Vector Space of Cooperative Games: Construction of Basis Related with Solutions Based on Marginal Contributions and Determination of Games with Predefined Allocations

R. Amer[1] and J.M. Giménez[2]

[1] Department of Applied Mathematics II, Technical University of Catalonia, ETSEIAT, Colom 11, 08222 Terrassa, Spain, rafel.amer@upc.edu
[2] Department of Applied Mathematics III, Technical University of Catalonia, EPSEM, Bases de Manresa 61, 08242 Manresa, Spain
jose.miguel.gimenez@upc.edu

Summary. Semivalues form a wide family of solutions for cooperative games that assign to each player a weighted sum of its marginal contributions to the coalitions. The Shapley value and the Banzhaf value belong to the family of semivalues. In this work, all semivalues that admit a basis related with the concept of potential are determined, obtaining an explicit expression for its games. Also, for each semivalue whose potential basis has been found, a method to construct all cooperative games with a predefined payoff vector is offered.

1 Introduction

Probabilistic values as solution concept for cooperative games were introduced in [11]. The payoff that assigns a probabilistic value to each player is a weighted sum of marginal contributions to the coalitions, where the weighting coefficients form a probabilistic distribution over the coalitions he/she is a member.

A type of probabilistic values is formed by the semivalues that were defined in [5]. In this case the weighting coefficients are independent of the players and they only depend on the coalition size. Semivalues represent a natural generalization of both the Shapley value [10] and the Banzhaf value [2, 9]. Many properties of these solutions can extend to the set of semivalues. For instance, the potential, which was introduced in [7] for the Shapley value. The potential of the Shapley value assigns to each game and all its restricted games a number recursively obtained, so that the marginal contribution of each player to the potential coincides with the payoff to the player by the

A.D. Fitt et al. (eds.), *Progress in Industrial Mathematics at ECMI 2008*, Mathematics in Industry 15, DOI 10.1007/978-3-642-12110-4_112,

Shapley value. In a similar way, Dragan [3] defines a potential for the Banzhaf value, as well as a potential for every semivalue on cooperative games [4].

Indeed in [3], it is obtained for the Banzhaf value a series of new concepts and properties already known for the Shapley value. Among these concepts we find a particular basis for the vector space of cooperative games whose determination is directly related with the potential. This basis allows to solve an inverse problem for the Banzhaf value: find all games with a predefined payoff vector. For each vector space of cooperative games, this basis is know as potential basis.

The main purpose of this work consists in finding all games with a pre-established allocation for the greater possible number of semivalues. In a similar way to the Banzhaf value, the process passes through the potential basis, but now it has two levels: (1) determine the semivalues for which a potential basis can be obtained and (2) construct the games of the basis according to the weighting coefficients of each semivalue.

Our inverse problem for semivalues coincides with the resolution of a non-homogeneous system of linear equations. In a classical way, we obtain the solution as a sum of the general solution for the homogeneous system, the so-called *null space*, and a particular solution for the non-homogeneous system, *the short game*. In both cases, the potential basis plays an essential role, since its games are $\{0,1\}$-valued for the potential; these values easily allow to modulate the payoff vector according to predefined allocations.

2 Preliminaries

A *cooperative game* with transferable utility is a pair (N, v), where N is a finite set of *players* and $v : 2^N \to \mathbb{R}$ is the so-called *characteristic function*, which assigns to every *coalition* $S \subseteq N$ a real number $v(S)$, the *gain* or *worth* of coalition S, and satisfies the natural condition $v(\emptyset) = 0$. With G_N we denote the set of all cooperative games on N. For a given set of players N, we identify each game (N, v) with its characteristic function v.

With the usual operations, addition $(v_1+v_2)(S) = v_1(S)+v_2(S)$, and product $(\lambda v)(S) = \lambda v(S),\ \lambda \in \mathbb{R}$, the set G_N has structure of real vector space. For every nonempty coalition T, the *unity game* 1_T is defined by $1_T(S) = 1$ if $S = T$ and 0 otherwise. The family of unity games $\{1_T \,|\, \emptyset \neq T \subseteq N\}$ forms a basis in G_N and the dimension of G_N as real vector space is $2^n - 1$.

A function $\psi : G_N \to \mathbb{R}^N$ is called a *solution* and it represents a method to measure the negotiation strength of the players in the game. The payoff vector space $\mathbb{R}^N$ is also called the allocation space. In order to calibrate the importance of each player $i \in N$ in a cooperative game (N, v), we can look at his/her marginal contribution to the coalitions, $v(S) - v(S \setminus \{i\})$. If these contributions are weighted by means of identical weights according to the coalition size, we obtain the solution concept known as *semivalue*, introduced and axiomatically characterized in [5].

The payoff to the players in a game $v \in G_N$ by a semivalue ψ is an average of marginal contributions of each player:

$$\psi_i[v] = \sum_{S \ni i} p_s^n [v(S) - v(S \setminus \{i\})] \quad \forall i \in N \ (s = |S|),$$

where the weighting coefficients $(p_s^n)_{s=1}^n$ verify $\sum_{s=1}^n \binom{n-1}{s-1} p_s^n = 1$ and $p_s^n \geq 0$ for $1 \leq s \leq n$. With $Sem(G_N)$ we denote the set of all semivalues on G_N.

Given a semivalue $\psi \in Sem(G_N)$, $|N| = n$, with weighting coefficients $(p_s^n)_{s=1}^n$, the recursively obtained numbers

$$p_s^m = p_s^{m+1} + p_{s+1}^{m+1} \quad 1 \leq s \leq m < n,$$

define a *induced semivalue* ψ^m (see [4]) on the space of cooperative games with m players. Adding the own semivalue, the so-called *family of induced semivalues by* ψ in spaces of cooperative games with less than or equal n players is formed by $\psi^m \in Sem(G_M)$ with $1 \leq m \leq n$.

If the initial semivalue on G_N is the Shapley value, $p_s^n = 1/[n\binom{n-1}{s-1}]$, the Banzhaf value, $p_s^n = 1/2^{n-1}$, or *binomial semivalues* as they are defined in [1], $p_s^n = \alpha^{s-1}(1-\alpha)^{n-s}$, $\alpha \in (0,1)$, then the induced semivalues are also of the same initial types.

Definition 1. *Let us suppose* $\psi \in Sem(G_N)$ *with weighting coefficients* $(p_s^n)_{s=1}^n$. *The potential of game* v *restricted to coalition* $T \subseteq N$, $T \neq \emptyset$, *according to semivalue* ψ *is defined by*

$$P_\psi(T, v) = \sum_{S \subseteq T} p_s^t v(S) \quad \forall T \subseteq N.$$

We find this definition in [4]. It generalizes the potential for the Shapley value introduced in [7] and also the potential for the Banzhaf value in [3]. The above definition verifies the condition of potential, i.e.,

$$P_\psi(T, v) - P_\psi(T \setminus \{i\}, v) = \psi_i^t[T, v] \quad \forall T \subseteq N,\ |T| \geq 2,$$

and, for $|T| = 1$: $P_\psi(\{i\}, v) = v(\{i\}) = \psi_i^1[\{i\}, v] \quad \forall i \in N$.

3 Potential Basis for Semivalues

A basis in the game space G_N is potential basis with respect to a solution concept that has a potential if the components of every game $v \in G_N$ in this basis agree with the potentials of game (N, v) and its restricted games (T, v), $T \subset N$, $T \neq \emptyset$. We find the potential basis for the Banzhaf value in [3]. Now, we rewrite this definition for any semivalue.

Definition 2. *Let ψ be a semivalue on G_N. A basis in G_N $\{v_S \in G_N \,|\, S \subseteq N, S \neq \emptyset\}$ is potential basis with respect to semivalue ψ iff:*

$$\forall v \in G_N, \quad v = \sum_{S \subseteq N} \alpha_S v_S \;\Rightarrow\; \alpha_S = P_\psi(S, v).$$

Lemma 1. *A basis in the game space G_N $\{v_S \in G_N \,|\, S \subseteq N, S \neq \emptyset\}$ is potential basis with respect to semivalue ψ on G_N if and only if for every $S \subseteq N$, $S \neq \emptyset$, $P_\psi(S, v_S) = 1$; $\quad P_\psi(T, v_S) = 0 \quad \forall T \subseteq N$, $T \neq S$.*

Proposition 1. *Let us suppose $\psi \in Sem(G_N)$ with weighting coefficients $(p_s^n)_{s=1}^n$. Every game $v \in G_N$ can be recursively reconstructed from the potential P_ψ if and only if $p_n^n > 0$. Then, the recursive expression is:*

$$v(T) = \frac{1}{p_t^t}\Big[P_\psi(T, v) - \sum_{S \subset T} p_s^t v(S)\Big] \quad \forall T \subseteq N,\ 2 \leq |T| \leq n,$$

and $\quad v(\{i\}) = P_\psi(\{i\}, v) \quad \forall i \in N.$

According to Lemma 1, the games in the so-called potential basis are characterized by the potentials of their restricted games. By means of Proposition 1 we can reconstruct these games from their potentials and obtain an explicit expression for them.

Proposition 2. *[6] Let ψ be a semivalue on G_N whose last weighting coefficient is $p_n^n > 0$. If P_ψ denote the potential of ψ, for every $S \subseteq N$, $S \neq \emptyset$, there exists a unique game $c_{\psi,S} \in G_N$ with $P_\psi(S, c_{\psi,S}) = 1$ and $P_\psi(T, c_{\psi,S}) = 0$ $\forall T \subseteq N$, $T \neq S$, which has like explicit expression:*

$$c_{\psi,S}(T) = \begin{cases} (-1)^{t-s} \sum_{h=0}^{t-s} \binom{t-s}{h} \dfrac{(-1)^h}{p_{t-h}^{t-h}} & \text{if } T \supseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1. *[6] If ψ is a semivalue on G_N with last weighting coefficient $p_n^n > 0$, then the family of games $C_\psi = \{c_{\psi,S} \in G_N \;|\; S \subseteq N, S \neq \emptyset\}$ is potential basis in the vector space G_N with respect to the semivalue ψ.*

4 Inverse Problem for Semivalues

The potential for the games in a potential basis only takes values 0 and 1; it leads to simple allocations for these games, as we can see in the next Lemma.

Lemma 2. *Let us suppose $\psi \in Sem(G_N)$ with last weighting coefficient $p_n^n > 0$. If e_j, $1 \leq j \leq n$, are the unit vectors in the standard basis for $\mathbb{R}^n$, for the games of a potential basis $c_{\psi,S}$, $S \subseteq N$, $S \neq \emptyset$, we have:*

(a) $\psi[N, c_{\psi,N}] = \sum_{j=1}^{n} e_j$;

(b) $\psi[N, c_{\psi,N\setminus\{j\}}] = -e_j \quad \forall j \in N$;

(c) $\psi[N, c_{\psi,S}] = 0 \quad \forall S \subset N,\ 1 \leq |S| \leq n-2$.

Definition 3. *Let ψ be a semivalue on G_N. We call null space by ψ to the vector subspace of games in G_N that obtain payoff vector 0 according to semivalue ψ.*

$$NS(\psi) = \{v \in G_N \mid \psi[N, v] = 0\}.$$

Games in a null space are a solution for our inverse problem in a particular case. By means of a vector treatment, for semivalues with non-null last weighting coefficient, the next property shows the solution for the homogeneous inverse problem and, as a result, we can solve the general inverse problem.

Proposition 3. *Let us suppose $\psi \in Sem(G_N)$ with last weighting coefficient $p_n^n >\ 0$, then* $\dim(NS(\psi)) = 2^n - n - 1$ *and a basis for $NS(\psi)$ is formed by*

$$\left\{ c_{\psi,N} + \sum\nolimits_{j=1}^{n} c_{\psi,N\setminus\{j\}}\,, \quad c_{\psi,S} \mid 1 \leq |S| \leq n-2 \right\}.$$

Corollary 1. *For a given semivalue $\psi \in Sem(G_N)$ with last weighting coefficient $p_n^n > 0$ and a given payoff vector $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^N$, the solution for the equation*

$$\psi[N, v] = \eta, \tag{1}$$

has by expression:

$$v = \sum_{S \subseteq N,\, 1 \leq |S| \leq n-2} \lambda_S\, c_{\psi,S} + \lambda_N \left[c_{\psi,N} + \sum\nolimits_{j \in N} c_{\psi,N\setminus\{j\}} \right] - \sum_{j \in N} \eta_j\, c_{\psi,N\setminus\{j\}},$$

where λ_N, λ_S, $1 \leq |S| \leq n-2$, are freedom degrees of the set of solutions; for every selection, the numbers λ_N, λ_S are the potentials of game v on N and games v restricted to S, $1 \leq |S| \leq n-2$, respectively.

Definition 4. *We call short game that verifies equation (1) to the particular solution obtained by imposing $\lambda_N = 0$ and $\lambda_S = 0$ for $1 \leq |S| \leq n-2$; we denote it by $\overline{v}_\psi$.*

$$\overline{v}_\psi = -\sum_{j \in N} \eta_j\, c_{\psi,N\setminus\{j\}}.$$

Game $\overline{v}_\psi$ is a linear combination of games $c_{\psi,N\setminus\{j\}}$, $j \in N$, that only take non-null values on the coalitions $N \setminus \{j\}$ and N. An explicit expression for the short game $\overline{v}_\psi$ is:

$$\overline{v}_\psi = \frac{1}{p_{n-1}^n + p_n^n} \left[-\sum_{j \in N} \eta_j\, 1_{N\setminus\{j\}} + \frac{p_{n-1}^n}{p_n^n} \Big(\sum_{j \in N} \eta_j \Big) 1_N \right].$$

Theorem 2. *For a given semivalue ψ defined on game space G_N with last weighting coefficient $p_n^n > 0$ and a given vector $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^N$, the general solution of the non-homogeneous equation $\psi[N, v] = \eta$ is obtained as a sum of the general solution of the homogeneous equation $\psi[N, v] = 0$ and one particular solution of the non-homogeneous equation, i.e.,*

$$v = NS(\psi) + \overline{v}_\psi.$$

Acknowledgement. Research partially supported by Grant SGR 2005-00651 of the Catalonia Government (*Generalitat de Catalunya*) and Grant MTM 2006-06064 of the Education and Science Spanish Ministry and the European Regional Development Fund.

References

1. Amer, R., Giménez, J.M.: Modification of semivalues for games with coalition structures. Theor. Decis. **54**, 185–205 (2003)
2. Banzhaf, J.F.: Weighted voting doesn't work: a mathematical analysis. Rutgers Law Rev. **19**, 317–343 (1965)
3. Dragan, I.: New mathematical properties of the Banzhaf value. TR#300, Department of Mathematics, University of Texas at Arlington, 1–22 (1995)
4. Dragan, I.: Potential and consistency for semivalues of finite cooperative TU games. Int. J. Math. Game Theory Algebra **9**, 85–97 (1999)
5. Dubey, P., Neyman, A., Weber, R.J.: Value theory without efficiency. Math. Oper. Res. **6**, 122–128 (1981)
6. Giménez, J.M.: Contributions to the study of solutions for cooperative games (in Spanish). Ph.D. thesis, Technical University of Catalonia, Spain (2001)
7. Hart, S., Mas-Colell, A.: The potential of the Shapley value. In: Roth, A.E. (ed.) The Shapley Value: Essays in Honor of L.S. Shapley, pp. 127–137. Cambridge University Press, Cambridge (1988)
8. Owen, G.: Multilinear extensions of games. Manage. Sci. **18**, 64–79 (1972)
9. Owen, G.: Multilinear extensions and the Banzhaf value. Nav. Res. Log. Quarterly **22**, 741–750 (1975)
10. Shapley, L.S.: A value for n-person games. In: Kuhn, H.W., Tucker, A.W. (eds.) Contributions to the Theory of Games II, pp. 307–317. Princeton University Press, Princeton (1953)
11. Weber, R.J.: Probabilistic values for games. In: Roth, A.E. (ed.) The Shapley Value: Essays in Honor of L.S. Shapley, pp. 101–119. Cambridge University Press, Cambridge (1988)