Boundary Control of Radiative Transfer Equations for Application in Radiotherapy Planning

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Summary. A radiotherapy treatment planning problem is formulated as a boundary control problem constrained by the radiative transfer equation. Optimality conditions for the radiative transfer equation as well as for the SP_1 approximation are stated. The latter are solved numerically.

1 Introduction

Mathematical methods play an increasing role in medicine, especially in cancer therapy. Several special journal issues have been devoted to cancer modeling and treatment, cf. [2–4, 7] among others. While until recently, treatment planning was done by an experienced physician "by hand", computer-aided treatment planning systems based on optimization algorithms currently enter into clinical practice, cf. [14] and references therein.

The use of ionizing radiation is one of the main tools in the therapy of cancer. The aim of radiation treatment is to deposit enough energy in cancer cells so that they are destroyed. On the other hand, healthy tissue around the cancer cells should be harmed as little as possible. Furthermore, some regions at risk, like the spinal chord, should receive almost no radiation at all. Most dose calculation algorithms in clinical use rely on the Fermi–Eyges theory of radiation which is insufficient at inhomogeneities, e.g. void-like spaces like the lung. We start with a Boltzmann transport model for the radiation which accurately describes all physical interactions, and based on this model we develop a direct optimization approach based on adjoint equations. Until recently, dose calculation using a Boltzmann transport equation has not attracted much attention in the medical physics community. This access is based on deterministic transport equations of radiative transfer. Similar to Monte Carlo simulations it relies on a rigorous model of the physical interactions in human tissue that can in principle be solved exactly. Monte Carlo simulations are

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widely used, but it has been argued that a grid-based Boltzmann solution should have the same computational complexity [5]. Electron and combined photon and electron radiation were studied in the context of inverse therapy planning cf. [17, 18] and most recently [19]. Furthermore, several neutral particle codes have been applied to the dose calculation problem, see [12] for a review.

This work is part of an ongoing project on dose calculation methods and optimal treatment planning based on Boltzmann transport equations. A consistent model of combined photon and electron radiation was developed [13] that includes the most important physical interactions. In [8], an approximate partial differential equation model was designed. A step toward time-dependent control in the case of moving patients was done in [9]. In the present work we extend results from [10], where distributed control was considered analytically and numerically, to boundary control.

2 Radiotherapy Planning as a Boundary Control Problem

Consider a part of the patient's body which contains the region of the cancer cells. We assume that this part of the body can be described as a convex, open, bounded domain Z in $\mathbb{R}^3$. Furthermore, we assume that Z has a smooth boundary with outward normal vector n. The direction, into which the electron is moving is given by $\omega \in S^2$, where S^2 is the unit sphere in three dimensions. To formulate boundary conditions, we define the in- and outgoing boundaries as

$$\partial Z^{\pm} := \{(x, \omega) \in \partial Z \times S^2 : n(x) \cdot \omega > (<) 0\}.$$

We consider particle transport modeled by the Boltzmann equation for the particle density $\psi(x, \omega)$ as

$$\omega \cdot \nabla_x \psi(x, \omega) + \sigma_t(x, t)\psi(x, \omega) = \sigma_s(x) \int_{S^2} s(x, \omega \cdot \omega')\psi(x, \omega')d\omega' \quad (1)$$

with

$$\psi(x, \omega) = q(x, \omega) \text{ on } \partial Z^-.$$

For the sake of simplicity, we neglect the energy dependence of ψ. Here, $\psi(x, t, \omega) \cos\theta dA d\omega$ is the number of electrons that pass through an area dA at point x into a solid angle $d\omega$ around ω at time t. The angle θ is the angle between ω and dA. The total cross section $\sigma_t(x, t)$ is the sum of absorption cross section $\sigma_a(x, t)$ and total scattering cross section $\sigma_s(x, t)$. The scattering phase function is normalized,

$$2\pi \int_{-1}^{1} s(x, \mu)d\mu = 1. \quad (2)$$

From the physical interpretation, we have that ψ, q, σ_t, σ_s and s are non-negative quantities. The detailed interactions of electrons with atoms give rise to complicated explicit formulas for the scattering coefficient, see e.g. [13].

The problem of external beam radiotherapy is to determine a boundary condition $q \geq 0$ in some optimal way. A number of functionals and methods have been devised to describe the effect of radiation on biological tissue, cf. the extensive lists of references in the reviews [6] and [16]. It is clear that the amount of destroyed cells in a small volume, be they cancer or healthy cells, is not directly proportional to the dose

$$D(x) = \int_{S^2} \psi(x, \omega) d\omega \tag{3}$$

deposited in that volume. However, no single accepted type of model has emerged yet. Moreover, current biological models require input parameters which are not known exactly [16]. This is why the authors of [16] opted not to investigate these models but rather to focus on some general mathematical cost functionals. A quadratic objective function together with nonlinear constraints was identified as the most versatile model. Thus we try to find a boundary value q such that the quadratic deviation from a prescribed dose $\bar{D}$ becomes minimal. The ideal dose curve of course has a certain fixed value in the tumour tissue and is zero elsewhere. We write

$$\bar{D}(x) = \int_{S^2} \bar{\psi}(x, \omega) d\omega \tag{4}$$

and introduce a weight function α depending on space, which penalizes deviations from the desired dose in normal tissue, tumour tissue and regions at risk differently. We want to minimize

$$J_1(D) = \int_Z \frac{\alpha}{2} (D - \bar{D})^2 dx \tag{5}$$

Furthermore, we include a penalty term proportional to the applied external source in the minimization process to prevent trivial solutions. We use a simple penalization as

$$J_2(q) := \int_{\partial Z^-} \frac{\beta}{2} (n \cdot \omega) q(s, \omega)^2 ds d\omega. \tag{6}$$

The problem of teletherapy reads then: Find $q \geq 0$ such that

$$J_1(D) + J_2(q) \tag{7}$$

is minimal, subject to the radiative transfer equation (1), and the relation (3) between ψ and the dose. This is an optimization problem constrained by an integro-differential equation.

Along the lines of [10], we can introduce a Lagrangian to this problem and formally derive the following first-order optimality conditions: The radiative transfer equation is

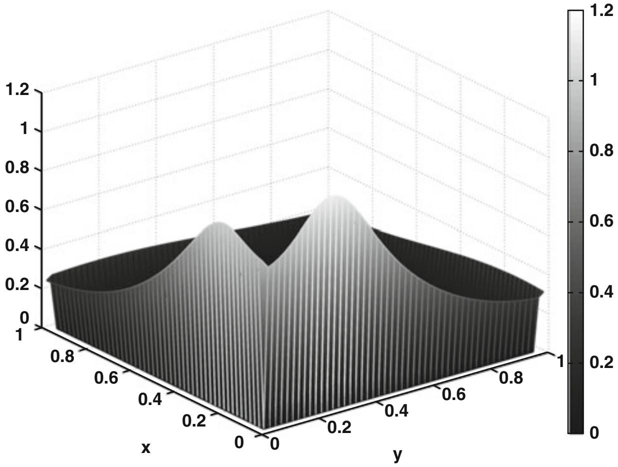

Fig. 1. Boundary control q

$$\omega\nabla_x\psi + \sigma_t\psi = \sigma_s \int_{S^2} s\psi d\omega' \quad \text{in } Z \text{ with } \psi = q \text{ on } \partial Z^-. \tag{8a}$$

The Lagrange multiplier λ satisfies a backward transfer equation

$$-\omega\nabla_x\lambda + \sigma_t\lambda = \sigma_s \int_{S^2} s\lambda + \alpha(\psi - \bar{\psi})d\omega \quad \text{in } Z \text{ with } \lambda = 0 \text{ on } \partial Z^+. \tag{8b}$$

The gradient information, which is necessary for any efficient optimization algorithm in this case, is encoded in λ. It can be used directly in the optimality condition for q:

$$q = \left(q + \lambda - \alpha \int_{Z^-} n \cdot \omega q d\omega\right)^+ \quad \text{on } \partial Z^-. \tag{8c}$$

Here, $\xi^+ = \max(\xi, 0)$.

3 Numerical Results

We apply the optimize-then-discretize approach to the control law. In our 2D simulations, we use the Simplified P_1 (SP_1) approximation [15]. The unknown is the dose

$$D(x) = \int_{S^2} \psi(x, \omega) d\omega. \tag{9}$$

The SP_1 approximation for the radiative transfer equation [11] reads for isotropic scattering

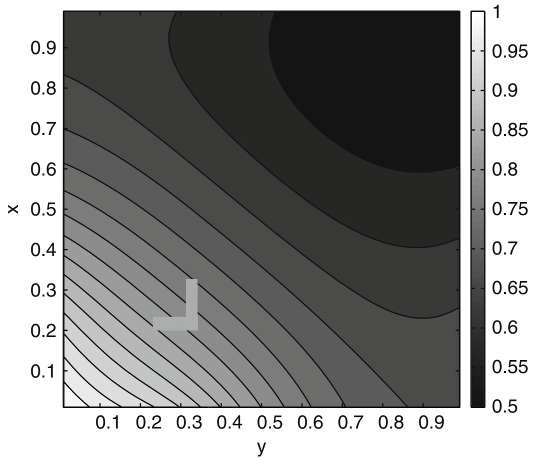

Fig. 2. Contour plot of the optimal dose distribution

$$-\nabla\frac{1}{3\sigma_t}\nabla D+(\sigma_t-\sigma_s)D=0. \tag{10}$$

The boundary conditions are

$$n\cdot\nabla D=\frac{3}{2}\sigma_t\left(l_1(q)-D\right),\quad \text{where}\quad l_1(q)=-4\int_{\partial Z^-}n\omega q(s,\omega)dsd\omega. \tag{11}$$

From the SP_1 approximation and the cost functional (5) and (6), we obtain the optimality system using SP_1 asymptotic as

$$\nabla\frac{1}{3\sigma_t}\nabla\lambda^{(0)}-(\sigma_t-\sigma_s)\lambda^{(0)}=-4\pi\alpha(D-\bar{D}),\ \text{with}\ n\cdot\nabla\lambda^{(0)}=-\frac{3}{2}\sigma_t\lambda^{(0)}\ \text{on}\ \partial Z^+. \tag{12}$$

The gradient equation is

$$\lambda^{(0)}-\frac{2}{3\sigma_t}n\cdot\nabla\lambda^{(0)}=-2\pi\beta l_1(q). \tag{13}$$

To demonstrate the feasibility of our approach, we consider the unit square $[0,1]^2$ with the parameters [1] $\sigma_s=0.05$ and $\sigma_t=0.5$. The domain contains an L-shaped tumour (Fig. 1). We numerically compute the solution to the SP_1-approximation of the optimality system. We consider the functional given by (5) and (6). When solving the optimal control problem we set $\beta=0$ (no regularization for the boundary control) and $\alpha=1$. All computations are done on a 50×50 grid. We solve the optimality system by using a projected gradient method. The optimal boundary control and the dose in the optimal state are shown in Figs. 1 and 2, respectively. Since the tumour is located in the bottom left corner, both maxima of the control are reasonable. The dose basically falls off exponentially from the boundary, thus here a better dose distribution cannot be expected. In terms of performance, because of the high dimensionality of the optimization problem, we expect our adjoint-based method to outperform all black box methods using numerical gradients.

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