A Study of Riesz Space-Valued Non-additive Measures

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Abstract. This paper gives a short survey of our recent developments in Riesz space-valued non-additive measure theory and contains the following topics: the Egoroff theorem, the Lebesgue theorem, the Riesz theorem, the Lusin theorem, and the Alexandroff theorem.

1 Introduction

In 1974, Sugeno [30] introduced the notion of fuzzy measure and integral to evaluate non-additive or non-linear quality in systems engineering. In the same year, Dobrakov [3] independently introduced the notion of submeasure from mathematical point of view to refine measure theory further. Fuzzy measures and submeasures are both special kinds of non-additive measures, and their studies have stimulated engineers' and mathematicians' interest in non-additive measure theory [2, 25, 33].

The study of non-additive measures deeply depends on the order in the range space in which the measures take values. In fact, a non-additive measure is defined as a *monotone* set function which vanishes at the empty set, and not a few features of non-additive measures, such as the order continuity and the continuity from above and below, concern the order on the range space. The Riesz space is a real vector space with partial ordering compatible with the structure of the vector space, and at the same time, it is a lattice. Therefore, it is a natural attempt to discuss the existing theory of real-valued non-additive measures in a Riesz space. Typical examples of Riesz spaces are the n-dimensional Euclidean space $\mathbb{R}^n$, the functions space $\mathbb{R}^\Lambda$ with non-empty set Λ, the Lebesgue functions spaces $\mathscr{L}_p[0,1]$ $(0 \leq p \leq \infty)$, and their ideals.

When we try to develop non-additive measure theory in a Riesz space, along with the non-additivity of measures, there is a tough technical hurdle to overcome, that

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V.-N. Huynh et al. (Eds.): Integrated Uncertainty Management and Applications, AISC 68, pp. 91–102.
springerlink.com

is, the ε-argument, which is useful in calculus, does not work in a general Riesz space. Recently, it has been recognized that, as a substitute for the ε-argument, certain smoothness conditions, such as the weak σ-distributivity, the Egoroff property, the weak asymptotic Egoroff property, and the multiple Egoroff property, should be imposed on a Riesz space to succeed in extending fundamental and important theorems in non-additive measure theory to Riesz space-valued measures. Thus, the study of Riesz space-valued measures will go with some smoothness conditions on the involved Riesz space.

This paper gives a short survey of our recent developments in Riesz space-valued non-additive measure theory and contains the following topics: the Egoroff theorem, the Lebesgue theorem, the Riesz theorem, the Lusin theorem, and the Alexandroff theorem. All the results in this paper, together with their proofs and the related problems, have been already appeared in [8, 9, 10, 11, 12, 13, 14, 15, 16], so that herein there are no new contributions to Riesz space-valued non-additive measure theory. The interested readers may obtain more information on the above topics and their related problems, such as Riesz space-valued Choquet integration theory, from the cited literatures in the reference of this paper. See [28] for some other ordering structures on Riesz spaces and lattice ordered groups, and their relation to measure and integration theory.

2 Notation and Preliminaries

In this section, we recall some basic definitions on Riesz spaces and Riesz space-valued non-additive measures. Denote by $\mathbb{R}$ and $\mathbb{N}$ the set of all real numbers and the set of all natural numbers, respectively.

2.1 Riesz Space

The real vector space V is called an *ordered vector space* if V is partially ordered in such a manner that the partial ordering is compatible with the vector structure of V, that is, (i) $u \le v$ implies $u+w \le v+w$ for every $w \in V$, and (ii) $u \ge 0$ implies $cu \ge 0$ for every $c \in \mathbb{R}$ with $c \ge 0$. The ordered vector space V is called a *Riesz space* if for every pair u and v in V, the supremum $\sup(u,v)$ and the infimum $\inf(u,v)$ with respect to the partial ordering exist in V.

Let V be a Riesz space. Denote by V^+ the set of all positive elements of V. Let $D := \{u_t\}_{t\in T}$ be a set of elements of V and $u \in V$. We write $\sup D = u$ or $\sup_{t\in T} u_t = u$ to mean that there exists a supremum of D and equal to u. The meaning of $\inf D = u$ or $\inf_{t\in T} u_t = u$ is analogous. We say that V is *Dedekind complete* (respectively, *Dedekind σ-complete*) if every non-empty (respectively, countable, non-empty) subset of V which is bounded from above has a supremum.

Let $\{u_n\}_{n\in\mathbb{N}} \subset V$ be a sequence and $u \in V$. We write $u_n \downarrow u$ to mean that it is decreasing and $\inf_{n\in\mathbb{N}} u_n = u$. The meaning of $u_n \uparrow u$ is analogous. We say that $\{u_n\}_{n\in\mathbb{N}}$ *converges in order* to u and write $u_n \to u$ if there is a sequence $\{p_n\}_{n\in\mathbb{N}} \subset V$ with $p_n \downarrow 0$ such that $|u_n - u| \le p_n$ for all $n \in \mathbb{N}$. The order convergence can be defined for nets $\{u_\alpha\}_{\alpha\in\Gamma}$ of elements of V in an obvious way. A Riesz space V is said to be *order separable* if every set in V possessing a supremum contains an at most countable subset having the same supremum.

The following smoothness conditions on a Riesz space have been already introduced in [23] and [34]. Denote by Θ the set of all mappings from $\mathbb{N}$ into $\mathbb{N}$, which is ordered and directed upwards by pointwise partial ordering, that is, $\theta_1 \leq \theta_2$ is defined as $\theta_1(i) \leq \theta_2(i)$ for all $i \in \mathbb{N}$.

Definition 2.1. Let V be a Riesz space.

(i) A double sequence $\{u_{i,j}\}_{(i,j)\in\mathbb{N}^2} \subset V$ is called a *regulator* in V if it is order bounded, and $u_{i,j} \downarrow 0$ for each $i \in \mathbb{N}$, that is, $u_{i,j} \geq u_{i,j+1}$ for each $i, j \in \mathbb{N}$ and $\inf_{j\in\mathbb{N}} u_{i,j} = 0$ for each $i \in \mathbb{N}$.
(ii) We say that V has *the Egoroff property* if, for any regulator $\{u_{i,j}\}_{(i,j)\in\mathbb{N}^2}$ in V, there is a sequence $\{p_k\}_{k\in\mathbb{N}} \subset V$ with $p_k \downarrow 0$ such that, for each $(k,i) \in \mathbb{N}^2$, one can find $j(k,i) \in \mathbb{N}$ satisfying $u_{i,j(k,i)} \leq p_k$ [23].
(iii) Let V be Dedekind σ-complete. We say that V is *weakly σ-distributive* if, for any regulator $\{u_{i,j}\}_{(i,j)\in\mathbb{N}^2}$ in V, it holds that $\inf_{\theta\in\Theta} \sup_{i\in\mathbb{N}} u_{i,\theta(i)} = 0$ [34].

See [23] for unexplained terminology and more information on Riesz spaces.

2.2 *Riesz Space-Valued Non-additive Measures*

Throughout the paper, we assume that V is a Riesz space and $(X, \mathscr{F})$ is a measurable space, that is, $\mathscr{F}$ is a σ-field of subsets of a non-empty set X.

Definition 2.2. A set function $\mu : \mathscr{F} \to V$ is called a *non-additive measure* if $\mu(\emptyset) = 0$ and $\mu(A) \leq \mu(B)$ whenever $A, B \in \mathscr{F}$ and $A \subset B$.

We collect some *continuity* conditions of non-additive measures.

Definition 2.3. Let $\mu : \mathscr{F} \to V$ be a non-additive measure.

(i) μ is said to be *continuous from above* if $\mu(A_n) \downarrow \mu(A)$ whenever $\{A_n\}_{n\in\mathbb{N}} \subset \mathscr{F}$ and $A \in \mathscr{F}$ satisfy $A_n \downarrow A$.
(ii) μ is said to be *continuous from below* if $\mu(A_n) \uparrow \mu(A)$ whenever $\{A_n\}_{n\in\mathbb{N}} \subset \mathscr{F}$ and $A \in \mathscr{F}$ satisfy $A_n \uparrow A$.
(iii) μ is said to be *continuous* if it is continuous from above and below.
(iv) μ is said to be *strongly order continuous* if it is continuous from above at measurable sets of measure zero, that is, $\mu(A_n) \downarrow 0$ whenever $\{A_n\}_{n\in\mathbb{N}} \subset \mathscr{F}$ and $A \in \mathscr{F}$ satisfy $A_n \downarrow A$ and $\mu(A) = 0$ [17].
(v) μ is said to be *order continuous* if it is continuous from above at the empty set, that is, $\mu(A_n) \downarrow 0$ whenever $\{A_n\}_{n\in\mathbb{N}} \subset \mathscr{F}$ satisfies $A_n \downarrow \emptyset$.
(vi) μ is said to be *strongly order totally continuous* if $\inf_{\alpha\in\Gamma} \mu(A_\alpha) = 0$ whenever a net $\{A_\alpha\}_{\alpha\in\Gamma} \subset \mathscr{F}$ and $A \in \mathscr{F}$ satisfy $A_\alpha \downarrow A$ and $\mu(A) = 0$ [24].

The following are some *quasi-additivity* conditions of non-additive measures.

Definition 2.4. Let $\mu : \mathscr{F} \to V$ be a non-additive measure.

(i) μ is said to be *subadditive* if $\mu(A \cup B) \le \mu(A) + \mu(B)$ for all $A, B \in \mathscr{F}$.
(ii) μ is said to be *null-additive* if $\mu(A \cup B) = \mu(A)$ whenever $A, B \in \mathscr{F}$ and $\mu(B) = 0$.
(iii) μ is said to be *weakly null-additive* if $\mu(A \cup B) = 0$ whenever $A, B \in \mathscr{F}$ and $\mu(A) = \mu(B) = 0$.
(iv) μ is said to be *autocontinuous from above* if $\mu(A \cup B_n) \to \mu(A)$ whenever $A \in \mathscr{F}$, and $\{B_n\}_{n \in \mathbb{N}} \subset \mathscr{F}$ is a sequence with $\mu(B_n) \to 0$.
(v) μ is said to be *autocontinuous from below* if $\mu(A \setminus B_n) \to \mu(A)$ whenever $A \in \mathscr{F}$, and $\{B_n\}_{n \in \mathbb{N}} \subset \mathscr{F}$ is a sequence with $\mu(B_n) \to 0$.
(vi) μ is said to be *autocontinuous* if it is autocontinuous from above and below.
(vii) μ is said to be *uniformly autocontinuous from above* if, for any sequence $\{B_n\}_{n \in \mathbb{N}} \subset \mathscr{F}$ with $\mu(B_n) \to 0$, there is a sequence $\{p_n\}_{n \in \mathbb{N}} \subset V$ with $p_n \downarrow 0$ such that $\mu(A \cup B_n) \le \mu(A) + p_n$ for all $A \in \mathscr{F}$ and $n \in \mathbb{N}$.
(viii) μ is said to be *uniformly autocontinuous from below* if, for any sequence $\{B_n\}_{n \in \mathbb{N}} \subset \mathscr{F}$ with $\mu(B_n) \to 0$, there is a sequence $\{p_n\}_{n \in \mathbb{N}} \subset V$ with $p_n \downarrow 0$ such that $\mu(A) \le \mu(A \setminus B_n) + p_n$ for all $A \in \mathscr{F}$ and $n \in \mathbb{N}$.
(ix) μ is said to be *uniformly autocontinuous* if it is uniformly autocontinuous from above and below.

3 The Egoroff Theorem

The classical theorem of Egoroff [4] is one of the most fundamental and important theorems in measure theory. This asserts that almost everywhere convergence implies almost uniform convergence (and hence convergence in measure) and gives a key to handle a sequence of measurable functions. However, it is known that the Egoroff theorem does not valid in general for non-additive measures.

Recently, Murofushi *et al.* [24] discovered a necessary and sufficient condition, called the Egoroff condition, which assures that the Egoroff theorem is still valid for non-additive measures, and indicated that the continuity of a non-additive measure is one of the sufficient conditions for the Egoroff condition; see also [18, 19, 21, 22]. Those conditions can be naturally described for Riesz space-valued non-additive measures.

Definition 3.1. Let $\mu : \mathscr{F} \to V$ be a non-additive measure.

(i) A double sequence $\{A_{m,n}\}_{(m,n) \in \mathbb{N}^2} \subset \mathscr{F}$ is called a *μ-regulator* in $\mathscr{F}$ if it satisfies the following two conditions:
  (i) $A_{m,n} \supset A_{m,n'}$ whenever $m, n, n' \in \mathbb{N}$ and $n \le n'$.
  (ii) $\mu\left(\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} A_{m,n}\right) = 0$.
(ii) We say that μ satisfies *the Egoroff condition* if $\inf_{\theta \in \Theta} \mu\left(\bigcup_{m=1}^{\infty} A_{m,\theta(m)}\right) = 0$ for any μ-regulator $\{A_{m,n}\}_{(m,n) \in \mathbb{N}^2}$ in $\mathscr{F}$.

Definition 3.2. Let $\mu : \mathscr{F} \to V$ be a non-additive measure. Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of $\mathscr{F}$-measurable, real-valued functions on X and f also such a function.

(i) $\{f_n\}_{n\in\mathbb{N}}$ is said to converge μ*-almost everywhere* to f if there is a set $E \in \mathscr{F}$ with $\mu(E) = 0$ such that $f_n(x)$ converges to $f(x)$ for all $x \in X - E$.
(ii) $\{f_n\}_{n\in\mathbb{N}}$ is said to converge μ*-almost uniformly* to f if there is a decreasing net $\{E_\alpha\}_{\alpha\in\Gamma} \subset \mathscr{F}$ with $\mu(E_\alpha) \downarrow 0$ such that f_n converges to f uniformly on each set $X - E_\alpha$.
(iii) $\{f_n\}_{n\in\mathbb{N}}$ is said to converge *in* μ*-measure* to f if, for any $\varepsilon > 0$, there is a sequence $\{p_n\}_{n\in\mathbb{N}} \subset V$ with $p_n \downarrow 0$ such that $\mu(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}) \leq p_n$ for all $n \in \mathbb{N}$.
(iv) We say that *the Egoroff theorem holds for* μ if, for any sequence $\{f_n\}_{n\in\mathbb{N}}$ of $\mathscr{F}$-measurable, real-valued functions on X converging μ-almost everywhere to such a function f on X, it converges μ-almost uniformly to the same limit f.

The following theorem gives a Riesz space version of [24, Proposition 1].

Theorem 3.1. *Let* $\mu : \mathscr{F} \to V$ *be a non-additive measure. Then,* μ *satisfies the Egoroff condition if and only if the Egoroff theorem holds for* μ.

Li [21, Theorem 1] proved that the Egoroff theorem holds for any continuous real-valued non-additive measure. Its proof is essentially based on the ε-argument which does not work in a general Riesz space. Therefore, it seems that, as a substitute for the ε-argument, some smoothness conditions should be introduced and imposed on a Riesz space to obtain successful analogues of the Egoroff theorem for Riesz space-valued non-additive measures. The following is one of our new smoothness conditions on a Riesz space by which we will develop Riesz space-valued non-additive measure theory.

Definition 3.3. Consider a multiple sequence $u^{(m)} := \{u_{n_1,\ldots,n_m}\}_{(n_1,\ldots,n_m)\in\mathbb{N}^m}$ of elements of V for each $m \in \mathbb{N}$. Let $u \in V^+$.

(i) A sequence $\{u^{(m)}\}_{m\in\mathbb{N}}$ of the multiple sequences is called a *u-multiple regulator* in V if, for each $m \in \mathbb{N}$ and $(n_1, \ldots, n_m) \in \mathbb{N}^m$, the multiple sequence $u^{(m)}$ satisfies the following two conditions:

  (i) $0 \leq u_{n_1} \leq u_{n_1,n_2} \leq \cdots \leq u_{n_1,\ldots,n_m} \leq u$.
  (ii) Letting $n \to \infty$, then $u_n \downarrow 0$, $u_{n_1,n} \downarrow u_{n_1}, \ldots,$ and $u_{n_1,\ldots,n_m,n} \downarrow u_{n_1,\ldots,n_m}$.

(ii) A u-multiple regulator $\{u^{(m)}\}_{m\in\mathbb{N}}$ in V is said to be *strict* if, for each $m \in \mathbb{N}$ and each $(n_1, \ldots, n_m), (n'_1, \ldots, n'_m) \in \mathbb{N}^m$, it holds that $u_{n_1,\ldots,n_m} \geq u_{n'_1,\ldots,n'_m}$ whenever $n_i \leq n'_i$ for all $i = 1, 2, \ldots, m$.
(iii) We say that V has *the weak asymptotic Egoroff property* if, for each $u \in V^+$ and each strict u-multiple regulator $\{u^{(m)}\}_{m\in\mathbb{N}}$, the following two conditions hold:

  (i) $u_\theta := \sup_{m\in\mathbb{N}} u_{\theta(1),\ldots,\theta(m)}$ exists for each $\theta \in \Theta$.
  (ii) $\inf_{\theta\in\Theta} u_\theta = 0$.

We are now ready to give a Riesz space version of [18, Theorem 1].

Theorem 3.2. *Let* $\mu : \mathscr{F} \to V$ *be a non-additive measure. Assume that* V *has the weak asymptotic Egoroff property. Then,* μ *satisfies the Egoroff condition whenever it is continuous.*

In [24], Murofushi *et al.* gave two sufficient conditions and one necessary condition for the validity of the Egoroff theorem for real-valued non-additive measures. One of the two sufficient conditions is strong order total continuity, and the necessary condition is strong order continuity. Further, they proved that, if X is countable, the Egoroff condition, strong order continuity, and strong order total continuity are all equivalent for any real-valued non-additive measure. These results can be easily extended to Riesz space-valued non-additive measures without assuming any smoothness conditions on the Riesz space by almost the same proof in [24]; see [9] for the precise statements of the above results.

To the contrary, it is not obvious to verify that another condition, that is, strong order continuity, together with property (S), remains sufficient for the validity of the Egoroff theorem for Riesz space-valued non-additive measures. We can give an affirmative answer for this problem by assuming that the Riesz space has the Egoroff property. Recall that a non-additive measure $\mu : \mathscr{F} \to V$ has *property* (*S*) if any sequence $\{A_n\}_{n\in\mathbb{N}} \subset \mathscr{F}$ with $\mu(A_n) \to 0$ has a subsequence $\{A_{n_k}\}_{k\in\mathbb{N}}$ such that $\mu\left(\bigcap_{k=1}^{\infty}\bigcup_{i=k}^{\infty} A_{n_i}\right) = 0$ [31].

Theorem 3.3. *Let $\mu : \mathscr{F} \to V$ be a non-additive measure. Assume that V is Dedekind σ-complete and has the Egoroff property. Then, μ has the Egoroff condition whenever it is strongly order continuous and has property* (*S*)*.*

When the Riesz space V is assumed to be weakly σ-distributive, which is a weaker smoothness than having the Egoroff property, the following version of the Egoroff theorem holds.

Theorem 3.4. *Let $\mu : \mathscr{F} \to V$ be a non-additive measure. Assume that V is Dedekind σ-complete and weakly σ-distributive. Then, μ satisfies the Egoroff condition whenever it is uniformly autocontinuous from above, strongly order continuous, and continuous from below.*

4 The Lebesgue and the Riesz Theorem

Other important theorems concerning the convergence of measurable functions, such as the Lebesgue theorem and the Riesz theorem, can be also extended to Riesz space-valued non-additive measures.

Theorem 4.1. *Let $\mu : \mathscr{F} \to V$ be a non-additive measure. Then, μ is strongly order continuous if and only if the Lebesgue theorem holds for μ, that is, for any sequence $\{f_n\}_{n\in\mathbb{N}}$ of $\mathscr{F}$-measurable, real-valued functions on X converging almost everywhere to such a function f on X, it converges in μ-measure to f.*

Theorem 4.2. *Let $\mu : \mathscr{F} \to V$ be a non-additive measure. Assume that V has the Egoroff property. Then, μ has property* (*S*) *if and only if the Riesz theorem holds for μ, that is, for any sequence $\{f_n\}_{n\in\mathbb{N}}$ of $\mathscr{F}$-measurable, real-valued functions on X converging in μ-measure to such a function f on X, it has a subsequence converging almost everywhere to f.*

5 The Lusin Theorem

The regularity of measures on topological spaces serves as a bridge between measure theory and topology. It gives a tool to approximate general Borel sets by more tractable sets such as closed or compact sets. The well-known Lusin theorem, which is useful for handling the continuity and the approximation of measurable functions, was proved by the help of the regularity of measures.

In non-additive measure theory, Li and Yasuda [20] recently proved that every weakly null-additive, continuous Borel non-additive measure on a metric space is regular, and the Lusin theorem is still valid for such measures. In this section, we extend those results to Riesz space-valued non-additive measures. To this end, we will introduce another new smoothness condition on a Riesz space, called the multiple Egoroff property, that strengthen the weak asymptotic Egoroff property.

Definition 5.1. Consider a multiple sequence $u^{(m)} := \{u_{n_1,\ldots,n_m}\}_{(n_1,\ldots,n_m)\in\mathbb{N}^m}$ of elements of V for each $m \in \mathbb{N}$. We say that V has *the multiple Egoroff property* if, for each $u \in V^+$ and each strict u-multiple regulator $\{u^{(m)}\}_{m\in\mathbb{N}}$, the following two conditions hold:

(i) $u_\theta := \sup_{m\in\mathbb{N}} u_{\theta(1),\ldots,\theta(m)}$ exists for each $\theta \in \Theta$.
(ii) There is a sequence $\{\theta_k\}_{k\in\mathbb{N}}$ of elements of Θ such that $u_{\theta_k} \to 0$.

The multiple Egoroff property and the weak asymptotic Egoroff property are variants of the Egoroff property that was thoroughly studied in [23, Chapter 10].

We now go back to the regularity of non-additive measures. Throughout this section, we assume that S is a Hausdorff space. Denote by $\mathscr{B}(S)$ the σ-field of all Borel subsets of S, that is, the σ-field generated by the open subsets of S. A non-additive measure defined on $\mathscr{B}(S)$ is called a *Borel non-additive measure* on S.

Definition 5.2. Let μ be a V-valued Borel non-additive measure on S. We say that μ is *regular* if, for each $A \in \mathscr{B}(S)$, there are sequences $\{F_n\}_{n\in\mathbb{N}}$ of closed sets and $\{G_n\}_{n\in\mathbb{N}}$ of open sets such that $F_n \subset A \subset G_n$ for all $n \in \mathbb{N}$ and $\mu(G_n \setminus F_n) \to 0$ as $n \to \infty$.

Theorem 5.1. *Let S be a metric space. Assume that V has the multiple Egoroff property. Every weakly null-additive, continuous V-valued Borel non-additive measure on S is regular.*

The Lusin theorem in non-additive measure theory was given by [20, Theorem 4]. The following is its Riesz space-valued counterpart.

Theorem 5.2. *Let S be a metric space. Let μ be a weakly null-additive, continuous V-valued Borel non-additive measure on S. Assume that V has the multiple Egoroff property and is order separable. Let f be a Borel measurable, real-valued function on S. Then, there is an increasing sequence $\{F_n\}_{n\in\mathbb{N}}$ of closed sets such that $\mu(S \setminus F_n) \downarrow 0$ as $n \to \infty$ and f is continuous on each set F_n.*

6 The Alexandroff Theorem

A classical theorem of A.D. Alexandroff [1] states that every finitely additive, regular measure on a field of subsets of a compact Hausdorff space is countably additive. This result was extended in Riečan [27] and Hrachovina [6] for Riesz space-valued compact measures, and in Volauf [32] for lattice group-valued compact measures. The counterpart of the Alexandroff theorem in non-additive measure theory can be found in Wu and Ha [35, Theorem 3.2], which asserts that every uniformly autocontinuous, Radon non-additive measure on a complete separable metric space is continuous (unfortunately, Theorem 2.1 of [35] was proved incorrectly; see [36]). The purpose of this section is to give successful analogues of those results for Riesz space-valued non-additive measures. Recall that $(X, \mathscr{F})$ is a measurable space.

Definition 6.1. Let $\mu : \mathscr{F} \to V$ be a non-additive measure.

(i) A non-empty family $\mathscr{K}$ of subsets of X is called a *compact system* if, for any sequence $\{K_n\}_{n\in\mathbb{N}} \subset \mathscr{K}$ with $\bigcap_{n=1}^{\infty} K_n = \emptyset$, there is $n_0 \in \mathbb{N}$ such that $\bigcap_{i=1}^{n_0} K_i = \emptyset$.
(ii) We say that μ is *compact* if there is a compact system $\mathscr{K}$ such that, for each $A \in \mathscr{F}$, there are sequences $\{K_n\}_{n\in\mathbb{N}} \subset \mathscr{K}$ and $\{B_n\}_{n\in\mathbb{N}} \subset \mathscr{F}$ such that $B_n \subset K_n \subset A$ for all $n \in \mathbb{N}$ and $\mu(A \setminus B_n) \to 0$.

Remark 6.1. Our definition of the compactness of a measure is stronger than that of [6, Definition 1]. In fact, they coincide if V is Dedekind σ-complete, weakly σ-distributive, and order separable.

Theorem 6.1. *Let $\mu : \mathscr{F} \to V$ be a non-additive measure. Assume that V has the weak asymptotic Egoroff property. Then, μ is continuous whenever it is compact and autocontinuous.*

We also have the following version if we assume the weak σ-distributivity on the Riesz space V, which is a weaker smoothness than the weak asymptotic Egoroff property, and assume the uniform autocontinuity of the measure μ, which is a stronger quasi-additivity than the autocontinuity.

Theorem 6.2. *Let V be Dedekind σ-complete. Let $\mu : \mathscr{F} \to V$ be a non-additive measure. Assume that V is weakly σ-distributive. Then, μ is continuous whenever it is compact and uniformly autocontinuous.*

7 Radon Non-additive Measures

In this section, we establish some properties of Radon non-additive measures and the close connection to their continuity. Recall that S is a Hausdorff space and $\mathscr{B}(S)$ is the σ-field of all Borel subsets of S.

Definition 7.1. Let μ be a V-valued Borel non-additive measure on S. We say that μ is *Radon* if, for each $A \in \mathscr{B}(S)$, there are sequences $\{K_n\}_{n\in\mathbb{N}}$ of compact sets and $\{G_n\}_{n\in\mathbb{N}}$ of open sets such that $K_n \subset A \subset G_n$ for all $n \in \mathbb{N}$ and $\mu(G_n \setminus K_n) \to 0$ as $n \to \infty$. We also say that μ is *tight* if there is a sequence $\{K_n\}_{n\in\mathbb{N}}$ of compact sets such that $\mu(S \setminus K_n) \to 0$ as $n \to \infty$.

Proposition 7.1. *Let μ be a V-valued Borel non-additive measure on S. Assume that μ is weakly null-additive and continuous from above. Then, μ is Radon if and only if it is regular and tight.*

Since the family of all compact subsets of a Hausdorff space is a compact system, the compactness of a non-additive measure follows from its Radonness. Thus, by Theorems 6.1 and 6.2 we have

Theorem 7.1. *Let μ be a V-valued Borel non-additive measure on S.*

1. *Assume that V has the weak asymptotic Egoroff property. Then, μ is continuous whenever it is Radon and autocontinuous.*
2. *Assume that V is Dedekind σ-complete and weakly σ-distributive. Then, μ is continuous whenever it is Radon and uniformly autocontinuous.*

Recently, Li and Yasuda [20, Theorem 1] proved that every weakly null-additive, continuous real-valued non-additive measure on a metric space is regular. The following is its Riesz space version.

Theorem 7.2. *Let S be a metric space. Assume that V has the multiple Egoroff property. Every weakly null-additive, continuous V-valued Borel non-additive measure on S is regular.*

It is known that every finite Borel measure on a complete or locally compact, separable metric space is Radon; see [26, Theorem 3.2] and [29, Theorems 6 and 9, Chapter II, Part I]. Its counterpart in non-additive measure theory can be found in [35, Theorem 2.3], which states that every uniformly autocontinuous, continuous Borel non-additive measure on a complete separable metric space is Radon. The following theorem contains those previous results; see also [7, Theorem 12].

Theorem 7.3. *Let S be a complete or locally compact, separable metric space. Assume that V has the multiple Egoroff property. Every weakly null-additive, continuous V-valued Borel non-additive measure on S is Radon.*

We end this section by establishing a close connection between Radonness and continuity of a non-additive measure. The following generalizes Theorems 2.3 and 3.2 of [35].

Theorem 7.4. *Let S be a complete or locally compact, separable metric space. Let μ be an autocontinuous V-valued Borel non-additive measure on S. Assume that V has the multiple Egoroff property. Then, μ is Radon if and only if it is continuous.*

8 Examples

We first give a typical and useful example of Riesz space-valued non-additive measures satisfying some specific properties.

Example 8.1. Denote by $\mathscr{L}_0[0,1]$ the Dedekind complete Riesz space of all equivalence classes of Lebesgue measurable, real-valued functions on $[0,1]$. Let K be a Lebesgue integrable, real-valued function on $[0,1]^2$ with $K(s,t) \geq 0$ for almost all $(s,t) \in [0,1]^2$. Define a vector-valued set function by $\lambda(A)(s) := \int_A K(s,t)dt$ for every Borel subset A of $[0,1]$ and almost all $s \in [0,1]$. Then λ is an $\mathscr{L}_0[0,1]$-valued *order countably additive* Borel measure on $[0,1]$, that is, it holds that $\sum_{k=1}^n \lambda(A_k) \to \lambda(A)$ whenever $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of mutually disjoint Borel subsets of $[0,1]$ with $A = \bigcup_{n=1}^\infty A_n$. Let $\Phi : \mathscr{L}_0[0,1] \to \mathscr{L}_0[0,1]$ be an increasing mapping with $\Phi(0) = 0$. Put $\mu(A) := \Phi(\lambda(A))$ for every Borel subset A of $[0,1]$.

1. The $\mathscr{L}_0[0,1]$-valued Borel measure μ may be non-additive whenever Φ is not additive. A typical example of such Φ can be defined by $\Phi(f) := \sqrt{f} + f^2$ for all $f \in \mathscr{L}_0[0,1]$.
2. We say that the Φ is *σ-continuous from above* if $\Phi(u_n) \downarrow \Phi(u)$ whenever a sequence $\{u_n\}_{n\in\mathbb{N}}$ and u in $\mathscr{L}_0[0,1]$ satisfy $u_n \downarrow u$. The σ-continuity of Φ from below can be defined analogously. Then, μ is continuous from above (respectively, from below) whenever Φ is σ-continuous from above (respectively, from below). We can also give some examples of $\mathscr{L}_0[0,1]$-valued Borel non-additive measures that do *not* have the above continuity.

Next we give some examples of Riesz spaces having our smoothness conditions. Let $(T, \mathscr{T}, \nu)$ be a σ-finite measure space. Denote by $\mathscr{L}_0(\nu)$ the Riesz space of all equivalence classes of ν-measurable, real-valued functions on T. Let $0 < p < \infty$. Denote by $\mathscr{L}_p(\nu)$ the ideal of all elements $f \in \mathscr{L}_0(\nu)$ such that $\int_T |f|^p d\nu < \infty$, and by $\mathscr{L}_\infty(\nu)$ the ideal of all elements $f \in \mathscr{L}_0(\nu)$ that are ν-essentially bounded.

Example 8.2

(i) The following Riesz spaces have the multiple Egoroff property, so that they have the weak asymptotic Egoroff property, the Egoroff property, and are weakly σ-distributive.

   (i) Every Banach lattice having order continuous norm.
   (ii) The Dedekind complete Riesz space s of all real sequences with coordinate wise ordering and its ideals ℓ_p $(0 < p \leq \infty)$.
   (iii) The Dedekind complete Riesz spaces $\mathscr{L}_p(\nu)$ $(0 \leq p \leq \infty)$.

(ii) Let Λ be a non-empty set. The Dedekind complete Riesz space $\mathbb{R}^\Lambda$ of all real-valued functions on Λ has the weak asymptotic Egoroff property. However, there is an uncountable set Λ such that $\mathbb{R}^\Lambda$ does not have the Egoroff property [5, Example 4.2], and hence does not have the multiple Egoroff property.

(iii) Let V and W be Riesz spaces with W Dedekind complete. Assume that W has the weak asymptotic Egoroff property. The Dedekind complete Riesz space $\mathscr{L}_b(V,W)$ of all order bounded, linear operators from V into W has the weak asymptotic Egoroff property.

(iv) The Riesz space $C[0,1]$ of all continuous, real-valued functions on $[0,1]$ has neither the weak asymptotic Egoroff property nor the Egoroff property. On the other hand, the Riesz space $\mathbb{R}^2$ with lexicographical order does not have the weak asymptotic Egoroff property, but has the Egoroff property.

9 Conclusion

A short survey of our recent developments in Riesz space-valued non-additive measure theory has been carried out. Such a study goes with smoothness conditions on the involved Riesz space, because the ε-argument, which is useful in the existing theory of real-valued non-additive measures, does not work well in a general Riesz space. Typical examples of Riesz spaces satisfying our smoothness conditions are the Lebesgue function spaces $\mathscr{L}_p[0,1]$ $(0 < p \leq \infty)$, so that the established results could be instrumental when developing non-additive extension of the theory of p-th order stochastic processes and fuzzy number-valued measure theory.

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