Aggregation of Quasiconcave Functions*

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Abstract. Aggregation of information is important in many fields, ranging from engineering and economics to artificial intelligence and decision making processes. Aggregation refers to the process of combining a number of values into a single value so that the final result of aggregation takes into account, in a given form, all individual values under consideration. In decision making processes the values to be aggregated are typically preference or satisfaction degrees. This paper could serve as a theoretical background for applications mainly in the area of decision analysis, decision making or decision support.

1 Introduction

Aggregation refers to the process of combining values into a single value so that the final result of aggregation takes into account, in a given form, all individual values under consideration. In decision making, values to be aggregated are typically preference or satisfaction degrees. A preference degree, for example $v(A,B)$, tells to what extent an alternative A is preferred to an alternative B. This way, however, will not be followed here. In this paper the values are understood and interpreted as satisfaction degrees which express to what extent a given alternative is satisfactory with respect to a given criterion - a given real-valued function, or as a kind of distance to a prototype which may represent the ideal alternative for the decision maker. Depending on concrete applications, values to be aggregated can be also interpreted as degrees of confidence in the fact that a given alternative is true, or as experts' opinions, similarity degrees, etc.; see, for example, [1].

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Once some values on a scale (for example, on the unit interval $[0,1]$) are given, we can aggregate them and obtain a new value defined on the same scale. This can be done in many different ways according to what is expected from such mappings. They are usually called aggregation operators, and they can be roughly divided into three classes, each possessing very distinct behavior and semantics, see [5].

Operators of the first class, *conjunctive type operators*, combine values as if they were related by a logical "and" operation. In other words, the result of combination is high if all individual values are high. Triangular norms are typical examples of conjunctive type aggregations.

On the other hand, *disjunctive type operators* combine values as an "or" operation, so that the result of aggregation is high if some of the values are high. The most common examples of disjunctive type operators are triangular conorms.

Between conjunctive and disjunctive type operators, there is room for the third class of aggregation operators, which are often called *averaging type operators*. They are usually located between minimum and maximum, which are the bounds of the t-norms and t-conorms. Averaging type operators have the property that low values of some criteria can be compensated by high values of the other criteria functions.

There are of course other operators which do not fit into any of these classes.

2 Definition and Basic Properties

When aggregating data in applications, we assign uniquely to each tuple of elements a real number. For this purpose, both t-norms and t-conorms are rather special operators on the unit interval $[0,1]$.

Definition 2.1. A function $T : [0,1]^2 \to [0,1]$ that is commutative, associative, nondecreasing in every variable and satisfies the boundary condition $T(a,1) = a$ for all $a \in [0,1]$, is called the *triangular norm* or *t-norm*. The most popular t-norms are defined as follows:

$$T_M(a,b) = \min\{a,b\}, \tag{1}$$

$$T_P(a,b) = a.b, \tag{2}$$

$$T_L(a,b) = \max\{0, a+b-1\}. \tag{3}$$

$$T_D(a,b) = \begin{cases} \min\{a,b\} & \text{if } \max\{a,b\} = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

They are called Minimum t-norm T_M, Product t-norm T_P, Lukasiewicz t-norm T_L and Drastic product T_D.

A class of functions closely related to the class of t-norms is the class of functions $S : [0,1]^2 \to [0,1]$ defined as follows.

Definition 2.2. A function $S : [0,1]^2 \to [0,1]$ that is commutative, associative, nondecreasing in every variable and satisfies the boundary condition $S(a,0) = a$ for all $a \in [0,1]$, is called the *triangular conorm* or *t-conorm*.

The functions S_M, S_P, S_L and S_D defined for $a,b \in [0,1]$ by

$$S_M(a,b) = \max\{a,b\}, \tag{5}$$
$$S_P(a,b) = a+b-a\cdot b, \tag{6}$$
$$S_L(a,b) = \min\{1,a+b\}, \tag{7}$$
$$S_D(a,b) = \begin{cases} \max\{a,b\} & \text{if } \min\{a,b\}=0 \\ 1 & \text{otherwise.} \end{cases} \tag{8}$$

are typical t-conorms. Often, S_M, S_P, S_L and S_D are called the maximum, probabilistic sum, bounded sum and drastic sum, respectively.

It can easily be verified that for each t-norm T, the function $T^* : [0,1]^2 \to [0,1]$ defined for all $a,b \in [0,1]$ by $T^*(a,b) = 1 - T(1-a,1-b)$ is a t-conorm. The converse statement is also true. Namely, if S is a t-conorm, then the function $S^* : [0,1]^2 \to [0,1]$ defined for all $a,b \in [0,1]$ by $S^*(a,b) = 1 - S(1-a,1-b)$ is a t-norm. The t-conorm T^* and t-norm S^* are called *dual* to the t-norm T and t-conorm S, respectively. It can easily be verified that

$$T_M^* = S_M,\ T_P^* = S_P,\ T_L^* = S_L,\ T_D^* = S_D. \tag{9}$$

Using the commutativity and associativity of t-norms, we extend them (and analogously t-conorms) to more than two arguments by the following formula

$$T^{n-1}(x_1,x_2,\ldots,x_n) = T(T^{n-2}(x_1,x_2,\ldots,x_{n-1}),x_n), \tag{10}$$

where $T^1(x_1,x_2) = T(x_1,x_2)$.

A triangular norm T is said to be *strict* if it is continuous and strictly monotone. It is said to be *Archimedian* if for all $x,y \in (0,1)$ there exists a positive integer n such that $T^{n-1}(x,\ldots,x) < y$. Notice that if T is strict, then T is Archimedian.

There exist other useful operations related to or generalizing t-norms or t-conorms, either on the unit interval or on an arbitrary closed subinterval $[a,b]$ of the extended real line. Because of the natural correspondence between $[a,b]$ and $[0,1]$, each result for operations on the interval $[a,b]$ can be transformed into a result for operators on $[0,1]$, and vice versa. Therefore, the discussion about aggregation operators on $[0,1]$ is sufficiently general, at least from the theoretical point of view.

Definition 2.3. An *aggregation operator* A is a sequence $\{A_n\}_{n=1}^{\infty}$ of mappings (called *aggregating mappings*) $A_n : [0,1]^n \to [0,1]$ satisfying the following properties:

(i) $A_1(x) = x$ for each $x \in [0,1]$;
(ii) $A_n(x_1,x_2,\ldots,x_n) \le A_n(y_1,y_2,\ldots,y_n)$ whenever $x_i \le y_i$ for every $i = 1,2,\ldots,n$, and every $n = 2,3,\ldots$;
(iii) $A_n(0,0,\ldots,0) = 0$ and $A_n(1,1,\ldots,1) = 1$ for every $n = 2,3,\ldots$.

Condition (i) means that A_1 is an identity unary operation, condition (ii) says that aggregating mapping A_n is nondecreasing in all of its arguments x_i, and condition (iii) represents natural boundary requirements. Some other mathematical properties can be requested from an aggregation operators, we list some of them in the following definition.

Definition 2.4. Let $A = \{A_n\}_{n=1}^{\infty}$ be an aggregation operator.

(i) The aggregation operator A is called *commutative*, *idempotent, nilpotent*, *strictly monotone* or *continuous* if, for each $n \geq 2$, the aggregating mapping A_n is commutative, idempotent, nilpotent, strictly monotone or continuous, respectively. The aggregation operator A is called *strict* if A_n is strictly monotone and continuous for all $n \geq 2$.

(ii) The aggregation operator A is called *associative* if, for all $m,n \geq 2$ and all tuples $(x_1,x_2,\ldots,x_m) \in [0,1]^m$ and $(y_1,y_2,\ldots,y_n) \in [0,1]^n$, we have

$$A_{m+n}(x_1,x_2,\ldots,x_m,y_1,y_2,\ldots,y_n) \\ = A_2(A_m(x_1,x_2,\ldots,x_m),A_n(y_1,y_2,\ldots,y_n)).$$

(iii) The aggregation operator A is called *decomposable* if, for all $m,n \geq 2$ and all tuples $(x_1,\ldots,x_m) \in [0,1]^m$ and $(y_1,\ldots,y_n) \in [0,1]^n$, we have

$$A_{m+n}(x_1,\ldots,x_m,y_1,\ldots,y_n) \\ = A_{m+n}(A_m(x_1,\ldots,x_m),\ldots,A_m(x_1,\ldots,x_m),y_1,\ldots,y_n) \tag{11}$$

where, in the right side, the term $A_m(x_1,x_2,\ldots,x_m)$ occurs m times.

(iv) The aggregation operator A is called *compensative* if, for $n \geq 2$ and for all tuples $(x_1,x_2,\ldots,x_n) \in [0,1]^n$, the following inequalities hold:

$$T_M(x_1,x_2,\ldots,x_n) \leq A_n(x_1,x_2,\ldots,x_n) \leq S_M(x_1,x_2,\ldots,x_n). \tag{12}$$

We have already seen that the commutativity and associativity make it possible to extend t-norms and t-conorms to n-ary operations, with $n > 2$. Therefore, a sequence $\{T^n\}_{n=1}^{\infty}$, where T^1 is the identity mapping, defines an aggregation operator, and T^n are its aggregating mappings. For the sake of simplicity, when there is no danger of a confusion, we call this aggregation operator also a t-norm and denote it by the original symbol T. In other words, when speaking about a t-norm T or t-conorm S as an aggregation operator, we always have in mind the corresponding sequence $\{T^n\}_{n=1}^{\infty}$ or $\{S^n\}_{n=1}^{\infty}$, respectively. Recall also that, for the same reason, we shall sometimes omit the index n in the aggregating mappings A_n. Considering this convention in the following propositions, we obtain some characterizations of the previously defined properties. Each t-norm and each t-conorm is a commutative and associative aggregation operator. The minimum T_M is the only idempotent t-norm, but it is not strict. The product norm T_P is strict, but not nilpotent. Lukasiewicz t-norm T_L is both strict and nilpotent. The drastic product T_D is nilpotent, but not continuous, see [3].

Analogous properties hold for t-conorms S_M, S_P, S_L and S_D. A transformation of an aggregation operator by means of a monotone bijection from $[0,1]$ to $[0,1]$ yields again an aggregation operator. We have the following proposition the proof of which is elementary.

Proposition 2.1. *Let* $A = \{A_n\}_{n=1}^{\infty}$ *be an aggregation operator and let* $\psi : [0,1] \to [0,1]$ *be a strictly increasing or strictly decreasing bijection. Then* $A^{\psi} = \{A_n^{\psi}\}_{n=1}^{\infty}$ *defined by* $A_n^{\psi}(x_1,x_2,\ldots,x_n) = \psi^{-1}(A_n(\psi(x_1),\ldots,\psi(x_n)))$ *for all* $n = 1,2,\ldots$ *and all tuples* $(x_1,x_2,\ldots,x_n) \in [0,1]^n$*, is an aggregation operator.*

Continuity of aggregation operators play an important role in applications. The following proposition shows that for continuity of commutative aggregation operators it is sufficient that they are continuous in a single variable only. The proof of the following two propositions can be found in [4].

Proposition 2.2. *Let* $A = \{A_n\}_{n=1}^{\infty}$ *be a commutative aggregation operator. The operator A is continuous if and only if, for each* $n = 1,2,\ldots$*, the mapping* A_n *is continuous in its first variable* x_1*; that is, if, for each n and* $x_2,\ldots,x_n \in [0,1]$*, the function* $A(\cdot,x_2,\ldots,x_n)$ *of single variable is continuous on* $[0,1]$.

Notice that a completely analogous proposition holds for the upper and lower semicontinuity. Also notice that, by monotonicity of an aggregation operator A, the left (right) continuity of A is equivalent to the LSC (USC) of A, and that the left and right continuity mean exactly the interchangeability of the supremum and infimum, respectively, with the application of the aggregation operator.

3 Averaging Aggregation Operators

Between conjunctive and disjunctive type operators, t-norms and t-conorms, there is a room for another class of aggregation operators of averaging type. They are located between minimum and maximum satisfying inequalities (12). Averaging type operators have the property that low values of some criteria can be compensated by high values of the other criteria.

Perhaps, even more popular aggregation operators than t-norms and t-conorms are the means: the *arithmetic mean* $M = \{M_n\}_{n=1}^{\infty}$, the *geometric mean* $G = \{G_n\}_{n=1}^{\infty}$, the *harmonic mean* $H = \{H_n\}_{n=1}^{\infty}$ and the *root-power mean* $M^{(\alpha)} = \{M_n^{(\alpha)}\}_{n=1}^{\infty}$, given by, respectively,

$$M_n(x_1,x_2,\ldots,x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i, \tag{13}$$

$$G_n(x_1,x_2,\ldots,x_n) = \left(\prod_{i=1}^{n} x_i\right)^{1/n}, \tag{14}$$

$$H_n(x_1,x_2,\ldots,x_n) = \frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}, \tag{15}$$

$$M_n^{(\alpha)}(x_1,x_2,\ldots,x_n) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{\alpha}\right)^{1/\alpha}, \quad \alpha \neq 0. \tag{16}$$

All these operators are commutative, idempotent and continuous, none of them is associative. The root-power mean operators $M^{(\alpha)}$, $\alpha \geq 0$, are strict, whereas G and H are not strict. Notice that $M = M^{(1)}$ and $H = M^{(-1)}$.

The next proposition says that the operators (13) - (16) are all compensative. It says even more, namely, that the class of idempotent aggregation operators is exactly the same as the class of compensative ones. The proof of this result is elementary and can be found in [2].

Proposition 3.1. *An aggregation operator is idempotent if and only if it is compensative.*

The following proposition clarifies the relationships between some other properties introduced in Definition 2.4. The proof can be found also in [2].

Proposition 3.2. *Let $A = \{A_n\}_{n=1}^{\infty}$ be a continuous and commutative aggregation operator. Then A is compensative, strict and decomposable, if and only if for all $x_1, x_2, \ldots, x_n \in [0,1]$*

$$A_n(x_1, x_2, \ldots, x_n) = \psi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\psi(x_i)\right), \tag{17}$$

with a continuous strictly monotone function $\psi : [0,1] \to [0,1]$.

The aggregation operator (17) is called the *generalized mean*. It covers a wide range of popular means including those of (13) - (16). The minimum T_M and the maximum S_M are the only associative and decomposable compensative aggregation operators.

4 Concave, Quasiconcave and Starshaped Functions

In this section and the following sections we shall deal with our main problem, that is, the aggregation of generalized quasiconcave functions. First, we will look for sufficient conditions that secure some properties of quasiconcavity. For a more detailed treatment of concavity and some of its generalizations, see [4].

The concepts of concavity, convexity, quasiconcavity, quasiconvexity and quasi-monotonicity of a function $f : \mathbf{R}^n \to \mathbf{R}$ can be introduced in several ways. The following definitions will be most suitable for our purpose.

Definition 4.1. Let X be a nonempty subset of $\mathbf{R}^n$. A function $f : X \to \mathbf{R}$ is called

(i) *concave on X (CA)* if

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y) \tag{18}$$

for every $x, y \in X$ and every $\lambda \in (0,1)$ with $\lambda x + (1-\lambda)y \in X$ (resp. *convex on* X if $-f$ is concave on X);

(ii) *strictly concave on* X if

$$f(\lambda x+(1-\lambda)y) > \lambda f(x)+(1-\lambda)f(y) \tag{19}$$

for every $x,y \in X$, $x \neq y$ and every $\lambda \in (0,1)$ with $\lambda x+(1-\lambda)y \in X$ (resp. *strictly convex on* X if $-f$ is strictly concave on X);

(iii) *semistrictly concave on* X if f is concave on X and (19) holds for every $x,y \in X$ and every $\lambda \in (0,1)$ with $\lambda x+(1-\lambda)y \in X$ such that $f(x) \neq f(y)$ (resp. *semistrictly convex on* X if $-f$ is semistrictly concave on X).

Definition 4.2. Let X be a nonempty subset of $\mathbf{R}^n$. A function $f : X \to \mathbf{R}$ is called

(i) *quasiconcave on X (QCA)* if

$$f(\lambda x+(1-\lambda)y) \geq \min\{f(x), f(y)\}$$

for every $x,y \in X$ and every $\lambda \in (0,1)$ with $\lambda x+(1-\lambda)y \in X$ (resp. *quasiconvex on* X if $-f$ is quasiconcave on X);

(ii) *strictly quasiconcave on* X if

$$f(\lambda x+(1-\lambda)y) > \min\{f(x), f(y)\} \tag{20}$$

for every $x,y \in X$, $x \neq y$ and every $\lambda \in (0,1)$ with $\lambda x+(1-\lambda)y \in X$ (resp. *strictly quasiconvex on* X if $-f$ is strictly quasiconcave on X);

(iii) *semistrictly quasiconcave on* X if f is quasiconcave on X and (20) holds for every $x,y \in X$ and every $\lambda \in (0,1)$ with $\lambda x+(1-\lambda)y \in X$ such that $f(x) \neq f(y)$ (resp. *semistrictly quasiconvex on* X if $-f$ is semistrictly quasiconcave on X).

Notice that in Definitions 4.1 and 4.2 the set X is not required to be convex. If in the above definitions the set X is convex, then we obtain the usual definition of (strictly) quasiconcave and (strictly) quasiconvex functions. Observe that if a function is (strictly) concave and (strictly) convex on X, then it is (strictly) quasiconcave and (strictly) quasiconvex on X, respectively, but not vice-versa.

In Definitions 4.1 and 4.2 we introduced concepts of semistrictly CA functions and semistrictly QCA functions, respectively. The former (the latter) is stronger than the concept of a CA function (QCA function), and weaker than the concept of a strictly CA function (strictly QCA function).

We shall need the following generalization of convexity of sets and functions.

Definition 4.3. Let X be a subset of $\mathbf{R}^n$, $y \in X$. The set X is *starshaped from* y if, for every $x \in X$, the convex hull of the set $\{x,y\}$ is included in X. The set of all points $y \in X$ such that X is starshaped from y is called the *kernel* of X and it is denoted by $\mathrm{Ker}(X)$. The set X is said to be a *starshaped set* if $\mathrm{Ker}(X)$ is nonempty, or X is empty.

Clearly, X is starshaped if there is a point $y \in X$ such that X is starshaped from y. From the geometric viewpoint, if there exists a point y in X such that for every other point x from X the whole linear segment connecting the points x and y belongs to X, then X is starshaped. Evidently, every convex set is starshaped. For a convex set X, we have $\mathrm{Ker}(X) = X$. Moreover, in the 1-dimensional space $\mathbf{R}$, convex sets and starshaped sets coincide.

To introduce starshaped functions, we begin with the following, well known, characterization of quasiconcave and quasiconvex functions.

Proposition 4.1. *Let X be a convex subset of $\mathbf{R}^n$. A function $f : X \to \mathbf{R}$ is quasiconcave on X if and only if all its upper-level sets are convex subsets of $\mathbf{R}^n$. Likewise, f is quasiconvex on X if and only if all its lower-level sets are convex subsets of $\mathbf{R}^n$.*

Proposition 4.1 suggests a way of generalization of quasiconcave and quasiconvex functions. Replacing all convex upper-level sets $U(f,\alpha)$ and convex lower-level sets $L(f,\alpha)$ in Proposition 4.1 by starshaped sets, we obtain the following generalization of quasiconcave and quasiconvex functions.

Definition 4.4. Let X be a starshaped subset of $\mathbf{R}^n$. A function $f : X \to \mathbf{R}$ is called

(i) *upper-starshaped on X (US)* if its upper-level sets $U(f,\alpha)$ are starshaped subsets of $\mathbf{R}^n$ for all $\alpha \in \mathbf{R}$;
(ii) *lower-starshaped on X (LS)* if its lower-level sets $L(f,\alpha)$ are starshaped subsets of $\mathbf{R}^n$ for all $\alpha \in \mathbf{R}$;
(iii) *monotone-starshaped on X (MS)* if it is both lower-starshaped and upper-starshaped on X.

It is obvious that if a function $f : X \to \mathbf{R}^n$ is upper-starshaped on X, then the function $-f$ is lower-starshaped on X, and vice-versa. From the fact that each convex set is starshaped it follows that each quasiconcave (quasiconvex) function is upper-starshaped (lower-starshaped). Moreover, each quasimonotone function is monotone-starshaped. Evidently, the classes of quasiconcave (quasiconvex) functions and upper-starshaped (lower-starshaped) functions coincide on $\mathbf{R}$. In more dimensions it is not true, see [4].

5 T-Quasiconcave Functions

In contrast to the previous section, we now restrict our attention to functions defined on $\mathbf{R}^n$ with range in the unit interval $[0,1]$ of real numbers. Such functions can be interpreted as membership functions of fuzzy subsets of $\mathbf{R}^n$. We therefore use several terms and some notation of fuzzy set theory. However, it should be pointed out that such functions arise in more contexts. In what follows, the Greek letter μ, sometimes with an index, denotes a function that maps $\mathbf{R}^n$ into the closed unit interval $[0,1]$ in $\mathbf{R}$.

We have introduced quasiconcave (semi)strictly quasiconcave, quasiconvex and (semi)strictly quasiconvex functions in Definition 4.1. First, we generalize Definition 4.1 by using triangular norms and conorms.

Definition 5.1. Let X be a nonempty convex subset of $\mathbf{R}^n$, T be a triangular norm, and S be a triangular conorm. A function $\mu : \mathbf{R}^n \to [0,1]$ is called

(i) *T-quasiconcave* on X if

$$\mu(\lambda x + (1-\lambda y)) \geq T(\mu(x), \mu(y)) \tag{21}$$

for every $x, y \in X$, $x \neq y$ and $\lambda \in (0,1)$;

(ii) *strictly T-quasiconcave* on X if

$$\mu(\lambda x + (1-\lambda)y) > T(\mu(x), \mu(y)) \tag{22}$$

for every $x, y \in X$, $x \neq y$ and $\lambda \in (0,1)$;

(iii) *semistrictly T-quasiconcave* on X if (21) holds for every $x, y \in X$, $x \neq y$ and $\lambda \in (0,1)$ and (22) holds for every $x, y \in X$ and $\lambda \in (0,1)$ such that $\mu(x) \neq \mu(y)$;

(iv) *S-quasiconvex* on X if

$$\mu(\lambda x + (1-\lambda y)) \leq S(\mu(x), \mu(y)) \tag{23}$$

for every $x, y \in X$, $x \neq y$ and $\lambda \in (0,1)$;

(v) *strictly S-quasiconvex* on X if

$$\mu(\lambda x + (1-\lambda)y) < S(\mu(x), \mu(y)) \tag{24}$$

for every $x, y \in X$, $x \neq y$ and $\lambda \in (0,1)$;

(vi) *semistrictly S-quasiconvex* on X if (23) holds for every $x, y \in X$, $x \neq y$ and $\lambda \in (0,1)$ and (24) holds for every $x, y \in X$ and $\lambda \in (0,1)$ such that $\mu(x) \neq \mu(y)$;

(vii) *(strictly, semistrictly) (T,S)-quasimonotone* on X, provided μ is (strictly, semistrictly) T-quasiconcave and (strictly) S-quasiconvex on X, respectively;

(viii) *(strictly, semistrictly) T-quasimonotone* on X if μ is (strictly, semistrictly) T-quasiconcave and (strictly, semistrictly) T^*-quasiconvex on X, where T^* is the dual t-conorm to T.

6 Aggregation of Functions

Obviously, the class of quasiconcave functions that map $\mathbf{R}^n$ into $[0,1]$ is exactly the class of T_M-quasiconcave functions and the class of quasiconvex functions that map $\mathbf{R}^n$ into $[0,1]$ is exactly the class of S_M-quasiconvex functions. Similarly, the class of quasimonotone functions that map $\mathbf{R}^n$ into $[0,1]$ is exactly the class of (T_M, S_M)-quasimonotone functions. As $S_M = T_M^*$, we have, by (viii) in Definition 5.1, that this is the class of T_M-quasimonotone functions. Moreover, since the minimum triangular norm T_M is the maximal t-norm, and the drastic product T_D is the minimal t-norm, we have the following consequence of Definition 5.1.

Proposition 6.1. *Let X be a nonempty convex subset of $\mathbf{R}^n$, μ be a function, $\mu : \mathbf{R}^n \to [0,1]$, and T be a triangular norm.*

(i) If μ is (strictly, semistrictly) quasiconcave on X, then μ is (strictly, semistrictly) T-quasiconcave on X , respectively.

(ii) If μ is (strictly, semistrictly) T-quasiconcave on X, then μ is also (strictly, semistrictly) T_D-quasiconcave on X, respectively.

Analogously, the maximum triangular conorm S_M is the minimal conorm and the drastic sum S_D is the maximal conorm, hence Proposition 6.1 can be reformulated for S-quasiconvex functions.

It is easy to show that there exist T-quasiconcave functions that are not quasiconcave (see [4]), and there exist strictly or semistrictly T-quasiconcave functions that are not strictly or semistrictly quasiconcave. Nevertheless, in the one-dimensional Euclidean space $\mathbf{R}$, the following proposition is of some interest.

Proposition 6.2. *Let X be a nonempty convex subset of $\mathbf{R}$, let T be a triangular norm, and let $\mu : \mathbf{R} \to [0,1]$ be upper-normalized in the sense that $\mu(\bar{x}) = 1$ for some $\bar{x} \in X$. If μ is (strictly, semistrictly) T-quasiconcave on X, then μ is (strictly, semistrictly) quasiconcave on X.*

Analogous propositions are valid for S-quasiconvex functions and for T-quasimonotone functions.

Proposition 6.3. *Let X be a nonempty convex subset of $\mathbf{R}$, let S be a triangular conorm, and let $\mu : \mathbf{R} \to [0,1]$ be lower-normalized in the sense that $\mu(\hat{x}) = 0$ for some $\hat{x} \in X$. If μ is (strictly, semistrictly) S-quasiconvex on X, then μ is (strictly, semistrictly) quasiconvex on X.*

To prove Proposition 6.3 we shall use the following relationship between T-quasiconcave and S-quasiconvex functions.

Proposition 6.4. *Let X be a nonempty convex subset of $\mathbf{R}^n$, let T be a triangular norm and let $\mu : \mathbf{R}^n \to [0,1]$ be (strictly, semistrictly) T-quasiconcave on X. Then $\mu^* = 1 - \mu$ is (strictly, semistrictly) T^*-quasiconvex on X, where T^* is the t-conorm dual to T.*

Proof. The proof follows directly from Definition 5.1 and the relation $T^*(a,b) = 1 - T(1-a, 1-b)$.

The following proposition is a consequence of Propositions 6.2 and 6.3.

Proposition 6.5. *Let X be a nonempty convex subset of $\mathbf{R}$, let T and S be a t-norm and t-conorm, respectively, and let $\mu : \mathbf{R} \to [0,1]$ be normalized. If μ is (strictly, semistrictly) (T,S)-quasimonotone on X, then μ is (strictly, semistrictly) quasimonotone on X.*

In what follows we shall use the above relationship between T-quasiconcave and T^*-quasiconvex functions restricting ourselves only to T-quasiconcave functions. Usually, with some exceptions, the dual formulation for S-quasiconvex functions will not be explicitly mentioned. It turns out that the assumption of (upper, lower)-normality of μ is essential for the validity of Propositions 6.2 and 6.3.

Proposition 6.6. *Let X be a nonempty convex subset of $\mathbf{R}^n$, let T and T' be t-norms and let $\mu_i : \mathbf{R}^n \to [0,1]$, $i = 1,2$, be T-quasiconcave on X. If T' dominates T, then $\varphi : \mathbf{R}^n \to [0,1]$ defined by $\varphi(x) = T'(\mu_1(x), \mu_2(x))$, is T-quasiconcave on X.*

Proof. As μ_i, $i = 1,2$, are T-quasiconcave on X, we have $\mu_i(\lambda x + (1-\lambda)y) \geq T(\mu_i(x), \mu_i(y))$ for every $\lambda \in [0,1]$ and $x, y \in X$. By monotonicity of T', we obtain

$$\begin{aligned}\varphi(\lambda x + (1-\lambda)y) &= T'(\mu_1(\lambda x + (1-\lambda)y), \mu_2(\lambda x + (1-\lambda)y)) \\ &\geq T'(T(\mu_1(x), \mu_1(y)), T(\mu_2(x), \mu_2(y))).\end{aligned} \tag{25}$$

Using the fact that T' dominates T, we obtain

$$\begin{aligned}&T'(T(\mu_1(x), \mu_1(y)), T(\mu_2(x), \mu_2(y))) \\ &\quad \geq T(T'(\mu_1(x), \mu_2(x)), T'(\mu_1(y), \mu_2(y))) = T(\varphi(x), \varphi(y)).\end{aligned} \tag{26}$$

Combining (25) and (26), we obtain the required result.

Corollary 6.1. *Let X be a convex subset of $\mathbf{R}^n$, let T be a t-norm, and let $\mu_i : \mathbf{R}^n \to [0,1]$, $i = 1,2$, be T-quasiconcave on X. Then $\varphi_i : \mathbf{R}^n \to [0,1]$, $i = 1,2$, defined by $\varphi_1(x) = T(\mu_1(x), \mu_2(x))$ and $\varphi_2(x) = T_M(\mu_1(x), \mu_2(x))$, are also T-quasiconcave on X.*

Proof. The proof follows from the preceding proposition and the evident fact that T dominates T and T_M dominates every t-norm T.

The following results of this type are also of some interest, for proofs, see [4].

Proposition 6.7. *Let X be a convex subset of $\mathbf{R}^n$, and let $\mu_i : \mathbf{R}^n \to [0,1]$, $i = 1,2,\ldots,m$, be upper normalized T_D-quasiconcave on X such that* $\mathrm{Core}(\mu_1) \cap \cdots \cap \mathrm{Core}(\mu_m) \neq \emptyset$. *Let $A_m : [0,1]^m \to [0,1]$ be an aggregating mapping. Then $\psi : \mathbf{R}^n \to [0,1]$ defined by $\psi(x) = A_m\big(\mu_1(x), \ldots, \mu_m(x)\big)$ is upper-starshaped on X.*

Proposition 6.8. *Let X be a convex subset of $\mathbf{R}^n$, and let $\mu_i : \mathbf{R}^n \to [0,1]$, $i = 1,2,\ldots,m$, be upper normalized T_D-quasiconcave on X such that* $\mathrm{Core}(\mu_1) = \cdots = \mathrm{Core}(\mu_m) \neq \emptyset$. *Let $A_m : [0,1]^m \to [0,1]$ be a strictly monotone aggregating mapping. Then $\psi : \mathbf{R}^n \to [0,1]$ defined for $x \in \mathbf{R}^n$ by $\psi(x) = A_m(\mu_1(x), \ldots, \mu_m(x))$ is T_D-quasiconcave on X.*

The above proposition allows for constructing new T_D-quasiconcave function on $X \subset \mathbf{R}^n$ from the original T_D-quasiconcave functions on $X \subset \mathbf{R}^n$ by using a strictly monotone aggregating operator, e.g., the t-conorm S_M. It is of interest to note that the condition $\mathrm{Core}(\mu_1) = \cdots = \mathrm{Core}(\mu_m) \neq \emptyset$ is essential for T_D-quasiconcavity of ψ in Proposition 6.8.

The following definition extends the concept of domination between two triangular norms to aggregation operators.

Definition 6.1. An aggregation operator $A = \{A_n\}_{n=1}^{\infty}$ *dominates* an aggregation operator $A' = \{A'_n\}_{n=1}^{\infty}$, if, for all $m \geq 2$ and all tuples $(x_1, x_2, \ldots, x_m) \in [0,1]^m$ and $(y_1, y_2, \ldots, y_m) \in [0,1]^m$, the following inequality holds

$$\begin{aligned}&A_m(A'_2(x_1, y_1), \ldots, A'_2(x_m, y_m)) \\ &\qquad \geq A'_2(A_m(x_1, x_2, \ldots, x_m), A_m(y_1, y_2, \ldots, y_m)).\end{aligned}$$

The following proposition generalizes Proposition 6.6.

Proposition 6.9. *Let X be a convex subset of $\mathbf{R}^n$, let $A = \{A_n\}_{n=1}^{\infty}$ be an aggregation operator, T be a t-norm and let $\mu_i : \mathbf{R}^n \to [0,1]$, $i = 1,2,\ldots,m$, be T-quasiconcave on X, and let A dominates T. Then $\varphi : \mathbf{R}^n \to [0,1]$ defined by $\varphi(x) = A_m(\mu_1(x),\ldots,\mu_m(x))$ is T-quasiconcave on X.*

Proof. As μ_i , $i = 1,2,\ldots,m$, are T-quasiconcave on X, we have $\mu_i(\lambda x + (1-\lambda)y) \geq T(\mu_i(x),\mu_i(y))$ for every $\lambda \in (0,1)$ and each $x,y \in X$. By monotonicity of aggregating mapping A_m, we obtain

$$\begin{aligned} &\varphi(\lambda x + (1-\lambda)y) \\ &\quad = A_m(\mu_1(\lambda x + (1-\lambda)y),\ldots,\mu_m(\lambda x + (1-\lambda)y)) \\ &\quad \geq A_m(T(\mu_1(x),\mu_1(y)),\ldots,T(\mu_m(x),\mu_m(y))). \end{aligned} \tag{27}$$

Using the fact that A dominates T, we obtain

$$\begin{aligned} &A_m(T(\mu_1(x),\mu_1(y)),\ldots,T(\mu_m(x),\mu_m(y))) \\ &\quad \geq T(A_m(\mu_1(x),\ldots,\mu_m(x)),A_m(\mu_1(y),\ldots,\mu_m(y))) \\ &\quad = T(\varphi(x),\varphi(y)), \end{aligned} \tag{28}$$

where $T = T^{(2)}$. Combining (27) and (28), we obtain the required result.

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