Filters on Commutative Residuated Lattices

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Abstract. In this short paper we define a *filter* of a commutative residuated lattice and prove that, for any commutative residuated lattice L, the lattice $Fil(L)$ of all filters of L is isomorphic to the congruence lattice $Con(L)$ of L, that is,

$$Fil(L) \cong Con(L).$$

1 Introduction

In the paper [3], it is proved that, for any commutative residuated lattice L, the congruence lattice $Con(L)$ is isomorphic to the lattice of all *convex subalgebras* of L, that is,

$$Con(L) \cong Sub_c(L).$$

But, when we investigate properties of BL-algebras and MV-algebras which are axiomatic extensions of commutative residuated lattices, we usually consider about *filters* of those algebras and get many results by use of filters. Thus it is convenient to consider some kind of filters on commutative residuated lattices instead of convex subalgebras to get uniform method. In this short paper, we define filters of a commutative residuated lattice L and prove that the lattice $Fil(L)$ of all filters of L is isomorphic to the lattice $Con(L)$ of all congruences on L, that is,

$$Fil(L) \cong Con(L).$$

An example in the paper shows that filters are different from convex subalgebras.

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2 Preliminaries

At first we recall the definition of commutative residuated lattices. By a *commutative residuated lattice* (CRL), we mean an algebraic structure $(L, \wedge, \vee, \odot, \rightarrow, e)$, where

1. $(L, \wedge, \vee)$ is a lattice;
2. $(L, \odot, e)$ is a commutative monoid with a unit element e;
3. For all $x, y, z \in L$, $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$.

Let L be a CRL. The following is familiar ([1, 2, 3, 4]).

Proposition 2.1. *For all $x, y, z \in L$, we have*

(i) $x \leq y \Longleftrightarrow e \leq x \rightarrow y$
(ii) $x \rightarrow (y \rightarrow z) = x \odot y \rightarrow z = y \rightarrow (x \rightarrow z)$
(iii) $x \odot (x \rightarrow y) \leq y$
(iv) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$
(v) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$

A subset $C \subseteq L$ is called a *convex subalgebra* of L if it satisfies the conditions:

(C1) C is order-convex, that is, if $a, b \in C$ and $a \leq x \leq b$ then $x \in C$;
(C2) C is a subalgebra of L, that is, if $x, y \in C$ then $x \wedge y, x \vee y, x \odot y, x \rightarrow y \in C$.

By $Sub_c(L)$, we mean the class of all convex subalgebras of L. It is easy to show that $Sub_c(L)$ is a lattice. Moreover if we denote the class of all congruences on L by $Con(L)$, then it is proved in [3] that

Theorem 2.1. *For any commutative residuated lattice L,*

$$Con(L) \cong Sub_c(L).$$

In the present paper we define filters which are different from convex subalgebras and prove the same characterization theorem by congruence lattices. A subset $F \subseteq L$ is called a *filter* if

(F1) $e \in F$;
(F2) If $x, y \in F$ then $x \odot y \in F$;
(F3) If $x, y \in F$ then $x \wedge y \in F$;
(F4) If $x \in F$ and $x \leq y$ then $y \in F$.

Let L be a commutative residuated lattice. It is clear that a set $[e) = \{x \in L \mid e \leq x\}$ is the least proper filter of L. For any filter F of L, we define a relation $\mathscr{C}(F)$ on L as follows:

$$(x, y) \in \mathscr{C}(F) \Longleftrightarrow x \rightarrow y, y \rightarrow x \in F.$$

Proposition 2.2. *If F is a filter then $\mathscr{C}(F)$ is a congruence.*

Conversely, for any congruence θ on L, we define a subset $\mathscr{F}(\theta)$ by

$$\mathscr{F}(\theta) = \{x \in L \mid \exists u \in e/\theta; u \leq x\}.$$

For that subset $\mathscr{F}(\theta)$, we can show the next result.

Proposition 2.3. *$\mathscr{F}(\theta)$ is a filter of L.*

It is easy to show that

Proposition 2.4. *Let F, G be filters and θ, φ be congruences. Then we have*

(1) If $F \subseteq G$ then $\mathscr{C}(F) \subseteq \mathscr{C}(G)$.
(2) If $\theta \subseteq \varphi$ then $\mathscr{F}(\theta) \subseteq \mathscr{F}(\varphi)$.

For a congruence $\mathscr{C}(F)$ determined by a filter F and a filter $\mathscr{F}(\theta)$ done by a congruence θ, it is a natural question whether $F = \mathscr{F}(\mathscr{C}(F))$ and $\theta = \mathscr{C}(\mathscr{F}(\theta))$. Concerning to the question we have an affirmative solution.

Theorem 2.2. *For any filter F and congruence θ, we have*

(1) $F = \mathscr{F}(\mathscr{C}(F))$
(2) $\theta = \mathscr{C}(\mathscr{F}(\theta))$

Proof. We only show the case of (1) $F = \mathscr{F}(\mathscr{C}(F))$. The other case can be proved similarly. To prove $F = \mathscr{F}(\mathscr{C}(F))$, suppose that $x \in \mathscr{F}(\mathscr{C}(F))$. There exists u such that $(u,e) \in \mathscr{C}(F)$ and $u \leq x$. Since $u = e \to u \in F$ and $u \leq x$, we have $x \in F$ and thus $\mathscr{F}(\mathscr{C}(F)) \subseteq F$. Conversely, if we take $x \in F$, since $e \in F$, then we have $e \wedge x \in F$. Since $e \wedge x \leq e$, we have $e \leq e \wedge x \to e$ and $e \wedge x \to e \in F$. Moreover, $e \to e \wedge x = e \wedge x \in F$. This means that $(e \wedge x, e) \in \mathscr{C}(F)$ and $e \wedge x \in e/\mathscr{C}(F)$. It follows from $e \wedge x \leq x$ that $x \in \mathscr{F}(\mathscr{C}(F))$ and hence that $F \subseteq \mathscr{F}(\mathscr{C}(F))$. Thus we have $F = \mathscr{F}(\mathscr{C}(F))$.

From the result above, considering maps $\xi : Fil(L) \to Con(L)$ defined by $\xi(F) = \mathscr{C}(F)$ and $\eta : Con(L) \to Fil(L)$ by $\eta(\theta) = \mathscr{F}(\theta)$, we conclude that these maps are isomorphism to each other.

Theorem 2.3. *For every commutative residuated lattice L, we have*

$$Fil(L) \cong Con(L).$$

3 Negative Cone

Let $L^- = \{x \mid x \leq e\}$. The subset L^- is called a *negative cone* in [3]. In this section, we show that each filter can be represented by a subset of negative cone.

Lemma 3.1. *For $S \subseteq L^-$, the set $\{x \mid \exists n, \exists s_i \in S; s_1 \odot s_2 \odot \cdots \odot s_n \leq x\}$ is a smallest filter including S.*

We hereby denote the smallest filter including a subset S by $[S)$, that is, by $[S)$ we mean the filter generated by S. For any subset S of L, a subset $S^* = \{x \in S \,|\, x \leq e\}$ is called a *negative cone* of S. We show that any filter can be determined by its negative cone. Let F be a filter and $F^* = \{x \wedge e \,|\, x \in F\}$. Since F is the filter, it is obvious that F^* is the negative cone of F, that is, $\{x \wedge e \,|\, x \in F\} = \{x \in F \,|\, x \leq e\}$.

Theorem 3.1. $F = [F^*)$.

By use of negative cones, we can concretely describe a filter $F \vee G$ of filters F and G.

Proposition 3.1. *For all filters F and G,*

$$F \vee G = \{x \,|\, \exists n, m,\ \exists u_i \in F^*,\ v_j \in G^*; u_1 \odot \cdots \odot u_n \odot v_1 \odot \cdots \odot v_m \leq x\},$$

where F^* and G^* are negative cones of F and G, respectively.

We note that our filters are different from convex subalgebras defined in [3], as the following example shows. Let L be the non-distributive lattice $M_5 = \{0, a, b, e, 1\}$. It is clear that a sublattice $B = L - \{e\}$ is a Boolean algebra and any element $x \in B$ has a complement element x', for example $a' = b$ and $b' = a$. We define operations $\odot$ and $\to$ as follows:

$$x \odot y = \begin{cases} x \wedge y & (\text{ if } x, y \in B\ ) \\ y & (\text{ if } x = e\ ) \end{cases}$$

$$x \to y = \begin{cases} x' \vee y & (\text{ if } x, y \in B\ ) \\ x' & (\text{ if } x \in B \text{ and } y = e\ ) \\ y & (\text{ if } x = e\ ) \end{cases}$$

It is clear that $(L, \wedge, \vee, \odot, \to, e, 0, 1)$ is the commutative residuated lattice. In the example, $\{e\}$ is the convex subalgebra but not a filter, because the smallest filter is $\{e, 1\}$. Thus, filters are different from convex subalgebras. Concerning to the relation between convex subalgebras and filters, we have a following result.

Theorem 3.2. *For any commutative residuated lattice L, $Sub_c(L) = Fil(L)$ if and only if L is integral, that is, e is a greatest element of L.*

Proof. Considering the fact $Sub_c(L) \cong Con(L) \cong Fil(L)$, since a convex subalgebra and a filter corresponding to the least congruence $\omega = \{(x, x) \,|\, x \in L\}$ are identical, we have $\{e\} = \{x \in L \,|\, e \leq x\}$. This means that e is the greatest element of L. Because, since $e \leq e \vee x$ for every element $x \in L$, we have $e \vee x \in \{e\}$ and $e \vee x = e$, that is, $x \leq e$. It follows that e is the greatest element of L.

Conversely, if e is the greatest element then we have $x \odot y \leq x$ for all $x, y \in L$. It is clear that any convex subalgebra is also a filter. For any filter F, since $x \leq y \to x$, if $x, y \in F$ then $x \to y, y \to x \in F$. It follows that F is the convex subalgebra.

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