Chapter 6

Branch Points

In $\mathbb{R}^3$ any solution of Plateau's problem minimizing Dirichlet's integral D or, equivalently, the area functional A, is an immersion in the sense that it has no interior branch points. This fact can easily be proved for planar boundaries as we have seen earlier, while the corresponding result in $\mathbb{R}^n$ is false for $n \geq 4$ according to Federer's counterexample. Therefore it remains to prove the assertion for *nonplanar* minimizers. Here we describe a new method, due to A. Tromba, to exclude interior branch points for nonplanar relative minimizers of Dirichlet's integral D. This method is based on the observation that one can compute any higher derivative of Dirichlet's integral in the direction of so-called *(interior) forced Jacobi fields*, using methods of complex analysis such as power series expansions and Cauchy's integral theorem as well as the residue theorem. These Jacobi fields lie in the kernel of the second variation of D; they also play a fundamental role in the index theory and the Morse theory of minimal surfaces.

We begin by calculating the first five derivatives of Dirichlet's integral in the direction of special types of forced Jacobi fields, thereby establishing that relative minimizers of D cannot have certain kinds of interior branch points. These introductory calculations will be carried out in Section 6.1, together with an outline of the variational procedure to be used in the sequel. These calculations are made transparent by shifting the branch point that is studied into the origin, and by bringing the minimal surface into a *normal form* with respect to the branch point $w = 0$ with an *order* n. Then also the *index* m of this branch point can be defined, with $m > n$. Furthermore, $w = 0$ is called an *exceptional branch point* if there is an integer $\kappa > 1$ such that $m + 1 = \kappa(n + 1)$. It turns out that Tromba's method works perfectly in excluding nonexceptional branch points of relative minimizers of D, while the exclusion of exceptional branch points only succeeds for absolute minimizers of the area A in $\mathcal{C}(\Gamma)$. Since the general investigation is quite lengthy, we only discuss one of the several general cases that are possible for nonexceptional

U. Dierkes, S. Hildebrandt, A.J. Tromba, *Regularity of Minimal Surfaces*,
Grundlehren der mathematischen Wissenschaften 340,
DOI 10.1007/978-3-642-11700-8_6,

branch points (see Section 6.2). A comprehensive presentation of the method for all cases will be given in forthcoming work by A. Tromba.

In Section 6.1 it is described how the variations $\hat{Z}(t)$ of a minimal surface $\hat{X}$ are constructed by using interior forced Jacobi fields. This leads to the (rather weak) notion of a *weak minimizer of* D. Any absolute or weak relative minimizer of D in $\mathcal{C}(\Gamma)$ will be a weak D-minimizer, and the aim is to investigate whether such minimizers can have $w = 0$ as in interior branch point. This possibility is excluded if one can find an integer $L \geq 3$ and a variation $\hat{Z}(t)$ of $\hat{X}$, $|t| \ll 1$, such that $E(t) := D(\hat{Z}(t))$ satisfies

$$E^{(j)}(0) = 0 \quad \text{for } 1 \leq j \leq L-1, \quad E^{(L)}(0) < 0.$$

It will turn out that the existence of such an L depends on the order n and the index m of the branch point $w = 0$.

In Section 6.1, this idea is studied by investigating the third, fourth and fifth derivatives of $E(t)$ at $t = 0$. Here one meets fairly simple cases for testing the technique which show its efficiency. Furthermore, the difficulties are exhibited that will come up generally.

A case of general nature is treated in Section 6.2. Assuming that $n+1$ is even and $m+1$ is odd (whence $w = 0$ is nonexceptional) it will be seen that $E^{(m+1)}(0)$ can be made negative while $E^{(j)}(0) = 0$ for $1 \leq j \leq m$, and so $\hat{X}$ cannot be a weak minimizer of D.

In Section 6.3 we study boundary branch points of a minimal surface $\hat{X} \in \mathcal{C}(\Gamma)$ with a smooth boundary contour. In particular we show that $\hat{X}$ cannot be a minimizer of D in $\mathcal{C}(\Gamma)$ if it has a boundary branch point whose order n and index m satisfy the condition $2m-2 < 3n$ (Wienholtz's theorem).

Furthermore, in Sections 6.1 and 6.3 we exhibit geometric conditions which furnish bounds for the index of interior and boundary branch points. These estimates supplement the bounds on the order of branch points provided by the Gauss–Bonnet theorem.

6.1 The First Five Variations of Dirichlet's Integral, and Forced Jacobi Fields

In this chapter we take the point of view of Jesse Douglas and consider minimal surfaces as critical points of Dirichlet's integral within the class of harmonic surfaces $X : B \to \mathbb{R}^3$ that are continuous on the closure of the unit disk B and map $\partial B = S^1$ homeomorphically onto a closed Jordan curve Γ of $\mathbb{R}^3$. It will be assumed that Γ *is smooth of class* C^∞ *and nonplanar*. Then any minimal surface bounded by Γ will be a nonplanar surface of class $C^\infty(\overline{B}, \mathbb{R}^3)$, and so we shall be allowed to take directional derivatives (i.e. "variations") of any order of the Dirichlet integral along an arbitrary C^∞-smooth path through the minimal surface.

The first goal is to develop a *technique* which enables us to compute variations of any order of Dirichlet's integral, D, at an arbitrary minimal surface bounded by Γ, using complex analysis in form of Cauchy's integral theorem. This will be achieved by varying a given minimal surface via a one-parameter family of admissible harmonic mappings. Such harmonic variations will be generated by varying the boundary values of a given minimal surface in an admissible way and then extending the varied boundary values harmonically into B. From this point of view the admissible boundary maps $\partial B = S^1 \to \Gamma$ are the primary objects while their harmonic extensions $\overline{B} \to \mathbb{R}^3$ are of secondary nature. This calls for a change of notation: An admissible boundary map will be denoted by $X : \partial B \to \Gamma$, whereas $\hat{X}$ is the uniquely determined harmonic extension of X into B; i.e. $\hat{X} \in C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ is the solution of

$$\triangle \hat{X} = 0 \quad \text{in } B, \quad \hat{X}(w) = X(w) \quad \text{for } w \in \partial B.$$

Instead of $\hat{X}$ we will occasionally write HX or $H(X)$ for this extension, and

$$D(\hat{X}) := \frac{1}{2} \int_B \nabla \hat{X} \cdot \nabla \hat{X} \, du \, dv$$

is its Dirichlet integral.

In the sequel the main idea is to vary the boundary values X of a given minimal surface $\hat{X}$ in direction of a so-called *forced Jacobi field*, as this restriction will enable us to evaluate the variations of D at X by means of Cauchy's integral theorem. In order to explain what forced Jacobi fields are we first collect a few useful formulas.

Let us begin with an arbitrary mapping $X \in C^\infty(\partial B, \mathbb{R}^n)$ and its harmonic extension $\hat{X} \in C^\infty(\overline{B}, \mathbb{R}^3)$. Then $\hat{X}$ is of the form

$$\hat{X}(w) = \operatorname{Re} f(w), \tag{1}$$

where f is holomorphic on B and can be written as

$$f = \hat{X} + i\hat{X}^* \quad \text{with } \hat{X}_u = \hat{X}^*_v \text{ and } \hat{X}_v = -\hat{X}^*_u. \tag{2}$$

We also note that

$$f'(w) = 2\hat{X}_w(w) = \hat{X}_u(w) - i\hat{X}_v(w) \quad \text{in } B. \tag{3}$$

Conversely, *if f is holomorphic in B and $\hat{X} = \operatorname{Re} f$ then f' and $\hat{X}_w$ are related by the formula $f' = 2\hat{X}_w$;* in particular, *$\hat{X}_w$ is holomorphic in B.* This simple, but basic fact will be used repeatedly in later computations.

Let us introduce polar coordinates r, θ about the origin by $w = re^{i\theta}$, and set $\hat{Y}(r,\theta) = \hat{X}(re^{i\theta})$. Then a straight-forward computation yields

$$iw\hat{X}_w(w)\Big|_{w=e^{i\theta}} = \frac{1}{2}\Big[\hat{Y}_\theta(1,\theta) + i\hat{Y}_r(1,\theta)\Big] \tag{4}$$

whence

$$(5) \qquad 2\text{Re}\left\{iw\hat{X}_w(w)\right\}\Big|_{w=e^{i\theta}} = \hat{Y}_\theta(1,\theta) = \frac{\partial}{\partial\theta}X(e^{i\theta}) = Y_\theta(\theta)$$

since

$$\hat{Y}(1,\theta) = \hat{X}(e^{i\theta}) = X(e^{i\theta}) =: Y(\theta).$$

If $X \in C^\infty(S^1, \mathbb{R}^3)$ maps S^1 homeomorphically onto Γ then $Y_\theta(\theta)$ is tangent to Γ at $Y(\theta)$, i.e. $Y_\theta(\theta) \in T_{Y(\theta)}\Gamma$, and so the left-hand side of (5) is tangent to Γ.

Consider now a continuous function $\tau : \overline{B} \to \mathbb{C}$ that is meromorphic in B with finitely many poles in B, and that is real on ∂B. Then τ can be extended to a meromorphic function on an open set Ω with $\overline{B} \subset \Omega$, and τ is holomorphic in a strip containing ∂B. It follows from (5) that

$$(6) \qquad 2\text{Re}\left\{iw\hat{X}_w(w)\tau(w)\right\}\Big|_{w=e^{i\theta}} = \tau(e^{i\theta})Y_\theta(\theta) \in T_{Y(\theta)}\Gamma.$$

Suppose now that $\hat{X}$ is a minimal surface with finitely many branch points in $\overline{B}$. These points are the zeros of the function $F(w) := \hat{X}_w(w)$ which is of class C^∞ on $\overline{B}$ and holomorphic in B. If $\tau(w)$ has its poles at most at the (interior) zeros of the function $wF(w)$, and if the order of any pole does not exceed the order of the corresponding zero of $wF(w)$, then the function $K(w) := iw\hat{X}_w(w)\tau(w)$ is holomorphic in B and of class $C^\infty(\overline{B}, \mathbb{R}^3)$. We call $\hat{h} := \text{Re } K$ an **inner forced Jacobi field** $\hat{h} : \overline{B} \to \mathbb{R}^3$ **at** $\hat{X}$ with the **generator** τ.

In case that one wants to study boundary branch points of $\hat{X}$ it will be useful to admit factors $\tau(w)$ which are meromorphic on $\overline{B}$, real on ∂B, with poles at most at the zeros of $wF(w)$, the pole orders not exceeding the orders of the associated zeros of $wF(w)$. Then

$$(7) \qquad \hat{h} := \text{Re}\, K \quad \text{with } K(w) := iwF(w)\tau(w),\ w \in \overline{B},\ F := \hat{X}_w,$$

is said to be a (general) **forced Jacobi field** $\hat{h} : \overline{B} \to \mathbb{R}^3$ at the minimal surface $\hat{X}$, and τ is called the **generator** of $\hat{h}$.

The boundary values $\hat{h}|_{S^1}$ of a forced Jacobi field $\hat{h}$ are given by

$$(8) \qquad h(\theta) := \hat{h}(e^{i\theta}) = \text{Re}\, K(e^{i\theta}) = \frac{1}{2}\tau(e^{i\theta})Y_\theta(\theta), Y(\theta) := \hat{X}(\cos\theta, \sin\theta).$$

Using the asymptotic expansion of $F(w) = X_w(w)$ at a branch point $w_0 \in \overline{B}$ having the order $\lambda \in \mathbb{N}$, we obtain the factorization

$$(9) \qquad F(w) = (w - w_0)^\lambda G(w) \quad \text{with } G(w_0) \neq 0,$$

and, using Taylor's expansion in B or Taylor's formula on ∂B respectively, it follows that $G(w) = G(u,v)$ is a holomorphic function of w in B and a C^∞-function of $(u,v) \in \overline{B}$. It follows that *any forced Jacobi field* $\hat{h} : \overline{B} \to \mathbb{R}^3$ *is of class* $C^\infty(\overline{B}, \mathbb{R}^3)$ *and harmonic in* B.

Denote by $J(\hat{X})$ the linear space of forced Jacobi fields at $\hat{X}$, and let $J_0(\hat{X})$ be the linear subspace of inner forced Jacobi fields. The importance of $J(\hat{X})$ arises from the fact that *every forced Jacobi field* $\hat{h}$ *at* $\hat{X}$ *annihilates the second variation of* D, *i.e.*

$$\delta^2 D(\hat{X}, \hat{h}) = 0 \quad \textit{for all } \hat{h} \in J(\hat{X}).$$

This will be proved later in Section 6.3. In the present section we only deal with inner forced Jacobi fields, and so we only prove the weaker statement (cf. Proposition 1):

$$\delta^2 D(\hat{X}, \hat{h}) = 0 \quad \text{for all } \hat{h} \in J_0(X).$$

The existence of forced Jacobi fields arises from the group of conformal automorphisms of $\overline{B}$ and from the presence of branch points; the more branch points $\hat{X}$ has, and the higher their orders are, the more Jacobi fields appear—this explains the adjective 'forced'. To see the first statement we consider one-parameter families of conformal automorphisms $\varphi(\cdot, t), |t| < \epsilon, \epsilon > 0$ of $\overline{B}$ with

(10) $\quad w \mapsto \varphi(w, t) = w + t\eta(w) + o(t)$ and $\varphi(w, 0) = w, \dot{\varphi}(w, 0) = \eta(w)$.

Type I:

$$\varphi_1(w, t) = e^{i\alpha(t)} w$$

with $\alpha(t) \in \mathbb{R}, \alpha(0) = 0, \dot{\alpha}(0) = a$. Then $\varphi_1(w, t) = w + tiwa + o(t)$, and so

$$\eta_1(w) = iwa \quad \text{with } a \in \mathbb{R}.$$

Type II:

$$\varphi_2(w, t) := \frac{w + i\beta(t)}{1 - i\beta(t)w}$$

with $\beta(t) \in \mathbb{R}, \beta(0) = 0, \dot{\beta}(0) = b$.
Then $\varphi_2(w, t) = w + t\eta_2(w) + o(t)$ with $\eta_2(w) = ib + ibw^2$, and so

$$\eta_2(w) = iw\left(\frac{b}{w} + bw\right) \quad \text{with } b \in \mathbb{R}.$$

Type III:

$$\varphi_3(w, t) := \frac{w - \gamma(t)}{1 - \gamma(t)w}$$

with $\gamma(t) \in \mathbb{R}, \gamma(0) = 0, \dot{\gamma}(0) = c$.

Then $\varphi_3(w, t) = w + t\eta_3(w) + o(t)$ with $\eta_3(w) = -c + cw^2$, whence

$$\eta_3(w) = iw\left(\frac{ic}{w} - icw\right).$$

We set

$$(11)\qquad \tau_1(w) := a, \quad \tau_2(w) := b \cdot \left(\frac{1}{w} + w\right), \quad \tau_3(w) := c \cdot \left(\frac{i}{w} - iw\right),$$

with arbitrary constants $a, b, c \in \mathbb{R}$. For $w = e^{i\theta} \in \partial B$ we have

$$\tau_1(w) = a, \quad \tau_2(w) = 2b\cos\theta, \quad \tau_3(w) = -2c\sin\theta,$$

and so $\tau_j, j = 1, 2, 3$, are generators of the 'special' forced Jacobi field $\hat{h}_j := \operatorname{Re} K_j$, defined by

$$(12)\qquad K_j(w) := iwF(w)\tau_j(w), \quad w \in \overline{B},\ F := \hat{X}_w,$$

which are inner forced Jacobi fields for any minimal surface $\hat{X}$ bounded by Γ. If we vary $\hat{X}$ by means of $\varphi = \varphi_1, \varphi_2, \varphi_3$ with $\alpha := \operatorname{Re}\varphi,\ \beta := \operatorname{Im}\varphi$, i.e. $\varphi(w,t) = \alpha(u,v,t) + i\beta(u,v,t)$, setting

$$\hat{Z}(w,t) := \hat{X}(\varphi(w,t)) = \hat{X}(\alpha(u,v,t), \beta(u,v,t)),$$

we obtain

$$\begin{aligned}\frac{d}{dt}\hat{Z} &= \frac{d}{dt}\hat{X}\circ\varphi = \frac{d}{dt}\hat{X}(\alpha,\beta) = \hat{X}_u(\alpha,\beta)\dot{\alpha} + \hat{X}_v(\alpha,\beta)\dot{\beta}\\ &= 2\operatorname{Re}\hat{X}_w(\varphi)\dot{\varphi},\end{aligned}$$

and so

$$\frac{d}{dt}\hat{Z}\bigg|_{t=0} = 2\operatorname{Re}\{\hat{X}_w\dot{\varphi}(0)\}.$$

For $\varphi = \varphi_j$ we have $\dot{\varphi}(0) = \eta_j$, hence

$$(13)\qquad \frac{d}{dt}\hat{Z}(w,t)\bigg|_{t=0} = 2\operatorname{Re}\{iw\hat{X}_w(w)\tau_j(w)\} = 2\hat{h}_j(w).$$

Let us now generate variations $\hat{Z}(t)$, $|t| \ll 1$, of a minimal surface $\hat{X}$ using any inner forced Jacobi field $\hat{h} \in J_0(\hat{X})$. We write $\hat{Z}(t) = \hat{Z}(\cdot,t)$ for the variation of $\hat{X}$ and $Z(t)$ for the variation of the boundary values X of $\hat{X}$, and start with the definition of $Z(t)$. Then $\hat{Z}(t)$ will be defined as the harmonic extension of $Z(t)$, i.e.

$$(14)\qquad \hat{Z}(t) = H(Z(t)).$$

First we pick a smooth family $\gamma(t) = \gamma(\cdot,t), |t| < \delta$, of smooth mappings $\gamma(t) : \mathbb{R} \to \mathbb{R}$ with $\gamma(0) = \mathrm{id}_{\mathbb{R}}$ which are "shift periodic" with the period 2π, i.e.

$$(15)\qquad \gamma(\theta, 0) = \theta \quad \text{and} \quad \gamma(\theta + 2\pi, t) = \gamma(\theta, t) + 2\pi \quad \text{for } \theta \in \mathbb{R}.$$

Setting $\sigma(\theta, t) := \gamma(\theta, t) - \theta$ we obtain

$$\gamma(\theta, t) = \theta + \sigma(\theta, t) \quad \text{with } \sigma(\theta, 0) = 0 \text{ and } \sigma(\theta + 2\pi, t) = \sigma(\theta, t)$$

and

$$\gamma_\theta(\theta, t) = 1 + \sigma_\theta(\theta, t) = 1 + \sigma_{\theta t}(\theta, 0)t + o(t).$$

Choosing $\delta > 0$ sufficiently small it follows that

$$\gamma_\theta(\theta, t) > 0 \quad \text{for } (\theta, t) \in \mathbb{R} \times (-\delta, \delta).$$

Now we define the variation $\{Z(t)\}_{|t|<\delta}$ of X by

$$Z(e^{i\theta}, t) := X(e^{i\gamma(\theta,t)}) = \hat{X}(\cos\gamma(\theta, t), \sin\gamma(\theta, t)). \tag{16}$$

Then

$$\frac{\partial}{\partial t} Z(e^{i\theta}, t) = \left[-\hat{X}_u(e^{i\gamma(\theta,t)}) \sin\gamma(\theta, t) + \hat{X}_v(e^{i\gamma(\theta,t)}) \cos\gamma(\theta, t) \right] \gamma_t(\theta, t).$$

By (4) we have

$$ie^{i\theta} \hat{X}_w(e^{i\theta}) = \frac{1}{2} \left[X_\theta(\theta) + i\hat{X}_r(1, \theta) \right]$$

if we somewhat sloppily write $\hat{X}(r, \theta)$ for $\hat{X}(re^{i\theta})$ and $X(\theta)$ for $\hat{X}(1, \theta) = X(e^{i\theta})$. This leads to

$$-\hat{X}_u(e^{i\gamma(\theta,t)}) \sin\gamma(\theta, t) + \hat{X}_v(e^{i\gamma(\theta,t)}) \cos\gamma(\theta, t) = X_\theta(\gamma(\theta, t))$$

whence

$$\frac{\partial}{\partial t} Z(e^{i\theta}, t) = X_\theta(\gamma(\theta, t))\gamma_\theta(\theta, t) \cdot \frac{\gamma_t(\theta, t)}{\gamma_\theta(\theta, t)}.$$

On account of

$$Z(\theta, t) := Z(e^{i\theta}, t) = X(\gamma(\theta, t)) \tag{17}$$

we have

$$Z_\theta(\theta, t) = X_\theta(\gamma(t, \theta)) \cdot \gamma_\theta(\theta, t),$$

and so it follows that

$$\frac{\partial}{\partial t} Z(e^{i\theta}, t) = \frac{\partial}{\partial t} Z(\theta, t) = \frac{\partial}{\partial \theta} Z(\theta, t) \cdot \phi(\theta, t)$$

with

$$\phi(\theta, t) := \frac{\gamma_t(\theta, t)}{\gamma_\theta(\theta, t)}. \tag{18}$$

Defining the family $\{\phi(t)\}_{|t|<\delta}$ of 2π-periodic functions $\phi(t) : \mathbb{R} \to \mathbb{R}$ by $\phi(t) := \phi(\cdot, t)$, we have

(19) $$\frac{\partial}{\partial t}Z(t) = \phi(t)Z(t)_\theta =: h(t).$$

Now we consider the varied Dirichlet integral

(20) $$E(t) := D(\hat{Z}(t)) = \frac{1}{2}\int_B \nabla\hat{Z}(t)\cdot\nabla\hat{Z}(t)\,du\,dv.$$

Then

$$\frac{d}{dt}E(t) = \int_B \nabla\hat{Z}(t)\cdot\nabla\frac{d}{dt}\hat{Z}(t)\,du\,dv.$$

Since the operations $\frac{d}{dt}$ and H (i.e. $\hat{Z}$) commute, we have

$$\frac{d}{dt}\hat{Z}(t) = H\left(\frac{d}{dt}Z(t)\right)$$

and therefore

$$\frac{d}{dt}E(t) = \int_B \nabla\hat{Z}(t)\cdot\nabla H\left(\frac{d}{dt}Z(t)\right)\,du\,dv.$$

Since $\triangle\hat{Z}(t) = 0$, an integration by parts leads to

(21) $$\frac{d}{dt}E(t) = \int_0^{2\pi}\frac{\partial}{\partial r}\hat{Z}(t)\cdot h(t)\,d\theta \quad \text{with } h(t) = \frac{\partial}{\partial t}Z(t).$$

For brevity we write in the following computations $\hat{Z}$ instead of $\hat{Z}(t)$. We have

$$w\hat{Z}_w = \frac{1}{2}(\hat{Z}_r - i\hat{Z}_\theta)$$

if we write $\hat{Z}(r,\theta)$ for $\hat{Z}(w)|_{w=re^{i\theta}}$, cf. (4), and also

$$dw = iw\,d\theta \quad \text{for } w = e^{i\theta}\in\partial B.$$

Then on ∂B:

$$\begin{aligned} w\hat{Z}_w\cdot\hat{Z}_w\,dw &= i(w\hat{Z}_w)\cdot(w\hat{Z}_w)\,d\theta \\ &= \frac{i}{4}(\hat{Z}_r - i\hat{Z}_\theta)\cdot(\hat{Z}_r - i\hat{Z}_\theta)\,d\theta \\ &= \left[\frac{1}{2}\hat{Z}_r\cdot\hat{Z}_\theta - \frac{i}{4}(\hat{Z}_r\cdot\hat{Z}_r - \hat{Z}_\theta\cdot\hat{Z}_\theta)\right]\,d\theta, \end{aligned}$$

and so

$$2\mathrm{Re}[w\hat{Z}_w\cdot\hat{Z}_w\phi\,dw] = \hat{Z}_r\cdot\hat{Z}_\theta\phi\,d\theta \quad \text{on } \partial B.$$

Furthermore, $\hat{Z}_\theta = Z_\theta$ on ∂B as well as $h = \phi Z_\theta$ (see (19)), and so (21) leads to the formula

(22) $$\frac{d}{dt}E(t) = 2\mathrm{Re}\int_{S^1} w\hat{Z}(t)_w \cdot \hat{Z}(t)_w \phi(t)\,dw,$$

where the closed curve S^1 is positively oriented. This formula will be the starting point for calculating all higher order derivatives $\frac{d^n}{dt^n}E(t)$ and, in particular, of $\frac{d^n}{dt^n}E(0) := \frac{d^n}{dt^n}E(t)\big|_{t=0}$. In order to evaluate the latter expressions for any n, it will be essential that we can choose $\phi(t)$ and any number of t-derivatives of $\phi(t)$ in an arbitrary way. This is indeed possible according to the following result:

Lemma 1. *By a suitable choice of $\gamma(\theta,t) = \theta + \sigma(\theta,t)$ with $\sigma \in C^\infty$ on $\mathbb{R} \times (-\delta,\delta), \sigma(\theta,0) = 0$ and $\sigma(\theta+2\pi,t) = \sigma(\theta,t)$ we can ensure that the variation of the boundary values of the minimal surface $\hat{X}$, defined by $Z(\theta,t) := X(\gamma(\theta,t))$, leads to "test functions" $\phi(\theta,t)$ in formula (22) such that the functions*

$$\phi_\nu(\theta) := \frac{\partial^\nu}{\partial t^\nu}\phi(\theta,t)\big|_{t=0}, \quad \nu = 0,1,2,\dots,n,$$

can arbitrarily be prescribed as 2π-periodic functions of class C^∞.

Proof. Let us first check that, given $\phi_0,\phi_1,\dots,\phi_n$, the computation of σ, and so of γ, can be carried out in a formal way. Consider the Fourier expansion of the function $\sigma(\theta,t)$ which is to be determined:

(23) $$\sigma(\theta,t) = \frac{1}{2}a_0(t) + \sum_{k=1}^{\infty}[a_k(t)\cos k\theta + b_k(t)\sin k\theta].$$

From $\sigma(\theta,0) = 0$ it follows that

$$a_0(0) = a_k(0) = b_k(0) = 0 \quad \text{for } k \in \mathbb{N}.$$

Furthermore,

(24) $$\sigma_\nu(\theta) := \frac{\partial^\nu}{\partial t^\nu}\sigma(\theta,0) = \frac{1}{2}a_0^{(\nu)}(0) + \sum_{k=1}^{\infty}[a_k^{(\nu)}(0)\cos k\theta + b_k^{(\nu)}(0)\sin k\theta].$$

Hence if $D_t^\nu\sigma(\theta,0)$ are known for $\nu = 1,2,\dots,n$, one also knows all derivatives $D_\theta D_t^\nu\sigma(\theta,0) = \sigma_\nu'(\theta)$ from the defining equation (18) for σ which amounts to

$$\phi(\theta,t) = \frac{\sigma_t(\theta,t)}{1+\sigma_\theta(\theta,t)}.$$

By differentiation with respect to t we obtain

$$\begin{aligned}
\phi_t &= \frac{\sigma_{tt}}{1+\sigma_\theta} - \frac{\sigma_t\sigma_{\theta t}}{(1+\sigma_\theta)^2},\\
\phi_{tt} &= \frac{\sigma_{ttt}}{1+\sigma_\theta} - \frac{2\sigma_{tt}\sigma_{\theta t}}{(1+\sigma_\theta)^2} - \frac{\sigma_t\sigma_{\theta tt}}{(1+\sigma_\theta)^2} + \frac{2\sigma_t(\sigma_{t\theta})^2}{(1+\sigma_\theta)^3}
\end{aligned}$$

etc. Setting $t = 0$ and observing that $\sigma_\theta(\theta, 0) = 0$ it follows that

$$\begin{aligned}\sigma_1 &= \phi_0 = \phi, \\ \sigma_2 &= \phi_1 + \sigma_1\sigma_1', \\ \sigma_3 &= \phi_2 + 2\sigma_2\sigma_1' + \sigma_1\sigma_2' - 2\sigma_1(\sigma_1')^2, \\ &\dots \\ \sigma_{\nu+1} &= \phi_\nu + f_\nu(\sigma_1, \dots, \sigma_\nu, \sigma_1', \dots, \sigma_\nu').\end{aligned}$$

Here f_ν is a polynomial in the variables $\sigma_1, \dots, \sigma_\nu, \sigma_1', \dots, \sigma_\nu'$. This shows that, given $\phi_0, \phi_1, \dots, \phi_n$, we can successively determine $\sigma_1, \sigma_2, \dots, \sigma_{n+1}$. On account of (23) we then obtain

$$A_0^\nu := a_0^{(\nu)}(0), \quad A_k^\nu := a_k^{(\nu)}(0), \quad B_k^\nu := b_k^{(\nu)}(0) \quad \text{for } k \in \mathbb{N}.$$

Defining

$$a_k(t) := \sum_{\nu=1}^{n+1} \frac{1}{\nu!} A_k^\nu t^\nu, \quad b_k(t) := \sum_{\nu=1}^{n+1} \frac{1}{\nu!} B_k^\nu t^\nu,$$

equation (23) furnishes the function $\gamma(\theta, t) = \theta + \sigma(\theta, t)$ with the desired properties. Furthermore, the construction shows that this procedure leads to a C^∞-function σ that is 2π-periodic with respect to θ. □

Let us inspect a variation $\hat{Z}(t) = H(Z(t))$ of a minimal surface $\hat{X} \in C^\infty(\overline{B}, \mathbb{R}^3)$ as we have just discussed. It is the harmonic extension of a variation $Z(t)$ of the boundary values X of $\hat{X}$, given by (15) and (16). Clearly, $\hat{Z}(t)$ is not merely an "inner variation" of $\hat{X}$, generated as a reparametrization $\hat{X} \circ \sigma(t)$ with a perturbation $\sigma(t) = \mathrm{id}_{\overline{B}} + t\lambda + \cdots$ of the identity $\mathrm{id}_{\overline{B}}$ on $\overline{B}$, but the image $\hat{Z}(t)(B)$ will differ from the image $\hat{X}(B)$. Only the images $Z(t)(S^1)$ and $X(S^1)$ of the boundary $S^1 = \partial B$ will be the same set Σ, but described by different parametrizations $Z(t) : S^1 \to \Sigma$ and $X : S^1 \to \Sigma$.

Definition 1. *We call such a variation $\hat{Z}(t)$ a* **boundary preserving variation of** $\hat{X}$ *(for $|t| \ll 1$).*

Note: If $\hat{X} \in \mathcal{C}(\Gamma)$ then any boundary preserving variation $\hat{Z}(t)$ (with $|t| \ll 1$) lies in $\mathcal{C}(\Gamma)$.

Definition 2. *We say that $\hat{X}$* **is a weak relative minimizer of** D *(with respect to its own boundary) if $E(0) \le E(t)$ holds for any variation $E(t) = D(\hat{Z}(t))$ of D by an arbitrary boundary preserving variation $\hat{Z}(t)$ of $\hat{X}$ with $|t| \ll 1$.*

If $\hat{X} \in \mathcal{C}(\Gamma)$ is a weak relative minimizer of D in $\mathcal{C}(\Gamma)$ with respect to some C^k-norm on $\overline{B}$, then $\hat{X}$ clearly is a weak relative minimizer of D in the sense of Definition 2.

Let us return to formula (19) which states that

$$\frac{\partial}{\partial t} Z(t) = \phi(t) Z(t)_\theta .$$

According to (5) we have

$$Z(t)_\theta = 2\mathrm{Re}[iw\hat{Z}_w(w,t)]\big|_{w=e^{i\theta}},$$

and since ϕ is real-valued it follows that

(25) $$\frac{\partial}{\partial t} Z(\theta,t) = 2\mathrm{Re}[iw\hat{Z}_w(w,t)\phi(\theta,t)]\big|_{w=e^{i\theta}}.$$

Since $\frac{\partial}{\partial t}$ and the harmonic extension H commute we obtain

(26) $$\frac{\partial}{\partial t} \hat{Z}(t) = H\{2\mathrm{Re}[iw\hat{Z}(t)_w \phi(t)]\} \quad \text{in } \overline{B}$$

having for brevity dropped the w, except for the factor iw (as this would require a clumsy notation). Then, by

$$\frac{\partial}{\partial t}\frac{\partial}{\partial w}\hat{Z}(t) = \frac{\partial}{\partial w}\frac{\partial}{\partial t}\hat{Z}(t),$$

it follows that

(27) $$\frac{\partial}{\partial t}\hat{Z}(t)_w = \Big(H\{2\mathrm{Re}[iw\hat{Z}(t)_w\phi(t)]\}\Big)_w .$$

Now a straight-forward differentiation of (22) yields

(28) $$\begin{aligned}\frac{d^2}{dt^2}E(t) &= 4\mathrm{Re}\int_{S^1} w\left\{\frac{\partial\hat{Z}(t)}{\partial t}\right\}_w \cdot \hat{Z}(t)_w\phi(t)\,dw \\ &\quad + 2\mathrm{Re}\int_{S^1} w\hat{Z}(t)_w\cdot\hat{Z}(t)_w\phi_t(t)\,dw.\end{aligned}$$

From (22) and (28) we obtain

Proposition 1. *Since $\hat{X} = \hat{Z}(0)$ is a minimal surface we have*

(29) $$\frac{dE}{dt}(0) = 0$$

and

(30) $$\frac{d^2E}{dt^2}(0) = 4\mathrm{Re}\int_{S^1} w\left\{\frac{\partial\hat{X}}{\partial t}\right\}_w\cdot\hat{X}_w\tau\,dw$$

with $\tau := \phi(0)$. If τ is the generator of an inner forced Jacobi field attached to $\hat{X}$, then

$$\text{(31)} \qquad \frac{d^2E}{dt^2}(0) = 0.$$

This means that

$$\text{(32)} \qquad \delta^2 D(\hat{X}, \hat{h}) = 0 \quad \text{for all } \hat{h} \in J_0(\hat{X}),$$

i.e. for all inner forced Jacobi fields $\hat{h} = \mathrm{Re}[iwX_w(w)\tau(w)]$.

Proof. We have $\hat{X}_w \cdot \hat{X}_w = 0$ since $\hat{X}$ is a minimal surface, and so (29) and (30) are proved. Secondly, $\hat{h}$ is holomorphic in B, as it is an inner forced Jacobi field, and the w-derivative of any harmonic mapping is holomorphic whence $\{\frac{\partial \hat{X}}{\partial t}\}_w$ is holomorphic in B. Thus the integrand of $\int_{S^1}(\ldots)\,dw$ in (30) is holomorphic. Hence this integral vanishes, since Cauchy's integral theorem implies $\int_{\partial B_r(0)}(\ldots)\,dw = 0$ for any $r \in (0,1)$ and then $\int_{S^1}(\ldots)\,dw = \lim_{r\to 1-0}\int_{\partial B_r(0)}(\ldots)\,dw = 0$ as the integrand $(\ldots)$ is continuous (and even of class C^∞) on $\overline{B}$. □

Now we want to compute $\frac{d^3}{dt^3}E(t)$, and in particular $\frac{d^3E}{dt^3}(0)$ if $\tau = \phi(0)$ is the generator of an inner forced Jacobi field. Differentiating (28) it follows

$$\text{(33)} \qquad \begin{aligned} \frac{d^3}{dt^3}E(t) = {} & 4\mathrm{Re}\int_{S^1} w\left\{\frac{\partial \hat{Z}(t)}{\partial t}\right\}_w \cdot \left\{\frac{\partial \hat{Z}(t)}{\partial t}\right\}_w \phi(t)\,dw \\ & + 4\mathrm{Re}\int_{S^1} w\left\{\frac{\partial^2 \hat{Z}(t)}{\partial t^2}\right\}_w \cdot \hat{Z}(t)_w \phi(t)\,dw \\ & + 8\mathrm{Re}\int_{S^1} w\left\{\frac{\partial \hat{Z}(t)}{\partial t}\right\}_w \cdot \hat{Z}(t)_w \phi_t(t)\,dw \\ & + 2\mathrm{Re}\int_{S^1} w\hat{Z}(t)_w \cdot \hat{Z}(t)_w \phi_{tt}(t)\,dw. \end{aligned}$$

Proposition 2. *Since* $\hat{X} = \hat{Z}(0)$ *is a minimal surface we have*

$$\text{(34)} \qquad \frac{d^3E}{dt^3}(0) = -4\mathrm{Re}\int_{S^1} w^3 \hat{X}_{ww} \cdot \hat{X}_{ww}\tau^3\,dw$$

if $\tau := \phi(0)$ *is the generator of an inner forced Jacobi field at* $\hat{X}$.

Proof. The fourth integral in (33) vanishes at $t = 0$ since

$$\hat{Z}(0)_w \cdot \hat{Z}(0)_w = X_w \cdot X_w = 0.$$

The integrand of the second integral in (33) is

$$\left\{\frac{\partial^2 \hat{Z}}{\partial t^2}(0)\right\}_w \cdot w\hat{X}_w \tau(w)$$

which is holomorphic in B since the w-derivative of a harmonic mapping is holomorphic and $\hat{h} = \text{Re}[iw\hat{X}_w\tau]$ is an inner forced Jacobi field. So also the second integral in (33) vanishes on account of Cauchy's integral theorem. Next, using (27), we obtain

$$\left\{\frac{\partial}{\partial t}\hat{Z}(t)\right\}_w\Big|_{t=0} = 2\frac{\partial}{\partial w}H\left\{\text{Re}[iw\hat{X}_w\tau]\right\} = [iw\hat{X}_w\tau]_w. \tag{35}$$

This implies

$$\begin{aligned}&\left[w\left\{\frac{\partial}{\partial t}\hat{Z}(t)\right\}_w\cdot\hat{Z}(t)_w\right]\Big|_{t=0}\\ &= w[iw\hat{X}_w\tau]_w\cdot\hat{X}_w\\ &= iw\hat{X}_w\cdot\hat{X}_w\tau + iw^2\hat{X}_{ww}\cdot\hat{X}_w\tau + iw^2\hat{X}_w\cdot\hat{X}_w\tau_w = 0\end{aligned}$$

since $\hat{X}_w\cdot\hat{X}_w = 0$, which also yields $\hat{X}_{ww}\cdot\hat{X}_w = 0$. Thus

$$\left[w\left\{\frac{\partial}{\partial t}\hat{Z}(t)\right\}_w\cdot\hat{Z}(t)_w\right]\Big|_{t=0} = 0 \tag{36}$$

and so the third integral in (33) vanishes for $t = 0$. Finally, by (35),

$$\begin{aligned}&\left(\left\{\frac{\partial}{\partial t}\hat{Z}(t)\right\}_w\cdot\left\{\frac{\partial}{\partial t}\hat{Z}(t)\right\}_w\right)\Big|_{t=0}\\ &= [iw\hat{X}_w\tau]_w\cdot[iwX_w\tau]_w\\ &= [i\hat{X}_w\tau + iw\hat{X}_{ww}\tau + iw\hat{X}_w\tau_w]\cdot[i\hat{X}_w\tau + iw\hat{X}_{ww}\tau + iw\hat{X}_w\tau_w]\\ &= -w^2\hat{X}_{ww}\cdot\hat{X}_{ww}\tau^2,\end{aligned}$$

using again $\hat{X}_w\cdot\hat{X}_w = 0$ and $\hat{X}_w\cdot\hat{X}_{ww} = 0$, i.e.

$$\left(\left\{\frac{\partial}{\partial t}\hat{Z}(t)\right\}_w\cdot\left\{\frac{\partial}{\partial t}\hat{Z}(t)\right\}_w\right)\Big|_{t=0} = -w^2\hat{X}_{ww}\cdot\hat{X}_{ww}\tau^2. \tag{37}$$

Thus the first integral in (33) amounts to

$$-4\text{Re}\int_{S^1} w^3\hat{X}_{ww}\cdot\hat{X}_{ww}\tau^3\,dw.$$

□

In order to simplify notation we drop the t in (33) and write

$$\begin{aligned}\frac{d^3}{dt^3}E = \text{Re}\Bigg[&4\int_{S^1} w\hat{Z}_{tw}\cdot\hat{Z}_{tw}\phi\,dw + 4\int_{S^1} w\hat{Z}_{ttw}\cdot\hat{Z}_w\phi\,dw\\ &+8\int_{S^1} w\hat{Z}_{tw}\cdot\hat{Z}_w\phi_t\,dw + 2\int_{S^1} w\hat{Z}_w\cdot\hat{Z}_w\phi_{tt}\,dw\Bigg].\end{aligned}$$

Differentiation yields

$$(38)\quad \frac{d^4}{dt^4}E = \mathrm{Re}\Big[12\int_{S^1} w\hat{Z}_{ttw}\cdot\hat{Z}_{tw}\phi\,dw + 4\int_{S^1} w\hat{Z}_{tttw}\cdot\hat{Z}_w\phi\,dw + 12\int_{S^1} w\hat{Z}_{tw}\cdot\hat{Z}_{tw}\phi_t\,dw + 12\int_{S^1} w\hat{Z}_{ttw}\cdot\hat{Z}_w\phi_t\,dw + 12\int_{S^1} w\hat{Z}_{tw}\cdot\hat{Z}_w\phi_{tt}\,dw + 2\int_{S^1} w\hat{Z}_w\cdot\hat{Z}_w\phi_{ttt}\,dw\Big] = \mathrm{Re}[I_1+I_2+I_3+I_4+I_5+I_6].$$

We have $I_6(0)=0$ since $\hat{Z}_w(0)\cdot\hat{Z}_w(0)=\hat{X}_w\cdot\hat{X}_w=0$. Moreover, by Cauchy's theorem, $I_2(0)=0$ since both $\hat{Z}_{tttw}\big|_{t=0}=[\hat{Z}_{ttt}(0)]_w$ and $w\hat{X}_w\tau$ are holomorphic. On account of (36) we also get $I_5(0)=0$. Finally, taking (17) into account, we see that

$$I_3(0) = -12\int_{S^1} w^3\hat{X}_{ww}\cdot\hat{X}_{ww}\tau^2\phi_t(0)\,dw,$$

and we arrive at

Proposition 3. *Since $\hat{X}=\hat{Z}(0)$ is a minimal surface we have*

$$(39)\quad \frac{d^4E}{dt^4}(0) = 12\mathrm{Re}\int_{S^1}\hat{Z}_{ttw}(0)\cdot[w\hat{Z}_{tw}(0)\tau + w\hat{X}_w\phi_t(0)]\,dw - 12\mathrm{Re}\int_{S^1} w^3\hat{X}_{ww}\cdot\hat{X}_{ww}\tau^2\phi_t(0)\,dw,$$

provided that $\tau=\phi(0)$ is the generator of an inner forced Jacobi field at $\hat{X}$.

Finally, as an exercise, we even compute $\frac{d^5E}{dt^5}(0)$. Differentiating (38) it follows that

$$(40)\quad \frac{d^5E}{dt^5} = \mathrm{Re}\sum_{j=1}^{9} I_j$$

with

$$I_1 := 16\int_{S^1} w\hat{Z}_{tttw}\cdot\hat{Z}_{tw}\phi\,dw,\quad I_2 := 12\int_{S^1} w\hat{Z}_{ttw}\cdot\hat{Z}_{ttw}\phi\,dw,$$
$$I_3 := 4\int_{S^1} w\hat{Z}_{ttttw}\cdot\hat{Z}_w\phi\,dw,\quad I_4 := 16\int_{S^1} w\hat{Z}_{tttw}\cdot\hat{Z}_w\phi_t\,dw,$$
$$I_5 := 48\int_{S^1} w\hat{Z}_{ttw}\cdot\hat{Z}_{tw}\phi_t\,dw,\quad I_6 := 24\int_{S^1} w\hat{Z}_{ttw}\cdot\hat{Z}_w\phi_{tt}\,dw,$$
$$I_7 := 24\int_{S^1} w\hat{Z}_{tw}\cdot\hat{Z}_{tw}\phi_{tt}\,dw,\quad I_8 := 16\int_{S^1} w\hat{Z}_{tw}\cdot\hat{Z}_w\phi_{ttt}\,dw,$$
$$I_9 := 2\int_{S^1} w\hat{Z}_w\cdot\hat{Z}_w\phi_{tttt}\,dw.$$

$I_3(0)$ vanishes by Cauchy's theorem since both $\hat{Z}_{tttt}(0)_w$ and $w\hat{X}_w\tau$ are holomorphic provided that $\tau = \phi(0)$ is the generator of a forced Jacobi field at $\hat{X}$. Furthermore, $I_8(0) = 0$ because of (36), and $\hat{X}_w \cdot \hat{X}_w = 0$ implies $I_9(0) = 0$. Thus we obtain by (37):

Proposition 4. *Since $\hat{X}$ is a minimal surface we have*

$$\begin{aligned}\frac{d^5E}{dt^5}(0) = {} & 16\text{Re}\int_{S^1} \hat{Z}_{tttw}(0)\cdot[w\hat{Z}_{tw}(0)\tau + w\hat{X}_w\phi_t(0)]\,dw \\ & + 12\text{Re}\int_{S^1} Z_{ttw}(0)\cdot[w\hat{Z}_{ttw}(0)\tau \\ & + 4w\hat{Z}_{tw}(0)\phi_t(0) + 2w\hat{X}_w\phi_{tt}(0)]\,dw \\ & - 24\text{Re}\int_{S^1} w^3\hat{X}_{ww}\cdot\hat{X}_{ww}\tau^2\phi_{tt}(0)\,dw\end{aligned} \tag{41}$$

provided that $\tau = \phi(0)$ is the generator of an inner forced Jacobi field at $\hat{X}$.

Note also that in (39) and (41) we can express $\hat{Z}_{tw}(0)$ by (35) which we write as

$$\hat{Z}_{tw}(0) = [iw\hat{X}_w\tau]_w. \tag{42}$$

The values of $E''(0)$ and $E'''(0)$ in (30) and (34) depend only on $\tau = \phi(0)$ and not on any derivatives of $\phi(t)$ at $t = 0$; in this sense we say that $E''(0)$ and $E'''(0)$ are *intrinsic*. As we shall see later, this reflects important facts, namely: The Dirichlet integral D has an intrinsic second derivative d^2D, and an intrinsic third derivative d^3D in direction of forced Jacobi fields.

Let us try to show that a nonplanar weak relative minimizer $\hat{X}$ of D cannot have a branch point in $\overline{B}$. To achieve this goal, a somewhat naive approach would be to compute sufficiently many derivatives $E^{(j)}(0) := \frac{d^jE}{dt^j}(0)$ and to hope that one can find some first nonvanishing derivative, say, $E^{(L)}(0) \neq 0$, whereas $E^{(j)}(0) = 0$ for $j = 1, 2, \ldots, L-1$. Then Taylor's formula with Cauchy's remainder term yields

$$E(t) = E(0) + \frac{1}{L!}E^{(L)}(\vartheta t)t^L \quad \text{for } |t| \ll 1,\ 0 < \vartheta < 1,$$

that is,

$$D(\hat{Z}(t)) = D(\hat{X}) + \frac{1}{L!}E^{(L)}(\vartheta t)t^L,$$

and we infer for some t with $0 < |t| \ll 1$ that

(i) $D(\hat{Z}(t)) < D(\hat{X})$ if L odd $= 2\ell + 1 \geq 3$ and $E^{(2\ell+1)}(0) \neq 0$,

and

(ii) $D(\hat{Z}(t)) < D(\hat{X})$ if L even $= 2\ell \geq 4$ and $E^{(2\ell)}(0) < 0$.

Let us see under which assumption on $\hat{X}$ this approach works for $L = 3$. Note that an arbitrary branch point $w_0 \in B$ of a minimal surface $\hat{X}$ can be moved to the origin by means of a suitable conformal automorphism of $\overline{B}$. Hence it is sufficient for our purposes to show that a minimizer $\hat{X}$ of D in $\mathcal{C}(\Gamma)$ does not have $w = 0$ as a branch point. Therefore we shall from now on assume the following **normal form of a nonplanar minimal surface** $\hat{X}$ (cf. Vol. 1, Section 3.2):

$\hat{X}$ has $w = 0$ as a branch point of *order* n, i.e.

$$\hat{X}_w(w) = aw^n + o(w^n) \quad \text{as } w \to 0.$$

Choosing a suitable Cartesian coordinate system in $\mathbb{R}^3$ we may assume that $\hat{X}_w$ can be written as

$$\hat{X}_w(w) = (A_1 w^n + A_2 w^{n+1} + \cdots, R_m w^m + R_{m+1} w^{m+1} + \cdots), \quad m > n, \tag{43}$$

with $A_j \in \mathbb{C}^2, R_j \in \mathbb{C}, A_1 \neq 0$ and $R_m \neq 0$ for some integer m satisfying $m > n$; the number m is called *index* of the branch point $w = 0$ of $\hat{X}$ given in the normal form (43). Note that a surface $\hat{X}$ can also be brought into the normal form (43) (with $n = 0$) if $\hat{X}$ is regular at $w = 0$.

Lemma 2. *The normal form* (43) *satisfies*

$$\begin{aligned} &A_1 \cdot A_1 = 0, \quad A_k = \lambda_k \cdot A_1 \quad \textit{for } k = 1, 2, \ldots, 2(m-n), \\ &A_1 \cdot A_{2m-2n+1} = -\frac{1}{2} R_m^2, \end{aligned} \tag{44}$$

and therefore

$$\hat{X}_{ww}(w) \cdot \hat{X}_{ww}(w) = (m-n)^2 R_m^2 w^{2m-2} + \cdots, \quad R_m \neq 0. \tag{45}$$

Proof. Equation (43) implies

$$\hat{X}_w(w) \cdot \hat{X}_w(w) = (w^{2n} p(w) + R_m^2 w^{2m}) + O(|w|^{2m+1}) \quad \text{as } w \to 0,$$

where $p(w)$ is a polynomial of degree 2ℓ in w with $\ell := m - n$ which is of the form

$$\begin{aligned} p(w) = \ & A_1 \cdot A_1 + 2A_1 \cdot A_2 w + (2A_1 \cdot A_3 + A_2 \cdot A_2) w^2 \\ & + (2A_1 \cdot A_4 + 2A_2 \cdot A_3) w^3 + (2A_1 \cdot A_5 + 2A_2 \cdot A_4 + A_3 \cdot A_3) w^4 \\ & + \cdots + (2A_1 \cdot A_{2\ell+1} + 2A_2 \cdot A_{2\ell} + \cdots + 2A_{\ell+2} \cdot A_\ell + A_{\ell+1} \cdot A_{\ell+1}) w^{2\ell} \\ = \ & c_0 + c_1 w + c_2 w^2 + \cdots + c_{2\ell} w^{2\ell}, \quad c_j \in \mathbb{C}. \end{aligned}$$

Since $\hat{X}_w \cdot \hat{X}_w = 0$ we obtain

$$c_0 = c_1 = \cdots = c_{2\ell-1} = 0, \quad c_{2\ell} + R_m^2 = 0.$$

Let $\langle A', A''\rangle := A' \cdot \overline{A''}$ be the Hermitian scalar product of two vectors $A', A'' \in \mathbb{C}^2$. The two equations $c_0 = 0$ and $c_1 = 0$ yield $A_1 \cdot A_1 = 0$ and $A_1 \cdot A_2 = 0$ which are equivalent to

$$\langle A_1, \overline{A}_1\rangle = 0 \quad \text{and} \quad \langle A_2, \overline{A}_1\rangle = 0.$$

Since $A_1 \neq 0$ and $\overline{A}_1 \neq 0$ this implies

$$A_2 = \lambda_2 A_1 \quad \text{for some } \lambda_2 \in \mathbb{C},$$

and so we also obtain

$$A_2 \cdot A_2 = \lambda_2^2 A_1 \cdot A_1 = 0.$$

On account of $c_2 = 0$ it follows $A_1 \cdot A_3 = 0$, and thus it follows

$$\langle A_1, \overline{A}_1\rangle = 0 \quad \text{and} \quad \langle A_3, \overline{A}_1\rangle = 0$$

whence

$$A_3 = \lambda_3 A_1 \quad \text{for some } \lambda_3 \in \mathbb{C},$$

and so

$$A_2 \cdot A_3 = \lambda_2 \lambda_3 A_1 \cdot A_1 = 0.$$

Then $c_3 = 0$ yields $A_1 \cdot A_4 = 0$, therefore

$$\langle A_1, \overline{A}_1\rangle = 0 \quad \text{and} \quad \langle A_4, \overline{A}_1\rangle = 0;$$

consequently

$$A_4 = \lambda_4 A_1 \quad \text{for some } \lambda_4 \in \mathbb{C}.$$

In this way we proceed inductively using $c_0 = 0, \dots, c_{2\ell-1} = 0$ and obtain $A_k = \lambda_k A_1$ for $k = 1, 2, \dots, 2(m-n)$. Since $A_1 \cdot A_1 = 0$ it follows that

$$A_j \cdot A_k = 0 \quad \text{for } 1 \leq j, k \leq 2(m-n). \tag{46}$$

Then the equation $c_{2\ell} + R_m^2 = 0$ implies $2A_1 \cdot A_{2\ell+1} + R_m^2 = 0$, i.e.

$$A_1 \cdot A_{2(m-n)+1} = -\frac{1}{2} R_m^2. \tag{47}$$

Furthermore, from

$$\hat{X}_w(w) = (A_1 w^n + A_2 w^{n+1} + \cdots + A_{2m-2n+1} w^{2m-n} + \cdots, R_m w^m + \cdots)$$

we infer

$$\begin{aligned}\hat{X}_{ww}(w) = {} & (nA_1 w^{n-1} + \cdots + (2m-n) A_{2m-2n+1} w^{2m-n-1} \\ & + \cdots, m R_m w^{m-1} + \cdots).\end{aligned}$$

Then (46) implies

$$\hat{X}_{ww}(w) \cdot \hat{X}_{ww}(w) = [2n(2m-n) A_1 \cdot A_{2m-2n+1} + m^2 R_m^2] w^{2m-2} + \cdots,$$

and by (47) we arrive at

$$\hat{X}_{ww}(w) \cdot \hat{X}_{ww}(w) = [-n(2m-n) R_m^2 + m^2 R_m^2] w^{2m-2} + \cdots,$$

which is equivalent to (45). □

Theorem 1. (D. Wienholtz). *Let $\hat{X}$ be a minimal surface in normal form with a branch point at $w = 0$ which is of order n and index m, $n < m$, and suppose that $2m - 2 < 3n$ (or, equivalently, $2m + 2 \le 3(n+1)$). Then we can choose a generator τ of a forced Jacobi field $\hat{h}$ such that $E^{(3)}(0) < 0$, and so $\hat{X}$ is not a weak relative minimizer of D.*

Proof. Define the integer k by

$$k := (2m + 2) - 2(n + 1).$$

Because of $m > n$ and $2m - 2 < 3n$ it follows that

$$1 < k \le n + 1.$$

Let

$$\tau_0 := cw^{-n-1} + \overline{c}w^{n+1}, \quad \tau_1 := cw^{-k} + \overline{c}w^k, \quad c \in \mathbb{C},$$

and set

(i) $\tau := \tau_0$ if $k = n + 1$;
(ii) $\tau := \epsilon\tau_0 + \tau_1$, $\epsilon > 0$, if $k < n + 1$.

In both cases τ is a generator of a forced Jacobi field at $\hat{X}$, since $w\hat{X}_w(w)$ has a zero of order $n + 1$ at $w = 0$, and Im $\tau = 0$ on ∂B. By (45) it follows for $w \in B$ that

$$w^3 \hat{X}_{ww}(w) \cdot \hat{X}_{ww}(w) = (m - n)^2 R_m^2 w^{2m+1} + \cdots ,$$

where $+\cdots$ always stands for higher order terms of a convergent power series. In case (i) one has

$$\tau^3(w) = c^3 w^{-3(n+1)} + \cdots ,$$

and so

$$w^3 \hat{X}_{ww}(w) \cdot \hat{X}_{ww}(w)\tau(w)^3 = (m - n)^2 R_m^2 c^3 w^{-1} + f(w),$$

where $f(w)$ is holomorphic in B and continuous on $\overline{B}$. Then formula (34) of Proposition 3 in conjunction with Cauchy's integral theorem yields

$$E^{(3)}(0) = -4\text{Re}[2\pi i(m - n)^2 R_m^2 c^3] \quad \text{if } k = n + 1.$$

With a suitable choice of $c \in \mathbb{C}$ we can arrange for $E^{(3)}(0) < 0$ since $R_m \ne 0$ and $(m - n)^2 \ge 1$.

In case (ii) we write $w^3 \hat{X}_{ww} \cdot \hat{X}_{ww}$ as

$$w^3 \hat{X}_{ww}(w) \cdot \hat{X}_{ww}(w) = (m - n)^2 R_m^2 w^{2m+1} + f(w),$$

where

$$f(w) := w^{2m+2} \sum_{j=0}^{\infty} a_j w^j, \quad a_j \in \mathbb{C}.$$

From

$$\tau^3 = \epsilon^3 \tau_0^3 + 3\epsilon^2 \tau_0^2 \tau_1 + 3\epsilon \tau_0 \tau_1^2 + \tau_1^3$$

it follows that

$$g(w) := w^3 \hat{X}_{ww}(w) \cdot \hat{X}_{ww}(w) \tau^3(w)$$

is meromorphic in B, continuous in $\{w : \rho < |w| \le 1\}$ for some $\rho \in (0,1)$, and its Laurent expansion at $w = 0$ has the residue

$$\mathrm{Res}_{w=0}(g) = 3\epsilon^2 c^3 (m-n)^2 R_m^2 + \epsilon^3 c^3 a_{n-k}, \quad 1 < k \le n.$$

Cauchy's residue theorem together with formula (34) of Proposition 3 then imply

$$E^{(3)}(0) = -4\mathrm{Re}\{2\pi i [3\epsilon^2 c^3 (m-n)^2 R_m^2 + \epsilon^3 c^3 a_{n-k}]\} \quad \text{for } k < n+1.$$

By an appropriate choice of $c \in \mathbb{C}$ and ϵ with $0 < \epsilon < 1$ we can achieve that $E^{(3)}(0) < 0$ also in case (ii). □

The following definition will prove to be very useful.

Definition 3. *Let $\hat{X}$ be a minimal surface in normal form having $w = 0$ as a branch point of order n and of index m. Then $w = 0$ is called an* **exceptional branch point** *if $m + 1 = \kappa(n+1)$ for some $\kappa \in \mathbb{N}$; necessarily $\kappa > 1$.*

Remark 1. If $2m - 2 < 3n$, i.e. $2(m+1) \le 3(n+1)$, then $w = 0$ is not exceptional, because $(m+1) = \kappa(n+1)$ with $\kappa > 1$ implies $2\kappa(n+1) \le 3(n+1)$ and therefore $2\kappa \le 3$ which is impossible for $\kappa \in \mathbb{N}$ with $\kappa > 1$.

Remark 2. Now we want to show that the notion "*$w = 0$ is an exceptional branch point*" is closely related to the notion "*$w = 0$ is a false branch point*". To this end we choose an arbitrary minimal surface $\hat{Z}(\zeta)$, $\zeta \in B$, in normal form without $\zeta = 0$ being a branch point, i.e. $\hat{Z} = \mathrm{Re}\, g$ where $g : B \to \mathbb{C}^3$ is holomorphic and of the form

$$g(\zeta) = \hat{Z}(0) + (B_0 \zeta + B_1 \zeta^2 + \cdots, C_\kappa \zeta^\kappa + \cdots), \quad B_0 \ne 0,\ C_\kappa \ne 0,\ \kappa > 1.$$

Consider a conformal mapping $w \mapsto \zeta = \varphi(w)$ from B into B with $\varphi(0) = 0$ which is provided by a holomorphic function

$$\varphi(w) = aw + \cdots, \quad a \ne 0,\ w \in B.$$

Then $\hat{X}(w) := \mathrm{Re}\, f(w)$ with $f(w) := g(\varphi^{n+1}(w))$, $w \in B$, is a minimal surface $\hat{X} : B \to \mathbb{R}^3$ such that $\hat{X}(0) = \hat{Z}(0)$ and

$$f(w) = \hat{X}(0) + (a^{n+1} B_0 w^{n+1} + \cdots,\ a^{\kappa(n+1)} C_\kappa w^{\kappa(n+1)} + \cdots).$$

Thus we obtain for $\hat{X}_w = \frac{1}{2} f'$ that

$$\hat{X}_w(w) = (A_1 w^n + \cdots, \; R_m w^m + \cdots), \quad A_1 \neq 0, \; R_m \neq 0,$$

and so $\hat{X}(w)$, $w \in B$, is a minimal surface in normal form which has the branch point $w = 0$ of order n and index $m := \kappa(n+1) - 1$, whence $w = 0$ is *exceptional*. Clearly $\hat{X}$ is obtained from the minimal immersion $\hat{Z}(\zeta)$ as a *false branch point* by setting $\hat{X} := \hat{Z} \circ \varphi^{n+1}$. As the "false parametrization" $\hat{X}$ of the regular surface $\mathcal{S} := \hat{Z}(B)$ is produced by an analytic expression $\zeta = \varphi^{n+1}(w)$ we call $w = 0$ an "*analytic false branch point*".

In Remark 1 we have noted that $w = 0$ cannot be "exceptional" if $2m - n < 3n$, and so it cannot be an "analytic false branch point".

It will be useful to have a **characterization of the nonexceptional branch points**, the proof of which is left to the reader.

Lemma 3. *The branch point $w = 0$ is nonexceptional if and only if one of the following two conditions is satisfied:*

(i) *There is an even integer L with*

$$(L-1)(n+1) < 2(m+1) < L(n+1). \tag{48}$$

(ii) *There is an odd integer L with*

$$(L-1)(n+1) < 2(m+1) \leq L(n+1). \tag{49}$$

We say that $w = 0$ satisfies condition (T_L) if either (48) with L even or (49) with L odd holds.

In Theorem 1 it was shown that $E^{(3)}(0)$ can be made negative if $2m - 2 < 3n$. Therefore *we shall now assume that* $2m - 2 \geq 3n$. It takes some experience to realize that the right approach to success lies in separating the two cases "$w = 0$ *is nonexceptional*" and "$w = 0$ *is exceptional*". Instead one might guess that the right generalization of Wienholtz's theorem consists in considering the cases

$$(C_L) \qquad (L-1)n \leq 2m - 2 < Ln, \quad L \in \mathbb{N} \text{ with } L \geq 3$$

and hoping that one can prove

$$E^{(j)}(0) = 0 \quad \text{for } 1 \leq j \leq L-1, \quad E^{(L)}(0) < 0$$

using appropriate choices of forced Jacobi fields in varying the minimal surface $\hat{X}$. Unfortunately this is not the case. To see what happens we study the two cases

$$(C_4) \qquad 3n \leq 2m - 2 < 4n$$

and

$$(C_5) \qquad 4n \le 2m - 2 < 5n$$

by computing $E^{(4)}(0)$ in the first case and $E^{(5)}(0)$ in the second one. We begin by treating special cases of (C_4) and (C_5), where we can proceed in a similar way as before with $E^{(3)}(0)$ for $2n \le 2m - 2 < 3n$.

The case (C_4) **with** $2m - 2 = 4p, p \in \mathbb{N}$.

Proposition 5. *If* $w\hat{Z}_{tw}(0)\tau + w\hat{X}_w\phi_t(0)$ *is holomorphic, then*

$$E^{(4)}(0) = -12\mathrm{Re}\int_{S^1} w^3 \hat{X}_{ww} \cdot \hat{X}_{ww}\tau^2\phi_t(0)\,dw. \tag{50}$$

Proof. Since $\hat{Z}_{ttw}(0)$ is holomorphic in B, the integrand of the first integral in (39) is holomorphic, and so this integral vanishes. □

Remark 3. In case (C_4) with $2m - 2 = 4p$ the branch point $w = 0$ is nonexceptional. To see this we note that $p < n$ whence

$$2m + 2 = 4(p + 1) < 4(n + 1)$$

and therefore

$$n + 1 < m + 1 < 2(n + 1).$$

Also note that $n = 1, 2, 3$ are not possible since $n = 1$ would imply $p < 1$; $n = 2$ would mean $p = 1$ whence $6 = 3n \le 4p = 4$; and $n = 3$ would imply $p \le 2$, and so $9 = 3n \le 4p = 8$. Finally $3n \le 4p$ and $n \ge 4$ yields $p \ge 3$.

Theorem 2. *If* $3n \le 2m - 2 = 4p < 4n$ *for some* $p \in \mathbb{N}$*, then one can find a variation* $\hat{Z}(t)$ *of* $\hat{X}$ *such that* $E^{(4)}(0) < 0$*, whereas* $E^{(j)}(0) = 0$ *for* $j = 1, 2, 3$.

Proof. First we want to choose $\tau = \phi(0)$ and $\phi_t(0)$ in such a way that the assumption of Proposition 5 is satisfied. To this end, set

$$\tau(w) := (a - ib)w^{-p-1} + (a + ib)w^{p+1},$$

which clearly is a generator of a forced Jacobi field. By (43) we get

$$\begin{aligned} &w\hat{X}_w(w)\tau(w) \\ &\quad = (a - ib)(A_1 w^{n-p} + A_2 w^{n-p+1} + \cdots + A_{2m-2n+1}w^{2m-n-p} + \cdots, \\ &\qquad R_m w^{m-p} + \cdots) + (a + ib)(A_1 w^{n+p+2} + \cdots, R_m w^{m+p+2} + \cdots). \end{aligned}$$

By (35) it follows

$$\begin{aligned} &w\hat{Z}_{tw}(w,0)\tau(w) = w[iw\hat{X}_w(w)\tau(w)]_w\tau(w) \\ &\quad = i(a - ib)^2((n - p)A_1 w^{n-2p-1} + (n - p + 1)A_2 w^{n-2p} + \cdots \\ &\qquad + (2m - n - p)A_{2m-2n+1}w^{2m-n-2p-1} + \cdots, (m - p)R_m w^{m-2p-1} + \cdots). \end{aligned}$$

Note that $2m-2=4p$ implies $m-2p-1=0$, whence $n-2p-1<0$ because of $m>n$, but $2m-n-2p-1=(m-2p-1)+(m-n)=m-n>0$. Thus the third component above has no pole, while the first (vectorial) component has a pole at least in the first term, but no pole anymore from the $(2m-2n+1)$-th term on. These poles will be removed by adding $w\hat{X}_w\phi_t(0)$ to $w\hat{Z}_{tw}(0)\tau$ with an appropriately chosen value of $\phi_t(0)$. We set $\phi_t(0)=\psi_1+\cdots+\psi_s$ where $\psi_1,\ldots,\psi_s$ are defined inductively. First set

$$\begin{aligned}\psi_1(w) := {} & -i(n-p)(a-ib)^2 w^{-2p-2} \\ & + i(n-p)(a+ib)^2 w^{2p+2}.\end{aligned}$$

Now $w\hat{Z}_{tw}(0)\tau + w\hat{X}_w\psi_1$ has no pole associated to A_1 while the poles associated to A_k, $1<k\le s$, are of the same order as before. Then we choose ψ_2 so that there is no pole associated to A_2, etc. The number s is the index of the last term $(n-p+s)A_{s+1}w^{n-2p+s-1}$ where $n-2p+s-1$ is ≥ 0 and $\le 2m-2n$. Note that

$$w\hat{X}_w(w) = (A_1 w^{n+1} + A_2 w^{n+2} + \cdots + A_{2m-2n+1} w^{2m-n+1} + \cdots, R_m w^{m+1} + \cdots)$$

and

$$A_1 \cdot A_k = 0 \quad \text{for } k=1,2,\ldots,2m-2n.$$

Therefore, $w\hat{X}_w\phi_t(0) = w\hat{X}_w\cdot[\psi_1+\psi_2+\cdots+\psi_s]$ removes all poles from $w\hat{Z}_{tw}(0)\tau$ and creates no new poles. Consequently $w\hat{Z}_{tw}(0)\tau + w\hat{X}_w\phi_t(0)$ is holomorphic, and so we have

$$E^{(4)}(0) = -12\mathrm{Re}\int_{S^1} w^3 \hat{X}_{ww}\cdot\hat{X}_{ww}\tau^2\phi_t(0)\,dw.$$

Formula (45) yields

$$w^3\hat{X}_{ww}(w)\cdot\hat{X}_{ww}(w) = (m-n)^2 R_m^2 w^{2m+1} + \cdots .$$

The leading term in $\phi_t(0)$ is that of ψ_1, and

$$\psi_1(w) = -i(n-p)(a-ib)^2 w^{-2p-2} + \cdots .$$

Furthermore,

$$\tau^2(w) = (a-ib)^2 w^{-2p-2} + \cdots ,$$

and so

$$\tau^2(w)\phi_t(w,0) = -i(a-ib)^4(n-p)w^{-4p-4} + \cdots .$$

Noticing that $2m+1=(2m+2)-1=4(p+1)-1$, and setting

$$\kappa := 12(m-n)^2(n-p) > 0$$

we obtain

$$E^{(4)}(0) = \kappa \text{Re}\left[i(a-ib)^4 R_m^2 \int_{S^1} \frac{dw}{w}\right] = -2\pi\kappa \text{Re}[(a-ib)^4 R_m^2]$$

and an appropriate choice of a and b yields $E^{(4)}(0) < 0$. Finally we note that $E^{(2)}(0) = 0$ and $E^{(3)}(0) = 0$ for the above choice of $\hat{Z}(t)$. The first statement follows from Proposition 1. To verify the second, we recall formula (34) from Proposition 2:

$$E^{(3)}(0) = -4\text{Re} \int_{S^1} w^3 \hat{X}_{ww} \cdot \hat{X}_{ww} \tau^3 \, dw.$$

From the preceding computations it follows that

$$w^3 \hat{X}_{ww}(w) \cdot \hat{X}_{ww}(w) \tau^3(w) = (m-n)^2 R_m^2 (a-ib)^3 w^{2m+1-3(p+1)} + \cdots ,$$

and, by assumption, $2m-2 = 4p$, whence

$$2m + 1 - 3(p+1) = 4p + 3 - 3(p+1) = p > 1;$$

therefore $E^{(3)}(0) = 0$. □

Remark 4. Under the special assumption that $2m - 2 = 4p$ we were able to carry out the program outlined above for $L = 4$. However, applying the method from Theorem 2 to cases when $2m - 2 \not\equiv 0 \mod 4$ one will get nowhere. Instead, trying another approach similar to that used in the proof of Theorem 1, one is able to handle the case (C_4) under the additional assumption $2m-2 \equiv 2 \mod 4$ by considering the next higher derivative, namely $E^{(5)}(0)$ instead of $E^{(4)}(0)$, cf. Theorem 4 stated later on. This seems to shatter the hope that one can always make $E^{(L)}(0)$ negative, with $E^{(j)}(0) = 0$ for $1 \le j \le L-1$, if (C_L) is satisfied. In fact, by studying assumption (C_5) we shall realize that (C_L) is probably not the appropriate classification for developing methods that in general lead to our goal. Rather, the case (C_5) will show us that one should distinguish between the cases "exceptional" and "nonexceptional" using the classification given in Lemma 3 to reach this purpose.

Let us mention that, assuming (C_4),the branch point $w = 0$ is nonexceptional according to Lemma 3, since $3n \le 2m - 2 < 4n$ implies

$$3(n+1) < 3n + 4 \le 2m + 2 < 4(n+1).$$

Let us now turn to the *investigation of* (C_5) *by means of the fifth derivative* $E^{(5)}(0)$.

Lemma 4. *If* $f(w) := w\hat{Z}_{tw}(0)\tau + w\hat{X}_w \phi_t(0)$ *is holomorphic, then*

$$\begin{aligned} &\hat{Z}_{ttw}(0) = \{iw[iw\hat{X}_w\tau]_w\tau + iw\hat{X}_w\phi_t(0)\}_w, \\ &\hat{Z}_{ttw}(0) \cdot \hat{X}_w = -\hat{Z}_{tw}(0) \cdot \hat{Z}_{tw}(0) = w^2 \hat{X}_{ww} \cdot \hat{X}_{ww}\tau^2. \end{aligned} \tag{51}$$

Proof. By (27) we have

$$\hat{Z}_{tw} = \{2H[\mathrm{Re}(iw\hat{Z}_w\phi)]\}_w$$

whence

$$\hat{Z}_{ttw} = \{2H[\mathrm{Re}(iw\hat{Z}_{tw}\phi + iw\hat{Z}_w\phi_t)]\}_w$$

and therefore

$$\begin{aligned}\hat{Z}_{ttw}(0) &= \{2H[\mathrm{Re}(if)]\}_w = \{if\}_w \\ &= \{iw\hat{Z}_{tw}(0)\tau + iw\hat{X}_w\phi_t(0)\}_w.\end{aligned}$$

By (35),

$$\hat{Z}_{tw}(0) = [iw\hat{X}_w\tau]_w,$$

and so

$$\hat{Z}_{ttw}(0) = \{iw[iw\hat{X}_w\tau]_w\tau + iw\hat{X}_w\phi_t(0)\}_w.$$

It follows that

$$Z_{ttw}(0)\cdot\hat{X}_w = \{iw[i\hat{X}_w\tau + iw\hat{X}_{ww}\tau + iw\hat{X}_w\tau_w]\tau + iw\hat{X}_w\phi_t(0)\}_w\cdot\hat{X}_w.$$

From $\hat{X}_w\cdot\hat{X}_w = 0$ one obtains $\hat{X}_w\cdot\hat{X}_{ww} = 0$, and then

$$\hat{X}_{www}\cdot\hat{X}_w = -\hat{X}_{ww}\cdot\hat{X}_{ww}.$$

This leads to

$$\begin{aligned}\hat{Z}_{ttw}(0)\cdot\hat{X}_w &= -w^2\hat{X}_{www}\cdot\hat{X}_w\tau^2 \\ &= w^2\hat{X}_{ww}\cdot\hat{X}_{ww}\tau^2 = -\hat{Z}_{tw}(0)\cdot\hat{Z}_{tw}(0),\end{aligned}$$

taking (37) into account. □

Proposition 4 and Lemma 4 imply

Proposition 6. *If* $f(w) := w\hat{Z}_{tw}(0)\tau + w\hat{X}_w\phi_t(0)$ *is holomorphic, then*

(52) $$E^{(5)}(0) = 12\mathrm{Re}\int_{S^1}[w\hat{Z}_{ttw}(0)\cdot\hat{Z}_{ttw}(0)\tau + 4w\hat{Z}_{ttw}(0)\cdot\hat{Z}_{tw}(0)\phi_t(0)]\,dw.$$

We are now going to discuss the envisioned program for the case (C_5) using the simplified form (52) for the fifth derivative $E^{(5)}(0)$. It will be useful to distinguish several subcases of (C_5):

(a) $5n \le 2m + 2$,
(b) $5n > 2m + 2$.

In case (a) we have $5n \le 2m + 2 < 5n + 4$, that is,

$$2m + 2 = 5n + \alpha, \quad 0 \le \alpha \le 3.$$

Therefore (a) consists of the four subcases

$$2m - 5n = 0, 1, -1, -2. \tag{53}$$

In case (b) we have $5n > 2m + 2$, and (C_5) implies $2m + 2 \ge n + 4$, whence $5n > n + 4$, and so we have $n > 1$ in case (b).

Case (a) allows an easy treatment based on the following representation of $2m + 2$ which we apply successively for $\alpha = 0, 1, 2, 3$ to deal with the four cases (53). We write

$$\alpha(n + 1) + \beta n = 2m + 2$$

with $\alpha := 2m + 2 - 5n$, $\beta := 5 - \alpha$ where $0 \le \alpha \le 3$ and $\beta \ge 2$. Then we choose

$$\tau := \tau_0 + \epsilon\tau_1, \quad \epsilon > 0,$$

where

$$\tau_0 := cw^{-n} + \overline{c}w^n, \quad \tau_1 := cw^{-n-1} + \overline{c}w^{n+1}, \quad c \in \mathbb{C}.$$

With an appropriate choice of $\phi_t(0)$ we obtain by an elimination procedure similar to the one used in the proof of Theorem 2 that $f := w\hat{Z}_{tw}(0)\tau + w\hat{X}_w\phi_t(0)$ is holomorphic. Here and in the sequel we omit the lengthy computations and merely state the results. As f is holomorphic one can use formula (52) for $E^{(5)}(0)$; we investigate the four different cases of (53) separately, but note that always

$$E^{(j)}(0) = 0, \quad j = 1, \dots, 4.$$

(I) $\boldsymbol{2m - 5n = 0}$, $1 \le n \le 4$. Only (i) $n = 2$ and (ii) $n = 4$ are possible. This leads to

(i) $n = 2,\ m = 5,\ (m + 1) = 2(n + 1)$, i.e. $w = 0$ is exceptional;
(ii) $n = 4,\ m = 10$, hence $m + 1 \not\equiv 0 \mod (n + 1)$, and so $w = 0$ is not exceptional.

For (i) we obtain $E^{(5)}(0) = 0 + o(\epsilon)$, whereas (ii) yields

$$E^{(5)}(0) = 12\mathrm{Re}[2\pi i \cdot 336 \cdot \epsilon^2 \cdot c^5 R_m^2] + o(\epsilon^2)$$

which can be made negative by appropriate choice of c. Thus the method is inconclusive for (i), but gives the desired result for (ii).

(II) $\boldsymbol{2m - 5n = 1}$, $1 \le n \le 4$. Then n necessarily either (i) $n = 1$ or (ii) $n = 3$. Here,

(i) $n = 1,\ m = 3,\ m + 1 = 2(n + 1)$, i.e. $w = 0$ is exceptional;
(ii) $n = 3,\ m = 8$, and $m+1 \not\equiv 0 \mod (n+1)$, hence $w = 0$ is not exceptional.

For (i) it follows that $E^{(5)}(0) = 0 + o(\epsilon^3)$, i.e. the method is inconclusive, while for (ii) one gets

$$E^{(5)}(0) = 12 \cdot \mathrm{Re}[2\pi i \cdot 250 \cdot \epsilon^3 \cdot c^5 R_m^2] + o(\epsilon^3),$$

and so $E^{(5)}(0) < 0$ for a suitable choice of c.

(III) $\boldsymbol{2m - 5n = -1}$, $1 \le n \le 4$. Then either (i) $n = 1$ or (ii) $n = 3$, i.e.

(i) $n = 1,\ m = 2$, and so $m+1 \not\equiv 0 \mod (n+1)$, i.e. $w = 0$ is not exceptional.
(ii) $n = 3,\ m = 7$, whence $m + 1 = 2(n + 1)$, i.e. $w = 0$ is exceptional.

For (i) we have $2m - 2 < 3n$, and this case was already dealt with in the positive sense by using $E^{(3)}(0)$, cf. Theorem 1. For (ii) the method is again inconclusive since one obtains

$$E^{(5)}(0) = 0 + o(\epsilon).$$

(IV) $\boldsymbol{2m - 5n = -2}$, $1 \le n \le 4$. Then either (i) $n = 2$ or (ii) $n = 4$, that is,

(i) $n = 2,\ m = 4$, whence $m+1 \not\equiv 0 \mod (n+1)$, i.e. $w = 0$ is not exceptional.
(ii) $n = 4,\ m = 9$, and so $m + 1 = 2(n + 1)$, i.e. $w = 0$ is exceptional.

In case (i) we have $3n = 2m - 2 < 4n$, i.e. condition (C_4) holds, and this case will be tackled by Theorem 4, to be stated later on. Case (ii) leads to $E^{(5)}(0) = 0 + o(1)$ as $\epsilon \to 0$ which is once again inconclusive.

Conclusion. *The method is inconclusive in all of the exceptional cases. In the nonexceptional cases it either leads to the positive result $E^{(5)}(0) < 0$ for appropriate choice of c, or one can apply the cases (C_3) or (C_4), and here one obtains the desired results $E^{(3)}(0) < 0$ or $E^{(4)}(0) < 0$ respectively (see Theorems 1 and 4).*

Now we turn to the case (b). We first note that (C_5) together with (b) implies $4(n + 1) \le 2m + 2 < 5n$. Hence either (i) $2(n + 1) = m + 1$, or (ii) $4(n + 1) < 2m + 2 < 5n$. Therefore, $w = 0$ is exceptional in case (i) and nonexceptional in case (ii). Furthermore we have

$$2m + 2 = 4n + k \quad \text{with } \le k < n,$$

where $k = 4$ is the case (i) and $4 < k < n$ is the case (ii).

In order to treat the case (b) which in some sense is the "*general subcase*" of (C_5) we use

$$\tau := c \cdot (\epsilon w^{-n} + w^{-k}) + \overline{c} \cdot (\epsilon w^n + w^k).$$

Choosing $\phi_t(0)$ appropriately we achieve that f is holomorphic, and so $E^{(5)}(0)$ is given by (52). Moreover, $E^{(j)}(0) = 0$ for $1 \le j \le 4$. It turns out that

$$E^{(5)}(0) = 12 \cdot \mathrm{Re}[2\pi i c^3 \epsilon^4 \gamma R_m^2] + o(\epsilon^4), \quad \epsilon > 0,$$

with

$$\gamma = (m-n)(k-4)^2 \left[\frac{5}{4}n + \frac{5}{8}(k-2)\right]$$

and $\gamma = 0$ in case (i), whereas $\gamma > 0$ in case (ii).

Thus the following result is established:

Theorem 3. *Suppose that* (C_5) *and* (b) *hold, hence* $4n+4 \le 2m+2 < 5n$. *This implies* $2m+2 = 4n+k$ *with* $4 \le k < n$. *For* $k = 4$ *the branch point* $w = 0$ *is exceptional, and the method is nonconclusive. If, however,* $4 < k < n$, *then* $\tau = \phi(0)$ *and* $\phi_t(0)$ *can be chosen in such a way that* $E^{(5)}(0) < 0$ *and* $E^{(j)}(0) = 0$ *for* $j = 1, \dots, 4$.

Next, we want to prove that the remaining cases of (C_4) lead to a conclusive result also for the remaining possibility $2m-2 \ne 4p$ for some $p \in \mathbb{N}$ with $1 \le p < n$. Because of $3n \le 2m-2 < 4n$ we can write $2m-2 = 4p+k$ with $0 < k < 4$ (the case $k = 0$ was treated before). Since k must be even, we are left with $k = 2$, and we recall that $w = 0$ is a nonexceptional branch point in the case (C_4).

Theorem 4. *Suppose that* $3n \le 2m-2 = 4p+2 < 4n$ *with* $1 \le p < n$ *holds (this is the subcase of* (C_4) *that was not treated in Theorem 2). Then* $\tau = \phi(0)$ *and* $\phi_t(0)$ *can be chosen in such a way that*

$$E^{(j)}(0) = 0 \quad \text{for } j = 1, \dots, 4, \quad E^{(5)}(0) < 0.$$

Proof. This follows with

$$\tau := c(w^{-k} + \epsilon w^{-p-1}) + \overline{c} \cdot (w^k + \epsilon w^{p+1}), \quad \epsilon > 0.$$

Then $E^{(j)}(0) = 0$ for $1 \le j \le 4$ and

$$E^{(5)}(0) = 12 \cdot \mathrm{Re}[2\pi i c^5 \epsilon^4 R_m^2 \gamma] + o(\epsilon^4),$$

where

$$\begin{aligned}\gamma := {} & (m-n)^2(m-2p-1)^2 + 4(m-n)^2(m-2p-1)(m-k-p) \\ & -8(n-p)(m-p)(m-n)(m-k-p) \\ & -4(m-n)(m-2p-1)[(n-p)(m-p+1) + (m-p)(2n-p-k+1)].\end{aligned}$$

Since $4(p+1) + k = 2m+2$ and

$$5(p+1) = 4p + k + p + (5-k) = (2m-2) + 3 + p \ge 2m+2$$

one can prove that $\gamma < 0$. Thus one can make $E^{(5)}(0) < 0$ for a suitable choice of c. □

Let us return to the case (C_4): $3n \le 2m-2 < 4n$ which splits into the two subcases $2m-2 \equiv 0 \bmod 4$ and $2m-2 \equiv 2 \bmod 4$. The first one was dealt with by $E^{(4)}(0)$, cf. Theorem 2, the second by $E^{(5)}(0)$, see Theorem 4. Combining both results we obtain

Theorem 5. *Let $\hat{X}$ be a minimal surface in normal form having the branch point $w=0$ with the order n and the index m such that (C_4) holds. Then $\hat{X}$ cannot be a weak minimizer of D.*

We want to give a new proof of this result which combines both cases into a single one. Note first that $3n \le 2m-2 < 4n$ is equivalent to $3(n+1)+1 \le 2m+2 < 4(n+1) = 3(n+1)+n+1$. Therefore $w=0$ is not exceptional, and

(54) $$2m+2 = 3(n+1)+r, \quad 1 \le r \le n.$$

The new approach consists in choosing the generator $\tau = \phi(0)$ as

(55) $$\tau = \tau_0 + \tau_1 \quad \text{with } \tau_0 := \epsilon c w^{-n-1} + \epsilon \overline{c} w^{n+1}, \; \tau_1 := c w^{-r} + \overline{c} w^r, \; c \in \mathbb{C}.$$

We need the following auxiliary result:

Lemma 5. *For any $\nu \in \mathbb{N}$ and $a \in \mathbb{C}$ we have*

(56) $$\{2H[\mathrm{Re}(aw^{-\nu})]\}_w = \nu \overline{a} w^{\nu-1} \quad \textit{on } \overline{B}.$$

Proof. On S^1 one has $w^{-\nu} = \overline{w}^\nu$ whence

$$aw^{-\nu} = a\overline{w}^\nu = \overline{\overline{a}w^\nu} \quad \text{on } S^1$$

and therefore

$$\mathrm{Re}(aw^{-\nu}) = \mathrm{Re}(\overline{a}w^\nu) \quad \text{on } S^1.$$

Consequently

$$2H[\mathrm{Re}(aw^{-\nu})] = 2H[\mathrm{Re}(\overline{a}w^\nu)] \quad \text{on } \overline{B}.$$

This implies

$$\{2H[\mathrm{Re}(aw^{-\nu})]\}_w = \{2H[\mathrm{Re}(\overline{a}w^\nu)]\}_w \quad \text{on } \overline{B}.$$

Finally, since $\overline{a}w^\nu$ is holomorphic in $\mathbb{C}$, it follows that

$$\{2H[\mathrm{Re}(\overline{a}w^\nu)]\}_w = \frac{d}{dw}(\overline{a}w^\nu) = \nu\overline{a}w^{\nu-1} \quad \text{on } \overline{B}. \qquad \square$$

Now we calculate $E^{(4)}(0)$ using the formulae (37) and (39):

(57) $$E^{(4)}(0) = 12\mathrm{Re}\int_{S^1} \hat{Z}_{ttw}(0)\cdot[w\hat{Z}_{tw}(0)\tau + w\hat{X}_w\phi_t(0)]\,dw + 12\mathrm{Re}\int_{S^1} w\hat{Z}_{tw}(0)\cdot\hat{Z}_{tw}(0)\phi_t(0)\,dw.$$

From

$$w\hat{X}_w = (A_1 w^{n+1} + \cdots + A_{2m-2n+1} w^{2m-n+1} + \cdots, R_m w^{m+1} + \cdots)$$

it follows that

$$\begin{aligned} &w\hat{X}_w \tau \\ &\quad = c\epsilon(A_1 + \cdots + A_{2m-2n+1} w^{2m-2n} + \cdots, R_m w^{m-n} + \cdots) \\ &\qquad + c(A_1 w^{n+1-r} + \cdots + A_{2m-2n+1} w^{2m-n-r+1} + \cdots, R_m w^{m+1-r} + \cdots) \\ &\qquad + g(w), \quad g(w) := w\hat{X}_w(w) \cdot [\epsilon\overline{c} w^{n+1} + \overline{c} w^r]. \end{aligned}$$

The expression $g(w)$ is "better" than the sum $T_1 + T_2$ of the first two terms T_1, T_2 on the right-hand side of this equation, in the sense that it is built in a similar way as $T_1 + T_2$ except that it is less singular. In the sequel this phenomenon will appear repeatedly, and so we shall always use a notation similar to the following:

$$w\hat{X}_w \tau = T_1 + T_2 + \langle \text{better} \rangle.$$

This sloppy notation will not do any harm since in the end we shall see that each of the two integrands in (57) possesses exactly one term of order w^{-1} as w-terms of least order, and no expression labelled "better" is contributing to them.

Using (35) one obtains

$$\begin{aligned} \hat{Z}_{tw}(0) = \; & ic\epsilon(A_2 + \cdots + (2m-2n) A_{2m-2n+1} w^{2m-2n-1} + \cdots, \\ & (m-n) R_m w^{m-n-1} + \cdots) \\ & + ic((n+1-r) A_1 w^{n-r} + \cdots \\ & + (2m-n+1-r) A_{2m-2n+1} w^{2m-n-r} \\ & + \cdots, (m+1-r) R_m w^{m-r} + \cdots) + \langle \text{better} \rangle. \end{aligned}$$

This implies

$$\begin{aligned} w\hat{Z}_{tw}(0)\tau = \; & ic^2\epsilon^2(A_2 w^{-n} + \cdots + (2m-2n) A_{2m-2n+1} w^{2m-3n-1} + \cdots, \\ & (m-n) R_m w^{m-2n-1} + \cdots) \\ & + ic^2\epsilon((n+1-r) A_1 w^{-r} + \cdots \\ & + (2m-n+1-r) A_{2m-2n+1} w^{2m-2n-r} \\ & + \cdots, (m+1-r) R_m w^{m-n-r} + \cdots) + \langle \text{better} \rangle. \end{aligned}$$

Recall that $A_k = \lambda_k A_1$ for $k = 1, \ldots, 2m-2n$. In order to remove all poles in the first two components of

$$f := w\hat{Z}_{tw}(0)\tau + w\hat{X}_w \phi_t(0)$$

one chooses $\phi_t(0)$ in a fashion similar to that used in the proof of Theorem 2:

$$\phi_t(0) := -ic^2\lambda_2\epsilon^2 w^{-2n-1} - ic^2\epsilon(n+1-r)w^{-n-1-r} + \cdots .$$

Then

$$\begin{aligned} f \,=\, & ic^2\epsilon^2(\cdots(2m-2n)A_{2m-2n+1}w^{2m-3n-1} \\ & + \cdots, (m-n)R_m w^{m-2n-1} + \cdots) \\ & + ic^2\epsilon(\cdots(2m-n+1-r)A_{2m-2n+1}w^{2m-2n-r} \\ & + \cdots, (m-n)R_m w^{m-n-r} + \cdots) \\ & + \langle \text{better} \rangle. \end{aligned}$$

Here and in the sequel, $\cdots$ stand for non-pole terms with coefficients A_j with $j \le 2m-2n$.

The first two components of f (i.e. the expressions before the commata) are holomorphic; the worst pole in the third component is the term with the power w^{m-2n-1}; note that

$$\gamma := m - 2n - 1 = \frac{1}{2}[(2m+2) - 4(n+1)] < 0.$$

Thus Lemma 5 yields

$$\{H[\mathrm{Re}(R_m w^\gamma)]\}_w = -\gamma \overline{R}_m w^{-\gamma-1}.$$

Using a formula established in the proof of Lemma 4 one obtains

$$\begin{aligned} \hat{Z}_{ttw}(0) \,=\, & -c^2\epsilon^2(\cdots(2m-2n)(2m-3n-1)A_{2m-2n+1}w^{2m-3n-2}, \\ & (m-n)(2n+1-m)\overline{R}_m w^{2n-m} + \cdots) \\ & - c^2\epsilon(\cdots(2m-n)(2m-2n-r)A_{2m-2n+1}w^{2m-2n-r-1} + \cdots, \\ & (m-n)(m-n-r)R_m w^{m-n-r-1}) + \langle \text{better} \rangle. \end{aligned}$$

It follows that

$$\begin{aligned} & \hat{Z}_{ttw}(0) \cdot [w\hat{Z}_{tw}(0)\tau + w\hat{X}_w\phi_t(0)] \\ & \quad = \{-ic^4\epsilon^3(m-n)^2(m-n-r)R_m^2 w^{-1} + \cdots\} + o(\epsilon^3) \end{aligned}$$

since

$$2m - 3n - r - 2 = (2m+2) - [3(n+1) + r] - 1 = -1. \tag{58}$$

A straight-forward calculation shows

$$\begin{aligned} & w\hat{Z}_{tw}(0) \cdot \hat{Z}_{tw}(0)\phi_t(0) \\ & \quad = \{ic^4\epsilon^3(m-n)^2(n+1-r)R_m^2 w^{-1} + \cdots\} + o(\epsilon^3). \end{aligned}$$

Thus one obtains by (57) that

$$E^{(4)}(0) = 12\epsilon^3 \mathrm{Re} \int_{S^1} ikc^4 R_m^2 \frac{dw}{w} + o(\epsilon^3)$$

with

$$k := (m-n)^2(n+1-r) - (m-n)^2(m-n-r).$$

Since

$$m - n - r = \frac{1}{2}\{(2m+2) - 2(n+1) - 2r\} = \frac{1}{2}(n+1-r)$$

it follows that

$$k = \frac{1}{2}(m-n)^2(n+1-r) > 0.$$

Hence, by suitable choice of $c \in \mathbb{C}$ one can achieve that $E^{(4)}(0) < 0$, while $E^{(j)}(0) = 0$ for $j = 1, 2, 3$. This concludes the new proof of Theorem 5. □

Finally we want to show that it often is possible to estimate the index m of an interior branch point w_0 of a minimal surface $\hat{X} \in \mathcal{C}(\Gamma)$ with the aid of a geometric condition on its boundary contour Γ. Following an idea by J.C.C. Nitsche, we use Radó's lemma for this purpose (cf. Vol. 1, Section 4.9), which states the following. *If $f \in C^0(\overline{B})$ is harmonic in B, $f(w) \not\equiv 0$ in B, and $\nabla^j f(w_0) = 0$ at $w_0 \in B$ for $j = 0, 1, \dots, m$, then f has at least $2(m+1)$ different zeros on ∂B.*

We can assume that the minimal surface $\hat{X}$ is transformed into the normal form with respect to the branch point $w_0 = 0$ having the index m. If the contour Γ is nonplanar, then $X^3(w) \not\equiv X_0^3 := X^3(0)$, whence $m < \infty$ and

$$X^3(w) = X_0^3 + \mathrm{Re}[cw^{m+1} + O(w^{m+2})] \quad \text{for } w \to 0$$

with $c \in \mathbb{C} \setminus \{0\}$. Hence $f := X^3 - X_0^3$ satisfies the assumptions of Radó's lemma, and therefore f has at least $2(m+1)$ different zeros on ∂B. Hence the plane $\Pi := \{(x^1, x^2, x^3) \in \mathbb{R}^3 : x^3 = X_0^3\}$ intersects Γ in at least $2(m+1)$ different points. If $m = \infty$ then even $\Gamma \subset \Pi$, and so we obtain:

Proposition 7. *If the minimal surface $\hat{X} \in \mathcal{C}(\Gamma)$ possesses a branch point $w_0 \in B$ with the index m, then there is a plane Π in $\mathbb{R}^3$ which intersects Γ in at least $2(m+1)$ different points. Consequently, if every plane in $\mathbb{R}^3$ intersects Γ in at most k different points, then the index m is bounded by*

$$2m + 2 \leq k.$$

This result motivates the following

Definition 4. *The* **cut number** *$c(\Gamma)$ of a closed Jordan curve Γ in $\mathbb{R}^3$ is the supremum of the number of intersection points of Γ with any (affine) plane Π in $\mathbb{R}^3$, i.e.*

$$c(\Gamma) := \sup\{\sharp(\Gamma \cap \Pi) : \ \Pi = \textit{affine plane in } \mathbb{R}^3\}. \tag{59}$$

It is easy to see that

$$4 \leq c(\Gamma) \leq \infty, \tag{60}$$

and for any nonplanar, real analytic, closed Jordan curve the cut number $c(\Gamma)$ *is finite.*

We can rephrase the second statement of Proposition 7 as follows:

Proposition 8. *The index* m *of any interior branch point of a minimal surface* $\hat{X} \in \mathcal{C}(\Gamma)$ *is bounded by*

$$2m + 2 \leq c(\Gamma). \tag{61}$$

If n is the order and m the index of some branch point, then $1 \leq n < m$. On the other hand, $c(\Gamma) = 4$ implies $m \leq 1$, and $c(\Gamma) = 6$ yields $m \leq 2$. Thus we obtain

Corollary 1. (i) *If* $c(\Gamma) = 4$ *then every minimal surface* $\hat{X} \in \mathcal{C}(\Gamma)$ *is free of interior branch points.*

(ii) *If* $c(\Gamma) = 6$ *then any minimal surface* $\hat{X} \in \mathcal{C}(\Gamma)$ *has at most simple interior branch points of index two; if* $\hat{X}$ *has an interior branch point, it cannot be a weak minimizer of* D *in* $\mathcal{C}(\Gamma)$.

Proof. (i) follows from $1 \leq n < m \leq 1$, which is impossible. (ii) $1 \leq n < m \leq 2$ implies $n = 1$ and $m = 2$ for an interior branch point w_0 of $\hat{X}$, whence $2n \leq 2m - 2 < 3$. Thus condition (C_3) is satisfied, and therefore the last assertion follows from Theorem 1. □

Corollary 2. *Let* $\hat{X} \in \mathcal{C}(\Gamma)$ *be a minimal surface with an interior branch point of order* n, *and suppose that the cut number of* Γ *satisfies* $c(\Gamma) \leq 4n+3$. *Then* $\hat{X}$ *is not a weak minimizer of* D *in* $\mathcal{C}(\Gamma)$.

Proof. By (61) we have

$$2m + 2 \leq 4n + 3;$$

hence either

$$2n + 4 \leq 2m + 2 < 3n + 4 \Leftrightarrow 2n \leq 2m - 2 < 3n$$

or

$$3n + 4 \leq 2m + 2 < 4n + 4 \Leftrightarrow 3n \leq 2m - 2 < 4n$$

hold true, i.e. either (C_3) or (C_4) are fulfilled. In the first case the assertion follows from Theorem 1, in the second from Theorem 5. □

6.2 The Theorem for $n+1$ Even and $m+1$ Odd

In this section we want to show that *a (nonplanar) weak relative minimizer* $\hat{X}$ *of Dirichlet's integral* D *that is given in the normal form cannot have* $w=0$ *as a branch point if its order* n *is odd and its index* m *is even.* Note that such a branch point is **nonexceptional** since $n+1$ cannot be a divisor of $m+1$. We shall give the proof only under the assumptions $n \geq 3$ since $n=1$ is easily dealt with by a method presented in a forthcoming book by A. Tromba. (Moreover it would suffice to treat the case $m \geq 6$ since $2m-2 < 3n$ is already treated by the Wienholtz theorem. So $2m \geq 3n+2 \geq 11$, i.e. $m \geq 6$ since m is even.)

The Strategy of the Proof

The strategy to find the first nonvanishing derivative of $E(t)$ at $t=0$ that can be made negative consists in the following four steps:

(I) Guess the candidate L for which $E^{(L)}(0) < 0$ can be achieved with a suitable choice of the generator $\tau = \phi(0)$.

(II) Select $D_t^\beta \phi(0)$, $\beta \geq 1$, so that the lower order derivatives $E^{(j)}(0), j = 1, 2, \ldots, L-1$ vanish, $(D_t^\beta := \frac{\partial^\beta}{\partial t^\beta})$.

(III) Prove that

$$E^{(L)}(0) = \operatorname{Re} \int_{S^1} c^L k R_m^2 \frac{dw}{w} = \operatorname{Re}\{2\pi i c^L k R_m^2\},$$

where $c \neq 0$ is a complex number which can be chosen arbitrarily, and $k \in \mathbb{C}$ is to the computed.

(IV) Show that $k \neq 0$.

Remark 1. In order to achieve (II) one tries to choose $D_t^\beta \phi(0)$, $\beta \geq 1$, in such a way that the integrands of $E^{(j)}(0)$ for $j < L$ are free of any poles and, therefore, free of first-order poles. To see that this strategy is advisable, let us consider the case $L=5$; then we have to achieve $E^{(4)}(0) = 0$. Recall that $E^{(4)}(0)$ consists of two terms, one of which has the form

$$I := 12 \operatorname{Re} \int_{S^1} \{2H[\operatorname{Re} if]\}_w f \, dw,$$

where

$$f := w[iw\hat{X}_w \tau]_w \tau + w\hat{X}_w \phi_t(0).$$

Assume that f had poles, say,

$$f(w) = g(w) + h(w), \quad g(w) = \sum_{j \geq 1} a_j w^{-j}, \; h = \text{holomorphic in } B,$$

and $h \in C^0(\overline{B})$. Then, by Lemma 5 of Section 6.1,

$$\{2H[\mathrm{Re}\, if]\}_w(w) = g^*(w) + h'(w), \quad g^*(w) := -i \sum_{j\geq 1} j\overline{a}_j w^{j-1}.$$

Thus, $I = 12 \cdot \{I_1 + I_2 + I_3\}$, with

$$I_1 := \mathrm{Re} \int_{S^1} g^* g\, dw, \quad I_2 := \mathrm{Re} \int_{S^1} h' g\, dw, \quad I_3 := \mathrm{Re} \int_{S^1} (g^* h + h' h)\, dw.$$

The worst term is I_1; one obtains

$$I_1 = \mathrm{Re} \int_{S^1} \sum_{j,\ell\geq 1} (-ij\overline{a}_j w^{j-1} a_\ell w^{-\ell})\, dw = 2\pi \sum_{j\geq 1} j|a_j|^2 > 0$$

and $I_3 = 0$. Hence, in order to achieve $I = 0$, one would have to balance I_2 against $I_1 > 0$ which seems to be pretty hopeless.

Let us now apply the "strategy" to prove

Theorem 1. *Let $\hat{X}$ be a nonplanar minimal surface in normal form that has $w = 0$ as a branch point of odd order $n \geq 3$ and of even index $m \geq 4$. Then, by a suitable choice of $\tau = \phi(0)$ and $D_t^\beta \phi(0)$, one can achieve that*

$$E^{(m+1)}(0) < 0 \quad \text{and} \quad E^{(j)}(0) = 0 \quad \text{for } 1 \leq j \leq m.$$

Proof. Set $N := L - 1$, $M := L - (\alpha + \beta + 1) = N - (\alpha + \beta)$, hence $L - 1 = \alpha + \beta + M$. By Leibniz's formula,

$$D_t^N \{[\hat{Z}_w \cdot \hat{Z}_w]\phi\} = \sum_{\alpha=0}^{N-\beta} \sum_{\beta=0}^{N} \frac{N!}{\alpha!\beta!(N-\beta-\alpha)!} (D_t^{N-\beta-\alpha} \hat{Z}_w) \cdot (D_t^\alpha \hat{Z}_w) D_t^\beta \phi.$$

Since

$$D_t E(t) = 2\mathrm{Re} \int_{S^1} w\hat{Z}(t)_w \cdot \hat{Z}(t)_w \phi(t)\, dw,$$

we can use Leibniz's formula to compute $E^{(L)}(t)$ from

$$E^{(L)}(t) = 2\mathrm{Re} \int_{S^1} w D_t^N \{[\hat{Z}_w(t) \cdot \hat{Z}_w(t)]\phi(t)\}\, dw.$$

We choose $L := m + 1$; then $L \geq 5$ as we have assumed $m \geq 4$. It follows that

$$E^{(L)}(0) = J_1 + J_2 + J_3, \tag{1}$$

where the terms J_1, J_2, J_3 are defined as follows: Set

$$T^{\alpha,\beta} := w(D_t^\alpha \hat{Z}(0))_w D_t^\beta \phi(0). \tag{2}$$

Then,

(3) $$J_1 := 4\,\mathrm{Re}\int_{S^1}[D_t^{L-1}\hat{Z}(0)]_w\cdot(w\hat{X}_w\tau)\,dw + 4\cdot(L-1)\,\mathrm{Re}\int_{S^1}[D_t^{L-2}\hat{Z}(0)]_w f\,dw + 4\sum_{M>\frac{1}{2}(L-1)}^{L-3}\frac{(L-1)!}{M!(L-M-1)!}\,\mathrm{Re}\int_{S^1}[D_t^M\hat{Z}(0)]_w\cdot g_{L-M-1}\,dw,$$

$$f := T^{1,0}+T^{0,1} = w[\hat{Z}_t(0)]_w\tau + w\hat{X}_w\phi_t(0),$$

$$g_\nu := \sum_{\alpha+\beta=\nu} c^\nu_{\alpha\beta}T^{\alpha,\beta} \quad \text{with } c^\nu_{\alpha\beta} := \frac{\nu!}{\alpha!\beta!};$$

(4) $$J_2 := \sum_{M=2}^{\frac{1}{2}(L-1)}\frac{2(L-1)!}{M!M!}\,\mathrm{Re}\int_{S^1}[D_t^M\hat{Z}(0)]_w\cdot h_M\,dw + 2(L-1)(L-2)\,\mathrm{Re}\int_{S^1}[\hat{Z}_t(0)]_w\cdot T^{1,L-3}\,dw,$$

$$h_M := \sum_{\alpha=0}^{M}\psi(M,\alpha)\frac{M!}{\alpha!(L-1-M-\alpha)!}T^{\alpha,L-1-M-\alpha},$$

$$\psi(M,\alpha) := 1 \quad \text{for } \alpha = M, \quad \psi(M,\alpha) := 2 \quad \text{for } \alpha \neq M;$$

(5) $$J_3 := 4(L-1)\,\mathrm{Re}\int_{S^1}w\hat{Z}_{tw}(0)\cdot\hat{X}_wD_t^{L-2}\phi(0)\,dw + 2\mathrm{Re}\int_{S^1}w\hat{X}_w\cdot\hat{X}_wD_t^{L-1}\phi(0)\,dw.$$

We have $J_3 = 0$ since $\hat{X}_w\cdot\hat{X}_w = 0$ and $\hat{Z}_{tw}(0)\cdot\hat{X}_w = 0$ on account of formula (36) in 6.1.

Now we proceed as follows:

Step 1. We choose $\tau = \phi(0)$ and $D_t^\beta\phi(0)$ for $\beta \geq 1$ in such a way that f and g_{L-M-1} are holomorphic. Then the integrands of the three integrals in J_1 are holomorphic because all w-derivatives $[D_t^j\hat{Z}(0)]_w$ of the harmonic functions $D_t^j\hat{Z}(t)$ are holomorphic. Then it follows that $J_1 = 0$, and thus we have

(6) $$E^{(L)}(0) = J_2.$$

Step 2. Then it will be shown that $E^{(L)}(0)$ reduces to the single term

(7) $$E^{(L)}(0) = \frac{2\cdot m!}{\left(\frac{m}{2}\right)!\left(\frac{m}{2}\right)!}\,\mathrm{Re}\int_{S^1}w[D_t^{m/2}\hat{Z}(0)]_w\cdot[D_t^{m/2}\hat{Z}(0)]_w\tau\,dw$$

which can be calculated explicitly; it will be shown that

$$E^{(L)}(0) = \frac{2 \cdot m!}{\left(\frac{m}{2}\right)!\left(\frac{m}{2}\right)!} \mathrm{Re}(2\pi i \cdot \kappa \cdot R_m^2), \tag{8}$$

where κ is the number

$$\kappa := i^{L-1}(a-ib)^L (m-1)^2 (m-3)^2 \cdots 3^2 \cdot 1^2 \tag{9}$$

if the generator $\tau = \phi(0)$ is chosen as

$$\tau(w) := (a-ib)w^{-2} + (a+ib)w^2. \tag{10}$$

For a suitable choice of $(a-ib)$ one obtains $E^{(L)}(0) < 0$. Furthermore the construction will yield $E^{(j)}(0) = 0$ for $1 \le j \le L-1$.

Before we carry out this program for general $n \ge 3$, $m \ge 4$, $n =$ odd, $m =$ even, we explain the procedure for the simplest possible case: $n = 3$ and $m = 4$.

From the normal form for $\hat{X}_w$ with the order n and the index m of the branch point $w = 0$ we obtain

$$w\hat{X}_w = (A_1 w^{n+1} + \cdots + A_{2m-2n+1} w^{2m-n+1} + \cdots, R_m w^{m+1} + \cdots). \tag{11}$$

Choosing τ according to (10) it follows from

$$[\hat{Z}_t(0)]_w = (iw\hat{X}_w \tau)_w$$

that

$$\begin{aligned} [\hat{Z}_t(0)]_w &\\ = (a-ib)(i(n-1)A_1 w^{n-2} &+ inA_2 w^{n-1} + \cdots \\ &+ i(2m-n-1)A_{2m-2n+1} w^{2m-n-2}, i(m-1)R_m w^{m-2} + \cdots) \\ &+ \langle \text{better} \rangle. \end{aligned} \tag{12}$$

Here, $\langle$better$\rangle$ stands again for terms that are similarly built as those in the preceding expression but whose w-powers attached to corresponding coefficients are of higher order. Then

$$\begin{aligned} w[\hat{Z}_t(0)]_w \tau &\\ = (a-ib)^2(i(n-1)A_1 w^{n-3} &+ inA_2 w^{n-2} + \cdots \\ &+ i(2m-n-1)A_{2m-2n+1} w^{2m-n-3} + \cdots, i(m-1)R_m w^{m-3} + \cdots) \\ &+ \langle \text{better} \rangle. \end{aligned} \tag{13}$$

Since this term is holomorphic we have the freedom to set $\phi_t(0) = 0$. Then $f(w) = w\hat{Z}_{tw}(0)\tau + w\hat{X}_w\phi_t(0)$ is holomorphic, and Proposition 6 in Section 6.1 yields

$$(14) \qquad E^{(5)}(0) = 12\mathrm{Re}\int_{S^1} w\hat{Z}_{ttw}(0)\cdot\hat{Z}_{ttw}(0)\tau\,dw.$$

(This follows of course also from the general formulas stated above.)

From formula (51) of Lemma 4 in Section 6.1 we get

$$\hat{Z}_{ttw}(0) = \{iw[iw\hat{X}_w\tau]_w\tau\}_w = i\{w\hat{Z}_{tw}(0)\tau\}_w,$$

and so

$$(15) \qquad \begin{aligned}\hat{Z}_{ttw}(0) &= -(a-ib)^2((n-1)(n-3)A_1 w^{n-4} + \cdots \\ &\quad + (2m-n-1)(2m-n-3)A_{2m-2n+1}w^{2m-n-4} \\ &\quad + \cdots, (m-1)(m-3)R_m w^{m-4} + \cdots) + \langle\text{better}\rangle.\end{aligned}$$

Since $n-3=0$ and $m=4$, this leads to

$$(16) \qquad \hat{Z}_{ttw}(0)\cdot\hat{Z}_{ttw}(0) = (a-ib)^4(m-1)^2(m-3)^2R_m^2 + \cdots,$$

and by (14) we obtain for $L = m+1 = 5$:

$$(17) \qquad \begin{aligned}E^{(L)}(0) &= E^{(5)}(0) = 12\cdot\mathrm{Re}\int_{S^1}(a-ib)^5(m-1)^2(m-3)^2R_m^2\frac{dw}{w} \\ &= 12\cdot\mathrm{Re}[2\pi i(a-ib)^5(m-1)^2(m-3)^2R_m^2], \quad m=4.\end{aligned}$$

Now we turn to the **general case** of an odd $n \geq 3$ and an even index $m \geq 4$.

Step 1. *The pole removal technique to make the expressions f and g_{L-M-1} in the integral J_1 holomorphic.*

We have already seen that $f(w)$ is holomorphic if we set $\phi_t(0) = 0$. In fact, we set

$$(18) \qquad D_t^\beta\phi(0) = 0 \quad \text{for } 1 \leq \beta \leq \frac{n-1}{2} \text{ and for } \beta > \frac{1}{2}(L-3)$$

and prove the following

Lemma 1. *By the pole-removing technique we can inductively choose $D_t^\beta\phi(0)$ for $\beta \leq \frac{1}{2}(L-3)$ such that g_ν is holomorphic for $\nu = 0,1,\ldots,\frac{1}{2}(L-3)$. Then the derivative $[D_t^\gamma\hat{Z}(0)]_w$ is not only holomorphic, but can be obtained in the form*

$$(19) \qquad [D_t^\gamma\hat{Z}(0)]_w = \{ig_{\gamma-1}\}_w \quad \textit{for } \gamma = 1,2,\ldots,\frac{1}{2}(L-1).$$

Suppose this result were proved. Since in J_1 there appear only g_ν with $\nu = L-M-1$ where $\frac{1}{2}(L-1) < M \leq L-3$, i.e. $2 \leq \nu \leq \frac{1}{2}(L-3)$, all integrands in J_1 were indeed holomorphic, and so $J_1 = 0$. Thus it remains to prove Lemma 1.

Proof of Lemma 1. By definition we have

(20) $$g_\nu = \sum_{\alpha+\beta=\nu} c^\nu_{\alpha\beta} T^{\alpha,\beta}, \quad T^{\alpha,\beta} := w[D_t^\alpha \hat{Z}(0)]_w D_t^\beta \phi(0),$$

and $\phi(0) = \tau$.

The expressions $w[D_t^\alpha \hat{Z}(0)]_w \tau$ have no pole for $\alpha \le \frac{n-1}{2}$, and we make the important observation that there are numbers c, c' such that

$$w[D_t^{\frac{n-1}{2}} \hat{Z}(0)]_w \tau = (cA_1 + \cdots, c' R_m w^{m-n} + \cdots).$$

Thus, a pole in $w[D_t^\alpha \hat{Z}(0)]_w \tau$ may arise at first for $\alpha = \frac{1}{2}(n+1)$; then we have, say

(21) $$w[D_t^{\frac{1}{2}(n+1)} \hat{Z}(0)]_w \tau = (cA_2 w^{-1} + \cdots, c' R_m w^{m-n-2} + \cdots).$$

This requires a nonzero $D_t^{\frac{n+1}{2}} \phi(0)$ in case that $cA_2 \ne 0$ if we want to make $g_{\frac{1}{2}(n+1)}$ pole-free. Now we go on and discuss the pole removal for $\nu = \frac{1}{2}(n+3), \frac{1}{2}(n+5), \ldots, \frac{1}{2}(L-3)$.

Observation 1. Since m is even, n is odd, and $m > n$, we have

(22) $$m = n + (2k+1), \quad k = 0, 1, 2, \ldots,$$

and therefore

(23) $$\frac{1}{2}(L-3) = \frac{1}{2}(m-2) = \frac{1}{2}(n+2k-1).$$

Thus, for $m = n+1$, all g_ν with $2 \le \nu \le \frac{1}{2}(L-3)$ are pole-free if we set $D_t^\beta \phi(0) = 0$ for all $\beta \ge 1$; cf. (18). For $m = n+3$, we have to choose $D_t^\beta \phi(0)$ appropriately for $\beta = \frac{1}{2}(n+1)$ while the other $D_t^\beta \phi(0)$ are taken to be zero. For $m = n+5$, we must also choose $D_t^\beta \phi(0)$ appropriately for $\beta = \frac{1}{2}(n+3)$ whereas the other $D_t^\beta \phi(0)$ are set to be zero. In this way we proceed inductively and choose $D_t^\beta \phi(0)$ in a suitable way for $\beta = \frac{1}{2}(n+1), \frac{1}{2}(n+3), \ldots, \frac{1}{2}(n+2k-1)$ in case that $m = n + 2k + 1$ while all other $D_t^\beta \phi(0)$ are taken to be zero according to (18).

Observation 2. The pole-removal procedure would only stop for some g_ν with $\frac{1}{2}(n+1) \le \nu \le \frac{1}{2}(L-3)$ if the w-power attached to $A_{2m-2n+1}$ became negative. We have to check that this does not happen for $\nu \le \frac{1}{2}(L-3)$. Since at the α-th stage in defining $[D_t^\alpha \hat{Z}(0)]_w$ the w-powers have been reduced by 2α, we must check that the terms $T^{\alpha,\beta}$ have no poles connected with $A_{2m-2n+1}$ if $\alpha + \beta \le \frac{1}{2}(L-3)$. Looking first only at $T^{\alpha,0} = w[D_t^\alpha \hat{Z}(0)]_w \tau$ for $\alpha \le \frac{1}{2}(L-3)$, we must have

$$2m-n-2\alpha = 2m-n+1-2(\alpha+1) \geq 0 \quad \text{for } \alpha \leq \frac{1}{2}(L-3),$$

which is true since

$$2m-n+1-2\cdot\frac{1}{2}(L-1) = m-n+1 > 0.$$

We must also check that during the process no pole is introduced into the third complex component. Again we first look at $T^{\alpha,0}$ for $\alpha \leq \frac{1}{2}(L-3)$. Then the order of the w-power at the R_m-term is

$$m-2\alpha-1 = (m+1)-2(\alpha+1) \geq (m+1)-(L-1) = 1,$$

and so there is no pole.

Let us now look at the pole-removal procedure. For $m=n+1$ all g_ν with $2 \leq \nu \leq \frac{1}{2}(L-3)$ are pole-free if we assume (18). If $m=n+3$ we have to make $g_{\frac{1}{2}(n+1)}$ pole-free. To this end it suffices to choose $D_t^{\frac{1}{2}(n+1)}\phi(0)$ appropriately; it need have a pole at most of order $(n+2)$ in order to remove a possible pole of $T^{\alpha,0}, \alpha = \frac{1}{2}(n+1)$, cf. (21).

If $m=n+5$, we have to choose $D_t^\beta\phi(0)$ appropriately for $\beta = \frac{1}{2}(n+1)$ and $\beta = \frac{1}{2}(n+3)$. The derivative $D_t^{\frac{1}{2}(n+1)}\phi(0)$ will be taken as before, while $D_t^{\frac{1}{2}(n+3)}\phi(0)$ is to be chosen in such a way that

$$g_{\frac{1}{2}(n+3)} = T^{\frac{1}{2}(n+3),0} + T^{1,\frac{1}{2}(n+1)} + T^{0,\frac{1}{2}(n+3)}$$

becomes holomorphic. Since

$$\begin{aligned} T^{1,\frac{1}{2}(n+1)} &= w[\hat{Z}_t(0)]_w D_t^{\frac{1}{2}(n+1)}\phi(0) \\ &= (i(n-1)(a-ib)A_1 w^{n-1} + \cdots, \\ &\quad\; i(m-1)(a-ib)R_m w^{m-1} + \cdots)D_t^{\frac{1}{2}(n+1)}\phi(0) \\ &= (cA_1 w^{-3} + \cdots, c'R_m w^{m-n-3} + \cdots) \end{aligned}$$

with some constants c, c', the derivative $D_t^{\frac{1}{2}(n+3)}\phi(0)$ in

$$T^{0,\frac{1}{2}(n+3)} = w\hat{X}_w D_t^{\frac{1}{2}(n+3)}\phi(0)$$

should have a pole of order $n+4$, while a pole of lower order than $n+4$ is needed to remove a possible singularity in the first term $T^{\frac{1}{2}(n+3),0} = w[D_t^{\frac{1}{2}(n+3)}\hat{Z}(0)]_w\tau$.

In this way we can proceed inductively choosing the poles of $D_t^\beta\phi(0)$ always at most of order

$$n+2\left(\beta-\frac{n-1}{2}\right) = 2\beta+1 \quad \text{for } \frac{1}{2}(n+1) \leq \beta \leq \frac{1}{2}(L-3). \tag{24}$$

This is the crucial estimate on the order of the pole of $D_t^\beta\phi(0)$ in order to ensure that these derivatives play no role in the final calculations.

Observation 3. Consider the last complex component of

$$g_{\frac{1}{2}(n+1)} = w[D_t^{\frac{1}{2}(n+1)}\hat{Z}(0)]_w\tau + w\hat{X}_w D_t^{\frac{1}{2}(n+1)}\phi(0).$$

The lowest w-power attached to R_m in the first term is $1+m-(n+1)-2 = m-n-2 \geq 1$ (since in this case $m \geq n+3$ according to Observation 1). The lowest w-power associated to R_m in the second term is $1+m-(n+2) = m-n-1 > m-n-2$. Continuing inductively we see that the lowest w-power attached to R_m in any g_ν arises from $\tau = \phi(0)$ and not from any $D_t^\beta\phi(0)$. □

This ends the proof of Step 1, and we have found that $E^{(L)}(0) = J_2$. Now we come to

Step 2. The integral J_2 is a linear combination of the real parts of the integrals

$$(25) \qquad I_{\alpha\gamma\beta} := \int_{S^1} w[D_t^\alpha\hat{Z}(0)]_w \cdot [D_t^\gamma\hat{Z}(0)]_w D_t^\beta\phi(0)\, dw,$$

where $1 \leq \alpha, \gamma \leq \frac{1}{2}(L-1)$ and $\beta = (L-1)-\alpha-\gamma$. Then we have

$$(26) \qquad \beta = 0 \quad \text{if and only if} \quad \alpha = \gamma = \frac{1}{2}(L-1) = \frac{m}{2}.$$

This implies

$$(27) \qquad J_2 = \frac{2\cdot m!}{(\frac{m}{2})!(\frac{m}{2})!}\mathrm{Re}\int_{S^1} w[D_t^{\frac{m}{2}}\hat{Z}(0)]_w \cdot [D_t^{\frac{m}{2}}\hat{Z}(0)]_w\tau\, dw$$

because of the following

Lemma 2. *We have*

$$(28) \qquad I_{\alpha\gamma\beta} = 0 \quad \textit{for } 1 \leq \alpha,\gamma \leq \frac{1}{2}(L-1) \textit{ and } 1 \leq \beta = m-\alpha-\gamma.$$

Proof. Let us first show that the product of the last complex components of $[D_t^\alpha\hat{Z}(0)]_w$ and $[D_t^\gamma\hat{Z}(0)]_w$ and of $wD_t^\beta\phi(0)$ have a zero integral. In fact, this product has the form

$$\begin{aligned}&\text{const}(wR_m w^{m-2\alpha}\cdot R_m w^{m-2\gamma}+\cdots)(w^{-2\beta-1}+\cdots)\\&\quad= \text{const}R_m^2 w^{1+2m-2(\alpha+\beta+\gamma)-1}+\cdots = \text{const}\cdot R_m^2+\cdots\end{aligned}$$

since $\alpha+\beta+\gamma = L-1 = m$.

The same holds true for the scalar product of the first two complex components, multiplied by $wD_t^\beta\phi(0)$. To see this we assume without loss of generality that $\alpha \geq \gamma$. Denote by $P^{\alpha\gamma}$ the expression

$$P^{\alpha\gamma} := w[C_1^\alpha\cdot C_1^\gamma + C_2^\alpha\cdot C_2^\gamma],$$

where C_1^α, C_2^α and C_1^γ, C_2^γ are the first two complex components of $[D_t^\alpha \hat{Z}(0)]_w$ and $[D_t^\gamma \hat{Z}(0)]_w$ respectively.

Case 1. If $2\gamma \le 2\alpha < n$ then

$$\begin{aligned} P^{\alpha\gamma} &= w(\text{const}A_j w^{n-2\alpha} + \cdots + \text{const}A_{2m-2n+1} w^{2m-n-2\alpha} + \cdots) \\ &\quad \cdot (\text{const}A_\ell w^{n-2\gamma} + \cdots + \text{const}A_{2m-2n+1} w^{2m-n+\gamma} + \cdots) \end{aligned}$$

with $j, \ell < 2m - 2n + 1$.

Case 2. If $2\gamma < n < 2\alpha$ then

$$\begin{aligned} P^{\alpha\gamma} &= w(\text{const}A_j + \cdots + \text{const}A_{2m-2n+1} w^{2m-n} + \cdots) \\ &\quad \cdot (\text{const}A_\ell w^{n-2\gamma} + \cdots + \text{const}A_{2m-2n+1} w^{2m-n-2\gamma} + \cdots) \end{aligned}$$

with $j, \ell < 2m - 2n + 1$.

Case 3. If $n < 2\alpha$ and $n < 2\gamma$ then

$$\begin{aligned} P^{\alpha\gamma} &= w(\text{const}A_j + \cdots + \text{const}A_{2m-2n+1} w^{2m-n-2\alpha} + \cdots) \\ &\quad \cdot (\text{const}A_\ell + \cdots + \text{const}A_{2m-2n+1} w^{2m-n-2\gamma} + \cdots). \end{aligned}$$

Let $\mu(\alpha, \gamma)$ be the lowest w-power appearing in $P^{\alpha\gamma} D_t^\beta \phi(0)$. Recalling $\alpha + \beta + \gamma = m$ we obtain the following results:

Case 1.

$$\begin{aligned} \mu(\alpha, \gamma) &= 1 + 2m - 2\gamma - 2\alpha - 2\beta - 1 \\ &= 2 + 2m - 2(\alpha + \beta + \gamma + 1) \\ &= 2 + 2m - 2(m+1) = 0. \end{aligned}$$

Case 2. $\mu(\alpha, \gamma)$ is either zero as in Case 1, or

$$\begin{aligned} \mu(\alpha, \gamma) &= 1 + 2m - n - 2\gamma - 2\beta - 1 \\ &= 2 + 2m - n - 2(\gamma + \beta + 1) \\ &= 2 + 2m - n - 2(m + 1 - \alpha) = 2\alpha - n > 0. \end{aligned}$$

Case 3. As in Case 2 we have $\mu(\alpha, \gamma) > 0$.

This proves $I_{\alpha\gamma\beta} = 0$ for $1 \le \alpha, \gamma \le \frac{m}{2}$ and $1 \le \beta = m - \alpha - \gamma$, which yields Lemma 2. □

Thus we have arrived at (27), and a straight-forward computation leads to (8) and (9); so the proof of Theorem 1 is complete. □

6.3 Boundary Branch Points

In this section we first show that Dirichlet's integral possesses intrinsic second and third derivatives at a minimal surface $\hat{X}$ on the tangent space $T_X M$ of $M := H^2(\partial B, \mathbb{R}^n)$ of $X = \hat{X}|_{\partial B}$ on the space $J(\hat{X})$ of forced Jacobi fields for $\hat{X}$. These results will also be used in Vol. 3, Chapters 5 and 6. In particular it will be seen that $J(\hat{X})$ is a subspace of the kernel of the Hessian $D^2E(X)$ of Dirichlet's integral $E(X)$ defined in (1) below, and an interesting formula (see (16)) for the second variation of Dirichlet's integral is derived.

Secondly we prove that, for a sufficiently smooth contour Γ in $\mathbb{R}^3$, not only the *order*, but also the *index* of a boundary branch point of a minimal surface $X \in \mathcal{C}(\Gamma)$ can be estimated in terms of the *total curvature* of Γ if curvature and torsion of Γ are nowhere zero.

Finally we prove Wienholtz's theorem, which states a condition under which a minimizer for Plateau's problem cannot possess a boundary branch point. In particular we show: If n is the order and m the index of a boundary branch point of $\hat{X}$ such that $2m-2 < 3n$ (equivalently $2m+2 \le 3(n+1)$) then $\hat{X}$ cannot be a minimizer of Dirichlet's integral or of area. The key idea of the proof will be to recompute the third derivative of Dirichlet's integral, D, in an intrinsic way on $J(\hat{X})$, thereby showing that the formula for $E^{(3)}(0) = \frac{d^3}{dt^3} D(\hat{Z}(t))\big|_{t=0}$ derived in Section 6.1 is valid in the presence of boundary branch points as well.

Towards these goals, we first show that if the boundary contour $\Gamma \subset \mathbb{R}^n$ is of class $D^{\boldsymbol{r}+7}, \boldsymbol{r} \ge 3$, the space $\mathcal{H}^{5/2}_\Gamma(\overline{B}, \mathbb{R}^n)$ of harmonic surfaces from $\overline{B}$ into $\mathbb{R}^n$, mapping $S^1 = \partial B$ to Γ, is a $C^{\boldsymbol{r}}$ manifold, in fact, a $C^{\boldsymbol{r}}$-submanifold of the space $\mathcal{H}^{5/2}(\overline{B}, \mathbb{R}^n)$ of harmonic mappings from $\overline{B}$ into $\mathbb{R}^n$. Instead of the dimension $n = 3$ we do this for arbitrary dimension n, since this result is necessary for the index theorem to be derived in Chapter 5 of Vol. 3. Here it is essential that we operate in the context of a manifold since the third derivative of any real-valued C^3-smooth function is seen to be well defined as a trilinear form on the kernel of the Hessian of this function at any critical point. As in Chapters 5 and 6 of Vol. 3 we shall use the symbol D for the total derivative or the Fréchet derivative. Therefore we need another notation for Dirichlet's integral; instead of D we employ the symbol E and consider E as a function of boundary values $X : S^1 \to \mathbb{R}^n$ (instead of their harmonic extension $\hat{X}$), i.e.

$$(1) \qquad E(X) := \frac{1}{2} \int_B (\hat{X}_u \cdot \hat{X}_u + \hat{X}_v \cdot \hat{X}_v)\, du\, dv \quad \text{for } X \in H^{1/2}(S^1, \mathbb{R}^n).$$

It is a well-known fact that $\mathbb{R}^n$ carries a $C^{\boldsymbol{r}+6}$-Riemannian metric g with respect to which Γ is totally geodesic, i.e. any g-geodesic $\sigma : (-1,1) \to \mathbb{R}^n$ with $\sigma(0) \in \Gamma$ and $\sigma'(0) \in T_{\sigma(0)}\Gamma$ remains on Γ. Let $(p, v) \mapsto \exp_p v$ denote the exponential map of g; it is of class $C^{\boldsymbol{r}+4}$. Via harmonic extension we identify the space

$$M := H^2(S^1, \Gamma)$$

of H^2-maps from S^1 to Γ with the space $\mathcal{H}^{5/2}_\Gamma(\overline{B}, \mathbb{R}^n)$. In order to show that M is a submanifold of $H^2(S^1, \mathbb{R}^n)$ we need to identify the tangent space $T_X M$ for $X \in H^2(S^1, \Gamma)$. (In Vol. 3, Chapters 5 and 6, we shall denote M by $\mathcal{N}_\alpha$ if Γ is given by $\Gamma = \alpha(S^1)$.)

Definition 1. *We define the tangent space $T_X M$ of M at $X \in H^2(S^1, \Gamma)$ as*

$$T_X M := \{Y \in H^2(S^1, \mathbb{R}^n) : Y(e^{i\theta}) \in T_{X(e^{i\theta})}\Gamma,\ \theta \in \mathbb{R}\}.$$

Clearly $T_X M$ is a Hilbert subspace of $H^2(S^1, \mathbb{R}^n)$. Our goal is to show that the map

$$\Phi(Y)(s) := \exp_{X(s)} Y(s), \quad s = e^{i\theta},$$

is a local $C^{\boldsymbol{r}}$-diffeomorphism about the zero $0 \in H^2(S^1, \mathbb{R}^n)$ mapping a neighbourhood of zero in $T_X M$ onto a neighbourhood of X in M. Towards this goal we have:

Theorem 1. *If $\varphi \in C^{\boldsymbol{r}+3}(\mathbb{R}^n, \mathbb{R}^n)$, then $\Phi : H^2(S^1, \mathbb{R}^n) \to H^2(S^1, \mathbb{R}^n)$ defined by $\Phi(Y) := \varphi \circ Y$ is of class $C^{\boldsymbol{r}}$. Furthermore,*

$$D^m \Phi_Y(\lambda_1, \dots, \lambda_m)(s) = D^m \varphi_{Y(s)}(\lambda_1(0), \dots, \lambda_m(s)) \quad \textit{for } 0 \le m \le r.$$

The proof of this theorem will be a consequence of the following

Lemma 1. *Let $\mathcal{L}^m(\mathbb{R}^n, \mathbb{R}^n)$ be the space of m-linear maps from $\mathbb{R}^n$ into $\mathbb{R}^n$, and suppose that $f \in C^3(\mathbb{R}^n, \mathcal{L}^m(\mathbb{R}^n, \mathbb{R}^n))$. Then the map $F : H^2(S^1, \mathbb{R}^n) \to \mathcal{L}^m(H^2(S^1, \mathbb{R}^n), H^2(S^1, \mathbb{R}^n))$ defined by*

$$Y \mapsto F(Y)(\lambda_1, \dots, \lambda_m)(s) := f(Y(s))(\lambda_1(s), \dots, \lambda_m(s))$$

is continuous. Moreover, if $f \in C^4$ then $F \in C^1$, and the derivative of $Y \mapsto F(Y)$ is

$$\lambda \mapsto df(Y(s))(\lambda(s), \lambda_1(s), \dots, \lambda_m(s)).$$

Proof. Recall that $H^2(S^1, \mathbb{R}^n)$ is continuously and compactly embedded into $C^1(S^1, \mathbb{R}^n)$. Assume for simplicity that

$$\|\lambda_j\|_{H^2} \le 1, \quad \|Y\|_{H^2} < 2, \quad \|\tilde{Y}\|_{H^2} < 2,$$

and consider the difference

$$[F(Y) - F(\tilde{Y})](\lambda_1, \dots, \lambda_m)(s) = [f(Y(s)) - f(\tilde{Y}(s))](\lambda_1(s), \dots, \lambda_m(s)).$$

Then

$$
\begin{aligned}
&\frac{d}{ds}[F(Y) - F(\tilde{Y})](\lambda_1, \ldots, \lambda_m)(s) \\
&\quad = df(Y(s))(Y'(s))(\lambda_1(s), \ldots, \lambda_m(s)) - df(\tilde{Y}(s))(\tilde{Y}'(s))(\lambda_1(s), \ldots, \lambda_m(s)) \\
&\qquad + \sum_{j=1}^{m} [f(Y(s)) - f(\tilde{Y}(s))](\lambda_1(s), \ldots, \lambda_{j-1}(s), \lambda_j'(s), \lambda_{j+1}(s), \ldots, \lambda_m(s)) \\
&\quad = df(Y(s))(Y'(s) - \tilde{Y}'(s))(\lambda_1(s), \ldots, \lambda_m(s)) \\
&\qquad + [df(Y(s) - df(\tilde{Y}(s))](\tilde{Y}'(s))(\lambda_1(s), \ldots, \lambda_m(s)) \\
&\qquad + \sum_{j=1}^{m} [f(Y(s)) - f(\tilde{Y}(s))](\lambda_1(s), \ldots, \lambda_j'(s), \ldots, \lambda_m(s)).
\end{aligned}
$$

Since f is Lipschitz continuous, we have

$$
\begin{aligned}
\sup_s |f(Y(s)) - f(\tilde{Y}(s))| &\leq \text{const} \sup_s |Y(s) - \tilde{Y}(s)| \\
&\leq \text{const} \, \|Y - \tilde{Y}\|_{H^1},
\end{aligned}
$$

and therefore

$$
\begin{aligned}
&\left| \sum_{j=1}^{m} [f(Y(s)) - f(\tilde{Y}(s))](\lambda_1(s), \ldots, \lambda_j'(s), \ldots, \lambda_m(s)) \right| \\
&\quad \leq \text{ const} \sum_{j=1}^{m} ||Y - \tilde{Y}||_{H^1} |\lambda_j'(s)|,
\end{aligned}
$$

from which it follows that

$$
\left\| \sum_{j=1}^{m} [f(Y) - f(\tilde{Y})](\lambda_1, \ldots, \lambda_j', \ldots, \lambda_m) \right\|_{L^2} \leq \text{ const} \, \|Y - \tilde{Y}\|_{H^2}.
$$

Furthermore, the Lipschitz continuity of df implies

$$
\begin{aligned}
&\|df(Y)(Y' - \tilde{Y}')(\lambda_1, \ldots, \lambda_m)\|_{L^2} \leq \text{const} \, \|Y - \tilde{Y}\|_{H^2}, \\
&\|df(Y) - df(\tilde{Y})](\tilde{Y}')(\lambda_1, \ldots, \lambda_m)\|_{L^2} \leq \text{ const} \, \|Y - \tilde{Y}\|_{H^2}.
\end{aligned}
$$

Summarizing these estimates we obtain

$$
\left\| \frac{d}{ds}[F(Y) - \tilde{F}(Y)](\lambda_1, \ldots, \lambda_m) \right\|_{L^2} \leq \text{const} \, \|Y - \tilde{Y}\|_{H^2}.
$$

In the same manner we infer

$$
\left\| \frac{d^2}{ds^2}[F(Y) - \tilde{F}(Y)](\lambda_1, \ldots, \lambda_m) \right\|_{L^2} \leq \text{ const} \, \|Y - \tilde{Y}\|_{H^2},
$$

since f, df, and $d^2 f$ are Lipschitz continuous, using

$$
\begin{aligned}
&\frac{d^2}{ds^2}[F(Y) - F(\tilde{Y})](\lambda_1, \dots, \lambda_m)(s) \\
&\quad = d^2 f(Y(s))(Y'(s))(Y'(s) - \tilde{Y}'(s))(\lambda_1(s), \dots, \lambda_m(s)) \\
&\qquad + df(Y(s))(Y''(s) - \tilde{Y}''(s))(\lambda_1(s), \dots, \lambda_m(s)) \\
&\qquad + \sum_{j=1}^{m} df(Y(s))(Y'(s) - \tilde{Y}'(s))(\lambda_1(s), \dots, \lambda_j'(s), \dots, \lambda_m(s)) \\
&\qquad + [d^2 f(Y(s))(Y'(s)) - d^2 f(\tilde{Y}(s))(\tilde{Y}'(s))](\tilde{Y}'(s))(\lambda_1(s), \dots, \lambda_m(s)) \\
&\qquad + [df(Y(s)) - df(\tilde{Y}(s))](\tilde{Y}''(s))(\lambda_1(s), \dots, \lambda_m(s)) \\
&\qquad + \sum_{j=1}^{m} [df(Y(s)) - df(\tilde{Y}(s))](Y'(s))(\lambda_1(s), \dots, \lambda_j'(s), \dots, \lambda_m(s)) \\
&\qquad + \sum_{j=1}^{m} [f(Y(s)) - f(\tilde{Y}(s))](\lambda_1(s), \dots, \lambda_j''(s), \dots, \lambda_m(s)) \\
&\qquad + \sum_{j,k=1, j<k}^{m} [f(Y(s)) - f(\tilde{Y}(s))](\lambda_1(s), \dots, \lambda_j'(s), \dots, \lambda_k'(s), \dots, \lambda_m(s)) \\
&\qquad + \sum_{j=1}^{m} [df(Y(s))(Y'(s)) - df(\tilde{Y})(\tilde{Y}'(s))](\lambda_1(s), \dots, \lambda_j'(s), \dots, \lambda_m(s)).
\end{aligned}
$$

The estimates above prove that F maps $H^2(S^1, \mathbb{R}^n)$ continuously into the space

$$\mathfrak{L}^m(H^2(S^1, \mathbb{R}^n), H^2(S^1, \mathbb{R}^m)).$$

If $f \in C^4$ then $df \in C^3$ and $d^2 f \in C^2$, and Taylor's theorem yields

$$f(u + h) - f(u) - df(u)h = r(u, h)(h, h),$$

where

$$r(u, h)(h, h) := \int_0^1 (1 - t)[d^2 f(u + th) - d^2 f(u)](h, h)\, dt.$$

Since f is in C^4 we obtain

$$\|r(u, h)(h, h)\|_{H^2} \leq \text{ const } \|h\|^2_{H^2} \quad \text{for } \|h\|_{H^2} \leq 1.$$

This shows that the mapping F is differentiable, and its derivative $DF(Y)$ at $Y \in H^2(S^1, \mathbb{R}^n)$ is given by

$$(DF(Y)h)(s) = df(Y(s))h(s).$$

Since $df \in C^3$, the first part of the lemma yields $DF \in C^0$. □

Proof of Theorem 1. Applying Lemma 1 to $f = d^m \varphi$ succesively to $m = 0, 1, \dots, \boldsymbol{r} - 1$, we infer that $D\Phi, D^2\Phi, \dots, D^{\boldsymbol{r}}\Phi$ exist and are continuous. □

Theorem 2. *$M = H^2(S^1, \Gamma)$ is a $C^{\boldsymbol{r}}$-submanifold of $H^2(S^1, \mathbb{R}^n)$.*

Proof. Since $H^2(S^1, \mathbb{R}^n) \subset C^1(S, \mathbb{R}^n)$, the set M is closed in $H^2(S^1, \mathbb{R}^n)$. Consider the map $Y \mapsto \Phi(Y)$ defined by

$$\Phi(Y)(s) := \exp_{X(s)} Y(s) \quad \text{for } X \in H^2(S^1, \Gamma),$$

which is of class $C^{\boldsymbol{r}}$ by virtue of Theorem 1.

Since $\Phi(0)$ is the identity map, the inverse function theorem implies that Φ is a local $C^{\boldsymbol{r}}$-diffeomorphism about 0. Moreover, as the Riemannian metric g is totally geodesic with respect to Γ, we see that Φ maps $T_X M$ into M. Since Φ is also locally invertible, it provides a coordinate chart for M as a submanifold of $H^2(S^1, \mathbb{R}^n)$. □

Before we can apply the preceding results to Plateau's problem we need an abstract functional analytic reasoning which shows that a C^3-function $E : M \to \mathbb{R}$ on a $C^{\boldsymbol{r}}$-smooth submanifold M of a Hilbert space $\mathcal{H}, \boldsymbol{r} \geq 3$, possesses intrinsic first, second, and third order derivatives for any critical point x of E (i.e. $DE(x) = 0$). To prove this we need a few prerequisites.

By $E \in C^3(M)$ we mean that E extends to a C^3-map on a neighbourhood of every point $x \in M$. Equivalently we can use coordinate charts as follows. From the definition of a submanifold it follows that about each point $x \in M$ there is a $C^{\boldsymbol{r}}$-diffeomorphism $\rho : \mathcal{V} \to \mathcal{V}'$ from a neighbourhood $\mathcal{V}$ of x in $\mathcal{H}$ onto a neighbourhood $\mathcal{V}'$ of 0 in $\mathcal{H}$ with $\rho(x) = 0$ such that $\rho(\mathcal{V} \cap M)$ is an open subset of a fixed subspace $\mathcal{H}_0$ of $\mathcal{H}$. Then "$E \in C^3(M)$" means that $E \circ \psi$ is of class C^3 for any such chart $(\rho, \mathcal{V})$ where ψ is the inverse of ρ. For $x \in M$ with the image $0 = \rho(x)$ we define the tangent space $T_x M$ of M at x by

$$T_x M := D\psi(0)[\mathcal{H}_0] \subset \mathcal{H},$$

i.e. as the image of $\mathcal{H}_0$ under the mapping provided by the derivative $D\psi(0)$. This definition of $T_x M$ does not depend on the choice of the chart $(\rho, \mathcal{V})$.

As each $h \in T_x M$ can be written as $h = D\psi(0)\tilde{h}$ with $\tilde{h} \in \mathcal{H}_0$, we define

$$DE(x)h := D(E \circ \psi)(0)\tilde{h},$$

which again can be shown to be independent of the choice of the chart.

A point $x \in M$ is a critical point of $E : M \to \mathbb{R}$ if $DE(x) = 0$. At a critical point x of E there is a well-defined bilinear form

$$D^2 E(x) : T_x M \times T_x M \to \mathbb{R}$$

defined by

$$D^2 E(x)(h, k) := D^2(E \circ \psi)(0)(\tilde{h}, \tilde{k}) \quad \text{for}$$

$$h = D\psi(0)\tilde{h}, \quad k = D\psi(0)\tilde{k}; \quad \tilde{h}, \tilde{k} \in \mathcal{H}_0.$$

This is the Hessian (bilinear form), which again does not depend on the choice of the chart $(\rho, \mathcal{V})$, as we will shortly show. Surprisingly, there is also a third intrinsic derivative $D^3E(x)$, but this is intrinsically defined only on the kernel K_x of $D^2E(x)$, i.e. on

$$K_x := \{h \in T_xM : D^2E(x)(h,k) = 0 \text{ for all } k \in T_xM\}.$$

Let us state this formally as

Theorem 3. *At a critical point x of $E \in C^3(M)$ there is an intrinsically defined*[1] *second derivative $D^2E(x) : T_xM \times T_xM \to \mathbb{R}$, and a third derivative $D^3E(x) : K_x \times K_x \times K_x \to \mathbb{R}$ defined as a trilinear map on the kernel K_x of $D^2E(x)$.*

To prove this we have to show that, with respect to any transition map $\varphi : U \to U$ on $U \subset M$ fixing the critical point $x \in U$ of E, the second and third derivative of $E \circ \varphi$ depend only on the first derivative of φ and are independent of $D^2\varphi(x)$ and $D^3\varphi(x)$. Since we may choose the critical point x as the origin 0, the theorem is a consequence of the following

Lemma 2. *Let U be an open subset of a Hilbert space and suppose that $0 \in U$ is a critical point of $E \in C^3(U)$. Assume also that K is the kernel of the Hessian of E at 0 and $\varphi : U \to U$ is a C^3-diffeomorphism of U onto itself with $\varphi(0) = 0$. Then*

$$D^2(E \circ \varphi)(0)(k_1, k_2) = D^2E(0)(D\varphi(0)k_1, D\varphi(0)k_2),$$

and furthermore, if $D\varphi(0)k_j \in K, j = 1, 2, 3,$ then

$$D^3(E \circ \varphi)(0)(k_1, k_2, k_3) = D^3E(0)(D\varphi(0)k_1, D\varphi(0)k_2, D\varphi(0)k_3).$$

Proof. Repeatedly using the chain rule we see that

(i) $D(E \circ \varphi)(x)(h) = DE(\varphi(x))D\varphi(x)h,$

(ii) $$D^2(E \circ \varphi)(x)(h,k) = D^2E(\varphi(x))(D\varphi(x)h, D\varphi(x)k) + DE(\varphi(x))D^2\varphi(x)(h,k).$$

(iii) $$\begin{aligned} D^3(E \circ \varphi)(x)(h,k,\ell) = {} & D^3E(\varphi(x))(D\varphi(x)h, D\varphi(x)k, D\varphi(x)\ell) \\ & + D^2E(\varphi(x))(D^2\varphi(x)(h,\ell), D\varphi(x)k) \\ & + D^2E(\varphi(x))(D\varphi(x)h, D^2\varphi(x)(k,\ell)) \\ & + D^2E(\varphi(x))(D^2\varphi(x)(h,k), D\varphi(x)\ell) \\ & + DE(\varphi(x))D^3\varphi(x)(h,k,\ell). \end{aligned}$$

[1] An intrinsic derivative $D^*f(x)$ of a map $f : M \to \mathbb{R}$ on a subspace σ of the tangent space T_xM is an r-linear form $\sigma^+ \to \mathbb{R}$ of $\sigma^r = \sigma \times \cdots \times \sigma$ which is defined independently of the choice of any coordinate chart.

Set $k_1 := h, k_2 := k, k_3 := \ell$ and note that $DE(0) = 0$. Then the first assertion follows from (ii) and $\varphi(0) = 0$. The second claim is a consequence of (iii) noting that $\varphi(0) = 0$, $DE(0) = 0$, and by assumption $D\varphi(0)k_j \in K$, $1 \le j \le 3$. □

Now we shall apply the preceding result to Dirichlet's integral $E : H^2(S^1, \mathbb{R}^n) \to \mathbb{R}$ defined by (1). Recall the assumption $\Gamma \in C^{\boldsymbol{r}+7}, \boldsymbol{r} \ge 3$. By Theorem 2 it follows that $M := H^2(S^1, \Gamma)$ is a $C^{\boldsymbol{r}}$-submanifold of $H^2(S^1, \mathbb{R}^n)$, and since $E : H^2(S^1, \mathbb{R}^n) \to \mathbb{R}$ is of class C^∞, it follows immediately that the restriction $E|M$ is of class $C^{\boldsymbol{r}}$. Let us simply write E instead of $E|M$, i.e. we view E as a function of class $C^r(M)$.

We wish now to calculate the intrinsic third derivative in the direction of certain specific elements of the kernel of $D^2E(X) : T_XM \times T_XM \to \mathbb{R}$, namely the forced Jacobi fields, in the case that $X \in H^2(S^1, \Gamma)$ is a minimal surface. By the results of Chapter 2 we know that $\hat{X} \in C^{\boldsymbol{r}+6,\alpha}(\overline{B}, \mathbb{R}^n)$ and therefore also $X \in C^{\boldsymbol{r}+6,\alpha}(S^1, \mathbb{R}^n)$ for all $\alpha \in (0,1)$.

Besides assuming that $\Gamma \in C^{\boldsymbol{r}+7}$ we make another standing assumption on Γ, namely that the total curvature $\int_\Gamma \kappa\, ds$ of Γ satisfies

$$\int_\Gamma \kappa\, ds \le \frac{1}{3}\pi \boldsymbol{r}, \tag{2}$$

which implies $\boldsymbol{r} \ge 6$. Then the generalized Gauss–Bonnet formula (19) of Section 2.11 implies

$$2\pi \sum_{w_j \in B} \nu(w_j) + \pi \sum_{\zeta_k \in \partial B} \nu(\zeta_k) + 2\pi \le \frac{1}{3}\pi \boldsymbol{r},$$

where $\nu(w_j)$ are the orders of the interior branch points w_j of a (branched) minimal surface $\hat{X} \in \mathcal{C}(\Gamma)$, and $\nu(\zeta_k)$ are the orders of its boundary branch points, $k = 1, \dots, q$. Suppose that $q \ge 1$. Then

$$\nu(\zeta_k) \le \boldsymbol{r}/3 - 2. \tag{3}$$

Recall the definition of a *forced Jacobi field* of a minimal surface $\hat{X} : \overline{B} \to \mathbb{R}^3$ which we now generalize to a minimal surface $\hat{X} : \overline{B} \to \mathbb{R}^n$ with $n \ge 3$ which has the interior branch points $w_1, \dots, w_p$ and the boundary branch points $\zeta_1, \dots, \zeta_q$. The *generator* τ of a forced Jacobi field $\hat{Y}$ for $\hat{X}$ is a meromorphic function on $\overline{B}$ with poles possibly at $w = 0$ and at the branch points of $\hat{X}$ whose orders are at most $\nu(w_j)$ at $w_j \ne 0$, $\nu(0) + 1$ at $w = 0$, $\nu(\zeta_j)$ at ζ_j, and which is real on ∂B. Then the *forced Jacobi field* $\hat{Y}$ of $\hat{X}$ with the generator τ is a mapping $\hat{Y} : \overline{B} \to \mathbb{R}^n$ of the form

$$\hat{Y} = 2\beta\, \mathrm{Re}(iw\hat{X}_w\tau) \quad \text{with } \beta \in \mathbb{R},$$

and

$$Y = \beta X_\theta \tau|_{S^1} : S^1 \to \mathbb{R}^n$$

are its boundary values. From the regularity of $\hat{X}$ and (3) we infer as in 6.1 that certainly $Y \in H^2(S^1, \mathbb{R}^n)$, $\hat{Y}_w \in C^0(\overline{B}, \mathbb{R}^n)$, and clearly $Y \in T_X M$. The space of forced Jacobi fields of $\hat{X}$ is denoted by $J(\hat{X})$.

We shall show that the forced Jacobi fields are in the kernel of the Hessian of $E : M \to \mathbb{R}$, and we will compute the second and third derivative of E in these directions. In Chapter 5 of Vol. 3 we shall describe how the forced Jacobi fields were discovered.

Computation of D^2E and D^3E.
Let $\Omega(p) : \mathbb{R}^n \to T_p\Gamma$ be the $C^{\boldsymbol{r}+6}$-smooth orthogonal projection of $\mathbb{R}^n$ onto the tangent space $T_p\Gamma$ for $p \in \Gamma$. We extend $\Omega(p)$ to a $C^{\boldsymbol{r}+6}$-smooth mapping $p \mapsto \Omega(p)$ from $\mathbb{R}^n$ into $\mathfrak{L}(\mathbb{R}^n, \mathbb{R}^n)$. We then can write the first derivative of E at $X \in M = H^2(S^1, \Gamma)$ as

$$(4) \qquad DE(X) = \int_{S^1} \langle \Omega(X)\hat{X}_r, h\rangle \, d\theta, \quad \hat{X}_r = \text{radial derivative of } \hat{X}.$$

A slight generalization of Theorem 1 yields that $X \to \Omega(X)$ belongs to $C^r(M, H^2(S^1, \mathfrak{L}(\mathbb{R}^n, \mathbb{R}^n)))$, $M = H^2(S^1, \Gamma)$, if we take Theorem 2 into account. Clearly, *X is a critical point of E if and only if*

$$(5) \qquad \Omega(X)\hat{X}_r = 0.$$

$\hat{X}$ will be a solution to Plateau's problem if X is also a monotonic map from S^1 onto Γ.

The derivative of $\Omega(X)\hat{X}_r$ is given by

$$(6) \qquad h \mapsto \Omega(X)\hat{h}_r + D\Omega(X)h[\hat{X}_r],$$

and so the Hessian of E is

$$(7) \qquad D^2E(X)(h,k) = \int_{S^1} \langle \Omega(X)\hat{h}_r + D\Omega(X)h[\hat{X}_r], k\rangle \, d\theta.$$

It follows that *the kernel of* (6) *is just the kernel of the Hessian $D^2E(X)$ of E at X.*

Claim: *The forced Jacobi fields of X lie in the kernel of $D^2E(X)$.* To see this we first note that

$$(8) \qquad |X_\theta|^2 \Omega(X)m = \langle m, X_\theta\rangle X_\theta \quad \text{for } m \in \mathbb{R}^n.$$

Differentiating this in direction of a tangent vector $h \in T_X M$, $M = H^2(S^1, \Gamma)$, we obtain

$$(9) \qquad 2\langle X_\theta, h_\theta\rangle \Omega(X)[m] + |X_\theta|^2 D\Omega(X)(h)[m] = \langle m, h_\theta\rangle X_\theta + \langle m, X_\theta\rangle h_\theta.$$

Thus the kernel of (6) is the kernel of

$$h \mapsto |X_\theta|^{-2}\{\langle \hat{X}_r, h_\theta\rangle X_\theta + \langle \hat{X}_r, X_\theta\rangle h_\theta - 2\langle X_\theta, h_\theta\rangle \Omega(X)\hat{X}_r\} + \Omega(X)\hat{h}_r.$$

From (5) we infer

$$\langle \hat{X}_r, X_\theta\rangle = 0 \quad \text{and} \quad \Omega(X)\hat{X}_r = 0,$$

and (8) yields

$$\Omega(X)\hat{h}_r = |X_\theta|^{-2}\langle \hat{h}_r, X_\theta\rangle X_\theta.$$

Thus h is in the kernel of (6) if and only if

$$|X_\theta|^{-2}\{\langle \hat{X}_r, h_\theta\rangle X_\theta + \langle X_\theta, \hat{h}_r\rangle X_\theta\} = 0$$

that is, if and only if

$$\langle \hat{X}_r, h_\theta\rangle + \langle X_\theta, \hat{h}_r\rangle = 0, \tag{10}$$

since the zeros of $X_\theta(\theta)$ are isolated because of the asymptotic expansion of $\hat{X}_w$ at branch points $w_0 \in \overline{B}$.

On $S^1 = \partial B$ we have

$$iw\hat{X}_w = \frac{1}{2}(X_\theta + i\hat{X}_r), \quad iw\hat{h}_w = \frac{1}{2}(h_\theta + i\hat{h}_r),$$

implying that

$$\langle \hat{X}_r, h_\theta\rangle + \langle X_\theta, \hat{h}_r\rangle = -4\,\mathrm{Im}\{w^2\langle \hat{X}_w, \hat{h}_w\rangle\}. \tag{11}$$

If $\hat{h}$ is a forced Jacobi field we have

$$h = \beta X_\theta \tau|_{S^1} \quad \text{and} \quad \hat{h} = 2\,\mathrm{Re}(\beta iw\hat{X}_w\tau)$$

with $\beta \in \mathbb{R}$ and τ the generator of $\hat{h}$. Since $w\hat{X}_w\tau$ is holomorphic on $\overline{B}$, it follows

$$\hat{h}_w = \beta[iw\hat{X}_w\tau]_w.$$

Hence, if $w \in \overline{B}$ is not a branch point of $\hat{X}$, we obtain

$$\hat{h}_w(w) = \beta[i\hat{X}_w(w)\tau + iw\hat{X}_{ww}(w)\tau(w) + iw\hat{X}_w(w)\tau_w(w)].$$

On the other hand, a minimal surface $\hat{X}$ satisfies

$$\langle \hat{X}_w, \hat{X}_w\rangle = 0$$

and therefore also

$$\langle \hat{X}_w(w), \hat{h}_w(w)\rangle = 0$$

if $w \in \overline{B}$ is not a branch point of $\hat{X}$, and by continuity of $\hat{h}_w$ on $\overline{B}$ it follows

$$\langle \hat{X}_w, \hat{h}_w\rangle = 0 \quad \text{if } \hat{h} \in J(\hat{X}). \tag{12}$$

From (10), (11) and (12) we infer that for a forced Jacobi field $\hat{h}$ its boundary values h lie in the kernel of (6) and therefore in the kernel K_X of the Hessian $D^2E(X)$. This proves the claim, and we have established

Proposition 1. *If $\hat{X}$ is a minimal surface with $X \in M = H^2(S^1, \Gamma)$ then the boundary values h of any $\hat{h} \in J(\hat{X})$ lie in the kernel K_X of the Hessian $D^2E(X)$ of E at X, that is, $h \in T_X M$ and*

$$D^2E(X)(h,k) = 0 \quad \textit{for all } k \in T_X M.$$

Remark 1. We would like to point out that $D^2E(X)$ has been defined for *branched* minimal surfaces without making normal variations of $\hat{X}$.

Before we compute $D^3E(X)$ we give a geometric interpretation of

$$D^2E(X)(h,h) = \delta^2 E(X,h),$$

i.e. of the second variation of E at X in direction of $h \in T_X M$. An integration by parts yields

$$\begin{aligned} \int_{\overline{B}} \nabla\hat{h}\cdot\nabla\hat{h}\,du\,dv &= \int_{S^1}\langle \hat{h}_r, h\rangle\,d\theta - \int_B \langle \Delta\hat{h},\hat{h}\rangle\,du\,dv \\ &= \int_{S^1}\langle \hat{h}_r, h\rangle\,d\theta \end{aligned} \tag{13}$$

since $\Delta\hat{h} = 0$. Away from branch points on S^1 we set

$$h = aX_\theta \quad \text{and} \quad b = \langle \hat{h}_r, X_\theta\rangle.$$

By (8) we have

$$\Omega(X)\hat{h}_r = |X_\theta|^{-2}\langle \hat{h}_r, X_\theta\rangle X_\theta,$$

and so

$$\langle h, \Omega(X)\hat{h}_r\rangle = \langle aX_\theta, bX_\theta\rangle |X_\theta|^{-2} = ab = \langle \hat{h}_r, aX_\theta\rangle = \langle \hat{h}_r, h\rangle$$

and by continuity it follows

$$\langle \hat{h}_r, h\rangle = \langle h, \Omega(X)\hat{h}_r\rangle \quad \text{on } S^1.$$

On account of (7) and (13) it follows that

$$D^2E(X)(h,h) = \int_B |\nabla\hat{h}|^2\,du\,dv + \int_{S^1}\langle h, D\Omega(X)h[\hat{X}_r]\rangle\,d\theta. \tag{14}$$

In order to simplify the boundary term we return to (9) where we insert $m = \hat{X}_r$. Since $\langle \hat{X}_r, X_\theta\rangle = 0$ we have $\Omega(X)\hat{X}_r = 0$ on S^1, and so two terms in (9) vanish. We are left with

$$D\Omega(X)h[\hat{X}_r] = |X_\theta|^{-2}\langle \hat{X}_r, h_\theta\rangle X_\theta.$$

Since $h = aX_\theta$ (away from branch points), we have

$$h_\theta = aX_{\theta\theta} + a_\theta X_\theta$$

whence

$$\langle \hat{X}_r, h_\theta \rangle = a\langle \hat{X}_r, X_{\theta\theta}\rangle.$$

This implies

$$\begin{aligned}
&\langle h, D\Omega(X)h[\hat{X}_r]\rangle \\
&\quad = |X_\theta|^{-2}\langle aX_\theta, a\langle \hat{X}_r, X_{\theta\theta}\rangle X_\theta\rangle = a^2\langle \hat{X}_r, X_{\theta\theta}\rangle \\
&\quad = |h|^2|X_\theta|^{-2}\langle \hat{X}_r, X_{\theta\theta}\rangle = |h|^2 k_g,
\end{aligned}$$

where

(15) $$k_g := |X_\theta|^{-2}\langle \hat{X}_r, X_{\theta\theta}\rangle$$

is the *signed geodesic curvature of Γ in the minimal surface $\hat{X}$, i.e. the interior product of the curvature vector of Γ with the unit vector $|\hat{X}_r|^{-1}\hat{X}_r$,* since $|X_\theta| = |\hat{X}_r|$ on S^1.

Thus we infer from (14) the following result which was independently obtained by R. Böhme and A. Tromba:

Proposition 2. *If $\hat{X}$ is a minimal surface with $X \in M = H^2(S^1, \Gamma)$ then, for any $h \in T_X M$, we obtain*

(16) $$D^2E(X)(h,h) = \int_B |\nabla \hat{h}|^2\, du\, dv + \int_{S^1} k_g |h|^2\, d\theta,$$

where k_g is the signed geodesic curvature (15) of the boundary contour Γ in the minimal surface $\hat{X}$.

Now we proceed to compute the intrinsic third derivative $D^3E(X)$. Let us return to formula (9) which will be differentiated in direction of a vector $k \in T_X M$. This yields

$$\begin{aligned}
&2\langle h_\theta, k_\theta\rangle \Omega(X)m + 2\langle X_\theta, h_\theta\rangle D\Omega(X)[k]m \\
&\qquad + 2\langle X_\theta, k_\theta\rangle D\Omega(X)[h]m + |X_\theta|^2 D^2\Omega(X)(h,k)m \\
&\quad = \langle m, h_\theta\rangle k_\theta + \langle m, k_\theta\rangle h_\theta.
\end{aligned}$$

Choosing $m := \hat{X}_r$ we see that

$$\begin{aligned}
&2\langle X_\theta, h_\theta\rangle D\Omega(X)(k)[\hat{X}_r] + 2\langle X_\theta, k_\theta\rangle D\Omega(X)(h)[\hat{X}_r] \\
&\quad + |X_\theta|^2 D^2\Omega(X)(h,k)[\hat{X}_r] = \langle \hat{X}_r, h_\theta\rangle k_\theta + \langle \hat{X}_r, k_\theta\rangle h_\theta.
\end{aligned}$$

By (7) we may write for h, k in the kernel of $D^2E(X)$ (and therefore in the kernel of (6))

(17) $$D\Omega(X)(h)[\hat{X}_r] = -\Omega(X)\hat{h}_r,\ \ D\Omega(X)(k)[\hat{X}_r] = -\Omega(X)\hat{k}_r,$$

then obtaining

$$(18)\qquad -2\langle X_\theta, h_\theta\rangle \varOmega(X)\hat{k}_r - 2\langle X_\theta, k_\theta\rangle \varOmega(X)\hat{h}_r + |X_\theta|^2 D^2\varOmega(X)(h,k)[\hat{X}_r] = \langle \hat{X}_r, h_\theta\rangle k_\theta + \langle \hat{X}_r, k_\theta\rangle h_\theta.$$

Setting in (9) $m = \hat{k}_r$ we get

$$(19)\qquad 2\langle X_\theta, h_\theta\rangle \varOmega(X)\hat{k}_r + |X_\theta|^2 D\varOmega(X)[h]\hat{k}_r = \langle \hat{k}_r, h_\theta\rangle X_\theta + \langle \hat{k}_r, X_\theta\rangle h_\theta.$$

Commuting h and k it follows also

$$(20)\qquad 2\langle X_\theta, k_\theta\rangle \varOmega(X)\hat{h}_r + |X_\theta|^2 D\varOmega(X)[k]\hat{h}_r = \langle \hat{h}_r, k_\theta\rangle X_\theta + \langle \hat{h}_r, X_\theta\rangle k_\theta.$$

Adding (19) and (20) to (18) we see that

$$(21)\qquad |X_\theta|^2 D^2\varOmega(X)(h,k)\hat{X}_r + |X_\theta|^2 D\varOmega(X)[h]\hat{k}_r + |X_\theta|^2 D\varOmega(X)[k]\hat{h}_r = \langle \hat{X}_r, h_\theta\rangle k_\theta + \langle \hat{X}_r, k_\theta\rangle h_\theta + \langle \hat{h}_r, k_\theta\rangle X_\theta + \langle \hat{h}_r, X_\theta\rangle k_\theta + \langle \hat{k}_r, h_\theta\rangle X_\theta + \langle \hat{k}_r, X_\theta\rangle h_\theta.$$

By (10) we have

$$\langle X_\theta, \hat{h}_r\rangle = -\langle \hat{X}_r, h_\theta\rangle \quad \text{and} \quad \langle X_\theta, \hat{k}_r\rangle = -\langle \hat{X}_r, k_\theta\rangle.$$

Therefore (21) reduces to

$$(22)\qquad |X_\theta|^2\{D^2\varOmega(X)(h,k)\hat{X}_r + D\varOmega(X)[h]\hat{k}_r + D\varOmega(X)[k]\hat{h}_r\} = \{\langle \hat{h}_r, k_\theta\rangle + \langle \hat{k}_r, h_\theta\rangle\}X_\theta.$$

Suppose now that h, k, ℓ lie in the space $J(\hat{X})$ of forced Jacobi fields. By (7) we have

$$(22')\qquad D^2E(X)(h,\ell) = \int_{S^1} \langle D\varOmega(X)h[\hat{X}_r] + \varOmega(X)\hat{h}_r, \ell\rangle \, d\theta.$$

Differentiating this in direction of k it follows

$$(23)\qquad D^3E(X)(h,\ell,k) = \int_{S^1} \langle D^2\varOmega(X)(h,k)[\hat{X}_r] + D\varOmega[h]\hat{k}_r + D\varOmega(X)[k]\hat{h}_r, \ell\rangle \, d\theta,$$

which by (22) yields

$$(24)\qquad D^3E(X)(h,\ell,k) = \int_{S^1} \{\langle \hat{h}_r, k_\theta\rangle + \langle \hat{k}_r, h_\theta\rangle\}|X_\theta|^{-2}\langle X_\theta, \ell\rangle \, d\theta.$$

Actually there are two more terms on the right-hand side of (24) which come from the derivatives ℓ' and h' of ℓ and h. We have to shows that these terms are zero if ℓ and h are forced Jacobi fields. The additional ℓ'-term is

$$\int_{S^1} \langle D\Omega(X)h[\hat{X}_r] + \Omega(X)\hat{h}_r, \ell' \rangle \, d\theta.$$

It vanishes since

$$D\Omega(X)h[\hat{X}_r] + \Omega(X)\hat{h}_r = 0,$$

as h is a forced Jacobi field.
The second additional term becomes

$$\int_{S^1} \langle h', (\widehat{\lambda X_\theta})_r - (\lambda \hat{X}_r)_\theta \rangle \, d\theta$$

if we write $\ell = \lambda X_\theta = \text{Re}\{\lambda i w \hat{X}_w\}$ and integrate by parts. But ℓ is holomorphic in B. and so the Cauchy–Riemann equations yield

$$-\frac{\partial}{\partial \theta}(\widehat{\lambda X_\theta}) + \frac{\partial}{\partial r}(\widehat{\lambda X_\theta}) = 0.$$

This equation extends to the boundary $S^1 = \partial B$, and so the second additional term vanishes too.

The two expressions (23) and (24) yield the intrinsic third derivative of E at X. We synonymously write

$$\begin{aligned} \frac{\partial E}{\partial h}(X) &= DE(X)h, \\ \frac{\partial^2 E}{\partial h \partial k}(X) &= D^2E(X)(h,k), \\ \frac{\partial^3 E(X)}{\partial h \partial \ell \partial k} &= D^3E(X)(h,\ell,k). \end{aligned} \tag{25}$$

Suppose that $h, k, \ell \in J(\hat{X})$ have the generators τ, ρ, λ; we shall write τ, ρ, λ also for the boundary values $\tau|_{S^1}, \rho|_{S^1}, \lambda|_{S^1}$:

$$\begin{aligned} h(\theta) &= \tau(\theta)X_\theta(\theta), \quad \text{so} \quad & \hat{h}(w) &= 2\text{Re}(iw\tau(w)\hat{X}_w(w)), \\ k(\theta) &= \rho(\theta)X_\theta(\theta), & \hat{k}(w) &= 2\text{Re}(iw\rho(w)\hat{X}_w(w)), \\ \ell(\theta) &= \lambda(\theta)X_\theta(\theta), & \hat{\ell}(w) &= 2\text{Re}(iw\lambda(w)\hat{X}_w(w)). \end{aligned} \tag{26}$$

Then (24) becomes

$$D^3E(X)(h,\ell,k) = \int_{S^1} \{\langle \hat{h}_r, k_\theta \rangle + \langle \hat{k}_r, h_\theta \rangle\} \lambda(\theta) \, d\theta. \tag{27}$$

On S^1 we have $d\theta = \frac{dw}{iw}$ and

$$2w\hat{h}_w = \hat{h}_r - ih_\theta, \quad 2w\hat{k}_w = \hat{k}_r - ik_\theta$$

whence

$$\langle \hat{h}_r, k_\theta \rangle + \langle \hat{k}_r, h_\theta \rangle = -4\,\mathrm{Im}(w^2 \hat{h}_w \hat{k}_w \rangle.$$

Furthermore,

$$\begin{aligned} \hat{h}_w &= (iw\hat{X}_w\tau)_w = i(w\tau\hat{X}_{ww} + \hat{X}_w\tau + w\hat{X}_w\tau_w), \\ \hat{k}_w &= (iw\hat{X}_w\rho)_w = i(w\rho\hat{X}_{ww} + \hat{X}_w\rho + w\hat{X}_w\rho_w). \end{aligned}$$

Since $\hat{X}_w \cdot \hat{X}_w = 0$ and $\hat{X}_w \cdot \hat{X}_{ww} = 0$ it follows that

$$w^2\hat{h}_w\hat{k}_w = -w^4\tau\rho\hat{X}_{ww} \cdot \hat{X}_{ww}$$

and consequently

$$\langle \hat{h}_r, k_\theta \rangle + \langle \hat{k}_r, h_\theta \rangle = 4\,\mathrm{Im}(w^4\tau\rho\hat{X}_{ww} \cdot \hat{X}_{ww}).$$

This implies

$$\begin{aligned} D^3E(X)(h,\ell,k) &= 4\int_{S^1} \mathrm{Im}(w^4\tau\rho\hat{X}_{ww} \cdot \hat{X}_{ww})\lambda\,d\theta \\ &= 4\,\mathrm{Im}\int_{S^1} w^4\tau\rho\lambda\hat{X}_{ww} \cdot \hat{X}_{ww}\,d\theta \\ &= 4\,\mathrm{Im}\int_{S^1} w^4\tau\rho\lambda\hat{X}_{ww} \cdot \hat{X}_{ww}\frac{dw}{iw}, \end{aligned}$$

and we arrive at

$$\begin{aligned} D^3E(X)(h,\ell,k) &= -4\,\mathrm{Re}\int_{S^1} w^3\tau\rho\lambda\hat{X}_{ww} \cdot \hat{X}_{ww}\,dw \\ &= 4\int_{S^1} \mathrm{Im}(w^4\tau\rho\lambda\hat{X}_{ww} \cdot \hat{X}_{ww})\,d\theta. \end{aligned} \tag{28}$$

It follows from (23) that the right-hand side of (28) is the integral of a continuous function. If we wish to apply the residue theorem to evaluate the integral in (28) we have to get a better grip to the integrand. To this end we impose an *additional standing assumption:* $n = 3$*, i.e. we consider boundary contours only in* $\mathbb{R}^3$.

First we wish to understand what the generators τ of forced Jacobi fields for a minimal surface $\hat{X}$ with a boundary branch point $w_0 \in S^1$ are. By means of a rotation we can move w_0 to the point $w = 1$. Thus we make the following further standing **assumption:**

$\hat{X} \in \mathcal{C}(\Gamma)$ *is a minimal surface in the unit disk* B *with the boundary branch point* $w = 1$ *of order* n*, and the boundary contour* $\Gamma \in C^2$ *has a total curvature* $\kappa(\Gamma) := \int_\Gamma \kappa(s)\,ds$ *satisfying* $3\kappa(\Gamma) \le \tau\boldsymbol{r}$*. It is also assumed that*

$\Gamma \in C^{\boldsymbol{r}+7}, \boldsymbol{r} \geq 2$, which implies $\hat{X} \in C^{\boldsymbol{r}+6,\beta}(\overline{B}, \mathbb{R}^3)$, $0 < \beta < 1$, and $n \leq \boldsymbol{r}/3 - 2$.

It is easy to verify that

$$\tau(w) := \beta \left(i \frac{w+1}{w-1} \right)^{\ell}, \quad \beta \in \mathbb{R}, \tag{29}$$

is a meromorphic function on $\overline{B}$ with a pole of order ℓ at $w = 1$ such that $\tau(w) \in \mathbb{R}$ for $w \in S^1 \setminus \{1\}$. If $\ell \leq n$ then $\hat{X}_w(w)\tau(w)$ is holomorphic in B and at least continuous on $\overline{B}$ since we have the asymptotic expansion

$$\begin{gathered} \hat{X}_w(w) = a(w-1)^n + o(|w-1|^n) \quad \text{as } w \to 1,\ w \in \overline{B} \setminus \{1\} \\ \text{with } a \in \mathbb{C}^3,\ a \neq 0, \text{ and } a \cdot a = 0. \end{gathered} \tag{30}$$

Thus τ generates a forced Jacobi field for $\hat{X}$. Consider the conformal mapping $\varphi : \overline{B} \setminus \{-1\} \to \overline{\mathcal{H}}$, defined by

$$w \mapsto z = \varphi(w) := -i \frac{w-1}{w+1}, \quad w \in \overline{B} \setminus \{-1\}, \tag{31}$$

which maps $B = \{w \in \mathbb{C} : |w| < 1\}$ onto the upper halfplane

$$\mathcal{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

and takes $S^1 \setminus \{-1\}$ onto the real line $\mathbb{R}$ such that $\varphi(1) = 0, \varphi(i) = 1, \varphi(-1) = \infty$. The inverse $\psi := \varphi^{-1}$ is given by

$$z \mapsto w = \psi(z) := \frac{1+iz}{1-iz}. \tag{32}$$

We write $z = x + iy$ with $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$, while $w = u + iv, u = \operatorname{Re} w, v = \operatorname{Im} w$. From (31) we infer

$$\frac{1}{z} = i \frac{w+1}{w-1}$$

and so

$$\sigma := \tau \circ \psi = \frac{\beta}{z^{\ell}}. \tag{33}$$

Transforming the minimal surface $\hat{X}(w)$ to the new parameter z, we obtain

$$\hat{Y}(z) := \hat{X}(\psi(z)) \tag{34}$$

which has the branch point $z = 0$ on $\mathbb{R} = \partial\mathcal{H}$ with the asymptotic expansion

$$\begin{aligned} &\hat{Y}_z(z) = bz^n + o(|z|^n) \quad \text{as } z \to 0,\ z \in \overline{\mathcal{H}} \setminus \{0\} \\ &b \in \mathbb{C}^3 \setminus \{0\}, \quad b \cdot b = 0. \end{aligned}$$

Choosing a suitable coordinate system in $\mathbb{R}^3$ we may assume that $\hat{Y}_z(z)$ can be written in the normal form

$$\hat{Y}_z(z) = \tilde{A}_1 z^n + o(z^n) \tag{35}$$

with $\tilde{A}_1 = (a_1 + ib_1)$; $a_1, b_1 \in \mathbb{R}^3$, $|a_1|^2 = |b_1|^2 \neq 0$; $a_1 \cdot b_1 = 0$, $a_1 = (n+1)\alpha e_1$, $e_1 = (1, 0, 0)$, $\alpha > 0$, where a_1, b_1 span the tangent space to $\hat{X}$ at $X(1)$. Let us recall that the order of any boundary branch point is even; thus we can set

$$n = 2\nu \quad \text{with } \nu \in \mathbb{N}. \tag{36}$$

Now we wish to write $\hat{Y}_z$ in the more specific form

$$\hat{Y}_z(z) = (A_1 z^n + \cdots + A_{m-n+1} z^m + O(|z|^{m+1}), \quad R_m z^m + O(|z|^{m+1})) \tag{37}$$

with

$$R_m \neq 0. \tag{38}$$

By Taylor's theorem and (35) we can achieve (37) for any $m \in \mathbb{N}$ with $m > n$ and such that $\hat{Y} \in C^{m+2}(\overline{\mathcal{H}}, \mathbb{R}^3)$.

However, it is not at all a priori obvious that one can achieve also (38). This fact is ensured by the following

Proposition 3. *Suppose that $\hat{Y} \in C^{3n+6}(\overline{\mathcal{H}}, \mathbb{R}^3)$ and that both the torsion $\boldsymbol{\tau}$ and the curvature $\boldsymbol{\kappa}$ of Γ are nonzero. Then there is an $m \in \mathbb{N}$ with $n + 1 < m + 1 \leq 3(n+1)$ such that*

$$\hat{Y}_z^3(z) = R_m z^m + O(|z|^{m+1}) \quad \text{for } |z| \ll 1 \text{ and } R_m \neq 0. \tag{39}$$

Proof. Otherwise we have

$$\hat{Y}_z^3(z) = O(|z|^{3n+3}). \tag{40}$$

Let $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$ be the local representation of Γ with respect to its arc-length parameter s such that $\gamma(0) = \hat{Y}(0)$ and $\gamma'(0) = e_1$. By (35) and (36) we have

$$\hat{Y}_x(x, 0) = (n+1)\alpha e_1 x^n + O(x^{n+1}), \quad n = 2\nu,$$

and so s and x are related by $s = \sigma(x)$ with

$$\sigma'(x) = |Y_x(x)| = [(n+1)\alpha x^n + O(x^{n+1})],$$

whence

$$\sigma(x) = \alpha x^{n+1} + O(x^{n+2}) \quad \text{as } x \to 0. \tag{41}$$

Then $Y(x) = \gamma(\sigma(x))$ for $|x| \ll 1$, and therefore the third component Y^3 of Y is given by

$$Y^3(x) = \gamma_3(\sigma(x)) = \gamma_3(\alpha x^{n+1} + O(x^{n+2})) \quad \text{for } x \to 0.$$

Because of (40) we have $Y^3_x(x) = O(x^{3n+3})$ as $x \to 0$, which implies

$$Y^3(x) = O(x^{3n+4}) \quad \text{as } x \to 0. \tag{42}$$

On the other hand

$$\gamma(s) = \gamma'(0)s + O(s^2) \quad \text{as } s \to 0.$$

Consequently

$$Y^3(x) = \gamma_3'(0)\alpha x^{n+1} + O(x^{n+2}) \quad \text{as } x \to 0.$$

On account of (42) and $\alpha > 0$ it follows $\gamma_3'(0) = 0$. Thus we can write

$$\gamma_3(s) = \frac{1}{2}\gamma_3''(0)s^2 + O(s^3) \quad \text{as } s \to 0,$$

which implies

$$Y^3(x) = \frac{1}{2}\gamma_3''(0)\alpha^2 x^{2n+2} + O(x^{2n+3}) \quad \text{as } x \to 0.$$

By (42) and $\alpha > 0$ we obtain $\gamma_3''(0) = 0$, and we have

$$\gamma_3(s) = \frac{1}{6}\gamma_3'''(0)s^3 + O(s^4) \quad \text{as } s \to 0.$$

Hence,

$$Y^3(x) = \frac{1}{6}\gamma_3'''(0)\alpha^3 x^{3n+3} + O(x^{3n+4}) \quad \text{as } x \to 0,$$

and then $\gamma_3'''(0) = 0$ on account of (42) and $\alpha > 0$. Thus we have found

$$\gamma_3'(0) = 0, \quad \gamma_3''(0) = 0, \quad \gamma_3'''(0) = 0,$$

and so the three vectors $\gamma'(0), \gamma''(0), \gamma'''(0)$ are linearly dependent. This will contradict our assumption $\kappa(s) \neq 0$ and $\tau(s) \neq 0$. To see this we introduce the Frenet triple $\boldsymbol{T}(s), \boldsymbol{N}(s), \boldsymbol{B}(s)$ of the curve Γ satisfying $\boldsymbol{T} = \gamma', \boldsymbol{T}' = \gamma'', \boldsymbol{T}'' = \gamma'''$, and

$$\begin{aligned} \boldsymbol{T}' &= \quad\quad \boldsymbol{\kappa N} \\ \boldsymbol{N}' &= -\boldsymbol{\kappa T} \quad\quad + \boldsymbol{\tau B} \\ \boldsymbol{B}' &= \quad\quad -\boldsymbol{\tau N}. \end{aligned}$$

Then $\boldsymbol{T}_3(0) = 0, \boldsymbol{T}_3'(0) = 0, \boldsymbol{T}_3''(0) = 0$, and from $\boldsymbol{T}' = \boldsymbol{\kappa N}$ and $\boldsymbol{\kappa} \neq 0$ it follows that $\boldsymbol{N}_3(0) = 0$. Since

$$\boldsymbol{N}' = \left(\frac{1}{\boldsymbol{\kappa}}\right)' \boldsymbol{T}' + \frac{1}{\boldsymbol{\kappa}}\boldsymbol{T}''$$

we obtain $\boldsymbol{N}_3'(0) = 0$ whence $\boldsymbol{\tau}(0)\boldsymbol{B}_3(0) = 0$. Because of $\boldsymbol{\tau} \neq 0$ it follows that $\boldsymbol{B}_3(0) = 0$, and so $\boldsymbol{T}(0), \boldsymbol{N}(0), \boldsymbol{B}(0)$ are linearly dependent. This is a contradiction since $(\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B})$ is an orthonormal frame, hence the assumption (40) is impossible. □

Remark 2. Note that $n \leq \boldsymbol{r}/3 - 2$ implies $3n + 6 \leq \boldsymbol{r} < \boldsymbol{r} + 7$. Thus the assumption $\hat{Y} \in C^{3n+6}(\overline{\mathcal{H}}, \mathbb{R}^3)$ is certainly satisfied if we assume $3\kappa(\Gamma) \leq \pi \boldsymbol{r}$ and $\Gamma \in C^{\boldsymbol{r}+7}$. Thus we have a lower bound on $\boldsymbol{r}$ and upper bounds on n and m. We call the number m in (39) with $n < m < 3n + 3$ the *index* of the boundary branch point $z = 0$ of $\hat{Y}$, or of the boundary branch point $w = 1$ of $\hat{X}$.

Assumption. *In what follows we assume that the assumptions and therefore also the conclusions of Proposition 3 are satisfied.*

Proposition 4. *If $m + 1 \not\equiv 0 \bmod (n+1)$ (i.e. if $z = 0$ is not an exceptional branch point of $\hat{Y}$) then the coefficient R_m in (39) satisfies*

$$\operatorname{Re} R_m = 0, \tag{43}$$

i.e. R_m is purely imaginary, and therefore

$$R_m^2 < 0 \tag{44}$$

since $R_m \neq 0$. If we write (39) in the form

$$Y_z^3(z) = R_m z^m + R_{m+1} z^{m+1} + R_{m+2} z^{m+2} + o(|z|^{m+2}) \quad \textit{for } |z| \ll 1 \tag{45}$$

and if $2m - 2 < 3n$, then we in addition obtain that

$$\operatorname{Re} R_{m+1} = 0 \quad \textit{and, if } n > 2, \textit{ also} \quad \operatorname{Re} R_{m+2} = 0. \tag{46}$$

Finally, independent of any assumption on m, we have

$$A_j = \mu_j A_1, \quad j = 1, \ldots, \min\{n+1, 2m - 2n\}, \ \textit{with } \mu_j \in \mathbb{R} \tag{47}$$

for the coefficients A_j in the expansion (37).

Remark 3. The relations (47) are in some sense a strengthening of the equations

$$A_j = \lambda_j A_1, \quad j = 1, \ldots, 2m - 2n, \ \text{with } \lambda_j \in \mathbb{C}$$

which hold at an interior branch point $w = 0$ of a minimal surface $\hat{X}$ in normal form.

Proof of Proposition 4. (i) From (45) we infer

$$Y^3(x) = \operatorname{Re} \left(\frac{R_m}{m+1} x^{m+1} + \frac{R_{m+1}}{m+2} x^{m+2} + \frac{R_{m+2}}{m+3} x^{m+3} + o(x^{m+3}) \right) \quad \text{for } x \to 0. \tag{48}$$

On the other hand,

$$Y^3(x) = \gamma_3(\alpha x^{n+1} + o(x^{n+1}))$$

and $\gamma(0) = 0, \gamma'(0) = e_3$ whence also $\gamma_3(0) = \gamma_3'(0) = 0$. As pointed out before it is then impossible that both $\gamma_3''(0) = 0$ and $\gamma_3'''(0) = 0$ because this would imply that $\boldsymbol{T}(0), \boldsymbol{N}(0), \boldsymbol{B}(0)$ are linearly dependent. Thus we obtain

$$\gamma_3(s) = \frac{1}{k!}\gamma^{(k)}(0)s^k + O(s^{k+1}) \quad \text{as } s \to 0, \gamma^{(k)}(0) \neq 0,$$

for $k = 2$ or $k = 3$. Therefore

$$Y^3(x) = \frac{1}{k!}\gamma_3^{(k)}(0)\alpha^k x^{k(n+1)} + o(x^{k(n+1)}) \quad \text{as } x \to 0. \tag{49}$$

Comparing (48) and (49) it follows that $\operatorname{Re} R_m \neq 0$ implies $m+1 = k(n+1)$ for $k = 2$ or $k = 3$, which is excluded by assumption. Thus $\operatorname{Re} R_m = 0$, and we have

$$\begin{aligned} Y^3(x) &= \operatorname{Re}\left(\frac{R_{m+1}}{m+2}x^{m+2} + \frac{R_{m+2}}{m+3}x^{m+3} + o(x^{m+3})\right) \\ &= \frac{1}{k!}\gamma^k(0)\alpha^k x^{k(n+1)} + o(x^{k(n+1)}) \quad \text{as } x \to 0. \end{aligned} \tag{50}$$

Suppose now that $2m - 2 < 3n$, which is equivalent to

$$2m \leq 3n \tag{51}$$

since n is even, and so

$$m + 2 < m + 3 \leq \frac{3}{2}n + 3 < 3(n+1).$$

Thus, for $k = 3$, equation (50) can only hold if

$$\operatorname{Re} R_{m+1} = 0 \quad \text{and} \quad \operatorname{Re} R_{m+2} = 0.$$

Furthermore, (51) yields also

$$m + 2 < m + 3 \leq \frac{3}{2}n + 3 = (2n+2) + \left(1 - \frac{n}{2}\right) \begin{cases} = 2n+2 & \text{and} \quad n = 2 \\ < 2n+2 & \phantom{\text{and}} \quad n > 2. \end{cases}$$

Hence it follows in this case that always $\operatorname{Re} R_{m+1} = 0$ while $\operatorname{Re} R_{m+2} = 0$ holds for $n > 2$.

(ii) From $Y_x(x) = 2\operatorname{Re}\hat{Y}_z(x, 0)$, (50) and (37) it follows that

$$Y_x(x) = 2\operatorname{Re}(A_1 x^n + \cdots + A_{n+1}x^{2n} + o(x^{2n}), o(x^{2n}))$$

whence

$$Y(x) = 2\operatorname{Re}\left(\frac{A_1}{n+1}x^{n+1} + \cdots + \frac{A_{n+1}}{2n+1}x^{2n+1} + o(x^{2n+1}), o(x^{2n+1})\right).$$

Furthermore,

$$\gamma(s) = e_1 s + O(s^2) \quad \text{as } s \to 0$$

and

$$\sigma(x) = b_1 x^{n+1} + \cdots + b_{n+1} x^{2n+1} + o(x^{2n+1}) \quad \text{as } x \to 0$$

with $b_1, \ldots, b_{n+1} \in \mathbb{R}$, $\alpha e_1 = b_1 e_1 = \frac{2}{n+1} \operatorname{Re} A_1$. Then

$$Y(x) = \gamma(\sigma(x)) = (b_1 x^{n+1} + \cdots + b_{n+1} x^{2n+1}) e_1 + O(x^{2n+2}).$$

Comparing the coefficients we get

$$2 \operatorname{Re} A_j = (n+j) b_j e_1 \quad \text{with } \alpha = b_1 > 0 \text{ for } 1 \le j \le n+1.$$

Then $\operatorname{Re} A_j = \frac{(n+j)b_j}{(n+1)\alpha} \operatorname{Re} A_1$, and so

$$\operatorname{Re} A_j = \mu_j \operatorname{Re} A_1 \quad \text{for } j = 2, \ldots, n+1$$

with

$$\mu_j := \frac{n+j}{n+1} \frac{b_j}{\alpha}, \quad 2 \le j \le n+1.$$

Set $A_j := a_j + i b_j$; $a_j := \operatorname{Re} A_j$, $b_j := \operatorname{Im} A_j \in \mathbb{R}^n$. We know from 6.1 that $A_j = \lambda_j A_1$ for $j = 1, \ldots, 2m - 2n$ with $\lambda_j \in \mathbb{C}$ hence

$$a_j = (\operatorname{Re} \lambda_j) a_1 - (\operatorname{Im} \lambda_j) b_1 \quad \text{for } 2 \le j \le 2m - 2n$$

and

$$a_j = \mu_j a_1 \quad \text{for } 2 \le j \le n+1.$$

From $|\hat{Y}_x| = |\hat{Y}_y|$ it follows that $|b_1| = |a_1| = \frac{n+1}{2}\alpha > 0$, and $\hat{Y}_x \cdot \hat{Y}_y = 0$ yields $a_1 \cdot b_1 = 0$; thus we obtain $\operatorname{Im} \lambda_j = 0$ for $j = 2, \ldots, n+1$ whence $\lambda_j = \mu_j \in \mathbb{R}$ and $A_j = \mu_j A_1$ for $1 \le j \le \min\{n+1, 2m-2n\}$. □

Let us now return to formula (28) for $D^3 E(X)(h, k, \ell)$ in the direction of forced Jacobi fields (with the boundary values) h, k, ℓ; note that (28) is symmetric in h, k, ℓ. We already know that (28) is the integral of a continuous function; but we need to understand (28) at a level where we can apply the residue theorem. To this end we consider the conformal mapping (31) defined by

$$w \mapsto z = \varphi(w) := -i \frac{w-1}{w+1}, \quad w \in \overline{B} \setminus \{-1\},$$

which has the derivative

$$\varphi'(w) = \frac{-2i}{(w+1)^2}. \tag{52}$$

Using the inverse

$$z \mapsto w = \psi(z) := \frac{1+iz}{1-iz}$$

we obtain

(53) $$\varphi'(\psi(z)) = \frac{-i}{2}(1 - iz)^2,$$

or sloppily

$$\frac{dz}{dw} = -\frac{i}{2}(1 - iz)^2.$$

From (34) we get $\hat{X}(w) = \hat{Y}(\varphi(w))$, whence

$$\hat{X}_{ww} = \hat{Y}_{zz}(\varphi)(\varphi')^2 + \hat{Y}_z(\varphi)\varphi''.$$

From $\hat{Y}_z \cdot \hat{Y}_z = 0$ it follows $\hat{Y}_z \cdot \hat{Y}_{zz} = 0$, and then

(54) $$\hat{X}_{ww} \cdot \hat{X}_{ww} = \hat{Y}_{zz}(\varphi) \cdot \hat{Y}_{zz}(\varphi)(\varphi')^4,$$

which we sloppily write

$$\hat{X}_{ww} \cdot \hat{X}_{ww} = \hat{Y}_{zz} \cdot \hat{Y}_{zz} \left(\frac{dz}{dw}\right)^4.$$

Lemma 3. *Assuming $2m - 2 < 3n$ (i.e. $2m \le 3n$) we obtain the Taylor expansion*

(55) $$(\hat{Y}_{zz} \cdot \hat{Y}_{zz})(z) = \sum_{j=0}^{s} Q_j z^{2m-2+j} + R(z)$$

with $s := (3n - 1) - (2m - 2) = (3n - 2m) + 1 \ge 1$, $R(z) = O(z^{3n})$, where $Q_0 := (m - n)^2 R_m^2 < 0$ and $\operatorname{Im} Q_j = 0$ for $0 \le j \le s$.

Proof. From $2m - 2 < 3n$ we infer $2m \le 3n$ since n is even. Thus $s \ge 1$ and $2m - 2n + 1 \le n + 1$. Consider the Taylor expansion

$$\hat{Y}_z(z) = (A_1 z^n + A_2 z^{n+1} + \cdots, R_m z^m + R_{m+1} z^{m+1} + \cdots),$$

where "$+\cdots$" indicates further z-powers plus a remainder term. As for interior branch points we have

(56) $$A_1 \cdot A_{2m-2n+1} = -R_m^2/2$$

and

(57) $$A_2 \cdot A_{2m-2n+1} + A_1 \cdot A_{2m-2n+2} = -R_m R_{m+1}.$$

By (44) we have $R_m^2 < 0$ whence $A_1 \cdot A_{2m-2n+1} \in \mathbb{R}$. Since $2 \le 2m - 2n \le n$ it follows $A_2 = \mu_2 A_1$ with $\mu_2 \in \mathbb{R}$ on account of (47). Then (56) implies $A_2 \cdot A_{2m-2n+1} \in \mathbb{R}$, and furthermore $R_m R_{m+1} \in \mathbb{R}$ in virtue of (44) and (46). Then (57) yields $A_1 \cdot A_{2m-2n+2} \in \mathbb{R}$, and we arrive at

$$\hat{Y}_{zz}(z) \cdot \hat{Y}_{zz}(z) = Q_0 z^{2m-2} + Q_1 z^{2m-1} + \cdots$$

with $Q_0 = (m-n)^2 R_m^2$, (see Section 6.1), and $Q_0 < 0$ as well as $Q_1 \in \mathbb{R}$, since Q_1 is a real linear combination of $A_1 \cdot A_{2m-2n+2}$, $A_2 \cdot A_{2m-2n+1}$, and $R_m R_{m+1}$. Suppose now that $s = 3n-2m+1 > 1$. In order to show $\operatorname{Im} Q_j = 0$ for $2 \le j \le s$, we note that by (54)

$$\tau\rho\lambda w^4 \hat{X}_{ww} \cdot \hat{X}_{ww} = \tau\rho\lambda \hat{Y}_{zz} \cdot \hat{Y}_{zz} \left(w \frac{dz}{dw} \right)^4,$$

where τ, ρ, λ are generators of forced Jacobi fields with the pole $w = 1$. Furthermore, by (52),

$$w \frac{dz}{dw} = \frac{-2iw}{(w+1)^2} = \frac{1+z^2}{2i}. \tag{58}$$

Thus

$$\operatorname{Im}(\tau\rho\lambda w^4 \hat{X}_{ww} \cdot \hat{X}_{ww}) = \frac{1}{16} \operatorname{Im}[\tau\rho\lambda(1+z^2)^4 \hat{Y}_{zz} \cdot \hat{Y}_{zz}]. \tag{59}$$

By (28) the left-hand side of (59) is a continuous function on S^1, and thus the right-hand side must be continuous in a neighbourhood of 0 in $\overline{\mathcal{H}}$ for all generators τ, ρ, λ of forced Jacobi fields $\hat{h}, \hat{k}, \hat{l}$ with poles at $w = 1$.

Suppose now that not all Q_j with $2 \le j \le s$ are real, $s = (3n-1)-(2m-2)$, and let J be the smallest of the indices $j \in \{2, \ldots, s\}$ with the property that $\operatorname{Im} Q_j \ne 0$. Then we choose λ, ρ, τ such that the sum of their pole orders at $w = 1$ equals $(J+1) + (2m-2) \le 3n$. Transforming λ, ρ, τ from w to z it follows for $z = x \in \mathbb{R} = \partial\mathcal{H}$ that

$$\begin{aligned} &\operatorname{Im}[\tau\rho\lambda(1+z^2)^4 \hat{Y}_{zz} \cdot \hat{Y}_{zz}]\Big|_{z=x\in\mathbb{R}} \\ &\quad = (1+x^2)^4 \beta_1 (\operatorname{Im} Q_J) \frac{1}{x} + \langle \text{terms continuous in } x \rangle, \end{aligned} \tag{60}$$

$\beta_1 \in \mathbb{R} \setminus \{0\}$. This is clearly not a continuous function unless $\operatorname{Im} Q_J = 0$, a contradiction, therefore no such J exists. □

Now we want to evaluate the integral in (28) by applying the residue theorem. To this and we state

Proposition 5. *Let τ be given by (29), and consider the function*

$$f(w) := \tau(w)^4 w^4 \hat{X}_{ww}(w) \cdot \hat{X}_{ww}(x), \quad w \in \overline{B}, \tag{61}$$

which has a continuous imaginary part on $S^1 = \partial B$. Then there is a meromorphic function $g(w)$ on $\overline{B}$ with a pole only at $w = 1$ such that

(i) $\operatorname{Im}[f(w) - g(w)] = 0$ *for $w \in S^1 = \partial B$;*

(ii) $f - g$ *is continous on $\overline{B}$.*

Proof. Setting $w = \psi(z) = (1+iz)/(1-iz)$ we obtain

$$f(\psi(z)) = \frac{1}{16}\tau(\psi(z))^3(1+z^2)^4\hat{Y}_{zz}(z)\cdot\hat{Y}_{zz}(z).$$

By (55) of Lemma 3 we see that, in a neighbourhood of $z=0$ in $\mathcal{H}$, we can write the right-hand side as

$$\sum_{j=0}^{s}\sum_{\ell_j}\tilde{\beta}_j\tilde{Q}_j z^{-l_j} + G(z)$$

with $\tilde{\beta}_j \in \mathbb{R}$, $\tilde{Q}_j \in \mathbb{R}$, $0 < l_j \le (3n-1)-(2m-2) = s$, and a continuous term $G(z)$. Set

$$\tilde{g}(z) := \sum_{j=0}^{s}\sum_{l_j=1}^{s}\tilde{\beta}_j\tilde{Q}_j z^{-l_j} \quad \text{for } z \in \overline{\mathcal{H}}\setminus\{0\}$$

and

$$g(w) := \tilde{g}(\varphi(w)) = \sum_{j=0}^{s}\sum_{l_j=1}^{s}\tilde{\beta}_j\tilde{Q}_j\left(i\frac{w+1}{w-1}\right)^{l_j}.$$

Clearly f and g satisfy (i) and (ii). □

Corollary 1. *We have*

$$\int_{S^1}[f(w)-g(w)]\,d\theta = -2\pi\,\mathrm{res}_{w=0}\frac{g(w)}{w}. \tag{62}$$

Proof. For $w = e^{i\theta} \in S^1$ we have $d\theta = dw/(iw)$, whence

$$\begin{aligned}\int_{S^1}[f(w)-g(w)]\,d\theta &= \int_{S^1}[f(w)-g(w)]\frac{dw}{iw}\\ &= 2\pi\,\mathrm{res}_{w=0}\left\{\frac{f(w)-g(w)}{w}\right\}\\ &= -2\pi\,\mathrm{res}_{w=0}\left\{\frac{g(w)}{w}\right\}\end{aligned}$$

since $f(w)/w$ is holomorphic at $w=0$. □

Since $\mathrm{Im}\,g = 0$ on S^1, we obtain

Corollary 2. *We have*

$$\mathrm{Im}\int_{S^1} f(w)\,d\theta = 2\pi\,\mathrm{Im}\,\mathrm{res}_{w=0}\left\{\frac{g(w)}{w}\right\}. \tag{63}$$

Furthermore we have

$$-4\text{Re}\{w^3\tau^3\hat{X}_{ww}\cdot\hat{X}_{ww}\,dw\} = (-4)\text{Re}\{iw^4\tau^3\hat{X}_{ww}\cdot\hat{X}_{ww}\}\,d\theta$$
$$= 4\text{Im}\{w^4\tau^3\hat{X}_{ww}\cdot\hat{X}_{ww}\}\,d\theta = 4\,\text{Im}\,f(w)\,d\theta.$$

Then (28) and Corollary 2 imply

$$D^3E(X)(h,h,h) = -8\pi\,\text{Im}\,\text{res}_{w=0}\left\{\frac{g(w)}{w}\right\}. \tag{64}$$

Remark 4. We note the following slight, but very useful generalization of the three preceding results. Namely, if $\hat{X}$ has other boundary branch points than $w = 1$ we are allowed to change τ by an additive term having poles of first order at these branch points. Then Proposition 5 as well as Corollaries 1 and 2 also hold for the new f defined by (61) and the modified τ. This observation is used in order to ensure that the forced Jacobi field $\hat{h}$ generated by τ produces a variation $\hat{Z}(t)$, $|t| \ll 1$, of $\hat{X}$ which is monotonic on $\partial B = S^1$.

Now we turn to evaluation of $D^3E(X)(h,h,h)$ using formula (64). We distinguish three possible cases: There is an $l \in N$ such that

(i) $2m - 1 = 3l$; then l is odd;

(ii) $2m - 2 = 3l$; in this case l is even;

(iii) $2m = 3l$; here l is again even.

Since $2m \le 3n$ it follows $l < n$ for (i) and (ii), whereas $l \le n$ in case (iii).

Case (i). Choose τ as

$$\begin{aligned} \tau &:= \beta\tau_1 + \epsilon\tau^* \quad \text{and} \quad \beta > 0,\ \epsilon > 0, \quad \text{and} \\ \tau_1 &= \left(i\frac{w+1}{w-1}\right)^l = \frac{1}{z^l}, \quad w \in \overline{B}\setminus\{1\}, \end{aligned} \tag{65}$$

$w = \psi(z), w \in \overline{B}\setminus\{-1\}, z \in \overline{\mathcal{H}}\setminus\{0\}$. We will choose τ^* as a meromorphic function that has poles of order 1 at the boundary branch points different from $w = 1$ or $z = 0$ respectively. Then close to $w = 1$ or $z = 0$ respectively we have

$$\tau^3 w^4\hat{X}_{ww}\cdot\hat{X}_{ww} = \frac{1}{16}\tau^3(1+z^2)^4\hat{Y}_{zz}\cdot\hat{Y}_{zz}$$
$$\overset{(55)}{=} \frac{\beta^3}{16}(m-n)^2R_m^2\frac{1}{z} + G(z) + O(\epsilon)$$

with a continuous $G(z)$.

Choose

$$g(w) = \frac{\beta^3}{16}(m-n)^2R_m^2\left(i\frac{w+1}{w-1}\right)$$

and let $\hat{h}(w) = \mathrm{Re}(iw\hat{X}_w(w)\tau(w))$ be the forced Jacobi field generated by $\tau, h := \hat{h}|_{S^1}$. Then by Proposition 5 and the Corollaries 1, 2 we obtain

$$(66) \quad D^3E(X)(h,h,h) = -\frac{8\pi}{16}\beta^3(m-n)^2R_m^2 \mathrm{Im}\left\{\mathrm{res}_{w=1}\frac{i}{w}\left(\frac{w+1}{w-1}\right)\right\}$$
$$= \frac{1}{2}\pi\beta^3(m-n)^2R_m^2 + O(\epsilon).$$

Since $R_m^2 < 0$ this yields for $0 < \epsilon \ll 1$ that

$$D^3E(X)(h,h,h) < 0.$$

Case (ii). Here we have $3l = 2m-2 < 3n$ whence $l < n$. Since both l and n are even we obtain $l+1 < n$ whence $n > 2$. Moreover, $2m-1 = 2(l+1)+(l-1)$. Set

$$(67) \qquad \tau := \epsilon\tau_1 + \beta\tau_2 + \epsilon^3\tau^*, \quad \beta > 0,\ \epsilon > 0,$$
$$\tau_1 := \left(i\frac{w+1}{w-1}\right)^{l+1},\ \tau_2 := \left(i\frac{w+1}{w-1}\right)^{l-1},\ \tau^* \text{ as in Case 1.}$$

Note also that both $l+1$ and $l-1$ are odd. We then have that

$$\tau^3 = \beta^3\tau_2^3 + 3\beta^2\tau_2^2\tau_1\epsilon + 3\beta\epsilon^2\tau_1^2\tau_2 + O(\epsilon^3)$$
$$= \beta^3z^{-2m+5} + 3\beta^2z^{-2m+3} + 3\beta\epsilon^2z^{-2m+1} + O(\epsilon^3)$$

for z close to zero, but this does not add a contribution to (64).

By the same procedure as in Case 1 we find for $\hat{h} = \mathrm{Re}(iw\hat{X}_w\tau)$ that

$$(69) \qquad D^3E(X)(h,h,h) = \frac{3}{2}\pi\epsilon^2\beta(m-n)^2R_m^2 + O(\epsilon^3),$$

which implies

$$D^3E(X)(h,h,h) < 0 \quad \text{for } 0 < \epsilon \ll 1.$$

Case (iii). Now we have $2m = 3l, l =$ even. We have two subcases.

(a) If $l = n$ we write $2m-1 = 2l + (l-1)$ and set

$$(70) \qquad \tau_1 := \left(i\frac{w+1}{w-1}\right)^{l-1}, \quad \tau_2 := \left(i\frac{w+1}{w-1}\right)^{l},$$
$$\tau := \beta\tau_1 + \epsilon\tau_2 + \epsilon^3\tau^*, \quad \beta > 0,\ \epsilon^3 > 0.$$

(b) If $l < n$ we write $2m-1 = 2(l-1) + (l+1)$ and set

$$(71) \qquad \tau_1 := \left(i\frac{w+1}{w-1}\right)^{l+1}, \quad \tau_2 := \left(i\frac{w+1}{w-1}\right)^{l-1},$$
$$\tau := \epsilon\tau_1 + \beta\tau_2 + \epsilon^3\tau^*.$$

Then our now established procedure yields
(72)

$$D^3E(X)(h,h,h) = \begin{cases} \dfrac{1}{2}3\pi\beta\epsilon^2(m-n)^2R_m^2 + O(\epsilon^3) & \text{in Subcase (a)}, \\[2ex] \dfrac{1}{2}3\pi\beta^2\epsilon(m-n)^2R_m^2 + O(\epsilon^2) & \text{in Subcase (b)}. \end{cases}$$

This again implies $D^3E(X)(h,h,h) < 0$ for $0 < \epsilon \ll 1$ and $\hat{h} = \mathrm{Re}(iw\hat{X}_w\tau)$.

Remark 5. The choice of τ^* has to be carried out in such a way that the variation $Z(t)$ of X produced by $\hat{h} = \mathrm{Re}(iw\hat{X}_w\tau)$ furnishes a monotonic mapping of $\partial B = S^1$ onto the boundary contour Γ. The details on how this can be achieved by the formulae (65), (67), (70) and (71) can be found in the thesis of D. Wienholtz [2]. The complete proof is technically quite involved and will here be omitted. We just sketch the intuitive idea underlying the proof; we shall argue only locally, identifying Γ with its tangent line, and writing $\Gamma \mathrel{\hat{=}} \mathbb{R}$. The boundary values $Y(x)$, $x \in \mathbb{R}$, of our minimal surface $\hat{Y}(z)$ are then interpreted as a mapping $Y : \mathbb{R} \to \mathbb{R}$ with $\hat{Y}(0) = 0$ where $z = 0$ is the boundary branch point of Y which we consider. Then we have

$$Y(x) = \frac{1}{n+1}a_n x^{n+1} + o(x^{n+1}) \quad \text{as } x \to 0$$

with $a_n > 0$ which shows that $Y(x)$ is (locally) monotone. Suppose now that $\tau(x) = \beta x^{-k}, k < n$, k odd, $\beta > 0$. Now define a one-parameter family $Z(t)$ of variations

$$Z(x,t) = Z(t)(x) = Y(x) + tY_x(x)\tau(x).$$

Then

$$\frac{\partial}{\partial x}Z(x,t) = a_n x^n + \beta t(n-k)x^{n-k-1} + o(x^n) + to(x^{n-k-1})$$

and we have

$$\frac{\partial}{\partial x}Z(x,t) > 0 \quad \text{for } 0 < |x| \ll 1 \text{ and } t > 0$$

since $[n-(k+1)]$ is even; thus $Z(x,t)$ is monotonic in x for $|x| \ll 1$ and $t > 0$.

In the actual proof one defines a variation

$$\tilde{Z}(t) := Y + tY_x\tau,$$

and then, using either the normal bundle projection for Γ or an exponential map, we project $\tilde{Z}(t)$ onto Γ, which defines $Z(t)$. The technical difficulty lies in showing that this variation remains monotonic near the branch point 0 for all $t \geq 0$. Global monotonicity for small $t \geq 0$ will follow from the compactness of S^1.

In conclusion we have

Theorem 4. (D. Wienholtz). *If $\hat{X}$ is a minimal surface in $\mathcal{C}(\Gamma)$ with $\Gamma \in C^{\boldsymbol{r}+7}, 3\int_\Gamma \kappa\, ds \le \pi \boldsymbol{r}$, having a boundary branch point of order n and index m satisfying the Wienholtz condition $2m - 2 < 3n$, then X cannot be an $H^2(S^1, \mathbb{R}^3)$-minimizer for Dirichlet's integral $E(X)$ defined by* (1)*, and thus $\hat{X}$ cannot be an $H^{5/2}(\overline{B}, \mathbb{R}^3)$-minimizer of area.*

Remark 6. There remains the question if one can use higher derivatives of E to show that minimizers $\hat{X}$ cannot have boundary branch points if Γ is taken to be sufficiently smooth. Proposition 3 implies that for this purpose it would suffice to consider at most seven derivatives of E if one assumes nonvanishing curvature and torsion of Γ. Focussing on nonexceptional branch points, merely six derivatives of E would suffice.

The exceptional case is even more challenging since we no longer have $\operatorname{Re} R_m = 0$.

6.4 Scholia

The solution of Plateau's problem presented by J. Douglas [12] and T. Radó [17] was achieved by a – very natural – redefinition of the *notion of a minimal surface* $X : \Omega \to \mathbb{R}^3$ which is also used in our book[2]: Such a surface is a harmonic and conformally parametrized mapping; but it is not assumed to be an immersion. Consequently X may possess branch points, and thus some authors speak of "branched immersions". This raises the question whether or not Plateau's problem always has a solution which is immersed, i.e. regular in the sense of differential geometry. Certainly there exist minimal surfaces with branch points; but one might conjecture that area minimizing solutions of Plateau's problem are free of (interior) branch points. To be specific, let Γ be a closed, rectifiable Jordan curve in $\mathbb{R}^3$, and denote by $\mathcal{C}(\Gamma)$ the class of disk-type surfaces $X : B \to \mathbb{R}^3$ bounded by Γ which was defined in Vol. 1, Section 4.2. Then one may ask: *Suppose that $X \in \mathcal{C}(\Gamma)$ is a disk-type minimal surface $X : \overline{B} \to \mathbb{R}^3$ which minimizes both A and D in $\mathcal{C}(\Gamma)$. Does X have branch points in B (or in $\overline{B}$)?*

Radó [17], pp. 791–795 gave a first answer to this question for some special classes of boundary contours Γ, using the following result:

If $X_w(w)$ vanishes at some point $w_0 \in B$ then any plane through the point $P_0 := X(w_0)$ intersects Γ in at least four distinct points.

This observation has the following interesting consequence: *Suppose that there is a straight line $\mathcal{L}$ in $\mathbb{R}^3$ such that any plane through $\mathcal{L}$ intersects Γ in at most two distinct points. Then any minimal surface $X \in \mathcal{C}(\Gamma)$ has no branch points in B.* In fact, for $P_0 \notin \mathcal{L}$, the plane Π determined by P_0 and $\mathcal{L}$

[2] We now denote a minimal surface by X and no longer by $\hat{X}$, i.e. we no longer emphasize the difference between a surface $\hat{X}$ and its boundary values X.

meets Γ in at most two points, and for $P_0 \in \mathcal{L}$ there are infinitely many such planes.

In particular: *If Γ has a simply covered star-shaped image under a (central or parallel) projection upon some plane Π_0, then any minimal surface $X \in \mathcal{C}(\Gamma)$ is free of branch points in B.*

Somewhat later, Douglas [15], pp. 733, 739, 753 thought that he had found a contour Γ with the property that any minimal surface $X \in \mathcal{C}(\Gamma)$ is branched, namely a curve whose orthogonal projection onto the x^1, x^2-plane is a certain closed curve with a double point. Radó [21], p. 109 commented on this assertion as follows: A curve Γ with this x^1, x^2-projection can be chosen in such a way that its x^1, x^3-projection is a simply covered star-shaped curve in the x^1, x^3-plane; thus no minimal surface in $\mathcal{C}(\Gamma)$ has a branch point.

In 1941, Courant [11] believed to have found a contour Γ for which some minimizer of Dirichlet's integral in $\mathcal{C}(\Gamma)$ has an interior branch point. This assertion is not correct, as Osserman [12], p. 567 pointed out in 1970. Moreover, in [12] he described an ingenious line of argumentation which seemed to exclude interior branch points for area minimizing solutions of Plateau's problem. For this purpose he distinguished between *true* and *false* branch points (cf. Osserman, [15], p. 154, Definition 6; and, more vaguely, [12], p. 558): A branch point is false, if the image of some neighbourhood of the branch point lies on a regularly embedded minimal surface; otherwise it is a true branch point. Osserman's treatment of the false branch points is incomplete,but contains essential ideas used by later authors, while his exclusion of true branch points is essentially complete (see also W.F. Pohl [1], Gulliver, Osserman, and Royden [1], p. 751, D. Wienholtz [1], p. 2). The principal ideas of Osserman in dealing with true branch points w_0 are the following: First, the geometric behaviour of the minimal surface X in the neighbourhood of w_0 is studied, yielding the existence of branch lines. Then a remarkable discontinuous parameter transformation G is introduced such that $\tilde{X} := X \circ G$ lies again in $\mathcal{C}(\Gamma)$ and has the same area as X, but in addition $\tilde{X}$ has a wedge, and so its area can be reduced by "smoothing out" the wedge. Osserman's definition of G is somewhat sloppy, but K. Steffen has kindly pointed out to us how this can be remedied and that the construction of the area reducing surface can rigorously be carried out.

Osserman's paper [12] was the decisive break-through in excluding true branch points for area minimizing minimal surfaces in $\mathbb{R}^3$, and it inspired the succeeding papers by R. Gulliver [2] and H.W. Alt [1,2], which even tackled the more difficult branch point problem for H-surfaces and for minimal surfaces in a Riemannian manifold (Gulliver). Nearly simultaneously, both authors published proofs of the assertion that area minimizing minimal surfaces in $\mathcal{C}(\Gamma)$ possess no interior branch points (and of the analogous statement for H-surfaces).

Gulliver's reasoning runs as follows: Let us assume that $w_0 = 0$ is an interior branch point of the minimal surface $X \in \overline{\mathcal{C}}(\Gamma), X : \overline{B} \to \mathbb{R}^3$. Then there is a neighbourhood $V \Subset B$ of 0 in which two oriented Jordan arcs

$\gamma_1, \gamma_2 \in C^1([0,1], B)$ exist with $\gamma_1(0) = \gamma_2(0) = 0, |\gamma_j'(0)| = 1, \gamma_1'(0) \neq \gamma_2'(0), X(\gamma_1(t)) \equiv X(\gamma_2(t))$, and such that $(X_u \wedge X_v)(\gamma_1(t)), (X_u \wedge X_v)(\gamma_2(t))$ are linearly independent for $0 < t \leq 1$. One can assume that ∂V is smooth, and that γ_1, γ_2 meet ∂V transversally at distinct points $\gamma_1(\epsilon), \gamma_2(\epsilon), 0 < \epsilon < 1$. Then there is a homeomorphism $F : \overline{B}_\epsilon \to \overline{V}$ with $F(it) = \gamma_1(t), F(-it) = \gamma_2(t)$ for $0 \leq t \leq \epsilon$, and $F \in C^2(\overline{B}_\epsilon \setminus \{0\})$ where $B_\epsilon := B_\epsilon(0) = \{w \in \mathbb{C} : |w| < \epsilon\}$. Define a discontinuous map $G : \overline{B}_\epsilon \to \overline{B}_\epsilon$ such that $\{it : 0 < t \leq 1\}$ and $\{-it : 0 < t \leq 1\}$ are mapped to i and $-i$ respectively; $\pm\epsilon/2$ are taken to zero; on the segments of discontinuity $[-\epsilon/2, 0]$ and $[0, \epsilon/2]$ are each given two linear mappings by limiting values under approach from the two sides; G is continuous on a neighbourhood of ∂B_ϵ with $G|_{\partial B_\epsilon} = \mathrm{id}_{\partial B_\epsilon}$; and G is conformal on each component of $B_\epsilon \setminus I_\epsilon \setminus$imaginary axis, where I_ϵ is the interval $[-\epsilon/2, \epsilon/2]$ on the real axis. Thus $X \circ F \circ G$ is continuous and piecewise C^2. Now define

$$\overline{X}(w) := \begin{cases} (X \circ F \circ G \circ F^{-1})(w) & \text{for } w \in V, \\ X(w) & \text{for } w \in \overline{B} \setminus V. \end{cases}$$

Then $\overline{X}$ is continuous and piecewise C^2, and $\overline{X} \in \mathcal{C}(\Gamma)$. The metric

$$ds^2 := \langle d\overline{X}, d\overline{X} \rangle = a\,du^2 + 2b\,du\,dv + c\,dv^2,$$

$$a := |\overline{X}_u|^2, \quad b := \langle \overline{X}_u, \overline{X}_v \rangle, \quad c := |\overline{X}_v|^2,$$

induced on B by pulling back the metric induced from $\mathbb{R}^3$ along $\overline{X}$ has bounded, piecewise smooth coefficients. "It follows from the uniformization theorem of Morrey ([1], Theorem 3) that there exists $T : B \to B$ with L^2 second derivatives, which is almost everywhere conformal from B with its usual metric to B with its induced metric, and T may be extended to a homeomorphism $\overline{B} \to \overline{B}$".

Now define $\tilde{X} := \overline{X} \circ T$; then $\tilde{X} \in \mathcal{C}(\Gamma), A(\tilde{X}) = A(X)$, and $\langle \tilde{X}_w, \tilde{X}_w \rangle = 0$ a.e. on B, and consequently

$$\inf_{\mathcal{C}(\Gamma)} D = \inf_{\mathcal{C}(\Gamma)} A = D(X) = A(X) = A(\tilde{X}) = D(\tilde{X}).$$

Thus $\tilde{X}$ is D-minimizing, and so its surface normal $\tilde{N}$ is continuous on B. On the other hand, the sets $\overline{X}(B)$ and $\tilde{X}(B)$ are the same, and so $\tilde{X}(B)$ has an edge, whence $\tilde{N}$ cannot be continuous, a contradiction.

This reasoning requires two comments. First, D. Wienholtz in his Diploma thesis [1], p. 3 (published as [2]), noted that Gulliver's discontinuous map $G : \overline{B}_\epsilon \to \overline{B}_\epsilon$ does not exist, since its existence contradicts Schwarz's reflection principle. A remedy of this deficiency is to set up another definition of G or T, such as used in Alt [1], pp. 360–361, or in Steffen and Wente [1], p. 218, or by a modification of the definition of G as in Gulliver and Lesley [1], p. 24.

Secondly, the application of one of Morrey's uniformization theorems from [1] is not immediately justified, as Theorem 3 of §2 requires besides $a, b, c \in L^\infty(B)$ the assumption

$$(*) \qquad ac - b^2 = 1,$$

and Theorem 3 of Moorey's §4 demands the existence of constants $\lambda_1, \lambda_2 \in \mathbb{R}$ with $0 < \lambda_1 \le \lambda_2$ such that

$$(**) \qquad \lambda_1[\xi^2 + \eta^2] \le a(w)\xi^2 + 2b(w)\xi\eta + c(w)\eta^2 \le \lambda_2[\xi^2 + \eta^2]$$

for all $(\xi, \eta) \in \mathbb{R}^2$ and for almost all $w \in B$. However, $\overline{X}(w) \equiv X(w)$ on $B \setminus V$, and X might have another branch point $w_0' \in B \setminus V$; then $a(w_0') = b(w_0') = c(w_0') = 0$, and so neither (*) nor (**) were satisfied.

This difficulty is overcome by assuring that $\overline{X}$ is quasiconformal in the sense that

$$|\overline{X}_u|^2 + |\overline{X}_v|^2 \le \kappa |\overline{X}_u \wedge \overline{X}_v| \quad \text{(a.e. on } B)$$

holds for some constant $\kappa > 0$. Then it follows

$$a, |b|, c \le \kappa\sqrt{ac - b^2},$$

and thus the quadratic form

$$d\sigma^2 := \alpha\, du^2 + 2\beta\, du\, dv + \gamma\, dv^2$$

with

$$\alpha := \frac{a}{\sqrt{ac - b^2}}, \quad \beta := \frac{b}{\sqrt{ac - b^2}}, \quad \gamma := \frac{c}{\sqrt{ac - b^2}}$$

satisfies $|\alpha|, |\beta|, |\gamma| \le \kappa$ and $\alpha\gamma - \beta^2 = 1$. Hence one can apply Morrey's first uniformization theorem (as quoted above), obtaining a homeomorphism T from $\overline{B}$ onto $\overline{B}$ with $T, T^{-1} \in H_2^1(B, B)$ such that the pull-back $T^*\, d\sigma^2$ is a multiple of the Euclidean metric ds_e^2, i.e.

$$T^*\, d\sigma^2 = \lambda\, ds_e^2$$

whence

$$T^*\, ds^2 = \tilde{\lambda}\, ds_e^2$$

with $\tilde{\lambda} := \lambda\sqrt{\tilde{a}\tilde{c} - \tilde{b}^2}$, $\tilde{a} := a \circ T$, $\tilde{b} := b \circ T$, $\tilde{c} := c \circ T$.

Now one can proceed for $\tilde{X} := \overline{X} \circ T$ as above. Alt's method to exclude true branch points (worked out in detail by D. Wienholtz [1,2]) eventually uses the same contradiction argument as Gulliver, namely to derive the existence of an energy minimizer $\tilde{X} \in \mathcal{C}(\Gamma)$ with a discontinuous normal $\tilde{N}$. The construction of $\tilde{X}$ is different from Gulliver's approach. Alt defines a new surface $\overline{X}$ on B_ϵ which is quasiconformal, and by reparametrization a new surface $\tilde{X} = \overline{X} \circ \tau$ is obtained which is energy minimizing with respect to its boundary values. Here Morrey's lemma on ϵ-conformal mappings is used as well as an elaboration of Lemma 9.3.3 in Morrey [8].

The nonexistence of false branch points for solutions X of Plateau's problem was proved by R. Gulliver [2], H.W. Alt [2], and then by Gulliver, Osserman, and Royden in their fundamental 1973-paper [1]. Here one only needs that $X|_{\partial B}$ is $1-1$, and this observation is used by Alt as well as by Gulliver, Osserman, and Royden, while Gulliver also employs the minimizing property of X. K. Steffen pointed out to us that Osserman's original paper [12] already contains significant contributions to the problem of excluding false branch points, and it even is satisfactory if, for some reason, an inner point of X cannot lie on the boundary curve Γ, say, if Γ lies on the surface of a convex body.

Furthermore, in Section 6 of their paper, Gulliver, Osserman, and Royden proved a rather general result on branched surfaces $X : \overline{B} \to \mathbb{R}^n$, $n \geq 2$, such that $X|_{\partial B}$ is injective, which implies the following: *A minimal surface $X \in \mathcal{C}(\Gamma)$ has no false boundary branch points* (see [1], pp. 799–809, in particular Theorem 6.16).

In 1973, R. Gulliver and F.D. Lesley [1] published the following result which we cite in a slightly weaker form: *If Γ is a real analytic and regular contour in $\mathbb{R}^3$, then any area minimizing minimal surface in $\mathcal{C}(\Gamma)$ has no boundary branch points.*

To prove this result they extend a minimizer X across the boundary of the parameter domain B as a minimal surface, so that a branch point w_0 on ∂B can be treated as an inner point. Then the same analysis of X in a small neighbourhood of w_0 can be carried out, and w_0 is either seen to be false or true. To exclude the possibility of a true branch point, they apply the method from Gulliver's paper [2], except that a new discontinuous "Osserman-type" mapping G is described, which is appropriate for this situation. In a different way, true boundary branch points for analytic Γ were excluded by B. White [24], see below.

The elimination of the possibility of false branch points in the Gulliver–Lesley paper is achieved by using results from the theory of "*branched immersions*", created by Gulliver, Osserman, and Royden.

The theory of branched immersions was extended by Gulliver [4,5,7] in such a way that it applies to surfaces of higher topological type (minimal surfaces and H-surfaces in a Riemannian manifold).

K. Steffen and H. Wente [1] showed in 1978 that minimizers of

$$E_Q(X) := \int_B \left[\frac{1}{2} |\nabla X|^2 + Q(X) \cdot (X_u \wedge X_v) \right] du\, dv$$

in $\mathcal{C}(\Gamma)$ subject to a volume constraint $V(X) = \text{const}$ with

$$V(X) := \frac{1}{3} \int_B X \cdot (X_u \wedge X_v)\, du\, dv$$

have no interior branch points. While their treatment of true branch points essentially follows Osserman [12], they simplified, in their special situation, the discussion of false branch points by Gulliver, Osserman, and Royden [1] and Gulliver [4].

In 1980, Beeson [2] showed that a minimal surface in $\mathcal{C}(\Gamma)$, given by a local Weierstrass representation, cannot have a true interior branch point if it is a C^1-local minimizer of D in $\mathcal{C}(\Gamma)$. (According to D. Wienholtz, Beeson's proof does not work for C^k-local minimizers with $k \geq 2$.) Motivated by the discovery of forced Jacobi fields, Beeson achieved this result by arguing that some first non-vanishing derivative must be negative.

Later on, in 1994, M. Micaleff and B. White [1] excluded the existence of true interior branch points for area minimizing minimal surfaces in a Riemannian 3-manifold, and in 1997, B. White [24] proved that an area minimizing minimal surface $X : \overline{B} \to \mathbb{R}^n, n \geq 3$, cannot have a true branch point on any part of ∂B which is mapped by X onto a real analytic portion of Γ, even if $n \geq 4$. This is quite surprising as X may have interior branch points if $n \geq 4$ (Federer's examples). However, White pointed out that, for any $k < \infty$, one can find C^k-curves Γ in $\mathbb{R}^4$ that bound area minimizing disk-type minimal surfaces with true boundary branch points, and Gulliver [11] found a C^∞-curve in $\mathbb{R}^6$ bounding an area minimizer with a true boundary branch point.

It is a major open question to decide whether or not an area minimizing minimal surface of disk-type in $\mathbb{R}^3$ *can have a boundary branch point assuming that it is bounded by a (regular)* C^k*- or* C^∞*-contour* Γ*, rather than by an analytic one.*

We furthermore mention the paper of H.W. Alt and F. Tomi [1] where the nonexistence of branch points for minimizers to certain free boundary problems is proved (see also Section 1.9 of this volume, Theorem 5), and the work of R. Gulliver and F. Tomi [1] where the absence of interior branch points for minimizers of higher genus is established. Specifically, they showed that such a minimizer $X : M \to N$ cannot possess false branch points if X induces an isomorphism on fundamental groups.

In 1977–81, R. Böhme and A. Tromba [1,2] showed that, *generically*, every smooth Jordan curve in $\mathbb{R}^n$, $n \geq 4$, bounds only immersed minimal surfaces, and admits only simple interior branch points for $n = 3$, but no boundary branch points. "Generic" means that there is an open and dense subset in the space of all sufficiently smooth $\alpha : S^1 \to \mathbb{R}^n$ defining a Jordan curve Γ, for which subset the assertion holds. This result is based on the Böhme–Tromba index theory, which is presented in Vol. 3.

A completely new method to exclude the existence of branch points for *minimal surfaces in* $\mathbb{R}^3$ *which are weak relative minimizers of* D was developed by A.J. Tromba [11] in 1993 by deriving an *intrinsic third derivative of* D *in direction of forced Jacobi fields.* He showed that if $X \in \mathcal{C}(\Gamma)$ has only simple interior branch points satisfying a *Schüffler condition* (a condition which by K. Schüffler [2] had been identified as generic), then the third variation of D can be made negative, while the first and second derivatives are zero, and so X cannot be a weak relative minimizer of D in $\mathcal{C}(\Gamma)$. D. Wienholtz in his Doctoral thesis [3] generalized Tromba's method to interior and boundary branch points of arbitrary order, satisfying a "Schüffler-type condition", by computing the third derivative of D in suitable directions generated by forced Jacobi fields. This work of Tromba and Wienholtz is described in Sections 6.1 and 6.3. We note that Wienholtz's results also refer to boundary branch points of minimal surfaces in $\mathbb{R}^n, n \geq 3$, but they do not apply to Gulliver's $\mathbb{R}^6$-example (see Wienholtz [3], p. 244). In forthcoming work by Tromba it will be shown how the ideas presented in Sections 6.1 and 6.2 can be used to exclude interior branch points for absolute minimizers of A in $\mathcal{C}(\Gamma)$.