A Survey of Fuzzy Integrals: Directions for Extensions

Haruki Imaoka

Abstract. This study describes four directions for extension of various fuzzy integrals based on the Sugeno integral and Choquet integral. Fuzzy theory as well as utility theory is covered as a motivation for these extensions. Further, a new general fuzzy integral is proposed.

1 Introduction

We have two objectives in surveying fuzzy integrals. First, we want to classify many proposed fuzzy integrals into four types and map their possible extensions. We are particularly interested in the Sugeno integral and the rediscovered Choquet integral, the two prominent fuzzy integrals. Second, we want to propose a new general fuzzy integral to combine three groups of fuzzy integrals into one type.

Theories and applications in various fields have produced a wide and rich stream of fuzzy integrals. Some studies cover these fuzzy measures and fuzzy integrals. For theoretical matters, a book by Grabisch, Murofushi, and Sugeno 2000a [13] and more recently a paper by Mesiar and Mesiarová 2008 [34] are recommendable. As for applications, the above book and more recently another paper by Grabisch and Labreuche 2008 [17] are recommendable. In the field of economics specifically, a book by Eichberger and Kelsey 2009 [8] is recommendable.

Here, we summarize many fuzzy integrals in literature into four types from the viewpoint of their directions for extension.

Type 1: Union of the two prominent Sugeno and Choquet integrals. There are three groups of fuzzy integrals. The first group contains integrals such as the Sugeno and Choquet integrals. Two more integrals named the Shilkret integral and the Opposite-Sugeno integral are touchstones to discern whether an integral in the

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second and third groups covers each of the integrals in the first group or not. In this direction, a new general fuzzy integral, named the two-copulas integral, is proposed. The next two directions for extension come from economics.

Type 2: Concave integrals for uncertainty aversion. Here are two ways to attach concavity onto integrals. One method is changing integrals and another one is restricting the fuzzy measures.

Type 3: Negative values for risk aversion. Here are two ways to treat negative functions named the Choquet integral and Šipoš integral. They are combined and extended.

Type 4: Nonmonotonic measure for welfare preference. This direction for extension comes from the welfare field. Some preferences in this field showed impartiality, i.e., the same incomes are better than bigger incomes but not the same. This extension requests studying nonmonotonic measures.

2 Theory

2.1 Fuzzy Integrals

A triplet $(X, 2^X, \mu)$ is a fuzzy measure space. A set function $\mu : 2^X \to [0,1]$ is called a *fuzzy measure* (Sugeno 1974 [55] a.k.a. capacity by Choquet 1953 [4] or nonadditive measure by Shackle 1949 [49]) if μ satisfies the following properties:

- On the boundary of μ, $\mu(\phi) = 0, \mu(X) = 1$.
- The monotonicity condition is $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$. [See Extension Type 4]

Some special fuzzy measures are as follows:

- additive: $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$.
- superadditive: $\mu(A \cup B) + \mu(A \cap B) \geq \mu(A) + \mu(B)$. [See Extension Type 2]
- subadditive: $\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B)$.
- conjugate: $\hat{\mu}(A) = \mu(X) - \mu(A^C)$. [See Extension Type 3]

Definition 2.1. A functional of $f : X \to [0,1]$ w.r.t. a fuzzy measure μ is called a *fuzzy integral* $I_\mu(f)$. [See Extension Type 1]

Here we define a *distribution function* of a function f. A distribution function $F : [0,1] \to [0,1]$ is defined by $F(\alpha) := \mu(\{x | f(x) > \alpha\})$. [See Extension Type 3]

Two prominent fuzzy integrals are defined by

$$\text{Sugeno} : S_\mu(f) := \underset{\alpha \in [0,1]}{Sup} [\alpha \wedge F(\alpha)] \qquad \text{(Sugeno 1974 [55])}$$

$$\text{Choquet} : C_\mu(f) := \int_0^1 F(\alpha) d\alpha \qquad \text{(Vitali 1925 [59], Choquet 1953 [4])}$$

The Choquet integral was rediscovered independently in different fields (Höhle 1982 [18], Schmeidler 1986 [46], Murofushi and Sugeno 1989 [38]) and has been widely used. [See Extension Type 1]

2.2 Binary Operations

Two symbols $\otimes$ and $\oplus$ are binary operations to make more general fuzzy integrals.

Two important multiplication-like binary operations from $[0,1]^2$ to $[0,1]$ are as follows:

- *T-norm* (Schweizer and Sklar 1961 [48]) satisfies:
  - Boundary conditions: $x \otimes 0 = 0 \otimes x = 0$, $x \otimes 1 = 1 \otimes x = x$.
  - Nondecreasing: $x \otimes y \leq u \otimes v$, whenever $x \leq u, y \leq v$.
  - Commutative: $x \otimes y = y \otimes x$.
  - Associative: $(x \otimes y) \otimes z = x \otimes (y \otimes z)$.
- *Copula* (Sklar 1959 [53]) satisfies:
  - Boundary conditions: $x \otimes 0 = 0 \otimes x = 0$, $x \otimes 1 = 1 \otimes x = x$.
  - 2-increasing: $x \otimes y - u \otimes y - x \otimes v + u \otimes v \geq 0$, whenever $x \leq u, y \leq v$.

If both t-norm and copula have a generator, which is an increasing function $\phi : [0,1] \to [0,1]$ and $\phi(0) = 0, \phi(1) = 1$ (here we show a product-type generator), the following equation holds: $x \otimes y = \phi^{-1}(\phi(x) \cdot \phi(y))$.

To compare the two binary operations, the following theorem is useful.

Theorem 2.1. *A t-norm is a copula if and only if it satisfies the Lipschitz condition $u \otimes y - x \otimes y \leq u - x$, whenever $x \leq u$ (Moynihan, Schweizer, and Sklar 1978 [37]).*

Here, we introduce three important binary operations, which are both copulas and t-norms.

1. Minimum copula (a.k.a. logical product); $x \wedge y :=$ minimum$\{x, y\}$.
2. Product copula (a.k.a. arithmetic product); $x \cdot y := xy$.
3. Łukasiewicz copula (a.k.a. bounded product); $x \circ y :=$ maximum$\{x + y - 1, 0\}$.

As for copulas, the following relation is known as the Fréchet-Hoeffding bounds: $x \circ y \leq x \otimes y \leq x \wedge y$. Frank's generator $\phi(t) = \frac{e^{qt}-1}{e^{q}-1}$ (Frank 1979 [10]) encompasses minimum, product, and Łukasiewicz copulas.

An addition-like binary operation named t-conorm (a.k.a. s-norm), denoted by $\oplus$, is defined by replacing only the boundary condition of t-norm as follows: Boundary conditions: $x \oplus 0 = 0 \oplus x = x$, $x \oplus 1 = 1 \oplus x = 1$. Sometimes a binary operation $\oplus$ is defined using the corresponding binary operation $\otimes$: $x \oplus y := 1 - (1 - x) \otimes (1 - y)$. The following equality is called the conjugate condition: $x \oplus y = x + y - x \otimes y$.

Using t-conorm, another important binary operation called a *pseudo difference* (Weber 1984 [64]) is defined as follows: $u -_{\oplus} v := \inf\{t \in [0,1] \,|\, v \oplus t \geq u\}$.

As for the maximum, $u -_{\vee} v := \inf\{t \in | v \vee t \geq u\} = u$, whenever $u \geq v$ is satisfied.

A special function denoted by $\mathbf{1}_A$, which satisfies $x \in A \Rightarrow f(x) = 1$ and $x \notin A \Rightarrow f(x) = 0$, is called a characteristic function of a subset A. There are two important relationships considered as inspectors used for binary operations: (Lower edge): $I_\mu(a \cdot \mathbf{1}_A) = a \otimes \mu(A), a \in [0,1]$; (Upper edge): $I_\mu(\mathbf{1}_B + a \cdot \mathbf{1}_{B^c}) = a \oplus \mu(B), a \in [0,1]$.

2.3 Extended Fuzzy Integrals

There are various fuzzy integrals, which are classified into three groups. The first group contains fixed fuzzy integrals such as the Sugeno integral and Choquet integral. The second one contains wider fuzzy integrals, which have one free binary operation. The third group contains the widest fuzzy integrals, which have two or more free binary operations.

- The first group:
  - Sugeno: $\underset{\alpha\in[0,1]}{Sup}\ [\alpha \wedge F(\alpha)]$ (Sugeno 1974 [55]).
  - Choquet: $\int_0^1 F(\alpha)d\alpha$ (Choquet 1953 [4]).
  - Shilkret: $\underset{\alpha\in[0,1]}{Sup}\ [\alpha \cdot F(\alpha)]$ (Shilkret 1971 [51]).
  - Opposite-Sugeno: See Imaoka in the next second group.
- The second group:
  - Weber: $\underset{\alpha\in[0,1]}{Sup}\ [\alpha \otimes F(\alpha)]$ (Weber 1984 [64]).
  - Mesiar: $\zeta^{-1}\left[\int_0^1 \zeta \circ F(\alpha)d\zeta(\alpha)\right]$, where an increasing function $\zeta : [0,1] \to [0,\infty]$ is used and $\zeta(0) = 0$ (Mesiar 1995 [33]).
  - Imaoka: $\int_0^1 C_x(\alpha, F(\alpha))d\alpha$, where $C_x(x,y) := \frac{\partial}{\partial x}C(x,y)$ is a partial derivative of a copula (Imaoka 1997 [19]). Vivona and Divari 2002 [60] studied this integral deeply. If the Łukasiewicz copula is used, it is called an Opposite-Sugeno integral, and the integral is a member of the first group. The Opposite-Sugeno integral does not follow the law of large numbers (Imaoka 2000 [20]).

Before explaining the third group, we introduce how to make a fuzzy integral. In the first step, a simple function is defined on X with range $R_n = \{x_1, x_2, \cdots, x_n\}$, where $0 < x_1 < x_2 \cdots < x_n < 1$ are satisfied. μ_i is defined by $\mu_i := F(x_i - 0)$ for a simple function. In the second step, an integral is defined by a supremum of simple functions.

- The third group:
  - Murofushi and Sugeno: $Sup\left\{\bigoplus_{i=1}^{n}(x_i \ominus x_{i-1}) \otimes \mu_i \,\middle|\, \forall n, \forall R_n, \mu_i \le F(x_i - 0)\right\}$ (Murofushi and Sugeno 1991 [39]). Benvenuti and Mesiar 2000 [2] proposed a similar pseudoadditive integral. The integral has three independent binary operations and covers both Weber and Mesiar integrals.

Note that Imaoka integral can be written by:

$$Sup\left\{\sum_{i=1}^{n}(x_i \otimes \mu_i - x_{i-1} \otimes \mu_i)\,|\forall n, \forall R_n, \mu_i \le F(x_i - 0)\right\}.$$

The Weber integral covers Sugeno and Shilkret. The Mesiar integral covers Choquet. The Imaoka integral covers Sugeno, Choquet, and Opposite-Sugeno. The Murofushi and Sugeno integral covers Sugeno, Choquet, and Shilkret.

A new fuzzy integral, named a two-copulas integral, which is a member of the third group is defined in two ways as follows, where $\odot \geq \otimes$ means $x \odot y \geq x \otimes y, \forall (x,y) \in [0,1]^2$:

$$\odot \geq \otimes : Sup \left\{ \sum_{i=1}^{n} (x_i \otimes \mu_i - x_{i-1} \odot \mu_i) \, | \forall n, \forall R_n, \mu_i \leq F(x_i - 0) \right\},$$
$$\odot \leq \otimes : Inf \left\{ \sum_{i=1}^{n} (x_i \otimes \mu_{i-1} - x_{i-1} \odot \mu_{i-1}) \, | \forall n, \forall R_n, \mu_i \geq F(x_i + 0) \right\}.$$

2.4 Discrete Fuzzy Integrals

If a function f is represented by a vector $\mathbf{X} = (x_1, x_2, \ldots, x_n)$, a fuzzy integral $I_\mu(\mathbf{X}) : [0,1]^n \to [0,1]$ is called a discrete fuzzy integral w.r.t. a (discrete) fuzzy measure μ, where a universal set X is defined as $X = \{1, 2, \cdots, n\}$.

Let $\pi(i)$ be a permutation so that $x_{\pi(1)} \leq x_{\pi(2)} \leq \cdots \leq x_{\pi(n)}$ is satisfied. A fuzzy measure is defined by $\mu_{\pi(i)} := \mu(\{\pi(i), \pi(i+1), \cdots, \pi(n)\})$. However, the succeeding calculation is not the same as the case of a simple function in general.

The integral is considered an aggregation function from n variables and 2^n parameters to one output. The Choquet integral produces an output, and the value is not selected from the input values and parameters in general. In contrast, the Sugeno integral produces an output selected from the input values and parameters. This fact is easily understood from Kandel's expression (Kandel and Byatt 1978 [21]) that the Sugeno integral is a weighted median (WM). The Choquet and Sugeno integrals are said to be quantitative and qualitative, or a ratio scale and ordinal scale, respectively. Here, we want to show some properties of the fuzzy integrals including the Choquet and Sugeno integrals.

1. Fuzzy integral $I_\mu(\mathbf{X})$ is continuous w.r.t. $\mathbf{X}$ and μ.
2. Nondecreasing: $I_\mu(\mathbf{X}) \leq I_\nu(\mathbf{Y})$, whenever $\mathbf{X} \leq \mathbf{Y}$ and $\mu \leq \nu$ are satisfied.
3. Idempotent: $I_\mu(a \cdot \mathbf{1}_X) = a, a \in [0,1]$. Note that this relation is equivalent to $\min_{i \in X} x_i \leq I_\mu(\mathbf{X}) \leq \max_{i \in X} x_i$ if property (2) is satisfied.
4. Indicating: $I_\mu(\mathbf{1}_A) = \mu(A)$. A discrete fuzzy integral can be considered a function from n-dimensional unit hypercube $[0,1]^n$ to $[0,1]$. From this relation, we can distinguish between the role of a fuzzy measure and a fuzzy integral. A fuzzy measure determines the output values of vertices before integration and a fuzzy integral plays an interpolation role (Grabisch 2004 [15]).
5. Coincidence regardless of the model when a fuzzy measure is logical: If a fuzzy measure satisfies a condition $\mu(A) \in \{0,1\}$, it is said to be a logical measure, and every fuzzy integral coincides with every other.

2.5 Characterization of Fuzzy Integrals

Instead of operational definitions of fuzzy integrals, definitions by characterizations have been studied. Two vectors $\mathbf{X}, \mathbf{Y}$ are called comonotonic if the following conditions are satisfied: $x_i \geq x_j \Rightarrow y_i \geq y_j, \forall (i,j) \in \{1, \cdots, n\}^2$.

Some special fuzzy integrals are as follows:

$$\begin{aligned}&\text{comonotone additive}: I_\mu(\mathbf{X}+\mathbf{Y}) = I_\mu(\mathbf{X}) + I_\mu(\mathbf{Y}).\\&\text{comonotone maxtive}: I_\mu(\mathbf{X}\vee\mathbf{Y}) = I_\mu(\mathbf{X}) \vee I_\mu(\mathbf{Y}).\\&\text{comonotone minitive}: I_\mu(\mathbf{X}\wedge\mathbf{Y}) = I_\mu(\mathbf{X}) \wedge I_\mu(\mathbf{Y}).\end{aligned}$$

Comonotone additivity and maxtivity (minitivity) are local characteristics of the Choquet and Sugeno integrals, respectively. Here we consider a restricted domain called the hypercorn, which contains $n+1$ strictly increasing sequential vertices such as characteristic vectors of $\phi \subset A_1 \subset \ldots \subset A_n = X$. The word *comonotone* is interpreted as "as for a limited hypercorn." Dellacherie 1970 [6] showed that the Choquet integral satisfies comonotone additivity. Inversely, Schmeidler 1986 [46] showed that comonotone additivity characterizes the Choquet integral, which is known as Schmeidler's representation theorem. This representational approach has been deeply studied (Denneberg 1997 [7], Narukawa, Murofushi, and Sugeno 2000 [41], Marichal 2000b [31]). Additivity $I_\mu(\mathbf{X}+\mathbf{Y}) = I_\mu(\mathbf{X}) + I_\mu(\mathbf{Y})$ is called Cauchy's functional equation and the solution is $I_\mu(\mathbf{X}) = \sum c_i x_i$, where c_i is an arbitrary constant if $I_\mu(\mathbf{X})$ satisfies one of the mild regularity conditions (continuity at a single point or monotonicity, etc.) (Alisina, Frank, and Schweizer 2006 [1]). Roughly speaking, additivity means linear interpolation. Marichal 2002 [32] pointed toward and Grabisch 2004 [15] showed that the Choquet integral is considered a parsimonious linear interpolation.

The Möbious transform is one of the most exciting transforms in both theoretical and practical studies on fuzzy measures and the Choquet integral. The Choquet integral w.r.t. Möbious transform (a.k.a. Lovász extension) can be considered a global expression (Lovász 1983 [29], Rota 1964 [45]).

In contrast, DeCampos and Bolanos 1992 [5] showed that maxtivity and minitivity are characterizations of the Sugeno integral. Some axiomatic characterization of the Sugeno integral have been deeply studied (Marichal 2000a [30]).

If we are interested in *uncertainty aversion* (Knight 1921 [25], Schmeidler 1989 [47]), there are other useful global properties of the fuzzy integrals:

$$\begin{aligned}&0 \le \lambda \le 1,\\&\text{concave}: I_\mu(\lambda\mathbf{X} + (1-\lambda)\mathbf{Y}) \ge \lambda I_\mu(\mathbf{X}) + (1-\lambda) I_\mu(\mathbf{Y}).\\&\text{convex}: I_\mu(\lambda\mathbf{X} + (1-\lambda)\mathbf{Y}) \le \lambda I_\mu(\mathbf{X}) + (1-\lambda) I_\mu(\mathbf{Y}).\end{aligned}$$

In the Choquet integral, there is a deep relationship between the properties of the fuzzy measure and the concave or convex properties.

Theorem 2.2. *A Choquet integral is concave if and only if a fuzzy measure is superadditive (Lovász 1983 [29], Schmeidler 1989 [47]).*

If a fuzzy measure satisfies the additivity, it is called a probability measure and is denoted by P instead of μ.

Theorem 2.3. *When a fuzzy measure μ is superadditive (its conjugate fuzzy measure $\hat{\mu}$ is subadditive), its core is nonempty and the Choquet integral of any random variable f is given by*

$$C_\mu(f) = \min_{P \in core(\mu)} C_P(f),$$
$$core(\mu) := \{P | \mu(A) \leq P(A) \leq \hat{\mu}(A), \forall A \subseteq X\}$$

(Shapley 1971 [50], Schmeidler 1986 [46]).

When f is modified by a utility function, this formulation is called the Max-Min Expected Utility (MMEU) (Gilboa and Schmeidler 1989 [11]).

3 Fuzzy Integrals and Utility Theory

3.1 The Choquet Integral in Economics

In the field of fuzzy theory, idempotency of the fuzzy integral is usually assumed since aggregation must be between a minimum and a maximum value. On the other hand, in the field of expected utility, it is natural to deny idempotency. Narukawa's Choquet-Stieltjes integral does not satisfy idempotency and is defined as follows:

$$\int_0^1 F(\alpha) d\psi(\alpha) \qquad \text{(Narukawa and Murofushi 2008 [43])}$$

When a nondecreasing function $\psi : [0,1] \to [0,1]$ is considered a utility function, it is called the Choquet Expected Utility (CEU) (Schmeidler 1989 [63], Wakker 2001 [63]).

Furthermore, if fuzzy measures are distorted from probability measures by a nondecreasing function $\xi : [0,1] \to [0,1]$ as follows:

$$F(\alpha) = \xi \circ P(\{x | f(x) > \alpha\}),$$

they are called the Rank-Dependent Expected Utility (RDEU) (Quiggin 1982 [44], Yaari 1987 [66]). CEU and RDEU were born from different streams, namely *uncertainty aversion* and *risk aversion*, but the formulation is almost the same (Wakker 1990 [62]).

3.2 Fuzzy Integrals for Negative Inputs

Negative functions and positive/negative functions have been studied. The Šipoš integral (Šipoš 1970 [52]) from $[-1,1]^n$ to $[-1,1]$ states:

$$\mathbf{X}^+ := \mathbf{X} \vee \mathbf{0}$$
$$\mathbf{X}^- := -\mathbf{X} \vee \mathbf{0}$$
$$SI_\theta(\mathbf{X}) := C_\theta(\mathbf{X}^+) - C_\theta(\mathbf{X}^-).$$

As for the Choquet integral from $[-1,1]^n$ to $[-1,1]$, the fuzzy integral for the negative part is a conjugate one:

$$C_\theta(\mathbf{X}) := C_\theta(\mathbf{X}^+) - C_{\hat{\theta}}(\mathbf{X}^-).$$

The Choquet and Šipoš integrals were named asymmetric and symmetric integrals, respectively (Denneberg 1997 [7]). Note that the rule "Coincidence regardless of the model when a fuzzy measure is logical," is only a rule when positive functions are considered, but is violated between the Choquet and Šipoš integrals. In the Šipoš integral, minimum :$[-1,1]^2 \to [-1,1]$ makes a big roof similar to the Sugeno integral, and maximum :$[-1,1]^2 \to [-1,1]$ makes a big roof similar to the Opposite-Sugeno integral.

Both integrals from $[-1,1]^n$ to $[-1,1]$ were extended using two independent fuzzy measures in the context of risk aversion, one for positive and another for negative, and were called the Cumulative Prospect Theory (CPT) (Kahneman and Tversky 1979 [22], Tversky and Kahneman 1992 [58]):

$$C_\theta(\mathbf{X}) := C_\theta(\mathbf{X}^+) - C_\sigma(\mathbf{X}^-).$$

As for the Sugeno integral, the negative function has been studied by Grabisch 2000b [14].

4 Further Four Directions for Extension

There are many new fuzzy integrals or related integrals. Here we list the integrals with direction types and short comments.

- Type 1: Union of two prominent integrals
  - Fuzzy measure-based fuzzy integral (Klement, Mesiar, and Pap 2004 [23]). Extension of Imaoka's extension.
  - Extremal fuzzy integral (Struk 2006 [54]). If the generalization is done from the viewpoint that fuzzy measures are also considered as input instead of parameters, the extremal integral appears.
  - Universal integral (Klement, Mesiar, and Pap 2007 [24]). To exclude the extremal integral, a universal integral is defined.
  - Two-fold integral (Torra 2003 [57], Narukawa and Torra 2003 [42]). This unites two prominent integrals, but the formulation and motivation are quite different from the above mentioned extensions.
- Type 2: Concave integrals for uncertainty aversion
  - Max-Min Expected Utility (MMEU)
    - · Concave integral (Lehrer [28], Teper 2009 [56]). This integral always produces a concave shape. If the fuzzy measure is superadditive, this integral coincides with the Choquet integral; otherwise, it produces higher roofs.
    - · Set-system Choquet integral (Faigle and Grabisch2009 [9]). This is another approach to make concave shapes. Fuzzy measures are restricted and discussed theoretically.
- Type 3: Negative values for risk aversion
  - Rank-Dependent Expected Utility (RDEU)

    · Two-fold integral (Torra 2003 [57], Narukawa and Torra 2003 [42]). In this Choquet extended integral, integrands are distorted by the Sugeno or Choquet integral.
    · The following two integrals extend a distribution function from $F(\alpha)$ to $\mu_\alpha(\{x|f(x)>\alpha\})$.
      Level dependent Choquet integral (Giove, Greco, and Matarazzo 2007 [12]).
      Level dependent Sugeno integral (Mesiar, Mesiarova, and Ahmad 2009 [35]).
  - Cumulative Prospect Theory (CPT)
    · Bi-capacities Choquet integral (Grabisch and Labreuche 2005 [16]). A negative fuzzy measure is defined in this study.
    · Bipolar Choquet-like integral (Labreuche and Grabisch 2006 [27]). Negative fuzzy measures are used in the Choquet integral. Partially bipolar Choquet integral (Kojadinovic and Labreuche 2009 [26]). A modified version of the above integral.
    · Asymmetric general Choquet integral (Mihailovic and Pap 2009 [36]). This integral is considered an extension of Mesiar integral.

- Type 4: Nonmonotonic measure for welfare preference
  There are some works where monotonicity of fuzzy measures is not required.
  - Murofushi, Sugeno, and Machida 1994 [40]
  - Waegenaere and Wakker 2001 [61]
  - Cardin and Giove 2008 [3]

5 Conclusions

Fuzzy integrals have been modified and extended by many researchers. The origins are the Sugeno and Choquet integrals. Since the adoption of the Choquet integral in expected utility theory has had significant influence, we surveyed many fuzzy integrals in order to make a map of directions for extending these integrals. From a theoretical point of view, two main streams exist. One is an extension to negative values of both integrands and fuzzy measures. Another stream is uniting the Sugeno and Choquet integrals. In this latter stream, a new general fuzzy integral was proposed.

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