Capacities, Set-Valued Random Variables and Laws of Large Numbers for Capacities

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Abstract. In this paper, we shall survey some connections between the theory of set-valued random variables and Choquet theory. We shall focus on investigating some results of the relationships between the distributions of set-valued random variables and capacities, and also some connections between the Aumann integral and the Choquet integral. Then we shall review some results on laws of large numbers (LLN's) for set-valued random variables and for capacities, and point out some relations between these two kinds of LLN's. Finally we shall give a new strong LLN of exchangeable random variables for capacities.

1 Introduction

It is well known that classical probability measures and linear mathematical expectations are powerful tools for dealing with stochastic phenomena. However, there are uncertain phenomena which can not be easily modeled by using additive measure and linear mathematical expectations in many applied areas. For example, economists have found the Allais paradox and the Ellsberg paradox (cf. [1, 15]) of the expected utility theory based on classical probability theory in financial economics. So it is necessary to examine non-additive measures and nonlinear expectations with their applications.

In 1953, Choquet [10] introduced concepts of capacities and the Choquet integral. Capacities are non-additive measures and the Choquet integral can be considered as one kind of nonlinear expectations with respect to capacities. Many papers developed the Choquet theory and its applications, for examples, see [9, 14, 18, 34, 37, 38, 45, 46]. In 1973, Sugeno [41] defined another nonlinear expectation with respect to non-additive measures, called Sugeno integral in literature. For more results

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about Sugeno integral, including connections between Choquet integral and Sugeno integral, one may refer to [19].

There is also another way to deal with uncertain phenomena, i.e., set-valued random variables (also called random sets, multifunctions, correspondences in literature) and the Aumann integral (cf. [4]). The start of the theory of random sets may be when Aumann used it to discuss the competitive equilibria problem [5]. The theory of set-valued random variables and set-valued stochastic processes and its applications were developed very deeply and extensively in the past 40 years. For instance, see [3, 7, 23, 24, 28, 30, 44, 48].

It is necessary to investigate further the connections between the theory of set-valued random variables and Choquet theory. In 1967, Dempster [13] introduced the concepts of upper and lower probabilities induced by a random set. Both of upper and lower probabilities are special capacities with good properties. The lower probability was called belief function by Nguyen in [33, 34]. Actually the lower and upper probabilities can be considered as lower and upper distributions of the random set. And also lower and upper distributions can be axiomatized as that we have done in classical probabilities. On the other hand, if given a capacity satisfying the axioms of lower or upper distribution, we can find a set-valued random variable such that its distribution is just equal to the given capacity. This result is called Choquet Theorem. It is one of bridges between the theory of random sets and the theory of Choquet theory (see Section 3, for details).

In this paper we focus on surveying some results of the relationships between the distributions of set-valued random variables and capacities, and also some connections between the Aumann integral and the Choquet integral. More interpretation of our motivation about why should we do such work can be seen at the beginning of Section 3.

The organization of the paper is as follows. In Section 2, we shall recall some basic concepts and results of capacities and the Choquet integral. In Section 3, we shall give definitions about set-valued random variables and the Aumann integral, discuss the relationships between totally monotone capacities and random sets, and then survey some connections between the Aumann integral and the Choquet integral.

On the other hand, as we know, laws of large numbers are the foundation for statistical inferences. In Section 4, we shall review some literature about laws of large numbers for random sets and also for capacities, and we shall point out their some connection. Finally we give a new strong law of large numbers of exchangeable random variable for capacities.

2 Preliminaries for Capacities and Choquet Integral

Assume that $(\mathfrak{X}, d)$ is a Polish space, $\mathscr{B}$ is its Borel σ-algebra and $\mathbf{P}$ is the set of all probabilities on $\mathscr{B}$, $\mathbb{R}$ is the set of all natural numbers.

Definition 2.1. A set function $\nu : \mathscr{B} \to [0,1]$ is called a (Choquet) capacity if it satisfies the following two conditions

(c1) $\nu(\emptyset) = 0$, $\nu(\mathfrak{X}) = 1$;
(c2) $\nu(A) \leq \nu(B)$ whenever $A \subseteq B$ and $A, B \in \mathscr{B}$.
The conjugate $\overline{\nu} : \mathscr{B} \to [0,1]$ of ν is defined by $\overline{\nu}(A) = 1 - \nu(A^c)$.

A capacity ν is convex if $\nu(A \bigcup B) \geq \nu(A) + \nu(B) - \nu(A \bigcap B)$ for all $A, B \in \mathscr{B}$. A capacity ν on $\mathscr{B}$ is totally monotone if for any $n \geq 2$, and any $\{A_1, \cdots, A_n\} \subseteq \mathscr{B}$,

$$\nu\Big(\bigcup_{i=1}^{n} A_i\Big) \geq \sum_{\emptyset \neq I \subseteq \{1,\cdots,n\}} (-1)^{|I|+1} \nu\Big(\bigcap_{i \in I} A_i\Big), \tag{1}$$

where $|I|$ is the cardinality of the set I. Obviously, a totally monotone capacity is convex.

A capacity ν on $\mathscr{B}$ is infinitely alternating if for any $n \geq 2$, and any $\{A_1, \cdots, A_n\} \subseteq \mathscr{B}$,

$$\nu\Big(\bigcap_{i=1}^{n} A_i\Big) \leq \sum_{\emptyset \neq I \subseteq \{1,\cdots,n\}} (-1)^{|I|+1} \nu\Big(\bigcup_{i \in I} A_i\Big), \tag{2}$$

It is easy to show that a capacity ν is infinitely alternating if and only if its conjugate $\overline{\nu}$ is totally monotone. Any probability on $\mathscr{B}$ are both totally monotone and infinitely alternating.

A capacity ν on $\mathscr{B}$ is continuous from below if $\nu(B_n) \uparrow \nu(B)$ for all sequences $B_n \in \mathscr{B}$, $B_n \uparrow B$. ν is continuous from above if $\nu(B_n) \downarrow \nu(B)$ for all sequences $B_n \in \mathscr{B}$, $B_n \downarrow B$. A capacity with both below and above continuous is called continuous.

A capacity ν is a mass if $\nu(A \bigcup B) = \nu(A) + \nu(B)$ for any $A, B \in \mathscr{B}$ with $A \bigcap B = \emptyset$.

A capacity ν is null-additive if $\nu(A \bigcup B) = \nu(A)$ for any $A, B \in \mathscr{B}$ such that $A \bigcap B = \emptyset$ and $\nu(B) = 0$. Notice that a convex capacity is null-additive if and only if $\nu(A) = 0$ implies $\nu(A^c) = 1$ for every $A \in \mathscr{B}$.

The core $C(\nu)$ of the capacity ν is defined as

$$C(\nu) = \{\mu \in \mathbf{P} : \mu(A) \geq \nu(A) \text{ for all } A \in \mathscr{B}\};$$

and the anti-core $AC(\nu)$ of ν is given by

$$AC(\nu) = \{\mu \in \mathbf{P} : \mu(A) \leq \nu(A) \text{ for all } A \in \mathscr{B}\}.$$

We have the following properties:

1) If ν is a convex capacity, we always have $C(\nu) \neq \emptyset$;
2) $AC(\overline{\nu}) = C(\nu)$.

In some literature (e.g. [36]), a capacity ν is called a balanced game if $C(\nu) \neq \emptyset$.

A random variable X on Ω is a (Borel) measurable function $X : (\Omega, \mathscr{B}) \to (\mathbb{R}, \mathscr{B}(\mathbb{R}))$, where $\mathscr{B}(\mathbb{R})$ is the Borel σ-field of $\mathbb{R}$.

The Choquet integral of a bounded random variable X with respect to the capacity ν is defined by

$$(C)\int X d\nu = \int_0^{+\infty} \nu(X > t) dt + \int_{-\infty}^{0} [\nu(X > t) - 1] dt.$$

If ν is a probability measure, $(C)\int X d\nu$ coincides with standard notion of integral. Notice that the Choquet integral is an asymmetric integral since the integral $(C)\int X d\nu$ is not equal to $-(C)\int -X d\nu$, so they are called lower and upper Choquet integrals respectively. In general, $(C)\int X d\nu \leq -(C)\int -X d\nu$.

If $C(\nu) \neq \emptyset$, we can introduce the upper and lower integrals of a random variable X given by

$$J_\nu(X) = \sup_{P \in C(\nu)} \int X dP, \quad I_\nu(X) = \inf_{P \in C(\nu)} \int X dP.$$

Then we have

$$(C)\int X d\nu \leq I_\nu(X) \leq J_\nu(X) \leq (C)\int X d\overline{\nu},$$

since $J_\nu(X) = -I_\nu(-X)$ and $(C)\int X d\overline{\nu} = -(C)\int -X d\nu$.

Sugeno introduced the concepts of fuzzy measure and fuzzy integral in [41]. Concerning the relationship between the Sugeno fuzzy integral and the Choquet integral, refer to [19]. Fuzzy measures and Choquet capacities are also called non-additive measures. For more general concepts and results, readers may refer to [14].

3 Some Connections between Theory of Set-Valued Random Variables and Choquet Theory

In this section, we shall discuss the relationships between the distributions of set-valued random variables and capacities. We shall also survey some connections between the Aumann integral and the Choquet integral. Assume that $(\Omega, \mathscr{A}, P)$ is a complete probability space. Firstly let us explain our motivation.

In classical statistics, all possible outcomes of a random experiment can be described by some random variable X or its probability distribution P_X. In practice, however, we often face the situation that we can not measure exactly the values of X, we can only get coarse data, that is, a multi-valued random variable F (we call it random set or set-valued random variable) such that $P(X \in F) = 1$ (X is an almost surely selection of F).

Let $A \subseteq \Omega$ be an event. A is said to occur if $X(\omega) \in A$. But if we can not observe $X(\omega)$, but only $F(\omega)$ is observed, then clearly we are even uncertainty about the occurrence of A. If $F(\omega) \subseteq A$, then clearly A occurs. So we quantify our *degree of belief* in occurrence of A by $\underline{P}_F(A) = P(F \subseteq A)$, which is less than the actual probability that A occurs, i.e. $\underline{P}_F(A) \leq P(X \in A)$, since X is an almost sure selection of F. This fact is a starting point of well-known Dempster-Shafer theory of evidence (cf. [13], [39]). $\underline{P}_F$ is also related with another concept called *a belief function* (cf. [33]), which is popular in the field of artificial intelligence.

On the other hand, if $F(\omega) \cap A \neq \emptyset$, then it is possible that A occurs. Since $P(X \in F) = 1$, we have almost sure $\{X \in A\} \subseteq \{F \cap A \neq \emptyset\}$ and hence $P(X \in A) \leq P(F \cap A \neq \emptyset)$. Thus, to quantify this possibility is to take $\overline{P}_F(A) = P(F \cap A \neq \emptyset)$. It seems to be consistent with the common sense that the possibilities are always larger than the probabilities since the possibilities tend to represent the optimistic assessments as opposed to beliefs.

From mathematical point of view, it relates to the lower distribution $\underline{P}_F$ and upper distribution $\overline{P}_F$ of the set-valued random variable F. For each $B \in \mathscr{B}$, we have $\overline{P}_F(B) = 1 - \underline{P}_F(B^c)$. Thus we only need to consider one of both upper and lower distributions. We notice that $\underline{P}_F$ is a special totally monotone capacity and $\overline{P}_F$ is a special infinitely alternating capacity. Thus, the Choquet integral with respect to $\underline{P}_F$ and $\overline{P}_F$ have some connection with the theory of set-valued random variables. Now we discuss this problem in details. We first review some notations and basic results about set-valued random variables and the Aumann integral.

3.1 Set-Valued Random Variables and the Aumann Integral

Assume that $\mathscr{P}_0(\mathfrak{X})$ is the family of all nonempty subsets of $\mathfrak{X}$, $\mathscr{G}(\mathfrak{X})$ is the class of all open sets of $\mathfrak{X}$, $\mathbf{K}(\mathfrak{X})$ (*reps.,* $\mathbf{K}_b(\mathfrak{X}), \mathbf{K}_k(\mathfrak{X})$, $\mathbf{K}_{kc}(\mathfrak{X})$) is the family of all nonempty closed (*reps.,* bounded closed, compact, compact convex) subsets of $\mathfrak{X}$, The Hausdorff metric on $\mathbf{K}(\mathfrak{X})$ is defined by

$$d_H(A,B) = \max\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\}. \tag{3}$$

The metric space $(\mathbf{K}_b(\mathfrak{X}), d_H)$ is complete but not separable in general. However, $\mathbf{K}_k(\mathfrak{X})$ and $\mathbf{K}_{kc}(\mathfrak{X})$ are complete and separable with respect to d_H (cf. [30]). For an A in $\mathbf{K}_b(\mathfrak{X})$, let $\|A\|_{\mathbf{K}} = d_H(\{0\},A)$, $\mathscr{B}_{d_H}(\mathbf{K}_k(\mathfrak{X}))$ be the Borel σ-field of $(\mathbf{K}_k(\mathfrak{X}), d_H)$ and similar notation for $\mathscr{B}_{d_H}(\mathbf{K}_b(\mathfrak{X}))$, and so on.

On the other hand, let $\mathscr{F}_L = \{I_*(G) \cap \mathbf{K}_k(\mathfrak{X}) : G \in \mathscr{G}(\mathfrak{X})\}$ and $\mathscr{F}_U = \{I^*(G) \cap \mathbf{K}_k(\mathfrak{X}) : G \in \mathscr{G}(\mathfrak{X})\}$, where $I_*(G) = \{A \in \mathscr{P}_0(\mathfrak{X}) : A \subseteq G\}$, $I^*(G) = \{A \in \mathscr{P}_0(\mathfrak{X}) : A \cap G \neq \emptyset\}$. And let $\sigma(\mathscr{F}_L), \sigma(\mathscr{F}_U)$ be the σ-fields induced by $\mathscr{F}_L, \mathscr{F}_U$ respectively. We have the following result (cf. [48]).

Theorem 3.1. $\mathscr{B}_{d_H}(\mathbf{K}_k(\mathfrak{X})) = \sigma(\mathscr{F}_L) = \sigma(\mathscr{F}_U)$.

Let $F : \Omega \to \mathbf{K}(\mathfrak{X})$. For any $A \in \mathscr{B}$, write

$$F^{-1}(A) = \{\omega \in \Omega : F(\omega) \cap A \neq \emptyset\},$$

$$F_{-1}(A) = \{\omega \in \Omega : F(\omega) \subseteq A\},$$

and the graph of F

$$G(F) = \{(\omega, x) \in \Omega \times \mathfrak{X} : x \in F(\omega)\}.$$

A set-valued mapping $F : \Omega \to \mathbf{K}(\mathfrak{X})$ is called a set-valued random variable (or random set) if, for each open subset O of $\mathfrak{X}$, $F^{-1}(O) \in \mathscr{A}$. In [30], authors summarized the following equivalent definitions of random sets.

Theorem 3.2. *The following statements are equivalent:*

(i) *F is a set-valued random variable;*
(ii) *for each $C \in \mathbf{K}(\mathfrak{X})$, $F^{-1}(C) \in \mathscr{A}$;*
(iii) *for each $B \in \mathscr{B}$, $F^{-1}(B) \in \mathscr{A}$;*

(iv) *$\omega \mapsto d(x, F(\omega))$ is a measurable function for each $x \in \mathfrak{X}$, where $d(x,C) = \inf\{d(x,y) : y \in C\}$ for $C \subseteq \mathfrak{X}$;*

(v) *$G(F)$ is $\mathscr{A} \times \mathscr{B}$-measurable.*

Furthermore, if F takes values in $\mathbf{K}_k(\mathfrak{X})$, then F is a set-valued random variable if and only if F is $\mathscr{A}$-$\mathscr{B}_{d_H}(\mathbf{K}_k(\mathfrak{X}))$ measurable.

Now we give the concepts of selections of set-valued random variables.

Definition 3.1. An $\mathfrak{X}$-valued measurable function $f : \Omega \to \mathfrak{X}$ is called a *selection* of a set-valued mapping $F : \Omega \to \mathbf{K}(\mathfrak{X})$ if $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$. A measurable function $f : \Omega \to \mathfrak{X}$ is called an *almost surely selection* of F if $P\{\omega \in \Omega : f(\omega) \in F(\omega)\} = 1$.

Let $L^1[\Omega; \mathfrak{X}]$ be the class of integrable $\mathfrak{X}$-valued random variables, $S(F)$ be the class of all selections of F and S_F^1 the class of almost surely and integrable selections of F, i.e.

$$S_F^1 = \{f \in L^1[\Omega; \mathfrak{X}] : f(\omega) \in F(\omega), a.e.\}.$$

Then we have the following result.

Theorem 3.3. *Under our assumptions in this paper, $S(F) \neq \emptyset$ for any set-valued random variable; $S_F^1 \neq \emptyset$ if and only if $d(0, F(\omega)) \in L^1[\Omega; [0,\infty)]$.*

For a set-valued random variable F, the expectation of F, denoted by $(A)\int F dP$, is defined by

$$(A)\int F dP = \Big\{ \int_\Omega f dP : f \in S_F^1 \Big\}, \tag{4}$$

This integral was first introduced by Aumann [4], called the Aumann integral in literature.

Remark 1. (1) In general, $(A)\int F dP$ is not closed when F takes closed set values. But if $\mathfrak{X} = \mathbb{R}^d$, the d-dimensional Euclidean space and F takes compact set values, $(A)\int F dP$ is compact.

(2) If P is nonatomic, then $\mathrm{cl}((A)\int F dP)$ is convex.

The above results can be found in [23, 30].

3.2 Capacities, Upper and Lower Distributions of Set-Valued Random Variables

In the classical probability, an $\mathfrak{X}$-valued random variable (or $\mathfrak{X}$-valued element) $f : \Omega \to \mathfrak{X}$ induces a probability distribution P_f on $\mathscr{B}$ defined by

$$P_f(B) = P(f^{-1}(B)), \quad B \in \mathscr{B}.$$

In a similar way, for a random set F, we have the concepts of upper distribution $\overline{P}_F$ and lower distribution $\underline{P}_F$, defined as

$$\overline{P}_F(B) = P(F^{-1}(B)), \quad \underline{P}_F(B) = P(F_{-1}(B)), \quad B \in \mathscr{B}. \tag{5}$$

In the special case of a random variable, i.e. $F = f : \Omega \to \mathfrak{X}$, we have that $\overline{P}_F(B) = \underline{P}_F(B)$ for each $B \in \mathscr{B}$. Thus, $\underline{P}_F$ reduces to the standard probability distribution P_f. Dempster called $\overline{P}_F, \underline{P}_F$ *upper probability, lower probability* respectively in [13]. Nguyen called $\overline{P}_F$ the distribution function of F in [34].

Obviously, $\overline{P}_F$ and $\underline{P}_F$, in general, are non-additive, they are capacities, and $\overline{P}_F$ is the conjugate of $\underline{P}_F$. Thus we only need to state the properties of $\underline{P}_F$. We have the following theorems (cf. [7, 8, 34]).

Theorem 3.4. $\underline{P}_F$ *have the following properties*

(i) $\underline{P}_F(\emptyset) = 0, \quad \underline{P}_F(\mathfrak{X}) = 1$;
(ii) If $B_n \downarrow B$ *with* $B_n, B \in \mathscr{B}$, *then* $\underline{P}_F(B_n) \downarrow \underline{P}_F(B)$;
(iii) $\underline{P}_F$ *is totally monotone.*
If, in addition, F *takes values in* $\mathbf{K}_k(\mathfrak{X})$, *then*
(iv) $\underline{P}_F$ *is regular, i.e.*

$$\underline{P}_F(B) = \sup\{\underline{P}_F(C) : C \subseteq B, C \in \mathbf{K}(\mathfrak{X})\}$$

$$= \inf\{\underline{P}_F(G) : B \subseteq G, G \in \mathscr{G}(\mathfrak{X})\}$$

for any $B \in \mathscr{B}$.
(v) $\underline{P}_F$ *is tight, i.e.*

$$\underline{P}_F(B) = \sup\{\underline{P}_F(K) : K \subseteq B, K \in \mathbf{K}_k(\mathfrak{X})\}$$

for any $B \in \mathscr{B}$.

From the above discussion, we know that for any given random set F, the lower distribution $\underline{P}_F$ induced by F is a totally monotone and continuous from above capacity. On the other hand, for any given totally monotone and continuous from above capacity ν on $(\mathfrak{X}, \mathscr{B})$, dose there exist a probability space $(\Omega, \mathscr{A}, P)$ and a set-valued random variable F on Ω such that $\nu = \underline{P}_F$? The answer is positive and it is called the Choquet Theorem.

Theorem 3.5. *If* ν *is a totally monotone and continuous from above capacity on* $\mathscr{B}$, *there exists a set-valued random variable* $F : [0,1] \to \mathbf{K}(\mathfrak{X})$ *such that* $\nu = \underline{P}_F$, *where* $[0,1]$ *is endowed with the Lebesgue* σ*-algebra and the Lebesgue measure.*

3.3 Some Connections between Aumann Integral and Choquet Integral

For any given set-valued random variable F, its selection set $S(F)$ is a family of $\mathfrak{X}$-valued random variables. For each $f \in S(F)$, we can get the probability distribution P_f. Thus, we obtain a set of probabilities $\mathbf{P}_F =: \{P_f : f \in S(F)\}$ and $\mathbf{P}_F \subseteq \mathbf{P}$.

Theorem 3.6. *[12] If* F *is* $\mathscr{A}$-$\mathscr{B}_{d_H}(\mathbf{K}(\mathfrak{X}))$*-measurable set-valued random variable, then* $\underline{P}_F$ *is attainable on* $\mathscr{G}(\mathfrak{X}) \cup \mathbf{K}(\mathfrak{X})$, *i.e.*

$$\underline{P}_F(A) = \min\{P_f(A) : f \in S(F)\}, \quad A \in \mathscr{G}(\mathfrak{X}) \cup \mathbf{K}(\mathfrak{X}). \tag{6}$$

Next we have the connection theorem between the selection set $\mathbf{P}_F$ and core of $\underline{P}_F$ (cf. [8]).

Theorem 3.7. *If F is a compact set-valued random variable, then*

$$C(\underline{P}_F) = \overline{\text{co}}(\mathbf{P}_F) \tag{7}$$

where $\overline{\text{co}}$ means the weak*-closed convex hull in $\mathbf{P}$.

Furthermore, if P is nonatomic, then

$$C(\underline{P}_F) = \text{cl}(\mathbf{P}_F) \tag{8}$$

Theorem 3.8. *Assume that $X : \mathfrak{X} \to \mathbb{R}$ is Borel measurable and bounded, $F : \Omega \to \mathbf{K}_k(\mathfrak{X})$ is a set-valued random variable, the composition $X \circ F$ is given by $(X \circ F) = X(F(\omega))$ for any $\omega \in \Omega$. Then*

$$(A)\int (X \circ F) dP = \Big\{ \int X dP_f : f \in S(F) \Big\}. \tag{9}$$

In particular,

$$\inf (A)\int (X \circ F) dP = (C)\int X d\underline{P}_F, \quad \sup (A)\int (X \circ F) dP = (C)\int X d\overline{P}_F, \tag{10}$$

and moreover the inf *(resp.,* sup*) is attained if X is lower (resp., upper) Weierstrass.*

Remark 2. (1) A Borel function $X : \mathfrak{X} \to \mathbb{R}$ is lower (*resp.*, upper) Weierstrass if it attains infimum (*resp.*, supermum) on each $K \in \mathbf{K}_k(\mathfrak{X})$. All simple Borel functions and all lower (*resp.*, upper) semicontinuous functions are lower (*resp.*, upper) Weierstrass.

(2) If P is nonatomic, from Remark 1 and the above theorem, we have

$$(A)\int (X \circ F) dP = \Big[(C)\int X d\underline{P}_F, (C)\int X d\overline{P}_F \Big], \tag{11}$$

when X is both lower and upper Weierstrass.

4 Laws of Large Numbers for Random Sets and for Capacities

In this Section, we shall firstly survey some results on laws of large numbers (LLN's) for set-valued random variables and for capacities, and point out some connections between these two kinds of LLN's. Then we shall give a new strong law of large numbers of exchangeable random variables for capacities.

There are many different kinds of LLN's for random sets. Here we only list some of them. The first LLN was proved in [3] for independent identically distributed (i.i.d.) compact random variables in the sense of Hausdorff metric d_H, where the basic space is the d-dimensional Euclidean space $\mathbb{R}^d$. After this work, LLN's were obtained for i.i.d. compact random sets in a separable Banach space in [26, 35]. Taylor and his coauthors contributed a lot in the area of LLN's. We mention here that in 1985, Taylor and Inoue proved Chung's type LLN's and weighted sums type LLN's for compact set-valued random sets in [42, 43]. For more results, refer to their summary paper in [44].

For general closed set-valued random variables, Artstein and Hart [2] proved LLN's in $\mathbb{R}^d$ and Hiai obtained LLN's in a separable Banach space in Kuratowski-Mosco sense. In some papers, Kuratowski-Mosco convergence is called Painlevé-Kuratowsk convergence in the special case of $\mathbb{R}^d$. Fu and Zhang [17] obtained LLN's for set-valued random variables with slowly varying weights in the sense of d_H. There are also some extension results of LLN's from set-valued to fuzzy set-valued random variables [11, 17, 20, 21, 25, 27, 29].

Now we cite some results of LLN's for real-valued random variables $X : \mathfrak{X} \to \mathbb{R}$ with respect to capacities. In [32], Marinacci proved a strong LLN for i.i.d. continuous random sequences with respect to a totally monotone and continuous capacity, and a weak LLN but with respect to a convex and continuous capacity under the assumption that $\mathfrak{X}$ is a compact space. In his proofs, he mainly used some very good properties and techniques of capacities and the Choquet integral. In [31], Maccheroni and Marinacci obtained a strong LLN for under weaker conditions in a separable Banach space $\mathfrak{X}$. The proof is quite short by using the Choquet Theorem and the result of strong LLN for set-valued random variables. In [36], Rebille obtained a Markov type LLN and a Bienayme-Tchebichev type LLN for a balanced game under some other conditions of variances, where he used the core of ν to define variance and covariance of random variables.

Now we state a new strong LLN of exchangeable random variables for capacities. To do it, we firstly introduce the concept of exchangeable random variables.

Definition 4.1. Random variables $X_i : \mathfrak{X} \to \mathbb{R}$, $i = 1, 2, \cdots, n$, are called exchangeable with respect to a capacity ν if $(X_{\pi_1}, \cdots, X_{\pi_n})$ has the same joint distribution as $(X_1, \cdots, X_n)$ for every permutation $\pi = (\pi_1, \cdots, \pi_n)$ of $(1, \cdots, n)$, i.e.,

$$\nu(X_1 \in B_1, \cdots, X_n \in B_n) = \nu(X_{\pi_1} \in B_1, \cdots, X_{\pi_n} \in B_n)$$

for any $B_1, \cdots, B_n \subseteq \mathscr{B}$. An infinite sequence of random variables $\{X_n : n \geq 1\}$ is said to be exchangeable if every finite subset of $\{X_n : n \geq 1\}$ consists of exchangeable random variables.

Theorem 4.1. *Assume that ν is a totally monotone and continuous capacity on $\mathscr{B}$, and $\{X_n : n \geq 1\}$ a sequence of bounded, exchangeable and identically distributed random variables, it is parwise incorrected, and for each random variable X_i is either continuous or simple, then*

$$\nu\Big(\Big\{\omega \in \Omega : \mathbf{E}[X_1] \leq \liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} X_j(\omega) \leq \limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} X_j(\omega) \leq -\mathbf{E}[-X_1]\Big\}\Big) = 1,$$

where $\mathbf{E}[X_1] = (C)\int X_1 d\nu$.

Remark 3. (1) If ν is null-additive, under the assumptions of theorem we also have

$$\nu\Big(\Big\{\omega \in \Omega : \liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} X_j(\omega) < \mathbf{E}[X_1]\Big\}\Big) = 0,$$

and

$$\nu\Big(\Big\{\omega \in \Omega : \limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} X_j(\omega) > -\mathbf{E}[-X_1]\Big\}\Big) = 0$$

(2) When ν is a probability measure we have $\mathbf{E}[X] = -\mathbf{E}[-X] = E[X]$. Thus, in this case our result reduces to the classical LLN for exchangeable real-valued random variables

$$\nu\Big(\Big\{\omega \in \Omega : \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} X_j(\omega) = E[X]\Big\}\Big) = 1.$$

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