Chapter 9
Fractal Geometry of Architecture

Fractal Dimension as a Connection Between Fractal Geometry and Architecture

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Abstract In Fractals smaller parts and the whole are linked together. Fractals are self-similar, as those parts are, at least approximately, scaled-down copies of the rough whole. In architecture, such a concept has also been known for a long time. Not only architects of the twentieth century called for an overall idea that is mirrored in every single detail, but also Gothic cathedrals and Indian temples offer self-similarity. This study mainly focuses upon the question whether this concept of self-similarity makes architecture with fractal properties more diverse and interesting than Euclidean Modern architecture. The first part gives an introduction and explains Fractal properties in various natural and architectural objects, presenting the underlying structure by computer programmed renderings. In this connection, differences between the fractal, architectural concept and true, mathematical Fractals are worked out to become aware of limits. This is the basis for dealing with the problem whether fractal-like architecture, particularly facades, can be measured so that different designs can be compared with each other under the aspect of fractal properties. Finally the usability of the Box-Counting Method, an easy-to-use measurement method of Fractal Dimension is analyzed with regard to architecture.

9.1 Fractal Concepts in Nature and Architecture

9.1.1 From the Language of Fractals to Classification

For a long period of time, nature has been an inspiration for architects, which implies copying natural forms, translating them into floral ornamentation or using underlying structures found in nature for static optimization and many other possibilities

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of translation. Fractal Geometry has provided scientists with an improved approach to analyzing and generating natural forms. In 1975, the computer-scientist Benoit Mandelbrot introduced the term Fractal to describe irregular, non-smooth curves, and to distinguish self-similar non-smooth structures from smooth Euclidean ones. When he writes that clouds are not spheres, mountains are not cones or bark is not smooth, this shows very clearly that Euclidean geometry lacks the capability to describe natural objects [1]. With his concept of Fractal Geometry, Mandelbrot has increased and broadened our insight in nature. Over two thousand years, our environment had primarily been described in terms of classical Euclidean Geometry – the geometry of simple shapes – and people had focused on a simplified view of nature. Fractal Geometry, however, offers methods to describe and produce nature-like objects directly, using the underlying structure rather than describing them with simple forms or reducing the overall form by dividing it into more simple, smooth components. The language of Fractals enables us to describe the twisted, rough and irregular surfaces of our environment by a few simple rules without reducing their complexity. The major question in this article is whether the language of Fractals may also be applied to architecture.

The language of Fractals can be illustrated with the help of the Barnsley fern, a computer generated fern named after the American mathematician Michael Barnsley (Fig. 9.1). It is supposed to resemble the Black Spleenwort, Asplenium adiantum-nigrum. At first sight, the shape of the natural fern can only be described precisely by defining each detail in an extremely time-consuming process. This type of description could be compared with a non-compressed computer image, saving information about every single dot in large files. Describing a facade in that way would mean providing information on size and position of every single architectural element, including roof, doors, windows, window-strips and columns down to even small-sized ornaments. Barnsley used the iterated function system (IFS) to generate an image of the fern with only four relatively simple transformation rules. Four different configurations of translating, reducing and rotating an initial rectangle or converting it into a single line – and the right fine-tuning – are the basis for a fern-like image. In that process the overall information is reduced to a few underlying construction rules – the algorithm is then the language that is used to describe Fractals. Even though the complex output may not be an exact copy of its natural

Fig. 9.1 Barnsley fern

counterpart, the character is the same and, by fine-tuning, the image can become a very close approximation. How can we draw an analogy between that approach and methods found in architecture? Some architects used basic ideas and basic motifs as a designing tool. Horizontality for instance can be regarded as the basic idea of Robie House by Frank Lloyd Wright – not only for a first impression but also for a deeper understanding.

Focusing on details of the fern we become aware of its fractal characteristics. No matter which part is analyzed, it looks like the whole or, putting it differently, the same characteristics can be discovered on each level of scale – the basic settings of the configuration are present in each part. This phenomenon is called self-similarity, and one of the main properties of Fractals is as follows: smaller parts of an image are hierarchically linked to the whole. Analyzing nature in terms of Fractals shows that nature, from mountains to coastlines and down to plants, is based on self-similarity – it is self-similarity that makes nature so fascinating from the large to the small. In case of Robie house horizontality is evident in the overall view as well as in the horizontally stretched roofs, window-strips and even in details such as long-stretched bricks. The basic idea of horizontality is the common denominator for all the individual components that form a complex whole. As the example of the fern illustrates, Fractal Geometry can be used to simulate complex natural objects with the help of simple algorithms, even if, in many cases, no clear rule of the object's development can be identified at first sight. That is, the right configuration of translating, reducing and rotating has to be found first. But as soon as the right configuration has been detected, complex natural objects can be described with the help of Fractal Geometry. This may also be true for complex artificial objects.

In architecture Mandelbrot distinguished buildings of the Beaux-Arts, which are close to Fractal Geometry, from buildings by Mies van der Rohe, which he calls scale-bound throwback to Euclid [1]. It seems that Euclidean Geometry of Modern architecture – throughout this study the term of Modern Architecture is used for those styles using the design vocabulary of simplification of form and the elimination of ornament – is mostly reduced to a few elementary objects. Even if every single architectural element of a smooth architectural design has to be defined by its size and position, the resulting data set remains manageable. But, would this mean that the Fractal concept is not valid for classical Modern architecture? At first sight, no connection between different elements can be found that might reduce the data set to a set of a few construction rules or to basic ideas. But on closer observation there are such rules, for instance rules developed out of systems of proportion that are applied to all elements from the overall facade to the very small detail. Fractal-like architecture is characterized by the presence of various details of several different sizes that are linked together. Le Corbusier demonstrates how this connection can even be based on a certain angle only, e.g. at the facade of a villa he designed in 1916 [2]. There a specific angle defines the diagonal of the overall facade whose numerous parallels together with their perpendicular lines determine the dimension of elements of the second order such as doors, windows down to certain details. Le Corbusier created a similar example with House Ozenfant in 1923. In those examples all elements such as windows and individual parts but also the whole are based on a system of proportion. The only difference between these examples and architecture that is regarded as more fractal-like is the limited range of scales.

If architecture is analyzed from the point of view of Fractal Geometry, an index of coherence can be introduced. On the very low end, there is an absolutely smooth plane then, an empty rectangle, which is not fractal but belongs to Euclidean Geometry. The index of coherence is increased if fractal properties such as self-similarity and others, which will be described in the following section, are present. Facades belonging to this section are then called fractal-like, because they are not Fractals in a mathematical sense, but offer fractal properties within a limited range of scales. In case of such facades, single architectural components of different sizes are combined, leading to a consistent overall composition. At the upper end of the index of coherence, we can find architectural examples with many unlinked details, hence confusing patterns. "Unlinked details" means that architectural components of different sizes are not interrelated, for instance by a formal basic idea. That means while approaching such a building the observer is confronted with constantly changing and often confusing new impressions.

The systems of proportion at the facade of House Ozenfant are only applied within a restricted range of scales and, in addition to that, the facade looks rather smooth. Those are reasons that put the facade of House Ozenfant on the lower end of the index of coherence, while Robie House can be placed in the middle because of its self-similar characteristics.

9.2 Fractals: A Definition from a Mathematical and an Architectural Point of View

Fractals can be explained by their properties. These include roughness, self-similarity, development through iterations, infinite complexity, dependence on starting conditions, they are common to nature and their Hausdorff dimension exceeds their topological dimension. A simplified definition of the Hausdorff dimension can be illustrated by covering a set of points of finite expansion in a three dimensional space with a minimal number of balls $N(R)$ of radius (R). Decreasing the radius will enlarge the number of spheres. The Hausdorff dimension (d_{H}) is then defined by

$$d_{\mathrm{H}} = -\lim_{\mathrm{R}\to 0} \frac{\log(N)}{\log(R)}.$$

9.2.1 Roughness and Length Measurement

Roughness can be described very well if we look at the coastline of Norway, where Fjords with their sub-bays, inlets, cliffs and rocks lead to a very fractional border in contrast to a circle offering a smooth border. This also becomes evident in connection with length measurement. Benoit Mandelbrot introduced Fractal geometry with the question: "How long is the coastline of Britain" [1]. This is not a trivial question. For length measurement, different maps of various scales can be used, which represent different distances between the observer and the coastline. A large scale may correspond to the view out of an aeroplane hundreds of meters above ground, while a small scale may correspond to the impression an observer gets when walking along the coastline on foot. If we keep in mind the example of the Fjords, it is obvious that each time a smaller part of the coastline, represented on a map of smaller scale, is analyzed, new sub-bays will become visible, which could not be identified in the previous large-scale map. The length measurement of the coastline on a map using a larger scale presents a rough image of the real coastline as it only includes the larger bays. If the same section of the coastline is measured on a small-scale map offering a more detailed version of reality, the larger bays mentioned before are indented by smaller bays so that the measured length increases. In other words, length depends on the details presented on the map used, or on the measuring devices in the real world. In contrast to the Fjords, if the borderline of a circle is analyzed in the same way, the length measurement will follow a different behaviour. If we use smaller and smaller measuring devices, length will tend to a limiting value very quickly (approximation through polygon with "N" edges). This is because no additional details will be presented while zooming in. The difference between Fractals and Euclidean objects (e.g. squares and circles) then lies in the fact that length measurement fails for the first ones.

What does length increase look like in a mathematical fractal such as the Koch curve, a famous representative of a classical Fractal? The construction rule of the Koch curve starts with a straight line of length one, which is called the initiator. This line is then divided into three equal parts (Fig. 9.2). The middle part is replaced by an equilateral triangle, whose base line is removed. This adjustment of four smaller lines is called the generator of the Koch curve. As the resulting structure consists of four scaled-down copies (four lines) of 1/3rd of the initiator (one line of the initial size one), the overall size increases by 4/3rd. In the next step, each of the four new lines is again replaced by the generator, resulting in 16 scaled-down copies. The scaling factor in relation to the initial line is now 1/9th. That means the length increase is 4/3rd by 4/3rd or in other words 16/9th. From this it was not only followed that the length increases by 4/3rd from one iteration – that is a step of replacement – to the next, but also that the offsets of the curve increase – a higher degree of roughness can be identified. For the third iteration length increase – in relation to the initiator – is 4/3rd by 4/3rd by 4/3rd or 4/3 above 3. Expressed in a more general way this leads to the following equation

$$L_{(i)} = (N_{(1)} \times s_{(1)})^{i}$$

with (i) indicating the number of iterations, $(L_{(\mathrm{i})})$ the length of the curve after (i) iterations, $(N_{(1)})$ the number of single lines of the generator and $(s_{(1)})$ the reduction factor of the generator, being one third for the Koch curve. From one iteration to the

Fig. 9.2 Koch curve

next, the curve gets longer and longer, hence more and more twisted, which means it is getting rougher from iteration to iteration. Because mathematical Fractals are results of infinite iterations, the final length is infinite.

9.2.2 Scale Range and Distance

As the above example of coastlines shows, scale range and distance determine the degree of roughness and detail the viewer is able to see. These factors also influence a viewer's perception of a building. From far away, the viewer will only perceive very large components, the silhouette and significant edges. This view corresponds to an elevation of large scale that only includes a few details. Approaching the building, the observer's attention begins to focus upon the sequence of base, middle or roof part. Then windows and doors or the rhythm of columns are the most prominent parts. Zooming to one of the next levels it turns out that (e.g.) windows consist of smaller details such as window frames and window handles, but even walls offer smaller parts such as face bricks or tiles. Thinking about grain of wood, the cascade of details will come down to the material itself. For Salingaros, the size of the smallest detail has to correspond to the smallest perceivable scale [3]. This clearly shows that facades are in general not smooth Euclidean two-dimensional planes, which is per se valid for Modern architecture as well as organic architecture or any other style. Elevations as two-dimensional translations of facades are rather something between lines, defined by edges, and the plane, defining the surface. Before architecture can be analyzed and buildings compared with each other from the point of view of coherence between elements of different sizes, or more precisely with regard to self-similarity, it is important to define the range of scale first – it has to be standardized. The range correlates with the distance of the observer while approaching the building and the smallest possible detail that can be perceived from certain distances.

9.2.3 Self-Similarity: An Important Attribute of Fractals

Self-similarity is an important property of Fractals. In mathematical terms, two objects are similar if their corresponding angles are identical and their corresponding sides are in proportion regardless of their size. An object is called self-similar if one or more of its parts look like the whole. The parts can be exactly similar or approximately similar. The Koch curve is exactly self-similar because each part is an exact, scaled-down copy of the whole. The final Koch curve is the result of applying similarity transformations (scaling, translation, rotation) to the initial object, which modifies proportion by the same factor. The resulting scaled-down copy may be rotated or transformed while the shape remains the same. If the pieces of the object are scaled down by different amounts in different directions, the fractal is then called self-affine instead of self-similar. When simulating nature the factor of chance has to be added. The new structures are called statistically self-similar, that is whenever small copies, looking like the whole, have variations [4]. Parts show the same statistical properties at many different levels of scale. This is important for describing natural objects, but also architecture. When we, for instance, examine certain parts of coastlines, they do not just represent scaled-down, transformed and rotated copies of the whole, but show a similar character and degree of irregularity. Cliffy coastlines offer the same strong irregularity from the very large to the very small level of scale, similar to gently twisted coastlines offering the same softness all over as well.

Geometric shapes with fractal properties had already been known long before the term Fractals was introduced and long before all the facts about Fractals were combined to form a theory. The Koch curve for instance – a continuous curve without tangents – was already presented in 1904. In connection with length increase, it was indicated that the number of components (single lines) of smaller scales increases from one iteration to next. In architecture, fractal properties had also been used consciously but also unconsciously, which is true for Gothic cathedrals as well as for Robie house by Frank Lloyd Wright [16, 17]. Carl Bovill already drew an analogy between writings of Frank Lloyd Wright and the fractal concept [5]. According to Frank Lloyd Wright, nature should be a source of inspiration. Carl Bovill demonstrates that for Frank Lloyd Wright the underlying organizing structure of nature – which is finally the fractal concept – is of particular interest. The fundamental idea of bringing the characteristics of a horizontal environment, the Prairie, into the building by Frank Lloyd Wright, Robie House, is implemented on many levels of scale, from the wide overhanging roof and stretched storeys over window strips and horizontal parts of walls to the design of the horizontal and straight joints [15]. The number of components that are evident on a specific level of scale increases from the overall view over the level of scale that includes windows down to ranges of scale on which details of bricks and material become visible. Analogously, a cauliflower consists of a smaller number of similar scaled-down components that again consist of many scaled-down components. In this sense examples such as Robie House can be called fractal-like architecture. Architecture is called fractal-like if the whole and all other formal elements are derived from one basic idea and, by that, a simple, specific form characterizes the expression of the building. Parts are then reflections of the whole, the formal elements are held together in scale and character [5]. Eero Saarinen also pointed out that a building should follow the strong, simple concept of the whole – each part has to be an active component of a certain overall theme. This is valid irrespective of the point of time when decisions to follow such an overall theme are taken: Right from the start with regard to ground plan or construction-system or at a later stage, when detailed elements such as colours inside the house or even door-handles are in the architect's focus of interest [6].

9.2.4 Architectural Examples

There are different approaches to self-similarity in architecture: Self-similarity in architecture may arise from a basic proportional system, from a basic form or from an overall idea. To give another example, Rietveld-Schröder house in Utrecht (Netherlands, 1924) offers coherence from the large elements to the small ones [17]. The architect Gerrit Rietveld translated the use of form of the Stijl, with basic forms and basic colours, into architecture. Those basic forms are taken from Euclidean Geometry and include straight lines, planes and slabs that are detached from each other and seem to glide past – the floating space playing an important role [7]. The building is characterised by large openings, horizontal and vertical elements and an intersection between inside and outside elements. The cascade of architectural components starts with basic spatial and constructive structure, formed by few main elements of large white slabs (windows and doors are cut out). Then, on a smaller level of scale smaller slabs for balconies and canopies follow, while linear elements such as rainwater pipes add vertical and horizontal accents. Inside, an analogy can be drawn between the basic elements and window-frames, sliding and revolving panels and their linear hanging. Coming even closer, the basic elements are repeated in the design of the furniture, from chair to modular cupboard. The translation of basic elements into smaller levels of scale connects the furniture to the architecture around. This example illustrates that even though the number of iterations is limited, the basic idea of separating components by means of form and colour and the composition of planes and linear accents is nevertheless evident within all levels of scale.

There is one more example we should deal with: in 1956, Bruce Goff constructed the Joe Price studio in Bartlesville, Oklahoma. The basic idea of construction was focused on 60° angles and their multiplication or subdivision to hexagons and triangles [8]. Once again self-similarity occurs on the basis of a simple component, a certain angle. Variations of the basic element can be found from the large elements, including the roof, with the basic shapes forming a twisted outline, down to the small size of decorative elements. Inside in turn, the ceiling, the walls, the hexagonal sitting hole, which is a hollow in the floor, and certain details are based on triangular shapes. The individual components of different sizes are variations of the basic theme and therefore offer coherence within all levels of scale. Such variation arises from the material used and from purpose such as deflecting sound in the music room.

The work of Antoni Gaudí offers fractal-like architecture as well [16]. Although buildings by Antoni Gaudí appear very complex, they are nevertheless coherent. This is not a surprise because Gaudí pointed out the importance of dealing with details to produce a complete work of art. As an example the use of curved construction stones and nature-like, organic forms can be found all over Sagrada

Familia from outside to inside from large elements to small details. Gaudí was interested in the forces of nature that act behind the surfaces rather than in the shape, hence the surface itself [9]. Inspired by nature, Antonio Gaudí introduced catenary shapes as idealized forms of arches [10]. He developed catenarian-models and cable-models with sandbags illustrating the reverted interrelation of forces for columns and pillars. The result was then an upside down model of the structure in the building. Gaudí found out that the use of parabolic arches and inclined buttresses could withstand the forces involved. On smaller levels of scale, hyperbolic paraboloids can be detected in the vaults but also at the base of columns [10]. Antoni Gaudí used cone, cylinder, simple hyperboloids, hyperbolic paraboloids to design non-smooth architectural components. The analysis of a small part of the building will make the observer think about the whole, and the building is coherent in form and character.

Self-similarity alone, however, does not define a Fractal. Considering that a line between two points can also be divided into smaller parts that are scaled-down copies of the whole, the line nevertheless does not increase in length from one iteration to the next, and there are no additional details offered either when zooming in. Or, zooming in on a circle will show a more and more straightened part of the circumference – apart from irregularities arising from the drawing itself – but does not offer additional details. The latter two examples belong to Euclidean Geometry because no further details arise when zooming in.

9.2.5 Developed Through Iteration

Generalized for a first approach, Gothic cathedrals can be described by verticality and light-flooded interior rooms, which was made possible by pointed arch, flying buttress and ribbed vault. They are an expression of unity between single components, inside and outside, and the whole: In numerous functional and decorative elements such as windows, portals, baldachins, pinnacles, attics, Gothic gables and tabernacles, references to the whole can be identified. Examples such as the Gothic windows of the Southwest tower of the cathedral of Cologne or of the Angel Choir of the Cathedral in Lincoln definitely provide excellent examples to describe how Fractals can be constructed. The overall shape, a pointed arch, defines the initiator, which is then replaced by the generator, a pointed arch of the same size vaulting two smaller ones of half the size. In the next step, each pointed arch is again replaced by the generator and so forth. Figure 9.3a shows a series with the initiator that is the starting object, the generator that is the replacing rule and some iterations. Although the replacing rule for the Gothic window may be applied infinite times in theory, in the real world there are some restrictions arising from the material used and from the usability as a window. The built version tries to find a harmonious balance between a slight big opening and construction rules and therefore already ends after a few levels of replacement or iterations, respectively. Furthermore, the algorithm may be adopted with regard to the reduction of the diameter of the shafts.

Such replacement procedures can also be identified as underlying construction elements of Indian temples, such as the Sikbam of a Jaina temple on Mount Girnaz [11]. The second image in Fig. 9.3b shows an output of a construction rule that is derived from this example. The basic curved shape of the whole temple, rising out of a square, has undergone a transformation in size and position, bringing up four additional similar shapes at the sides of the main part. Each of these four new elements is then transformed in the same way, resulting in a complex whole, where all parts are formally linked with the whole. Figure 9.3b shows the initiator and the fourth iteration, each of three different configurations of such an insertion rule. The basic transformation rule, scaling down the initial shape and positioning it at each side, is adjusted in the way that different reduction factors and movements are used. It is also conceivable that the reduction factor varies within the same iteration by the factor of chance. Using different basic shapes (initiator) and adjustments, the basic rule may then lead to high rising as well as vaulted structures (Fig. 9.4a, b).

The output of such computer-generated self-similar structures cannot be used for architectural design without changing or adapting it with regard to the limits imposed by the material used, construction, function and environmental influences. Such computer outputs are only useful to show how self-similar patterns in connection with architecture can be generated in a simplified way. If facade generating programs are developed to compute facades with visual depth, the question of fabrication and usability has to be considered. Such facade generating programs might start with a simple box, dividing it in (n) by (m) boxes, which are removed or added in certain ways. Single boxes vary in their expansion to the front, so that variation in depth is achieved. Then the new boxes are manipulated in the same way, using the same parameters or slightly adopted ones with, e.g. random

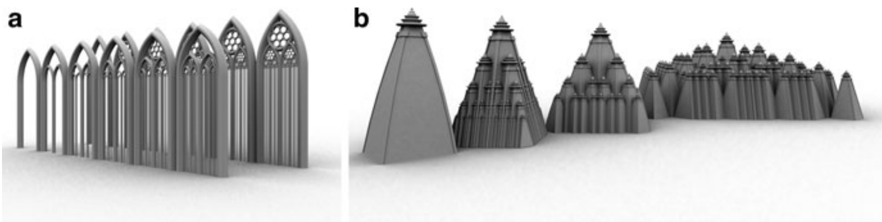

Fig. 9.3 Various examples of geometric output – (**a**) Gothic window algorithm, (**b**) temple algorithm

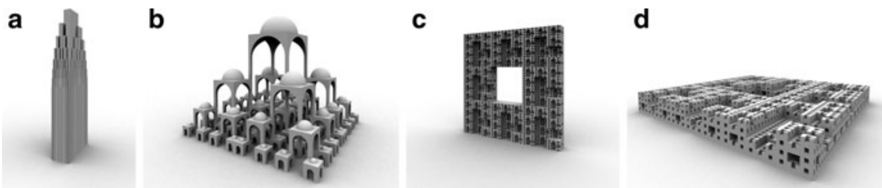

Fig. 9.4 Different insertion rules – (**a**) high rising, (**b**) vaulted, (**c**, **d**) window element

factors for expansion. Figures 9.4c, d are based on such a rule and offer self-similar patterns after a few iterations. Windows are either included in such algorithms by cutting out middle parts or such outputs are simply arranged around existing cuts. In the first case, one large cut in the facade is surrounded by smaller ones, surrounded by even smaller cuts and so on.

9.2.6 Differences Between Architectural and Mathematical Fractals

The main difference between nature or fractal-like architecture and mathematical Fractals is the limited number of iterations. Although mathematical Fractals are theoretical constructions offering infinitely small parts, self-similarity of nature and architecture only exists for a limited range of scale. It could be said that nature and architecture only uses a limited number of iterations. This will be illustrated by comparing the Sierpinski triangle, a mathematical fractal, with Castel del Monte, situated in the Apulia region, by the Holy Roman Emperor Frederick II. The Sierpinski triangle can be simulated by starting with an equilateral triangle as the initiator. This initiator is then replaced by a generator consisting of three equilateral triangles, each scaled-down copies of the initiator by the factor of 0.5. All corners of the initiator act as fixed points for one of those three new triangles when they are scaled down. In other words an equilateral triangle is cut out in the middle of the initiator where the corners of this cut out triangle are situated on the middle points of the sides of the initiator. For the next iterations, each triangle is subjected to the same procedure, leading to nine triangles. In theory, this is repeated infinite times leading to a real Fractal, where even the smallest part is a scaled-down copy of the whole. In case of Castel del Monte, which offers the beginning of a replacing algorithm, the difference between mathematics and architecture becomes evident. The basic octagonal shape is complemented by further octagons at its corners. For the next iteration, each of these octagons would be complemented by further octagons at their corners and so on. But the "real" Castel del Monte only offers the first iteration. Nevertheless, the circumference of Castel del Monte increases from the initiator to the first iteration and offers a rough outline. Together with the rough stonewall this distinguishes Castel del Monte from smooth surfaces with a lower index of coherence.

9.2.7 Fractals as a Design Aid

The competition contribution for Cardiff opera house in Wales by Greg Lynn in 1994 mandates a new concept for waterfront urban space that is nonetheless in conformity with the history of the site and Cardiff's waterfront [12]. The analysis of the coastline for self-similar structures from the large down to the small level of scale (water's edge being captured by land) together with the oval form of the basin was translated into a rule with the starting body being replaced by three parts of different sizes and orientation. Some iterations of this Fractal basic shape in combination with the basin defined the character of the opera house next to it – the opera house is then a continuation of Cardiff's waterfront. The output was adapted with regard to function, construction and form. Volumes were rearranged because of requirements of the foyer, auditorium and acoustic properties, stages, studios, offices and other purposes of the opera house.

In general, fractals can also be used as a basic design for breaking open an otherwise straight line. For example, the characteristics of a natural coastline can be simulated by a fractal, if the right configuration or the right insertion rules are found. The twisting of the Fractal then leads to a coastline section, which is based on a more environmentally appropriate scale. What can be learnt from such attempts – forming buildings and border-lines? In general, using such Fractal-based designs means that the architect looks for a rule coming close to the one that is inherent in the environment on a larger scale to continue the cascade of similar character down to the size of harbors or buildings and even to human level connecting smaller scales to the whole. Man-made interaction will then continue the natural characteristics of the environment and in turn will not restrict the range of scale. The basic Fractal nevertheless has to be adapted with regard to usability concerning function and costs, but also with regard to the fact that the result may again consist of straight parts though on an even smaller scale.

9.2.8 Fractals Are Common to Nature

Dealing with Fractal Geometry also means focusing on nature: Clouds, bark and trees are not smooth but rough, and snowflakes or the distribution of stars offer self-similarity. Different fractal methods, which take advantage of self-similarity, underline the connection between nature and fractals [20]. The right algorithms produce models of plants, mountains, crystals and entire natural scenes. Self-similarity may in turn be the reason why nature seems so fascinating from the large elements to the small ones and why forms of nature seem better balanced to us than Euclidean smooth shapes. If so, this explains why Gothic cathedrals, rural houses and organic buildings, which rather contain self-similarity, are so fascinating to many observers. Frequently those buildings offer details, which are prominent on different levels of scale and are consistent with each other and the whole. Consequently, while we are approaching the building, new details of smaller size get into the focus of our attention reminding us of the whole. Because of variation of components, which differentiates self-similarity from self-sameness, they also remain diversified. Assuming that those parts are exact, scaled-down copies of the whole, this would mean the observer can judge from looking at the whole what the detail will exactly be like. Because of variation he can envisage what comes next but may be slightly surprised and confirmed at the same time. Mandelbrot believes that fractal art is more acceptable because it imitates nature to make the observer guess its rules and is therefore more familiar to us [1].

9.2.9 The Factor Chance

Nature-like images produced by strict self-similar construction rules are often too clinically "perfect" to imitate their originals to their last consequence. Local influences such as temperature, wind, waves or nutritive substances, which deform the objects, are missing. Effects of such influences can be imitated by the factor of chance. This can be illustrated by means of the tree algorithm (Fig. 9.5a). Starting from a stem of certain diameter and length, three or any other number of scaled down copies of it are moved to the end of the stem and rotated in different directions. Each of these branches is again copied, scaled down, moved to the end and rotated in different directions. This rule is applied to a couple of iterations. If the rotation is chosen randomly between certain limits as well as the length of each branch, the resulting image resembles nature even more closely. The basic shape is a smooth Euclidean cylinder, but the object itself becomes rougher from one iteration to the next, while smaller pieces are added. On the one hand, the resulting tree can be described by defining length, position and direction of each single branch, which produces large data quantities. On the other hand, the resulting tree can also be described just by its insertion rule.

With the factor of chance, even the Koch curve can be turned into a natural-looking coastline. Figure 9.5b shows three such Koch curves added to one Koch Island. In the basic rule, the middle vertex of the generators of each Koch curve looks to the outside and so does every replacing step. To get a more nature-like model, the middle vertex is also allowed to look to the inside. The choice whether it is pointed to the outside or inside is chosen randomly. After some iterations, the resulting curve is not as clinical as the origin without the factor of chance, but nevertheless offers the same characteristics and length increase. Both examples, the tree

Fig. 9.5 (**a**) Simulation of a tree, (**b**) Koch Island with factor of chance, (**c**) simulation of a mountain

algorithm and the coastline, indicate the importance of variation for natural-looking images. This has been taken into account for certain Fractal methods producing images of plants, trees, mountains and even planets. The rule used for such a method can be very simple, while the output will look extremely complex. This can be illustrated with a program imitating mountains, which can be implemented into a CAD package quite easily. The starting image to generate such a mountain is a triangle. Then the midpoints of each of the three sides of the triangle are marked. These points are then moved up or down by chance and by a certain factor. The higher the factor, the rougher the resulting mountain is. In the next step, each point is connected to a new triangle with its neighbouring points. These triangles are then subjected to the same rule as before (Fig. 9.5c). Parts are not identical with the whole but the overall character remains the same from one zoom level to the next – analyzing small parts will offer similar structures as the whole. From this follows that, the other way round, underlying construction rules of natural objects are difficult to identify because of variation, which have been here simulated by the factor of chance.

9.3 From Simulation to Measurement

Fractal methods generating architectural structures, as they have been introduced with the Gothic window, the Indian temple and the facade designer, are only a first approach to Fractal architecture. All those attempts have in common that their algorithm is very simple, just demonstrating basic rules of existing buildings, simulating them with the help of a computer and producing outputs of different shapes. This first attempt can be made more sophisticated with the factor chance to develop symmetry breaking designs such as those that may result from environmental restrictions or adjustments because of internal function. Including parameters that simulate the influence of daylight and shadow, compactness, functional fitness or resources directly will increase computing time. With the help of a computer, a large number of different alternatives can be generated from which the architect can finally make his choice. This choice can then be modified and developed further as in the case of the competition contribution for Cardiff opera in England by Greg Lynn.

9.3.1 Curdling

Several examples of fractal-like architecture from Gothic buildings to Robie house and simplified architectural simulations have not only shown that, similar to nature, architecture possesses fractal properties up to a certain degree, but also that those properties are difficult to describe because of modification by certain influences. Consequently, for comparing works of architecture with each other – with regard to the degree they are fractal-like – a consistent measurement method has to be developed. If self-similarity is present in architecture, then this is also expressed by a similar distribution of architectural elements from one level of scale to the

Fig. 9.6 Curdling

next – each level of scale has its elements of specific size that again contains smaller details of similar distribution. Such connections between different levels of scale can be demonstrated with the help of the so-called curdling, introduced by Benoit Mandelbrot to demonstrate the process that produces a disconnected set of points with nevertheless clustered characteristics [5]. The process is called curdling since originally uniform mass distribution clogs together forming many small regions of high density [4]. There, structures are generated by the factor chance where zooming in and analyzing parts will offer similar characteristics like the whole. Finally, such connection between many levels of scale can be given by a set of values, characterizing the cluster. The measurement method of these values will now be introduced with curdling.

"Curdling" was the name Mandelbrot coined for a procedure that produces a random fractal dust in two dimensions [1]. With curdling, the starting object, a simple square, is divided into a grid. In our example, the grid consists of three by three cells. Tossing a coin determines whether a cell is deleted or again divided into a three by three grid. The coin can also be replaced by a probability factor. High probabilities lead to a higher chance for cells to remain and consequently to a higher 'density'. Figure 9.6 shows some iterations for different probabilities.

Considering the probability of one out of nine, this means that mathematically one cell will remain after the first iteration. That also means that eight out of nine cells are deleted. Because of the element of chance, in practice none may be chosen as well, which stops the algorithm, but also more than one is possible. What happens to the number of cells when using the probability two out of nine? Mathematically, after the first iteration only two of nine cells remain and seven are deleted. For the second iteration, both remaining cells are again divided into a three by three grid. Again in each grid, two cells remain and the others are deleted. This increases the total number of remaining cells up to two by two, hence four. In the next iteration, each of these four cells is again divided into a three by three grid. Again two cells of each grid remain, increasing the total number of remaining cells to two by two by two, hence two above three. Simplified, this leads to the equation

$$N_{(i)} = (N_{(1)})^i$$

giving the connection between increasing numbers of remaining cells and iteration. There ($N_{(i)}$) is the total number of remaining cells after (i) iterations and ($N_{(1)}$) the number of remaining cells of the generator. For the probability three out of nine, the number of remaining boxes after three iterations is then three above three, hence $N_{(3)} = 27$. For the same iteration but for the probability eight out of nine it is $N_{(3)} = 512$.

Simulations use a random generator of a certain probability that determines for each cell whether it remains or not. Therefore, different simulations of the same probability may lead to different numbers of remaining cells. The two examples on the right in Fig. 9.6 show two different results for the probability of two out of nine, the middle two for the probability six out of nine and the two on the left for the probability eight out of nine. These examples indicate that the remaining cells after (i) iterations vary for one and the same probability. For simulations, the simple connection between the number of remaining cells and iteration of the previous equation is then no more valid.

Considering that the grid represents a building site, the same algorithm may simulate distribution of buildings. This time the grid does not have to be regular and the algorithm will be stopped at a proper size, that of buildings. Then cells of the last iteration are moved from the border defining streets in between. Once again this is just a formal simulation using one basic rule only, but it will produce choices of different distributions as discussion bases. More practicable models can be developed, enlarging the basic rule by additional parameters accounting for different influences.

9.3.2 Fractal Dimension

For indicating consistency between different iterations, we have to look for a comparison between the increase in number of remaining cells and the reduction factor. With the Koch curve, the number of single lines $(N_{(i)})$ and the reduction factor $(s_{(i)})$ are both increasing/decreasing by the same index: number of parts after the first iteration $(N_{(1)}) = 4$, the reduction factor of these parts compared with the size of the initiator $(s_{(1)}) = (1/3)$; number of parts after the second iteration $(N_{(2)}) = 4^2$, $(s_{(2)}) = (1/3)^2$; $(N_{(3)}) = 4^3$, $(s_{(3)}) = (1/3)^3$. From this it was derived that there exists a connection between the number of single lines after (i) iterations $(N_{(i)})$ and the reduction factor $(s_{(i)})$. This connection is shown in the equation

$$N_{(i)} = \left(\frac{1}{S_{(i)}}\right)^{D_s}$$

introducing index (D_s), which is the self-similar dimension. Modified, this leads to

$$D_S = \frac{\log(N_{(i)})}{\log\left(\frac{1}{S_{(i)}}\right)},$$

which equals (D_s) 1.26 for the Koch curve with four lines of one third each after the first iteration $(i) = 1$.

With the theoretical output of curdling, self-similar dimension can be calculated in the same way. Using the example from above – tiling each cell into three by three smaller cells – with a probability of two thirds, this means that mathematically six cells will remain ($N_{(1)}$) after the first iteration. The reduction factor of the grid-size is one third ($s_{(1)} = 1/3$). Inserting these values into the last equation, the self-similar dimension (D_s) equals 1.631. With the probability of one ninth, mathematically only one cell remains after the first iteration while the reduction factor is again one third. This leads to the value zero, equal to the topological dimension given for a dot. At the other extreme with the probability of nine ninth all nine cells remain, resulting in the value two, which equals the topological dimension for a plane.

What is of interest next is how the value (we will call it the index of coherence), defining the connection between the number of single elements and the reduction factor, behaves throughout the range of scales. With curdling, this means to compare the increase in the number of remaining cells with the decrease of the reduction factor. Using once more the example from above with a grid of three by three cells, mathematically the number of remaining cells after the first iteration is six for the probability of two thirds and the reduction factor is one third. Then the second iteration, using equation $N_{(i)} = (N_{(1)})^i$, increases the number of cells to 36 with the reduction factor decreasing to one ninth. Between two iterations, the differences of both, number of cells as well as reduction factor, can be examined. Consequently, the equation for calculating D_s is rearranged, leading to the equation

$$N_{(i)} - N_{(i-1)} = \left(\frac{1}{S_{(i)}}\right)^{D_B} - \left(\frac{1}{S_{(i-1)}}\right)^{D_B}.$$

The difference of cells is compared with the difference of the inverse reduction factors above the value of index, this time called (D_B) with regard to analyzing boxes (we called them cells). This equation is rearranged to

$$D_B = \frac{\log(N_{(i)}) - \log(N_{(i-1)})}{\log\left(\frac{1}{S_{(i)}}\right) - \log\left(\frac{1}{S_{(i-1)}}\right)}.$$

Inserting the mathematically calculated values of our example with a probability of 2/3rd – that is for the number of remaining cells $N_{(1)} = 6$ with the reduction factor $s_{(1)} = 1/3$ for the first iteration and $N_{(2)} = 36$ with $s_{(2)} = 1/9$ for the second iteration – (D_B) again equals 1.631. Analyzing the third and fourth example from Fig. 9.6, both using a random calculator with the probability of 2/3rd, the results change. While the reduction factor remains the same, the number of remaining cells changes. That is $N_{(1)} = 7$ for the first example after the first iteration and $N_{(2)} = 47$ after the second iteration. The calculated value D_B between first and second iteration then equals 1.73. D_B of the second example equals 1.75 with $N_{(1)} = 7$ and $N_{(2)} = 48$.

So both values are slightly higher than the result inserting the mathematically calculated values. Using the last equation, a set of index values (D_B) can be given for a certain number of iterations. Each iteration is compared with the next one, which leads to a certain (D_B) from the first to the second iteration, to another from the second to the third and so forth. Because of the consistency between iterations, self-similar structures will then offer similar values throughout the range of scales.

9.3.3 Perception and Distance

Comparing buildings with each other means that their appearance has to be standardized first, including influences of color, shadow but also depth of details – equals the number of iterations or levels of scale. For a first approach, just black and white elevations will be analyzed, which means removing all influences other than depth of details. It is the set of index values (D_B) for a certain range of scale that is of interest. Because two-dimensional plans will be analyzed, a first step of standardization is to define the translation of the original building into the elevation. There the smallest detail presented in the elevation depends on the distance of the viewing point from the building in reality and on the human eye. This derives from the fact that the smallest possible detail depends on the reading field, which is inside a cone of 0° 1' [13]. With the aid of trigonometric function, the relationship between detail and distance can be given by

$$\text{Minimum Size of Detail} > \text{Distance to Detail} \times \text{tangents}\left(0^\circ 1'\right).$$

This means that for a given distance of 10 m, the size of the smallest perceivable part is approximately 3 mm. Since the smallest perceivable detail depends on the distance between the observer and the building, consequently the distance defines what should be presented in the elevation. From the smallest detail on, all architectural elements of larger size have to be included and translated to the elevation as significant edges. Only then coherence between these levels of scale or sets of different sizes of architectural elements can be analyzed, but also only then buildings can be compared with each other by their elevations, standardized by the observer's distance. The distance to the building, in turn, should be chosen in the way that its whole extent can be perceived, mainly its vertical extent. Maertens gives an indication, where a distance equal to the relevant height of a building is appropriate to view details of the object [13]. Then the uppermost part is within an angle of 45° above horizon. A distance of double height equals 27° and there the whole building can be viewed for itself. Finally at a distance of the observer of three times the height, the building will become one with its environment. From these angles, the distance may then be derived.

9.4 Fractal Dimension and Architecture

Fractal curves are the result of insertion rules after infinite iterations. The Peano curve, a representative of a classical fractal, is an endless, twisted curve between two points that does not exceed a certain space. As it can be separated by removing just one point from the set, it is said to have a topological dimension of one. But on closer observation, it passes through the two-dimensional plane completely – it offers area-filling property. The Koch curve again does not fill the two-dimensional plane but is also endlessly twisted. Since length is infinite, a point on such curves cannot be defined by only giving the distance. But also a mountain, being rough, does not fill the three-dimensional space completely. Such structures can then be characterised in a better way by their Fractal Dimension. The Fractal Dimension expresses how fast a fractal curve tends to infinity from one iteration to the next or how completely a fractal appears to fill space.

Facades are rough surfaces that consist of cuts, different architectural elements of different sizes and details. Hence, they are no flat smooth two-dimensional planes. Likewise, their expressions on paper, elevations are more than a one-dimensional line that defines the silhouette, but they also do not fill the plane where they are situated in completely. They are rough structures and Fractal Dimension is then the expression of the degree of this roughness, which means how much texture an object has [5]. With fractal-like architecture, self-similarity cannot simply be identified by rescaling parts transforming them into the whole again. Then Fractal Dimension is an adequate possibility to describe such structures, where coherence of roughness can be analyzed by calculating a set of index values (D_{B}) for a certain range of scale.

9.4.1 Fractal Dimension and Approaching a Building

Expressed in a different way, Fractal Dimension is the degree of mixture of order and surprise. For objects that are visualized on paper, Fractal Dimension can be measured by the Box-Counting Method. This measurement method was first applied to architecture by Carl Bovill [5, 19]. First, a grid is put over the object to be measured. The grid-size is defined by the number of boxes across the bottom row of the grid, that is the numbers of boxes in x-direction (1/s). Its inverse value then defines the scale of the grid (s). Then those boxes, which cover relevant parts of the elevation, are counted. Relevant parts are the outline, the roof, windows, doors, walls, but also certain details. The depth of details of the analyzed elevation depends on the scale of plan. For the first grid-size ($1/s_1$), the number of boxes that contain relevant parts is defined as (N_1). For measurement, the grid-size is then reduced to ($1/s_2$), and the number of boxes that contain relevant parts is counted again (N_2). Finally, the Box-Counting Dimension between two scales of grid-size is calculated by the relationship between the difference of the logarithms of the number of boxes that contain relevant parts and the difference of the logarithms of grid-size as given in the equation

$$D_{\mathrm{B}(1-2)} = \frac{\log(N_{(S2)}) - \log(N_{(S1)})}{\log\left(\frac{1}{S_2}\right) - \log\left(\frac{1}{S_1}\right)}.$$

The Box-Counting Dimension D_B is another special application of Mandelbrot's Fractal Dimension [4, 14]. The Box-Counting Method compares the roughness – represented by lines – between different grid-sizes and thus allows measuring the complexity of a structure across certain sizes of details. Equivalence exists between the scale of the elevation, the scale of the grid and coming closer to a building. In the first case, a large scale of the elevation only gives an impression of the building. In this case, larger grid-sizes are used for measurement. Then reducing the scale of elevation, which means including smaller details, will allow us to identify more and more details. This asks for smaller grid-sizes. The same is true when approaching the building in reality. On the level of scale of far distance, smaller details are faded out, because they cannot be perceived. Consequently, they have to be excluded from the elevation for measuring the Box-Counting Dimension. Then from a shorter distance, bigger architectural elements such as windows and doors are perceived that could not be distinguished before, followed by window-frames and door-handles. For measuring the Box-Counting Dimension for this distance, the elevation has to include these components. If the building follows the Fractal concept, the kind of roughness nevertheless remains the same for all steps. In the logic of Fractal Geometry, they are linked together by the depth of similar roughness.

9.4.2 Results of Measurement

The behaviour of the relationship between grid-size and number of boxes that contain relevant parts is analyzed in a double logarithmic graph, where the slope of the replacing line defines the Box-Counting Dimension for a certain range of grid-sizes. The result for many different measurements of Robie House by Frank Lloyd Wright remains between 1.6 and 1.65, taking different elevations with different detail-richness into account: from an overview to plans of smaller scale, including details such as stained glass and brick. That means different elevations were used for measurement to include different distances. For the first distance only main edges such as outline, windows and doors were included. Then approaching the building, hence using smaller boxes, sections of the elevation were analysed, including stained glass and bricks. For all measurements, it is valid that the result depends on what is included in the elevation and how it is presented.

The double logarithmic graph illustrates that certain measuring points of Robie House are very close to their replacing lines. From this follows that, although Box-Counting Dimensions between two single grid-sizes may vary when approaching the building, the set of measurement points in the double-logarithmic graph is nevertheless stable over a large range. The slope of the replacing line then gives quite a significant value (D_B) for this range. Comparing these results with the Koch curve, whose Self-similar Dimension is known, it can be indicated that measurements for the Koch curve are even more stable. Nevertheless, while comparing different buildings with each other certain influences have to be dealt with [14, 18]. Some derive from the measurement method itself – that is dependence on starting position or overall grid-size – and others from range dependence of architectural elements. Architectural elements only emerge locally, which means they have a specific range of distance of the observer in which they are significantly present. In general, to minimize local influences a whole range of grid-sizes is analyzed rather than only two single levels. The standard deviation of the graph then gives the degree of coherence. For Robie House, it turned out that the measurement points in the double logarithmic graph are quite stable for a broad range hence indicating coherence for a broad range.

9.5 Conclusions and Outlook

Basically the fractal concept of architecture means that details of different sizes are kept together by a central rule or idea, respectively – avoiding monotony by using variation. In architecture, this concept is the reason why Gothic cathedrals and examples of the so-called organic architecture are so interesting and diversified. Modern architecture may also offer fractal properties but not for a broad range of scale. For measuring the presence and coherence of architectural elements across many levels of scale, the Box-Counting Method turned out to provide a first verifiable measurement method. The double logarithmic graph – grid-size vs. number of boxes covered – gives a first indication for similar density across certain scales. Although the resulting graph does not tell us anything about the quality of a building or about its form, it provides a first impression of the coherence between levels of scale as it is true for Robie House by Frank Lloyd Wright.

Future focus lies on further Box-Counting measurements mainly of different architectural styles for comparison of fractal-like works of architecture with representatives of Modern architecture. Those buildings where the single measurement points in the double logarithmic graph are very close to the replacing line will be analyzed more closely with regard to a possibly underlying Fractal concept. An interesting aspect will then, however, be in how far such a concept has some influence on architectural quality and on the acceptance of the building by observers.

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