A Multi-Class Mean-Field Model with Graph Structure for TCP Flows

C. Graham[1] and Ph. Robert[2]

[1] CNRS – École Polytechnique, Route de Saclay, 91128 Palaiseau, France
carl@cmap.polytechnique.fr

[2] INRIA Paris – Rocquencourt, Domaine de Voluceau, 78153 Le Chesnay, France
philippe.robert@inria.fr

Summary. A Markovian mean-field multi-class model for the interaction of several classes of permanent connections in a network is analyzed. Connections create congestion at the nodes they utilize, and adapt their throughput to the congestion they encounter in a way similar to the Transmission Control Protocol (TCP).

1 Introduction

The Internet can be described as a very large distributed system for data transmission, with self-adaptive capabilities to the different congestion events that regularly occur. In this paper, a packet level model of the self-adaptive behavior of data flows submitted to Additive Increase Multiplicative Decrease (AIMD) algorithms, similar to Transmission Control Protocol (TCP), is established and studied. Throughput grows linearly in the number of known successful packet transmissions. When a loss is detected, the throughput is sharply reduced by multiplication by some factor $r < 1$ (usually $1/2$).

Studies up to now usually consider a *single* node carrying *similar* connections, see e.g. Ott et al. [1], Adjih et al. [2], Baccelli et al. [3], Dumas et al. [4], and Guillemin et al. [5].

This proceeding announces *without proof* results in Graham and Robert [6], still work in progress at the time of ECMI 2008, in which the interaction due to the simultaneous transmission of *several* classes of permanent connections is rigorously analyzed. A class of connections is characterized, in particular, by the set of nodes it uses, and how, at those nodes, the connections create some congestion and adapt to the total congestion encountered.

For mean-field limit proofs for systems of statistically *indistinguishable* objects, assuming mean-field limit convergence of initial conditions, Sznitman [7] has developed compactness-uniqueness methods, as well as coupling methods between the system and an i.i.d. system. Mean-field studies of stochastic communication networks have been performed notably by Dobrushin and his co-authors, see Karpelevitch et al. [8]. See also Graham [9].

A.D. Fitt et al. (eds.), *Progress in Industrial Mathematics at ECMI 2008*,
Mathematics in Industry 15, DOI 10.1007/978-3-642-12110-4_13,

The model of interest here features *dissimilar* connections classified in a finite number of *classes* according to their characteristics. Few convergence proofs for such *multi-class* systems exist, and those in Graham and Méléard [10] require a structure lacking here. So, Graham and Robert [6] develop a coupling method which extends the methods in Sznitman [7], yielding more tractable non-linear limit equations.

The scope is then to study the equilibrium behavior of the *limit* system, and hopefully to establish that the equilibrium behavior for a finite number of connections converges to it. This can be seen as the inversion of limits

$$\lim_{N\to\infty}\lim_{t\to\infty} = \lim_{t\to\infty}\lim_{N\to\infty}$$

where t and N are time and size parameters.

We refer to Graham and Robert [6] for a more complete introduction with a survey of the literature in the domain, and rigorous proofs.

2 The Markovian Network Model

The network has $J \ge 1$ nodes and accommodates $K \ge 1$ classes of sizes $N_k \ge 1$ for $1 \le k \le K$ of permanent connections (or flows, streams, etc.). Let

$$N = (N_1, \dots, N_K), \qquad |N| = N_1 + \cdots + N_K .$$

We study the connection *transmission rate*, governed by the *window size* restricting the quantity of data allowed to be in transit at one time.

An *allocation matrix* $A = (A_{jk}, 1 \le j \le J, 1 \le k \le K)$ describes the utilization of nodes by the connections. We have $A_{jk} \ge 0$, and if $w_{n,k} \ge 0$ is the state of the n-th class k connection, its utilization of node j is given by $A_{jk} w_{n,k}$. The total utilization u_j of node j by the various connections is then

$$u_j = \sum_{k=1}^{K} \sum_{n=1}^{N_k} A_{jk} w_{n,k} , \qquad 1 \le j \le J .$$

An example is $A_{jk} = 1$ if a class k connection uses node j and else $A_{jk} = 0$.

The quantity u_j represents the level of congestion at node j, in particular the loss rate of a connection going through it will depend on it. There are functions $a_k : \mathbb{R}_+ \times \mathbb{R}_+^J \to \mathbb{R}_+$ and $b_k : \mathbb{R}_+ \times \mathbb{R}_+^J \to \mathbb{R}_+$ for $1 \le k \le K$, such that, when the resource utilization vector of the network is $u = (u_j, 1 \le j \le J)$ and the state of a class k connection is w_k then,

- This state increases linearly at rate $a_k(w_k, u)$.
- A loss occurs at rate $b_k(w_k, u)$ and causes a jump from w_k to $r_k w_k$.

A natural form for such functions (with slight abuse of notation) is

$$a_k(w_k, u) = a_k(u), \qquad b_k(w_k, u) = w_k \beta_k(u), \tag{1}$$

$$a_k(u) = \left(\tau_k + \sum_{j=1}^{J} t_{jk}(u_j) \right)^{-1}, \qquad \beta_k(u) = \delta_k + \sum_{j=1}^{J} d_{jk}(u_j), \tag{2}$$

where $\tau_k > 0$ can be interpreted as the round trip time (RTT) between source and destination, and $\delta_k \geq 0$ as the loss rate, of class k connections in a non-congested network, and $t_{jk}(u_j) \geq 0$ as the additional RTT delay and $d_{jk}(u_j) \geq 0$ as the additional loss rate at node j when its utilization is u_j.

The SDE Representation and the Mean-Field Scaling

The Markov process describing the state of the connections is given by

$$W^N(t) = (W^N_{n,k}(t), 1 \leq n \leq N_k, \, 1 \leq k \leq K), \qquad t \in \mathbb{R}_+,$$

where $W^N_{n,k}(t)$ is the state of the n-th connection of class k at time t. It can be represented by the solution of a stochastic differential equation (SDE): for $1 \leq k \leq K$ and $1 \leq n \leq N_k$,

$$dW^N_{n,k}(t) = a_k(W^N_{n,k}(t-), U^N(t-))\, dt \\ -(1 - r_k) W^N_{n,k}(t-) \int \mathbf{1}_{\{0 \leq z \leq b_k(W^N_{n,k}(t-), U^N(t-))\}} \mathcal{N}_{n,k}(dz, dt) \tag{3}$$

with $U^N(t) = (U^N_j(t), 1 \leq j \leq J)$ and

$$U^N_j(t) = \sum_{k=1}^{K} A_{jk} \sum_{n=1}^{N_k} W^N_{n,k}(t),$$

where $(\mathcal{N}_{n,k}, 1 \leq k \leq K, 1 \leq n \leq N_k)$ are independent Poisson processes with Lebesgue intensity measure on $\mathbb{R}^2_+$. Existence and uniqueness of solutions is classical if a_k is Lipschitz and b_k bounded, $1 \leq k \leq K$.

A scaling is used to reduce the high dimensionality of (3) in order to investigate its qualitative and quantitative properties. It is assumed that

$$N_k \to \infty, \quad \frac{N_k}{|N|} = \frac{N_k}{N_1 + \cdots + N_K} \to p_k, \qquad 1 \leq k \leq K, \tag{4}$$

where $p_k \geq 0$ and $p_1 + \cdots + p_K = 1$. The capacity is accordingly scaled by setting $\overline{U}^N = U^N/|N|$ in the functions a_k and b_k. We obtain the mean-field scaled SDE: for $1 \leq k \leq K$ and $1 \leq n \leq N_k$,

$$dW^N_{n,k}(t) = a_k(W^N_{n,k}(t-), \overline{U}^N(t-))\, dt \\ -(1 - r_k) W^N_{n,k}(t-) \int \mathbf{1}_{\{0 \leq z \leq b_k(W^N_{n,k}(t-), \overline{U}^N(t-))\}} \mathcal{N}_{n,k}(dz, dt) \tag{5}$$

with $\overline{U}^N(t) = (\overline{U}_j^N(t), 1 \le j \le J)$ and

$$\overline{U}_j^N(t) = \sum_{k=1}^{K} \frac{N_k}{|N|} A_{jk} \overline{W}_k^N(t) \quad \text{with} \quad \overline{W}_k^N(t) = \frac{1}{N_k} \sum_{n=1}^{N_k} W_{n,k}^N(t)\,.$$

This multi-class mean-field system interacts through the *scaled utilization* vector $\overline{U}^N(t)$, or the *scaled state* vector $\overline{W}^N(t) = (\overline{W}_k^N(t), 1 \le k \le K)$.

3 The Non-Linear Limit Process

When N goes to infinity, in view of (5), mean-field behavior is expected: the connection evolutions should become independent, and for class k connections should converge in law to that of $(W_k(t), t \ge 0)$, where the stochastic process $(W(t), t \ge 0) = \big((W_k(t), 1 \le k \le K), t \ge 0\big)$ solves the non-linear SDE

$$\begin{aligned} dW_k(t) &= a_k\left(W_k(t-), u_W(t)\right) dt \\ &\quad - (1 - r_k) W_k(t-) \int \mathbf{1}_{\{0 \le z \le b_k(W_k(t-), u_W(t))\}} \mathcal{N}_k(dz, dt) \end{aligned} \tag{6}$$

for $1 \le k \le K$, with $u_W(t) = (u_{W,j}(t), 1 \le j \le J)$ and

$$u_{W,j}(t) = \sum_{k=1}^{K} A_{jk} p_k \mathbf{E}(W_k(t)),$$

where $(\mathcal{N}_k, 1 \le k \le K)$ are i.i.d. Lebesgue intensity Poisson point processes.

In this *non-linear* SDE, the evolution of the process $(W(t), t \ge 0)$ depends not only on its instantaneous value but also on the mean utilization vector $u(t)$, or on the mean value $\mathbf{E}(W(t))$. Its infinitesimal generator depends at time t *on the law* of $W(t)$ itself, which thus solves non-linear equations.

We seek results valid for a_k and b_k of the form (1)–(2), where b_k has a quadratic behavior. To control the long-time evolution or the stationary behavior of $W(t)$, initial conditions cannot be assumed to be uniformly bounded, so that exponential and Gaussian moment assumptions are introduced.

Condition (C) *Holds for a family of random variables $\{X_0^\alpha, \alpha \in \mathcal{S}\}$ in $\mathbb{R}_+^K$, for (b_k), and for $\varepsilon > 0$ when at least one of the two conditions is satisfied:*

1. *for $1 \le k \le K$, the function $b_k : \mathbb{R}_+ \times \mathbb{R}_+^J \to \mathbb{R}_+$ is Lipschitz, and*

$$\sup_{\alpha \in \mathcal{S}} \mathbf{E}\big(\exp(\varepsilon \|X_0^\alpha\|)\big) < \infty\,,$$

2. *for $1 \le k \le K$, $b_k(w, u) = w\beta_k(u)$ and $\beta_k : \mathbb{R}_+^J \to \mathbb{R}_+$ is Lipschitz, and*

$$\sup_{\alpha \in \mathcal{S}} \mathbf{E}\big(\exp\left(\varepsilon \|X_0^\alpha\|^2\right)\big) < \infty\,.$$

Theorem 1. *If the functions $a_k : \mathbb{R}_+ \times \mathbb{R}_+^J \to \mathbb{R}_+$, $1 \le k \le K$ are bounded and Lipschitz and if Condition (C) holds for W_0, (b_k) and $\varepsilon > 0$, then there is pathwise existence and uniqueness of a solution $(W(t), t \ge 0)$ of the non-linear SDE (6) starting at W_0, with continuous dependence on the initial condition.*

4 The Mean-Field Limit for Converging Initial Data

The fundamental notions of exchangeability and chaoticity, see Aldous [11] and Sznitman [7], must be extended to such multi-class models. We use the notation $\lim_{N\to\infty}$ for the limit along an arbitrary subsequence of $N = (N_k)_{1\le k\le K} \in \mathbb{N}^K$ such that $\min_{1\le k\le K} N_k$ goes to infinity.

Definition 1. *The family of r.v. $(X_{n,k}, 1 \le n \le N_k,\, 1 \le k \le K)$ is* multi-exchangeable *if its law is invariant under permutation of the indexes* within *the classes: for $1 \le k \le K$ and all permutations σ_k of $\{1, \dots, N_k\}$, we have*

$$\mathcal{L}\big(X_{\sigma_k(n),k}, 1 \le n \le N_k,\, 1 \le k \le K\big) = \mathcal{L}\big(X_{n,k}, 1 \le n \le N_k,\, 1 \le k \le K\big)\,.$$

*A sequence $(X^N_{n,k}, 1 \le n \le N_k,\, 1 \le k \le K)$ of multi-class random variables indexed by $N = (N_k)_{1\le k\le K} \in \mathbb{N}^K$ is $P_1 \otimes \cdots \otimes P_K$-*multi-chaotic *if*

$$\lim_{N\to\infty} \mathcal{L}\big(X^N_{n,k}, 1 \le n \le m,\, 1 \le k \le K\big) = P_1^{\otimes m} \otimes \cdots \otimes P_K^{\otimes m}\,, \qquad \forall m \ge 1\,,$$

where P_k for $1 \le k \le K$ is a probability measure on $\mathbb{R}_+$.

The following theorem is the main result. It uses the topology of uniform convergence on compact sets for the sample path spaces.

Theorem 2. *In the mean-field scaling (4), if*

1. *The initial values $(W^N_{n,k}(0), 1 \le n \le N_k,\, 1 \le k \le K)$ are multi-exchangeable and $P_{1,0} \otimes \cdots \otimes P_{K,0}$-multi-chaotic, and*
2. *The functions $a_k : \mathbb{R}_+ \times \mathbb{R}_+^J \to \mathbb{R}_+$, $1 \le k \le K$, are bounded and Lipschitz, and Condition (C) holds for $\{W^N_1(0),\, N \in \mathbb{N}^K\}$, (b_k) and $\varepsilon > 0$,*

then, as N goes to infinity, the processes

$$\big((W^N_{n,k}(t), t \ge 0), 1 \le n \le N_k,\, 1 \le k \le K\big)$$

solving the SDE (5) with initial values $(W^N_{n,k}(0))$ are multi-exchangeable and P_W-multi-chaotic, where $P_W = P_{W_1} \otimes \cdots \otimes P_{W_K}$ is the law of the process $(W(t), t \ge 0) = ((W_k(t), t \ge 0), 1 \le k \le K)$, the solution of the non-linear SDE (6) with initial law $P_{1,0} \otimes \cdots \otimes P_{K,0}$.

5 Invariant Laws and a Fixed Point Equation

We consider the probability densities given for $0 < r < 1$ and $\rho > 0$ by

$$H_{r,\rho}(x) = \frac{\sqrt{2\rho/\pi}}{\prod_{n=0}^{+\infty}(1-r^{2n+1})} \sum_{n=0}^{+\infty} \frac{r^{-2n}}{\prod_{k=1}^{n}(1-r^{-2k})} e^{-\rho r^{-2n} x^2/2}, \quad x \in \mathbb{R}_+ ,$$

which have first moment (expected value)

$$\int_{x\geq 0} x H_{r,\rho}(x)\, dx = \sqrt{\rho}\psi(r), \qquad \psi(r) = \sqrt{\frac{2}{\pi}} \prod_{n=1}^{+\infty} \frac{1-r^{2n}}{1-r^{2n-1}} .$$

Theorem 3. *If the functions a_k and b_k, $1 \leq k \leq K$, are of the form (1) with $\beta_k > 0$, and a_k and β_k are Lipschitz functions and a_k is bounded, then the invariant laws for solutions $(W(t), t \geq 0)$ of (6) are in one-to-one correspondence with the solutions $u = (u_j)_{1\leq j\leq J} \in \mathbb{R}_+^J$ of the fixed point equation*

$$u_j = \sum_{k=1}^{K} A_{jk} p_k \psi(r_k) \sqrt{\frac{a_k(u)}{\beta_k(u)}}, \;\; 1 \leq j \leq J ,$$

and the invariant law corresponding to such a solution u^* has density $w = (w_k)_{1\leq k\leq K} \mapsto \prod_{k=1}^{K} H_{r_k,\rho_k}(w_k)$ with $\rho_k = a_k(u^*)/\beta_k(u^*)$, see above.

References

1. Ott, T.J., Kemperman, J.H.B., Mathis, M.: The stationary behavior of Ideal TCP Congestion Avoidance. Unpublished manuscript, August (1996)
2. Adjih, C., Jacquet, Ph., Vvedenskaya, N.: Performance evaluation of a single queue under multi-user TCP/IP connections. Tech. Report RR-4141, INRIA, March 2001, http://hal.archives-ouvertes.fr/docs/00/07/24/84/PDF/RR-4141.pdf (2001)
3. Baccelli, F., McDonald, D.R., Reynier, J.: A mean-field model for multiple TCP connections through a buffer implementing RED. Perform. Eval. **49**, 77–97 (2002)
4. Dumas, V., Guillemin, F., Robert, Ph.: A markovian analysis of additive-increase multiplicative-decrease (AIMD) algorithms. Adv. Appl. Probab. **34**(1), 85–111 (2002)
5. Guillemin, F., Robert, Ph., Zwart, B.: AIMD algorithms and exponential functionals. Ann. Appl. Probab. **14**(1), 90–117 (2004)
6. Graham, C., Robert, Ph.: Interacting Multi-class Transmissions in Large Stochastic Networks. Ann. Appl. Probab. **19**(6), 2334–2361 (2009)
7. Sznitman, A.S.: Topics in propagation of chaos, École d'été de Saint-Flour. Lecture Notes in Maths, vol. 1464, pp. 167–243. Springer, Berlin (1989)
8. Karpelevich, F.I., Pechersky, E.A., Suhov, Yu.M.: Dobrushin's approach to queueing network theory. J. Appl. Math. Stochastic Anal. **9**(4), 373–397 (1996) MR1429262 (98d:60182)

9. Graham, C.: Kinetic limits for large communication networks. In: Bellomo, N., Pulvirenti, M. (eds.) Modelling in Applied Sciences: A Kinetic Theory Approach, pp. 317–370. Birkhauser, Boston (2000)
10. Graham, C., Méléard, S.: Chaos hypothesis for a system interacting through shared resources. Probab. Theor. Relat. Field. **100**, 157–173 (1994)
11. Aldous, D.J.: Exchangeability and related topics, École d'été de Probabilités de Saint-Flour XIII. Lecture Notes in Mathematics, vol. 1117, pp. 1–198. Springer, New York (1985)