varepsilon_2 &\approx \varepsilon_2^{(0)} + \left.\frac{d\varepsilon_2}{dS^2}\right|_{S^2=0} S^2 + \frac{1}{2}\left.\frac{d^2\varepsilon_2}{d(S^2)^2}\right|_{S^2=0} (S^2)^2 + \cdots \\
&= \varepsilon_2^{(0)} + \frac{2(X^I_{,2} + \varphi_1^{(I)})\varphi_2^{(I)}}{G_{22}(1+\varepsilon_2^{(0)})} S^2 \\
&\quad + \frac{[2\varphi_2^{(I)}\varphi_2^{(I)} + 3(\delta_2^I + \varphi_1^{(I)})\varphi_3^{(I)}]}{G_{22}(1+\varepsilon_2^{(0)})} (S^2)^2 \\
&\quad - \frac{2[(X^I_{,2} + \varphi_1^{(I)})\varphi_2^{(I)}]^2}{G_{22}^2(1+\varepsilon_2^{(0)})^3} (S^2)^2 + \cdots;
\end{aligned} \tag{8.91}
$$

$$
\begin{aligned}
\gamma_{12} &\approx \gamma_{12}^{(0)} + \left.\frac{d\gamma_{12}}{dS^2}\right|_{S^2=0} S^2 + \frac{1}{2}\left.\frac{d^2\gamma_{12}}{d(S^2)^2}\right|_{S^2=0} (S^2)^2 + \cdots \\
&= \gamma_{12}^{(0)} + \frac{1}{\cos\gamma_{12}^{(0)}} \left\{ \frac{2(X^I_{,1} + u_{0,1}^{(I)})\varphi_2^{(I)} + (X^I_{,2} + \varphi_1^{(I)})\varphi_{1,1}^{(I)}}{\sqrt{G_{11}}\sqrt{G_{22}}(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})} \right. \\
&\quad \left. - \sin\gamma_{12}^{(0)} \left[\frac{(X^I_{,1} + u_{0,1}^{(I)})\varphi_{1,1}^{(I)}}{G_{11}(1+\varepsilon_1^{(0)})^2} + \frac{2(X^I_{,2} + \varphi_1^{(I)})\varphi_2^{(I)}}{G_{22}(1+\varepsilon_2^{(0)})^2} \right] \right\} S^2 + \cdots,
\end{aligned} \tag{8.92}
$$

where for $I = 1, 2$

$$
\begin{aligned}
\varepsilon_1^{(0)} &= \frac{1}{\sqrt{G_{11}}}\sqrt{(X^I_{,1} + u^I_{0,1})(X^I_{,1} + u^I_{0,1})} - 1 \\
&= \frac{1}{\sqrt{G_{11}}}\sqrt{(X^1_{,1} + u^1_{0,1})^2 + (X^2_{,1} + u^2_{0,1})^2} - 1,
\end{aligned} \tag{8.93}
$$

$$
\begin{aligned}
\varepsilon_2^{(0)} &= \frac{1}{\sqrt{G_{22}}}\sqrt{(X^I_{,2} + \varphi_1^{(I)})(X^I_{,2} + \varphi_1^{(I)})} - 1 \\
&= \frac{1}{\sqrt{G_{22}}}\sqrt{(X^1_{,2} + \varphi_1^{(1)})^2 + (X^2_{,2} + \varphi_1^{(2)})^2} - 1,
\end{aligned} \tag{8.94}
$$

$$
\begin{aligned}
\gamma_{12}^{(0)} &= \sin^{-1}\frac{(X^I_{,1} + u^I_{0,1})(X^I_{,2} + \varphi_1^{(I)})}{\sqrt{G_{11}G_{22}}(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})} \\
&= \sin^{-1}\frac{(X^1_{,1} + u^1_{,1})(X^1_{,2} + \varphi_1^{(1)}) + (X^2_{,1} + u^2_{,1})(X^2_{,2} + \varphi_1^{(2)})}{\sqrt{G_{11}G_{22}}(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})}.
\end{aligned} \tag{8.95}
$$

The constitutive laws give the stresses on the deformed configuration, i.e.,

$$
\sigma_1 = f(\varepsilon_1, \gamma_{12}, t) \text{ and } \sigma_{12} = g(\varepsilon_1, \gamma_{12}, t). \tag{8.96}
$$

The internal forces and moments in the deformed beam are defined as

$$\begin{aligned} N_1 &= \int_A \sigma_1 \left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23}\right] dA, \\ M_3 &= \int_A \frac{\sigma_1 S^2}{1+\varphi_1^{(2)}}\left[\sqrt{G_{22}}(1+\varepsilon_3)(1+\varepsilon_2)^2 \cos\gamma_{23}\right] dA, \\ Q_1 &= \int_A \sigma_{12}\left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23}\right] dA. \end{aligned} \tag{8.97}$$

For convenience, the subscripts of the internal forces can be dropped. The internal force vectors are defined as

$$\begin{aligned} \mathbf{M} &\equiv M^I \mathbf{I}_I = M\mathbf{n}_3, \\ \boldsymbol{\mathcal{N}} &\equiv N^I \mathbf{I}_I = N\mathbf{n}_1 + Q\mathbf{n}_2, \\ {}^{\mathrm{N}}\mathbf{M} &\equiv {}^{\mathrm{N}}M^I \mathbf{I}_I = \mathbf{g}_1 \times \boldsymbol{\mathcal{N}}; \end{aligned} \tag{8.98}$$

$$\mathbf{g}_1 \equiv \frac{d\mathbf{r}}{dS} = (X_{,1}^I + u_{0,1}^I)\mathbf{I}_I \text{ and } {}^{\mathrm{N}}\mathbf{M} \equiv \frac{1}{dS} d\mathbf{r} \times \boldsymbol{\mathcal{N}}. \tag{8.99}$$

The components of the internal forces in the $\mathbf{I}_I$ -direction are

$$\begin{aligned} N^I &= N\mathbf{n}_1 \cdot \mathbf{I}_I + Q\mathbf{n}_2 \cdot \mathbf{I}_I = N\cos\theta_{(\mathbf{n}_1,\mathbf{I}_I)} + Q\cos\theta_{(\mathbf{n}_2,\mathbf{I}_I)} \\ &= \frac{N(X_{,1}^I + u_{0,1}^I)}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})} + \frac{Q(X_{,2}^I + \varphi_1^I)}{\sqrt{G_{22}}(1+\varepsilon_2^{(0)})}, \end{aligned} \tag{8.100}$$

$$\begin{aligned} M^I &= M\mathbf{n}_3 \cdot \mathbf{I}_I = M\cos\theta_{(\mathbf{n}_3,\mathbf{I}_I)} = \frac{M(X_{,3}^I + u_{(0),3}^I)}{1+\varepsilon_3^{(0)}}, \\ {}^{\mathrm{N}}M^1 &= (\mathbf{g}_1 \times \boldsymbol{\mathcal{N}}) \cdot \mathbf{I}_1 = (X_{,1}^2 + u_{0,1}^2)N^3 - (X_{,1}^3 + u_{0,1}^3)N^2, \\ {}^{\mathrm{N}}M^2 &= (\mathbf{g}_1 \times \boldsymbol{\mathcal{N}}) \cdot \mathbf{I}_2 = (X_{,1}^3 + u_{0,1}^3)N^1 - (X_{,1}^1 + u_{0,1}^1)N^3, \\ {}^{\mathrm{N}}M^3 &= (\mathbf{g}_1 \times \boldsymbol{\mathcal{N}}) \cdot \mathbf{I}_3 = (X_{,1}^1 + u_{0,1}^1)N^2 - (X_{,1}^2 + u_{0,1}^2)N^1. \end{aligned} \tag{8.101}$$

Due to $u_{0,3}^I = 0,\ X_{,\alpha}^3 = 0,\ X_{,3}^1 = X_{,3}^2 = 0,\ u^3 = 0$ and $\varepsilon_3^{(0)} = 0$, the following equations are achieved, i.e.,

$$\begin{aligned} &{}^{\mathrm{N}}M^3 = Q\frac{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})}{\sqrt{G_{22}}(1+\varepsilon_2^{(0)})}, \text{ and } {}^{\mathrm{N}}M^1 = {}^{\mathrm{N}}M^2 = 0. \\ &M^1 = M^2 = 0 \text{ and } M^3 = M. \end{aligned} \tag{8.102}$$

Based on the deformed middle surface in the Lagrangian coordinates, the equations of motion for the deformed beam are

$$\begin{aligned} \boldsymbol{\mathcal{N}}_{,1} + \mathbf{q} &= \rho\mathbf{u}_{0,tt} + I_3\boldsymbol{\varphi}_{1,tt}, \\ \mathbf{M}_{,1} + {}^{\mathrm{N}}\mathbf{M} + \mathbf{m} &= I_3\mathbf{u}_{0,tt} + J_3\boldsymbol{\varphi}_{1,tt}; \end{aligned} \tag{8.103}$$

and the scalar expressions are for $I = 1, 2$

$$\begin{aligned} N^I_{,1} + q^I &= \rho u^I_{0,tt} + I_3 \varphi^{(I)}_{1,tt}, \\ M^3_{,1} + {}^{\mathrm{N}}M^3 + m^3 &= I_3 u^1_{0,tt} + J_3 \varphi^{(1)}_{1,tt} \end{aligned} \tag{8.104}$$

where $\rho = \int_A \rho_0 dA$, $I_3 = \int_A \rho_0 X^2 dA$ and $J_3 = \int_A \rho_0 (X^2)^2 dA$. With Eqs.(8.89) and (8.100)–(8.102), the foregoing equation gives

$$\begin{aligned} &\left[\frac{N(X^1_{,1} + u^1_{0,1})}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)} - \frac{Q(X^2_{,1} + u^2_{0,1})}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)} \right]_{,1} + q^1 = \rho u^1_{0,tt} + I_3 \varphi^{(1)}_{1,tt}, \\ &\left[\frac{N(X^2_{,1} + u^2_{0,1})}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)} + \frac{Q(X^1_{,1} + u^1_{0,1})}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)} \right]_{,1} + q^2 = \rho u^2_{0,tt} + I_3 \varphi^{(2)}_{1,tt}, \end{aligned} \tag{8.105}$$

$$M_{,1} + Q \frac{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)}{\sqrt{G_{22}}} + m^3 = I_3 u^1_{0,tt} + J_3 \varphi^{(1)}_{1,tt}. \tag{8.106}$$

As in Eqs.(8.61)–(8.65), the force and displacement continuity and boundary conditions can be given as follows.

The force condition at a point $\mathscr{P}_k$ with $S^1 = S^1_k$ is

$$\begin{aligned} &{}^{-}\mathbf{N}(S^1_k) + {}^{+}\mathbf{N}(S^1_k) + \mathbf{F}_k = 0, \\ &{}^{-}N^I(S^1_k) = {}^{+}N^I(S^1_k) + F^I_k \quad (I = 1, 2). \end{aligned} \tag{8.107}$$

The force boundary condition at the boundary point $\mathscr{P}_r$ is

$$\mathbf{N}(S^1_r) + \mathbf{F}_r = 0 \ \text{ or } \ N^I(S^1_r) + F^I_r = 0 \quad (I = 1, 2). \tag{8.108}$$

If there is a concentrated moment at a point $\mathscr{P}_k$ with $S^1 = S_k$, the corresponding moment boundary condition is

$$\begin{aligned} &{}^{-}\mathbf{M}(S^1_k) + {}^{+}\mathbf{M}(S^1_k) + \mathbf{M}_k = 0, \\ &{}^{-}M^I(S^1_k) = {}^{+}M^I(S^1_k) + M^I_k \quad (I = 3). \end{aligned} \tag{8.109}$$

The moment boundary condition at the boundary point $\mathscr{P}_k$ is

$$\mathbf{M}(S^1_r) + \mathbf{M}_r = 0 \ \text{or} \ M^I(S^1_r) + M^I_r = 0 \quad (I = 3). \tag{8.110}$$

The displacement continuity and boundary conditions are the same as in Eq.(8.68). From the sign convention, the positive "+" in the second equation of Eq.(8.89) was adopted.

8.3.2. A nonlinear theory based on the curvilinear coordinates

The strain based on the change in length of $d\mathbf{R}$ per unit length gives

$$\varepsilon_\alpha = \frac{|d\underset{\alpha}{\mathbf{r}}| - |d\underset{\alpha}{\mathbf{R}}|}{|d\underset{\alpha}{\mathbf{R}}|} = \sqrt{1+\frac{2E_{\alpha\alpha}}{G_{\alpha\alpha}}} - 1 = \frac{1}{\sqrt{G_{\alpha\alpha}}}\sqrt{G_{\alpha\alpha}+2E_{\alpha\alpha}} - 1$$

$$= \frac{\sqrt{G_{\beta\beta}}}{\sqrt{G_{\alpha\alpha}}}\sqrt{(\delta_\alpha^\beta + u_{;\alpha}^\beta)(\delta_\alpha^\beta + u_{;\alpha}^\beta)} - 1, \tag{8.111}$$

where the Lagrangian strain tensor $E_{\alpha\beta}$ to the initial configuration is

$$E_{\alpha\beta} = \frac{1}{2}(u_{\alpha;\beta} + u_{\alpha;\beta} + u_{;\alpha}^\gamma u_{\gamma;\beta})$$

$$= \frac{1}{2}\left[(\delta_\alpha^\gamma + u_{;\alpha}^\gamma)(\delta_\beta^\delta + u_{;\beta}^\delta)G_{\gamma\delta} - G_{\alpha\beta}\right]. \tag{8.112}$$

Similarly, the angles between $\mathbf{n}_\alpha$ and $\mathbf{n}_\beta$ *before* and *after* deformation are expressed by $\Theta_{(\mathbf{N}_\alpha,\mathbf{N}_\beta)} = \pi/2$ and $\theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)}$, i.e.,

$$\cos\theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)} = \frac{d\underset{\alpha}{\mathbf{r}}\cdot d\underset{\beta}{\mathbf{r}}}{|d\underset{\alpha}{\mathbf{r}}||d\underset{\beta}{\mathbf{r}}|} = \frac{G_{\alpha\beta}+2E_{\alpha\beta}}{\sqrt{(G_{\alpha\alpha}+2E_{\alpha\alpha})(G_{\beta\beta}+2E_{\beta\beta})}}$$

$$= \frac{(\delta_\alpha^\gamma + u_{;\alpha}^\gamma)(\delta_\beta^\delta + u_{;\beta}^\delta)G_{\gamma\delta}}{\sqrt{G_{\alpha\alpha}G_{\beta\beta}}(1+\varepsilon_\alpha)(1+\varepsilon_\beta)}, \tag{8.113}$$

and the shear strain is

$$\gamma_{\alpha\beta} \equiv \Theta_{(\mathbf{N}_\alpha,\mathbf{N}_\beta)} - \theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)}$$

$$= \sin^{-1}\frac{G_{\alpha\beta}+2E_{\alpha\beta}}{\sqrt{(G_{\alpha\alpha}+2E_{\alpha\alpha})(G_{\beta\beta}+2E_{\beta\beta})}},$$

$$= \sin^{-1}\frac{(\delta_\alpha^\gamma + u_{;\alpha}^\gamma)(\delta_\beta^\delta + u_{;\beta}^\delta)G_{\gamma\delta}}{\sqrt{G_{\alpha\alpha}G_{\beta\beta}}(1+\varepsilon_\beta)(1+\varepsilon_\beta)}. \tag{8.114}$$

The direction cosine of the rotation without summation on α and β is

$$\cos\theta_{(\mathbf{N}_\alpha,\mathbf{n}_\beta)} = \frac{d\underset{\alpha}{\mathbf{R}}\cdot d\underset{\beta}{\mathbf{r}}}{|d\underset{\alpha}{\mathbf{R}}||d\underset{\beta}{\mathbf{r}}|} = \frac{G_{\alpha\beta}+u_{\alpha;\beta}}{\sqrt{G_{\alpha\alpha}}\sqrt{G_{\beta\beta}+2E_{\beta\beta}}}$$

$$= \frac{G_{\alpha\beta}+u_{\alpha;\beta}}{\sqrt{G_{\alpha\alpha}G_{\gamma\gamma}}\sqrt{(\delta_\beta^\gamma + u_{;\beta}^\gamma)(\delta_\beta^\gamma + u_{;\beta}^\gamma)}}$$

$$= \frac{G_{\alpha\beta} + u_{\alpha;\beta}}{\sqrt{G_{\alpha\alpha}G_{\beta\beta}}(1+\varepsilon_\beta)} = \frac{(\delta^\gamma_\beta + u^\gamma_{;\alpha})G_{\gamma\alpha}}{\sqrt{G_{\alpha\alpha}G_{\beta\beta}}(1+\varepsilon_\beta)}. \tag{8.115}$$

In addition, the change ratio of areas *before* and *after* deformation are given by

$$\frac{\underset{\alpha\beta}{da}}{\underset{\alpha\beta}{dA}} = (1+\varepsilon_\alpha)(1+\varepsilon_\beta)\sin\theta_{(\mathbf{n}_\alpha,\mathbf{n}_\beta)} \tag{8.116}$$

where $\underset{\alpha\beta}{da} = |\underset{\alpha}{d\mathbf{r}} \times \underset{\beta}{d\mathbf{r}}|$ and $\underset{\alpha\beta}{dA} = |\underset{\alpha}{d\mathbf{R}} \times \underset{\beta}{d\mathbf{R}}|$.

From assumption (ii),

$$u^\Lambda = u_0^\Lambda(S,t) + \sum\nolimits_{n=1}^{\infty}(S^2)^n \varphi_n^{(\Lambda)}(S,t) \quad \text{for } \Lambda=1,2. \tag{8.117}$$

No displacements exist in the direction of S^3 (i.e., $u^3 = 0$). From the Kirchhoff's assumptions, under bending only, if the cross section is normal to the neutral surface before deformation, then after deformation, the deformed cross section is still normal to the deformed neutral surface. Thus, equation (8.117) becomes

$$u^\Lambda = u_0^\Lambda(S,t) + S^2\varphi^{(\Lambda)}(S,t) \quad (\alpha = 1, 2). \tag{8.118}$$

From Assumptions (iii) and (iv),

$$\begin{aligned}(\delta_2^\Lambda + u_{;2}^\Lambda)(\delta_2^\Lambda + u_{;2}^\Lambda)G_{\Lambda\Lambda} &= G_{22},\\ (\delta_1^\Lambda + u_{;1}^\Lambda)(\delta_2^\Gamma + u_{;2}^\Gamma)G_{\Lambda\Gamma} &= 0.\end{aligned} \tag{8.119}$$

With $u^3 = 0$ and Eq.(8.117), the Taylor series expansion of Eq.(8.119) gives for the zero-order of S^2:

$$\begin{aligned}G_{\Lambda\Lambda}(\delta_2^{(\Lambda)} + \varphi_1^{(\Lambda)})(\delta_2^{(\Lambda)} + \varphi_1^{(\Lambda)}) &= G_{22},\\ (\delta_1^\Lambda + u_{0;1}^\Lambda)(\delta_2^\Gamma + \varphi_1^{(\Gamma)})G_{\Lambda\Gamma} &= 0.\end{aligned} \tag{8.120}$$

Form the foregoing equations,

$$\begin{aligned}\varphi_1^{(1)} &= \mp\frac{u_{0;1}^2 G_{22}}{\sqrt{G_{11}}\sqrt{(1+u_{0;1}^1)^2 G_{11} + (u_{0;1}^2)^2 G_{22}}},\\ \varphi_1^{(2)} &= \pm\frac{(1+u_{0;1}^1)\sqrt{G_{11}}}{\sqrt{(1+u_{0;1}^1)^2 G_{11} + (u_{0;1}^2)^2 G_{22}}} - 1.\end{aligned} \tag{8.121}$$

From the sign convention, the positive "+" in the second equation of Eq.(8.121) will be adopted. Similarly, one obtains $\varphi_n^{(\alpha)}$ ($n = 1, 2, \ldots$ and $\alpha = 1, 2$). The approximate strains for the curved beam in the curvilinear coordinates are:

$$\begin{aligned}\varepsilon_1 &\approx \varepsilon_1^{(0)} + \left.\frac{d\varepsilon_1}{dS^2}\right|_{S^2=0} S^2 + \frac{1}{2}\left.\frac{d^2\varepsilon_1}{d(S^2)^2}\right|_{S^2=0}(S^2)^2 + \cdots \\ &= \varepsilon_1^{(0)} + \frac{(\delta_1^\Lambda + u_{0;1}^\Lambda)\varphi_{1;1}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{11}(1+\varepsilon_1^{(0)})} S^2 \\ &\quad + \frac{1}{2}\left\{\frac{[2(\delta_1^\Lambda + u_{0;1}^\Lambda)\varphi_{2;1}^{(\Lambda)}] + \varphi_{1;1}^{(\Lambda)}\varphi_{1;1}^{(\Lambda)}}{G_{11}(1+\varepsilon_1^{(0)})} G_{\Lambda\Lambda}\right. \\ &\quad \left. - \frac{[(\delta_1^\Lambda + u_{0;1}^\Lambda)\varphi_{1;1}^{(\Lambda)}]^2 G_{\Lambda\Lambda}^2}{G_{11}^2(1+\varepsilon_1^{(0)})^3}\right\}(S^2)^2 + \cdots, \end{aligned} \tag{8.122}$$

$$\begin{aligned}\varepsilon_2 &\approx \varepsilon_2^{(0)} + \left.\frac{d\varepsilon_2}{dS^2}\right|_{S^2=0} S^2 + \frac{1}{2}\left.\frac{d^2\varepsilon_2}{d(S^2)^2}\right|_{S^2=0}(S^2)^2 + \cdots \\ &= \varepsilon_2^{(0)} + \frac{2(X_{,2}^\Lambda + \varphi_1^{(\Lambda)})\varphi_2^{(\Lambda)} G_{\Lambda\Lambda}}{G_{22}(1+\varepsilon_2^{(0)})} S^2 \\ &\quad + \left\{\frac{[2\varphi_2^{(\Lambda)}\varphi_2^{(\Lambda)} + 3(\delta_2^\Lambda + \varphi_1^{(\Lambda)})\varphi_3^{(\Lambda)}]G_{\Lambda\Lambda}}{G_{22}(1+\varepsilon_2^{(0)})}\right. \\ &\quad \left. - \frac{2[(X_{,2}^\Lambda + \varphi_1^{(\Lambda)})\varphi_2^{(\Lambda)}]^2 G_{\Lambda\Lambda}^2}{G_{22}^2(1+\varepsilon_2^{(0)})^3}\right\}(S^2)^2 + \cdots, \end{aligned} \tag{8.123}$$

$$\begin{aligned}\gamma_{12} &\approx \gamma_{12}^{(0)} + \left.\frac{d\gamma_{12}}{dS^2}\right|_{S^2=0} S^2 + \frac{1}{2}\left.\frac{d^2\gamma_{12}}{d(S^2)^2}\right|_{S^2=0}(S^2)^2 + \cdots \\ &= \gamma_{12}^{(0)} + \frac{1}{\cos\gamma_{12}^{(0)}}\left\{\frac{(\delta_1^\Lambda + u_{0;1}^{(\Lambda)})\varphi_1^{(\Lambda)} + (\delta_2^\Lambda + \varphi_1^{(\Lambda)})\varphi_{1;1}^{(\Lambda)}}{\sqrt{G_{11}}\sqrt{G_{22}}(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})} G_{\Lambda\Lambda}\right. \\ &\quad \left. - \sin\gamma_{12}^{(0)}\left[\frac{(\delta_1^\Lambda + u_{0;1}^{(\Lambda)})\varphi_{1;1}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{11}(1+\varepsilon_1^{(0)})^2} + \frac{2(\delta_2^\Lambda + \varphi_1^{(\Lambda)})\varphi_2^{(\Lambda)} G_{\Lambda\Lambda}}{G_{22}(1+\varepsilon_2^{(0)})^2}\right]\right\} S^2 + \cdots, \end{aligned} \tag{8.124}$$

$$\begin{aligned}\varepsilon_1^{(0)} &= \frac{1}{\sqrt{G_{11}}}\sqrt{(\delta_1^\Lambda + u_{0;1}^\Lambda)(\delta_1^\Lambda + u_{0;1}^\Lambda)G_{\Lambda\Lambda}} - 1 \\ &= \frac{1}{\sqrt{G_{11}}}\sqrt{(1+u_{0;1}^1)^2 G_{11} + (u_{0;1}^2)^2 G_{22}} - 1, \end{aligned} \tag{8.125}$$

$$\begin{aligned}\varepsilon_2^{(0)} &= \frac{1}{\sqrt{G_{22}}}\sqrt{(\delta_2^\Lambda + \varphi_1^{(\Lambda)})(\delta_2^\Lambda + \varphi_1^{(\Lambda)})G_{\Lambda\Lambda}} - 1 \\ &= \frac{1}{\sqrt{G_{22}}}\sqrt{(\varphi_1^{(1)})^2 G_{11} + (1+\varphi_1^{(2)})^2 G_{22}} - 1, \end{aligned} \tag{8.126}$$

$$\gamma_{12}^{(0)} = \sin^{-1} \frac{(\delta_1^\Lambda + u_{0;1}^\Lambda)(\delta_2^\Gamma + \varphi_1^{(\Gamma)})G_{\Lambda\Gamma}}{\sqrt{G_{11}G_{22}}(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})}$$

$$= \sin^{-1} \frac{(1+u_{0;1}^1)\varphi_1^{(1)}G_{11} + u_{0;1}^2(1+\varphi_1^{(2)})G_{22}}{\sqrt{G_{11}G_{22}}(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})}. \tag{8.127}$$

As in Eq.(8.96), the stresses on the deformed configuration can be defined by the constitutive laws, i.e.,

$$\sigma_1 = f(\varepsilon_1, \gamma_{12}, t) \text{ and } \sigma_{12} = g(\varepsilon_1, \gamma_{12}, t). \tag{8.128}$$

The internal forces and moments in the deformed beam are defined as

$$\begin{aligned} N_1 &= \int_A \sigma_1 \left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23}\right] dA, \\ M_3 &= \int_A \sigma_1 \frac{S^2}{1+\varphi_1^{(2)}} \left[(1+\varepsilon_3)(1+\varepsilon_2)^2 \cos\gamma_{23}\right] dA, \\ Q_1 &= \int_A \sigma_{12} \left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23}\right] dA. \end{aligned} \tag{8.129}$$

For convenience, the subscripts of the internal forces can be dropped again. The internal force vectors are defined as

$$\begin{aligned} \mathbf{M} &\equiv M^3\mathbf{N}_3 = M\mathbf{n}_3, \\ \mathcal{N} &\equiv N^\Lambda \mathbf{N}_\Lambda = N\mathbf{n}_1 + Q\mathbf{n}_2, \\ {}^{\mathbf{N}}\mathbf{M} &\equiv {}^{\mathbf{N}}M^\Lambda \mathbf{N}_\Lambda = \mathbf{g}_1 \times \mathcal{N}, \end{aligned} \tag{8.130}$$

where

$$\mathbf{g}_1 \equiv \frac{d\mathbf{r}}{dS} = (\delta_1^\Lambda + u_{0;1}^\Lambda)\sqrt{G_{\Lambda\Lambda}}\mathbf{N}_\Lambda \text{ and } {}^{\mathbf{N}}\mathbf{M} \equiv \frac{1}{dS} d\mathbf{r} \times \mathcal{N}. \tag{8.131}$$

The components of the internal forces in the $\mathbf{G}_\Lambda$-direction are

$$\begin{aligned} N^\Lambda &= N\mathbf{n}_1 \cdot \mathbf{N}_\Lambda + Q\mathbf{n}_2 \cdot \mathbf{N}_\Lambda = N\cos\theta_{(\mathbf{n}_1,\mathbf{N}_\Lambda)} + Q\cos\theta_{(\mathbf{n}_2,\mathbf{N}_\Lambda)} \\ &= \frac{N(\delta_1^\Gamma + u_{0;1}^\Gamma)G_{\Gamma\Lambda}}{\sqrt{G_{\Lambda\Lambda}G_{11}}(1+\varepsilon_1^{(0)})} + \frac{Q(\delta_2^\Gamma + \varphi_1^{(\Gamma)})G_{\Gamma\Lambda}}{\sqrt{G_{\Lambda\Lambda}G_{22}}(1+\varepsilon_2^{(0)})}, \end{aligned} \tag{8.132}$$

$$\begin{aligned} M_1^\Lambda &= M\mathbf{n}_3 \cdot \mathbf{N}_\Lambda = M\cos\theta_{(\mathbf{n}_3,\mathbf{N}_\Lambda)} = \frac{MG_{3\Lambda}}{\sqrt{G_{\Lambda\Lambda}G_{33}}}, \\ {}^{\mathbf{N}}M^1 &= (\mathbf{g}_1 \times \mathcal{N}) \cdot \mathbf{N}_1 = u_{0;1}^2\sqrt{G_{22}}N^3 - u_{0;1}^3\sqrt{G_{33}}N^2, \\ {}^{\mathbf{N}}M^2 &= (\mathbf{g}_1 \times \mathcal{N}) \cdot \mathbf{N}_2 = u_{0;1}^3\sqrt{G_{33}}N^1 - (1+u_{0;1}^1)\sqrt{G_{11}}N^3, \\ {}^{\mathbf{N}}M^3 &= (\mathbf{g}_1 \times \mathcal{N}) \cdot \mathbf{N}_3 = (1+u_{0;1}^1)\sqrt{G_{11}}N^2 - u_{0;1}^2\sqrt{G_{22}}N^1. \end{aligned} \tag{8.133}$$

If $u^{\Lambda}_{(0),3} = u^3 = 0,\ \varepsilon_3^{(0)} = 0$, $G_{\Lambda 3} = 0$ ($\Lambda \neq 3$) and $G_{33} = 1$, then

$$
\begin{aligned}
&M_1^1 = M_1^2 = 0,\quad M_1^3 = M,\\
&{}^{\mathrm{N}}M^1 = {}^{\mathrm{N}}M^2 = 0,\\
&{}^{\mathrm{N}}M^3 = \frac{Q}{(1+\varepsilon_2^{(0)})}\frac{\sqrt{G_{11}}}{\sqrt{G_{22}}}(1+\varepsilon_1^{(0)}).
\end{aligned}
\tag{8.134}
$$

Based on the deformed middle surface in the Lagrangian coordinates, the equations of motion for the deformed beam are

$$
\begin{aligned}
\mathcal{N}_{,1} + \mathbf{q} &= \rho \mathbf{u}_{0,tt} + I_3 \boldsymbol{\varphi}_{1,tt},\\
\mathbf{M}_{,1} + {}^{\mathrm{N}}\mathbf{M} + \mathbf{m} &= I_3 \mathbf{u}_{0,tt} + J_3 \boldsymbol{\varphi}_{1,tt};
\end{aligned}
\tag{8.135}
$$

and for $(\Lambda = 1, 2)$,

$$
\begin{aligned}
N^{\Lambda}_{;1} + q^{\Lambda} &= \rho u^{\Lambda}_{0,tt} + I_3 \varphi^{(\Lambda)}_{1,tt},\\
M^3_{;1} + {}^{\mathrm{N}}M^3 + m^3 &= I_3 u^1_{0,tt} + J_3 \varphi^{(1)}_{1,tt},
\end{aligned}
\tag{8.136}
$$

or

$$
\begin{aligned}
&\left[\frac{N(1+u^1_{0;1})}{1+\varepsilon_1^{(0)}} + \frac{Q\varphi_1^{(1)}\sqrt{G_{11}}}{\sqrt{G_{22}}(1+\varepsilon_2^{(0)})}\right]_{;1} + q^1 = \rho u^1_{0,tt} + I_3 \varphi^{(1)}_{1,tt},\\
&\left[\frac{N u^2_{0;1}\sqrt{G_{22}}}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})} + \frac{Q(1+\varphi_1^{(2)})}{1+\varepsilon_2^{(0)}}\right]_{;1} + q^2 = \rho u^2_{0,tt} + I_3 \varphi^{(2)}_{1,tt},\\
&M_{;1} + Q\frac{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})}{\sqrt{G_{22}}} + m^3 = I_3 u^1_{0,tt} + J_3 \varphi^{(1)}_{1,tt},
\end{aligned}
\tag{8.137}
$$

where $\rho = \int_A \rho_0 dA$, $I_3 = \int_A \rho_0 S^2 dA$ and $J_3 = \int_A \rho_0 (S^2)^2 dA$.

As in Eqs.(8.106)–(8.109), the force and displacement continuity and boundary conditions can be given. The force condition at a point $\mathscr{P}_k$ with $S^1 = S^1_k$ is

$$
\begin{aligned}
&{}^{-}\mathbf{N}(S^1_k) + {}^{+}\mathbf{N}(S^1_k) + \mathbf{F}_k = 0,\\
&{}^{-}N^{\Lambda}(S^1_k) = {}^{+}N^{\Lambda}(S^1_k) + F^{\Lambda}_k \quad (\Lambda = 1, 2).
\end{aligned}
\tag{8.138}
$$

The force boundary condition at the boundary point $\mathscr{P}_r$ is

$$
\mathbf{N}(S^1_r) + \mathbf{F}_r = 0 \ \text{ or } \ N^{\Lambda}(S^1_r) + F^{\Lambda}_r = 0 \quad (\Lambda = 1, 2). \tag{8.139}
$$

If there is a concentrated moment at $\mathscr{P}_k$ with $S^1 = S^1_k$, the corresponding moment boundary condition is

$$^{-}\mathbf{M}(S_k^1)+{}^{+}\mathbf{M}(S_k^1)+\mathbf{M}_k=0,$$
$$^{-}M^{\Lambda}(S_k^1)={}^{+}M^{\Lambda}(S_k^1)+M_k^{\Lambda}\quad(\Lambda=3). \tag{8.140}$$

The moment boundary condition at the boundary point $\mathscr{P}_r$ is

$$\mathbf{M}(S_r^1)+\mathbf{M}_r=0 \text{ or } M^{\Lambda}(S_r^1)+M_r^{\Lambda}=0 \quad (\Lambda=3). \tag{8.141}$$

The displacement continuity and boundary conditions are similar to Eq. (8.68). i.e., $u_{k-}^{\Lambda}=u_{k+}^{\Lambda}$ and $u_r^{\Lambda}=B_r^{\Lambda}$ ($\Lambda=1,2$).

8.4. A nonlinear theory of straight rods

Consider a nonlinear rod in the initial configuration to be straight, which requires that the initial curvature and torsion should be zero ($\kappa(S)=0$ and $\tau(S)=0$). Thus, let $S^I=X^I$, $G_{IJ}=0$ and $G_{II}=1$ ($I,J=1,2,3$). The three dimensional displacements, strains, the directional cosine of rotation and the change rate of the area are given in Eqs.(8.29)–(8.35). It is assumed that the coordinate X^1 is along the longitudinal direction of rod and the other two coordinates X^2 and X^3 are on the cross section of the rod with the direction of X^1. The coordinates for the deformed straight rod are (s^1,s^2,s^3). As in the thin beam theory, the widths of rod in two directions of X^2 and X^3 are very small compared to the length of the rod in direction of X^1, the elongation in the two directions of X^2 and X^3 should be very small, which can be neglected. Based on the aforementioned reasons, the assumptions for thin rods are adopted.

(i) The elongations in the two directions of X^2 and X^3 are very small (i.e., $\varepsilon_2\approx 0$ and $\varepsilon_3\approx 0$).
(ii) Under bending only, the neutral surface is not deformed (i.e., $\gamma_{12}\approx 0$ and $\gamma_{13}\approx 0$).

Choose an arbitrary coordinate frame as (X^1,X^2,X^3) and the coordinate of X^1 goes through the *centroid* on the cross section of the rod. The centroid curve of the deformed rod is along the coordinate of s^1 in the coordinates of (s^1,s^2,s^3), as shown in Fig.8.4. The external forces on the rod can be given as in Eqs.(8.36)–(8.40). Under the torque, the rod possesses torsion $\tau(s^1)=\tau(s)$ in the direction of s^1. The transverse forces off the shear center produces the torques included in m^1 and ${}^kM^1$. Compared to the longitudinal length S, X^2 and X^3 on the cross section are very small. From assumption (i), three displacements $u^I=u^I(S,X^2,X^3)$ can be expressed by the Taylor series as

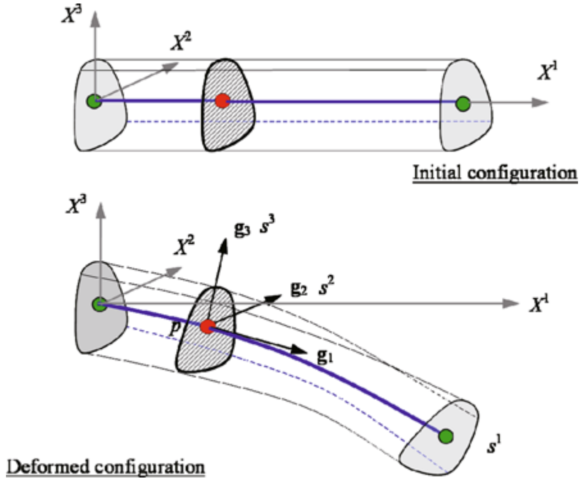

Fig. 8.4 A straight rod with initial and deformed configuration.

$$\begin{aligned} u^I &= u_0^I(S,t) + \sum_{n=1}^{\infty}(X^2)^n \varphi_n^{(I)}(S,t) + \sum_{n=1}^{\infty}(X^3)^n \theta_n^{(I)}(S,t) \\ &\quad + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(X^2)^m (X^3)^n \vartheta_{mn}^{(I)}(S,t), \end{aligned} \tag{8.142}$$

where

$$\begin{aligned} \varphi_n^{(I)} &= \frac{1}{n!}\frac{\partial^n u^I}{\partial (X^2)^n}\bigg|_{(X^2,X^3)=(0,0)}, \\ \theta_n^{(I)} &= \frac{1}{n!}\frac{\partial^n u^I}{\partial (X^3)^n}\bigg|_{(X^2,X^3)=(0,0)}, \\ \vartheta_{mn}^{(I)} &= \frac{1}{m!n!}\frac{\partial^{m+n} u^I}{\partial (X^2)^m \partial (X^3)^n}\bigg|_{(X^2,X^3)=(0,0)}. \end{aligned} \tag{8.143}$$

From Eqs.(8.30) and (8.33), the approximate six strains are

$$\begin{aligned} \varepsilon_1 &\approx \varepsilon_1^{(0)} + \frac{\partial \varepsilon_1}{\partial X^2}\bigg|_{(X^2,X^3)=(0,0)} X^2 + \frac{\partial \varepsilon_1}{\partial X^3}\bigg|_{(X^2,X^3)=(0,0)} X^3 \\ &\quad + \frac{1}{2!}\frac{\partial^2 \varepsilon_1}{\partial (X^2)^2}\bigg|_{(X^2,X^3)=(0,0)} (X^2)^2 + \frac{1}{2!}\frac{\partial^2 \varepsilon_1}{\partial (X^3)^2}\bigg|_{(X^2,X^3)=(0,0)} (X^3)^2 \\ &\quad + \frac{\partial^2 \varepsilon_1}{\partial X^2 \partial X^3}\bigg|_{(X^2,X^3)=(0,0)} X^2 X^3 + \cdots \\ &= \varepsilon_1^{(0)} + \frac{(\delta_1^I + u_{0,1}^I)\varphi_{1,1}^{(I)}}{1+\varepsilon_1^{(0)}} X^2 + \frac{(\delta_1^I + u_{0,1}^I)\theta_{1,1}^{(I)}}{1+\varepsilon_1^{(0)}} X^3 \end{aligned}$$

$$+\frac{1}{2}\left\{\frac{[2(\delta_1^I+u_{0,1}^I)\varphi_{2,1}^{(I)}]+\varphi_{1,1}^{(I)}\varphi_{1,1}^{(I)}}{1+\varepsilon_1^{(0)}}-\frac{[(\delta_1^I+u_{0,1}^I)\varphi_{1,1}^{(I)}]^2}{(1+\varepsilon_1^{(0)})^3}\right\}(X^2)^2$$

$$+\frac{1}{2}\left\{\frac{[2(\delta_1^I+u_{0,1}^I)\theta_{2,1}^{(I)}]+\theta_{1,1}^{(I)}\theta_{1,1}^{(I)}}{1+\varepsilon_1^{(0)}}-\frac{[(\delta_1^I+u_{0,1}^I)\theta_{1,1}^{(I)}]^2}{(1+\varepsilon_1^{(0)})^3}\right\}(X^3)^2$$

$$+\left[\frac{(\delta_1^I+u_{0,1}^I)\vartheta_{11,1}^{(I)}}{1+\varepsilon_1^{(0)}}+\frac{\varphi_{1,1}^{(I)}\theta_{1,1}^{(I)}}{1+\varepsilon_1^{(0)}}\right]X^2X^3+\cdots, \tag{8.144}$$

$$\varepsilon_2\approx\varepsilon_2^{(0)}+\left.\frac{\partial\varepsilon_2}{\partial X^2}\right|_{(X^2,X^3)=(0,0)}X^2+\left.\frac{\partial\varepsilon_2}{\partial X^3}\right|_{(X^2,X^3)=(0,0)}X^3$$

$$+\frac{1}{2!}\left.\frac{\partial^2\varepsilon_2}{\partial(X^2)^2}\right|_{(X^2,X^3)=(0,0)}(X^2)^2+\frac{1}{2!}\left.\frac{\partial^2\varepsilon_2}{\partial(X^3)^2}\right|_{(X^2,X^3)=(0,0)}(X^3)^2$$

$$+\left.\frac{\partial^2\varepsilon_2}{\partial X^2\partial X^3}\right|_{(X^2,X^3)=(0,0)}X^2X^3+\cdots$$

$$=\varepsilon_2^{(0)}+\frac{2(\delta_2^I+\varphi_1^{(I)})\varphi_2^{(I)}}{1+\varepsilon_2^{(0)}}X^2+\frac{(\delta_2^I+\varphi_1^{(I)})\vartheta_{11}^{(I)}}{1+\varepsilon_2^{(0)}}X^3$$

$$+\left\{\frac{[2\varphi_2^{(I)}\varphi_2^{(I)}+3(\delta_2^I+\varphi_1^{(I)})\varphi_3^{(I)}]}{1+\varepsilon_2^{(0)}}-\frac{2[(\delta_2^I+\varphi_1^{(I)})\varphi_2^{(I)}]^2}{(1+\varepsilon_2^{(0)})^3}\right\}(X^2)^2$$

$$+\frac{1}{2}\left\{\frac{\vartheta_{11}^{(I)}\vartheta_{11}^{(I)}}{1+\varepsilon_2^{(0)}}-\frac{[(\delta_2^I+\varphi_1^{(I)})\vartheta_{11}^{(I)}]^2}{2(1+\varepsilon_2^{(0)})^3}\right\}(X^3)^2$$

$$+\left[\frac{4(\delta_2^I+\varphi_1^{(I)})\vartheta_{21}^{(I)}}{1+\varepsilon_2^{(0)}}+\frac{2\varphi_2^{(I)}\vartheta_{11}^{(I)}}{1+\varepsilon_2^{(0)}}\right]X^2X^3+\cdots; \tag{8.145}$$

$$\varepsilon_3\approx\varepsilon_3^{(0)}+\left.\frac{\partial\varepsilon_3}{\partial X^2}\right|_{(X^2,X^3)=(0,0)}X^2+\left.\frac{\partial\varepsilon_3}{\partial X^3}\right|_{(X^2,X^3)=(0,0)}X^3$$

$$+\frac{1}{2!}\left.\frac{\partial^2\varepsilon_3}{\partial(X^2)^2}\right|_{(X^2,X^3)=(0,0)}(X^2)^2+\frac{1}{2!}\left.\frac{\partial^2\varepsilon_3}{\partial(X^3)^2}\right|_{(X^2,X^3)=(0,0)}(X^3)^2$$

$$+\left.\frac{\partial^2\varepsilon_3}{\partial X^2\partial X^3}\right|_{(X^2,X^3)=(0,0)}X^2X^3+\cdots$$

$$=\varepsilon_3^{(0)}+\frac{(\delta_3^I+\theta_1^{(I)})\vartheta_{11}^{(I)}}{1+\varepsilon_3^{(0)}}X^2+\frac{2(\delta_3^I+\theta_1^{(I)})\theta_2^{(I)}}{1+\varepsilon_3^{(0)}}X^3$$

$$+\frac{1}{2}\left\{\frac{\vartheta_{11}^{(I)}\vartheta_{11}^{(I)}}{1+\varepsilon_3^{(0)}}-\frac{[(\delta_2^I+\theta_1^{(I)})\vartheta_{11}^{(I)}]^2}{2(1+\varepsilon_3^{(0)})^3}\right\}(X^2)^2$$

$$+\left\{\frac{[2\theta_2^{(I)}\theta_2^{(I)}+3(\delta_3^I+\theta_1^{(I)})\theta_3^{(I)}]}{1+\varepsilon_3^{(0)}}-\frac{2[(\delta_3^I+\theta_1^{(I)})\theta_2^{(I)}]^2}{(1+\varepsilon_3^{(0)})^3}\right\}(X^3)^2$$

$$+\left[\frac{4(\delta_3^I+\theta_1^{(I)})\vartheta_{12}^{(I)}}{1+\varepsilon_3^{(0)}}+\frac{2\theta_2^{(I)}\vartheta_{11}^{(I)}}{1+\varepsilon_3^{(0)}}\right]X^2X^3+\cdots; \tag{8.146}$$

$$\gamma_{12}\approx\gamma_{12}^{(0)}+\left.\frac{\partial\gamma_{12}}{\partial X^2}\right|_{(X^2,X^3)=(0,0)}X^2+\left.\frac{\partial\gamma_{12}}{\partial X^3}\right|_{(X^2,X^3)=(0,0)}X^3$$

$$+\frac{1}{2!}\left.\frac{\partial^2\gamma_{12}}{\partial(X^2)^2}\right|_{(X^2,X^3)=(0,0)}(X^2)^2+\frac{1}{2!}\left.\frac{\partial^2\gamma_{12}}{\partial(X^3)^2}\right|_{(X^2,X^3)=(0,0)}(X^3)^2$$

$$+\left.\frac{\partial^2\gamma_{12}}{\partial X^2\partial X^3}\right|_{(X^2,X^3)=(0,0)}X^2X^3+\cdots$$

$$=\gamma_{12}^{(0)}+\frac{1}{\cos\gamma_{12}^{(0)}}\left\{\frac{2(\delta_1^I+u_{0,1}^{(I)})\varphi_2^{(I)}+(\delta_2^I+\varphi_1^{(I)})\varphi_{1,1}^{(I)}}{(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})}\right.$$

$$\left.-\sin\gamma_{12}^{(0)}\left[\frac{(\delta_1^I+u_{0,1}^{(I)})\varphi_{1,1}^{(I)}}{(1+\varepsilon_1^{(0)})^2}+\frac{2(\delta_2^I+\varphi_1^{(I)})\varphi_2^{(I)}}{(1+\varepsilon_2^{(0)})^2}\right]\right\}X^2$$

$$+\frac{1}{\cos\gamma_{12}^{(0)}}\left\{\frac{(\delta_2^I+\varphi_1^{(I)})\theta_{1,1}^{(I)}+(\delta_1^I+u_{0,1}^{(I)})\vartheta_{11}^{(\alpha)}}{(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})}\right.$$

$$\left.-\sin\gamma_{12}^{(0)}\left[\frac{(\delta_1^I+u_{0,1}^{(I)})\theta_{1,1}^{(I)}}{(1+\varepsilon_1^{(0)})^2}+\frac{(\delta_2^I+\varphi_1^{(I)})\vartheta_{11}^{(I)}}{(1+\varepsilon_2^{(0)})^2}\right]\right\}X^3+\cdots; \tag{8.147}$$

$$\gamma_{13}\approx\gamma_{13}^{(0)}+\left.\frac{\partial\gamma_{13}}{\partial X^2}\right|_{(X^2,X^3)=(0,0)}X^2+\left.\frac{\partial\gamma_{12}}{\partial X^3}\right|_{(X^2,X^3)=(0,0)}X^3$$

$$+\frac{1}{2!}\left.\frac{\partial^2\gamma_{13}}{\partial(X^2)^2}\right|_{(X^2,X^3)=(0,0)}(X^2)^2+\frac{1}{2!}\left.\frac{\partial^2\gamma_{13}}{\partial(X^3)^2}\right|_{(X^2,X^3)=(0,0)}(X^3)^2$$

$$+\left.\frac{\partial^2\gamma_{13}}{\partial X^2\partial X^3}\right|_{(X^2,X^3)=(0,0)}X^2X^3+\cdots$$

$$=\gamma_{13}^{(0)}+\frac{1}{\cos\gamma_{13}^{(0)}}\left\{\frac{(\delta_3^I+\theta_1^{(I)})\varphi_{1,1}^{(I)}+(\delta_1^I+u_{0,1}^{(I)})\vartheta_{11}^{(I)}}{(1+\varepsilon_1^{(0)})(1+\varepsilon_3^{(0)})}\right.$$

$$\left.-\sin\gamma_{13}^{(0)}\left[\frac{(\delta_1^I+u_{0,1}^{(I)})\varphi_{1,1}^{(I)}}{(1+\varepsilon_1^{(0)})^2}+\frac{(\delta_3^I+\theta_1^{(I)})\vartheta_{11}^{(I)}}{(1+\varepsilon_3^{(0)})^2}\right]\right\}X^2$$

$$+\frac{1}{\cos\gamma_{13}^{(0)}}\left\{\frac{2(\delta_1^I+u_{0,1}^{(I)})\theta_2^{(I)}+(\delta_3^I+\theta_1^{(I)})\vartheta_{1,1}^{(I)}}{(1+\varepsilon_1^{(0)})(1+\varepsilon_3^{(0)})}\right.$$

$$-\sin\gamma_{13}^{(0)}\left[\frac{(\delta_1^I+u_{0,1}^{(I)})\vartheta_{1,1}^{(I)}}{(1+\varepsilon_1^{(0)})^2}+\frac{2(\delta_3^I+\theta_1^{(I)})\theta_2^{(I)}}{(1+\varepsilon_3^{(0)})^2}\right]\Bigg\}X^3+\cdots; \tag{8.148}$$

$$\begin{aligned}
\gamma_{23} &\approx \gamma_{23}^{(0)}+\left.\frac{\partial\gamma_{23}}{\partial X^2}\right|_{(X^2,X^3)=(0,0)}X^2+\left.\frac{\partial\gamma_{23}}{\partial X^3}\right|_{(X^2,X^3)=(0,0)}X^3\\
&+\frac{1}{2!}\left.\frac{\partial^2\gamma_{23}}{\partial(X^2)^2}\right|_{(X^2,X^3)=(0,0)}(X^2)^2+\frac{1}{2!}\left.\frac{\partial^2\gamma_{23}}{\partial(X^3)^2}\right|_{(X^2,X^3)=(0,0)}(X^3)^2\\
&+\left.\frac{\partial^2\gamma_{23}}{\partial X^2\partial X^3}\right|_{(X^2,X^3)=(0,0)}X^2X^3+\cdots\\
&=\gamma_{23}^{(0)}+\frac{1}{\cos\gamma_{23}^{(0)}}\Bigg\{\frac{2(\delta_3^I+\theta_1^{(I)})\varphi_2^{(I)}+(\delta_2^I+\varphi_1^{(I)})\vartheta_{11}^{(I)}}{(1+\varepsilon_2^{(0)})(1+\varepsilon_3^{(0)})}\\
&-\sin\gamma_{23}^{(0)}\left[\frac{2(\delta_2^I+\varphi_1^{(I)})\varphi_2^{(I)}}{(1+\varepsilon_2^{(0)})^2}+\frac{(\delta_3^I+\theta_1^{(I)})\vartheta_{11}^{(I)}}{(1+\varepsilon_3^{(0)})^2}\right]\Bigg\}X^2\\
&+\frac{1}{\cos\gamma_{23}^{(0)}}\Bigg\{\frac{2(\delta_2^I+\varphi_1^{(I)})\theta_2^{(I)}+(\delta_3^I+\theta_1^{(I)})\vartheta_{11}^{(I)}}{(1+\varepsilon_2^{(0)})(1+\varepsilon_3^{(0)})}\\
&-\sin\gamma_{23}^{(0)}\left[\frac{(\delta_2^I+\varphi_1^{(I)})\vartheta_{11}^{(I)}}{(1+\varepsilon_2^{(0)})^2}+\frac{2(\delta_3^I+\theta_1^{(I)})\theta_2^{(I)}}{(1+\varepsilon_3^{(0)})^2}\right]\Bigg\}X^3+\cdots,
\end{aligned} \tag{8.149}$$

where

$$\begin{aligned}
\varepsilon_1^{(0)} &= \sqrt{(\delta_1^I+u_{0,1}^I)(\delta_1^I+u_{0,1}^I)}-1,\\
\varepsilon_2^{(0)} &= \sqrt{(\delta_2^I+\varphi_1^{(I)})(\delta_2^I+\varphi_1^{(I)})}-1,\\
\varepsilon_3^{(0)} &= \sqrt{(\delta_3^I+\theta_1^{(I)})(\delta_3^I+\theta_1^{(I)})}-1,\\
\gamma_{12}^{(0)} &= \sin^{-1}\frac{(\delta_1^I+u_{0,1}^I)(\delta_2^I+\varphi_1^{(I)})}{(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})},\\
\gamma_{13}^{(0)} &= \sin^{-1}\frac{(\delta_1^I+u_{0,1}^I)(\delta_2^I+\theta_1^{(I)})}{(1+\varepsilon_1^{(0)})(1+\varepsilon_3^{(0)})},\\
\gamma_{23}^{(0)} &= \sin^{-1}\frac{(\delta_2^I+\varphi_1^{(I)})(\delta_3^I+\theta_1^{(I)})}{(1+\varepsilon_2^{(0)})(1+\varepsilon_3^{(0)})}.
\end{aligned} \tag{8.150}$$

From Assumptions (i) and (ii), consider the zero order of the Taylor series of the six strains to give

$$\varepsilon_2^{(0)}=0,\ \varepsilon_3^{(0)}=0;\ \gamma_{12}^{(0)}=0,\ \gamma_{13}^{(0)}=0,\ \gamma_{23}^{(0)}=0. \tag{8.151}$$

The deformed rod for $X^2 = X^3 = 0$ satisfies the following relation:

$$(1+\varepsilon_1^{(0)})^2 = (\delta_1^I + u_{0,1}^I)(\delta_1^I + u_{0,1}^I). \tag{8.152}$$

Note that one assumed $\varepsilon_1^{(0)} = 0$, which is not adequate (e.g., Novozhilov, 1953). Equation (8.152) implies that only 1-dimenisonal membrane force in the rod is considered. From Eqs.(8.151) and (8.152),

$$\begin{aligned}
\frac{1}{(1+\varepsilon_1^{(0)})^2}(\delta_1^I + u_{0,1}^I)(\delta_1^I + u_{0,1}^I) &= 1,\\
(\delta_2^I + \varphi_1^{(I)})(\delta_2^I + \varphi_1^{(I)}) &= 1,\\
(\delta_3^I + \theta_1^{(I)})(\delta_3^I + \theta_1^{(I)}) &= 1;\\
(\delta_1^I + u_{0,1}^I)(\delta_2^I + \varphi_1^{(I)}) &= 0,\\
(\delta_1^I + u_{0,1}^I)(\delta_3^I + \theta_1^{(I)}) &= 0,\\
(\delta_2^I + \varphi_1^{(I)})(\delta_3^I + \theta_1^{(I)}) &= 0.
\end{aligned} \tag{8.153}$$

Using the zero order terms of X^2 and X^3 in Eq.(8.34), the direction cosine matrix $((l_{ij})_{3\times 3})$ is given by

$$\begin{aligned}
\cos\theta_{(\mathbf{n}_1,\mathbf{I}_I)} &= l_{1I} = \frac{\delta_1^I + u_{0,1}^I}{1+\varepsilon_1^{(0)}},\\
\cos\theta_{(\mathbf{n}_2,\mathbf{I}_I)} &= l_{2I} = \frac{\delta_2^I + \varphi_1^{(I)}}{1+\varepsilon_2^{(0)}},\\
\cos\theta_{(\mathbf{n}_3,\mathbf{I}_I)} &= l_{3I} = \frac{\delta_3^I + \theta_1^{(I)}}{1+\varepsilon_3^{(0)}}.
\end{aligned} \tag{8.154}$$

From the geometrical relations, the nine directional cosines must satisfy the trigonometric relations without summation on $\alpha = 1, 2, 3$ as

$$l_{\alpha I} l_{\alpha I} = 1 \ \text{ for } I=1, 2, 3 \tag{8.155}$$

and for $\alpha, \beta = 1, 2, 3$ and $\alpha \neq \beta$,

$$l_{\alpha I} l_{\beta I} = 0. \tag{8.156}$$

As aforesaid, only the three rotations of rod are considered. Thus, the unknowns $\varphi_1^{(I)}$ and $\theta_1^{(I)}$ ($I = 1, 2, 3$) can be determined by the three Euler angles (Φ, Ψ and Θ). The Euler angles Φ and Ψ rotates around the axes of X^2 and X^3, respectively, and the Euler angle Θ rotates around the axis of X^1, as sketched in Fig.8.5. Due to bending, the first rotation around the axis of X^2 is to form $(\bar{X}^1, \bar{X}^2, \bar{X}^3)$ in Fig.8.5(a). The second rotation around the axis of $\bar{X}^3$ gives

$(\overline{\overline{X}}^1, \overline{\overline{X}}^2, \overline{\overline{X}}^3)$, as shown in Fig.8.5(b). The last rotation around the axis of $\overline{\overline{X}}^1$ is because of the torsion, and the final state of the rod in the frame of $(\overline{\overline{\overline{X}}}^1, \overline{\overline{\overline{X}}}^2, \overline{\overline{\overline{X}}}^3)$ in Fig.8.5(c) gives the coordinates (s^1, s^2, s^3) for the deformed rod. The rotation deformation is the same as the rotation given by Eq.(8.153). The rotation matrices are

$$\mathbf{R}^1 = \begin{bmatrix} \cos\Phi & 0 & -\sin\Phi \\ 0 & 1 & 0 \\ \sin\Phi & 0 & \cos\Phi \end{bmatrix}, \tag{8.157}$$

$$\mathbf{R}^2 = \begin{bmatrix} \cos\Psi & \sin\Psi & 0 \\ -\sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{8.158}$$

$$\mathbf{R}^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Theta & \sin\Theta \\ 0 & -\sin\Theta & \cos\Theta \end{bmatrix}. \tag{8.159}$$

From the above rotations, the directional cosine matrix ($\boldsymbol{l} = (l_{ij})_{3\times 3}$) is

$$\boldsymbol{l} = \mathbf{R}^3\mathbf{R}^2\mathbf{R}^1 = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ l_{12} & l_{22} & l_{32} \\ l_{13} & l_{23} & l_{33} \end{bmatrix}, \tag{8.160}$$

where

$$\begin{aligned}
l_{11} &= \cos\Phi\cos\Psi, \\
l_{21} &= -\cos\Phi\sin\Psi\cos\Theta + \sin\Phi\sin\Theta, \\
l_{31} &= \cos\Phi\sin\Psi\sin\Theta + \sin\Phi\cos\Theta, \\
l_{12} &= \sin\Psi, \\
l_{22} &= \cos\Psi\cos\Theta, \\
l_{32} &= -\cos\Psi\sin\Theta, \\
l_{13} &= -\sin\Phi\cos\Psi, \\
l_{23} &= \sin\Phi\sin\Psi\cos\Theta + \cos\Phi\sin\Theta, \\
l_{33} &= -\sin\Phi\sin\Psi\sin\Theta + \cos\Phi\cos\Theta.
\end{aligned} \tag{8.161}$$

Compared with Eq.(8.153), equations (8.156) and (8.161) give

$$\begin{aligned}
u^1_{0,1} &= (1+\varepsilon^{(0)}_1)\cos\Phi\cos\Psi - 1, \\
\varphi^{(1)}_1 &= -\cos\Phi\sin\Psi\cos\Theta + \sin\Phi\sin\Theta,
\end{aligned}$$

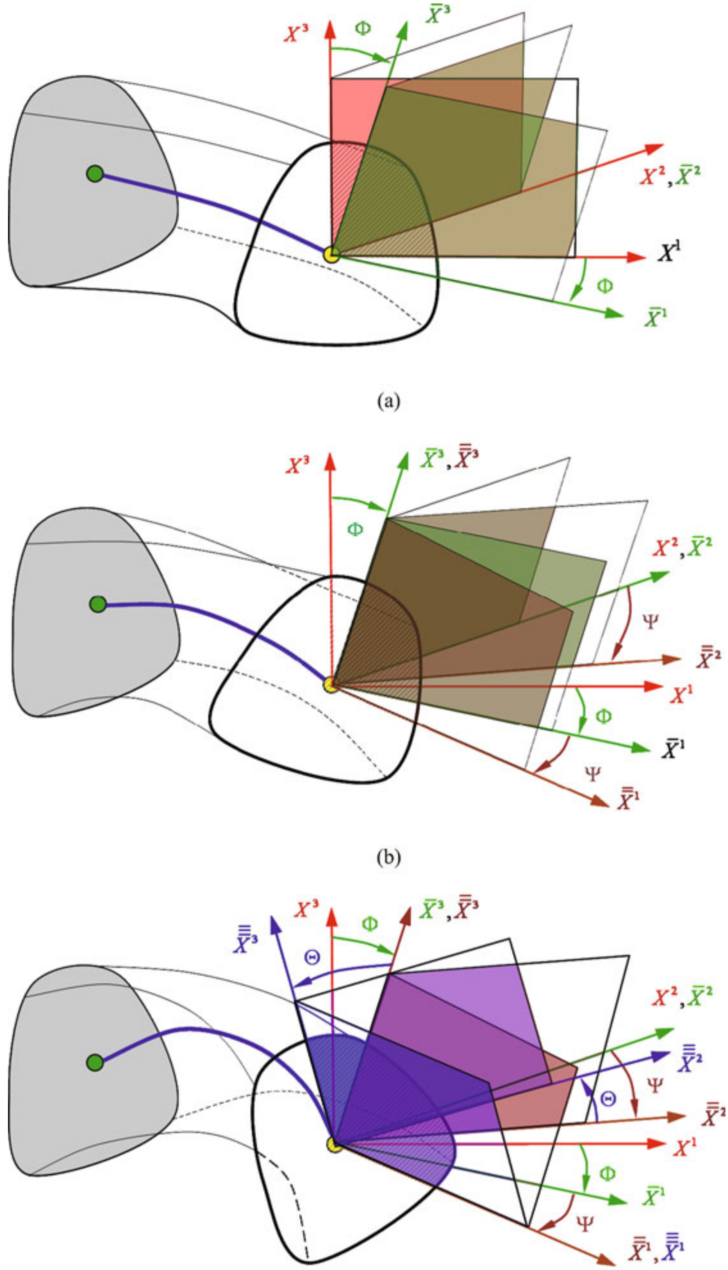

Fig. 8.5 Euler angles of rod rotation caused by bending and torsion: (a) the initial (red) to first rotation (green), (b) the first to second rotation (brown), (c) from the second to the last rotation (blue). (color plot in the book end)

$$\begin{aligned}
\theta_1^{(1)} &= \cos\Phi\sin\Psi\sin\Theta + \sin\Phi\cos\Theta,\\
u_{0,1}^2 &= (1+\varepsilon_1^{(0)})\sin\Psi,\\
\varphi_1^{(2)} &= \cos\Psi\cos\Theta - 1,\\
\theta_1^{(1)} &= -\cos\Psi\sin\Theta,\\
u_{0,1}^3 &= -(1+\varepsilon_1^{(0)})\sin\Phi\cos\Psi,\\
\varphi_1^{(3)} &= \sin\Phi\sin\Psi\cos\Theta + \cos\Phi\sin\Theta,\\
\theta_1^{(1)} &= -\sin\Phi\sin\Psi\sin\Theta + \cos\Phi\cos\Theta - 1.
\end{aligned} \tag{8.162}$$

The first, fourth and seventh equations of the foregoing equation give

$$\begin{aligned}
\cos\Psi &= \pm\frac{\sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2}}{1+\varepsilon_1^{(0)}},\\
\sin\Psi &= \frac{u_{0,1}^2}{1+\varepsilon_1^{(0)}};\\
\cos\Phi &= \pm\frac{1+u_{0,1}^1}{\sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2}},\\
\sin\Phi &= \mp\frac{u_{0,1}^3}{\sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2}},
\end{aligned} \tag{8.163}$$

and

$$\begin{aligned}
\varphi_1^{(1)} &= \mp\frac{1}{\sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2}}\left[\frac{u_{0,1}^2(1+u_{0,1}^1)}{1+\varepsilon_1^{(0)}}\cos\Theta + u_{0,1}^3\sin\Theta\right],\\
\theta_1^{(1)} &= \pm\frac{1}{\sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2}}\left[\frac{u_{0,1}^2(1+u_{0,1}^1)}{1+\varepsilon_1^{(0)}}\sin\Theta - u_{0,1}^3\cos\Theta\right],\\
\varphi_1^{(2)} &= \pm\frac{\sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2}}{1+\varepsilon_1^{(0)}}\cos\Theta - 1,\\
\theta_1^{(2)} &= \mp\frac{\sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2}}{1+\varepsilon_1^{(0)}}\sin\Theta,\\
\varphi_1^{(3)} &= \mp\frac{1}{\sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2}}\left[\frac{u_{0,1}^3 u_{0,1}^2}{1+\varepsilon_1^{(0)}}\cos\Theta - (1+u_{0,1}^1)\sin\Theta\right],\\
\theta_1^{(3)} &= \pm\frac{1}{\sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2}}\left[\frac{u_{0,1}^3 u_{0,1}^2}{1+\varepsilon_1^{(0)}}\sin\Theta + (1+u_{0,1}^1)\cos\Theta\right] - 1.
\end{aligned} \tag{8.164}$$

If $\Theta = 0$, the rotation about the longitudinal axis disappears. So only the bending rotation exists. This case reduces to the pure bending of the rod as discussed in Section 8.2.

From Eq.(8.154), the directional cosine vectors are defined as

$$\boldsymbol{l}_\alpha = l_{\alpha I}\mathbf{I}_I \quad \text{for } \alpha = 1, 2, 3. \tag{8.165}$$

Thus, the change ratio of the directional cosines along the deformed rod is

$$\frac{d\boldsymbol{l}_\alpha}{ds} = \frac{dl_{\alpha I}}{ds}\mathbf{I}_I = \frac{dl_{\alpha I}}{(1+\varepsilon_1^{(0)})dS}\mathbf{I}_I \quad \text{for } \alpha = 1, 2, 3 \tag{8.166}$$

The three vectors form a instantaneous, rotational coordinate frame, and the rotation ratio vector about three axes are defined as

$$\boldsymbol{\omega} = \omega_I \mathbf{I}_I. \tag{8.167}$$

From rigid-body dynamics (e.g., Goldstein et al., 2002), the change ratio of the directional cosines along the deformed rod can be computed in an analogy way, i.e.,

$$\begin{aligned}\frac{d\boldsymbol{l}_\alpha}{ds} &= \frac{dl_{\alpha I}}{ds}\mathbf{I}_I = \frac{dl_{\alpha I}}{(1+\varepsilon_1^{(0)})dS}\mathbf{I}_I = \boldsymbol{\omega}\times\boldsymbol{l}_\alpha \\ &= \begin{vmatrix} \mathbf{I}_1 & \mathbf{I}_2 & \mathbf{I}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ l_{\alpha 1} & l_{\alpha 2} & l_{\alpha 3} \end{vmatrix} = e_{IJK}\,\omega_J l_{\alpha K}\mathbf{I}_I,\end{aligned} \tag{8.168}$$

for $\alpha = 1, 2, 3$ and $I, J, K \in \{1, 2, 3\}$ with $I \neq J \neq K \neq I$. From the foregoing equation, the rotation ratio components are given by

$$e_{IJK}\frac{dl_{\alpha I}}{ds}l_{\alpha K} = e_{IJK}\frac{dl_{\alpha I}}{(1+\varepsilon_1^{(0)})dS}l_{\alpha K} = \omega_J. \tag{8.169}$$

In other words, the foregoing equation is expressed by

$$\begin{aligned}\omega_1 &= e_{I1K}\frac{dl_{\alpha I}}{ds}l_{\alpha K} = -\frac{dl_{\alpha 2}}{ds}l_{\alpha 3} = -\frac{dl_{\alpha 2}}{(1+\varepsilon_1^{(0)})dS}l_{\alpha 3}, \\ \omega_2 &= e_{I2K}\frac{dl_{\alpha I}}{ds}l_{\alpha K} = -\frac{dl_{\alpha 3}}{ds}l_{\alpha 1} = -\frac{dl_{\alpha 3}}{(1+\varepsilon_1^{(0)})dS}l_{\alpha 1}, \\ \omega_3 &= e_{I3K}\frac{dl_{\alpha I}}{ds}l_{\alpha K} = -\frac{dl_{\alpha 1}}{ds}l_{\alpha 2} = -\frac{dl_{\alpha 1}}{(1+\varepsilon_1^{(0)})dS}l_{\alpha 2}.\end{aligned} \tag{8.170}$$

From Eq.(8.161), the foregoing equations gives

$$\omega_1 = \frac{1}{1+\varepsilon_1^{(0)}}\left(\sin\Phi\frac{d\Psi}{dS} + \cos\Psi\cos\Phi\frac{d\Theta}{dS}\right),$$

$$\omega_2 = \frac{1}{1+\varepsilon_1^{(0)}}(\frac{d\Psi}{dS} + \sin\Psi \frac{d\Theta}{dS}),$$
$$\omega_3 = \frac{1}{1+\varepsilon_1^{(0)}}(\cos\Phi \frac{d\Psi}{dS} - \sin\Phi\cos\Psi \frac{d\Theta}{dS}); \tag{8.171}$$

or

$$\omega_1 = \sin\Phi \frac{d\Psi}{ds} + \cos\Psi\cos\Phi \frac{d\Theta}{ds},$$
$$\omega_2 = \frac{d\Psi}{ds} + \sin\Psi \frac{d\Theta}{ds}, \tag{8.172}$$
$$\omega_3 = \cos\Phi \frac{d\Psi}{ds} - \sin\Phi\cos\Psi \frac{d\Theta}{ds}.$$

Notice that one often assumes $dS = ds$ in Love (1944), which is not adequate for large deformation.

On the other hand, using Eq.(8.25), the particle location on the deformed rod is expressed by

$$\mathbf{r} = (X^I + u^I)\mathbf{I}_I. \tag{8.173}$$

Because $X^1 = S$, X^2 and X^3 are independent of S. From Eq.(8.3), the base vector along the longitudinal direction of the deformed rod is given by

$$\tilde{\mathbf{g}}_1 = \tilde{g}_1^I \mathbf{I}_I = (\delta_1^I + u_{,1}^I)\mathbf{I}_I, \tag{8.174}$$

and the corresponding unit vector is

$$\tilde{\mathbf{n}}_1 = \frac{\tilde{\mathbf{g}}_1}{|\tilde{\mathbf{g}}_1|} = \frac{\delta_1^I + u_{,1}^I}{\sqrt{(\delta_1^K + u_{,1}^K)(\delta_1^K + u_{,1}^K)}}\mathbf{I}_I = \frac{\delta_1^I + u_{,1}^I}{1+\varepsilon_1}\mathbf{I}_I. \tag{8.175}$$

For $X^2 = X^3 = 0$,

$$\tilde{\mathbf{n}}_1 = \frac{\delta_1^I + u_{0,1}^I}{\sqrt{(\delta_1^K + u_{0,1}^K)(\delta_1^K + u_{0,1}^K)}}\mathbf{I}_I = \frac{\delta_1^I + u_{0,1}^I}{1+\varepsilon_1^{(0)}}\mathbf{I}_I. \tag{8.176}$$

Note that $\tilde{\mathbf{n}}_1 = \mathbf{n}_1$. The base vector in the principal normal direction of the deformed rod is

$$\tilde{\mathbf{g}}_2 = \frac{d\tilde{\mathbf{n}}_1}{ds} = \tilde{g}_2^I \mathbf{I}_1, \tag{8.177}$$

where

$$\tilde{g}_2^I \equiv \frac{1}{\tilde{g}_{11}^2}\left[u_{,11}^I(\delta_1^K + u_{,1}^K)(\delta_1^K + u_{,1}^K) - (\delta_1^I + u_{,1}^I)(\delta_1^K + u_{,1}^K)u_{,11}^K\right]. \tag{8.178}$$

The unit principal normal vector is

$$\tilde{\mathbf{n}}_2 = \frac{\tilde{\mathbf{g}}_2}{|\tilde{\mathbf{g}}_2|} = \frac{\tilde{\mathbf{g}}_2}{\sqrt{\tilde{g}_{22}}} = \frac{\tilde{\mathbf{g}}_2}{\tilde{\kappa}(S)} = \frac{\tilde{g}_2^I}{\tilde{\kappa}(S)}\mathbf{I}_I, \tag{8.179}$$

and from Eq.(8.9), the curvature of the deformed rod becomes

$$\begin{aligned}\tilde{\kappa}(S) &= |\tilde{\mathbf{g}}_2| = \sqrt{\tilde{g}_{22}} = \sqrt{\left(X_{,ss}^I + u_{,ss}^I\right)\left(X_{,ss}^I + u_{,ss}^I\right)} \\ &= \frac{\sqrt{(u_{,11}^I u_{,11}^I)(\delta_1^K + u_{,1}^K)(\delta_1^K + u_{,1}^K) - [(\delta_1^I + X_{,1}^I)u_{,11}^I]^2}}{\tilde{g}_{11}^{3/2}}.\end{aligned} \tag{8.180}$$

For $X^2 = X^3 = 0$,

$$\begin{aligned}\tilde{g}_2^I = \frac{1}{\tilde{g}_{11}^2}\Big[&u_{0,11}^I(\delta_1^K + u_{0,1}^K)(\delta_1^K + u_{0,1}^K) \\ &- (\delta_1^I + u_{0,1}^I)(\delta_1^K + u_{0,1}^K)u_{0,11}^K\Big].\end{aligned} \tag{8.181}$$

$$\tilde{\kappa}(S) = \frac{\sqrt{(u_{0,11}^I u_{0,11}^I)(\delta_1^K + u_{0,1}^K)(\delta_1^K + u_{0,1}^K) - [(\delta_1^I + u_{0,1}^I)u_{0,11}^I]^2}}{\tilde{g}_{11}^{3/2}}. \tag{8.182}$$

From Eq.(8.11), the unit bi-normal vector is obtained by

$$\tilde{\mathbf{n}}_3 = \tilde{\mathbf{n}}_1 \times \tilde{\mathbf{n}}_2 = \tilde{g}_3^I \mathbf{I}_I, \tag{8.183}$$

with

$$\tilde{g}_3^I = e_{IJK}\frac{\tilde{g}_1^J}{\sqrt{g_{11}}}\frac{\tilde{g}_2^K}{\tilde{\kappa}(S)} \text{ and } \tilde{g}_{33} = 1. \tag{8.184}$$

Because of the axial rotation, the rotation vector along the longitudinal arc length $s^1 = s$ of the deformed rod is

$$\boldsymbol{\omega} = \tilde{\kappa}(S)\mathbf{n}_3 + \tilde{\tau}(S)\mathbf{n}_1 = \omega_I \mathbf{I}_I, \tag{8.185}$$

where the torsion of the deformed rod is computed from Eq.(8.13), i.e.,

$$\tilde{\tau}(S) = \frac{e_{IJK}(\delta_1^I + u_{,1}^I)u_{,11}^J u_{,111}^K}{(u_{,11}^I u_{,11}^I)[(\delta_1^K + u_{,1}^K)(\delta_1^K + u_{,1}^K)] - [(\delta_1^I + u_{,1}^I)u_{,11}^I]^2}. \tag{8.186}$$

For $X^2 = X^3 = 0$, the foregoing equation is rewritten as

$$\tilde{\tau}(S) = \frac{e_{IJK}(\delta_1^I + u_{0,1}^I)u_{0,11}^J u_{0,111}^K}{(u_{0,11}^I u_{0,11}^I)[(\delta_1^K + u_{0,1}^K)(\delta_1^K + u_{0,1}^K)] - [(\delta_1^I + u_{0,1}^I)u_{0,11}^I]^2}. \tag{8.187}$$

From Eq.(8.185),

$$\begin{aligned} \tilde{\kappa}(S) &= \omega_I \mathbf{I}_I \cdot \tilde{\mathbf{n}}_3 = \omega_I \tilde{g}_3^I, \\ \tilde{\tau}(S) &= \omega_I \mathbf{I}_I \cdot \tilde{\mathbf{n}}_1 = \omega_I \tilde{g}_2^I, \end{aligned} \tag{8.188}$$

or

$$\omega_I = \tilde{\kappa}(S)\mathbf{n}_3 \cdot \mathbf{I}_I + \tilde{\tau}(S)\mathbf{n}_1 \cdot \mathbf{I}_I = \tilde{\kappa}(S)\tilde{g}_3^I + \tilde{\tau}(S)\tilde{g}_1^I. \tag{8.189}$$

However, from Eq.(8.163), $d\Psi / dS$ and $d\Phi / dS$ are determined by

$$\begin{aligned} \frac{d\Psi}{dS} &= \pm \frac{u_{0,11}^2[(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2] - u_{0,1}^2[(1+u_{0,1}^1)u_{0,11}^1 + (u_{0,1}^3)u_{0,11}^3]}{(1+\varepsilon_1^{(0)})^2\sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2}}, \\ \frac{d\Phi}{dS} &= -\frac{u_{0,11}^3(1+u_{0,1}^1) - u_{0,11}^1 u_{0,1}^3}{(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2}; \end{aligned} \tag{8.190}$$

or

$$\begin{aligned} \frac{d\Psi}{ds} &= \pm \frac{u_{0,11}^2[(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2] - u_{0,1}^2[(1+u_{0,1}^1)u_{0,11}^1 + (u_{0,1}^3)u_{0,11}^3]}{(1+\varepsilon_1^{(0)})^3\sqrt{(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2}}, \\ \frac{d\Phi}{ds} &= -\frac{u_{0,11}^3(1+u_{0,1}^1) - u_{0,11}^1 u_{0,1}^3}{(1+\varepsilon_1^{(0)})[(1+u_{0,1}^1)^2 + (u_{0,1}^3)^2]}. \end{aligned} \tag{8.191}$$

Substitution of Eqs.(8.163) and (8.190) into Eqs.(8.171) and (8.188) gives Θ and $d\Theta / dS$ when the initial twisting about the longitudinal direction of s is zero ($\Theta_0 = 0$).

As before, the constitutive laws for deformed rods give the corresponding resultant stresses for ($\alpha = 1, 2, 3$) as

$$\sigma_{1\alpha} = f_\alpha(\varepsilon_1, \gamma_{12}, \gamma_{13}, t). \tag{8.192}$$

The internal forces and moments in the deformed rod are defined as

$$\begin{aligned} N_1 &= \int_A \sigma_{11}\left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23}\right]dA, \\ Q_2 &= \int_A \sigma_{12}\left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23}\right]dA, \\ Q_3 &= \int_A \sigma_{13}\left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23}\right]dA, \\ M_3 &= \int_A \sigma_{11}\frac{X^2}{1+\varphi_1^{(2)}}\left[(1+\varepsilon_3)(1+\varepsilon_2)^2\cos\gamma_{23}\right]dA, \end{aligned}$$

$$M_2 = -\int_A \sigma_{11} \frac{X^3}{1+\theta_1^{(3)}} \left[(1+\varepsilon_3)(1+\varepsilon_2)^2 \cos\gamma_{23}\right] dA,$$

$$T_1 = \int_A \left[\sigma_{12} \frac{X^3(1+\varepsilon_3)}{1+\theta_1^{(3)}} - \sigma_{13} \frac{X^2(1+\varepsilon_2)}{1+\varphi_1^{(2)}} \right] \left[(1+\varepsilon_3)(1+\varepsilon_2)\cos\gamma_{23}\right] dA. \tag{8.193}$$

For convenience, the notations ($Q_2 \equiv N_2$, $Q_3 \equiv N_3$ and $T_1 \equiv M_1$) are used.

$$\begin{aligned} \mathbf{M} &\equiv M^I \mathbf{I}_I = M_\alpha \mathbf{n}_\alpha, \\ \boldsymbol{\mathcal{N}} &\equiv N^I \mathbf{I}_I = N_\alpha \mathbf{n}_\alpha, \\ {}^{\mathrm{N}}\mathbf{M} &\equiv {}^{\mathrm{N}}M^I \mathbf{I}_I = \mathbf{g}_1 \times \boldsymbol{\mathcal{N}}, \end{aligned} \tag{8.194}$$

where

$$\mathbf{g}_1 \equiv \frac{d\mathbf{r}}{dS} = (\delta_1^I + u_{0,1}^I)\mathbf{I}_I \text{ and } {}^{\mathrm{N}}\mathbf{M} \equiv \frac{1}{dS} d\mathbf{r} \times \boldsymbol{\mathcal{N}}. \tag{8.195}$$

With Eq.(8.154), the components of the internal forces in the $\mathbf{I}_I$-direction are

$$\begin{aligned} N^I &= N_\alpha \mathbf{n}_\alpha \cdot \mathbf{I}_I = N_\alpha \cos\theta_{(\mathbf{n}_\alpha, \mathbf{I}_I)} \\ &= \frac{N_1(\delta_1^I + u_{0,1}^I)}{1+\varepsilon_1^{(0)}} + \frac{N_2(\delta_2^I + \varphi_1^I)}{1+\varepsilon_2^{(0)}} + \frac{N_3(\delta_3^I + \theta_1^I)}{1+\varepsilon_3^{(0)}}, \end{aligned} \tag{8.196}$$

$$\begin{aligned} M^I &= M_\alpha \mathbf{n}_\alpha \cdot \mathbf{I}_I = M_\alpha \cos\theta_{(\mathbf{n}_\alpha, \mathbf{I}_I)} \\ &= \frac{M_1(\delta_1^I + u_{0;1}^I)}{1+\varepsilon_1^{(0)}} + \frac{M_2(\delta_2^I + \varphi_1^I)}{1+\varepsilon_2^{(0)}} + \frac{M_3(\delta_3^I + \theta_1^I)}{1+\varepsilon_3^{(0)}}, \end{aligned} \tag{8.197}$$

$${}^{\mathrm{N}}M^I = (\mathbf{g}_1 \times \boldsymbol{\mathcal{N}}) \cdot \mathbf{I}_I = e_{IJK}(\delta_1^J + u_{0,1}^J) N^K.$$

Using the external forces as in Eqs.(8.34)−(8.38), equations of motion on the deformed rod are given by

$$\begin{aligned} \boldsymbol{\mathcal{N}}_{,1} + \mathbf{q} &= \rho \mathbf{u}_{0,tt} + I_3 \boldsymbol{\varphi}_{1,tt} + I_2 \boldsymbol{\theta}_{1,tt}, \\ \mathbf{M}_{,1} + {}^{\mathrm{N}}\mathbf{M} + \mathbf{m} &= \mathscr{J}\boldsymbol{\Theta}_{,tt}, \end{aligned} \tag{8.198}$$

where

$$\begin{aligned} \mathscr{J}\boldsymbol{\Theta}_{,tt} &= [(I_3 u_{0,tt}^3 - I_2 u_{0,tt}^2) + (J_{22}\varphi_{1,tt}^{(3)} - J_{23}\varphi_{1,tt}^{(2)}) \\ &\quad + (J_{23}\theta_{1,tt}^{(3)} - J_{33}\theta_{1,tt}^{(2)})]\mathbf{I}_1 + (I_3 u_{0,tt}^1 + J_{23}\varphi_{1,tt}^{(1)} + J_{22}\theta_{1,tt}^{(1)})\mathbf{I}_2 \\ &\quad - (I_2 u_{0,tt}^1 + J_{22}\varphi_{1,tt}^{(1)} + J_{23}\theta_{1,tt}^{(1)})\mathbf{I}_3, \end{aligned} \tag{8.199}$$

$$\rho = \int_A \rho_0 dA, \;\; I_2 = \int_A \rho_0 X^3 dA, \;\; I_3 = \int_A \rho_0 X^2 dA,$$

$$\begin{aligned} J_{22} &= \int_A \rho_0 (X^3)^2 dA, \\ J_{33} &= \int_A \rho_0 (X^2)^2 dA, \\ J_{23} &= \int_A \rho_0 (X^2)(X^3) dA, \end{aligned} \tag{8.200}$$

and the scalar expressions are for $I = 1, 2, 3$,

$$\begin{aligned} N^I_{,1} + q^I &= \rho u^I_{0,tt} + I_3 \varphi^{(I)}_{1,tt} + I_2 \theta^{(I)}_{1,tt}, \\ M^I_{,1} + {}^{\mathrm{N}}M^I + m^I &= \mathscr{J} \boldsymbol{\Theta}_{,tt} \cdot \mathbf{I}_I. \end{aligned} \tag{8.201}$$

or

$$\begin{aligned} &\left[\frac{N_1(\delta^I_1 + u^I_{0,1})}{1+\varepsilon^{(0)}_1} + \frac{N_2(\delta^I_2 + \varphi^I_1)}{1+\varepsilon^{(0)}_2} + \frac{N_3(\delta^I_3 + \theta^I_1)}{1+\varepsilon^{(0)}_3} \right]_{,1} \\ &+ q^I = \rho u^I_{0,tt} + I_3 \varphi^{(I)}_{1,tt} + I_2 \theta^{(I)}_{1,tt}; \\ &\left[\frac{M_1(\delta^I_1 + u^I_{0,1})}{1+\varepsilon^{(0)}_1} + \frac{M_2(\delta^I_2 + \varphi^I_1)}{1+\varepsilon^{(0)}_2} + \frac{M_3(\delta^I_3 + \theta^I_1)}{1+\varepsilon^{(0)}_3} \right]_{,1} \\ &+ e_{IJK}(\delta^J_1 + u^J_{0,1}) N^K + m^I = \mathscr{J} \boldsymbol{\Theta}_{,tt} \cdot \mathbf{I}_I. \end{aligned} \tag{8.202}$$

The force condition at a point $\mathscr{P}_k$ with $X^1 = X^1_k$ is

$$\begin{aligned} {}^{-}\mathbf{N}(X^1_k) + {}^{+}\mathbf{N}(X^1_k) + \mathbf{F}_k &= 0, \\ {}^{-}N^I(X^1_k) = {}^{+}N^I(X^1_k) + F^I_k \quad (I &= 1, 2, 3). \end{aligned} \tag{8.203}$$

The force boundary condition at the boundary point $\mathscr{P}_r$ is

$$\begin{aligned} \mathbf{N}(X^1_r) + \mathbf{F}_r &= 0, \\ N^I(X^1_r) + F^I_r = 0 \quad (I &= 1, 2, 3). \end{aligned} \tag{8.204}$$

If there is a concentrated moment at a point $\mathscr{P}_k$ with $X^1 = X^1_k$, the corresponding moment boundary condition is

$$\begin{aligned} {}^{-}\mathbf{M}(X^1_k) + {}^{+}\mathbf{M}(X^1_k) + \mathbf{M}_k &= 0, \\ {}^{-}M^I(X^1_k) = {}^{+}M^I(X^1_k) + M^I_k \quad (I &= 1, 2, 3). \end{aligned} \tag{8.205}$$

The moment boundary condition at the boundary point $\mathscr{P}_r$ is

$$\begin{aligned} \mathbf{M}(X^1_r) + \mathbf{M}_r &= 0, \\ M^I(X^1_r) + M^I_r = 0 \quad (I &= 1, 2, 3). \end{aligned} \tag{8.206}$$

The displacement continuity and boundary conditions are

$$u_{k-}^{I} = u_{k+}^{I} \text{ and } u_{r}^{I} = B_{r}^{I} \quad (I = 1, 2, 3). \tag{8.207}$$

The rod theory can be reduced to the Cosserat theory of rods (e.g., E. and F. Cosserat,1909; Ericksen and Truesdell, 1958; Whitman and DeSilva,1969).

8.5. Nonlinear curved rods

Consider an arbitrary coordinates as (S^1, S^2, S^3) on the cross section of the rod. The deformed curve of the rod is shown in Fig.8.6. The coordinate S^1 is along the longitudinal direction of rod and the other two coordinates S^2 and S^3 are on the cross section of the rod. Without loss of generality, S^2 and S^3 are collinear to $\mathbf{N}_2$ and $\mathbf{N}_3$ for the curvature and torsion directions of the curve, respectively. The coordinates for the deformed rod are (s^1, s^2, s^3). Since the widths of rod in two directions of S^2 and S^3 are very small compared to the length of the rod, the elongations in the two directions of S^2 and S^3 should be very small, which are ignorable. Thus, the following assumptions will be adopted.

(i) The elongations in the two directions of S^2 and S^3 are very small (i.e., $\varepsilon_2 \approx 0$ and $\varepsilon_3 \approx 0$).

(ii) Under bending only, the neutral surface is not deformed (i.e., $\gamma_{12} \approx 0$ and $\gamma_{13} \approx 0$).

As in Eqs.(8.69) and (8.74), consider external distributed forces and moments on the initial configuration for ($I, \Lambda = 1, 2, 3$) as

$$\mathbf{q} = q^I \mathbf{I}_I = q^\Lambda \mathbf{N}_\Lambda \text{ and } \mathbf{m} = m^I \mathbf{I}_I = m^\Lambda \mathbf{N}_\Lambda \tag{8.208}$$

and concentrated forces on the initial configuration at a point $S^1 = S_k$,

$$\mathbf{F}_k = F_k^I \mathbf{I}_I = F_k^\Lambda \mathbf{N}_\Lambda \text{ and } \mathbf{M}_k = M_k^I \mathbf{I}_I = M_k^\Lambda \mathbf{N}_\Lambda. \tag{8.209}$$

Thus, one obtains the relations, i.e.,

$$\begin{aligned} F^\Lambda |_{S^1=S} &= F^I \mathbf{I}_I \cdot \mathbf{N}_\Lambda = F^I \cos\theta_{(\mathbf{I}_I, \mathbf{N}_\Lambda)}, \\ M^\Lambda |_{S^1=S} &= M^I \mathbf{I}_I \cdot \mathbf{N}_\Lambda = M^I \cos\theta_{(\mathbf{I}_I, \mathbf{N}_\Lambda)}. \end{aligned} \tag{8.210}$$

The displacement vectors on the initial configuration are

$$\mathbf{R}(S^1) = X^I(S^1)\mathbf{I}_I, \mathbf{R}(S) = X^I(S)\mathbf{I}_I \text{ and } \mathbf{R}_k = X_k^I \mathbf{I}_I. \tag{8.211}$$

The internal forces and moments for $(S^1 > S_k^1)$ are

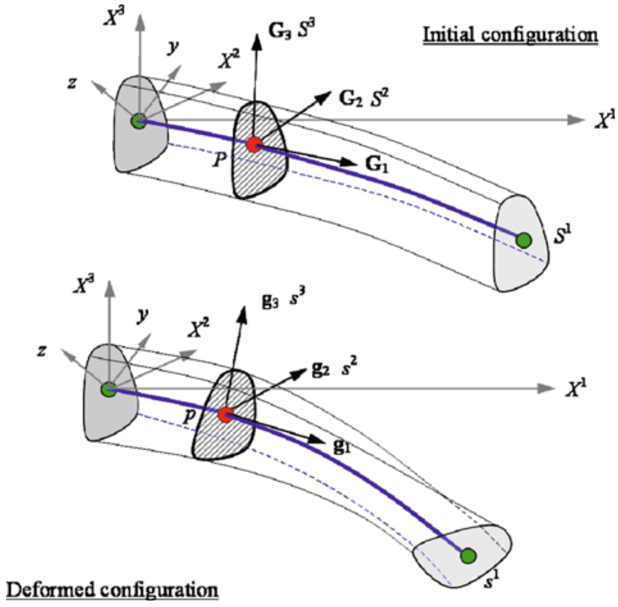

Fig. 8.6 A curved rod with initial and deformed configuration.

$$\begin{aligned}
\mathbf{F}\,|_{S^1=S} &= \sum\nolimits_{k=1}\mathbf{F}_k + \int_0^S \mathbf{q}\,dS^1, \\
\mathbf{M}\,|_{S^1=S} &= \sum\nolimits_{k=1}\mathbf{M}_k + \int_0^S \mathbf{m}\,dS^1 \\
&\quad + \sum\nolimits_{k=1}(\mathbf{R}(S)-\mathbf{R}_k)\times {}^k\mathbf{F} + \int_0^S (\mathbf{R}(S)-\mathbf{R}(S^1))\times\mathbf{q}\,dS^1;
\end{aligned} \tag{8.212}$$

or for $I, J, K = 1, 2, 3\ \ (I \neq J \neq K \neq I)$,

$$\begin{aligned}
F^I\Big|_{S^1=S} &= \sum\nolimits_{k=1} {}^kF^I + \int_0^S q^I dS^1, \\
M^I\Big|_{S^1=S} &= \sum\nolimits_{k=1} M_k^I + \int_0^S m^I dS^1 \\
&\quad + \sum\nolimits_{k=1} e_{IJK}(X_S^J - X_k^J)F_k^K + \int_0^S e_{IJK}(X_S^J - X_k^J)q^K dS^1.
\end{aligned} \tag{8.213}$$

8.5.1. A curved rod theory based on the Cartesian coordinates

As in Eqs.(8.79)–(8.84) for the strains of the 3-D deformed beam, the exact strain for the 3-D deformed rods can be obtained. Similar to Eq. (8.142), the displacement field for any fiber of the deformed rod at a position $\mathbf{R}$ is assumed by

$$u^I = u_0^I(S,t) + \sum_{n=1}^{\infty}(S^2)^n \varphi_n^{(I)}(S,t) + \sum_{n=1}^{\infty}(S^3)^n \theta_n^{(I)}(S,t)$$
$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(S^2)^m (S^3)^n \vartheta_{mn}^{(I)}(S,t), \tag{8.214}$$

where $S^1 = S$, and u_0^I is displacements of the centroid curve of the rod for $S^2 = S^3 = 0$, and $X^I = X^I(S^1)$. The coefficients of the higher order terms $\varphi_n^{(I)}$, $\theta_n^{(I)}$ and $\vartheta_{mn}^{(I)}(S)$ ($m, n = 1, 2, \cdots$) are from the Taylor series expansion, i.e.,

$$\varphi_n^{(I)} = \frac{1}{n!}\frac{\partial^n u^I}{\partial (S^2)^n}\bigg|_{(S^2,S^3)=(0,0)},$$
$$\theta_n^{(I)} = \frac{1}{n!}\frac{\partial^n u^I}{\partial (S^3)^n}\bigg|_{(S^2,S^3)=(0,0)}, \tag{8.215}$$
$$\vartheta_{mn}^{(I)} = \frac{1}{m!n!}\frac{\partial^{m+n} u^I}{\partial (S^2)^m \partial (S^3)^n}\bigg|_{(S^2,S^3)=(0=0)}.$$

Substitution of Eq.(8.214) into Eqs.(8.79)–(8.82) gives

$$\varepsilon_1 \approx \varepsilon_1^{(0)} + \frac{\partial \varepsilon_1}{\partial S^2}\bigg|_{(S^2,\,S^3)=(0,0)} S^2 + \frac{\partial \varepsilon_1}{\partial S^3}\bigg|_{(S^2,\,S^3)=(0,\,0)} S^3$$
$$+\frac{1}{2!}\frac{\partial^2 \varepsilon_1}{\partial (S^2)^2}\bigg|_{(S^2,\,S^3)=(0,\,0)} (S^2)^2 + \frac{1}{2!}\frac{\partial^2 \varepsilon_1}{\partial (S^3)^2}\bigg|_{(S^2,\,S^3)=(0,\,0)} (S^3)^2$$
$$+\frac{\partial^2 \varepsilon_1}{\partial S^2 \partial S^3}\bigg|_{(S^2,\,S^3)=(0,\,0)} S^2 S^3 + \cdots$$
$$= \varepsilon_1^{(0)} + \frac{(X_{,1}^I + u_{0,1}^I)\varphi_{1,1}^{(I)}}{G_{11}(1+\varepsilon_1^{(0)})} S^2 + \frac{(X_{,1}^I + u_{0,1}^I)\theta_{1,1}^{(I)}}{G_{11}(1+\varepsilon_1^{(0)})} S^3$$
$$+\frac{1}{2}\left\{\frac{[2(X_{,1}^I + u_{0,1}^I)\varphi_{2,1}^{(I)}] + \varphi_{1,1}^{(I)}\varphi_{1,1}^{(I)}}{G_{11}(1+\varepsilon_1^{(0)})} - \frac{[(X_{,1}^I + u_{0,1}^I)\varphi_{1,1}^{(I)}]^2}{G_{11}^2[(1+\varepsilon_1^{(0)})]^3}\right\}(S^2)^2$$
$$+\frac{1}{2}\left\{\frac{[2(X_{,1}^I + u_{0,1}^I)\theta_{2,1}^{(I)}] + \theta_{1,1}^{(I)}\theta_{1,1}^{(I)}}{G_{11}(1+\varepsilon_1^{(0)})} - \frac{[(X_{,1}^I + u_{0,1}^I)\theta_{1,1}^{(I)}]^2}{G_{11}^2[(1+\varepsilon_1^{(0)})]^3}\right\}(S^3)^2$$
$$+\left[\frac{(X_{,1}^I + u_{0,1}^I)\vartheta_{11,1}^{(I)}}{G_{11}(1+\varepsilon_1^{(0)})} + \frac{\varphi_{1,1}^{(I)}\theta_{1,1}^{(I)}}{G_{11}(1+\varepsilon_1^{(0)})}\right] S^2 S^3 + \cdots, \tag{8.216}$$

$$\varepsilon_2 \approx \varepsilon_2^{(0)} + \frac{\partial \varepsilon_2}{\partial S^2}\bigg|_{(S^2,\,S^3)=(0,\,0)} S^2 + \frac{\partial \varepsilon_2}{\partial S^3}\bigg|_{(S^2,\,S^3)=(0,\,0)} S^3$$
$$+\frac{1}{2!}\frac{\partial^2 \varepsilon_2}{\partial (S^2)^2}\bigg|_{(S^2,\,S^3)=(0,\,0)} (S^2)^2 + \frac{1}{2!}\frac{\partial^2 \varepsilon_2}{\partial (X^3)^2}\bigg|_{(S^2,\,S^3)=(0,\,0)} (S^3)^2$$

$$
\begin{aligned}
&+\frac{\partial^2 \varepsilon_2}{\partial S^2 \partial S^3}\bigg|_{(S^2,\, S^3)=(0,\,0)} S^2 S^3 + \cdots \\
&= \varepsilon_2^{(0)} + \frac{2(X^I_{,2}+\varphi_1^{(I)})\varphi_2^{(I)}}{G_{22}(1+\varepsilon_2^{(0)})} S^2 + \frac{(X^I_{,2}+\varphi_1^{(I)})\vartheta_{11}^{(I)}}{G_{22}(1+\varepsilon_2^{(0)})} S^3 \\
&+ \left\{ \frac{[2\varphi_2^{(I)}\varphi_2^{(I)} + 3(X^I_{,2}+\varphi_1^{(I)})\varphi_3^{(I)}]}{G_{22}(1+\varepsilon_2^{(0)})} - \frac{2[(X^I_{,2}+\varphi_1^{(I)})\varphi_2^{(I)}]^2}{G_{22}^2(1+\varepsilon_2^{(0)})^3} \right\} (S^2)^2 \\
&+ \frac{1}{4}\left\{ \frac{2\vartheta_{11}^{(I)}\vartheta_{11}^{(I)}}{G_{22}(1+\varepsilon_2^{(0)})} - \frac{[(X^I_{,2}+\varphi_1^{(I)})\vartheta_{11}^{(I)}]^2}{G_{22}^2(1+\varepsilon_2^{(0)})^3} \right\} (S^3)^2 \\
&+ \left[\frac{4(X^I_{,2}+\varphi_1^{(I)})\vartheta_{21}^{(I)}}{G_{22}(1+\varepsilon_2^{(0)})} + \frac{2\varphi_2^{(I)}\vartheta_{11}^{(I)}}{G_{22}(1+\varepsilon_2^{(0)})} \right] S^2 S^3 + \cdots; \qquad (8.217)
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_3 &\approx \varepsilon_3^{(0)} + \frac{\partial \varepsilon_3}{\partial S^2}\bigg|_{(S^2,\, S^3)=(0,0)} S^3 + \frac{\partial \varepsilon_3}{\partial S^3}\bigg|_{(S^2,\, S^3)=(0,0)} S^3 \\
&+ \frac{1}{2!}\frac{\partial^2 \varepsilon_3}{\partial (S^2)^2}\bigg|_{(S^2,\, S^3)=(0,0)} (S^2)^2 + \frac{1}{2!}\frac{\partial^2 \varepsilon_3}{\partial (S^3)^2}\bigg|_{(S^2,\, S^3)=(0,0)} (S^3)^2 \\
&+ \frac{\partial^2 \varepsilon_3}{\partial S^2 \partial S^3}\bigg|_{(S^2,\, S^3)=(0,\,0)} S^2 S^3 + \cdots \\
&= \varepsilon_3^{(0)} + \frac{(X^I_{,3}+\theta_1^{(I)})\vartheta_{11}^{(I)}}{G_{33}(1+\varepsilon_3^{(0)})} S^2 + \frac{2(X^I_{,3}+\theta_1^{(I)})\theta_2^{(I)}}{G_{33}(1+\varepsilon_3^{(0)})} S^3 \\
&+ \frac{1}{2}\left\{ \frac{\vartheta_{11}^{(I)}\vartheta_{11}^{(I)}}{G_{33}(1+\varepsilon_3^{(0)})} - \frac{[(X^I_{,3}+\theta_1^{(I)})\vartheta_{11}^{(I)}]^2}{2G_{33}^2(1+\varepsilon_3^{(0)})^3} \right\} (S^2)^2 \\
&+ \left\{ \frac{[2\theta_2^{(I)}\theta_2^{(I)} + 3(X^I_{,3}+\theta_1^{(I)})\theta_3^{(I)}]}{G_{33}(1+\varepsilon_3^{(0)})} - \frac{2[(X^I_{,3}+\theta_1^{(I)})\theta_2^{(I)}]^2}{G_{33}^2(1+\varepsilon_3^{(0)})^3} \right\} (S^3)^2 \\
&+ \left[\frac{4(X^I_{,3}+\theta_1^{(I)})\vartheta_{12}^{(I)}}{G_{33}(1+\varepsilon_3^{(0)})} + \frac{2\theta_2^{(I)}\vartheta_{11}^{(I)}}{G_{33}(1+\varepsilon_3^{(0)})} \right] S^2 S^3 + \cdots; \qquad (8.218)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{12} &\approx \gamma_{12}^{(0)} + \frac{\partial \gamma_{12}}{\partial S^2}\bigg|_{(S^2,\, S^3)=(0,\,0)} S^2 + \frac{\partial \gamma_{12}}{\partial S^3}\bigg|_{(S^2,\, S^3)=(0,\,0)} S^3 \\
&+ \frac{1}{2!}\frac{\partial^2 \gamma_{12}}{\partial (S^2)^2}\bigg|_{(S^2,\, S^3)=(0,\,0)} (S^2)^2 + \frac{1}{2!}\frac{\partial^2 \gamma_{12}}{\partial (S^3)^2}\bigg|_{(S^2,\, S^3)=(0,\,0)} (S^3)^2 \\
&+ \frac{\partial^2 \gamma_{12}}{\partial S^2 \partial S^3}\bigg|_{(S^2,\, S^3)=(0,\,0)} S^2 S^3 + \cdots
\end{aligned}
$$

$$
\begin{aligned}
&= \gamma_{12}^{(0)} + \frac{1}{\cos\gamma_{12}^{(0)}} \left\{ \frac{2(X_{,1}^{I} + u_{0,1}^{(I)})\varphi_2^{(I)} + (X_{,2}^{I} + \varphi_1^{(I)})\varphi_{1,1}^{(I)}}{\sqrt{G_{11}G_{22}}(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})} \right. \\
&\left. - \sin\gamma_{12}^{(0)} \left[\frac{(X_{,1}^{I} + u_{0,1}^{(I)})\varphi_{1,1}^{(I)}}{G_{11}(1+\varepsilon_1^{(0)})^2} + \frac{2(X_{,2}^{I} + \varphi_1^{(I)})\varphi_2^{(I)}}{G_{22}(1+\varepsilon_2^{(0)})^2} \right] \right\} S^2 \\
&+ \frac{1}{\cos\gamma_{12}^{(0)}} \left\{ \frac{(X_{,2}^{I} + \varphi_1^{(I)})\theta_{1,1}^{(I)} + (X_{,1}^{I} + u_{0,1}^{(I)})\vartheta_{11}^{(I)}}{\sqrt{G_{11}G_{22}}(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})} \right. \\
&\left. - \sin\gamma_{12}^{(0)} \left[\frac{(X_{,1}^{I} + u_{0,1}^{(I)})\theta_{1,1}^{(I)}}{G_{11}(1+\varepsilon_1^{(0)})^2} + \frac{(X_{,2}^{I} + \varphi_1^{(I)})\vartheta_{11}^{(I)}}{G_{22}(1+\varepsilon_2^{(0)})^2} \right] \right\} S^3 + \cdots; \qquad (8.219)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{13} &\approx \gamma_{13}^{(0)} + \left.\frac{\partial\gamma_{13}}{\partial S^2}\right|_{(S^2,\,S^3)=(0,\,0)} S^2 + \left.\frac{\partial\gamma_{13}}{\partial S^3}\right|_{(S^2,\,S^3)=(0,\,0)} S^3 \\
&+ \frac{1}{2!}\left.\frac{\partial^2\gamma_{13}}{\partial(S^2)^2}\right|_{(S^2,\,S^3)=(0,\,0)} (S^2)^2 + \frac{1}{2!}\left.\frac{\partial^2\gamma_{13}}{\partial(S^3)^2}\right|_{(S^2,\,S^3)=(0,\,0)} (S^3)^2 \\
&+ \left.\frac{\partial^2\gamma_{13}}{\partial S^2\partial S^3}\right|_{(S^2,\,S^3)=(0,\,0)} S^2S^3 + \cdots \\
&= \gamma_{13}^{(0)} + \frac{1}{\cos\gamma_{13}^{(0)}} \left\{ \frac{(X_{,3}^{I} + \theta_1^{(I)})\varphi_{1,1}^{(I)} + (X_{,1}^{I} + u_{0,1}^{(I)})\vartheta_{11}^{(I)}}{\sqrt{G_{11}G_{33}}(1+\varepsilon_1^{(0)})(1+\varepsilon_3^{(0)})} \right. \\
&\left. - \sin\gamma_{13}^{(0)} \left[\frac{(X_{,1}^{I} + u_{0,1}^{(I)})\varphi_{1,1}^{(I)}}{G_{11}(1+\varepsilon_1^{(0)})^2} + \frac{(X_{,3}^{I} + \theta_1^{(I)})\vartheta_{11}^{(I)}}{G_{33}(1+\varepsilon_3^{(0)})^2} \right] \right\} S^2 \\
&+ \frac{1}{\cos\gamma_{13}^{(0)}} \left\{ \frac{2(X_{,1}^{I} + u_{0,1}^{(I)})\theta_2^{(I)} + (X_{,3}^{I} + \theta_1^{(I)})\vartheta_{1,1}^{(I)}}{\sqrt{G_{11}G_{33}}(1+\varepsilon_1^{(0)})(1+\varepsilon_3^{(0)})} \right. \\
&\left. - \sin\gamma_{13}^{(0)} \left[\frac{(X_{,1}^{I} + u_{0,1}^{(I)})\vartheta_{1,1}^{(I)}}{G_{11}(1+\varepsilon_1^{(0)})^2} + \frac{2(X_{,3}^{I} + \theta_1^{(I)})\theta_2^{(I)}}{G_{33}(1+\varepsilon_3^{(0)})^2} \right] \right\} S^3 + \cdots; \qquad (8.220)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{23} &\approx \gamma_{23}^{(0)} + \left.\frac{\partial\gamma_{23}}{\partial S^2}\right|_{(S^2,\,S^3)=(0,\,0)} S^2 + \left.\frac{\partial\gamma_{23}}{\partial S^3}\right|_{(S^2,\,S^3)=(0,\,0)} S^3 \\
&+ \frac{1}{2}\left.\frac{\partial^2\gamma_{23}}{\partial(S^2)^2}\right|_{(S^2,\,S^3)=(0,\,0)} (S^2)^2 + \frac{1}{2}\left.\frac{\partial^2\gamma_{23}}{\partial(S^3)^2}\right|_{(S^2,\,S^3)=(0,\,0)} (S^3)^2 \\
&+ \left.\frac{\partial^2\gamma_{23}}{\partial S^2\partial S^3}\right|_{(S^2,\,S^3)=(0,\,0)} S^2S^3 + \cdots
\end{aligned}
$$

$$
\begin{aligned}
&= \gamma_{23}^{(0)} + \frac{1}{\cos\gamma_{23}^{(0)}} \left\{ \frac{2(X_{,3}^I + \theta_1^{(I)})\varphi_2^{(I)} + (X_{,2}^I + \varphi_1^{(I)})\vartheta_{11}^{(I)}}{\sqrt{G_{22}G_{33}}\,(1+\varepsilon_2^{(0)})(1+\varepsilon_3^{(0)})} \right. \\
&\quad \left. - \sin\gamma_{23}^{(0)} \left[\frac{2(X_{,2}^I + \varphi_1^{(I)})\varphi_2^{(I)}}{G_{22}(1+\varepsilon_2^{(0)})^2} + \frac{(X_{,3}^I + \theta_1^{(I)})\vartheta_{11}^{(I)}}{G_{33}(1+\varepsilon_3^{(0)})^2} \right] \right\} S^2 \\
&\quad + \frac{1}{\cos\gamma_{23}^{(0)}} \left\{ \frac{2(X_{,2}^I + \varphi_1^{(I)})\theta_2^{(I)} + (X_{,3}^I + \theta_1^{(I)})\vartheta_{11}^{(I)}}{(1+\varepsilon_2^{(0)})(1+\varepsilon_3^{(0)})} \right. \\
&\quad \left. - \sin\gamma_{23}^{(0)} \left[\frac{(X_{,2}^I + \varphi_1^{(I)})\vartheta_{11}^{(I)}}{G_{22}(1+\varepsilon_2^{(0)})^2} + \frac{2(X_{,3}^I + \theta_1^{(I)})\theta_2^{(I)}}{G_{33}(1+\varepsilon_3^{(0)})^2} \right] \right\} S^3 + \cdots, \qquad (8.221)
\end{aligned}
$$

where

$$
\begin{aligned}
\varepsilon_1^{(0)} &= \frac{1}{\sqrt{G_{11}}} \sqrt{(X_{,1}^I + u_{0,1}^I)(X_{,1}^I + u_{0,1}^I)} - 1, \\
\varepsilon_2^{(0)} &= \frac{1}{\sqrt{G_{22}}} \sqrt{(X_{,2}^I + \varphi_1^{(I)})(X_{,2}^I + \varphi_1^{(I)})} - 1, \\
\varepsilon_3^{(0)} &= \frac{1}{\sqrt{G_{33}}} \sqrt{(X_{,3}^I + \theta_1^{(I)})(X_{,3}^I + \theta_1^{(I)})} - 1, \\
\gamma_{12}^{(0)} &= \sin^{-1} \frac{(X_{,1}^I + u_{0,1}^I)(X_{,2}^I + \varphi_1^{(I)})}{\sqrt{G_{11}G_{22}}\,(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})}, \\
\gamma_{13}^{(0)} &= \sin^{-1} \frac{(X_{,1}^I + u_{0,1}^I)(X_{,3}^I + \theta_1^{(I)})}{\sqrt{G_{11}G_{33}}\,(1+\varepsilon_1^{(0)})(1+\varepsilon_3^{(0)})}, \\
\gamma_{23}^{(0)} &= \sin^{-1} \frac{(X_{,2}^I + \varphi_1^{(I)})(X_{,3}^I + \theta_1^{(I)})}{\sqrt{G_{22}G_{33}}\,(1+\varepsilon_2^{(0)})(1+\varepsilon_3^{(0)})}.
\end{aligned}
\qquad (8.222)
$$

From Assumptions (i) and (ii), consider the zero order terms of the Taylor series of the five strains to give

$$
\varepsilon_2^{(0)} = 0,\ \varepsilon_3^{(0)} = 0;\ \gamma_{12}^{(0)} = 0,\ \gamma_{13}^{(0)} = 0,\ \gamma_{23}^{(0)} = 0. \qquad (8.223)
$$

The stretch of the deformed rod for $S^2 = S^3 = 0$ satisfies

$$
(1+\varepsilon_1^{(0)})^2 = \frac{1}{G_{11}} (X_{,1}^I + u_{0,1}^I)(X_{,1}^I + u_{0,1}^I). \qquad (8.224)
$$

Equation (8.224) implies that only 1-dimenisonal stretch is considered as in the cable. Similarly, from the higher order terms of the Taylor series of the six strains, the relations for the unknowns in displacement field can be obtained. From Eqs.(8.222) and (8.223),

$$\begin{aligned}
&\frac{1}{G_{11}(1+\varepsilon_1^{(0)})^2}(X^I_{,1}+u^I_{0,1})(X^I_{,1}+u^I_{0,1})=1,\\
&\frac{1}{G_{22}}(X^I_{,2}+\varphi_1^{(I)})(X^I_{,2}+\varphi_1^{(I)})=1,\\
&\frac{1}{G_{33}}(X^I_{,3}+\theta_1^{(I)})(X^I_{,3}+\theta_1^{(I)})=1;\\
&(X^I_{,1}+u^I_{0,1})(X^I_{,2}+\varphi_1^{(I)})=0,\\
&(X^I_{,1}+u^I_{0,1})(X^I_{,3}+\theta_1^{(I)})=0,\\
&(X^I_{,2}+\varphi_1^{(I)})(X^I_{,3}+\theta_1^{(I)})=0.
\end{aligned} \tag{8.225}$$

Using the zero order terms of S^2 and S^3 in Eq.(8.83), the directional cosine matrix ($(l_{ij})_{3\times 3}$) is

$$\begin{aligned}
\cos\theta_{(\mathbf{n}_1,\mathbf{I}_I)}&=l_{1I}=\frac{X^I_{,1}+u^I_{0,1}}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})},\\
\cos\theta_{(\mathbf{n}_2,\mathbf{I}_I)}&=l_{2I}=\frac{X^I_{,2}+\varphi_1^{(I)}}{\sqrt{G_{22}}(1+\varepsilon_2^{(0)})},\\
\cos\theta_{(\mathbf{n}_3,\mathbf{I}_I)}&=l_{3I}=\frac{X^I_{,3}+\theta_1^{(I)}}{\sqrt{G_{33}}(1+\varepsilon_3^{(0)})}.
\end{aligned} \tag{8.226}$$

From the geometrical relations, the nine directional cosines must satisfy trigonometric relations in Eqs.(8.155) and (8.156). As in Fig.8.5, consider the initial Euler angles (Φ_0, Ψ_0 and Θ_0) rotating about the axes of X^1, X^2 and X^3, respectively. The Euler angles of the deformed rod are (Φ, Ψ and Θ). As same as in Eq.(8.157)-(8.159) gives the direction cosine matrix ($\boldsymbol{l}=(l_{ij})_{3\times 3}$) in Eqs.(8.160) and (8.161) for the deformed rod. Compared with Eq.(8.235), with Eq.(8.156), equation (8.161) for the deformed rod gives

$$\begin{aligned}
&\frac{X^1_{,1}+u^1_{0,1}}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})}=\cos\Phi\cos\Psi,\\
&\frac{X^1_{,2}+\varphi_1^{(1)}}{\sqrt{G_{22}}}=-\cos\Phi\sin\Psi\cos\Theta+\sin\Phi\sin\Theta,\\
&\frac{X^1_{,3}+\theta_1^{(1)}}{\sqrt{G_{33}}}=\cos\Phi\sin\Psi\sin\Theta+\sin\Phi\cos\Theta,\\
&\frac{X^2_{,1}+u^2_{0,1}}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})}=\sin\Psi,
\end{aligned}$$

$$\begin{aligned}
\frac{X^2_{,2}+\varphi^{(2)}_1}{\sqrt{G_{22}}} &= \cos\Psi\cos\Theta, \\
\frac{X^2_{,3}+\theta^{(2)}_1}{\sqrt{G_{33}}} &= -\cos\Psi\sin\Theta, \\
\frac{X^3_{,1}+u^3_{0,1}}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)} &= -\sin\Phi\cos\Psi, \\
\frac{X^3_{,2}+\varphi^{(3)}_1}{\sqrt{G_{22}}} &= \sin\Phi\sin\Psi\cos\Theta+\cos\Phi\sin\Theta, \\
\frac{X^3_{,3}+\theta^{(3)}_1}{\sqrt{G_{33}}} &= -\sin\Phi\sin\Psi\sin\Theta+\cos\Phi\cos\Theta.
\end{aligned} \tag{8.227}$$

The first, fourth and seventh equations of Eq.(8.227) give

$$\begin{aligned}
\cos\Psi &= \pm\frac{\sqrt{(X^1_{,1}+u^1_{0,1})^2+(X^3_{,1}+u^3_{0,1})^2}}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)} = \pm\frac{\Delta}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)}, \\
\sin\Psi &= \frac{X^2_{,1}+u^2_{0,1}}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)}; \\
\cos\Phi &= \pm\frac{X^1_{,1}+u^1_{0,1}}{\sqrt{(X^1_{,1}+u^1_{0,1})^2+(X^3_{,1}+u^3_{0,1})^2}} = \pm\frac{X^1_{,1}+u^1_{0,1}}{\Delta}, \\
\sin\Phi &= \mp\frac{X^3_{,1}+u^3_{0,1}}{\sqrt{(X^1_{,1}+u^1_{0,1})^2+(X^3_{,1}+u^3_{0,1})^2}} = \mp\frac{X^3_{,1}+u^3_{0,1}}{\Delta},
\end{aligned} \tag{8.228}$$

and

$$\begin{aligned}
\frac{X^1_{,2}+\varphi^{(1)}_1}{\sqrt{G_{22}}} &= \mp\frac{1}{\Delta}\Big(\frac{\Delta_{12}}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)}\cos\Theta+(X^1_{,1}+u^3_{0,1})\sin\Theta\Big), \\
\frac{X^1_{,3}+\theta^{(1)}_1}{\sqrt{G_{33}}} &= \pm\frac{1}{\Delta}\Big(\frac{\Delta_{12}}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)}\sin\Theta-(X^3_{,1}+u^3_{0,1})\cos\Theta\Big), \\
\frac{X^2_{,2}+\varphi^{(2)}_1}{\sqrt{G_{22}}} &= \pm\frac{\Delta}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)}\cos\Theta, \\
\frac{X^2_{,3}+\theta^{(2)}_1}{\sqrt{G_{33}}} &= \mp\frac{\Delta}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)}\sin\Theta, \\
\frac{X^3_{,2}+\varphi^{(3)}_1}{\sqrt{G_{22}}} &= \mp\frac{1}{\Delta}\Big(\frac{\Delta_{23}}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)}\cos\Theta-(X^1_{,1}+u^1_{0,1})\sin\Theta\Big),
\end{aligned}$$

$$\frac{X^3_{,3}+\theta^{(3)}_1}{\sqrt{G_{33}}}=\pm\frac{1}{\Delta}\left(\frac{\Delta_{23}}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)}\sin\Theta+(X^1_{,1}+u^1_{0,1})\cos\Theta\right), \tag{8.229}$$

where

$$\begin{aligned}\Delta&=\sqrt{(X^1_{,1}+u^1_{0,1})^2+(X^3_{,1}+u^3_{0,1})^2},\\ \Delta_{12}&=(X^2_{,1}+u^2_{0,1})(X^1_{,1}+u^1_{0,1}),\\ \Delta_{23}&=(X^3_{,1}+u^3_{0,1})(X^2_{,1}+u^2_{0,1}).\end{aligned} \tag{8.230}$$

If $\Theta=\Theta_0$, the rotation about the longitudinal axis disappears. So only the bending rotation exists, which is the pure bending of the rod as in Section 8.3.1.

From Eq.(8.165), the change ratio of the directional cosines along the deformed rod in Eq.(8.166) becomes

$$\frac{\boldsymbol{dl}_\alpha}{ds}=\frac{dl_{\alpha I}}{ds}\mathbf{I}_I=\frac{dl_{\alpha I}}{\sqrt{\mathrm{G}_{11}}(1+\varepsilon^{(0)}_1)dS}\mathbf{I}_I \quad \text{for } \alpha=1,2,3. \tag{8.231}$$

As in Eq.(8.168), the change ratio of the directional cosines along the deformed rod can be computed by

$$\frac{\boldsymbol{dl}_\alpha}{ds}=\varepsilon_{IJK}\omega_J l_{\alpha K}\mathbf{I}_I, \tag{8.232}$$

where the rotation ratio components in Eq.(8.232) are

$$\omega_J=e_{IJK}\frac{dl_{\alpha I}}{ds}l_{\alpha K}=e_{IJK}\frac{dl_{\alpha I}}{\sqrt{\mathrm{G}_{11}}(1+\varepsilon^{(0)}_1)dS}l_{\alpha K}. \tag{8.233}$$

From Eq.(8.161), the foregoing equations give

$$\begin{aligned}\omega_1&=\frac{1}{\sqrt{\mathrm{G}_{11}}(1+\varepsilon^{(0)}_1)}\left(\sin\Phi\frac{d\Psi}{dS}+\cos\Psi\cos\Phi\frac{d\Theta}{dS}\right),\\ \omega_2&=\frac{1}{\sqrt{\mathrm{G}_{11}}(1+\varepsilon^{(0)}_1)}\left(\frac{d\Psi}{dS}+\sin\Psi\frac{d\Theta}{dS}\right),\\ \omega_3&=\frac{1}{\sqrt{\mathrm{G}_{11}}(1+\varepsilon^{(0)}_1)}\left(\cos\Phi\frac{d\Psi}{dS}-\sin\Phi\cos\Psi\frac{d\Theta}{dS}\right);\end{aligned} \tag{8.234}$$

similar to Eq.(8.171).

In an alike fashion, using Eq.(8.25), the particle location on the deformed, curved rod can be expressed by

$$\mathbf{r}=(X^I+u^I)\mathbf{I}_I. \tag{8.235}$$

The corresponding base vector of the deformed, curved rod is

$$\tilde{\mathbf{g}}_1 = \tilde{g}_1^I \mathbf{I}_I = (X_{,1}^I + u_{,1}^I)\mathbf{I}_I, \tag{8.236}$$

and the unit vector for the deformed, curved rod is

$$\tilde{\mathbf{n}}_1 = \frac{\tilde{\mathbf{g}}_1}{|\tilde{\mathbf{g}}_1|} = \frac{X_{,1}^I + u_{,1}^I}{\sqrt{(X_{,1}^I + u_{,1}^I)(X_{,1}^I + u_{,1}^I)}}\mathbf{I}_I = \frac{X_{,1}^I + u_{,1}^I}{\sqrt{G_{11}}(1+\varepsilon_1)}\mathbf{I}_I. \tag{8.237}$$

For $S^2 = S^3 = 0$,

$$\tilde{\mathbf{n}}_1 = \frac{\tilde{\mathbf{g}}_1}{|\tilde{\mathbf{g}}_1|} = \frac{X_{,1}^I + u_{0,1}^I}{\sqrt{(X_{,1}^I + u_{0,1}^I)(X_{,1}^I + u_{0,1}^I)}}\mathbf{I}_I = \frac{X_{,1}^I + u_{0,1}^I}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})}\mathbf{I}_I. \tag{8.238}$$

The base vector in the principal normal direction is as in Eq.(8.177), i.e.,

$$\tilde{\mathbf{g}}_2 = \frac{d\tilde{\mathbf{n}}_1}{ds} = \tilde{g}_2^I \mathbf{I}_I, \tag{8.239}$$

where

$$\begin{aligned}\tilde{g}_2^I = \frac{1}{g_{11}^2}\Big[&(X_{,11}^I + u_{,11}^I)(X_{,1}^K + u_{,1}^K)(X_{,1}^K + u_{,1}^K) \\ &-(X_{,1}^I + u_{,1}^I)(X_{,1}^K + u_{,1}^K)(X_{,11}^K + u_{,11}^K)\Big].\end{aligned} \tag{8.240}$$

Thus, the unit principal normal vector in Eq.(8.179) can be rewritten, i.e.,

$$\tilde{\mathbf{n}}_2 = \frac{\tilde{\mathbf{g}}_2}{|\tilde{\mathbf{g}}_2|} = \frac{\tilde{\mathbf{g}}_2}{\sqrt{\tilde{g}_{22}}} = \frac{\tilde{\mathbf{g}}_2}{\tilde{\kappa}(S)} = \frac{\tilde{g}_2^I}{\tilde{\kappa}(S)}\mathbf{I}_I, \tag{8.241}$$

where the curvature of the deformed rod is

$$\begin{aligned}&\tilde{\kappa}(S) = |\tilde{\mathbf{g}}_2| = \sqrt{\tilde{g}_{22}} = \sqrt{(X_{,ss}^I + u_{,ss}^I)(X_{,ss}^I + u_{,ss}^I)} = \frac{1}{\tilde{g}_{11}^{3/2}}\sqrt{\Xi_1}, \\ &\Xi_1 = (X_{,11}^I + u_{,11}^I)(X_{,11}^I + u_{,11}^I)(X_{,1}^K + u_{,1}^K)(X_{,1}^K + u_{,1}^K) \\ &\quad -[(X_{,1}^I + X_{,1}^I)(X_{,11}^I + u_{,11}^I)]^2.\end{aligned} \tag{8.242}$$

For $S^2 = S^3 = 0$,

$$\begin{aligned}\tilde{g}_2^I = \frac{1}{\tilde{g}_{11}^2}\Big[&(X_{,11}^I + u_{0,11}^I)(X_{,1}^K + u_{0,1}^K)(X_{,1}^K + u_{0,1}^K) \\ &-(X_{,1}^I + u_{0,1}^I)(X_{,1}^K + u_{0,1}^K)(X_{,11}^K + u_{0,11}^K)\Big],\end{aligned} \tag{8.243}$$

$$\begin{aligned}\Xi_1 = {} &(X_{,11}^I + u_{0,11}^I)(X_{,11}^I + u_{0,11}^I)(X_{,1}^K + u_{0,1}^K)(X_{,1}^K + u_{0,1}^K) \\ &-[(X_{,1}^I + u_{0,1}^I)(X_{,11}^I + u_{0,11}^I)]^2.\end{aligned} \tag{8.244}$$

The rotation vector of the deformed, curved rod can be expressed as in Eq.(8.185), i.e.,

$$\boldsymbol{\omega} = \tilde{\kappa}(S)\tilde{\mathbf{n}}_3 + \tilde{\tau}(S)\tilde{\mathbf{n}}_1 = \omega_I \mathbf{I}_I \tag{8.245}$$

where the torsion of the deformed rod is

$$\tilde{\tau}(S) = \frac{e_{IJK}(X^I_{,1} + u^I_{,1})(X^I_{,11} + u^J_{,11})(X^K_{,111} + u^K_{,111})}{\Xi_1}. \tag{8.246}$$

The foregoing equation for $S^2 = S^3 = 0$ becomes

$$\tilde{\tau}(S) = \frac{e_{IJK}(X^I_{,1} + u^I_{0,1})(X^I_{,11} + u^J_{0,11})(X^K_{,111} + u^K_{0,111})}{\Xi_1}. \tag{8.247}$$

From Eq.(8.245), equations similar to Eqs.(8.188) and (8.189) are:

$$\begin{aligned} \tilde{\kappa}(S) &= \omega_I \mathbf{I}_I \cdot \mathbf{n}_3 = \omega_I \tilde{g}^I_3, \\ \tilde{\tau}(S) &= \omega_I \mathbf{I}_I \cdot \mathbf{n}_1 = \omega_I \tilde{g}^I_2, \end{aligned} \tag{8.248}$$

or

$$\begin{aligned} \omega_I &= \tilde{\kappa}(S)\mathbf{n}_3 \cdot \mathbf{I}_I + \tilde{\tau}(S)\mathbf{n}_1 \cdot \mathbf{I}_I \\ &= \tilde{\kappa}(S)\tilde{g}^I_3 + \tilde{\tau}(S)\tilde{g}^I_1. \end{aligned} \tag{8.249}$$

From Eq.(8.238), $d\Psi / dS$ and $d\Phi / dS$ are determined by

$$\begin{aligned} \frac{d\Psi}{dS} &= \pm\frac{(X^2_{,11} + u^2_{0,11})\Delta^2 - [\Delta_{12}(X^2_{,11} + u^1_{0,11}) + \Delta_{23}(X^3_{,11} + u^3_{0,11})]}{G_{11}(1+\varepsilon^{(0)}_1)^2\Delta}, \\ \frac{d\Phi}{dS} &= -\frac{(X^3_{,11} + u^3_{0,11})(X^1_{,1} + u^1_{0,1}) - (X^1_{,11} + u^1_{0,11})(X^3_{,1} + u^3_{0,1})}{\Delta^2}; \end{aligned} \tag{8.250}$$

or

$$\begin{aligned} \frac{d\Psi}{ds} &= \pm\frac{(X^2_{,11} + u^2_{0,11})\Delta^2 - [\Delta_{12}(X^2_{,11} + u^1_{0,11}) + \Delta_{23}(X^3_{,11} + u^3_{0,11})]}{\left[\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)\right]^3\Delta}, \\ \frac{d\Phi}{ds} &= -\frac{(X^3_{,11} + u^3_{0,11})(X^1_{,1} + u^1_{0,1}) - (X^1_{,11} + u^1_{0,11})(X^3_{,1} + u^3_{0,1})}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)\Delta^2}. \end{aligned} \tag{8.251}$$

Substitution of Eqs.(8.228) and (8.250) into Eqs.(8.234) and (8.248) gives Θ and $d\Theta / dS$.

As in Eq.(8.192), the constitutive laws for deformed rods give the corresponding resultant stresses, and the internal forces and moments in the deformed rod are in the form of Eq.(8.193), i.e.,

$$\begin{aligned}
N_1 &= \int_A \sigma_{11}\left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23}\right]dA,\\
Q_2 &= \int_A \sigma_{12}\left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23}\right]dA,\\
Q_3 &= \int_A \sigma_{13}\left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23}\right]dA,\\
M_3 &= \int_A \sigma_{11}\frac{S^2\sqrt{G_{22}}}{1+\varphi_1^{(2)}}\left[(1+\varepsilon_3)(1+\varepsilon_2)^2\cos\gamma_{23}\right]dA,\\
M_2 &= -\int_A \sigma_{11}\frac{S^3\sqrt{G_{33}}}{1+\theta_1^{(3)}}\left[(1+\varepsilon_3)^2(1+\varepsilon_2)\cos\gamma_{23}\right]dA,\\
T_1 &= \int_A [\sigma_{12}\frac{S^3(1+\varepsilon_3)\sqrt{G_{33}}}{1+\theta_1^{(3)}}-\sigma_{13}\frac{S^2(1+\varepsilon_2)\sqrt{G_{22}}}{1+\varphi_1^{(2)}}]\\
&\quad\times\left[(1+\varepsilon_3)(1+\varepsilon_2)\cos\gamma_{23}\right]dA.
\end{aligned}\tag{8.252}$$

The notations $Q_2 \equiv N_2$, $Q_3 \equiv N_3$, and $T_1 \equiv M_1$ are used again. The internal forces are expressed as in Eq.(8.194), i.e.,

$$\begin{aligned}
\mathbf{M} &\equiv M^I\mathbf{I}_I = M_\alpha\mathbf{n}_\alpha,\\
\boldsymbol{\mathcal{N}} &\equiv N^I\mathbf{I}_I = N_\alpha\mathbf{n}_\alpha,\\
{}^{\mathrm{N}}\mathbf{M} &\equiv {}^{\mathrm{N}}M^I\mathbf{I}_I = \mathbf{g}_1\times\boldsymbol{\mathcal{N}},
\end{aligned}\tag{8.253}$$

where

$$\mathbf{g}_1 \equiv \frac{d\mathbf{r}}{dS} = (X_{,1}^I + u_{0,1}^I)\mathbf{I}_I \text{ and } {}^{\mathrm{N}}\mathbf{M} \equiv \frac{1}{ds}d\mathbf{r}\times\boldsymbol{\mathcal{N}}.\tag{8.254}$$

With Eq.(8.225), the components of the internal forces in the $\mathbf{I}_I$-direction are

$$\begin{aligned}
N^I &= N_\alpha\mathbf{n}_\alpha\cdot\mathbf{I}_I = N_\alpha\cos\theta_{(\mathbf{n}_\alpha,\mathbf{I}_I)}\\
&= \frac{N_1(X_{,1}^I+u_{0,1}^I)}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})}+\frac{N_2(X_{,2}^I+\varphi_1^I)}{\sqrt{G_{22}}(1+\varepsilon_2^{(0)})}+\frac{N_3(X_{,3}^I+\theta_1^I)}{\sqrt{G_{33}}(1+\varepsilon_3^{(0)})},
\end{aligned}\tag{8.255}$$

$$\begin{aligned}
M^I &= M_\alpha\mathbf{n}_\alpha\cdot\mathbf{I}_I = M_\alpha\cos\theta_{(\mathbf{n}_\alpha,\mathbf{I}_I)}\\
&= \frac{M_1(X_{,1}^I+u_{0,1}^I)}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})}+\frac{M_2(X_{,2}^I+\varphi_1^I)}{\sqrt{G_{22}}(1+\varepsilon_2^{(0)})}+\frac{M_3(X_{,3}^I+\theta_1^I)}{\sqrt{G_{33}}(1+\varepsilon_3^{(0)})},\\
{}^{\mathrm{N}}M^I &= (\mathbf{g}_1\times\boldsymbol{\mathcal{N}})\cdot\mathbf{I}_I = e_{IJK}(X_{,1}^J+u_{0,1}^J)N^K.
\end{aligned}\tag{8.256}$$

Using the external forces as in Eqs.(8.208)−(8.212), equations of motion on the deformed rod are given as in Eq.(8.198), i.e.,

$$\begin{aligned}&\mathcal{N}_{,1}+\mathbf{q}=\rho\mathbf{u}_{0,tt}+I_3\boldsymbol{\varphi}_{1,tt}+I_2\boldsymbol{\theta}_{1,tt},\\&\mathbf{M}_{,1}+{}^{\mathrm{N}}\mathbf{M}+\mathbf{m}=\mathscr{J}\boldsymbol{\Theta}_{,tt},\end{aligned}\tag{8.257}$$

where

$$\begin{aligned}\mathscr{J}\boldsymbol{\Theta}_{,tt}&=[(I_3u^3_{0,tt}-I_2u^2_{0,tt})+(J_{22}\varphi^{(3)}_{1,tt}-J_{23}\varphi^{(2)}_{1,tt})\\&\quad+(J_{23}\theta^{(3)}_{1,tt}-J_{33}\theta^{(2)}_{1,tt})]\mathbf{I}_1+(I_3u^1_{0,tt}+J_{23}\varphi^{(1)}_{1,tt}+J_{22}\theta^{(1)}_{1,tt})\mathbf{I}_2\\&\quad-(I_2u^1_{0,tt}+J_{22}\varphi^{(1)}_{1,tt}+J_{23}\theta^{(1)}_{1,tt})\mathbf{I}_3,\end{aligned}\tag{8.258}$$

and the scalar expressions are for $I=1,2,3$,

$$\begin{aligned}&N^I_{,1}+q_I=\rho u^I_{(0),tt}+I_3\varphi^{(I)}_{1,tt}+I_2\theta^{(I)}_{1,tt};\\&M^I_{,1}+{}^{\mathrm{N}}M^I+m^I=(\mathscr{J}\boldsymbol{\Theta}_{,tt})\cdot\mathbf{I}_I.\end{aligned}\tag{8.259}$$

or

$$\begin{aligned}&\left[\frac{N_1(X^I_{,1}+u^I_{0,1})}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)}+\frac{N_2(X^I_{,2}+\varphi^I_1)}{\sqrt{G_{22}}(1+\varepsilon^{(0)}_2)}+\frac{N_3(X^I_{,3}+\theta^I_1)}{\sqrt{G_{33}}(1+\varepsilon^{(0)}_3)}\right]_{,1}\\&+q^I=\rho u^I_{(0),tt}+I_3\varphi^{(I)}_{1,tt}+I_2\theta^{(I)}_{1,tt};\\&\left[\frac{M_1(X^I_{,1}+u^I_{0,1})}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)}+\frac{M_2(X^I_{,2}+\varphi^I_1)}{\sqrt{G_{22}}(1+\varepsilon^{(0)}_2)}+\frac{M_3(X^I_{,3}+\theta^I_1)}{\sqrt{G_{33}}(1+\varepsilon^{(0)}_3)}\right]_{,1}\\&+e_{IJK}(X^J_{,1}+u^J_{0,1})N^K+m^I=(\mathscr{J}\boldsymbol{\Theta}_{,tt})\cdot\mathbf{I}_I.\end{aligned}\tag{8.260}$$

From Assumption (i), one has $\varepsilon^{(0)}_2=\varepsilon^{(0)}_3=0$. In addition, the force and moment balance conditions at any point $\mathscr{P}_k$, and the force boundary conditions are given in Eqs.(8.203)–(8.206), and the displacement continuity and boundary conditions are the same as in Eq.(8.207).

8.5.2. A curved rod theory based on the curvilinear coordinates

In this section, the curved rod theory on the curvilinear coordinates is discussed in an analogy way as in the Cartesian coordinates. The strains for 3-D deformed beam in Eqs.(8.110)–(8.116) can be used for the 3-D deformed rod. Similar to Eq.(8.214), the displacement field for any fiber of the deformed rod at a position $\mathbf{R}$ is assumed by

$$\begin{aligned}u^\Lambda&=u^\Lambda_0(S,t)+\sum\nolimits_{n=1}^{\infty}(S^2)^n\varphi^{(\Lambda)}_n(S,t)+\sum\nolimits_{n=1}^{\infty}(S^3)^n\theta^{(\Lambda)}_n(S,t)\\&\quad+\sum\nolimits_{m=1}^{\infty}\sum\nolimits_{n=1}^{\infty}(S^2)^m(S^3)^n\vartheta^{(\Lambda)}_{mn}(S,t),\end{aligned}\tag{8.261}$$

where $S^1 = S$ and u_0^Λ ($\Lambda = 1, 2, 3$) are displacements of centroid curve of the rod for $S^2 = S^3 = 0$. The coefficients of the Taylor series expansion $\varphi_n^{(\Lambda)}$, $\theta_n^{(\Lambda)}$ and $\vartheta_{mn}^{(\Lambda)}$ ($m, n = 1, 2, \cdots$) are

$$\begin{aligned}
\varphi_n^{(I)} &= \frac{1}{n!}\frac{\partial^n u^I}{\partial (S^2)^n}\bigg|_{(S^2,\,S^3)=(0,0)}, \\
\theta_n^{(I)} &= \frac{1}{n!}\frac{\partial^n u^I}{\partial (S^3)^n}\bigg|_{(S^2,\,S^3)=(0,0)}, \\
\vartheta_{mn}^{(I)} &= \frac{1}{m!n!}\frac{\partial^{m+n} u^I}{\partial (S^2)^m \partial (S^3)^n}\bigg|_{(S^2,\,S^3)=(0,0)}.
\end{aligned} \tag{8.262}$$

Because $G_{\Lambda\Gamma} = 0$ ($\Lambda, \Gamma \in \{1, 2, 3\}$ and $\Lambda \neq \Gamma$), the Taylor series expansion of six strains are given as follows:

$$\begin{aligned}
\varepsilon_1 \approx{}& \varepsilon_1^{(0)} + \frac{\partial \varepsilon_1}{\partial S^2}\bigg|_{(S^2,\,S^3)=(0,0)} S^2 + \frac{\partial \varepsilon_1}{\partial S^3}\bigg|_{(S^2,\,S^3)=(0,\,0)} S^3 \\
&+ \frac{1}{2!}\frac{\partial^2 \varepsilon_1}{\partial (S^2)^2}\bigg|_{(S^2,\,S^3)=(0,0)} (S^2)^2 + \frac{1}{2!}\frac{\partial^2 \varepsilon_1}{\partial (S^3)^2}\bigg|_{(S^2,\,S^3)=(0,\,0)} (S^3)^2 \\
&+ \frac{\partial^2 \varepsilon_1}{\partial S^2 \partial S^3}\bigg|_{(S^2,\,S^3)=(0,0)} S^2 S^3 + \cdots \\
={}& \varepsilon_1^{(0)} + \frac{(\delta_1^\Lambda + u_{0;1}^\Lambda)\varphi_{1,1}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{11}(1+\varepsilon_1^{(0)})} S^2 + \frac{(\delta_1^\Lambda + u_{0;1}^\Lambda)\theta_{1;1}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{11}(1+\varepsilon_1^{(0)})} S^3 \\
&+ \frac{1}{2}\left\{\frac{[2(\delta_1^\Lambda + u_{0;1}^\Lambda)\varphi_{2;1}^{(\Lambda)} + \varphi_{1;1}^{(\Lambda)}\varphi_{1;1}^{(\Lambda)}]G_{\Lambda\Lambda}}{G_{11}(1+\varepsilon_1^{(0)})} - \frac{[(\delta_1^\Lambda + u_{0;1}^\Lambda)\varphi_{1;1}^{(\Lambda)}]^2 G_{\Lambda\Lambda}^2}{G_{11}^2(1+\varepsilon_1^{(0)})^3}\right\}(S^2)^2 \\
&+ \frac{1}{2}\left\{\frac{[2(\delta_1^\Lambda + u_{0;1}^\Lambda)\theta_{2;1}^{(\Lambda)} + \theta_{1;1}^{(\Lambda)}\theta_{1;1}^{(\Lambda)}]G_{\Lambda\Lambda}}{G_{11}(1+\varepsilon_1^{(0)})} - \frac{[(\delta_1^\Lambda + u_{0;1}^\Lambda)\theta_{1;1}^{(\Lambda)}]^2 G_{\Lambda\Lambda}^2}{G_{11}^2(1+\varepsilon_1^{(0)})^3}\right\}(S^3)^2 \\
&+ \left[\frac{(\delta_1^\Lambda + u_{0;1}^\Lambda)\vartheta_{11;1}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{11}(1+\varepsilon_1^{(0)})} + \frac{\varphi_{1;1}^{(\Lambda)}\theta_{1;1}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{11}(1+\varepsilon_1^{(0)})}\right] S^2 S^3 + \cdots,
\end{aligned} \tag{8.263}$$

$$\begin{aligned}
\varepsilon_2 \approx{}& \varepsilon_2^{(0)} + \frac{\partial \varepsilon_2}{\partial S^2}\bigg|_{(S^2,\,S^3)=(0,\,0)} S^2 + \frac{\partial \varepsilon_2}{\partial S^3}\bigg|_{(S^2,\,S^3)=(0,\,0)} S^3 \\
&+ \frac{1}{2!}\frac{\partial^2 \varepsilon_2}{\partial (S^2)^2}\bigg|_{(S^2,\,S^3)=(0,\,0)} (S^2)^2 + \frac{1}{2!}\frac{\partial^2 \varepsilon_2}{\partial (X^3)^2}\bigg|_{(S^2,\,S^3)=(0,\,0)} (S^3)^2 \\
&+ \frac{\partial^2 \varepsilon_2}{\partial S^2 \partial S^3}\bigg|_{(S^2,\,S^3)=(0,\,0)} S^2 S^3 + \cdots
\end{aligned}$$

$$
\begin{aligned}
&= \varepsilon_2^{(0)} + \frac{2(\delta_2^\Lambda + \varphi_1^{(\Lambda)})\varphi_2^{(\Lambda)} G_{\Lambda\Lambda}}{G_{22}(1+\varepsilon_2^{(0)})} S^2 + \frac{(\delta_2^\Lambda + \varphi_1^{(\Lambda)})\vartheta_{11}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{22}(1+\varepsilon_2^{(0)})} S^3 \\
&\quad + \left\{ \frac{[2\varphi_2^{(\Lambda)}\varphi_2^{(\Lambda)} + 3(\delta_2^\Lambda + \varphi_1^{(\Lambda)})\varphi_3^{(\Lambda)}] G_{\Lambda\Lambda}}{G_{22}(1+\varepsilon_2^{(0)})} - \frac{2[(\delta_2^\Lambda + \varphi_1^{(\Lambda)})\varphi_2^{(\Lambda)}]^2 G_{\Lambda\Lambda}^2}{G_{22}^2(1+\varepsilon_2^{(0)})^3} \right\} (S^2)^2 \\
&\quad + \frac{1}{2} \left\{ \frac{\vartheta_{11}^{(\Lambda)}\vartheta_{11}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{22}(1+\varepsilon_2^{(0)})} - \frac{[(\delta_2^\Lambda + \varphi_1^{(\Lambda)})\vartheta_{11}^{(\Lambda)}]^2 G_{\Lambda\Lambda}^2}{2G_{22}^2(1+\varepsilon_2^{(0)})^3} \right\} (S^3)^2 \\
&\quad + \left[\frac{4(\delta_2^\Lambda + \varphi_1^{(\Lambda)})\vartheta_{21}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{22}(1+\varepsilon_2^{(0)})} + \frac{2\varphi_2^{(\Lambda)}\vartheta_{11}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{22}(1+\varepsilon_2^{(0)})} \right] S^2 S^3 + \cdots; \qquad (8.264)
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_3 &\approx \varepsilon_3^{(0)} + \left.\frac{\partial \varepsilon_3}{\partial S^2}\right|_{(S^2,\,S^3)=(0,\,0)} S^2 + \left.\frac{\partial \varepsilon_3}{\partial S^3}\right|_{(S^2,\,S^3)=(0,\,0)} S^3 \\
&\quad + \frac{1}{2!}\left.\frac{\partial^2 \varepsilon_3}{\partial (S^2)^2}\right|_{(S^2,\,S^3)=(0,\,0)} (S^2)^2 + \frac{1}{2!}\left.\frac{\partial^2 \varepsilon_3}{\partial (S^3)^2}\right|_{(S^2,\,S^3)=(0,\,0)} (S^3)^2 \\
&\quad + \left.\frac{\partial^2 \varepsilon_3}{\partial S^2 \partial S^3}\right|_{(S^2,\,S^3)=(0,\,0)} S^2 S^3 + \cdots \\
&= \varepsilon_3^{(0)} + \frac{(\delta_3^\Lambda + \theta_1^{(\Lambda)})\vartheta_{11}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{33}(1+\varepsilon_3^{(0)})} S^2 + \frac{2(\delta_3^\Lambda + \theta_1^{(\Lambda)})\theta_2^{(\Lambda)} G_{\Lambda\Lambda}}{G_{33}(1+\varepsilon_3^{(0)})} S^3 \\
&\quad + \frac{1}{2} \left\{ \frac{\vartheta_{11}^{(\Lambda)}\vartheta_{11}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{33}(1+\varepsilon_3^{(0)})} - \frac{[(\delta_2^\Lambda + \theta_1^{(\Lambda)})\vartheta_{11}^{(\Lambda)}]^2 G_{\Lambda\Lambda}^2}{2G_{33}^2(1+\varepsilon_3^{(0)})^3} \right\} (S^2)^2 \\
&\quad + \left\{ \frac{[2\theta_2^{(\Lambda)}\theta_2^{(\Lambda)} + 3(\delta_3^\Lambda + \theta_1^{(\Lambda)})\theta_3^{(\Lambda)}] G_{\Lambda\Lambda}}{G_{33}(1+\varepsilon_3^{(0)})} - \frac{2[(\delta_3^\Lambda + \theta_1^{(\Lambda)})\theta_2^{(\Lambda)}]^2 G_{\Lambda\Lambda}^2}{(1+\varepsilon_3^{(0)})^3} \right\} (S^3)^2 \\
&\quad + \left[\frac{4(\delta_3^\Lambda + \theta_1^{(\Lambda)})\vartheta_{12}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{33}(1+\varepsilon_3^{(0)})} + \frac{2\theta_2^{(\Lambda)}\vartheta_{11}^{(\Lambda)} G_{\Lambda\Lambda}}{G_{33}(1+\varepsilon_3^{(0)})} \right] S^2 S^3 + \cdots; \qquad (8.265)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{12} &\approx \gamma_{12}^{(0)} + \left.\frac{\partial \gamma_{12}}{\partial S^2}\right|_{(S^2,\,S^3)=(0,\,0)} S^2 + \left.\frac{\partial \gamma_{12}}{\partial S^3}\right|_{(S^2,\,S^3)=(0,\,0)} S^3 \\
&\quad + \frac{1}{2!}\left.\frac{\partial^2 \gamma_{12}}{\partial (S^2)^2}\right|_{(S^2,\,S^3)=(0,\,0)} (S^2)^2 + \frac{1}{2!}\left.\frac{\partial^2 \gamma_{12}}{\partial (S^3)^2}\right|_{(S^2,\,S^3)=(0,\,0)} (S^3)^2 \\
&\quad + \left.\frac{\partial^2 \gamma_{12}}{\partial S^2 \partial S^3}\right|_{(S^2,\,S^3)=(0,\,0)} S^2 S^3 + \cdots \\
&= \gamma_{12}^{(0)} + \frac{1}{\cos \gamma_{12}^{(0)}} \left\{ \frac{2(\delta_1^\Lambda + u_{0;1}^{(\Lambda)})\varphi_2^{(\Lambda)} + (\delta_2^\Lambda + \varphi_1^{(\Lambda)})\varphi_{1;1}^{(\Lambda)}}{\sqrt{G_{11}G_{22}}(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})} G_{\Lambda\Lambda} \right.
\end{aligned}
$$

$$-\sin\gamma_{12}^{(0)}\left[\frac{(\delta_1^{\Lambda}+u_{0;1}^{(\Lambda)})\varphi_{1;1}^{(\Lambda)}}{G_{11}(1+\varepsilon_1^{(0)})^2}G_{\Lambda\Lambda}+\frac{2(\delta_2^{\Lambda}+\varphi_1^{(\Lambda)})\varphi_2^{(\Lambda)}}{G_{22}(1+\varepsilon_2^{(0)})^2}G_{\Lambda\Lambda}\right]\Bigg\}S^2$$

$$+\frac{1}{\cos\gamma_{12}^{(0)}}\left\{\frac{(\delta_2^{\Lambda}+\varphi_1^{(\Lambda)})\theta_{1;1}^{(\Lambda)}+(\delta_1^{\Lambda}+u_{0;1}^{(\Lambda)})\vartheta_{11}^{(\alpha)}}{\sqrt{G_{11}G_{22}}(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})}G_{\Lambda\Lambda}\right.$$

$$\left.-\sin\gamma_{12}^{(0)}\left[\frac{(\delta_1^{\Lambda}+u_{0;1}^{(\Lambda)})\theta_{1;1}^{(\Lambda)}G_{\Lambda\Lambda}}{G_{11}(1+\varepsilon_1^{(0)})^2}+\frac{(\delta_2^{\Lambda}+\varphi_1^{(\Lambda)})\vartheta_{11}^{(\Lambda)}}{G_{22}(1+\varepsilon_2^{(0)})^2}G_{\Lambda\Lambda}\right]\right\}S^3+\cdots; \quad (8.266)$$

$$\gamma_{13}\approx\gamma_{13}^{(0)}+\left.\frac{\partial\gamma_{13}}{\partial S^2}\right|_{(S^2,\,S^3)=(0,\,0)}S^2+\left.\frac{\partial\gamma_{12}}{\partial S^3}\right|_{(S^2,\,S^3)=(0,\,0)}S^3$$

$$+\frac{1}{2!}\left.\frac{\partial^2\gamma_{13}}{\partial(S^2)^2}\right|_{(S^2,\,S^3)=(0,\,0)}(S^2)^2+\frac{1}{2!}\left.\frac{\partial^2\gamma_{13}}{\partial(S^3)^2}\right|_{(S^2,\,S^3)=(0,\,0)}(S^3)^2$$

$$+\left.\frac{\partial^2\gamma_{13}}{\partial S^2\partial S^3}\right|_{(S^2,\,S^3)=(0,\,0)}S^2S^3+\cdots$$

$$=\gamma_{13}^{(0)}+\frac{1}{\cos\gamma_{13}^{(0)}}\left\{\frac{(\delta_3^{\Lambda}+\theta_1^{(\Lambda)})\varphi_{1;1}^{(\Lambda)}+(\delta_1^{\Lambda}+u_{0;1}^{(\Lambda)})\vartheta_{11}^{(\Lambda)}}{\sqrt{G_{11}G_{33}}(1+\varepsilon_1^{(0)})(1+\varepsilon_3^{(0)})}G_{\Lambda\Lambda}\right.$$

$$\left.-\sin\gamma_{13}^{(0)}\left[\frac{(\delta_1^{\Lambda}+u_{0;1}^{(\Lambda)})\varphi_{1;1}^{(\Lambda)}}{G_{11}(1+\varepsilon_1^{(0)})^2}+\frac{(\delta_3^{\Lambda}+\theta_1^{(\Lambda)})\vartheta_{11}^{(\Lambda)}}{G_{33}(1+\varepsilon_3^{(0)})^2}\right]G_{\Lambda\Lambda}\right\}S^2$$

$$+\frac{1}{\cos\gamma_{13}^{(0)}}\left\{\frac{2(\delta_1^{\Lambda}+u_{0;1}^{(\Lambda)})\theta_2^{(\Lambda)}+(\delta_3^{\Lambda}+\theta_1^{(\Lambda)})\vartheta_{1;1}^{(\Lambda)}}{\sqrt{G_{11}G_{33}}(1+\varepsilon_1^{(0)})(1+\varepsilon_3^{(0)})}G_{\Lambda\Lambda}\right.$$

$$\left.-\sin\gamma_{13}^{(0)}\left[\frac{(\delta_1^{\Lambda}+u_{0,1}^{(\Lambda)})\vartheta_{1,1}^{(\Lambda)}}{G_{11}(1+\varepsilon_1^{(0)})^2}+\frac{2(\delta_3^{\Lambda}+\theta_1^{(\Lambda)})\theta_2^{(\Lambda)}}{G_{33}(1+\varepsilon_3^{(0)})^2}\right]G_{\Lambda\Lambda}\right\}S^3+\cdots; \quad (8.267)$$

$$\gamma_{23}\approx\gamma_{23}^{(0)}+\left.\frac{\partial\gamma_{23}}{\partial S^2}\right|_{(S^2,\,S^3)=(0,\,0)}S^2+\left.\frac{\partial\gamma_{23}}{\partial S^3}\right|_{(S^2,\,S^3)=(0,\,0)}S^3$$

$$+\frac{1}{2!}\left.\frac{\partial^2\gamma_{23}}{\partial(S^2)^2}\right|_{(S^2,\,S^3)=(0,\,0)}(S^2)^2+\frac{1}{2!}\left.\frac{\partial^2\gamma_{23}}{\partial(S^3)^2}\right|_{(S^2,\,S^3)=(0,\,0)}(S^3)^2$$

$$+\left.\frac{\partial^2\gamma_{23}}{\partial S^2\partial S^3}\right|_{(S^2,\,S^3)=(0,\,0)}S^2S^3+\cdots$$

$$=\gamma_{23}^{(0)}+\frac{1}{\cos\gamma_{23}^{(0)}}\left\{\frac{2(\delta_3^{\Lambda}+\theta_1^{(\Lambda)})\varphi_2^{(\Lambda)}+(\delta_2^{\Lambda}+\varphi_1^{(\Lambda)})\vartheta_{11}^{(\Lambda)}}{\sqrt{G_{22}G_{33}}(1+\varepsilon_2^{(0)})(1+\varepsilon_3^{(0)})}G_{\Lambda\Lambda}\right.$$

$$-\sin\gamma_{23}^{(0)}\left[\frac{2(\delta_2^{\Lambda}+\varphi_1^{(\Lambda)})\varphi_2^{(\Lambda)}}{G_{22}(1+\varepsilon_2^{(0)})^2}+\frac{(\delta_3^{\Lambda}+\theta_1^{(\Lambda)})\vartheta_{11}^{(\Lambda)}}{G_{33}(1+\varepsilon_3^{(0)})^2}\right]G_{\Lambda\Lambda}\Bigg\}S^2$$

$$+\frac{1}{\cos\gamma_{23}^{(0)}}\Bigg\{\frac{2(\delta_2^{\Lambda}+\varphi_1^{(\Lambda)})\theta_2^{(\Lambda)}+(\delta_3^{\Lambda}+\theta_1^{(\Lambda)})\vartheta_{11}^{(\Lambda)}}{\sqrt{G_{22}G_{33}}(1+\varepsilon_2^{(0)})(1+\varepsilon_3^{(0)})}G_{\Lambda\Lambda}$$

$$-\sin\gamma_{23}^{(0)}\left[\frac{(\delta_2^{\Lambda}+\varphi_1^{(\Lambda)})\vartheta_{11}^{(\Lambda)}}{G_{22}(1+\varepsilon_2^{(0)})^2}+\frac{2(\delta_3^{\Lambda}+\theta_1^{(\Lambda)})\theta_2^{(\Lambda)}}{G_{33}(1+\varepsilon_3^{(0)})^2}\right]G_{\Lambda\Lambda}\Bigg\}S^3+\cdots, \tag{8.268}$$

where

$$\begin{aligned}
\varepsilon_1^{(0)}&=\frac{\sqrt{G_{\Lambda\Lambda}}}{\sqrt{G_{11}}}\sqrt{(\delta_1^{\Lambda}+u_{0;1}^{\Lambda})(\delta_1^{\Lambda}+u_{0;1}^{\Lambda})}-1,\\
\varepsilon_2^{(0)}&=\frac{\sqrt{G_{\Lambda\Lambda}}}{\sqrt{G_{22}}}\sqrt{(\delta_2^{\Lambda}+\varphi_1^{(\Lambda)})(\delta_2^{\Lambda}+\varphi_1^{(\Lambda)})}-1,\\
\varepsilon_3^{(0)}&=\frac{\sqrt{G_{\Lambda\Lambda}}}{\sqrt{G_{33}}}\sqrt{(\delta_3^{\Lambda}+\theta_1^{(\Lambda)})(\delta_3^{\Lambda}+\theta_1^{(\Lambda)})}-1,\\
\gamma_{12}^{(0)}&=\sin^{-1}\frac{(\delta_1^{\Lambda}+u_{0;1}^{\Lambda})(\delta_2^{\Lambda}+\varphi_1^{(\Lambda)})G_{\Lambda\Lambda}}{\sqrt{G_{11}G_{22}}(1+\varepsilon_1^{(0)})(1+\varepsilon_2^{(0)})},\\
\gamma_{13}^{(0)}&=\sin^{-1}\frac{(\delta_1^{\Lambda}+u_{0;1}^{\Lambda})(\delta_3^{\Lambda}+\theta_1^{(\Lambda)})G_{\Lambda\Lambda}}{\sqrt{G_{11}G_{33}}(1+\varepsilon_1^{(0)})(1+\varepsilon_3^{(0)})},\\
\gamma_{23}^{(0)}&=\sin^{-1}\frac{(\delta_2^{\Lambda}+\varphi_1^{(\Lambda)})(\delta_3^{\Lambda}+\theta_1^{(\Lambda)})G_{\Lambda\Lambda}}{\sqrt{G_{22}G_{33}}(1+\varepsilon_2^{(0)})(1+\varepsilon_3^{(0)})}.
\end{aligned} \tag{8.269}$$

From Assumptions (i) and (ii), the zero order term of the Taylor series of the five strains gives

$$\varepsilon_2^{(0)}=0,\ \varepsilon_3^{(0)}=0;\ \gamma_{12}^{(0)}=0,\ \gamma_{13}^{(0)}=0,\ \gamma_{23}^{(0)}=0. \tag{8.270}$$

The stretch of the deformed rod for $S^2=S^3=0$ satisfies

$$(1+\varepsilon_1^{(0)})^2=\frac{G_{\Lambda\Lambda}}{G_{11}}(\delta_1^{\Lambda}+u_{0;1}^{\Lambda})(\delta_1^{\Lambda}+u_{0;1}^{\Lambda}), \tag{8.271}$$

From Eqs.(8.269)-(8.271),

$$\frac{G_{\Lambda\Lambda}}{G_{11}(1+\varepsilon_1^{(0)})^2}(\delta_1^{\Lambda}+u_{0;1}^{\Lambda})(\delta_1^{\Lambda}+u_{0;1}^{\Lambda})=1,$$

$$\frac{G_{\Lambda\Lambda}}{G_{22}}(\delta_1^{\Lambda}+\varphi_1^{(\Lambda)})(\delta_2^{\Lambda}+\varphi_1^{(\Lambda)})=1,$$

$$\begin{aligned}
&\frac{G_{\Lambda\Lambda}}{G_{33}}(\delta_3^\Lambda+\theta_1^{(\Lambda)})(\delta_3^\Lambda+\theta_1^{(\Lambda)})=1;\\
&(\delta_1^\Lambda+u_{0;1}^\Lambda)(\delta_2^\Lambda+\varphi_1^{(\Lambda)})G_{\Lambda\Lambda}=0,\\
&(\delta_1^\Lambda+u_{0;1}^\Lambda)(\delta_3^\Lambda+\theta_1^{(\Lambda)})G_{\Lambda\Lambda}=0,\\
&(\delta_2^\Lambda+\varphi_1^{(\Lambda)})(\delta_3^\Lambda+\theta_1^{(\Lambda)})G_{\Lambda\Lambda}=0.
\end{aligned}\tag{8.272}$$

Using the first order terms of S^2 and S^3 in Eq.(8.115), the direction cosine matrix $((l_{ij})_{3\times 3})$ is

$$\begin{aligned}
\cos\theta_{(\mathbf{n}_1,\mathbf{N}_\Lambda)}&=l_{1\Lambda}=\frac{(\delta_1^\Lambda+u_{0;1}^\Lambda)\sqrt{G_{\Lambda\Lambda}}}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})},\\
\cos\theta_{(\mathbf{n}_2,\mathbf{N}_\Lambda)}&=l_{2\Lambda}=\frac{(\delta_2^\Lambda+\varphi_1^{(\Lambda)})\sqrt{G_{\Lambda\Lambda}}}{\sqrt{G_{22}}(1+\varepsilon_2^{(0)})},\\
\cos\theta_{(\mathbf{n}_3,\mathbf{N}_\Lambda)}&=l_{3\Lambda}=\frac{(\delta_3^\Lambda+\theta_1^{(\Lambda)})\sqrt{G_{\Lambda\Lambda}}}{\sqrt{G_{33}}(1+\varepsilon_3^{(0)})}.
\end{aligned}\tag{8.273}$$

As in Fig.8.5, the unknowns $\varphi_1^{(I)}$ and $\theta_1^{(I)}$ ($I=1,2,3$) can be determined by the three Euler angles (Φ, Ψ and Θ). Similarly, the Euler angles Φ and Ψ rotates around the axes of S^2 and S^3, respectively, and the Euler angle Θ rotates around the axis of S^1. Due to bending, the first rotation around the axis of S^2 is to form $(\bar{S}^1,\bar{S}^2,\bar{S}^3)$. The second rotation around the axis of $\bar{S}^3$ gives $(\bar{\bar{S}}^1,\bar{\bar{S}}^2,\bar{\bar{S}}^3)$. The last rotation around the axis of $\bar{\bar{S}}^1$ gives a frame of $(\bar{\bar{\bar{S}}}^1,\bar{\bar{\bar{S}}}^2,\bar{\bar{\bar{S}}}^3)$ for the final state of the rod, which is the coordinates (s^1,s^2,s^3). The direction-cosines give

$$\begin{aligned}
&\frac{1+u_{0;1}^1}{(1+\varepsilon_1^{(0)})}=\cos\Phi\cos\Psi,\\
&\frac{\varphi_1^{(1)}\sqrt{G_{11}}}{\sqrt{G_{22}}}=-\cos\Phi\sin\Psi\cos\Theta+\sin\Phi\sin\Theta,\\
&\frac{\theta_1^{(1)}\sqrt{G_{11}}}{\sqrt{G_{33}}}=\cos\Phi\sin\Psi\sin\Theta+\sin\Phi\cos\Theta;\\
&\frac{u_{0;1}^2\sqrt{G_{22}}}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})}=\sin\Psi,\\
&\varphi_1^{(2)}=\cos\Psi\cos\Theta-1,\\
&\frac{\theta_1^{(2)}\sqrt{G_{22}}}{\sqrt{G_{33}}}=-\cos\Psi\sin\Theta;
\end{aligned}\tag{8.247a}$$

$$\frac{u^3_{0;1}\sqrt{G_{33}}}{\sqrt{G_{11}}\,(1+\varepsilon^{(0)}_1)} = -\sin\Phi\cos\Psi,$$
$$\frac{\varphi^{(3)}_1\sqrt{G_{33}}}{\sqrt{G_{22}}} = \sin\Phi\sin\Psi\cos\Theta + \cos\Phi\sin\Theta, \tag{8.274b}$$
$$\theta^{(3)}_1 = -\sin\Phi\sin\Psi\sin\Theta + \cos\Phi\cos\Theta - 1.$$

The first, fourth and seventh equations in Eq.(8.274) yield

$$\cos\Psi = \pm\frac{\sqrt{(1+u^1_{0;1})^2 G_{11} + (u^3_{0;1})^2 G_{33}}}{\sqrt{G_{11}}\,(1+\varepsilon^{(0)}_1)} = \pm\frac{\Delta}{\sqrt{G_{11}}\,(1+\varepsilon^{(0)}_1)},$$
$$\sin\Psi = \frac{u^2_{0;1}\sqrt{G_{22}}}{\sqrt{G_{11}}\,(1+\varepsilon^{(0)}_1)};$$
$$\cos\Phi = \pm\frac{(1+u^1_{0;1})\sqrt{G_{11}}}{\sqrt{(1+u^1_{0,1})^2 G_{11} + (u^3_{0,1})^2 G_{33}}} = \pm\frac{(1+u^1_{0;1})\sqrt{G_{11}}}{\Delta}, \tag{8.275}$$
$$\sin\Phi = \mp\frac{u^3_{0;1}\sqrt{G_{33}}}{\sqrt{(1+u^1_{0;1})^2 G_{11} + (u^3_{0;1})^2 G_{33}}} = \mp\frac{u^3_{0;1}\sqrt{G_{33}}}{\Delta},$$

and

$$\frac{\sqrt{G_{11}}}{\sqrt{G_{22}}}\varphi^{(1)}_1 = \mp\frac{1}{\Delta}\left[\frac{\Delta_{12}}{\sqrt{G_{11}}\,(1+\varepsilon^{(0)}_1)}\cos\Theta + u^3_{0;1}\sqrt{G_{33}}\sin\Theta\right],$$
$$\frac{\sqrt{G_{11}}}{\sqrt{G_{33}}}\theta^{(1)}_1 = \pm\frac{1}{\Delta}\left[\frac{\Delta_{12}}{\sqrt{G_{11}}\,(1+\varepsilon^{(0)}_1)}\sin\Theta - u^3_{0;1}\sqrt{G_{33}}\cos\Theta\right],$$
$$\varphi^{(2)}_1 = \pm\frac{\Delta}{\sqrt{G_{11}}\,(1+\varepsilon^{(0)}_1)}\cos\Theta - 1;$$
$$\frac{\sqrt{G_{22}}}{\sqrt{G_{33}}}\theta^{(2)}_1 = \mp\frac{\Delta}{\sqrt{G_{11}}\,(1+\varepsilon^{(0)}_1)}\sin\Theta, \tag{8.276}$$
$$\frac{\sqrt{G_{33}}}{\sqrt{G_{22}}}\varphi^{(3)}_1 = \mp\frac{1}{\Delta}\left[\frac{\Delta_{23}}{\sqrt{G_{11}}\,(1+\varepsilon^{(0)}_1)}\cos\Theta - (1+u^1_{0;1})\sqrt{G_{11}}\sin\Theta\right],$$
$$\theta^{(3)}_1 = \pm\frac{1}{\Delta}\left[\frac{\Delta_{23}}{\sqrt{G_{11}}\,(1+\varepsilon^{(0)}_1)}\sin\Theta + (1+u^1_{0;1})\sqrt{G_{11}}\cos\Theta\right] - 1,$$

where

$$\begin{aligned}\Delta &= \sqrt{(1+u_{0;1}^1)^2 G_{11} + (u_{0;1}^3)^2 G_{33}},\\ \Delta_{12} &= (u_{0;1}^2)(1+u_{0;1}^1)\sqrt{G_{11}G_{22}},\\ \Delta_{23} &= u_{0;1}^3 u_{0;1}^2 \sqrt{G_{22}G_{33}}.\end{aligned} \tag{8.277}$$

If $\Theta = 0$, the rotation about the longitudinal axis disappears. So only the bending rotation exists, which is the pure bending of the rod as in Section 8.3.2.

From Eq.(8.165), the change ratio of the directional cosines along the deformed rod in Eq.(8.166) becomes

$$\frac{\boldsymbol{dl}_\alpha}{ds} = \frac{d(l_{\alpha\Lambda}\mathbf{G}_\Lambda)}{ds} = \frac{l_{\alpha\Lambda;1}}{\sqrt{\mathrm{G}_{11}}(1+\varepsilon_1^{(0)})}\mathbf{G}_\Lambda \quad \text{for } \alpha=1, 2, 3. \tag{8.278}$$

Similar to Eq.(8.168), the change ratio of the directional cosines along the deformed rod can be computed by

$$\frac{\boldsymbol{dl}_\alpha}{ds} = \varepsilon_{\Lambda\Gamma\mathrm{K}}\,\omega_\Gamma l_{\alpha\mathrm{K}}\mathbf{G}_\Lambda, \tag{8.279}$$

where the rotation ratio components in Eq.(8.169) are computed by

$$\omega_\Gamma = \varepsilon_{\Lambda\Gamma\mathrm{K}}\frac{dl_{\alpha\Lambda;1}}{\sqrt{\mathrm{G}_{11}}(1+\varepsilon_1^{(0)})}l_{\alpha\mathrm{K}}. \tag{8.280}$$

From Eq.(8.161), the foregoing equations give

$$\begin{aligned}\omega_1 &= \frac{1}{\sqrt{\mathrm{G}_{11}}(1+\varepsilon_1^{(0)})}\left[(\frac{d\Psi}{dS}\sin\Phi + \frac{d\Theta}{dS}\cos\Psi\cos\Phi) - \Gamma_{13}^2\right],\\ \omega_2 &= \frac{1}{\sqrt{\mathrm{G}_{11}}(1+\varepsilon_1^{(0)})}\left[(\frac{d\Psi}{dS} + \frac{d\Theta}{dS}\sin\Psi) - \Gamma_{11}^3\right],\\ \omega_3 &= \frac{1}{\sqrt{\mathrm{G}_{11}}(1+\varepsilon_1^{(0)})}\left[(\frac{d\Psi}{dS}\cos\Phi - \frac{d\Theta}{dS}\sin\Phi\cos\Psi) - \Gamma_{12}^1\right];\end{aligned} \tag{8.281}$$

similar to Eq.(8.171). Note that for the orthogonal curvilinear coordinates, one has $\Gamma_{\Lambda\Gamma}^{\mathrm{K}} = 0$ ($\Lambda \neq \Gamma \neq \mathrm{K} \neq \Lambda$), $\Gamma_{\Lambda\Lambda}^{\Gamma} = -G_{\Lambda\Lambda,\Gamma}/2G_{\Gamma\Gamma}$ ($\Lambda \neq \Gamma$) and $\Gamma_{\Lambda\Gamma}^{\Lambda} = G_{\Lambda\Lambda,\Gamma}/2G_{\Lambda\Lambda}$ (no summation on Λ).

In an alike fashion, using Eq.(8.25), the particle location on the deformed, curved rod can be expressed by

$$\mathbf{r} = (S^\Lambda + u^\Lambda)\mathbf{G}_\Lambda. \tag{8.282}$$

The corresponding base vector of the deformed, curved rod is

$$\tilde{\mathbf{g}}_1 = \tilde{g}_1^\Lambda \mathbf{G}_\Lambda = (\delta_1^\Lambda + u_{;1}^\Lambda)\mathbf{G}_\Lambda, \tag{8.283}$$

and the unit vector for the deformed, curved rod is

$$\tilde{\mathbf{n}}_1 = \frac{\tilde{\mathbf{g}}_1}{|\tilde{\mathbf{g}}_1|} = \frac{\delta_1^\Lambda + u_{;1}^\Lambda}{\sqrt{(\delta_1^\Gamma + u_{;1}^\Gamma)(\delta_1^\Gamma + u_{;1}^\Gamma)}}\mathbf{G}_\Lambda = \frac{\delta_1^\Lambda + u_{;1}^\Lambda}{\sqrt{G_{11}}(1+\varepsilon_1)}\mathbf{G}_\Lambda. \tag{8.284}$$

For $S^2 = S^3 = 0$,

$$\tilde{\mathbf{n}}_1 = \frac{\tilde{\mathbf{g}}_1}{|\tilde{\mathbf{g}}_1|} = \frac{\delta_1^\Lambda + u_{0;1}^\Lambda}{\sqrt{(\delta_1^\Gamma + u_{0;1}^\Gamma)(\delta_1^\Gamma + u_{0;1}^\Gamma)}}\mathbf{G}_\Lambda = \frac{\delta_1^\Lambda + u_{0;1}^\Lambda}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})}\mathbf{G}_\Lambda. \tag{8.285}$$

The base vector in the principal normal direction of the deformed, curved rod is as in Eq.(8.177), i.e.,

$$\tilde{\mathbf{g}}_2 = \frac{d\tilde{\mathbf{n}}_1}{ds} = \tilde{g}_2^\Lambda \mathbf{G}_\Lambda, \tag{8.286}$$

where

$$\tilde{g}_2^\Lambda = \frac{1}{\sqrt{G_{11}}(1+\varepsilon_1)}\left[\frac{\delta_1^\Lambda + u_{;1}^\Lambda}{\sqrt{G_{11}}(1+\varepsilon_1)}\right]_{;1}. \tag{8.287}$$

Thus, the unit principal normal vector in Eq.(8.179) can be rewritten, i.e.,

$$\tilde{\mathbf{n}}_2 = \frac{\tilde{\mathbf{g}}_2}{|\tilde{\mathbf{g}}_2|} = \frac{\tilde{\mathbf{g}}_2}{\sqrt{\tilde{g}_{22}}} = \frac{\tilde{\mathbf{g}}_2}{\tilde{\kappa}(S)} = \frac{\tilde{g}_2^\Lambda}{\tilde{\kappa}(S)}\mathbf{G}_\Lambda, \tag{8.288}$$

where the curvature of the deformed rod is

$$\begin{aligned} &\tilde{\kappa}(S) = |\mathbf{g}_2| = \sqrt{\tilde{g}_{22}} = \sqrt{g_2^\Lambda g_2^\Lambda}, \\ &\tilde{g}_{22} = \frac{1}{G_{11}(1+\varepsilon_1)^2}\left[\frac{\delta_1^\Lambda + u_{;1}^\Lambda}{\sqrt{G_{11}}(1+\varepsilon_1)}\right]_{;1}\left[\frac{\delta_1^\Lambda + u_{;1}^\Lambda}{\sqrt{G_{11}}(1+\varepsilon_1)}\right]_{;1}. \end{aligned} \tag{8.289}$$

For $S^2 = S^3 = 0$,

$$\tilde{g}_2^\Lambda = \frac{1}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})}\left[\frac{\delta_1^\Lambda + u_{0;1}^\Lambda}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})}\right]_{;1}, \tag{8.290}$$

$$\tilde{g}_{22} = \frac{1}{G_{11}(1+\varepsilon_1^{(0)})^2}\left[\frac{\delta_1^\Lambda + u_{0;1}^\Lambda}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})}\right]_{;1}\left[\frac{\delta_1^\Lambda + u_{0;1}^\Lambda}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})}\right]_{;1}. \tag{8.291}$$

The rotation vector of the deformed curved rod can be expressed as in Eq.(8.186), i.e.,

$$\boldsymbol{\omega} = \tilde{\kappa}(S)\mathbf{n}_3 + \tilde{\tau}(S)\mathbf{n}_1 = \omega_\Lambda \mathbf{N}_\Lambda \tag{8.292}$$

where the torsion of the deformed rod is

$$\tilde{\tau}(S) = e_{IJK} \frac{(\delta_1^{\Lambda} + u_{;1}^{\Lambda})}{\sqrt{G_{11}}(1+\varepsilon_1)} \frac{\tilde{g}_2^{\Gamma}}{\tilde{\kappa}(S)} \left[\frac{\tilde{g}_2^{\mathrm{K}}}{\tilde{\kappa}(S)} \right]_{;1} G_{\Lambda}^{I} G_{\Gamma}^{J} G_{\mathrm{K}}^{K}. \tag{8.293}$$

For $S^2 = S^3 = 0$, the foregoing equation becomes

$$\tilde{\tau}(S) = e_{IJK} \frac{(\delta_1^{\Lambda} + u_{0;1}^{\Lambda})}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})} \frac{\tilde{g}_2^{\Gamma}}{\tilde{\kappa}(S)} \left[\frac{\tilde{g}_2^{\mathrm{K}}}{\tilde{\kappa}(S)} \right]_{;1} G_{\Lambda}^{I} G_{\Gamma}^{J} G_{\mathrm{K}}^{K}. \tag{8.294}$$

From Eq.(8.292),

$$\begin{aligned} \tilde{\kappa}(S) &= \omega_{\Lambda} \mathbf{N}_{\Lambda} \cdot \tilde{\mathbf{n}}_3 = \omega_{\Lambda} \tilde{g}_3^{\Lambda}, \\ \tilde{\tau}(S) &= \omega_{\Lambda} \mathbf{N}_{\Lambda} \cdot \tilde{\mathbf{n}}_1 = \omega_{\Lambda} \tilde{g}_2^{\Lambda}, \end{aligned} \tag{8.295}$$

or

$$\begin{aligned} \omega_{\Lambda} &= \tilde{\kappa}(S) \mathbf{n}_3 \cdot \mathbf{N}_{\Lambda} + \tilde{\tau}(S) \mathbf{n}_1 \cdot \mathbf{N}_{\Lambda} \\ &= \tilde{\kappa}(S) \tilde{g}_3^{\Lambda} + \tilde{\tau}(S) \tilde{g}_1^{\Lambda}. \end{aligned} \tag{8.296}$$

From Eq.(8.275), $d\Psi / dS$ and $d\Phi / dS$ are determined by

$$\begin{aligned} \frac{d\Psi}{dS} &= \pm \frac{1}{G_{11}(1+\varepsilon_1^{(0)})^2 \Delta} \left\{ \Delta^2 \frac{d}{dS}(u_{0;1}^2 \sqrt{G_{22}}) \right. \\ &\quad \left. - \Delta_{12} \frac{d}{dS}\left[(1+u_{0;1}^1)\sqrt{G_{11}} \right] - \Delta_{23} \frac{d}{dS}(u_{0;1}^3 \sqrt{G_{33}}) \right\}, \\ \frac{d\Phi}{dS} &= -\frac{1}{\Delta^2} \left\{ (1+u_{0;1}^1)\sqrt{G_{11}} \frac{d}{dS}(u_{0;1}^3 \sqrt{G_{33}}) \right. \\ &\quad \left. - (u_{0;1}^3 \sqrt{G_{33}}) \frac{d}{dS}\left[(1+u_{0;1}^1)\sqrt{G_{11}} \right] \right\}. \end{aligned} \tag{8.297}$$

Substitution of Eqs.(8.275) and (8.297) into Eqs.(8.281) and (8.295) gives Θ and $d\Theta / dS$.

As in Eq.(8.193), the constitutive laws for deformed rods give the corresponding resultant stresses. Similar to Eq.(8.194), the internal forces and moments in the deformed rod are defined by

$$\begin{aligned} N_1 &= \int_A \sigma_{11} \left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23} \right] dA, \\ Q_2 &= \int_A \sigma_{12} \left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23} \right] dA, \\ Q_3 &= \int_A \sigma_{13} \left[(1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23} \right] dA, \\ M_3 &= \int_A \sigma_{11} \frac{S^2}{1+\varphi_1^{(2)}} \left[(1+\varepsilon_3)(1+\varepsilon_2)^2 \cos\gamma_{23} \right] dA, \end{aligned} \tag{8.298a}$$

$$M_2 = -\int_A \sigma_{11} \frac{S^3}{1+\theta_1^{(3)}} \left[(1+\varepsilon_3)^2 (1+\varepsilon_2) \cos\gamma_{23} \right] dA,$$

$$T_1 = \int_A \left[\sigma_{12} \frac{S^3(1+\varepsilon_3)}{1+\theta_1^{(3)}} - \sigma_{13} \frac{S^2(1+\varepsilon_2)}{1+\varphi_1^{(2)}} \right] \times \left[(1+\varepsilon_3)(1+\varepsilon_2) \cos\gamma_{23} \right] dA. \tag{8.298b}$$

The notations $Q_2 \equiv N_2$, $Q_3 \equiv N_3$, and $T_1 \equiv M_1$ are used again. The internal forces are expressed as in Eq.(8.195), i.e.,

$$\begin{aligned} \mathbf{M} &\equiv M^\Lambda \mathbf{N}_\Lambda = M_\alpha \mathbf{n}_\alpha, \\ \boldsymbol{\mathcal{N}} &\equiv N^\Lambda \mathbf{N}_\Lambda = N_\alpha \mathbf{n}_\alpha, \\ {}^{\mathbf{N}}\mathbf{M} &\equiv {}^{\mathbf{N}}M^\Lambda \mathbf{N}_\Lambda = \mathbf{g}_1 \times \boldsymbol{\mathcal{N}}, \end{aligned} \tag{8.299}$$

where

$$\mathbf{g}_1 \equiv \frac{d\mathbf{r}}{dS} = (\delta_1^\Lambda + u_{0;1}^\Lambda) G_{\Lambda\Lambda} \mathbf{N}_\Lambda \text{ and } {}^{\mathbf{N}}\mathbf{M} \equiv \frac{1}{dS} d\mathbf{r} \times \boldsymbol{\mathcal{N}}. \tag{8.300}$$

With Eq.(8.273), the components of the internal forces in the $\mathbf{G}_\Lambda$-direction are

$$\begin{aligned} N^\Lambda &= N_\alpha \mathbf{n}_\alpha \cdot \mathbf{N}_\Lambda = N_\alpha \cos\theta_{(\mathbf{n}_\alpha, \mathbf{N}_\Lambda)} \\ &= \left[\frac{N_1(\delta_1^\Lambda + u_{0;1}^\Lambda)}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})} + \frac{N_2(\delta_2^\Lambda + \varphi_1^{(\Lambda)})}{\sqrt{G_{22}}(1+\varepsilon_2^{(0)})} + \frac{N_3(\delta_3^\Lambda + \theta_1^{(\Lambda)})}{\sqrt{G_{33}}(1+\varepsilon_3^{(0)})} \right] \sqrt{G_{\Lambda\Lambda}}, \end{aligned} \tag{8.301}$$

$$\begin{aligned} M^\Lambda &= M_\alpha \mathbf{n}_\alpha \cdot \mathbf{N}_\Lambda = M_\alpha \cos\theta_{(\mathbf{n}_\alpha, \mathbf{N}_\Lambda)} \\ &= \left[\frac{M_1(\delta_1^\Lambda + u_{0;1}^\Lambda)}{\sqrt{G_{11}}(1+\varepsilon_1^{(0)})} + \frac{M_2(\delta_2^\Lambda + \varphi_1^{(\Lambda)})}{\sqrt{G_{22}}(1+\varepsilon_2^{(0)})} + \frac{M_3(\delta_3^\Lambda + \theta_1^{(\Lambda)})}{\sqrt{G_{33}}(1+\varepsilon_3^{(0)})} \right] \sqrt{G_{\Lambda\Lambda}}, \\ {}^{\mathbf{N}}M^\Lambda &= (\mathbf{g}_1 \times \boldsymbol{\mathcal{N}}) \cdot \mathbf{N}_\Lambda = e_{\Lambda\Gamma\mathrm{K}} (\delta_1^\Gamma + u_{0;1}^\Gamma) \sqrt{G_{\Gamma\Gamma}} N^{\mathrm{K}}. \end{aligned} \tag{8.302}$$

Using the external forces as in Eqs.(8.209)−(8.213), equations of motion on the deformed rod are given as in Eq.(8.267), i.e.,

$$\begin{aligned} \boldsymbol{\mathcal{N}}_{,S} + \mathbf{q} &= \rho \mathbf{u}_{0,tt} + I_3 \boldsymbol{\varphi}_{1,tt} + I_2 \boldsymbol{\theta}_{1,tt}, \\ \mathbf{M}_{,S} + {}^{\mathbf{N}}\mathbf{M} + \mathbf{m} &= \mathscr{J} \boldsymbol{\Theta}_{,tt}, \end{aligned} \tag{8.303}$$

where

$$\begin{aligned} \mathscr{J} \boldsymbol{\Theta}_{,tt} = {} & [(I_3 u_{0,tt}^3 - I_2 u_{0,tt}^2) + (J_{22} \varphi_{1,tt}^{(3)} - J_{23} \varphi_{1,tt}^{(2)}) \\ & + (J_{23} \theta_{1,tt}^{(3)} - J_{33} \theta_{1,tt}^{(2)})] \mathbf{N}_1 + (I_3 u_{0,tt}^1 + J_{23} \varphi_{1,tt}^{(1)} + J_{22} \theta_{1,tt}^{(1)}) \mathbf{N}_2 \\ & - (I_2 u_{0,tt}^1 + J_{22} \varphi_{1,tt}^{(1)} + J_{23} \theta_{1,tt}^{(1)}) \mathbf{N}_3, \end{aligned} \tag{8.304}$$

and the scalar expressions are for $\Lambda = 1, 2, 3$,

$$\begin{aligned} &N^{\Lambda}_{;1} + q^{\Lambda} = \rho u^{\Lambda}_{(0),tt} + I_3 \varphi^{(\Lambda)}_{1,tt} + I_2 \theta^{(\Lambda)}_{1,tt}, \\ &M^{\Lambda}_{;1} + {}^{\mathbf{N}}M^{\Lambda} + m^{\Lambda} = (\mathscr{J}\boldsymbol{\Theta}_{,tt}) \cdot \mathbf{N}_{\Lambda}, \end{aligned} \tag{8.305}$$

or

$$\begin{aligned} &\left\{\left[\frac{N_1(\delta^{\Lambda}_1 + u^{\Lambda}_{0;1})}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)} + \frac{N_2(\delta^{\Lambda}_2 + \varphi^{(\Lambda)}_1)}{\sqrt{G_{22}}(1+\varepsilon^{(0)}_2)} + \frac{N_3(\delta^{\Lambda}_3 + \theta^{(\Lambda)}_1)}{\sqrt{G_{33}}(1+\varepsilon^{(0)}_3)}\right]\sqrt{G_{\Lambda\Lambda}}\right\}_{;1} \\ &+ q^{\Lambda} = \rho u^{\Lambda}_{0,tt} + I_3 \varphi^{(\Lambda)}_{1,tt} + I_2 \theta^{(\Lambda)}_{1,tt}, \\ &\left\{\left[\frac{M_1(\delta^{\Lambda}_1 + u^{\Lambda}_{0;1})}{\sqrt{G_{11}}(1+\varepsilon^{(0)}_1)} + \frac{M_2(\delta^{\Lambda}_2 + \varphi^{(\Lambda)}_1)}{\sqrt{G_{22}}(1+\varepsilon^{(0)}_2)} + \frac{M_3(\delta^{\Lambda}_3 + \theta^{(\Lambda)}_1)}{\sqrt{G_{33}}(1+\varepsilon^{(0)}_3)}\right]\sqrt{G_{\Lambda\Lambda}}\right\}_{;1} \\ &+ e_{\Lambda\Gamma K}(\delta^{\Gamma}_1 + u^{\Gamma}_{0;1})\sqrt{G_{\Gamma\Gamma}} N^{K} + m^{\Lambda} = (\mathscr{J}\boldsymbol{\Theta}_{,tt}) \cdot \mathbf{N}_{\Lambda}. \end{aligned} \tag{8.306}$$

The force condition at a point $\mathscr{P}_k$ with $S^1 = S^1_k$ is

$$\begin{aligned} &{}^{-}\mathbf{N}(S^1_k) + {}^{+}\mathbf{N}(S^1_k) + \mathbf{F}_k = 0, \\ &{}^{-}N^{\Lambda}(S^1_k) = {}^{+}N^{\Lambda}(S^1_k) + F^{\Lambda}_k \quad (\Lambda = 1, 2, 3). \end{aligned} \tag{8.307}$$

The force boundary condition at the boundary point $\mathscr{P}_r$ is

$$\mathbf{N}(S^1_r) + \mathbf{F}_r = 0 \ \text{ or } \ N^{\Lambda}(S^1_r) + F^{\Lambda}_r = 0 \quad (\Lambda = 1, 2, 3). \tag{8.308}$$

If there is a concentrated moment at a point $\mathscr{P}_k$ with $S^1 = S^1_k$, the corresponding moment boundary condition is given by

$$\begin{aligned} &{}^{-}\mathbf{M}(S^1_k) + {}^{+}\mathbf{M}(S^1_k) + \mathbf{M}_k = 0, \\ &{}^{-}M^{\Lambda}(S^1_k) = {}^{+}M^{\Lambda}(S^1_k) + M^{\Lambda}_k \quad (\Lambda = 1, 2, 3). \end{aligned} \tag{8.309}$$

The moment boundary condition at the boundary point $\mathscr{P}_r$ is

$$\mathbf{M}(S^1_r) + \mathbf{M}_r = 0 \ \text{ or } \ M^{\Lambda}(S^1_r) + M^{\Lambda}_r = 0 \quad (\Lambda = 1, 2, 3). \tag{8.310}$$

The displacement continuity and boundary conditions are similar to Eq. (8.207), i.e., $u^{\Lambda}_{k-} = u^{\Lambda}_{k+}$ and $u^{\Lambda}_r = B^{\Lambda}_r$ ($\Lambda = 1, 2, 3$).

References

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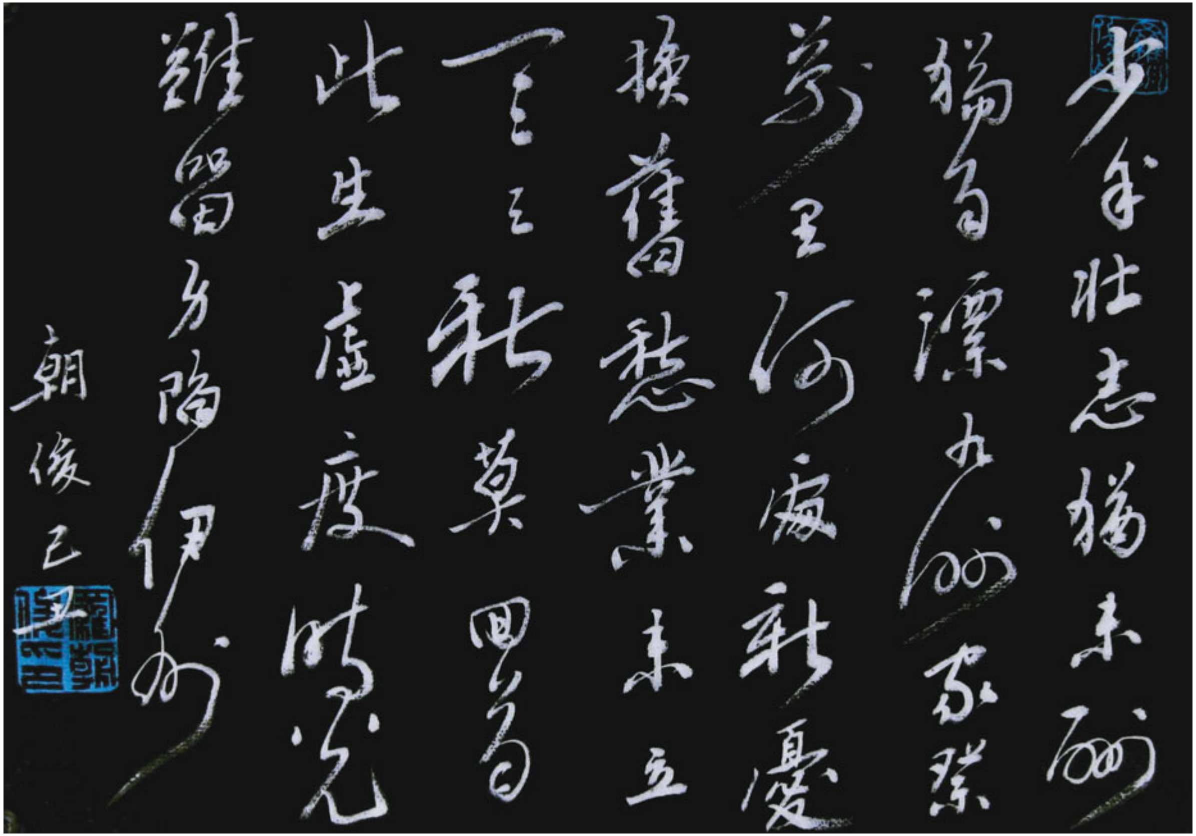


Homesick and disappointment without any achievements accompanies my life.

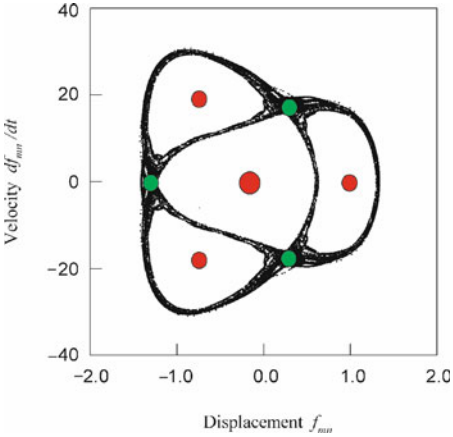
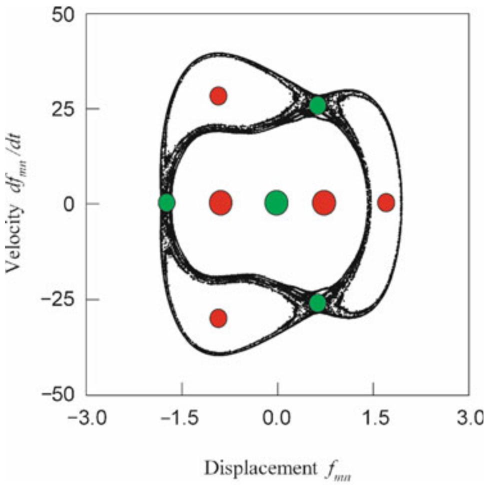
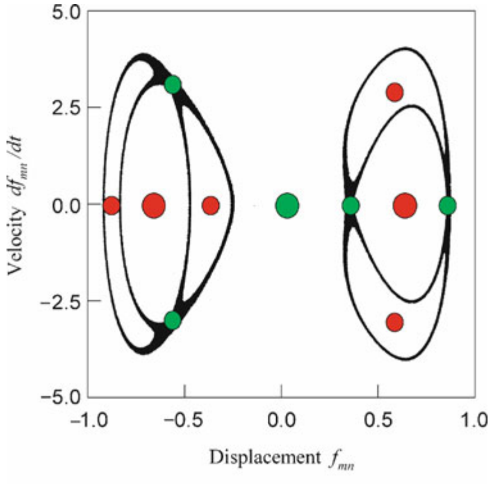


Fig. 6.9 Chaotic wave motion in the resonant separatrix zone of the third order of the axially traveling, pre-buckled plate. Red and green points are center and hyperbolic equilibrium points for resonant separatrix, respectively. ($f_{mn}^0 = 0.5398$, $\dot{f}_{mn}^0 = 18.7908$, $\Omega = 57.1181$, $Q_0 = 610 > Q_0^{\min} \approx 600$, $c = 15$ m/s, $E = 2 \times 10^{11}$ N/m^2, $\rho_0 = 7.8 \times 10^3$ kg/m^3, $\mu = 0.3$, $l = 2.0$ m, $b = m$, $h = 2$ mm, $m = n = 1$, $P = 200$ N, $c = 15$ m/s.)

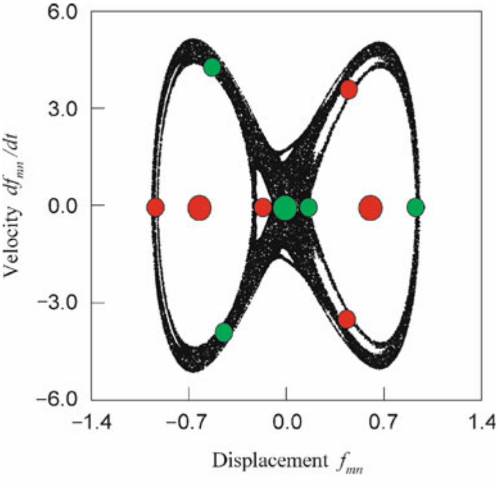


(a)

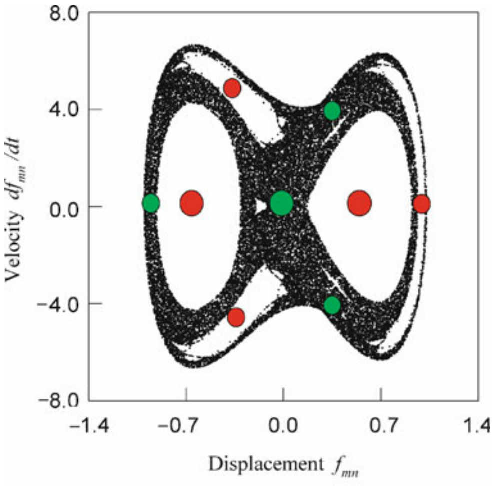


(b)

Fig. 6.10 Chaotic wave motion in the resonant separatrix zone of the axially traveling, post-buckled plate: (a) $E_0 > 0$ ($f^0_{mn} = 0.7376$, $\dot{f}^0_{mn} = 28.6717$, $\Omega = 60.6762$, $Q_0 = 270 > Q_0^{\min} \approx 265$) and (b) $E_0 < 0$ ($f^0_{mn} = -0.5617$, $\dot{f}^0_{mn} = -0.3099$) in the left well and ($f^0_{mn} = 0.3615$, $\dot{f}^0_{mn} = 0$) in the right well. ($\Omega = 25.4274$, $Q_0 = 4.5 > Q_0^{\min} \approx 3.5$. Red and green points are center and hyperbolic equilib rium points for resonant separatrix, respectively. ($E = 2 \times 10^{11}$ N/m^2, $\rho_0 = 7.8 \times 10^3$ kg/m^3, $\mu = 0.3$, $l = 2.0$ m, $b = 1.0$ m, $h = 2$ mm, $m = n = 1$, $P = 200$ N, $c = 35$ m/s.)



(a)



(b)

Fig. 6.11 Chaotic wave motions in the homoclinic separatrix zone of the axially traveling, post-buckled plate: (a) for the resonance of the second order at $E_0 < 0$ embedded and (b) for the resonance of the third order at $E_0 > 0$ embedded. Red and green points are center and hyperbolic equilibrium points for resonant separatrix, respectively. ($Q_0 = 1.2 > Q_0^{cr} \approx 1.05$, $Q_0 = 3.5 > Q_0^{cr} \approx 3.2$, $\Omega = 20.0$, $f_{mn}^0 = \dot{f}_{mn}^0 = 0$, $E = 2 \times 10^{11}$ N/m^2, $\rho = 7.8 \times 10^3$ kg/m^3, $\mu = 0.3$, $l = 2.0$ m, $b = 1.0$ m, $h = 2$ mm, $m = n = 1$, $P = 200$ N, $c = 35$ m/s.)

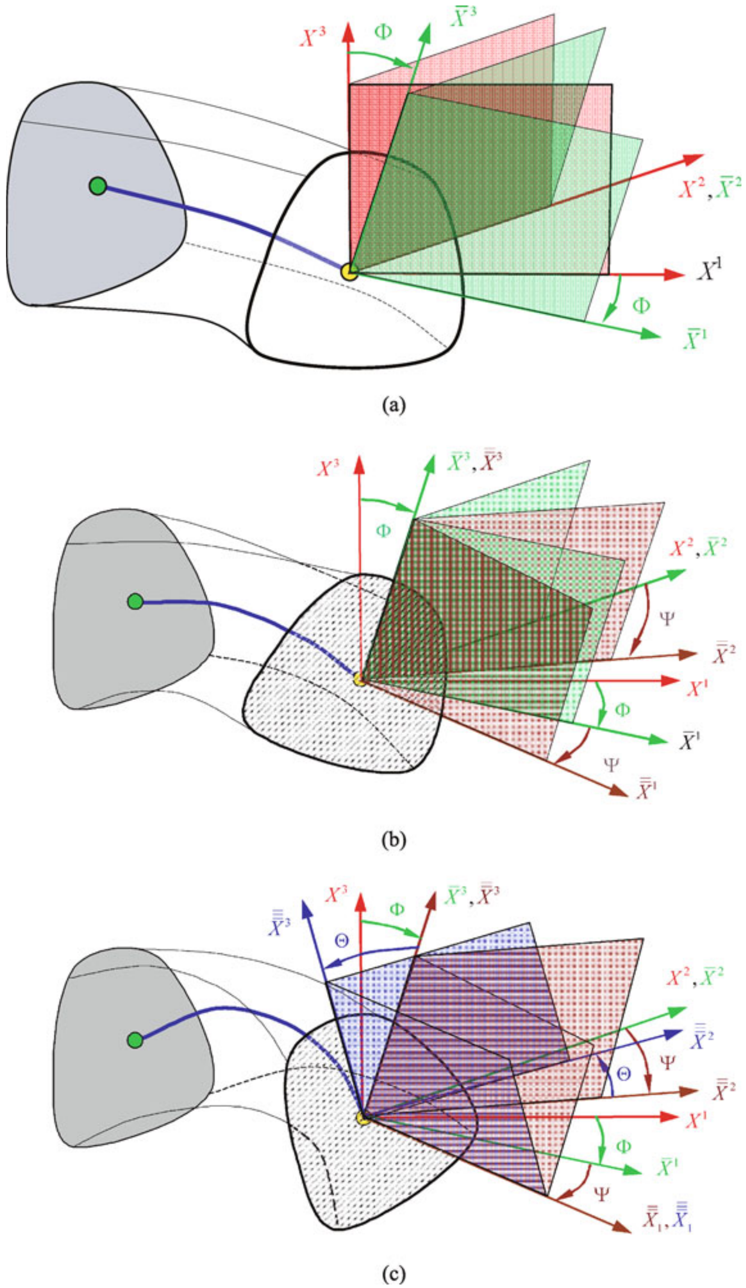


Fig. 8.5 Euler angles of rod rotation caused by bending and torsion: (a) the initial (red) to first rotation (green), (b) the first to second rotation (brown), (c) from the second to the last rotation (blue).