Domain Decomposition Based Multirating and its Perspective in Circuit Simulation

Michael Striebel[1], Andreas Bartel[2], and Michael Günther[2]

[1] Chemnitz University of Technology, Research Group Mathematics in Industry and Technology, D-09107 Chemnitz, Germany,
michael.striebel@mathematik.tu-chemnitz.de

[2] Bergische Universität Wuppertal, Department of Mathematics, Chair of Applied Mathematics/Numerical Analysis, D-42097 Wuppertal, Germany,
bartel@math.uni-wuppertal.de, guenther@math.uni-wuppertal.de

Summary. Based on domain decomposition, multirate time integration takes into account largely different timescales. In this class, a mixed multirate scheme and its application to an arbitrary number of subsystems is outlined. Moreover, the matter of activity change and the connection to model order reduction is discussed.

1 Domain Decomposition as Modular Modelling

In PDE domain decomposition, a domain is split into different sub-domains. For the consistency of the overall problem, these sub-domains have to be linked via artificial boundary conditions and Lagrangian multipliers to match the solution at the boundaries. A similar approach is used naturally in circuit simulation packages. Here complex circuits are decomposed into different parts with respect to their function. This approach enables to model the sub-circuits separately. In contrast to spatial (sub-)domains within the PDE case, we have to deal here with (sub-)circuits described by their topology, and hence

- Matching boundary conditions are transferred to artificial voltage sources, which match node potentials at the boundaries of the sub-circuits.
- Branch currents through artificial voltage sources play the role of Lagrangian multipliers linking the sub-domains.

Applying charge oriented modified nodal analysis [4, 9] to a decomposed circuit with $r \in \mathbf{N}$ subsystems, the mathematical models yields the following type of coupled differential-algebraic equations (DAEs):

$$\left.\begin{aligned} 0 &= \mathcal{A}_\lambda \dot{y}_\lambda + f_\lambda(x_\lambda, t) + \mathcal{A}_{w_\lambda} w \\ 0 &= y_\lambda - q_\lambda(x_\lambda) \end{aligned}\right\} \quad \text{for} \quad \lambda = 1, \ldots, r, \tag{1a}$$

$$0 = \mathcal{A}_{w_1}^t x_1 + \cdots + \mathcal{A}_{w_r}^t x_r \,. \tag{1b}$$

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Here $x_\lambda \in \mathbf{R}^{n_\lambda}$ denotes the unknown node potentials and branch currents of voltage defining elements, $y_\lambda \in \mathbf{R}^{m_\lambda}$ contains the unknown charges and fluxes, and the functions q_λ, f_λ describe the contribution of the reactive and nonreactive components of the λth subsystem, whose topology determines the incidence matrix $\mathcal{A}_\lambda \in \{-1, 0, 1\}^{n_\lambda \times m_\lambda}$. The coupling quantity $w \in \mathbf{R}^M$ denotes the set of branch currents through artificial voltage sources. These are implicitly defined by the coupling equation (1b), the constitutive equations for the artificial voltage sources. Finally, the incidence matrices $\mathcal{A}_{w_\lambda} \in \{-1, 0, 1\}^{n_\lambda \times M}$ $(\lambda = 1, \ldots, r)$ relate w to the corresponding terminal of the respective subcircuit as input source.

In the following, we restrict ourself to the case where (1) states an index-1 problem, in the sense that:

(C1) The overall system (1a,1b) has index 1 w.r.t. $x_1, \ldots, x_r, w$.
(C2) All systems (1a) define index-1 systems w.r.t. x_λ (given w).

The latter holds, if there are neither CV-loops nor LI-cutsets in the subcircuits [3]. Virtual voltage sources can be associated with the coupling and we can show that (C1) holds if there are also no loops of capacitors, voltage sources and virtual voltage sources in the overall circuit [1,9].

Under these conditions (1) is equivalent [9] to the semi-explicit system:

$$\left.\begin{aligned} \dot{y}_\lambda &= f_\lambda(z_\lambda, w), \\ 0 &= h_\lambda(y_\lambda, z_\lambda, w), \end{aligned}\right\} \quad \text{for } \lambda = 1, \ldots, r, \tag{2a}$$

$$0 = g(z_1, \ldots, z_r)\,, \tag{2b}$$

where f_λ, h_λ are linear in w and y_λ, w, respectively and g is linear in $z_1, \ldots, z_r$. Notice the abuse of notation in y_λ, f_λ.

2 Mixed Multirate

The functional diversity in modularly modelled systems causes a heterogeneous distribution of activity. At each time the quantities "node potential" and "element currents" may show the tendency to change rapidly in some regions of the circuit whereas only minor fluctuation can be recognised in other parts. *Multirate methods* accommodate this behaviour and reduce computational expenses. The basic idea of these schemes is to prevent parts to be integrated more often than necessary to meet prescribed error tolerances. To this end, we associate activity levels to step size proposals and are able to split large systems in subsystems which operate on different time scales with differing optimal step sizes. Now, domain decomposition based multirating means to use these (differing) optimal step sizes for the respective subsystems to define an overall method.

For simplicity, we formulate the multirate method first for a coupled system (of *latent* y_L and *active* y_A variables) for ordinary differential equations:

$$\dot{y}_L = f_L(y_L, y_A), \qquad \dot{y}_A = f_A(y_L, y_A).$$

At the current time point t_0 (with given $y(t_0) = (y_L, y_A)^t = y_0$), we suppose that the latent part (subscript L) can be integrated with one macrostep $\mathcal{H}_L$ whereas a sequence of q microsteps $\mathcal{H}_{A,1}, \ldots, \mathcal{H}_{A,q}$ is needed for the active part (subscript A) to reach $t_0 + \mathcal{H}_L$. Numerically, a subset of systems proposing a large individual step size is identified as latent; the others demand a small step, and are therefore active.

Various approaches are being developed [4, 8, 11]. We concentrate on the application of Rosenbrock-Wanner (ROW), i.e., one-step, methods. A detailed discussion can be found in [9]. Here we just outline the basic principles. In its most general way the one-step formalism of this procedure is given by:

$$y_{L,1} = y_{L,0} + \sum_{i=1}^{s_L} b_i^L \cdot l_i^L, \tag{3a}$$

$$y_{A,\mu} = y_{A,\mu-1} + \sum_{i=1}^{s_A} b_i^A \cdot l_i^{A,\mu} \quad (\mu = 1, \ldots, q), \tag{3b}$$

$$l_i^L = \Phi_L(\mathcal{H}_L;\, y_{L,0}, Y_i^A, l_1^L, \ldots, l_{s_L}^L) \quad (i = 1, \ldots, s_L), \tag{3c}$$

$$l_i^{A,\mu} = \Phi_A(\mathcal{H}_{A,\mu};\, y_{A,\mu-1}, Y_i^{L,\mu}, l_1^{A,\mu}, \ldots, l_{s_A}^{A,\mu}) \quad (i = 1, \ldots, s_A), \tag{3d}$$

where $\Phi_\mathcal{L}$ denotes an $s_\mathcal{L}$ stage ROW scheme with coefficients $b^\mathcal{L}$, $A^\mathcal{L}$, $B^\mathcal{L}$, $\Gamma^\mathcal{L}$ ($\mathcal{L} = L, A$). For ROW schemes, (3c,d) determine the increments l_i^L, $l_i^{A,\mu}$ by linear relations. The coupling of latent and active subsystems is performed by the terms Y_i^A and $Y_i^{L,\mu}$, which will be defined from the increments l^L and l^A, respectively. Y_i^A and $Y_i^{L,\mu}$ signify the sampling of the fast variables on the coarse grid and vice versa.

Mixed multirate [2] is characterised by a "compound step" and a series of "later microsteps". In the former, the macrostep (3a) and the first microstep (3b) ($\mu = 1$) are computed at once. Y_i^A, $Y_i^{L,1}$ are determined in RK-like manner, which employs additionally: coupling coefficients $D^{AL}, D^{LA}, N^{AL}, N^{LA}$, and scaling of increments $l_i^{A,1}$ and l_i^L by the *step size ratio* $\mathfrak{m} = \frac{\mathcal{H}_L}{\mathcal{H}_{A,1}}$ and $\mathfrak{m}^{-1}$, respectively. For the later microsteps *dense output* formulae [6] are applied to get reasonable values $Y_i^{L,2}, \ldots, Y_i^{L,q}$.

2.1 Mixed Multirate for Circuit Simulation

In the case of coupled index-1 networks (C1-C2) with active and latent variables, we have to solve the equivalent semi-explicit DAE:

$$\begin{aligned} \dot{y}_L &= f_L(z_L, w) & \dot{y}_A &= f_A(z_A, w) \\ 0 &= h_L(y_L, z_L, w) & 0 &= h_A(y_A, z_A, w) \\ & & 0 &= g(z_L, z_A), \end{aligned} \tag{4}$$

where the coupling quantity w is assumed to be latent, too. Using a ROW scheme like (3) for (4), we have to add increments for the algebraic variables. For $s(=s_{\mathcal{L}})$ stages, weights $b^{\mathcal{L}}$, and increments $l^{\mathcal{L}}$, $k^{\mathcal{L}}$, p, we have (sloppily):

$$\begin{pmatrix} y_{L,1} \\ z_{L,1} \\ w_1 \end{pmatrix} = \begin{pmatrix} y_{L,0} \\ z_{L,0} \\ w_0 \end{pmatrix} + (b^L)^t \begin{pmatrix} l^L \\ k^L \\ p \end{pmatrix}, \quad \begin{pmatrix} y_{A,1} \\ z_{A,1} \end{pmatrix} = \begin{pmatrix} y_{A,0} \\ z_{A,0} \end{pmatrix} + (b^A)^t \begin{pmatrix} l^A \\ k^A \end{pmatrix}. \quad \text{(5a)}$$

According to (3c,d) the stage increments are defined by the linear system

$$M^\star \cdot \left(l_i^L,\ k_i^L \mid l_i^A, k_i^A \mid p_i\right)^t = \mathrm{RHS}_i, \quad \text{for } i = 1, \ldots, s \quad \text{(5b)}$$

with the system matrix $M^\star =$

$$\left(\begin{array}{cc|cc|c} \mathrm{I}_{y_L} & -\mathcal{H}_L\gamma^L\frac{\partial f_L}{\partial z_L} & & & -\mathcal{H}_L\gamma^L\frac{\partial f_L}{\partial w} \\ -\gamma^L\frac{\partial h_L}{\partial y_L} & -\gamma^L\frac{\partial h_L}{\partial z_L} & & & -\gamma^L\frac{\partial h_L}{\partial w} \\ \hline & & \mathrm{I}_{y_A} & -\mathcal{H}_A\gamma^A\frac{\partial f_A}{\partial z_A} & -\frac{1}{\mathfrak{m}}\cdot\mathcal{H}_A\nu^{AL}\frac{\partial f_A}{\partial w} \\ & & -\gamma^A\frac{\partial h_A}{\partial y_a} & -\gamma^A\frac{\partial h_A}{\partial z_A} & -\frac{1}{\mathfrak{m}}\cdot\nu^{AL}\frac{\partial h_A}{\partial w} \\ \hline & -\gamma^L\frac{\partial g}{\partial z_L} & & -\mathfrak{m}\cdot\nu^{LA}\frac{\partial g}{\partial z_A} & \end{array}\right)$$

and a right-hand side RHS_i depending on stepsizes $\mathcal{H}_L, \mathcal{H}_A$, step size ratio $\mathfrak{m}$, increments $l_j^{\mathcal{L}}, k_j^{\mathcal{L}}, p_j$ of the former steps $j = 1, \ldots, i-1$. As above, two ROW coefficient sets (labels L, A) and additional coupling coefficients are employed.

In the *later microsteps*, it remains to solve $[\dot{y}_A = f_A, 0 = h_A]$ for unknown y_A, z_A, where $w(t)$ enters the right-hand-side. As we introduced the coupling quantity w as an additional latent unknown, it is already computed (in the compound step) and a cheap approximation to $w(t_0 + \theta \cdot \mathcal{H}_L)$ is obtained via dense output formulae. For a detailed definition, see [9].

The *method's coefficients* have to be determined such that the accuracy of the local approximation is of a prescribed order. To this end, B-series for ODEs [5] are adapted to our coupled problem (2). As for mixed multirate methods, the order conditions depend on the step size ratio $\mathfrak{m}$. Therefore, the coefficients depend on this quantity and have to be computed during integration.

2.2 Hierarchical Mixed Multirate

Aiming a multirate method that can deal with an arbitrary amount of activity levels, *hierarchical mixed multirate* seems to be the most feasible approach. The main idea is to nest compound steps and later micro-steps in a way, that at each time merely a two-level multirate scheme is engaged. At any time point of integration each subsystem has either the status *asleep* or *latent* or *active*. A part is *asleep* if the last time point at which an approximation is

available is beyond the current one. The set of non-sleeping subsystems is split into latent and active subsets. Due to this decomposition, a *compound step* can be applied to the non-sleeping part. Otherwise (only active variables are present) *later microsteps* are executed. The sleeping subsystems contribute to the current step via dense-output.

2.3 Implementation and Trapping Events

For electric network descriptions (1), a hierarchical mixed multirate method of order 2 has been embedded into Qimonda's in-house simulator `titan`. Step size control is performed with an embedded scheme of order 1. Linear transformations are applied to this multirate algorithm (5) such that the resulting method can be used for the network problem (1) directly.

Great importance is attached to the problem of traversing signals, which can force sleeping subsystems to "wake up" during a macrostep (t_{n-1} to $t_n = t_{n-1} + \mathcal{H}_L$). This causes an a-posteriori rejection of that macrostep. The detection of such situations is based on comparing pin voltages of connected subsystems: any time point $t_{\text{wup}} \in (t_{n-1}, t_n)$ where the difference of the voltages computed from the non-sleeping part and the corresponding voltage computed by a dense-output formula applied to the sleeping part becomes too large, is considered a wake up point. Moreover, not the whole macrostep is restored, but a re-initialisation at t_{wup} is performed using again dense-output formulae to get appropriate initial values. For a detailed description of an industrial test case (chain of inverters), we refer to [10].

3 Connection of Multirate to Model Order Reduction

To quickly get evidence of the behaviour of complex circuits, simulation techniques need to be adapted. *Multirate* tackles this task from an algorithmic point of view by incorporating subsystems' behaviour in the numerical procedure (sampling) for the overall system (1a)–(1b). *Model order reduction* (MOR) starts on the modeling level. It seeks to replace the r subsystems (1a) of presumably high order (large number of unknowns) with order reduced models. This is achieved in the following way: given the input w, the λth substitute model with essential state variable $\hat{x}_\lambda$ returns an output $\hat{\mathcal{A}}^t_{w_\lambda} \hat{x}_\lambda$ which approximates the corresponding output $\mathcal{A}^t_{w_\lambda} x_\lambda$ of the full system sufficiently accurate. For nonlinear problems, MOR basically is done by scanning the full system in a training phase and extracting dominant information that determines the reduced substitute model; this is realised, e.g., in the trajectory piecewise linearisation approach (TPWL) [7].

There are some analogies of multirate and MOR that could be used to improve or merge both strategies. In multirate the latent part contributes to the later microsteps just in terms of the terminal quantities, i.e., the output

$\mathcal{A}^t_{w_\lambda} x_\lambda$. The corresponding values on the fine time grid are derived from information gathered on the coarse one. This can be viewed as training of a current- or voltage source replacing the large latent subcircuit. If the latent part was replaced by a reduced order model incorporating more dynamical effects, the procedure could become more stable and we can hope to act out the full multirate behaviour. Moreover, the combination of compound step and later microsteps can be regarded as on-the-fly training. If this can be transferred to MOR, it may pave the way to construct models that are less sensitive to varying input signals and which could be produced whenever needed.

4 Conclusion

A multirate scheme for circuit simulation that can deal with an arbitrary number of subsystems has been derived, where domain decomposition of large electrical circuits is achieved by introducing extra variables. The hierarchical multirate method has been embedded in a sophisticated industrial simulator.

Future tasks are a partitioning strategy and step size control tailored to multirate needs. Step size control should be improved as we want to combine very large with small steps. Higher order schemes and extensions to higher index problems are desirable. The perspective is to use analogies of multirate and model order reduction to combine and enhance both approaches.

References

1. Arnold, M., Günther, M.: Preconditioned dynamic iteration for coupled differential-algebraic systems. BIT, **41**(1), 1–25 (2001)
2. Bartel, A., Günther, M., Kværnø, A.: Multirate methods in electrical circuit simulation. In: Anile, A.M., Capasso, V., Greco, A. (eds.) Progress in Industrial Mathematics at ECMI 2000, Mathematics in Industry, vol. I, pp. 258–265. Springer, Berlin (2002)
3. Estévez Schwarz, D., Tischendorf, C.: Structural analysis for electric circuits and consequences for MNA. Int. J. Circ. Theor. Appl. **28**, 131–162 (2000)
4. Günther, M., Feldmann, U., ter Maten, E.J.W.: Modelling and discretization of circuit problems. In: Ciarlet, P.G., Schilders, W.H.A., ter Maten, E.J.W. (eds.) Numerical Methods in Electromagnetics, vol. XIII of Handbook of Numerical Analysis, pp. 523–659. Elsevier, North Holland (2005)
5. Hairer, E., Nørsett, S.P., Wanner, G.: Solving Ordinary Differential Equations I – Nonstiff Problems, Second revised Edition, Springer, Berlin (2000)
6. Hairer, E., Wanner, G.: Solving Ordinary Differential Equations II – Stiff and Differential-Algebraic Problems. Second revised edition, Springer, Berlin (1996)
7. Rewieński, M.J., White, J.: A trajectory piecewise-linear approach to model order reduction and fast simulation of nonlinear circuits and micromachined devices. IEEE Trans. CAD Int. Circ. Syst. **22**(2), 155–170 (2003)
8. Savcenco, V., Hundsdorfer, W.H., Verwer, J.G.: A multirate time stepping strategy for stiff odes. BIT **47**, 137–155 (2007)

9. Striebel, M.: Hierarchical Mixed Multirating for Distributed Integration of DAE Network Equations in Chip Design. Number 404 in Fortschritt-Berichte VDI Reihe 20. VDI-Verlag Düsseldorf (2006)
10. Striebel, M., Bartel, A., Günther, M.: A Multirate ROW-Scheme for Index-1 Network Equations, Appl. Numer. Math. **59**, pp. 800–814 (2009)
11. Verhoeven, A., El Guennouni, A., ter Maten, E.J.W., Mattheij, R.: Multirate methods for the transient analysis of electrical circuits. PAMM, Proceedings to GAMM 2005 **5**, 821–822 (2005)