Approximations in Concept Lattices*

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Abstract. Motivated by Rough Set Theory we describe an interval arithmetic on complete lattices. Lattice elements get approximated by *approximations* which are pairs consisting of a lower and an upper approximation. The approximations form a complete lattice again. We describe these lattices of approximations by formal contexts. Furthermore, we interpret the result for concept lattices as restricting the scope to a subcontext of *interesting* objects and attributes.

1 Introduction

Given a large, possibly infinite formal context (G, M, I) one usually has to handle a very large number of concepts which often yields to overloaded, unreadable order diagrams of the concept lattice. One practicable way to solve this problem is to use *nested line diagrams*. To build such a nested line diagram one splits the attribute set M into two not necessarily disjoint subsets M_1 and M_2, and embeds the concept lattice of (G, M, I) into the direct product of the concept lattices of the two subcontexts $(G, M_1, I \cap G \times M_1)$ and $(G, M_2, I \cap G \times M_2)$. The higher readability follows from the edge saving method to draw the direct product by copying the diagram of the second concept lattice into each node of the first one. Hence, when looking at the nested line diagram one has to look inside the big nodes when one is interested in the attributes from M_2, and one has to look at the *outer lattice* when one is interested in the attributes from M_1.

Just looking at the outer lattice of such a nested line diagram is equivalent to picking a subset $N \subseteq M$ of *interesting* attributes and looking at the concept lattice $\underline{\mathfrak{B}}(G, N, I \cap G \times N)$. What we are going to do is to additionaly pick a subset $H \subseteq G$ of *interesting* objects. Obviously, the restriction to the subcontext $(H, N, I \cap H \times N)$ yields to a smaller concept lattice, but one loses information about the interesting objects and attributes. An implication between interesting attributes that holds in (G, M, I) also holds in $(H, N, I \cap H \times N)$. But an implication $A \to B$ that holds in the subcontext does not necessarily have to hold in (G, M, I). One calls an object x *less general* than y in the context (G, M, I), if x has every attribute that y has. This gives rise to the object quasiorder defined by

$$x \sqsubseteq y \;:\Longleftrightarrow\; x^I \supseteq y^I.$$

If for $x, y \in H$ the object x is less general than y in (G, M, I), the object x is also less general than y in the extracted context $(H, N, I \cap H \times N)$. Hence, the

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object quasiorder of (H, N, J) is a quasiorder extension of the object quasiorder of (G, M, I) restricted to H.

In summary we make the unsurprising observation that the restriction to the subcontext $(H, N, I \cap H \times N)$ yields to the negative side effect of losing information from (G, M, I) about the interesting objects and attributes. In order to avoid these problems we describe a familiar, but slightly more sophisticated method of restricting ourselves to interesting objects and attributes. It is based on so-called *approximations* in the concept lattice of (G, M, I). Thereby, approximations are pairs of concepts consisting of a *lower* and an *upper* approximation.

2 Approximations

In this section we describe so-called *approximations* in complete lattices. Let $\mathbf{L} = (L, \leq)$ be a complete lattice and let K be a kernel system and let C be a closure system in $\mathbf{L}$, i.e., for every $S \subseteq K$ and $T \subseteq C$ it holds that $\bigvee S \in K$ and $\bigwedge T \in C$. Furthermore, let $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ be the respective kernel[1] and closure operators. Hence, for $x \in L$ it holds that

$$\lfloor x \rfloor = \bigvee\{k \in K \mid k \leq x\} \quad \text{and} \quad \lceil x \rceil = \bigwedge\{c \in C \mid x \leq c\}.$$

We call $\lfloor x \rfloor$ the **lower approximation** of x and $\lceil x \rceil$ the **upper approximation** of x. Furthermore, we call the pair $(\lfloor x \rfloor, \lceil x \rceil)$ the **approximation generated by** x.

As an example from Rough Set Theory one can take for $\mathbf{L}$ the powerset lattice $(\mathfrak{P}(U), \subseteq)$ where the set U is the so-called *universe*. The approximations result from an equivalence relation $\sim$ on U which usually describes *indiscernibility* of objects. For $X \subseteq U$ the approximations are defined as follows:

$$\lfloor X \rfloor := \{u \in U \mid \forall v \sim u : \ v \in X\},$$
$$\lceil X \rceil := \{u \in U \mid \exists v \sim u : \ v \in X\}.$$

In this example, the kernel system equals the closure system. They consist exactly of those subsets of U that are the union of $\sim$ equivalence classes, the so-called *crisp* sets. It is well known that in this example the generated approximations $(\lfloor X \rfloor, \lceil X \rceil)$ (with $X \subseteq U$) form a lattice if one orders them by component-wise set inclusion.

In the case where $\mathbf{L}$ is a powerset lattice and where $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ are arbitrary kernel and closure operators, the generated approximations do not necessarily form a lattice. In [2] Ganter suggested to investigate the complete sublattice of $K \times C$ that is *generated* by the generated approximations $(\lfloor X \rfloor, \lceil X \rceil)$. Hence, this sublattice of approximations might contain pairs that are not generated by a subset of U, see also [5]. It is now an obvious step forward to investigate this approach for arbitrary complete lattices $\mathbf{L}$. Since K and C form complete

[1] Instead of using the terms *kernel system* and *kernel operator* it is quite common to use *interior system* and *interior operator* instead.

lattices, also $K \times C$ does. Thereby, infimum and supremum of a subset $\{(k_t, c_t) \in K \times C \mid t \in T\}$ are given by

$$\bigvee_{t\in T}(k_t, c_t) = \Big(\bigvee_{t\in T} k_t, \lceil \bigvee_{t\in T} c_t \rceil\Big),$$
$$\bigwedge_{t\in T}(k_t, c_t) = \Big(\lfloor \bigwedge_{t\in T} k_t \rfloor, \bigwedge_{t\in T} c_t\Big).$$

Definition 1. *The complete sublattice $\Gamma := \Gamma_{K,C}$ of $K \times C$ that is generated by the pairs $(\lfloor x \rfloor, \lceil x \rceil)$ with $x \in L$ is called the lattice of* **approximations***. The kernel k is called the* **bottom** *and the closure c is called the* **top** *of an approximation (k, c).*

The notion of an approximation yields to an interval arithmetic on the complete lattice $\mathbf{L}$. The following Section 3 investigates so-called *maximal approximations* and the role of *complete tolerance relations*, which yields to a better understanding and to a further generalisation of the approximations. In [2] the author gave a contextual representation of Γ for the special case where $\mathbf{L}$ is a power set lattice $(\mathfrak{P}(U), \subseteq)$. He therefore used so-called P-products which are the formal concept analytic way to describe subdirect products of complete lattices. In Section 5 we propose a similar contextual representation for our more general case and discuss properties of the so-called *concept approximations* and of its representing context. Section 4 provides the needed notions and propositions from Formal Concept Analysis.

3 Maximal Approximations

Even though the approximations (k, c) are defined to be special pairs of lattice elements, one automatically interprets them as intervals

$$[k, c] = \{x \in L \mid k \leq x \leq c\}.$$

in $\mathbf{L}$. The reason for this interpretation is the simple fact that the bottom of an approximation is alway less or equal than the top. In other words one can say that $\Gamma_{K,C}$ is a subset of the order relation $\leq$. We say an approximation is **contained** in another one if its interpretation as an interval is a subset of the interval interpretation of the other approximation. Formally this **containment order** $\sqsubseteq$ on $\Gamma_{K,C}$ is defined by

$$(k_1, c_1) \sqsubseteq (k_2, c_2) :\Longleftrightarrow [k_1, c_1] \subseteq [k_2, c_2].$$

We call the maximal elements of the ordered set $(\Gamma_{K,C}, \sqsubseteq)$ the **maximal** approximations. Dually, we call an approximation **minimal** if it is a minimal element of $(\Gamma_{K,C}, \sqsubseteq)$. An approximation (k, c) with $k = c$ is called **crisp**. Obviously, crisp approximations are always minimal and minimal approximations are always generated by a lattice element. One can easily show using Zorn's lemma that every approximation is contained in a maximal approximation. We will receive this result as a byproduct of the observation that the maximal approximations interpreted as intervals are the *blocks* of a *complete tolerance relation* on $\mathbf{L}$.

Definition 2 ([3]). *A binary relation $\Theta \subseteq L \times L$ is called a* **complete tolerance relation** *on* $\mathbf{L}$ *if it is reflexive, symmetric and compatible with suprema and infima, i.e., for which $x_t \Theta y_t$ $(t \in T)$ always implies*

$$(\bigwedge_{t \in T} x_t)\Theta(\bigwedge_{t \in T} y_t) \quad \textit{and} \quad (\bigvee_{t \in T} x_t)\Theta(\bigvee_{t \in T} y_t).$$

Hence, a binary relation is a congruence relation iff it is transitive and a complete tolerance relation. If Θ is a complete tolerance relation on $\mathbf{L}$*, we define for $a \in L$*

$$a_\Theta := \bigwedge\{x \in L \mid a\Theta x\} \quad \textit{and} \quad a^\Theta := \bigvee\{x \in L \mid a\Theta x\}.$$

The intervals $[a]_\Theta := [a_\Theta, (a_\Theta)^\Theta]$ are called the **blocks** *of Θ.*

Proposition 1. *The blocks of a complete tolerance relation Θ are precisely the maximal subsets X of L with $x\Theta y$ for all $x, y \in X$.*

Proof. See [3] Proposition 55. □

In our setting of a given kernel system K and a given closure system C in $\mathbf{L}$ we get a canonical tolerance relation $\Theta_{K,C}$ by

$$x\, \Theta_{K,C}\, y \;:\Longleftrightarrow\; \exists (k, c) \in \Gamma_{K,C} : \; \{x, y\} \subseteq [k, c].$$

The following propositions clarify the role of this tolerance relation $\Theta_{K,C}$.

Proposition 2. *For $x, y \in L$ it holds that*

$$x\, \Theta_{K,C}\, y \iff (\lfloor x \wedge y \rfloor, \lceil x \vee y \rceil) \in \Gamma_{K,C}.$$

Hence, for $k \in K$ and $c \in C$ with $k \leq c$ it holds that

$$k\, \Theta_{K,C}\, c \iff (k, c) \in \Gamma_{K,C}.$$

In other words one can write

$$\Gamma_{K,C} \;=\; (K \times C) \;\cap \leq \cap\; \Theta_{K,C}.$$

Proof. The second statement directly follows from the first. The backward direction of the first statement holds trivially. Let $x\, \Theta_{K,C}\, y$. Hence, there is an approximation (k, c) with $\{x, y\} \subseteq [k, c]$. Then it holds that

$$(\lfloor x \wedge y \rfloor, \lceil x \vee y \rceil) = \Big((k, c) \vee (\lfloor x \wedge y \rfloor, \lceil x \wedge y \rceil)\Big) \,\wedge\, (\lfloor x \vee y \rfloor, \lceil x \vee y \rceil).$$

□

Proposition 3. *The relation $\Theta_{K,C}$ is a complete tolerance relation. The blocks of $\Theta_{K,C}$ are precisely the intervals $[k,c]$ where (k,c) is a maximal approximation.*

Proof. Obviously $\Theta_{K,C}$ is symmetric and reflexive. Let $(x_t, y_t) \in \Theta_{K,C}$ for $t \in T$. Then for every $t \in T$ there is an approximation (k_t, c_t) with $\{x_t, y_t\} \subseteq [k_t, c_t]$. Since

$$\bigvee_{t\in T}(k_t, c_t) = \Big(\bigvee_{t\in T} k_t, \lceil \bigvee_{t\in T} c_t \rceil\Big)$$

is an approximation with

$$\Big\{\bigvee_{t\in T} x_t, \bigvee_{t\in T} y_t\Big\} \subseteq \Big[\bigvee_{t\in T} k_t, \lceil \bigvee_{t\in T} c_t \rceil\Big]$$

it follows $(\bigvee_{t\in T} x_t)\, \Theta_{K,C}\, (\bigvee_{t\in T} y_t)$. Dually one shows that $\Theta_{K,C}$ is compatible with the infimum. Let $X := [k,c]$ be an interval belonging to a maximal approximation (k,c) and let y be a lattice element fulfilling $x\, \Theta_{K,C}\, y$ for all $x \in X$. We show that $y \in X$ follows which implies by Proposition 1 that X is a block. From Proposition 2 we get that $(\lfloor k \wedge y \rfloor, \lceil k \vee y \rceil)$ is an approximation. Hence, also

$$(k,c) \vee (\lfloor k \wedge y \rfloor, \lceil k \vee y \rceil) = (k, \lceil c \vee \lceil k \vee y \rceil \rceil)$$

is an approximation as well. Since (k,c) is maximal we infer $c = \lceil c \vee \lceil k \vee y \rceil \rceil$ which implies $y \leq c$. Dually one shows $k \leq y$. Altogether we get $y \in X$. For the backward direction one takes a block $X = [k,c]$ of $\Theta_{K,C}$ and shows that (k,c) is a maximal approximation. With Proposition 2 one can argue that k is a kernel, that c is a closure and that (k,c) is an approximation. The maximality of (k,c) follows from Proposition 1. □

For a given complete tolerance relation the least elements of the blocks always form a kernel system. Dually, the greatest elements form a closure system. These two systems are isomorphic to each other, which allows to define a canonical order on the blocks and to *factorise* **L** (see [3]). Obviously, the kernel and closure system given by the blocks of $\Theta_{K,C}$ are subsystems of K and C, respectively. We call Θ a (K,C)**-tolerance** on **L** if it is a complete tolerance relation on **L** satisfying $x_\Theta \in K$ and $x^\Theta \in C$ for every $x \in L$. The (K,C)-tolerances form a closure system on $L \times L$, i.e., they are closed under intersections.

Proposition 4. *The relation $\Theta_{K,C}$ is the smallest (K,C)-tolerance.*

Proof. By Proposition 2 it suffices to show that every approximation is contained in every (K,C)-tolerance, i.e., $\Gamma_{K,C} \subseteq \Theta$ for every (K,C)-tolerance Θ. One proves this by first showing that every generated approximation $(\lfloor x \rfloor, \lceil x \rceil)$ belongs to Θ. Afterwards one easily shows that Θ is closed under the supremum and infimum as it is defined on $K \times C$. □

One can think of the tolerance $\Theta_{K,C}$ as having the role to ensure that bottom and top of an approximation do not differ too much. If one wants to define on its own what *not too much* means one can use the following generalisation of the notion of an approximation.

Definition 3. *Let* $\mathbf{L}$ *be a complete lattice, let* K *be a kernel system in* $\mathbf{L}$*, let* C *be a closure system in* $\mathbf{L}$ *and let* Θ *be a* (K,C)*-tolerance. We put*

$$\Gamma_{K,C,\Theta} := (K \times C) \cap \leq \cap \Theta$$

and call the pairs from $\Gamma_{K,C,\Theta}$ *the* (K,C,Θ)**-approximations***. The notions* bottom, top, containment order, maximal, minimal *and* crisp *are defined analogously to the case of approximations where* $\Theta = \Theta_{K,C}$*.*

Proposition 5. $\Gamma_{K,C,\Theta}$ *is a complete sublattice of* $K \times C$*. For* $x, y \in L$ *it holds that*

$$x\Theta y \iff (\lfloor x \wedge y \rfloor, \lceil x \vee y \rceil) \in \Gamma_{K,C,\Theta}.$$

The blocks of Θ *are precisely the intervals* $[k,c]$ *where* (k,c) *is a maximal* (K,C,Θ)*-approximation.*

Proof. Let $(k_t, c_t) \in \Gamma_{K,C,\Theta}$ $(t \in T)$ and let

$$(k,c) := \bigvee_{t\in T}(k_t,c_t) = \Big(\bigvee_{t\in T} k_t, \lceil \bigvee_{t\in T} c_t \rceil\Big).$$

It obviously holds that $(k,c) \in K \times C \cap \leq$. Since Θ is a complete tolerance it follows $(\bigvee k_t, \bigvee c_t) \in \Theta$. Hence there is a block $[x,y]$ containing k and $\bigvee c_t$. Since furthermore Θ is a (K,C)-tolerance we infer $y \in C$ which implies $c \leq y$ and hence $(k,c) \in \Theta$. Dually one shows that $\Gamma_{K,C,\Theta}$ is closed under arbitrary infima. The rest can be shown similarly to the proofs of the Propositions 2 and 3. □

If $K = L$ and $C = L$, it follows $\Gamma_{K,C,\Theta} = \Theta \cap \leq$ for every complete tolerance relation Θ on $\mathbf{L}$. Thus, if one additionally chooses Θ to be the universal relation $L \times L$, it follows that the set of (K,C,Θ)-approximations equals the order relation of $\mathbf{L}$:

$$\Gamma_{K,C,\Theta} = \Gamma_{L,L,L\times L} = \leq .$$

4 Bonds and Block Relations

This section lists needed notions and propositions from Formal Concept Analysis. For a more detailed insight we refer the reader to [3]. Let $\mathbb{K} = (G,M,I)$ and $\mathbb{L} = (H,N,J)$ be formal contexts. A relation $B \subseteq G \times N$ is called a **bond** from $\mathbb{K}$ to $\mathbb{L}$ if every row of (G,N,B) is an intent of $\mathbb{L}$ and every column of (G,N,B) is an extent of $\mathbb{K}$. The set of all bonds from $\mathbb{K}$ to $\mathbb{L}$ is a closure system on $G \times N$. The respective closure operator is denoted by $(\cdot)^\beta$. Hence, for a relation $T \subseteq G \times N$ the closure T^β is the smallest bond from $\mathbb{K}$ to $\mathbb{L}$ that contains T.

Lemma 1. *For* $A \subseteq G$ *and* $B \subseteq N$ *it holds that*

$$A^{II} \times B^{JJ} \subseteq (A \times B)^\beta.$$

Proof. Let R be a bond from $\mathbb{K}$ to $\mathbb{L}$ with $A \times B \subseteq R$. It holds that $A^{II} \subseteq A^{RR}$ and $B \subseteq A^R$. Thus, it follows $B^{JJ} \subseteq A^{RJJ} = A^R$ and

$$A^{II} \times B^{JJ} \subseteq A^{RR} \times A^R \subseteq R.$$

□

The complete tolerance relations discussed in Section 3 have special bonds as its contextual counterpart, the so-called *block relations*. A relation $J \subseteq G \times M$ is called a **block relation** of the formal context (G, M, I) if it is a self-bond that contains I, i.e., if J is a bond from (G, M, I) to (G, M, I) with $I \subseteq J$.

Proposition 6. *The lattice of all block relations of (G, M, I) is isomorphic to the lattice of all complete tolerance relations on the concept lattice $\underline{\mathfrak{B}}(G, M, I)$. The map κ assigning to any complete tolerance relation Θ the block relation defined by*

$$g\kappa(\Theta)m :\Longleftrightarrow \gamma g\Theta(\gamma g \wedge \mu m) \quad (\Longleftrightarrow (\gamma g \vee \mu m)\Theta\mu m))$$

is an isomorphism. Conversely,

$$(A, B)\kappa^{-1}(J)(C, D) \Longleftrightarrow A \times D \cup C \times B \subseteq J$$

yields the complete tolerance to a block relation J.

Proof. See [3] Theorem 15. □

Corollary 1. *For a set U the lattice of all tolerance relations on the power set lattice $(\mathfrak{P}(U), \subseteq)$ is isomorphic to the lattice itself. The map τ assigning to any subset $X \subseteq U$ the complete tolerance relation $\tau(X)$ defined by*

$$(A, B) \in \tau(X) :\Longleftrightarrow A \cap X = B \cap X$$

is an isomorphism. Hence, every complete tolerance on a power set lattice is a congruence.

Proof. Follows from Proposition 6 since the block relations of the context $(U, U, \neq)$ are precisely the relations J_X with $X \subseteq U$ where

$$J_X := \{(x, y) \in U \times U \mid x = y \text{ implies } x \in X\}.$$

That τ is indeed an isomorphism is an elementary deduction from the definition of κ^{-1}. □

5 Concept Approximations

In this section we study the approximations from Section 2 on concept lattices. Obviously one can describe a kernel system K in a complete lattice by supremum-dense subsets of K, i.e., by subsets $T \subseteq K$ with

$$K = \{\bigvee S \mid S \subseteq T\}.$$

In the case of concept lattices we restrict ourselves to the kernel systems that are describable by object concepts. Since for a given concept it is always possible to extend the contexts object set in such a way that the concept is an object concept, this restriction is not a proper one regarding to the aim of describing arbitrary kernel systems in complete lattices. Dually we restrict ourself to closure systems given by subsets of the contexts attribute set.

Hence, we have the situation described in Section 1 where (G, M, I) is a **universal** context and where $(H, N, I \cap H \times N)$ is a subcontext called **selection**. Thereby the elements from H and from N are called the **interesting** objects and attributes, respectively. The subset $H \subseteq G$ yields to a kernel operator $\lfloor \cdot \rfloor_H$ on $\underline{\mathfrak{B}}(G, M, I)$ in the following canonical way:

$$\lfloor (A, B) \rfloor_H := ((A \cap H)^{II}, (A \cap H)^I).$$

Dually, $N \subseteq M$ yields to a closure operator via

$$\lceil (A, B) \rceil_N := ((B \cap N)^I, (B \cap N)^{II}).$$

In order to shorten our notations we define for sets A, B and for a relation R

$$R_{A,B} := R \cap A \times B.$$

Remark 1. For a concept $(A, B) \in \mathfrak{B}(G, M, I)$ the following three statements are equivalent:

(a) (A, B) is a kernel regarding to $\lfloor \cdot \rfloor_H$, i.e., $\lfloor (A, B) \rfloor_H = (A, B)$,
(b) B is an intent of $(H, M, I_{H,M})$,
(c) (A, B) is the supremum of object concepts γh with $h \in H$.

Dually, the following three statements are equivalent:

(d) (A, B) is a closure regarding to $\lceil \cdot \rceil_N$, i.e., $\lceil (A, B) \rceil_N = (A, B)$,
(e) A is an extent of $(G, N, I_{G,N})$,
(f) (A, B) is the infimum of attribute concepts μn with $n \in N$.

Hence, the kernel system K_H and the closure system C_N are the sets

$$K_H := \{(E^{II}, E^I) \mid E \subseteq H\} \text{ and}$$
$$C_N := \{(F^I, F^{II}) \mid F \subseteq N\}.$$

Ordered with the subconcept-superconcept order K_H and C_N are obviously isomorphic to the concept lattices of $(H, M, I_{H,M})$ and of $(G, N, I_{G,N})$, respectively. We denote the respective lattice of approximations with

$$\Gamma_{H,N} := \Gamma_{K_H, C_N}.$$

We call the pairs of concepts from $\Gamma_{H,N}$ **concept approximations**. For $E \subseteq H$ and $F \subseteq N$ we define

$$[\![E, F]\!] := ((E^{II}, E^I), (F^I, F^{II})).$$

Obviously, the pairs of the form $[\![E,F]\!]$ are exactly the pairs consisting of kernel in the first and a closure in the second component. It holds that

$$\begin{aligned} K_H \times C_N &= \{[\![E,F]\!] \mid E \subseteq H \text{ and } F \subseteq N\} \\ &= \{[\![E,F]\!] \mid E \in \mathrm{Ext}(H,M,I_{H,M}) \text{ and } F \in \mathrm{Int}(G,N,I_{G,N})\}, \end{aligned}$$

and for $E_t \in \mathrm{Ext}(H,M,I_{H,M})$ and $F_t \in \mathrm{Int}(G,N,I_{G,N})$ it holds that

$$\begin{aligned} \bigwedge_{t\in T} [\![E_t,F_t]\!] &= \Big[\!\Big[\bigcap_{t\in T} E_t, \big(\bigcup_{t\in T} F_t\big)^{II} \cap N\Big]\!\Big], \\ \bigvee_{t\in T} [\![E_t,F_t]\!] &= \Big[\!\Big[\big(\bigcup_{t\in T} E_t\big)^{II} \cap H, \bigcap_{t\in T} F_t\Big]\!\Big]. \end{aligned}$$

But which pairs of the form $[\![E,F]\!]$ are concept approximations? The bottom of a concept approximation $[\![E,F]\!]$ is a subconcept of the top, i.e, it holds that

$$(E^{II}, E^{I}) \leq (F^{I}, F^{II}).$$

This is equivalent to (E,F) being a preconcept, i.e., $E \times F \subseteq I$. But not all pairs $[\![E,F]\!]$ where (E,F) is a preconcept are concept approximations. Analogously to the approximations on complete lattices where certain complete tolerance relations played an important role, it will be certain block relations that play that role for the concept approximations. We call a relation J with $I \subseteq J \subseteq G \times M$ a (H,N)**-block relation** if it is a bond from $(G,N,I_{G,N})$ to $(H,M,I_{H,M})$. Hence, (H,N)-block relation are always block relations and the classical block relations are precisely the (G,M)-block relations.

Proposition 7. *The lattice of all (H,N)-block relations is isomorphic to the lattice of all (K_H,C_N)-tolerances on the concept lattice $\underline{\mathfrak{B}}(G,M,I)$. The map κ assigning to any (K_H,C_N)-tolerance Θ the (H,N)-block relation defined by*

$$g\kappa(\Theta)m :\iff \gamma g\Theta(\gamma g \wedge \mu m) \quad (\iff (\gamma g \vee \mu m)\Theta\mu m))$$

is an isomorphism. Conversely,

$$(A,B)\kappa^{-1}(J)(C,D) \iff A \times D \cup C \times B \subseteq J$$

yields the (K_H,C_N)-tolerance to a (H,N)-block relation J. For $A \subseteq G$ and $B \subseteq M$ the pair (A,B) is a concept of (G,M,J) if and only if $[(B^I,B),(A,A^I)]$ is a block of $\kappa^{-1}(J)$.

Proof. Obviously Proposition 7 is a mild generalisation of Proposition 6 and we just have to show that κ and κ^{-1} are well-defined. Let Θ be a (K_H,C_N)-tolerance and let $J := \kappa(\Theta)$ be the corresponding block relation. By [3] Corollary 57 the blocks of J are the intervals of the form $[(B^I,B),(A,A^I)]$ where (A,B) is a concept of (G,M,J). Hence, we get $(B^I,B) \in K_H$ and $(A,A^I) \in C_H$. By Remark 1 we get that B is an intent of $(H,M,I_{H,M})$ and that A is an extent of $(G,N,I_{G,N})$ for every $(A,B) \in \mathfrak{B}(G,M,J)$. Hence, J is a (H,N)-block relation.

Let now J be a (H, N)-block relation, let $\Theta := \kappa^{-1}(J)$ be the corresponding complete tolerance relation and let $(A, B) \in \mathfrak{B}(G, M, I)$. Then

$$(C, D) \ := \ (A, B)^{\Theta}$$

is the greatest concept from $\underline{\mathfrak{B}}(G, M, I)$ with $A \times D \cup C \times B \subseteq J$ which is equivalent to $A \times D \subseteq J$ and $C \times D \subseteq J$. The first condition holds trivially since (C, D) is a superconcept of (A, B) and hence $A \subseteq C = D^I \subseteq D^J$. Thus (C, D) is the greatest superconcept of (A, B) satisfying the second condition $C \times B \subseteq J$, which directly yields to $C = B^J$. Hence, $(C, D) = (B^J, B^{JI})$ is a closure from C_N because $B^J \in \mathrm{Ext}(G, N, I_{H,N})$. Dually one shows that $(A, B)_\Theta \in K_H$. The rest follows from [3] Corollary 57. □

Lemma 2. *The relation $R \subseteq G \times M$ defined by*

$$R \ := \bigcup_{(A,B) \in \mathfrak{B}(G,M,I)} (B \cap N)^I \times (A \cap H)^I$$

satisfies $I \subseteq R$ and $I^\beta = R^\beta$, where $(\cdot)^\beta$ denotes the bond closure operator for bonds from $(G, N, I_{G,N})$ to $(H, M, I_{H,M})$. Hence, R^β is the smallest (H, N)-block relation.

Proof. For $(g, m) \in I$ there is some $(A, B) \in \mathfrak{B}(G, M, I)$ with $(g, m) \in A \times B$. From $g \in A = B^I \subseteq (B \cap N)^I$ and $m \in B = A^I \subseteq (A \cap H)^I$ it follows $(g, m) \in R$.

Let T be a bond with $I \subseteq T$. We show $R \subseteq T$: For every $(A, B) \in \mathfrak{B}(G, M, I)$ it holds that

$$\begin{aligned}(B \cap N)^I \times (A \cap H)^I \ &= \ (A^I \cap N)^I \times (B^I \cap H)^I \\ &= \ A^{I_{G,N} I_{G,N}} \times B^{I_{H,M} I_{H,M}} \\ &\subseteq \ (A \times B)^\beta \\ &\subseteq \ I^\beta \ \subseteq \ T^\beta \ = \ T.\end{aligned}$$

Thereby the first inclusion follows from Lemma 1. Hence, a bond contains I iff it contains R. □

	M	N
H	$I_{H,M}$	$I_{H,N}$
G	I^β	$I_{G,N}$

Fig. 1. The context $\mathbb{A}_{H,N}$. Thereby I^β denotes the smallest bond from $(G, N, I_{G,N})$ to $(H, M, I_{H,M})$ containing I. For technical reasons we have to think of G and H as being replaced by disjoint copies. Analogously for M and N.

Theorem 1. *$\Gamma_{H,N}$ is isomorphic to the concept lattice of the context $\mathbb{A}_{H,N}$ displayed in Figure 1. An isomorphism is given by*

$$\varphi : \underline{\mathfrak{B}}(\mathbb{A}_{H,N}) \longrightarrow \Gamma_{H,N}$$
$$(A, B) \longmapsto [\![A \cap H, B \cap N]\!].$$

A pair of concepts $[\![E, F]\!]$ where $E \subseteq H$ and $F \subseteq N$ is a concept approximation if and only if $E \times F \subseteq I$ and $F^I \times E^I \subseteq I^\beta$.

Proof. One can prove this Theorem 1 by showing that $\mathbb{A}_{H,N}$ is the P-fusion of the two P-contexts $((H, M, I_{H,M}), \alpha_H)$ and $((G, N, I_{G,N}), \alpha_G)$, where $P := \mathfrak{B}(G, M, I)$,

$$\alpha_H(A, B) := (A \cap H, (A \cap H)^I) \quad \text{and} \quad \alpha_N(A, B) := ((B \cap N)^I, B \cap N).$$

For this approach one needs Lemma 2. For details regarding P-fusions and P-contexts see [3]. We leave out the details of the proof since Theorem 1 is a special case of Theorem 2. □

It turns out that the bonds I^β and $I_{H,N}$ correspond one-to-one to the maximal and to the minimal concept approximations, respectively. In order to describe this we have to refresh two basic notions from Formal Concept Analysis. The context $(H, N, I_{H,N})$ is called a **dense** subcontext of (G, M, I) if $\gamma[H]$ is $\bigvee$-dense and $\mu[N]$ is $\bigwedge$-dense in $\underline{\mathfrak{B}}(G, M, I)$. The context $(H, N, I_{H,N})$ is called a **compatible** subcontext of (G, M, I) if the pair $(A \cap H, B \cap N)$ is a concept of $(H, N, I_{H,N})$ for every concept (A, B) of (G, M, I).

Proposition 8. *The maximal concept approximations are precisely the pairs of the form*

$$\big((B^I, B), (A, A^I)\big) = [\![B^I \cap H, A^I \cap N]\!]$$

where $(A, B) \in \mathfrak{B}(G, M, I^\beta)$. The mapping $(A, B) \mapsto \big((B^I, B), (A, A^I)\big)$ is an order-embedding of $\underline{\mathfrak{B}}(G, M, I^\beta)$ into $\Gamma_{H,N}$. It is an isomorphism if and only if the selection $(H, N, I_{H,N})$ is a dense subcontext of (G, M, I).

Proof. The mentioned equivalence follows from [3] Corollary 57 (3.). Hence, the mapping is well-defined. That it is an order-embedding is elementary: for $(A_i, B_i) \in (G, M, I^\beta)$ it holds that

$$(A_1, B_1) \leq (A_2, B_2) \iff \big((B_1^I, B_1), (A_1, A_1^I)\big) \leq \big((B_2^I, B_2), (A_2, A_2^I)\big).$$

The rest is obvious, since $(H, N, I_{H,N})$ is dense in (G, M, I) iff $K_H = C_N = \mathfrak{B}(G, M, I)$ holds. □

Proposition 9. *For every $(E, F) \in \mathfrak{B}(H, N, I_{H,N})$ the pair $[\![E, F]\!]$ of concepts is a concept approximation. The approximations of the form $[\![E, F]\!]$ where (E, F) is a concept of $(H, N, I_{H,N})$ are precisely the minimal concept approximations. Furthermore, an approximation $[\![E, F]\!]$ is crisp if and only if $(F^I, E^I) \in \mathfrak{B}(G, M, I)$.*

Proof. Let $(E,F) \in \mathfrak{B}(H,N,I_{H,N})$. Using Theorem 1 it suffices to show that $E \times F \subseteq I$ and $F^I \subseteq E^I \subseteq I^\beta$. The first item obviously holds. The second follows from Lemma 1:

$$F^I \times E^I = (E^I \cap N)^I \times (F^I \cap H)^I = E^{I_{G,N} I_{G,N}} \times F^{I_{H,M} I_{H,M}} \subseteq (E \times F)^\beta \subseteq I^\beta.$$

Let $[\![E,F]\!]$ be a concept approximation. W.l.o.g. we assume $E \in \operatorname{Ext}(H,M,I_{H,M})$ and $F \in \operatorname{Int}(G,N,I_{G,N})$. If $[\![E,F]\!]$ is not minimal, there is an approximation $[\![Q,R]\!]$ with

$$(E^{II},E^I) \leq (Q^{II},Q^I) \leq (R^I,R^{II}) \leq (F^I,F^{II})$$

where at most one of the two outer inequations is a proper $<$. Hence, it follows $E \subsetneq Q$ or $F \subsetneq R$ which implies $E \times F \subsetneq Q \times R \subseteq I_{H,N}$. Thus, (E,F) is not a concept of $(H,N,I_{H,N})$. Let us otherwise suppose that $[\![E,F]\!]$ is minimal and that $(E,F) \notin \mathfrak{B}(H,N,I_{H,N})$. Then there is a concept $(Q,R) \in \mathfrak{B}(H,N,I_{H,N})$ with $E \times F \subsetneq Q \times R$. We know from above that $[\![Q,R]\!]$ is an approximation. It holds that $Q^I \subseteq E^I$ and $R^I \subseteq F^I$. Equality of the first subset relationship implies

$$E = E^{I_{H,M} I_{H,M}} = E^{II} \cap H = Q^{II} \cap H = Q.$$

Dually, the equality $F^I = R^I$ implies $F = R$. Hence, $[\![Q,R]\!]$ is a concept approximation that is properly contained in $[\![E,F]\!]$. But this contradicts the minimality of $[\![E,F]\!]$. The characterisation of the crisp concept approximations directly follows from $[\![E,F]\!] = \big((E^{II},E^I),(F^I,F^{II})\big)$. □

Proposition 10. *The subcontext* $(H,N,I_{H,N})$ *is dense in* $\mathbb{A}_{H,N}$ *if and only if it is a compatible subcontext of* (G,M,I).

Proof. Lemma 2 from [4] says that $(H,N,I_{H,N})$ is dense in $\mathbb{A}_{H,N}$ iff the following three equations hold:

(i) $\operatorname{Ext}(H,M,I_{H,M}) = \operatorname{Ext}(H,N,I_{H,N})$,
(ii) $\operatorname{Int}(G,N,I_{G,N}) = \operatorname{Int}(H,N,I_{H,N})$, and
(iii) $I^\beta = \bigcup\{F^I \times E^I \mid (E,F) \in \mathfrak{B}(H,N,I_{H,N})\}$.

Let $(H,N,I_{H,N})$ be a compatible subcontext of (G,M,I). Then (i) and (ii) obviously hold. Furthermore, for the relation R from *our* Lemma 2 it holds that

$$R = \bigcup_{(E,F)\in\mathfrak{B}(H,N,I_{H,N})} F^I \times E^I.$$

If we show that R already is a bond, it follows (iii). For $m \in M$ it holds that

$$m^R = \bigcup\{F^I \mid (E,F) \in \mathfrak{B}(H,N,I_{H,N}) \text{ and } m \in E^I\} = (m^I \cap H)^{I_{G,N} I_{G,N}}.$$

The second equality follows from the fact that $(m^I \cap H, (m^I \cap H)^I \cap N)$ is a concept of $(H,N,I_{H,N})$ with the property that for every $(E,F) \in \mathfrak{B}(H,N,I_{H,N})$ with $m \in E^I$ it holds that

$$F^I \subseteq ((m^I \cap H)^I \cap N)^I = (m^I \cap H)^{I_{G,N} I_{G,N}}.$$

Dually one shows that g^R is an intent of $(H, M, I_{H,M})$. Hence, R is indeed a bond. Let now $(H, N, I_{H,N})$ be a subcontext fulfilling (i), (ii) and (iii). We show that $(H, N, I_{H,N})$ is a compatible subcontext of (G, M, I) by applying Proposition 35 from [3]. Analogously to the previously mentioned one gets from (iii) that $m^{I^\beta} = (m^I \cap H)^{I_{G,N} I_{G,N}}$. For $n \in N$ it follows

$$n^I \subseteq n^{I^\beta} = ((n^I \cap H)^I \cap N)^I \subseteq (n^{II} \cap N)^I = n^I,$$

which yields to $n^{I^\beta} = n^I$. Let $h \in H$ and $m \in M$ with $(h, m) \notin I$. Then by (i) there is an attribute $n \in N$ with $m^I \cap H \subseteq m^I \cap H$ and $(h, m) \notin I$. This implies

$$m^I \subseteq m^{I^\beta} = (m^I \cap H)^{I_{G,N} I_{G,N}} \subseteq (n^I \cap H)^{I_{G,N} I_{G,N}} = n^I.$$

Dually one can prove that $(H, N, I_{H,N})$ also fulfills the second condition of [3] Proposition 35. □

Proposition 11. *There is a natural embedding of* $\underline{\mathfrak{B}}(H, N, I_{H,N})$ *into* $\Gamma_{H,N}$. *The mapping*

$$\begin{aligned} \psi : \underline{\mathfrak{B}}(H, N, I_{H,N}) &\longrightarrow \Gamma_{H,N} \\ (E, F) &\longmapsto [\![E, F]\!] \end{aligned}$$

is an order embedding. Furthermore, ψ *is an isomorphism if and only if* $(H, N, I_{H,N})$ *is a compatible subcontext of* (G, M, I).

Proof. That ψ is well-defined follows from Proposition 9. Obviously ψ is order-preserving. That it is also order-reversing is elementary: for two concepts (E_1, F_1) and (E_2, F_2) of $(H, N, I_{H,N})$ with $\psi(E_1, F_1) \leq \psi(E_2, F_2)$ it follows $F_1^I \subseteq F_2^I$ which implies $E_1 = F_1^I \cap H \subseteq F_2^I \cap H = E_2$. Thus it follows $(E_1, F_1) \leq (E_2, F_2)$. The inverse isomorphism of φ from Theorem 1 maps every concept approximation $[\![E, F]\!]$ where w.l.o.g. E is an extent of $(H, M, I_{H,M})$ and F is an intent of $(G, N, I_{G,N})$ to the concept

$$\varphi^{-1}([\![E, F]\!]) := (E \uplus F^I, E^I \uplus F)$$

of $\mathbb{A}_{H,N}$. Hence, in order to finish our proof it suffices to show by Proposition 10 that $(H, N, I_{H,N})$ is a dense subcontext of $\mathbb{A}_{H,N}$ if and only if the mapping

$$\begin{aligned} \chi : \underline{\mathfrak{B}}(H, N, I_{H,N}) &\longrightarrow \underline{\mathfrak{B}}(\mathbb{A}_{H,N}) \\ (E, F) &\longmapsto (E \uplus F^I, E^I \uplus F) \end{aligned}$$

is surjective. Let $(H, N, I_{H,N})$ be dense in $\mathbb{A}_{H,N}$ and let $(E \uplus F^I, E^I \uplus F)$ be an arbitrary concept of $\mathbb{A}_{H,N}$. We have to show that (E, F) is a concept of $(H, N, I_{H,N})$. Let $\boxplus$ denote the incidence relation of $\mathbb{A}_{H,N}$. Since $(H, N, I_{H,N})$ is dense it follows

$$E \uplus F^I = (E \uplus F^I)^{\boxplus\boxplus} = E^{\boxplus\boxplus},$$

which implies

$$E^I \uplus F = (E \uplus F^I)^{\boxplus} = E^{\boxplus} = E^I \uplus E^{I_{H,N}}.$$

Hence, it follows $F = E^{I_{H,N}}$. Dually one shows $F^{I_{H,N}} = E$. Let now χ be surjective and let (A, B) be a concept of $\mathbb{A}_{H,N}$. Then there is a concept (E, F) of $(H, N, I_{H,N})$ with $(A, B) = (E \uplus F^I, E^I \uplus F)$. It follows that

$$(A \cap H)^{\boxplus} = E^{\boxplus} = E^I \uplus E^{I_{H,N}} = E^I \uplus F = B = A^{\boxplus}.$$

Dually one shows $(B \cap N)^{\boxplus} = B^{\boxplus}$. Hence, by [3] Propostion 39 $(H, N, I_{H,N})$ is a dense subcontext of $\mathbb{A}_{H,N}$. □

In the following we answer the question on how to integrate the further generalised (K, C, Θ)-approximations from Section 3 in the previously described contextual representation. As Theorem 2 will show, the obvious answer is to replace the block relation I^β by arbitrary (H, N)-block relations.

Definition 4. *Let J be a (H, N)-block relation and let $\Theta_J := \kappa^{-1}(J)$ be the corresponding (K_H, C_N)-tolerance (see Proposition 7). We put*

$$\Gamma_{H,N,J} := \Gamma_{K_C,H_N,\Theta_J}$$

and call the pairs from $\Gamma_{H,N,J}$ the (H, N, J)**-approximations***.*

Increasing the block relation J obviously yields to greater but viewer maximal approximations, which again yields to increasing $\Gamma_{H,N,J}$.

	M	N
H	$I_{H,M}$	$I_{H,N}$
G	J	$I_{G,N}$

Fig. 2. The context $\mathbb{A}_{H,N,J}$, where J is a (H, N)-block relation

Theorem 2. *$\Gamma_{H,N,J}$ is isomorphic to the concept lattice of the context $\mathbb{A}_{H,N,J}$ displayed in Figure 2. An isomorphism is given by*

$$\varphi : \underline{\mathfrak{B}}(\mathbb{A}_{H,N,J}) \longrightarrow \Gamma_{H,N,J}$$
$$(A, B) \longmapsto [\![A \cap H, B \cap N]\!]$$

The inverse isomorphism is the mapping that maps every (H, N, J)-approximation $[\![E, F]\!]$ where w.l.o.g. $E \in Ext(H, M, I_{H,M})$ and $F \in Int(G, N, I_{G,N})$ to

$$\varphi^{-1}([\![E, F]\!]) := (E \uplus F^I, E^I \uplus F).$$

	◇	♡	♠	♣	7	8	9	Q	K	10	A	◇J	♡J	♠J	♣J
◇7	×				×	×	×	×	×	×	×	×	×	×	×
◇8	×					×	×	×	×	×	×	×	×	×	×
◇9	×						×	×	×	×	×	×	×	×	×
◇Q	×							×	×	×	×	×	×	×	×
◇K	×								×	×	×	×	×	×	×
◇10	×									×	×	×	×	×	×
◇A	×										×	×	×	×	×
♡7		×			×	×	×	×	×	×	×	×	×	×	×
♡8		×				×	×	×	×	×	×	×	×	×	×
♡9		×					×	×	×	×	×	×	×	×	×
♡Q		×						×	×	×	×	×	×	×	×
♡K		×							×	×	×	×	×	×	×
♡10		×								×	×	×	×	×	×
♡A		×									×	×	×	×	×
♠7			×		×	×	×	×	×	×	×	×	×	×	×
♠8			×			×	×	×	×	×	×	×	×	×	×
♠9			×				×	×	×	×	×	×	×	×	×
♠Q			×					×	×	×	×	×	×	×	×
♠K			×						×	×	×	×	×	×	×
♠10			×							×	×	×	×	×	×
♠A			×								×	×	×	×	×
♣7				×	×	×	×	×	×	×	×	×	×	×	×
♣8				×		×	×	×	×	×	×	×	×	×	×
♣9				×			×	×	×	×	×	×	×	×	×
♣Q				×				×	×	×	×	×	×	×	×
♣K				×					×	×	×	×	×	×	×
♣10				×						×	×	×	×	×	×
♣A				×							×	×	×	×	×
◇J												×	×	×	×
♡J													×	×	×
♠J														×	×
♣J															×

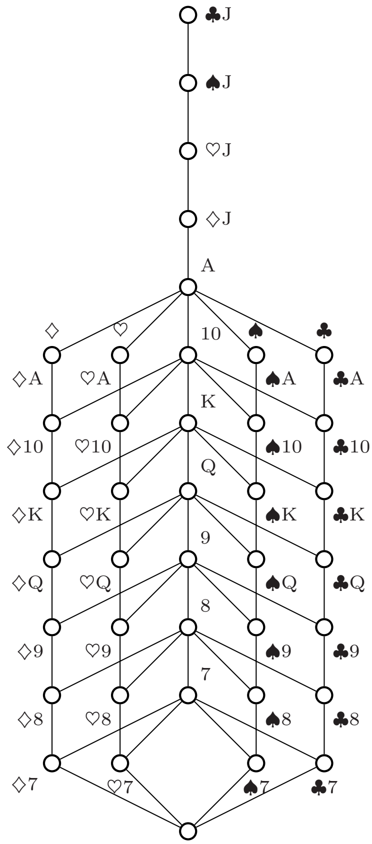

Fig. 3. The formal context (G, M, I) to the Skat example. Thereby Q stands for *queen*, K for *king*, A for *ace* and J for *jack*. The corresponding concept lattice has 40 concepts and is displayed on the right side. Note that the object concept of ♣J equals its attribute concept. The same holds for the other jacks. Hence, we do not have to label these concepts twice.

Proof. By [3] Theorem 32 the incidence relation $\boxplus$ of $\mathbb{A}_{H,N}$ is a closed subrelation in the sum context of $(H, M, I_{H,M})$ and $(G, N, I_{G,N})$. Hence, $\mathfrak{B}(\mathbb{A}_{H,N,J})$ is isomorphic to a complete sublattice $\mathfrak{S}$ of the direct product of $\underline{\mathfrak{B}}(H, M, I_{H,M})$ $(\cong K_H)$ and of $\underline{\mathfrak{B}}(G, N, I_{G,N})$ $(\cong C_N)$. An isomorphism is given via (see [3] Theorem 31)

$$\begin{aligned}\widehat{\varphi} : \underline{\mathfrak{B}}(\mathbb{A}_{H,N,J}) &\longrightarrow \mathfrak{S}\\ (A, B) &\longmapsto \big((A \cap H, B \cap M), (A \cap G, B \cap N)\big)\\ &\quad = \big((A \cap H, (A \cap H)^I), ((B \cap N)^I, B \cap N)\big)\end{aligned}$$

																◇	♡	♠	♣	Q	10	♠J
◇7	×				×	×	×	×	×	×	×	×	×	×	×	×				×	×	×
◇Q	×							×	×	×	×	×	×	×	×	×				×	×	×
♡7		×			×	×	×	×	×	×	×	×	×	×	×		×			×	×	×
♡Q		×						×	×	×	×	×	×	×	×		×			×	×	×
♠7			×		×	×	×	×	×	×	×	×	×	×	×			×		×	×	×
♠Q			×					×	×	×	×	×	×	×	×			×		×	×	×
♣7				×	×	×	×	×	×	×	×	×	×	×	×				×	×	×	×
♣Q				×				×	×	×	×	×	×	×	×				×	×	×	×
♡J													×	×	×							×
♠J														×	×							×
	×				×	×	×	×	×	×	×	×	×	×	×	×				×	×	×
	×				⊗	×	×	×	×	×	×	×	×	×	×	×				×	×	×
	×				⊗	⊗	×	×	×	×	×	×	×	×	×	×				×	×	×
	×				⊗	⊗	⊗	×	×	×	×	×	×	×	×	×				×	×	×
	×							⊗	×	×	×	×	×	×	×	×					×	×
	×							⊗	⊗	×	×	×	×	×	×	×					×	×
	×							⊗	⊗	⊗	×	×	×	×	×	×						×
		×			×	×	×	×	×	×	×	×	×	×	×		×			×	×	×
		×			⊗	×	×	×	×	×	×	×	×	×	×		×			×	×	×
		×			⊗	⊗	×	×	×	×	×	×	×	×	×		×			×	×	×
		×			⊗	⊗	⊗	×	×	×	×	×	×	×	×		×			×	×	×
		×						⊗	×	×	×	×	×	×	×		×				×	×
		×						⊗	⊗	×	×	×	×	×	×		×				×	×
		×						⊗	⊗	⊗	×	×	×	×	×		×					×
			×		×	×	×	×	×	×	×	×	×	×	×			×		×	×	×
			×		⊗	×	×	×	×	×	×	×	×	×	×			×		×	×	×
			×		⊗	⊗	×	×	×	×	×	×	×	×	×			×		×	×	×
			×		⊗	⊗	⊗	×	×	×	×	×	×	×	×			×		×	×	×
			×					⊗	×	×	×	×	×	×	×			×			×	×
			×					⊗	⊗	×	×	×	×	×	×			×			×	×
			×					⊗	⊗	⊗	×	×	×	×	×			×				×
				×	×	×	×	×	×	×	×	×	×	×	×				×	×	×	×
				×	⊗	×	×	×	×	×	×	×	×	×	×				×	×	×	×
				×	⊗	⊗	×	×	×	×	×	×	×	×	×				×	×	×	×
				×	⊗	⊗	⊗	×	×	×	×	×	×	×	×				×	×	×	×
				×				⊗	×	×	×	×	×	×	×				×		×	×
				×				⊗	⊗	×	×	×	×	×	×				×		×	×
				×				⊗	⊗	⊗	×	×	×	×	×				×			×
								⊗	⊗	⊗	⊗	×	×	×	×							×
								⊗	⊗	⊗	⊗	⊗	×	×	×							×
								⊗	⊗	⊗	⊗	⊗	⊗	×	×							×
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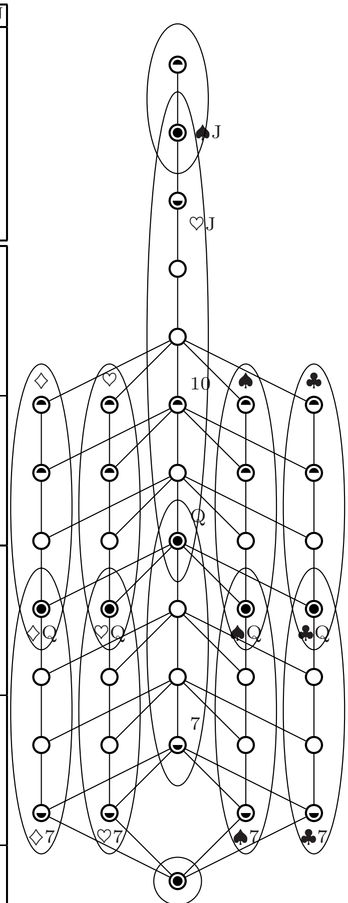

Fig. 4. The context $\mathbb{A}_{H,N}$ for our Skat example. We just labelled the interesting objects and attributes, which causes no problems since every concept approximation is of the form $[\![E, F]\!]$ with $E \subseteq H$ and $F \subseteq N$. The circled crosses $\otimes$ mark the pairs from I^β that do not belong to I. Note that the concept lattice of (G, M, I^β) is isomorphic to the lattice of all blocks; see Proposition 7. The right side shows a diagram of the concept lattice of (G, M, I), where just the interesting objects and attributes are labelled. The nodes having a filled lower half correspond to kernels. Closures are labelled by nodes that have a filled upper half. The 12 ellipse correspond to the blocks and hence to the maximal concept approximations.

Hence, for $E \in \text{Ext}(H, M, I_{H,M})$ and $F \in \text{Int}(G, N, I_{G,N})$ it holds that

$$\big((E, E^I), (F^I, F)\big) \in \mathfrak{S} \iff (E \uplus F^I, E^I \uplus F) \in \mathfrak{B}(\mathbb{A}_{H,N,J})$$
$$\iff E \times F \subseteq I_{H,N} \text{ and } F^I \times E^I \subseteq J$$

$$\begin{aligned}&\iff (E^{II}, E^{I}) \leq (F^{I}, F^{II}) \text{ and}\\ &\quad\;\; (E^{II}, E^{I})\kappa^{-1}(J)(F^{I}, F^{II})\\ &\iff [\![E, F]\!] \in \Gamma_{H,N,J}.\end{aligned}$$

Thereby the equivalence prior to the last one follows from Proposition 7 with the help of Lemma 1. Hence, $\mathfrak{S}$ is isomorphic to $\Gamma_{H,N,J}$ and the mapping φ indeed is an isomorphism. □

The attribute implications $A \to B$ with $A, B \subseteq N$ that hold in $\mathbb{A}_{H,N,J}$ are precisely the implications that hold in $(G, N, I_{G,N})$. Hence, these implications are precisely the attribute implications between interesting attributes that hold in the universal context (G, M, I). The dual statements holds for the interesting objects.

We close this section with a corollary from Proposition 7. It is a characterisation of the (K, C, Θ)-approximations for the case where $\mathbf{L}$ is a power set lattice.

Corollary 2. *Let $\mathcal{K}$ be a kernel system and $\mathcal{C}$ be a closure system* on *a set U. This means that $\mathcal{K}$ and $\mathcal{C}$ are a kernel system and a closure system* in *the power set lattice $\mathbf{L} := (\mathfrak{P}(U), \subseteq)$. Furthermore, let*

$$R := \{u \in U \mid \{u\} \in \mathcal{K} \textit{ and } U \setminus \{u\} \in \mathcal{C}\}$$

be the set of so-called **robust** *elements. Then the following statements hold:*

(1) A pair $(X, Y) \in \mathcal{K} \times \mathcal{C}$ is an approximation iff X is a subset of Y and $Y \setminus X$ does not contain a robust element.

(2) Let $\tau(S)$ for $S \subseteq U$ be the complete congruence relation on $\mathbf{L}$ from Corollary 1 defined by

$$(A, B) \in \tau(S) \;:\!\iff\; A \cap S = B \cap S.$$

Then $\tau(S)$ is a $(\mathcal{K}, \mathcal{C})$-tolerance on $\mathbf{L}$ if and only if $S \subseteq R$. Since every complete tolerance relation on $\mathbf{L}$ is of the form $\tau(S)$ this characterises the $(\mathcal{K}, \mathcal{C})$-tolerances.

	♢	♡	♠	♣	Q	10	♠J
♢7	×				×	×	×
♢Q	×				×	×	×
♡7		×			×	×	×
♡Q		×			×	×	×
♠7			×		×	×	×
♠Q			×		×	×	×
♣7				×	×	×	×
♣Q				×	×	×	×
♡J							×
♠J							×

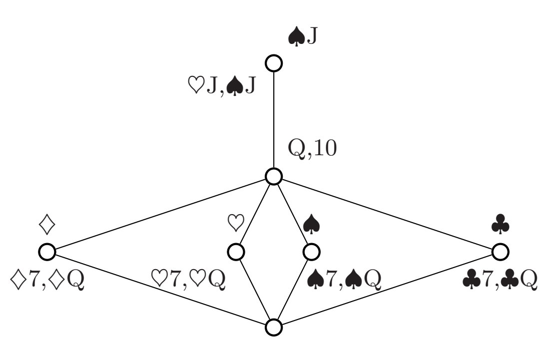

Fig. 5. The selection $(H, N, I_{H,N})$ and its concept lattice $\underline{\mathfrak{B}}(H, N, I_{H,N})$

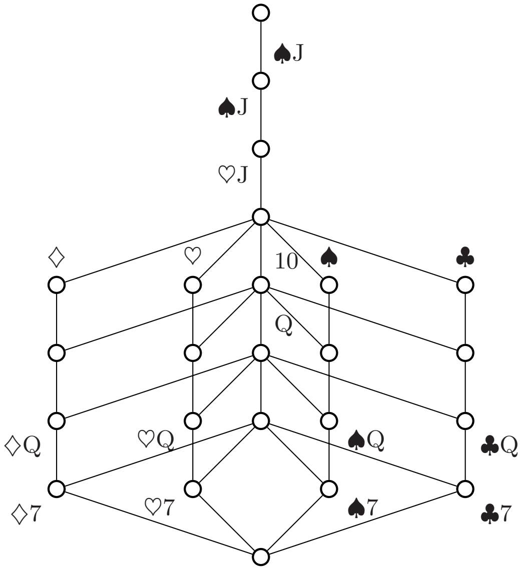

Fig. 6. The lattice of approximations $\Gamma_{H,N}$ for our Skat example. The corresponding context $\mathbb{A}_{H,N}$ is displayed in Figure 4. One reads the diagram as follows. Obviously the nodes represent the concept approximations $[\![E,F]\!]$. Similar to the reduced labelling of concept lattices, the elements from E are precisely the objects whose label can be found on the nodes below $[\![E,F]\!]$. Thereby *below* means that one can reach this node by going downwards along line paths in the diagram. Dually, the attributes from F are precisely the attributes labelling nodes above. As an example we take a look at the one unlabelled node at the very right. It represents the concept approximation $[\![E,F]\!] = \big((E^{II},E^{I}),(F^{I},F^{II})\big)$ with $E = \{$♣7, ♣Q$\}$ and $F = \{$10, ♣, ♣J$\}$.

(3) Let $S \subseteq R$. Then a pair $(X,Y) \in \mathcal{K} \times \mathcal{C}$ is a $(\mathcal{K},\mathcal{C},\tau(S))$-approximation iff $X \subseteq Y$ and $(Y \setminus X) \cap S = \emptyset$.

Proof. Statement (1) is from [5]. It is a special case of (3): By (2) $\tau(R)$ is the smallest $(\mathcal{K},\mathcal{C})$-tolerance and hence by Proposition 4 it holds that $\Theta_{\mathcal{K},\mathcal{C}} = \tau(R)$. Statement (3) follows from (2) since for $X \subseteq Y$ the equivalences

$$(X,Y) \in \tau(S) \iff X \cap S \supseteq Y \cap S \iff X \subseteq Y \cap S \iff (Y \setminus X) \cap S = \emptyset$$

hold. For $S \subseteq U$ the blocks of $\tau(S)$ are precisely the intervals of the form

$$[T,(U \setminus S) \cup T] \;=\; [T, U \setminus (S \setminus T)],$$

where $T \subseteq S$. If $S \subseteq R$ it follows $T \in \mathcal{K}$ and $U \setminus T \in \mathcal{C}$ for every $T \subseteq S$. Hence, $\tau(S)$ is a $(\mathcal{K},\mathcal{C})$-tolerance. If we otherise assume that $\tau(S)$ is a $(\mathcal{K},\mathcal{C})$-tolerance, it follows that $\{x\} \in \mathcal{K}$ (put $T := \{x\}$) and $U \setminus \{x\} \in \mathcal{C}$ (put $T := S \setminus \{x\}$) for every $x \in S$. □

6 An Example

Our example is a toy example. It deals with the German card game Skat. Skat is a three player game that is played with a card deck consisting of 32 cards. These 32 cards are the objects of the context (G, M, I) displayed in Figure 3. The attributes and the incidence relation are chosen in such a way that the object quasiorder reflects the cards standard hierarchy. This means that for two cards x and y it holds

$$x^I \supseteq y^I$$

if and only if card y *beats* card x. With *standard* we mean that just the four jacks are trump. Hence, one can think – with one little exception – of (G, M, I) as a scaled context resulting from a many-valued context with the two attributes *suite* and *value*. Thereby the values *diamonds* $\diamondsuit$*, hearts* $\heartsuit$*, spades* $\spadesuit$ and *clubs* $\clubsuit$ of the attribute *suit* are scaled nominally. The values of the second attribute *value* are scaled ordinally with the exception of the jacks. A jack is always trump which means that this card is above every non-jack in the cards hierarchy. The reader should note that furthermore the 10 beats the king of the same suit.

To start a Skat game each of the three players receives ten playing cards. The two remaining cards form the so-called *skat* and the player who wins the *bidding* process is allowed to use these two additional cards to build an improved combination of ten cards to play against his two opponents. From a players point of view the subset H of interesting cards might for instance be the ten cards he received at the beginning. Or maybe the interesting objects are the twelve cards he owns after winning the bidding for the skat. In order to receive a small lattice $\Gamma_{H,N}$ of approximations we chose the pretty regular set of playing cards

$$H := \{\diamondsuit 7,\ \diamondsuit \mathrm{Q},\ \heartsuit 7,\ \heartsuit \mathrm{Q},\ \spadesuit 7,\ \spadesuit \mathrm{Q},\ \clubsuit 7,\ \clubsuit \mathrm{Q},\ \heartsuit \mathrm{J},\ \spadesuit \mathrm{J}\}.$$

The choice of N might appear artificial, too. We took

$$N := \{\diamondsuit,\ \heartsuit,\ \spadesuit,\ \clubsuit,\ \mathrm{Q},\ 10,\ \spadesuit \mathrm{J}\}$$

which can be interpreted as coarsening the scale. The player might just be interested in the following questions: What is the suit of a given card? Is it weaker or equal than a queen or a 10 (of the same suit)? Is it weaker or equal than the jack of spades? The resulting selection $(H, N, I_{H,N})$ and its concept lattice is displayed in Figure 5. The Figures 4 and 6 show the context $\mathbb{A}_{H,N}$ and the corresponding lattice of approximations $\Gamma_{H,N}$.

Note that in our example the block relation I^β is relatively small, which yields to a relatively small number of concept approximations. It is for instance possible that the number of approximations exceeds the number of concepts of (G, M, I). But since the inequality

$$|\Gamma_{H,N}| \le |\mathfrak{B}(H, M, I_{H,M})| \cdot |\mathfrak{B}(G, N, I_{G,N})|$$

trivially holds, it follows that relatively small subcontexts tend to result in lattices of approximations that are noticeably smaller than the concept lattice $\underline{\mathfrak{B}}(G, M, I)$.

7 Conclusion

We introduced and discussed approximations in complete lattices and described them via formal contexts. Furthermore, we interpreted the result as restricting the view from a formal context to a subcontext without losing implicational knowledge about the selected objects and attributes.

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